Order, Algebra, and Structure:
Lattice-Ordered Groups and Beyond

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von
Almudena Colacito
von Italien

Leiter der Arbeit:
Prof. Dr. George Metcalfe
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Bern, July 2020 Almudena Colacito

Contents

INTRODUCTION

This thesis describes and examines some remarkable relationships existing between seemingly quite different properties—algebraic, order-theoretic, and structural—of ordered groups. On the one hand, it revisits the foundational aspects of the structure theory of lattice-ordered groups, contributing a novel systematization of its relationship with the theory of orderable groups. On the other hand, it branches off in new directions, probing the frontiers of several different areas of current research. More specifically, one of the main goals of this thesis is to suitably extend results that are proper to the theory of lattice-ordered groups to the realm of more general, related algebraic structures.

The interplay between order theory, algebra and structure theory will be a recurring theme in this thesis. One of the main contributions in this direction is a connection between validity in varieties of lattice-ordered groups, and orders on groups. A framework is also provided that allows for a systematic account of the relationship between orders and preorders on groups, and the structure theory of lattice-ordered groups. It has long been known that orders on groups and lattice-ordered groups may be viewed as two sides of the same coin. However, the results obtained in this thesis provide novel paradigmatic ways to study these connections, yielding various applications (e.g., decidability, orderability, generation results).

The second part of the thesis is concerned with more general algebraic structures; namely, distributive lattice-ordered monoids and residuated lattices. The theory of lattice-ordered groups provides the main source of inspiration for this thesis' contributions on these topics. Ordered groups also play a prominent role in the development of the algebraic study of logic. Although this connection is not pursued in this thesis, our interest in algebraic logic has prompted much of the research presented here.

THE STRUCTURE OF LATTICE-ORDERED GROUPS. The theory of lattice-ordered groups grew out of the groundbreaking work of Otto Hölder in 1901 on Archimedean ordered groups of magnitudes ([87]). Hölder's paper was followed in 1907 by Hans Hahn's fundamental article on ordered groups that may fail the Archimedean property ([82]). Lattice-ordered groups whose order is not necessarily total were studied by Frigyes Riesz, Hans Freudenthal, and Leonid Kantorovič, amongst others, with motivations coming from analysis. In this body of work from the 20's to the 40's, the underlying group was most often assumed to be Abelian. At the same time, in the United

States of America, Garret Birkhoff started a systematic investigation of not necessarily Abelian lattice-ordered groups ([11]) from the perspective of his newly created theory of general algebraic structures, nowadays called universal algebra. It was not until 1963, with Charles Holland's paper on lattice-ordered groups of order-automorphisms of a totally ordered set ([88]), that a substantial structure theory for general lattice-ordered groups began to emerge. Holland was a student of Paul Conrad, himself one of the most influential figures in the study of lattice-ordered groups. By 1970, when Conrad's fundamental paper on free lattice-ordered groups appeared ([40]), ordered groups had been established as an important area of research in mathematics.

A lattice-ordered group (briefly, ℓ-group) is an algebraic structure that consists of a group equipped with the binary meet and join operations of a lattice order compatible with the group multiplication. Compatibility means that the order relation is preserved by multiplication on the left and right; equivalently, the group operation distributes on both sides over meet and join. The additive group of continuous real-valued functions on any space, ordered pointwise, is an example of an Abelian ℓ-group. It is this class of examples that made Abelian ℓ-groups relevant to functional analysis in the first half of the twentieth century, as mentioned above. The group $\mathrm{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms of the reals can be lattice ordered pointwise. More generally, the group of order-preserving bijections $\mathrm{Aut}(\Omega)$ of any totally ordered set (briefly, chain) Ω ordered pointwise is an ℓ-group. Indeed, any ℓ-group embeds (as a sublattice subgroup) into such an ℓ-group for some chain. This result was proved in Holland's 1963 paper ([88]), and we refer to it as 'Holland's representation theorem'. The class of all ℓ-groups forms a variety (equivalently, an equationally definable class) in the sense of Birkhoff. This variety is generated by the ℓ-group $\mathrm{Aut}(\mathbb{R})$ (alternatively, $\mathrm{Aut}(\mathbb{Q})$); equivalently, the equational laws satisfied by the class of all ℓ-groups coincide with those valid just in $\mathrm{Aut}(\mathbb{R})$ (or $\mathrm{Aut}(\mathbb{Q})$). For example, it is clear that $\mathrm{Aut}(\mathbb{R})$ is distributive as a lattice. Therefore, the underlying lattice of any ℓ-group is distributive.

When we move from the study of equational laws to attempts to develop a structure theory of ℓ-groups, difficulties begin to emerge. Birkhoff's theory of general algebra suggests that we look at congruences. Similarly to what happens for groups, congruences in ℓ-groups are uniquely determined by the equivalence class of the identity element. More precisely, congruences are in one-to-one correspondence with certain subalgebras, namely normal order-convex sublattice subgroups, known as ℓ-ideals. According to Birkhoff's theory, the completely meet-irreducible ℓ-ideals are of special importance and, relatedly, so are the subdirectly irreducible ℓ-groups. These are the ℓ-groups having a minimum non-trivial ℓ-ideal. Unfortunately, it has long been recognized that subdirectly irreducible ℓ-groups may have a highly complex structure. This leads to the realization that convex sublattice subgroups are more relevant than ℓ-ideals for the structure theory of ℓ-groups. Like congruences, convex sublattice subgroups of any ℓ-group form an algebraic lattice. Prime subgroups are the finitely meet-irreducible elements of this lattice. In the development of a structure theory of ℓ-groups, prime subgroups replace completely meet-irreducible

ℓ-ideals. The process that allows us to decompose an ℓ-group into simpler components through the use of prime subgroups hinges crucially on the concept of a right order on a group.

ALGEBRA AND ORDER: RIGHT ORDERS ON GROUPS. A total order on a group is called a right order if it is preserved by multiplication on the right. In 1959, Conrad was the first to uncover an intrinsic relationship between the theory of ℓ-groups and the theory of right-ordered groups ([34]). This remarkable interplay has been widely studied ever since (see, e.g., [40, 129, 128, 3, 25]). We provide here some examples. Every ℓ-group is right-orderable as a group. Further, any right-orderable group is the subgroup of an ℓ-group. The lattice order of any ℓ-group can be obtained as the intersection of right orders. Moreover, the collection of all right orders on the free group over a set leads to a representation of the free ℓ-group over the same set.

In Chapter 1 we focus on this relationship between right orders and ℓ-groups, and establish a correspondence between subsets that do not extend to right orders on free groups, and valid ℓ-group equations (Corollary 1.3.2). That the equational theory of ℓ-groups is decidable was first proved in 1979 by Charles Holland and Stephen McCleary ([90]). Corollary 1.3.2 yields a new proof of this result by considering the problem, studied in 2009 by Adam Clay and Lawrence Smith ([27]), of deciding whether a finite subset of a free group extends to a right order. A correspondence is also established between validity of equations in varieties of representable ℓ-groups (equivalently, validity in classes of totally ordered groups) and subsets of relatively free groups (i.e., groups that are free relative to some classes of groups) that extend to orders on the group (Theorem 1.4.1). These results have a foundational nature, in the sense that their proofs do not use Holland's representation theorem or any other structural result for ℓ-groups. We use instead ordering theorems for groups as the basic ingredient.

A TOPOLOGICAL VIEW ON ORDER AND STRUCTURE. That right-orderable groups play a role in topological dynamics is indicated by the folklore fact that right-orderable countable groups are precisely those acting faithfully on the real line by orientation-preserving homeomorphisms. Decades after right orders became central to the theory of ℓ-groups, a new tool for the study of right-orderable groups in topological dynamics was introduced. At the beginning of the twenty-first century, Étienne Ghys and Adam Sikora independently topologized the set of right orders on a group, and studied the resulting topological space ([160]; cf. [69]), which is proved to be compact, Hausdorff, and zero-dimensional. Sikora's paper 'Topology on the spaces of orderings of groups', in particular, pioneered a new perspective on the interplay between topology and ordered groups. The topological space of right orders has been put to great use, for instance, by Dave Witte Morris in [136], where right orders on groups are applied to the study of amenable groups.

The use of topology in the theory of ℓ-groups goes back to the second half of the twentieth century. It is a notable fact that a topological space can be associated to any ℓ-group by considering its spectral space. The spectral space of an Abelian ℓ-group was introduced by Klaus Keimel in his doctoral dissertation (1971), as the set of its prime ℓ-ideals equipped with the hull-kernel topology. The notion of spectral space is not limited to the commutative setting, and can also be defined for an arbitrary ℓ-group, by considering the collection of its prime subgroups.

What we show in Chapter 3 is that the topological space of right orders on a group arises naturally from the study of ℓ-groups, as the subspace of minimal elements of a spectral space. Concretely, we show that the space of right orders on a group emerges from the ℓ-group freely generated by the group via a suitable application of Stone duality (Corollary 3.5.12; Theorem 3.5.18). As a byproduct, we provide a systematic, structural account of the relationship between right (pre)orders on a group and prime subgroups of ℓ-groups (Theorem 3.3.6). The connection we exhibit was previously identified in its basic form by McCleary in his paper on representations of free ℓ-groups by ordered permutation groups ([128]). The framework developed in Chapter 3, which may be viewed as a generalization and extension of McCleary's result, leads to a mathematically transparent description of how right orders on a group influence the structure of the ℓ-group freely generated by the group.

FORGETTING THE INVERSE: DISTRIBUTIVE LATTICE-ORDERED MONOIDS. Cayley's theorem for groups can be generalized to the context of semigroups and monoids in an obvious way: every monoid is isomorphic to a monoid of transformations of some set. Pursuing an analogous generalization in the setting of ℓ-groups, we consider order-preserving endomorphisms of chains. The monoid of such endomorphisms ordered pointwise is a distributive lattice-ordered monoid (briefly, distributive ℓ-monoid), in the sense that the monoid operation distributes over meet and join, and that the lattice reduct is distributive. Many of the important examples of ℓ-groups admit significant extensions to the monoid setting. For instance, given any topological space with a preorder, the set of bounded continuous monotone functions from the space to the real line, with monoid and lattice operations defined pointwise, is a commutative distributive ℓ-monoid.

In a paper from 1984, Marlow Anderson and Constance Edwards showed that any distributive ℓ-monoid is an ℓ-monoid of order-preserving endomorphisms on a chain ([2]), thereby extending Holland's representation theorem. One consequence is that the variety of distributive ℓ-monoids is generated by the class of ℓ-monoids of order-preserving endomorphisms on chains. We refine this result by proving that the variety of distributive ℓ-monoids is generated by the class of ℓ-monoids of order-preserving endomorphisms of finite chains (Theorem 4.2.2). Using the fact that every member of this class is finite, we show that the equational theory of distributive ℓ-monoids is decidable (Corollary 4.2.4).

The structure of distributive ℓ-monoids is not as well-understood as that of ℓ-groups, and the tools for a uniform treatment of these algebras are still lacking. In this thesis, we undertake a number of preliminary steps in this direction. As mentioned above, it is well-known that the class of right-orderable groups coincides with the class of subgroups of ℓ-groups. It may seem plausible that, analogously, the class of right-orderable monoids coincides with the class of submonoids of distributive ℓ-monoids. However, this is not the case. It is unclear at this stage what role right orders on monoids play in this theory. The relationship between distributive ℓ-monoids and right orders is briefly studied in this thesis, in the form of a correspondence between validity in distributive ℓ-monoids, and right orders on free monoids (Theorem 4.4.5; Theorem 4.4.6). A remarkable consequence of this is a neat connection between right orders on free monoids and right orders on free groups (Corollary 4.4.7).

The main contribution of this thesis to the theory of distributive ℓ-monoids is the result that the equational theory of ℓ-groups is a conservative extension of the equational theory of distributive ℓ-monoids (Theorem 4.4.3). This is especially interesting in view of the fact that, in contrast to the situation for ℓ-groups, finite distributive ℓ-monoids exist in abundance—indeed, as already mentioned, they generate the whole variety. Remarkably, an analogous result cannot be obtained for commutative distributive ℓ-monoids: this follows from a theorem of Vladimir Repnitskii, who in 1983 proved that the variety generated by the ordered monoid of integers is not finitely based ([149]). We extend this negative result here, showing that the same fact does not specialize to representable structures (Theorem 4.3.6). (In line with ℓ-group terminology, a distributive ℓ-monoid is representable if it is a subdirect product of totally ordered monoids.)

FROM LATTICE-ORDERED GROUPS TO RESIDUATED LATTICES. The notion of a residuated lattice was introduced by Morgan Ward and Robert Dilworth in 1939 ([173]). For any ring with unit, the set of its two-sided ideals partially ordered by inclusion forms a complete lattice, where meet is the intersection and join is the ideal addition. This lattice can be naturally equipped with monoid and residual operations, and was one of the motivations behind Ward and Dilworth's original notion of a residuated lattice. The modern notion studied here differs slightly from Ward and Dilworth's, and goes back to the work of Kevin Blount and Constantine Tsinakis ([12]). Residuated lattices provide a common abstraction of several fundamental classes of algebras: Boolean algebras, Heyting algebras, relation algebras, MV-algebras, and ℓ-groups, amongst others. This list alone suggests that residuated lattices are intimately related to logic.

A number of studies have provided compelling evidence of the importance of ℓ-groups in the investigation of residuated lattices. The term 'Conrad Program' traditionally refers to the approach that Conrad favoured in the study of ℓ-groups. The approach advocates the use of lattice-theoretic properties of the lattice of convex ℓ-subgroups in order to extract structural information about classes of ℓ-groups. Large parts of the Conrad Program can be profitably extended to the much wider class of e-cyclic residuated lattices (see, e.g., [115, 70, 15, 71, 116]), which includes all residuated lattices that are cancellative, divisible, or commutative. In this thesis, we use tools and results from the theory of ℓ-groups to obtain analogous results for cancellative residuated lattices.

The notion of nilpotency and the Hamiltonian property, both arising in the context of group theory, admit natural generalizations to the setting of ℓ-groups. The connection between nilpotent and Hamiltonian ℓ-groups goes beyond the fact that they can be seen as generalizations of Abelian ℓ-groups. Indeed, the most relevant facts about these two classes of structures are intertwined (see, e.g., [109, 92, 148, 13]). The Hamiltonian property has been more recently considered in the context of residuated lattices ([15]). A suitable notion of nilpotent residuated lattice is introduced in this thesis, as a natural generalization of the concept of nilpotent ℓ-group. The first main contribution of this thesis to the theory of residuated lattices is to extend relevant properties of nilpotent and Hamiltonian ℓ-groups (e.g., representability, existence of largest variety, failure of amalgamation) to nilpotent and Hamiltonian residuated lattices that are cancellative. In the absence of cancellativity, very little is known.

As we already pointed out, residuated lattices encompass a wide array of disparate mathematical structures. Therefore, it should not come as a surprise that neat descriptions of free objects are hard to come by. Motivated by the search for a better description of free objects, at least under some additional assumptions, we establish generation results for varieties of representable cancellative residuated lattices. (In line with ℓ-group terminology, a residuated lattice is representable if it is a subdirect product of residuated chains.) Inspired by analogous results holding for ℓ-groups, we show that certain varieties of representable cancellative residuated lattices are generated by totally ordered relatively free monoids (Theorem 5.6.5). These results are powerful tools, as a relatively free monoid endowed with a total order is a much simpler object than a free residuated lattice.

LOGIC AND ORDERED GROUPS. Substructural logics are weaker than classical logic, in that they may lack one or more of the structural rules of contraction, weakening and exchange in their Gentzen-style axiomatization. They encompass a large number of non-classical logics related to computer science, linguistics, philosophy, and many-valued reasoning. Residuated lattices are the algebraic counterparts of propositional substructural logics. We already remarked that ℓ-groups are an essential source of inspiration in the development of the theory of residuated lattices. In the past fifty years, ℓ-groups have also acquired increasing importance from a logical point of view. For example, a central result in the theory of MV-algebras is the categorical equivalence between the category of MV-algebras and the category of unital Abelian ℓ-groups ([138]). MV-algebras, unlike ℓ-groups, are a direct offspring of non-classical logic; they were introduced as the algebraic counterpart of Łukasiewicz many-valued logic.

From a logical perspective, the theory developed in Chapter 1 points to a connection between orderable groups and proof theory. In this sense, Chapter 1 can be seen as a prologue to the rest of the thesis, as some of the ordering theorems for groups contained therein have been implicitly motivated by proof-theoretic investigations. This thesis refrains from developing that connection. The interested reader can consult [31] and [29] for details. However, we offer some further remarks on the relationship between this research program and results in the thesis. Indeed, the ordering theorems discussed here, inductively characterizing when partial orders on a group extend to total orders, were used in [31] (cf. [29]) to devise proof systems for varieties of ℓ-groups, related to calculi already existing in the literature. In this regard, we mention the hypersequent calculi for Abelian ℓ-groups and related varieties obtained in 2005 by George Metcalfe, Nicola Olivetti and Dov Gabbay ([132]). Further, a successful attempt to obtain an analytic proof system for (non-Abelian) ℓ-groups is due to Nikolaos Galatos and George Metcalfe ([67]). However, the completeness proofs in all these papers are largely syntactic, using cut elimination or restricted quantifier elimination.

The search for uniform algebraic completeness proofs for analytic sequent and hypersequent calculi with respect to varieties of residuated lattices has been especially successful in recent years (e.g., [21, 66, 131, 133, 134]). However, the proposed methods, to which we shall refer as 'algebraic proof theory', do not encompass 'ordered group-like' structures, e.g., MV-algebras, varieties of cancellative residuated lattices. It was shown in a recent paper by Nikolaos Galatos and Peter Jipsen ([65]) that a similar successful treatment can be developed for varieties of distributive ℓ-monoids. We show here that the validity of any ℓ-group equation can be reduced to the validity of (finitely many) suitable 'inverse-free' equations (Theorem 4.4.11). This suggests a way to import tools from the proof theory of distributive ℓ-monoids into the theory of ℓ-groups. Therefore, the results obtained in this thesis, relating equational laws valid in ℓ-groups to those valid in distributive ℓ-monoids, have the potential to lead to the first algebraic proof of cut elimination for a Gentzen-style calculus for ℓ-groups and, more generally, to the development of a framework for a systematic study of the proof theory for ℓ-groups.

HOW TO READ THIS THESIS

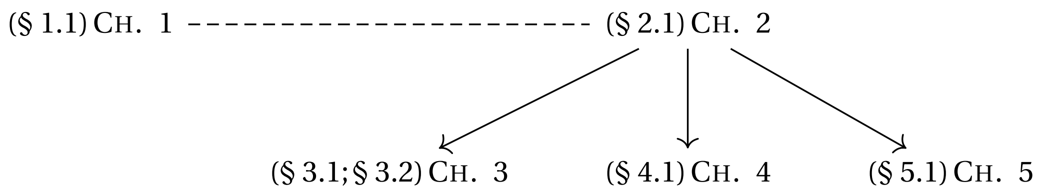

Figure 1: This diagram illustrates dependences and relations between the chapters. The parentheses refer to the sections where preliminaries can be found.

OVERVIEW OF THIS THESIS

We now describe the content of this thesis in detail. The first three chapters focus on the theory of ℓ-groups from the point of view of, respectively, order, algebra, and structure.

In CHAPTER 1, we begin with a brief overview of the relevant mathematical background on right-ordered groups, including some motivational examples and results from the literature. Algebraic (right) orderability criteria are described that stem from syntactic investigations of ℓ-groups; such inductive characterizations of subsets of groups that extend to (right) orders are the main focus of Section 1.2. These criteria are then used to show that (non-)valid equations in certain varieties of ℓ-groups correspond to subsets of right-orderable groups that extend to (right) orders. This correspondence, established in Section 1.3, leads to a new proof of the decidability of the equational theory of ℓ-groups. A correspondence is also established in the final section between validity of equations in varieties of representable ℓ-groups and subsets of relatively free groups that extend to orders on the group.

CHAPTER 2 is intended as a bridge between the ideas developed in Chapter 1 and the rest of the thesis. The goal of the chapter is to revisit from an algebraic perspective the results in the previous chapter. That is, in Section 2.2 the main theorems in Section 1.3 and Section 1.4 are obtained by using ℓ-group theory and general algebra. This algebraic account of the results allows for a broader analysis, encompassing classes of ℓ-groups that were not covered by the methods from the previous chapter. More precisely, the correspondence between order and validity is extended in Section 2.3 to the varieties of normal-valued and weakly Abelian ℓ-groups. The reader can also refer to Section 2.1 for a brief overview of the relevant mathematical background on ℓ-groups, which will be useful throughout the thesis.

The interplay between the theory of ℓ-groups and the theory of right-orderable groups is largely the focus of the first two chapters. This analysis is deepened in CHAPTER 3, where topology enters the stage. This chapter provides a systematic, structural account of the relationship between right (pre)orders on groups and prime subgroups of ℓ-groups. The topological space of right orders on a group is showed to arise naturally from the study of the ℓ-group freely generated by the group, as a subspace of its spectrum. The main correspondence result is developed and established in Section 3.3. As a consequence, we obtain a mathematically transparent description of how right orders on a group influence the structure of the ℓ-group freely generated by the group. Concretely, it is proved that the space of right orders on a group emerges from the ℓ-group freely generated by the group via a suitable application of Stone duality. The necessary background on the topological space of right orders on a group and on spectra of ℓ-groups is provided in Section 3.1 and Section 3.2, respectively.

In the remaining chapters, ℓ-groups are no longer the focus of the analysis, even though they still play a central role inspiring, conceptually and technically, the development of the theory. The two final chapters deal with two different generalizations of ℓ-groups; namely, distributive lattice-ordered monoids and cancellative residuated lattices, respectively.

CHAPTER 4 begins with a brief overview of the relevant mathematical background on distributive ℓ-monoids. The first preliminary section immediately highlights a few obvious similarities between distributive ℓ-monoids and ℓ-groups. However, in Section 4.2, distributive ℓ-monoids are proved to satisfy the finite model property—while all non-trivial ℓ-groups are infinite. The study of the interplay between classes of distributive ℓ-monoids and suitable classes of ℓ-groups is central throughout the chapter. The equations satisfied by distributive ℓ-monoids, in the language of the latter, are proved to be exactly the same equational properties satisfied by ℓ-groups. This is the main result, and can be found in Section 4.4. The interplay between algebra and order in the context of distributive ℓ-monoids is studied in the same section, where validity of equations in the variety of distributive ℓ-monoids is proved to correspond to subsets of the free monoid that extend to (right) orders. Subvarieties of distributive ℓ-monoids are studied in Section 4.3, where an axiomatization for the variety generated by totally ordered monoids is also provided.

In the final chapter of this thesis, CHAPTER 5, we study two classes of residuated lattices that extend and generalize commutative residuated lattices. The main goal of the chapter is to obtain for these classes representation and generation results that suitably generalize the analogous results holding in the setting of ℓ-groups. In particular, we show in Section 5.2 that nilpotent cancellative residuated lattices can be seen as nilpotent ℓ-groups endowed with a suitable modal operator. In Section 5.5, nilpotent cancellative residuated lattices are proved to share some similarities with commutative cancellative residuated lattices. First, they admit a simpler description for their congruences, as they are Hamiltonian. Second, the notion of representability collapses, over nilpotent cancellative residuated lattices, to the simpler notion of prelinearity. Total orders on monoids play a central role in this chapter as well, since any variety of representable cancellative integral residuated lattices defined by monoid equations is generated by those residuated totally ordered monoids which are finitely generated as monoids; in particular, it suffices to consider finitely generated relatively free monoids. These generation results are the main contribution of Section 5.6, which concludes the chapter.

Since most of the results obtained in the last two chapters grew out of analogous results for ℓ-groups, we emphasize throughout these chapters similarities and discrepancies between the different contexts.

SOURCES OF THE MATERIAL

Most of the results in this thesis have been obtained in collaboration with other researchers:

CHAPTER 1 is based on the paper [32], joint with George Metcalfe.

CHAPTER 2 is written for the purpose of this thesis.

CHAPTER 3 is based on the paper [30], joint with Vincenzo Marra.

CHAPTER 4 is partially based on joint work with Nikolaos Galatos and George Metcalfe.

CHAPTER 5 is based on the manuscript [33], jointly written with Constantine Tsinakis.

Some results and ideas are also taken from [31] (joint work with George Metcalfe) and [29] (joint work with Nikolaos Galatos and George Metcalfe).

This thesis is not meant to be self-contained, and we assume some familiarity with basic concepts from universal algebra, order theory, and topology. Universal algebra plays a central role, and the reader can find all the needed standard results in [18]. However, given that the notions and results appearing in this thesis range from order theory to topology and category theory, we have chosen to include some background in an appendix.

CHAPTER 1

A SYNTACTIC APPROACH TO ORDERS ON GROUPS

In 1959, Paul Conrad's paper 'Right-ordered groups' was the first to uncover an intrinsic relationship between ℓ-groups and groups equipped with a total order compatible with right multiplication. This remarkable interplay has been widely studied ever since, and examples of this interdependence can be found everywhere in the literature of both fields, e.g., every ℓ-group is right-orderable as a group, and any right-orderable group is the subgroup of an ℓ-group; also, the collection of all right orders on the free group over a set X leads to a representation of the free ℓ-group over X. (This perspective is treated in Chapters 2 and 3.)

The first aim of this chapter is to establish a correspondence between validity of equations in ℓ-groups and subsets of free groups that extend to right orders on the group. Here, $S \subseteq G$ is said to 'extend to a right order' on the group G provided that G admits a right order where every element of S is strictly positive. Throughout, we often write $s \leq t$ for the equation $s \wedge t \approx s$. We prove here that a finite subset $\{t_1, \ldots, t_n\}$ of a free group $F(X)$ extends to a right order if and only if the equation $\mathrm{e} \leq t_1 \vee \ldots \vee t_n$ is not valid in the variety of all ℓ-groups (Corollary 1.3.2). We then make use of the correspondence to obtain a new proof of the decidability of the equational theory of ℓ-groups, by appealing to an algorithm ([27]) that recognizes when a finite subset of a finitely generated free group extends to a right order.

The correspondence may be established using a theorem by Herbert A. Hollister ([91]; cf. [40, Theorem I]) that the lattice order of an ℓ-group is the intersection of right orders on its group reduct (see Section 2.2). However, we use here instead an inductive characterization of subsets of groups that extend to right orders (Theorem 1.2.2, closely related to a theorem of Conrad [34]). A correspondence is also established between validity of equations in varieties of representable ℓ-groups (equivalently, classes of totally ordered groups) and subsets of relatively free groups that extend to orders (sometimes called 'bi-orders') on the group.

We begin with an introduction to right orders and orders on groups, enriched by some relevant examples. This brings us to Section 1.2, where we develop the main tools for proving the correspondence results of Sections 1.3 and 1.4: namely, we provide ordering theorems for groups that stem from proof-theoretic investigations into ℓ-groups (see Section 1.5), and require very little structural theory for these algebras. The arguments rely on the following intuition: in constructing a proof of $e \leq t_1 \vee \ldots \vee t_n$, we obtain a (syntactic) description of the reason *why* the set $\{t_1, \ldots, t_n\}$ cannot be extended to a right order on $F(X)$ (Theorem 1.3.1).

This chapter is based on the paper [32]. The theory and terminology from order theory used in this chapter is reviewed in Appendix A.2.

1.1 WHEN CAN A GROUP BE RIGHT-ORDERED?

In this section, we introduce the concept of a right-orderable group, and briefly review the related literature. When possible, we refer to the original sources. Several examples and results included here will come up again later in this thesis.

Let G be a group. A partial order $\leq \subseteq G \times G$ is said to be right-invariant (resp., left-invariant) if for all $a, b, c \in G$, whenever $a \leq b$ then $ac \leq bc$ (resp., $ca \leq cb$). A right-invariant partial order on G is called a *partial right order on G*, and a *right order on G* if it is also total. If a group admits a right order, we call it *right-orderable*.[1] It is easy to see that the set of partial right orders on G is in one-to-one correspondence with the set of subsemigroups of G that omit e, via the map that associates to any such semigroup $C \subseteq G$ the relation: $a \leq_C b$ if and only if $ba^{-1} \in C \cup \{e\}$, for $a, b \in G$. The inverse of this bijection sends a partial right order $\leq$ to its *strict positive cone*

$$C = \{a \in G \mid e < a\}.$$

In this chapter, we deliberately confuse a partial right order on G with its strict positive cone. Right orders are identified with those semigroups omitting e and such that $C \cup C^{-1} = G \setminus \{e\}$. We write $(G, \leq)$ for a group G equipped with a right order $\leq$; equivalently, we write G_C for the group G endowed with the right order with strict positive cone $C \subseteq G$.

Remark 1.1.1. In this chapter, it is more convenient to identify a (partial) right order with its strict positive cone. However, it would be also possible to identify a partial right order $\leq$ on a group G with its *positive cone*

$$\{a \in G \mid e \leq a\}.$$

This too gives a bijection, whose inverse sends any submonoid C such that $C \cap C^{-1} = \{e\}$ to the partial right order defined by $a \leq_C b$ if and only if $ba^{-1} \in C$, for all $a, b \in G$. From this perspective, right orders correspond to all those partial right orders satisfying $C \cup C^{-1} = G$. This is the standpoint chosen in Chapter 3.

Right-orderable groups are torsion-free, i.e., for all $a \in G$ and $n \in \mathbb{N}^+$, $a^n = e$ only if $a = e$. The following result is due to Levi.

[1]We focus on right orders in this thesis, motivated by our interest in lattice-ordered groups; other authors prefer left orders, the difference being immaterial.

Proposition 1.1.2 ([117])**.** *An Abelian group is right-orderable if and only if it is torsion-free.*

Recall that a *lattice-ordered group* (briefly, *ℓ-group*) is an algebraic structure H, with operations $\cdot, \wedge, \vee, ^{-1}, \mathrm{e}$ such that $(H, \cdot, ^{-1}, \mathrm{e})$ is a group, $(H, \wedge, \vee)$ is a lattice, and the group operation distributes over the lattice operations, i.e., the following equations hold:

$$z(x \wedge y)w \approx zxw \wedge zyw$$

$$z(x \vee y)w \approx zxw \vee zyw.$$

Therefore, the class of all ℓ-groups is a variety, denoted by LG. The set of order-preserving bijections $\mathrm{Aut}(\Omega)$ of any chain Ω can be made into an ℓ-group, with group operation $f \cdot g$ defined as $g \circ f$, and pointwise lattice operations.

Proposition 1.1.3 ([34])**.** *A group G is right-orderable if and only if G acts faithfully on a chain by order-preserving bijections.*

Proof Sketch. For any right order C on the group G, the *right regular representation* of G_C

$$\begin{aligned} G &\xrightarrow{R_C} \mathrm{Aut}(G_C) \\ a &\longmapsto R_C(a)\colon b \mapsto ba \end{aligned} \tag{1.1}$$

is an embedding of G into the group $\mathrm{Aut}(G_C)$ of order-preserving bijections of G_C. On the other hand, every group $\mathrm{Aut}(\Omega)$ of order-preserving bijections of a chain Ω admits a right order; it suffices to well-order the chain $\Omega = \{a_\beta \mid \beta \in \delta\}$ with order type δ, and define a right order by $\mathrm{id}_\Omega \prec f$ if and only if $a_\gamma <_\Omega f(a_\gamma)$, where $\gamma = \min\{\beta \in \delta \mid f(a_\beta) \neq a_\beta\}$. □

Proposition 1.1.4. *A countable group G is right-orderable if and only if G acts faithfully on the real line by orientation-preserving homeomorphisms.*

(For the details of the proof, see, e.g., [69, Theorem 6.8].)

Proof Sketch. From right to left, it suffices to consider a countable dense sequence $\{r_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$, and define $\mathrm{id}_\mathbb{R} \prec f$ if and only if $r_m < f(r_m)$, where $m = \min\{n \in \mathbb{N} \mid f(r_n) \neq r_n\}$. The resulting relation $\preceq$ is a right order. Conversely, given a right order $\preceq$ on G, we can embed $(G, \preceq)$ into $\mathbb{R}$ as a (countable) chain (e.g., using a 'back-and-forth' argument). The group G acts naturally on the resulting subchain of $\mathbb{R}$, and hence to each element of G there is associated a different partial order-embedding of $\mathbb{R}$; such an action is then extended to an action on $\mathbb{R}$. □

A partial right order on a group G that is also left-invariant is called a *partial order on G*, and an *order on G* if it is total. The one-to-one correspondence between partial right orders on G and subsemigroups of G that omit e restricts to a correspondence between partial orders on G and those semigroups omitting e that are also normal, i.e., closed under group conjugation. If a group admits an order, we call it *orderable*.

Proposition 1.1.5 ([51])**.** *Any right order on an Abelian group is an order. Moreover, if every right order on a right-orderable group G is an order, then G is Abelian.*

The next result was proved independently by several authors (e.g., [159], [100, Theorem 1], [141, Corollary 3.3], [167], [103]; see also [9]).

Proposition 1.1.6. *Every free group is orderable.*

For any partial (right) order $\leq$ on a group G, its dual order $\leq^{\partial}$ (defined as in (A.42)) is also a partial (right) order on G; if C is the (strict) positive cone of $\leq$, then C^{-1} is the (strict) positive cone of $\leq^{\partial}$. A right order $\leq$ on a group G is called *Archimedean* if for any $a, b > \mathrm{e}$, there exists $n \in \mathbb{N}^{+}$ such that $b \leq a^{n}$.

The class of (right-)orderable groups is closed under taking direct products, isomorphisms, subgroups, and ultraproducts, and is therefore a quasivariety.

Example 1.1.7. The additive groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ are totally ordered by their usual orders. As a consequence, finitely generated free Abelian groups are orderable. Consider the free Abelian group $\mathbb{Z}^2$ over two generators—it is convenient to think of $\mathbb{Z}^2$ as embedded into $\mathbb{R}^2$.

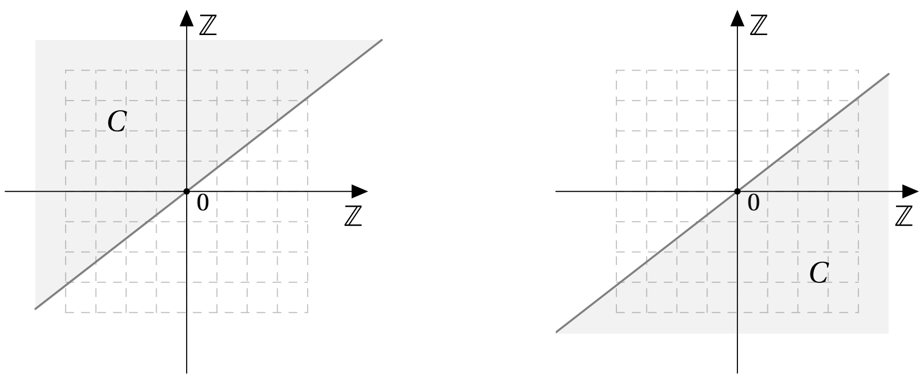

Figure 1.1: Two dual Archimedean orders on $\mathbb{Z}^2$

Easy computations confirm that every line $y = rx$, where $r \in \mathbb{R} \setminus \mathbb{Q}$, determines two (dual) Archimedean orders on $\mathbb{Z}^2$, depending on which half-plane is chosen to be the strict positive cone (see Figure 1.1). We identify every Archimedean order on $\mathbb{Z}^2$, determined by the line through the origin $y = rx$, with a point $(x, y) \in \mathbb{S}^1$ such that $\frac{x}{y} = r \in \mathbb{R} \setminus \mathbb{Q}$; its dual order is therefore identified with the point of $\mathbb{S}^1$ antipodal to (x, y). Similarly, every line $y = qx$, where $q \in \mathbb{Q}$, determines four orders on $\mathbb{Z}^2$. In fact, when r is rational, the resulting line has nontrivial intersection with $\mathbb{Z}^2$. Therefore, in order to obtain a (total) order, every point $a \in \mathbb{Z}^2$ on the line needs to be made either positive or negative. The resulting four options are illustrated in Figures 1.2 and 1.3.

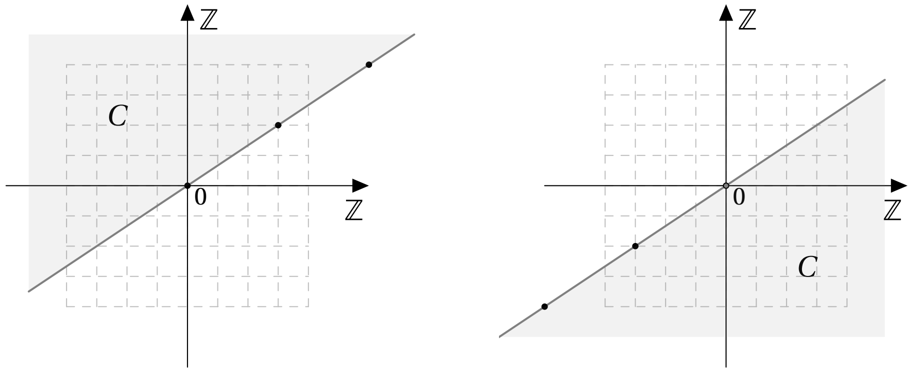

Figure 1.2: Two dual (lexicographic) orders on $\mathbb{Z}^2$

By the above reasoning, to every point $(x,y) \in \mathbb{S}^1$ such that $\frac{x}{y} = q \in \mathbb{Q}$, there remain associated two lexicographic orders induced by the line $y = qx$, whose duals are identified with the respective antipodal point on $\mathbb{S}^1$.

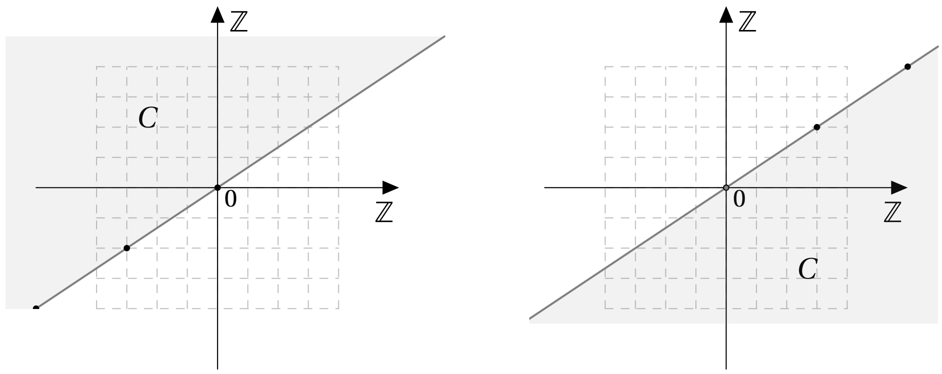

Figure 1.3: Two dual (lexicographic) orders on $\mathbb{Z}^2$

We have described precisely all the (right) orders on $\mathbb{Z}^2$; orders on $\mathbb{Z}^k$ for $k > 2$ can be described similarly. (The orders on $\mathbb{Z}^k$ were classified in [154] and [165].)

The example above describes an interesting classification of all the orders on $\mathbb{Z}^k$, for $k \geq 2$. Observe that, for instance, it is not possible to have a total order on $\mathbb{Z}^2$ that makes (x,y) and $(-nx,-ny)$ both strictly positive, for any $n \in \mathbb{N}^+$. In this case, it is standard to say that (x,y) and $(-nx,-ny)$ 'do not extend to' an order on $\mathbb{Z}^2$. For any group G and any given subset $S \subseteq G$, we say that S *extends to* a right order on G if there exists a strict positive cone C of a right order on G such that $S \subseteq C$; similarly, a subset $S \subseteq G$ extends to an order on G if $S \subseteq C$, for some strict positive cone C of an order on G. Also, a partial right order $\leq_1$ on G is said to *extend* the partial right order $\leq_2$ on G if $\leq_2 \subseteq \leq_1$; equivalently, if the strict positive cone C_2 of $\leq_2$ is included in the strict positive cone C_1 of $\leq_1$. We therefore say that a partial right order $\leq$ on a group

G extends to a (right) order on G if its strict positive cone C extends to a (right) order on G as a subset. The following two results are particularly interesting.

Proposition 1.1.8 ([118, 159, 62])**.** *Every partial order on a torsion-free Abelian group extends to an order.*

For the next result, see [123] (cf. [14, Theorem 2.2.4]; [5, Theorem 2], [151, Theorem 4]).

Proposition 1.1.9. *Every partial right order on a torsion-free nilpotent group extends to a right order. Further, every partial order on a torsion-free nilpotent group extends to an order.*

There is a substantial literature concerned with characterizing those subsets of a group that extend to (right) orders. A partial review of such literature is provided by the following results. For any group G, and any $S \subseteq G$, we write $\langle S \rangle$ for the subsemigroup of G generated by S, and $\langle\langle S \rangle\rangle$ for the normal subsemigroup of G generated by S. The following results emerge from straightforward applications of Zorn's Lemma.

Proposition 1.1.10 ([113, Lemma 3.1.1]; cf. [34, Theorem 2.2])**.** *A subset $S \subseteq G$ of a group G extends to a right order on G if and only if for every finite set of elements $\{a_1, \dots, a_n\} \subseteq G \setminus \{e\}$ there exist signs $\delta_1, \dots, \delta_n \in \{-1, 1\}$ such that*

$$e \notin \langle S \cup \{a_1^{\delta_1}, \dots, a_n^{\delta_n}\} \rangle.$$

Proposition 1.1.11 ([63, Theorem 1])**.** *A subset $S \subseteq G$ of a group G extends to an order on G if and only if for every finite set $\{a_1, \dots, a_n\} \subseteq G \setminus \{e\}$ there exist signs $\delta_1, \dots, \delta_n \in \{-1, 1\}$ such that*

$$e \notin \langle\langle S \cup \{a_1^{\delta_1}, \dots, a_n^{\delta_n}\} \rangle\rangle.$$

Remark 1.1.12. Note that, when G admits a right order, every $\{a\} \subseteq G \setminus \{e\}$ extends to a right order on G. Just observe that, if C is a right order on G and $a \notin C$, then $a \in C^{-1}$. Hence, a is positive in the dual order. The same is true for orderable groups, and orders.

Further examples of (right-)orderability conditions are the following.

Example 1.1.13. A group G is right-orderable if and only if for every $a \in G \setminus \{e\}$, there exists a partial right order P_a such that $a \in P_a$, and $G \setminus P_a$ is a subsemigroup of G ([34, Theorem 2.2]). Further, any partial right order P on a right-orderable group G such that $G \setminus P$ is a semigroup extends to a right order on G ([34, Theorem 2.3]).

Example 1.1.14. The analogue of Example 1.1.13 holds for orderable groups, and partial orders. Namely, a group G is orderable if and only if for every $a \in G \setminus \{e\}$, there exists a partial order P_a such that $a \in P_a$, and $G \setminus P_a$ is a subsemigroup of G ([143]). Further, any partial order P on an orderable group G such that $G \setminus P$ is a semigroup extends to an order on G.

1.2 ORDERING CONDITIONS: A SYNTACTIC PERSPECTIVE

This section is concerned with an inductive description of the set of subsets of a right-orderable group G that do not extend to a right order on G. Later, a similar description is provided of the subsets of an orderable group G that do not extend to orders on G. These descriptions share similarities with the right-orderability, and orderability conditions by, respectively, Conrad (see Example 1.1.13) and Ohnishi (see Example 1.1.14), and stem from proof-theoretic investigations of ℓ-groups. (For a broader discussion of the ideas underlying these results, see Section 1.5.)

Definition 1.2.1. For any group G, we define inductively for $n \in \mathbb{N}$:

$$\mathrm{R}_0(G) = \{S \subseteq G \mid S \cap S^{-1} \neq \emptyset\};$$
$$\mathrm{R}_{n+1}(G) = \mathrm{R}_n(G) \cup \{T \cup \{ab\} \mid T \cup \{a\}, T \cup \{b\} \in \mathrm{R}_n(G)\},$$

and set $\mathrm{R}(G) = \bigcup_{n\in\mathbb{N}} \mathrm{R}_n(G)$. That is, $\mathrm{R}(G)$ is the smallest subset of 2^G containing all subsets S such that $S \cap S^{-1} \neq \emptyset$, and with the property: if $T \cup \{a\}, T \cup \{b\} \in \mathrm{R}(G)$, then $T \cup \{ab\} \in \mathrm{R}(G)$.

Clearly, $\{\mathrm{R}_n(G)\}_{n\in\mathbb{N}}$ is an ascending chain of subsets of G. The set $\mathrm{R}(G)$ provides a tool for the study of the right-orderability of the group G, and it completely describes those subsets of G that (do not) extend to a right order on G.

Theorem 1.2.2. *A group G is right-orderable if and only if $\{a\} \notin \mathrm{R}(G)$ for all $a \in G \setminus \{\mathrm{e}\}$.*

Theorem 1.2.3. *For any right-orderable group G, the set $\mathrm{R}(G)$ consists precisely of those subsets of G that do not extend to a right order on G.*

In preparation for the proofs of Theorem 1.2.2 and Theorem 1.2.3, and in order to help the reader get acquainted with the intuition underlying $\mathrm{R}(G)$, here are a few examples.

Example 1.2.4. Observe that a subset of the additive group $\mathbb{Z}$ extends to a (right) order on $\mathbb{Z}$—of which there are just two: the standard order and its dual—if and only if it contains only strictly positive elements or only strictly negative elements. This should mean that $\mathrm{R}(\mathbb{Z})$ consists precisely of those subsets $S \subseteq \mathbb{Z}$ containing elements $m, n \in \mathbb{Z}$ such that $m < 0 < n$. For instance, consider $\{3,-5\} \subseteq \mathbb{Z}$. First, as $\{1,-1\} \in \mathrm{R}_0(\mathbb{Z})$, also $\{1,-2\} \in \mathrm{R}_1(\mathbb{Z})$. But $\{2,-2\} \in \mathrm{R}_0(\mathbb{Z}) \subseteq \mathrm{R}_1(\mathbb{Z})$, so $\{3,-2\} \in \mathrm{R}_2(\mathbb{Z})$; since $\{3,-3\} \in \mathrm{R}_0(\mathbb{Z}) \subseteq \mathrm{R}_2(\mathbb{Z})$, it follows that $\{3,-5\} \in \mathrm{R}_3(\mathbb{Z}) \subseteq \mathrm{R}(\mathbb{Z})$.

Remark 1.2.5. Since any $S \subseteq G$ occurring in $\mathrm{R}(G)$ must occur in $\mathrm{R}_n(G)$ for some $n \in \mathbb{N}$, there exists in this case a finite binary tree of subsets of G with root S and leaves in $\mathrm{R}_0(G)$ such that each non-leaf node is of the form $T \cup \{ab\}$, and has parent nodes $T \cup \{a\}, T \cup \{b\}$. The chain of reasoning in Example 1.2.4 can be displayed as a binary tree of finite sets of integers as follows:

$$\dfrac{\dfrac{}{\{3,-3\}} \qquad \dfrac{\dfrac{}{\{2,-2\}} \qquad \dfrac{\dfrac{}{\{1,-1\}} \quad \dfrac{}{\{1,-1\}}}{\{1,-2\}}}{\{3,-2\}}}{\{3,-5\}}$$

The next example is of particular interest, as it relates to both the main result of Section 1.3, and the earliest ideas from which the results in this chapter originated (see Section 1.5).

Example 1.2.6. Let $F(2)$ be the free group on two generators x and y. The following tree of finite subsets of $F(2)$ demonstrates that $S = \{xx, yy, x^{-1}y^{-1}\}$ is an element of the set $\mathrm{R}(F(2))$:

$$\dfrac{\dfrac{\overline{\{x,yy,x^{-1}\}}\quad\overline{\{x,yy,x^{-1}\}}}{\{xx,yy,x^{-1}\}}\qquad\dfrac{\overline{\{xx,y,y^{-1}\}}\quad\overline{\{xx,y,y^{-1}\}}}{\{xx,yy,y^{-1}\}}}{\{xx,yy,x^{-1}y^{-1}\}}$$

This should correspond to the fact that $S \subseteq F(2)$ does not extend to a right order on $F(2)$ (see Theorem 1.2.2) and the fact that $\mathrm{e} \leq xx \vee yy \vee x^{-1}y^{-1}$ holds in the variety LG of all ℓ-groups (see Corollary 1.3.2).

We begin the proof of Theorem 1.2.2 by showing that the members of the set $\mathrm{R}(G)$ do not extend to a right order on G. We make use of the characterization from Proposition 1.1.10.

Lemma 1.2.7. *For any group G, if $S \in \mathrm{R}(G)$, then S does not extend to a right order on G.*

Proof. Let G be a group, and $S \in \mathrm{R}(G)$. We show by induction on $k \in \mathbb{N}$ that if $S \in \mathrm{R}_k(G)$, then there exist $c_1,\dots,c_n \in G\setminus\{\mathrm{e}\}$ such that for every choice of signs $\delta_1,\dots,\delta_n \in \{-1,1\}$,

$$\mathrm{e} \in \langle S \cup \{c_1^{\delta_1},\dots,c_n^{\delta_n}\}\rangle.$$

For $S \in \mathrm{R}_0(G)$, we have $\{a, a^{-1}\} \subseteq S$ and hence, $\mathrm{e} \in \langle S\rangle$. For the inductive step, suppose $S \cup \{ab\} \in \mathrm{R}_{k+1}(G)$ as $S \cup \{a\}, S \cup \{b\} \in \mathrm{R}_k(G)$. By the induction hypothesis twice, we can find $c_1,\dots,c_n \in G\setminus\{\mathrm{e}\}$ such that for all $\delta_1,\dots,\delta_n \in \{-1,1\}$,

$$\mathrm{e} \in \langle S \cup \{a, c_1^{\delta_1},\dots,c_n^{\delta_n}\}\rangle \quad\text{and}\quad \mathrm{e} \in \langle S \cup \{b, c_1^{\delta_1},\dots,c_n^{\delta_n}\}\rangle.$$

But then, without loss of generality, for all $\delta_0,\delta_1,\dots,\delta_n \in \{-1,1\}$,

$$\mathrm{e} \in \langle S \cup \{ab, a^{\delta_0}, c_1^{\delta_1},\dots,c_n^{\delta_n}\}\rangle.$$

By Proposition 1.1.10, S does not extend to a right order on G, as was to be shown. □

We now move on to establishing some preliminary properties of $\mathrm{R}(G)$, for any group G.

Lemma 1.2.8. *For any group G, any $S, T \subseteq G$, and any $a, b \in G$:*

(a) $\{\mathrm{e}\} \in \mathrm{R}(G)$.

(b) *The set* $\mathrm{R}(G)$ *is upwards closed, i.e., if* $S \in \mathrm{R}(G)$ *and* $S \subseteq T$*, then* $T \in \mathrm{R}(G)$.

(c) *If* $S \in \mathrm{R}(G)$, *then there exists a finite* $S' \subseteq S$ *such that* $S' \in \mathrm{R}(G)$.

(d) *If* $S \cup \{ab\} \in \mathrm{R}(G)$, *then* $S \cup \{a, b\} \in \mathrm{R}(G)$.

(e) $S \in \mathrm{R}(G)$ *if and only if* $\langle S \rangle \in \mathrm{R}(G)$.

Proof. For (a), clearly $\{\mathrm{e}\} = \{\mathrm{e}, \mathrm{e}^{-1}\}$ and, by Definition 1.2.1, $\{\mathrm{e}\} \in \mathrm{R}(G)$. The claims in (b) and (c) follow by a straightforward induction on k such that $S \in \mathrm{R}_k(G)$. For (d), observe that if $S \cup \{ab\} \in \mathrm{R}(G)$, then, by (b), $S \cup \{a, ab\} \in \mathrm{R}(G)$. But also $S \cup \{a, a^{-1}\} \in \mathrm{R}(G)$, and hence, $S \cup \{a, b\} \in \mathrm{R}(G)$. Finally, for (e), that $S \in \mathrm{R}(G)$ implies $\langle S \rangle \in \mathrm{R}(G)$ follows directly from (b). Conversely, if $\langle S \rangle \in \mathrm{R}(G)$, we can apply (c) to obtain a finite subset $S' \subseteq \langle S \rangle$ such that $S' \in \mathrm{R}(G)$, and then use (b) to conclude $S' \subseteq S \cup S' \in \mathrm{R}(G)$. Now, observe that elements of S' are of the form $a_1^{\gamma_1} \cdots a_m^{\gamma_m}$ for $a_1, \ldots, a_m \in S$, and $\gamma_1, \ldots, \gamma_m \in \mathbb{N}$ and hence, by applying repeatedly (d), we get $S \in \mathrm{R}(G)$. □

To establish Theorem 1.2.2 and Theorem 1.2.3, we prove two preparatory results.

Lemma 1.2.9. *For any group* G, *and subset* $S \subseteq G$ *such that* $S \notin \mathrm{R}(G)$, *there exists a subsemigroup* T *of* G *extending* S *such that* $T \notin \mathrm{R}(G)$, *and* $G \setminus T$ *is a semigroup.*

Proof. Suppose $S \notin \mathrm{R}(G)$, and consider the set $\mathcal{U}$ of all subsemigroups of G extending S that are not contained in $\mathrm{R}(G)$, partially ordered by inclusion; i.e.,

$$\mathcal{U} = \{U \subseteq G \mid U \text{ is a subsemigroup of } G,\ S \subseteq U, \text{ and } U \notin \mathrm{R}(G)\}.$$

The set $\mathcal{U}$ is nonempty, as $\langle S \rangle \in \mathcal{U}$ by Lemma 1.2.8.(e). Further, if $\{U_i\}_{i \in I}$ is a chain in $\mathcal{U}$, we get $\bigcup_{i \in I} U_i \notin \mathrm{R}(G)$, since otherwise it would follow from Lemma 1.2.8.(c) and Lemma 1.2.8.(b) that $U_i \in \mathrm{R}(G)$ for some $i \in I$. Hence, every ascending chain has an upper bound in $\mathcal{U}$, and by Zorn's Lemma, there is a maximal element $T \in \mathcal{U}$. It remains to show that $G \setminus T$ is a semigroup. If $a, b \in G \setminus T$, then $T \subset T \cup \{a\}, T \cup \{b\}$, and therefore $T \cup \{a\}, T \cup \{b\} \in \mathrm{R}(G)$ by Lemma 1.2.8.(e), and by maximality of T. But then, $T \cup \{ab\} \in \mathrm{R}(G)$ by Definition 1.2.1, so we infer $ab \notin T$. □

Lemma 1.2.10. *For any group* G, *and* $S \subseteq G$ *satisfying* $S \notin \mathrm{R}(G)$ *and* $\{a\} \notin \mathrm{R}(G)$ *for all* $a \in G \setminus \{\mathrm{e}\}$, S *extends to a right order on* G.

Proof. Suppose $S \notin \mathrm{R}(G)$ and $\{a\} \notin \mathrm{R}(G)$ for all $a \in G \setminus \{\mathrm{e}\}$, and consider the set $\mathcal{U}$ of all subsemigroups U of G extending S such that $\mathrm{e} \notin U$ and $G \setminus U$ is a semigroup, partially ordered by inclusion. It follows from Lemma 1.2.9 that $\mathcal{U}$ is nonempty. Moreover, every ascending chain has an upper bound in $\mathcal{U}$, and by Zorn's Lemma, there is a maximal element $C \in \mathcal{U}$. We show that C is a right order on G extending S. Suppose for a contradiction that there exists $a \in G \setminus \{\mathrm{e}\}$ such that $a, a^{-1} \notin C$. By Lemma 1.2.9, the assumption $\{a\} \notin \mathrm{R}(G)$ yields a subsemigroup U_a of G containing a such that $U_a \notin \mathrm{R}(G)$ and $G \setminus U_a$ is a semigroup. In particular, $\mathrm{e} \notin U_a$. We claim that the maximality of C is contradicted by the set

$$D = C \cup \{b \in U_a \mid b, b^{-1} \notin C\}. \tag{1.2}$$

Observe first that D properly extends C and does not contain e. It remains to show that D and $G \setminus D$ are semigroups. For $b, c \in D$, there are a few cases. If $b, c \in C$, then $bc \in C \subseteq D$ since C is a semigroup; also, if $b, c \in U_a$ are such that $b, b^{-1}, c, c^{-1} \notin C$, then $bc \in U_a$ (as U_a is a semigroup) and $bc, c^{-1}b^{-1} \notin C$ (as $G \setminus C$ is a semigroup), so $bc \in D$. Suppose now that $c \in C$, and $b \in U_a$ is such that $b, b^{-1} \notin C$. Observe that $b^{-1}bc = c \in C$. Since $G \setminus C$ is a semigroup and $b^{-1} \notin C$, we must have $bc \in C \subseteq D$. To conclude, consider $b, c \notin D$. In particular, $b, c \notin C$, so $bc \notin C$. Now, there are three cases. First, if $b, c \notin U_a$, then $bc \notin U_a$ (as $G \setminus U_a$ is a semigroup) and hence, $bc \notin D$. Further, if $b, c \in U_a$, then, since $b, c \notin D$, we must have $b^{-1}, c^{-1} \in C$ by (1.2). So also $c^{-1}b^{-1} \in C$, and it follows that $bc \notin D$, since D is a semigroup omitting e. Finally, suppose without loss of generality that $b \in U_a$ and $c \notin U_a$. Since $b \in U_a$, $b \notin C$, and $b \notin D$, we must have $b^{-1} \in C$ by (1.2), or equivalently, $cc^{-1}b^{-1} \in C$. But then, as $c \notin C$ and $G \setminus C$ is a semigroup, we get $c^{-1}b^{-1} \in C$. Hence $bc \notin D$. Therefore, $C \cup \{e\}$ is a right order on G that extends S. □

Proof of Theorem 1.2.2. Let G be a group such that $\{a\} \notin \mathrm{R}(G)$ for all $a \in G \setminus \{e\}$. By applying Lemma 1.2.10 with $S = \emptyset$, we get that G admits a right order. Conversely, suppose G admits a right order, and pick $a \in G \setminus \{e\}$. Then, $\{a\}$ extends to a right order on G (see Remark 1.1.12), so $\{a\} \notin \mathrm{R}(G)$ by Lemma 1.2.7. □

Proof of Theorem 1.2.3. By Lemma 1.2.7, we only need to show that if $S \notin \mathrm{R}(G)$, then S extends to a right order on G. Since G is right-orderable, by Theorem 1.2.2 we have $\{a\} \notin \mathrm{R}(G)$ for all $a \in G \setminus \{e\}$, and we can apply Lemma 1.2.10 to conclude the proof. □

The orderability conditions described at the end of Section 1.1 do not assume the group under consideration to be right-orderable. However, this assumption is necessary for Theorem 1.2.3. This is easily seen, e.g., by observing that any finite group G is such that $\emptyset \notin \mathrm{R}(G)$, while $\emptyset$ does not extend to a right order on G (as G does not admit any right order).

We move on to specializing the above inductive description, to consider those subsets of a group G that do not extend to a (both left- and right-invariant) order on G. For this, it suffices to supplement the characterization for right orders with an extra condition, so to take into account that positive cones of orders must also be normal.

Definition 1.2.11. We define inductively for $n \in \mathbb{N}$:

$$\mathrm{B}_0(G) = \{S \subseteq G \mid S \cap S^{-1} \neq \emptyset\};$$
$$\mathrm{B}_{n+1}(G) = \mathrm{B}_n(G) \cup \{T \cup \{ab\} \mid T \cup \{a\}, T \cup \{b\} \in \mathrm{B}_n(G) \text{ or } T \cup \{ba\} \in \mathrm{B}_n(G)\},$$

and set $\mathrm{B}(G)$ to be $\bigcup_{n \in \mathbb{N}} \mathrm{B}_n(G)$.

The following two results are the analogues of Theorem 1.2.2 and Theorem 1.2.3, and they are concerned with orderability and orders.

Theorem 1.2.12. *A group G is orderable if and only if $\{a\} \notin \mathrm{B}(G)$ for all $a \in G \setminus \{e\}$.*

Theorem 1.2.13. *For any orderable group G, the set* $\mathrm{B}(G)$ *consists precisely of those subsets of G that do not extend to an order on G.*

Since the proofs proceed similarly to the ones of Theorem 1.2.2 and Theorem 1.2.3, they are omitted, and the remainder of this section is devoted to presenting a few clarifying examples.

Example 1.2.14. Let $K = \langle x, y \mid xyx^{-1}y \rangle$ be the fundamental group of the Klein bottle. Observe that $y^{-1} = xyx^{-1}$, and hence, $\{y, xyx^{-1}\} \in \mathrm{B}_0(K)$. But then, by Definition 1.2.11, we get $\{y, yx^{-1}x\} = \{y\} \in \mathrm{B}_1(K)$. This corresponds to the fact that K is not orderable (Theorem 1.2.12). We also remark that K does admit a right order (see [155, Theorem 5.2]; cf. [156, Theorem 3]).

The next example, similarly to Example 1.2.6, is concerned with the relation between orders on groups, and (representable) ℓ-groups, and will be fully justified in Sections 1.3 and 1.4.

Example 1.2.15. We consider again the free group $F(2)$ with generators x and y, and observe that $\{x, x^{-1}\} = \{x, x^{-1}y^{-1}y\} \in \mathrm{B}_0(F(2))$, and hence $\{x, yx^{-1}y^{-1}\} \in \mathrm{B}_1(F(2))$ by Definition 1.2.11. This should correspond to the fact that $\{x, yx^{-1}y^{-1}\}$ does not extend to an order on $F(2)$ and also the fact that $\mathrm{e} \leq x \vee yx^{-1}y^{-1}$ holds in all representable ℓ-groups, i.e., the variety generated by totally ordered groups (see Corollary 1.4.2). Note, however, that $\{x, yx^{-1}y^{-1}\} \notin \mathrm{R}(F(2))$, reflecting the fact that the set $\{x, yx^{-1}y^{-1}\}$ does extend to a right order on $F(2)$ and the fact that $\mathrm{e} \leq x \vee yx^{-1}y^{-1}$ does not hold in LG (see Corollary 1.3.2); in fact, such an equation suffices to axiomatize the variety Rep of representable ℓ-groups.

1.3 EQUATIONS IN LATTICE-ORDERED GROUPS, AND RIGHT ORDERS

Here, we use the results from Section 1.2 to establish a correspondence between subsets that do not extend to right orders on an arbitrary right-orderable group with presentation $\langle X \mid R \rangle$, and ℓ-group equations entailed by (the group equations determined by) the set R. We pay special attention to the case of valid ℓ-group equations ($R = \emptyset$) and subsets of free groups that do not extend to right orders.

We review here some general notation. Let K be a class of algebras of type $\mathcal{L}$, and $\Sigma \cup \{t \approx s\}$ a set of $\mathcal{L}$-equations. We write $\Sigma \models_{\mathsf{K}} t \approx s$ to mean that for any $A \in \mathsf{K}$ and any $\mathcal{L}$-homomorphism $\varphi\colon T^{\mathcal{L}}(X) \to A$ (sometimes called 'valuation'), whenever $\Sigma \subseteq \ker(\varphi)$, also $(t, s) \in \ker(\varphi)$. For simplicity, we identify $\Sigma \cup \{t, s\}$ with their representatives in the free object $F_{\mathsf{K}}(X)$ relative to K. If K is a quasivariety, it is known that, if the $\mathcal{L}$-congruence $\Theta(\Sigma)$ generated by Σ in $F_{\mathsf{K}}(X)$ is such that $F_{\mathsf{K}}(X)/\Theta(\Sigma) \in \mathsf{K}$ (congruences with this property are called 'relative congruences'), then $\Sigma \models_{\mathsf{K}} t \approx s$ if and only if $(t, s) \in \Theta(\Sigma)$. If the quasivariety is a variety V, any $\mathcal{L}$-congruence is a relative congruence, and hence $\Sigma \models_{\mathsf{V}} t \approx s$ if and only if $(t, s) \in \Theta(\Sigma)$ for all $\Sigma \cup \{t, s\}$. This can be equivalently formulated by considering $\alpha\colon F_{\mathsf{V}}(X) \twoheadrightarrow F_{\mathsf{V}}(X)/\Theta(\Sigma)$, and observing that $\Sigma \models_{\mathsf{V}} t \approx s$ if and only if $(t,s) \in \ker(\alpha)$.

We set $T(X)$ and $T^{\ell}(X)$ to be the term algebras over a set X for the languages of groups and ℓ-groups, respectively. We say that a group term[2] $t \in T(X)$ is reduced (or 'in reduced form') if it contains no adjacent symbols of the form x^{δ}, $x^{-\delta}$ for $\delta \in \{1,-1\}$. Clearly, every group term is equivalent in the variety of groups to exactly one reduced group term; we write t to denote both a group term in $T(X)$ and its reduced form in the free group $F(X)$. If G is an arbitrary group with presentation $\langle X \mid R\rangle$ for some $R \subseteq F(X)$, we then write $\alpha\colon F(X) \twoheadrightarrow G$ for the natural quotient map.

Theorem 1.3.1. *For any right-orderable group G with presentation $\langle X \mid R\rangle$, and any finite set of group terms $t_1,\dots,t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{\alpha(t_1),\dots,\alpha(t_n)\} \subseteq G$ does not extend to a right order on G.*
(2) $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_n$, *where* $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$.

Proof. (1) $\Rightarrow$ (2). Suppose that the set $\{\alpha(t_1),\dots,\alpha(t_n)\}$ does not extend to a right order on G. By Theorem 1.2.3,

$$\{\alpha(t_1),\dots,\alpha(t_n)\} \in \mathrm{R}(G) = \bigcup_{k\in\mathbb{N}} \mathrm{R}_k(G).$$

Then, we show by induction on $k \in \mathbb{N}$ that $\{\alpha(t_1),\dots,\alpha(t_n)\} \in \mathrm{R}_k(G)$ entails $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_n$. If $k=0$, we have $\{\alpha(t_1),\dots,\alpha(t_n)\} \in \mathrm{R}_0(G)$ if and only if $\alpha(t_i) = \alpha(t_j)^{-1}$ for some $1 \le i < j \le n$. This means that $(t_i, {t_j}^{-1}) \in \ker(\alpha)$ or equivalently, that $\Sigma \models_{\mathsf{LG}} t_i \approx {t_j}^{-1}$. Since $\mathsf{LG} \models \mathrm{e} \le x \vee x^{-1}$, we conclude $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_n$. For the inductive step, suppose

$$\{\alpha(t_1),\dots,\alpha(t_{n-1}),\alpha(u)\} \in \mathrm{R}_k(G) \ \text{ and } \ \{\alpha(t_1),\dots,\alpha(t_{n-1}),\alpha(v)\} \in \mathrm{R}_k(G),$$

where $t_n = uv$, and $\{\alpha(t_1),\dots,\alpha(t_{n-1}),\alpha(uv)\} \in \mathrm{R}_{k+1}(G)$. By the induction hypothesis twice,

$$\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_{n-1} \vee u \ \text{ and } \ \Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_{n-1} \vee v,$$

which entail $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \dots \vee t_{n-1} \vee uv$. (Recall that in any ℓ-group, if $\mathrm{e} \le x \vee y$ and $\mathrm{e} \le x \vee z$, then $\mathrm{e} \le x \vee yz$; see [67, Lemma 3.3].)

(2) $\Rightarrow$ (1). We proceed by contraposition. Let C be the positive cone of a right order on G such that $\alpha(t_1),\dots,\alpha(t_n) \in C$. Consider the order C^{∂} defined by $a \le_{\partial} b$ if and only if $b \le_C a$. Clearly, $\alpha(t_i)$ is strictly negative for each $1 \le i \le n$. Consider the ℓ-group $\mathrm{Aut}(G,\le_{\partial})$, and the valuation $\varphi\colon T^{\ell}(X) \to \mathrm{Aut}(G,\le_{\partial})$, obtained by extending the assignment

$$x \longmapsto \varphi(x)\colon \alpha(s) \mapsto \alpha(sx), \quad \text{for } \alpha(s) \in G.$$

[2]For consistency throughout the thesis, we write '(reduced) group term' to mean what is usually known in group theory as a '(reduced) group word'.

For all $r \in R$, and $\alpha(s) \in G$, it holds that $\varphi(r) = \mathrm{id}_G$, as

$$\begin{aligned}\varphi(r)(\alpha(s)) &= \alpha(sr)\\ &= \alpha(s)\alpha(r)\\ &= \alpha(s)\mathrm{e}\\ &= \alpha(s).\end{aligned}$$

Further, any $t \in T(X) \subseteq T^{\ell}(X)$ is sent to the order-automorphism $\alpha(s) \mapsto \alpha(st)$, for $\alpha(s) \in G$. Thus, as $\alpha(\mathrm{e}) = \mathrm{e}$, we have

$$\begin{aligned}\varphi(t_1 \vee \cdots \vee t_n)(\mathrm{e}) &= (\varphi(t_1) \vee \cdots \vee \varphi(t_n))(\mathrm{e})\\ &= \max_{\partial}\{\varphi(t_1)(\mathrm{e}), \dots, \varphi(t_n)(\mathrm{e})\}\\ &= \max_{\partial}\{\alpha(t_1), \dots, \alpha(t_n)\},\end{aligned}$$

and $\max_{\partial}\{\alpha(t_1), \dots, \alpha(t_n)\} <_{\partial} \mathrm{e}$, as $\alpha(t_i)$ is strictly negative for each $1 \le i \le n$. Therefore, since the order on $\mathrm{Aut}\,(G, \le_{\partial})$ is defined pointwise, we conclude

$$\mathrm{id}_G = \varphi(\mathrm{e}) \not\le \varphi(t_1 \vee \cdots \vee t_n);$$

equivalently, $\Sigma \not\models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \cdots \vee t_n$, as was to be shown. □

Corollary 1.3.2. *For any set $t_1, \dots, t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{t_1, \dots, t_n\} \subseteq F(X)$ does not extend to a right order on $F(X)$.*
(2) $\mathsf{LG} \models \mathrm{e} \le t_1 \vee \cdots \vee t_n$.

Proof. This is an immediate consequence of Theorem 1.3.1, by taking $R = \emptyset$. □

The claims that were made in Example 1.2.6 have now been mathematically justified.

Remark 1.3.3. By the distributivity properties of ℓ-groups, it is readily seen that any ℓ-group term is equivalent over LG to terms of the form $\bigwedge_{i \in I} \bigvee_{j \in J_i} t_{ij}$ where t_{ij} is a group term for each $i \in I$, $j \in J_i$. Therefore, the validity of an equation $s \approx t$ in the variety LG is equivalent to the validity of two equations $\mathrm{e} \le ts^{-1}$ and $\mathrm{e} \le st^{-1}$, whose right-hand sides are (equivalent in LG to) finite meets of finite joins. Hence, the validity of $s \approx t$ is equivalent to the validity of finitely many equations of the form

$$\mathrm{e} \le t_1 \vee \cdots \vee t_n,$$

where t_i are group terms for all $1 \le i \le n$. For this reason, Corollary 1.3.2 provides a full characterization of validity of equations in ℓ-groups.

In view of Remark 1.3.3, the result in Corollary 1.3.2 relates two decidability problems, i.e., the decidability of the equational theory of ℓ-groups, and the problem of deciding when a given finite subset of a finitely generated free group extends to a right order. The decidability of the former was first proved in 1979 by Holland and

McCleary ([90]), and a proof of the decidability of the latter is implicit in an article from 2009 ([27]).

For any finite set X with cardinality $k \in \mathbb{N}$, we write $F(k)$ to denote the free group over X. Let $|t|$ denote the length of a reduced term t in $F(k)$, and for $m \in \mathbb{N}$, let $F^*_m(k)$ denote the set of all elements of $F(k)$ of length at most m. Note that $F^*_m(k)$ is finite, and can be viewed as the m-ball of the Cayley graph of $F^*_m(k)$ relative to X. For any subset S of $F(k)$ not containing e, we say that S is an *m-truncated right order* on $F(k)$ if $S = \langle S\rangle \cap F^*_m(k)$, and for all $t \in F^*_{m-1}(k) \setminus \{e\}$, either $t \in S$ or $t^{-1} \in S$. The proof of the following result is the content of [27].

Theorem 1.3.4 ([27])**.** *For any $k \in \mathbb{N}$, a subset $S \subseteq F(k)$ extends to a right order on $F(k)$ if and only if S extends to an m-truncated right order on $F(k)$ for some $m \in \mathbb{N}$.*

The condition described above can be decided for finite S as follows. Consider m defined as

$$m = \max\{|t| \mid t \in S\}.$$

Extend S to the finite set S' obtained by adding st whenever s, t occur in the set constructed so far and $|st| \leq m$. Note that $S' = \langle S'\rangle \cap F^*_m(k)$. If $e \in S'$, then stop. Otherwise, given $t \in F^*_{m-1}(k) \setminus \{e\}$ such that $t \notin S'$ and $t^{-1} \notin S'$, add t to S' to obtain S_1 and t^{-1} to S' to obtain S_2, and repeat the process with these sets. This procedure eventually terminates, as $F^*_m(k)$ is finite. Hence, this algorithm can be used to decide whether a finite subset of a finitely generated free group extends to a right order. Thus, we obtain the following:

Corollary 1.3.5. *For any $k \in \mathbb{N}$, the problem of deciding if a finite subset of $F(k)$ extends to a right order is decidable.*

We bring to the reader's attention the fact that Corollary 1.3.5 provides, in view of Theorem 1.3.1, an alternative proof of the decidability of the equational theory of ℓ-groups. A further alternative proof will be obtained in Chapter 4.

Example 1.3.6. Consider the subset $S = \{xx, yy, x^{-1}y^{-1}\}$ of the free group $F(2)$ with generators $\{x, y\}$. By adding all products in $F^*_2(2)$ of members of S, we obtain

$$S' = \{xx, yy, x^{-1}y^{-1}, xy^{-1}, x^{-1}y, xy\}.$$

We consider all possible signs for $x, y \in F^*_1(2)$. If we add x^{-1} or y^{-1} to S' and take products, then clearly we obtain e (using xx or yy). Similarly, if we add x and y to S', then, taking products we obtain e (using $x^{-1}y^{-1}$). Hence, by Corollary 1.3.5, the set S does not extend to a right order on $F(2)$. Equivalently, by Theorem 1.3.1, we also obtain

$$\mathsf{LG} \models e \leq xx \vee yy \vee x^{-1}y^{-1}.$$

Example 1.3.7. Consider now the subset $S = \{xx, xy, yx^{-1}\}$ of the free group $F(2)$ with generators $\{x, y\}$. By adding all products in $F^*_2(2)$ of members of S, we obtain

$$S' = \{xx, xy, yx^{-1}, yx, yy\}.$$

We choose $x, y \in F_1^*(2)$ to be positive and get $\{xx, xy, yx^{-1}, yx, yy, x, y\}$, which is a 2-truncated right order on $F(2)$. Hence, by Corollary 1.3.5, the set S extends to a right order on $F(2)$ and

$$\mathsf{LG} \not\models \mathrm{e} \leq xx \vee xy \vee yx^{-1}.$$

We conclude the section with a final comment. Galatos and Metcalfe have proved that the equational theory of ℓ-groups is coNP-complete [67], and it follows that the problem of deciding whether or not a finite subset of a free group extends to a right order is also in the complexity class coNP. It is not known, however, if this latter problem is coNP-complete. Indeed, hardness is established for the equational theory of ℓ-groups using the fact that the equational theory of distributive lattices is coNP-complete ([98]).

1.4 ORDERS, AND VALIDITY IN TOTALLY ORDERED GROUPS

Now, we turn our attention to varieties generated by totally ordered groups; equivalently, varieties of representable ℓ-groups. More precisely, we make use of Theorem 1.2.13 to establish a correspondence between valid ℓ-group equations in varieties of representable ℓ-groups axiomatized by sets Σ of group equations, and subsets of free groups relative to the variety defined by Σ that (do not) extend to orders.

We call an ℓ-group *representable* if it is a subdirect product of totally ordered groups, and *Abelian* if its group reduct is Abelian. The class of representable ℓ-groups is a variety defined relative to LG by $\mathrm{e} \leq x \vee yx^{-1}y^{-1}$ ([118]). We also recall a useful quasiequation, which defines representable ℓ-groups relative to the variety of ℓ-groups (cf. [49, Proposition 47.1]):

$$x \wedge y \approx \mathrm{e} \quad \Longrightarrow \quad x \wedge z^{-1}yz \approx \mathrm{e}. \tag{1.3}$$

Abelian ℓ-groups form a subvariety Ab of the variety of representable ℓ-groups, defined relative to LG by $xy \approx yx$. We write Rep_Σ for the variety of ℓ-groups axiomatized relative to the variety of representable ℓ-groups by a set Σ of group equations; in particular, if $\Sigma = \emptyset$, then Rep_Σ is the variety Rep of all representable ℓ-groups. Set G_Σ to be the variety defined by Σ relative to the variety G of all groups, and consider the free group $F_\Sigma(X)$ over some set X relative to the variety G_Σ. We write t to denote both the element of $F_\Sigma(X)$, and the group term $t \in T(X)$. The proof of the following result resembles the one of Theorem 1.3.1, although they differ in some relevant details.

Theorem 1.4.1. *For any set Σ of group equations such that the relatively free group $F_\Sigma(X)$ is orderable, and any finite set of group terms $t_1, \dots, t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{t_1, \dots, t_n\} \subseteq F_\Sigma(X)$ does not extend to an order on $F_\Sigma(X)$.*

(2) $\mathsf{Rep}_\Sigma \models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$.

Proof. (1) ⇒ (2). Suppose that the set $\{t_1,\dots,t_n\} \subseteq F_\Sigma(X)$ does not extend to an order on $F_\Sigma(X)$. By Theorem 1.2.13,

$$\{t_1,\dots,t_n\} \in \mathrm{B}(F_\Sigma(X)) = \bigcup_{k\in\mathbb{N}} \mathrm{B}_k(F_\Sigma(X)).$$

The proof again proceeds by induction on $k \in \mathbb{N}$. We only consider the case where $\{t_1,\dots,t_{n-1},uv\} \in \mathrm{B}_{k+1}(F_\Sigma(X))$ since $\{t_1,\dots,t_{n-1},vu\} \in \mathrm{B}_k(F_\Sigma(X))$. By the induction hypothesis, we get $\mathsf{Rep}_\Sigma \models \mathrm{e} \le t_1 \vee\cdots\vee t_{n-1} \vee vu$, and infer $\mathsf{Rep}_\Sigma \models \mathrm{e} \le t_1 \vee\cdots\vee t_{n-1} \vee uv$. (It follows from (1.3) that $\mathrm{e} \le x \vee y$ entails $\mathrm{e} \le x \vee z^{-1}yz$ in representable ℓ-groups.)

(2) ⇒ (1). We proceed by contraposition. Let C be the positive cone of an order on $F_\Sigma(X)$ such that $t_1,\dots,t_n \in C$, and consider its dual order C^∂ on $F_\Sigma(X)$. Clearly, the totally ordered group $(F_\Sigma(X), \le_\partial)$ is an ℓ-group in Rep_Σ. Thus, consider the valuation $\varphi\colon T^\ell(X) \to (F_\Sigma(X), \le_\partial)$, obtained by extending the identity map $x \mapsto x$. As t_i is strictly negative for each $1 \le i \le n$,

$$\varphi(t_1 \vee\cdots\vee t_n) = \max_\partial\{\bar{t}_1,\dots,\bar{t}_n\} <_\partial \mathrm{e}.$$

Hence, we conclude $\varphi(\mathrm{e}) \not\le \varphi(t_1 \vee\cdots\vee t_n)$, and $\mathsf{Rep}_\Sigma \not\models \mathrm{e} \le t_1 \vee\cdots\vee t_n$ as was to be shown. □

Corollary 1.4.2. *For any set $t_1,\dots,t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{t_1,\dots,t_n\} \subseteq F(X)$ does not extend to an order on $F(X)$.*
(2) $\mathsf{Rep} \models \mathrm{e} \le t_1 \vee\cdots\vee t_n$.

Proof. This is an immediate consequence of Theorem 1.4.1, by taking $\Sigma = \emptyset$. □

The claims that were made in Example 1.2.15 have been now mathematically justified.

Example 1.4.3. Let $\Sigma = \{xyx^{-1}y \approx \mathrm{e}\}$. Then, $F_\Sigma(X)$ is the free Abelian group over X and, as such, it is torsion-free. By Proposition 1.1.8, the set $\{t_1,\dots,t_n\} \subseteq F_\Sigma(X)$ extends to an order on $F_\Sigma(X)$ if and only if it extends to a partial order on $F_\Sigma(X)$, i.e., $\mathrm{e} \notin \langle\{t_1,\dots,t_n\}\rangle$. The same argument applies to varieties of nilpotent groups of class $c \in \mathbb{N}^+$ (Proposition 1.1.9).

The differences between Theorem 1.4.1 and Theorem 1.3.1 raise the following question: would it be possible to prove the analogue of Theorem 1.4.1 for varieties of ℓ-groups defined by group equations, and free groups relative to the corresponding varieties of groups? Namely, suppose that V is axiomatized relative to LG by a set Σ of group equations such that the free group $F_\Sigma(X)$ relative to the variety G_Σ is right-orderable. Then, is it true for any finite set of group terms $t_1,\dots,t_n \in T(X)$ that $\{t_1,\dots,t_n\} \subseteq F_\Sigma(X)$ extends to a right order on $F_\Sigma(X)$ if and only if $\mathsf{V} \models \mathrm{e} \le t_1 \vee\cdots\vee t_n$ does not hold? The answer is negative, as the following example shows.

We call an ℓ-group *nilpotent of class* $c \in \mathbb{N}^+$ if its group reduct is nilpotent of class c; similarly, an ℓ-group is *nilpotent* if it is nilpotent of some class $c \in \mathbb{N}^+$.

Example 1.4.4. We consider here the free nilpotent group of class 2 over two generators $\{x, y\}$, and group terms xyx^{-1} and y^{-1}. As nilpotent ℓ-groups are representable ([109]; cf. [92, Theorem 4]), the variety N^2 of nilpotent ℓ-groups of class 2 is such that

$$\mathsf{N}^2 \models \mathrm{e} \leq xyx^{-1} \vee y^{-1}.$$

To give a negative answer to the question raised above, it suffices to exhibit a right order on the free nilpotent group of class 2 (over $\{x, y\}$) which makes the subset $\{xyx^{-1}, y^{-1}\}$ positive. For this, first observe that the free nilpotent group $F_2(x, y)$ of class 2 over two generators $\{x, y\}$ is isomorphic to the group $\mathrm{UT}_3(\mathbb{Z})$ of 3×3 upper unitriangular integral matrices (see, e.g., [105, Exercise 16.1.3]), via the group homomorphism obtained by extending the variable assignment

$$x \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad y \longmapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, the free nilpotent group of class 2 over two generators acts faithfully via order-preserving bijections on the chain $(\mathbb{Z}^3, \leq)$, where $\leq$ is the lexicographic order defined by

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \leq \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \iff z_1 < z_2 \text{ or } (z_1 = z_2 \text{ and } y_1 < y_2) \text{ or } (z_1 = z_2 \text{ and } y_1 = y_2 \text{ and } x_1 < x_2).$$

By Proposition 1.1.3, we can now define a right order on $\mathrm{UT}_3(\mathbb{Z})$ by considering a well-order on $\mathbb{Z}^3$ such that the least element is the vector $(0, -1, -2)$. We recall that the corresponding right order has positive cone $\{A \in \mathrm{UT}_3(\mathbb{Z}) \mid Av > v\}$, where $v = \min\{w \in \mathbb{Z}^3 \mid Aw \neq w\}$. Both the (image of the) term xyx^{-1} and the (image of the) term y^{-1} are positive in the resulting order. To see this, first observe that the element xyx^{-1} is identified via the isomorphism with the matrix

$$xyx^{-1} \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore, we conclude by observing that

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} > \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}.$$

1.5 CONCLUDING REMARKS

In this chapter, we used an inductive characterization of subsets of groups that extend to right orders to establish a correspondence between validity of equations in

ℓ-groups and subsets of free groups that extend to right orders on the group. A correspondence was also established between validity of equations in varieties of representable ℓ-groups and subsets of relatively free groups that extend to orders on the group. An immediate consequence of the correspondence established here is a new proof of the decidability of the equational theory of ℓ-groups.

The entanglement between the theory of ℓ-groups and the theory of right orderable groups is well-known, and has been explored in many ways in the literature of both fields. However, the explicit relationship between right orders on free groups and validity in ℓ-groups is new. The results obtained here lead to a new proof that $\mathrm{Aut}(\mathbb{R})$ generates the variety of all ℓ-groups; the advantage of this proof is that it does not use Holland's representation theorem or any other structural result for ℓ-groups. More precisely, the correspondence theorems obtained here follow from ordering theorems for groups that stem from proof-theoretic investigations into ℓ-groups, and require very little structure theory for these algebras. As a matter of fact, the theory developed here provides a connection between orderable groups, and proof theory.

Hypersequent calculi for Abelian ℓ-groups and related varieties go back to the work of George Metcalfe, Nicola Olivetti and Dov Gabbay ([132]; cf. [133]). A successful attempt to obtain an analytic proof system for (non-commutative) ℓ-groups is due to Nikolaos Galatos and George Metcalfe ([67]). However, the completeness proofs in all these papers are largely syntactic, using cut elimination or restricted quantifier elimination. It is interesting to observe that the ordering theorems discussed here, characterizing when partial orders on a group extend to total orders, can be used (and in fact, were used in [31]; cf. [29]) to devise hypersequent calculi for varieties of ℓ-groups. For instance, an analytic calculus for Abelian ℓ-groups, related to the calculus from [132], is obtained in [31] using the ordering theorem discussed in Proposition 1.1.8. Further, (non-analytic) calculi can be obtained from Corollaries 1.3.2 and 1.4.2 for ℓ-groups and representable ℓ-groups, respectively. It would be worth exploring the limits of this approach, by considering other classes of ℓ-groups, and suitable ordering theorems.

Problem 1. Use ordering theorems to obtain (analytic) calculi for other varieties of ℓ-groups.

The reason why the commutative setting is particularly well-behaved is precisely Proposition 1.1.8 (cf. Example 1.4.3). Related to this are what we sometimes call 'theorems of alternatives'. In [29], with Galatos and Metcalfe, we study varieties of commutative residuated lattices that behave similarly to Abelian ℓ-groups, in the sense that validity in such varieties is determined by their 'multiplicative fragment'. For example, in the case of Abelian ℓ-groups, validity in the whole variety is determined by validity in the class of torsion-free Abelian groups. Theorems of alternatives in this sense provide a systematic way to obtain hypersequent calculi for varieties of residuated lattices, from sequent calculi for their multiplicative fragments.

We mention here a related problem, central to the proof theory of ℓ-groups and, more generally, to the proof theory of those residuated lattices that are sufficiently similar to ℓ-groups. Considerable success has been enjoyed recently in obtaining uniform algebraic completeness proofs for analytic sequent and hypersequent calculi with respect to varieties of residuated lattices (e.g., [21, 66, 131, 133, 134]). These techniques, falling under the umbrella of 'algebraic proof theory', are based on the fact that cut-admissibility corresponds to closure under certain completions of corresponding varieties. These methods do not encompass, however, 'ordered group-like' structures: algebras with a group reduct such as ℓ-groups, and other algebras admitting representations via ordered groups (e.g., MV-algebras, varieties of cancellative residuated lattices), in view of the fact that these structures do not admit any completion.

Problem 2. Establish an algebraic proof of cut elimination for the hypersequent calculus for ℓ-groups obtained in [67].

Results obtained in Chapter 4 for distributive ℓ-monoids may provide a starting point for tackling this problem (see Theorem 4.4.11).

We already mentioned that a proof system for representable ℓ-groups can be obtained from Corollary 1.4.2. However, an analytic proof system for representable ℓ-groups is lacking. This is related to the fact that the decidability problem for equational theory of representable ℓ-groups is still unsolved.

Problem 3. Settle whether the equational theory of representable ℓ-groups (equivalently, the problem whether a finite subset of the free group extends to an order) is decidable or undecidable.

The methods currently available to prove the decidability of the equational theory of a given variety of ℓ-groups are not effective in the representable case (cf. [90, 127, 61, 67]). Moreover, the method developed by Adam Clay and Lawrence Smith ([27]) for deciding whether a finite subset of the free group extends to a right order seems unlikely to work for orders, and therefore Corollary 1.4.2 cannot be exploited to decide the equational theory of representable ℓ-groups.

CHAPTER 2

ORDERED GROUPS, ALGEBRAICALLY

This chapter is intended as a bridge between the ideas developed in Chapter 1 and the rest of the thesis, providing an algebraic account of the results obtained so far. Lattice-ordered groups play a central role in this work, since even when they are not the direct subject of study, their theory provides a major source of inspiration, both conceptually and technically (cf. Chapters 5 and 4). For this reason, the first section of the present chapter is devoted to a brief overview of the theory of ℓ-groups, from their structure theory to the celebrated Holland representation theorem. The theory introduced is then put into practice, and used to obtain and extend the results from Sections 1.3 and 1.4.

The syntactic approach described in Chapter 1 grew out of an attempt to provide a useful proof-theoretic account of ℓ-groups. The main advantage of such an approach is that deep results can be obtained without any knowledge of the structure theory of ℓ-groups. In this chapter we demonstrate that the results obtained in Chapter 1 hold in view of the fact that the structure of the free ℓ-group over a set X is completely determined by the right orders on the free group over X, and the latter generates the former as a distributive lattice. This interplay between right orders on a group, and the structure of the ℓ-group freely generated by such group is in fact the core of Chapter 3.

The chapter begins with an overview of the most important results from the theory of ℓ-groups, culminating with Holland's theorem that every ℓ-group acts faithfully on some chain by order-preserving bijections. Later, we revisit the results from Chapter 1, providing alternative proofs for Theorems 1.3.1 and 1.4.1. The main aim of these alternative proofs is to identify exactly which properties of the interplay between ℓ-groups and right-orderable groups allow Theorem 1.3.1 to hold, and similarly for representable ℓ-groups and orderable groups. In the process of identifying such properties, we emphasize the distinction between those properties that are particular to the theory of ℓ-groups, and those that hold in a more general algebraic context. Once these properties have been identified, we use them in the final part of the chapter to conclude that theorems analogous to Theorems 1.3.1 and 1.4.1 hold for further varieties of ℓ-groups (and suitable classes of right-orderable groups), namely normal-valued and weakly Abelian ℓ-groups. This shows that, at the present stage, the algebraic approach seems to have a broader scope than the syntactic approach.

The theory and terminology from order theory used in this chapter is reviewed in Appendix A.2.

2.1 THE STRUCTURE OF LATTICE-ORDERED GROUPS

We review the necessary background on ℓ-groups here, and will refer to this section throughout the rest of this thesis. Most proofs are omitted and, when necessary, we include a sketched intuition of the argument. However, the reader can find versions of the results and their proofs in any standard textbook on ℓ-groups (see, e.g., [10, 49, 112, 73]).

Congruences in groups are uniquely determined by the equivalence class of the identity element e, thereby corresponding to normal subgroups of the considered group. The same happens for ℓ-groups, where congruences are in one-to-one correspondence with specific subalgebras of the considered ℓ-group. (Note that a subalgebra of an ℓ-group H is a sublattice subgroup of H; we denote such subalgebras as ℓ-subgroups, as customary.) A *convex ℓ-subgroup* of an ℓ-group H is an ℓ-subgroup of H that is order-convex, and an *ℓ-ideal* is a convex ℓ-subgroup that is normal as a group. As such, every ℓ-ideal determines a group congruence by considering the equivalence relation induced by the right cosets. Indeed, this relation is an ℓ-group congruence. Conversely, for any congruence, the equivalence class of the group identity is an ℓ-ideal of the ℓ-group. This correspondence describes an isomorphism between the lattice Con H of congruences of H, and the lattice $\mathcal{NC}(H)$ of ℓ-ideals of H. Since it is a standard result that congruence lattices of universal algebras are algebraic (see, e.g., [18, Theorem 5.5]), we may conclude that $\mathcal{NC}(H)$ is an algebraic lattice. Indeed, the lattice $\mathcal{NC}(H)$ is the intersection of two algebraic sublattices of the lattice of subgroups of H, namely the normal subgroups and the convex ℓ-subgroups.

Convex ℓ-subgroups identify particular lattice congruences of ℓ-groups, namely those that are right (group) congruences—that is, compatible with multiplication on the right. Indeed, a convex ℓ-subgroup $\Bbbk$ induces a lattice congruence on the ℓ-group H, by considering: $a\theta b$ if and only if $\Bbbk a = \Bbbk b$. Then, the set of right cosets $H/\Bbbk$ is partially ordered by

$$\Bbbk a \leq \Bbbk b \iff a \wedge tb = a \text{ for some } t \in \Bbbk, \tag{2.1}$$

and, with such a partial order, $H/\Bbbk$ is in fact a lattice, with operations

$$\Bbbk a \wedge \Bbbk b = \Bbbk(a \wedge b) \text{ and } \Bbbk a \vee \Bbbk b = \Bbbk(a \vee b). \tag{2.2}$$

The reason why convex ℓ-subgroups are more relevant for the structure theory of ℓ-groups than normal convex ℓ-subgroups (=congruences) should be explained by the results in the next section. The key idea is based on the following facts: every ℓ-group H is an ℓ-subgroup of some Aut(Ω), where Ω is a suitable chain; furthermore, if H is subdirectly irreducible, then its resulting action on Ω is transitive (i.e., every element of Ω is the image of any other element of Ω under some function in H). When looking for representations of a given ℓ-group, considering convex ℓ-subgroups—specifically, those convex ℓ-subgroups inducing a totally ordered quotient—allows us to directly obtain a transitive representation of the factor quotients (see the map defined in (2.4), and Theorems 2.1.19 and 2.1.20).

Remark 2.1.1. As we will see in Chapter 3, when we restrict our attention to smaller varieties of ℓ-groups (e.g., representable ℓ-groups), it is not entirely necessary to consider convex ℓ-subgroups, since all the relevant properties of the structure are already encoded into the ℓ-ideals.

In what follows, we study the poset of convex ℓ-subgroups of any ℓ-group H. It is easy to see that the intersection of a set of convex ℓ-subgroups is again a convex ℓ-subgroup. Thus, the set $\mathcal{C}(H)$ of convex ℓ-subgroups partially ordered by inclusion is a complete lattice, where $\mathfrak{k}_1 \wedge \mathfrak{k}_2$ is the intersection $\mathfrak{k}_1 \cap \mathfrak{k}_2$ and $\mathfrak{k}_1 \vee \mathfrak{k}_2$ is the convex ℓ-subgroup generated by the union, i.e.,

$$\mathfrak{k}_1 \vee \mathfrak{k}_2 := \bigcap\{\mathfrak{k} \in \mathcal{C}(H) \mid \mathfrak{k}_1 \cup \mathfrak{k}_2 \subseteq \mathfrak{k}\}.$$

Moreover, the following useful facts are true for the resulting lattice.

Proposition 2.1.2 (cf. [49, Propositions 7.5 & 7.10])**.** *For any ℓ-group H, the poset $\mathcal{C}(H)$ of convex ℓ-subgroups is a distributive algebraic lattice, and a sublattice of the lattice of subgroups of H; i.e., the join of an arbitrary set of convex ℓ-subgroups is generated as a group by their union.*

The *positive cone* of an ℓ-group H is $H^+ = \{a \in H \mid a \geq \mathrm{e}\}$. The proof of Proposition 2.1.2 relies on the following beautiful result.

Proposition 2.1.3 (cf. [49, Theorem 3.11])**.** *For any ℓ-group H, and any $a, b_1, \ldots, b_n \in H^+$ such that $\mathrm{e} \leq a \leq b_1 b_2 \cdots b_n$, there exist $a_1, \ldots, a_n \in H^+$ such that $a = a_1 a_2 \cdots a_n$ and $a_i \leq b_i$, for each $1 \leq i \leq n$.*

This property is of key importance in the study of (lattices of) convex ℓ-subgroups, and is called the 'Riesz Decomposition Property' of ℓ-groups.

We write $\mathfrak{C}(S)$ to denote the convex ℓ-subgroup generated by $S \subseteq H$. If $a \in H$, we write $\mathfrak{C}(a)$ for $\mathfrak{C}(\{a\})$, and call it the *principal convex ℓ-subgroup* generated by a. If H is an ℓ-group and $a \in H$, the *absolute value* $|a| \in H^+$ of x is defined as $a \vee a^{-1}$. The following description of the convex ℓ-subgroup generated by a set is essential in the study of convex ℓ-subgroups.

Proposition 2.1.4 (cf. [49, Propositions 7.11])**.** *For any ℓ-group H, the convex ℓ-subgroup $\mathfrak{C}(S)$ generated by $S \subseteq H$ is*

$$\{a \in H \mid |a| \leq t \text{ for some } t \in \langle |S| \rangle_{\mathrm{e}}\}, \tag{2.3}$$

where $|S| = \{|s| \mid s \in S\}$, and $\langle T \rangle_{\mathrm{e}}$ is the submonoid generated by a subset T of H.

The description presented in Proposition 2.1.4 can be used to show that the set of principal convex ℓ-subgroups of H forms a sublattice of $\mathcal{C}(H)$.

Proposition 2.1.5 (cf. [49, Propositions 7.13 & 7.15])**.** *For any ℓ-group H, and $a, b \in H^+$, $c \in H$:*

(a) $\mathfrak{C}(c) = \mathfrak{C}(|c|) = \{h \in H \mid |h| \leq |c|^n$, *for some* $n \in \mathbb{N}^+\}$.

(b) $\mathfrak{C}(a \wedge b) = \mathfrak{C}(a) \wedge \mathfrak{C}(b) = \mathfrak{C}(a) \cap \mathfrak{C}(b)$ *and* $\mathfrak{C}(a \vee b) = \mathfrak{C}(a) \vee \mathfrak{C}(b)$.

We write $\mathcal{C}_p(H)$ for the sublattice of $\mathcal{C}(H)$ consisting of the principal convex ℓ-subgroups of H.

Proposition 2.1.6 (cf. [49] Proposition 7.16)**.** *For any ℓ-group H, the set $\mathcal{C}_p(H)$ of principal convex ℓ-subgroups of H consists precisely of the compact elements of $\mathcal{C}(H)$.*

It is readily seen that the principal convex ℓ-subgroups are precisely the finitely generated convex ℓ-subgroups.

Now we turn our attention back to ℓ-ideals, and describe a few properties that will be useful in Chapter 3. The set $\mathcal{NC}(H)$ of ℓ-ideals of H partially ordered by inclusion is a complete sublattice of $\mathcal{C}(H)$, and hence of the lattice of subgroups of H ([49, Theorem 8.7]). We write $\mathfrak{I}(S)$ to denote the ℓ-ideal generated by $S \subseteq H$. If $a \in H$, we write $\mathfrak{I}(a)$ for the *principal ℓ-ideal* $\mathfrak{I}(\{a\})$, and $\mathcal{NC}_p(H)$ for the collection of all principal ℓ-ideals of H.

Results similar to Propositions 2.1.4, 2.1.5, and 2.1.6 can be obtained for the ℓ-ideals of any ℓ-group H. We include proofs here, as we could not locate a convenient reference. We write $N(S)$ to denote the normal closure $\{b^{-1}ab \mid b \in H, a \in S\}$ of S.

Proposition 2.1.7. *For any ℓ-group H, the ℓ-ideal $\mathfrak{I}(S)$ generated by a subset $S \subseteq H$ is the convex ℓ-subgroup generated by the normal closure $N(S)$ of S; equivalently, $\mathfrak{I}(S) = \mathfrak{C}(N(S))$.*

Proof. By Proposition 2.1.4, $\mathfrak{C}(N(S)) = \mathfrak{C}(\langle |N(S)| \rangle_e)$, or equivalently,

$$\mathfrak{C}(N(S)) = \mathfrak{C}(\langle N(|S|) \rangle_e).$$

Clearly, $\mathfrak{C}(N(S)) \subseteq \mathfrak{I}(S)$. To conclude the proof, it suffices to show that $\mathfrak{C}(N(S))$ is normal. But this is immediate, since

$$|a| \leq \prod_I b_i^{-1} s_i b_i,$$

for $s_i \in |S|$, $b_i \in H$, and I finite, implies

$$|c^{-1}ac| = c^{-1}|a|c \leq c^{-1}\Big(\prod_I b_i^{-1} s_i b_i\Big)c = \prod_I c^{-1} b_i^{-1} s_i b_i c \in \langle N(|S|) \rangle_e,$$

and hence, $c^{-1}ac \in \mathfrak{C}(N(S))$ for every $c \in H$. □

Proposition 2.1.8. *For any ℓ-group H, and for any $a, b \in H^+$, $c \in H$:*

(a) $\mathfrak{I}(c) = \mathfrak{I}(|c|) = \{h \in H \mid |h| \leq \prod_{i \in I} w_i^{-1}|c|w_i$, *for* $w_i \in H$ *and index set I finite*$\}$.

(b) $\mathfrak{I}(a \wedge b) \subseteq \mathfrak{I}(a) \cap \mathfrak{I}(b)$ *and* $\mathfrak{I}(a \vee b) = \mathfrak{I}(a) \vee \mathfrak{I}(b)$.

Proof. (a) is immediate from Proposition 2.1.7. (b) follows from the fact that $\mathfrak{k} \in \mathcal{C}(H)$ contains elements $a, b \in H^+$ if and only if $a \vee b \in \mathfrak{k}$; also, $a \wedge b$ is contained in every convex ℓ-subgroup containing a or b. □

In light of Proposition 2.1.8, the set $\mathcal{NC}_p(H)$ of principal ℓ-ideals of an ℓ-group H partially ordered by inclusion is a $\vee$-semilattice with minimum $\mathfrak{I}(\mathrm{e}) = \{\mathrm{e}\}$, and a subsemilattice of $\mathcal{NC}(H)$.

Proposition 2.1.9. *For any ℓ-group H, the set $\mathcal{NC}_p(H)$ consists of the compact elements of $\mathcal{NC}(H)$.*

Proof. Pick a compact element $\mathfrak{k}$ of $\mathcal{NC}(H)$, and note that $\mathfrak{k} \subseteq \bigvee_{a\in\mathfrak{k}} \mathfrak{I}(a)$. By compactness, also $\mathfrak{k} \subseteq \mathfrak{I}(a_1) \vee \cdots \vee \mathfrak{I}(a_n)$, for some $a_1, \ldots, a_n \in \mathfrak{k}$. Hence, $\mathfrak{k} = \mathfrak{I}(a_1 \vee \cdots \vee a_n)$ by Proposition 2.1.8.(b). Conversely, take $\mathfrak{I}(a) \subseteq \bigvee_J \mathfrak{k}_j$ for some $a \in H$ and $\mathfrak{k}_j \in \mathcal{NC}(H)$. Then $|a| \in H^+$ equals $b_1 \cdots b_m$, for some $b_1, \ldots, b_m \in \bigcup_J \mathfrak{k}_j$. Thus, $\mathfrak{I}(a) \subseteq \mathfrak{k}_{j_1} \vee \cdots \vee \mathfrak{k}_{j_m}$, with $j_1, \ldots, j_m \in J$ such that $b_i \in \mathfrak{k}_{j_i}$. □

This result also follows in view of the correspondence between congruences and ℓ-ideals, as finitely generated congruences are precisely the compact congruences.

As the lattice $\mathcal{C}(H)$ is algebraic (see Proposition 2.1.2), its completely meet-irreducible elements play a fundamental role. We first consider the (finitely) meet-irreducible elements, and in particular the properties of the lattice quotients induced by such convex ℓ-subgroups. A convex ℓ-subgroup $\mathfrak{k}$ of an ℓ-group H is said to be proper if $\mathfrak{k} \neq H$.

Proposition 2.1.10 (cf. [49, Theorem 9.1])**.** *For any ℓ-group H and proper convex ℓ-subgroup $\mathfrak{p} \in \mathcal{C}(H)$, the following are equivalent:*

(1) $\mathfrak{p}$ *is a meet-irreducible element of the lattice* $\mathcal{C}(H)$.
(2) *For any $a, b \in H$, if $a \wedge b = \mathrm{e}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*
(3) *For any $a, b \in H$, if $a \wedge b \in \mathfrak{p}$, then either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.*
(4) *The lattice quotient $H/\mathfrak{p}$ is totally ordered.*

A convex ℓ-subgroup satisfying any of the equivalent conditions of Proposition 2.1.10 is traditionally called *prime*. Throughout, we write 'prime subgroup' to mean 'prime convex ℓ-subgroup', following usual practice in the literature. We also write Spec H for the set of prime subgroups of H partially ordered by inclusion. Therefore, the set Spec H consists precisely of the meet-irreducible elements of $\mathcal{C}(H)$.

A poset is a root system if the upper bounds of any one of its elements form a chain.

Proposition 2.1.11 (cf. [49] Theorem 9.8)**.** *For any ℓ-group H, the poset* Spec H *is a root system. Further,* Spec H *has minimal elements, and each $\mathfrak{p} \in$* Spec H *contains a minimal element.*

In any ℓ-group H, the minimal elements of Spec H are called minimal prime subgroups, and the set of minimal prime subgroups is denoted by Min H.

For any ℓ-group H, we adopt the standard notation $a \perp b$—read 'a and b are orthogonal'—to denote $|a| \wedge |b| = \mathrm{e}$, for $a, b \in H$. For $S \subseteq H$, we set

$$S^{\perp} = \{a \in H \mid a \perp b \text{ for all } b \in S\};$$

we write $S^{\perp\perp}$ instead of $(S^{\perp})^{\perp}$, and $a^{\perp}$ instead of $\{a\}^{\perp}$ for $a \in H$. It is clear that $a^{\perp} = |a|^{\perp}$ for every $a \in H$. A subset $T \subseteq H$ is a *polar* if it satisfies $T = T^{\perp\perp}$ or, equivalently, if there exists $S \subseteq H$ such that $T = S^{\perp}$. If $a \in H$, the set $a^{\perp\perp}$ is called the *principal polar* generated by a; clearly, $a^{\perp\perp}$ is the inclusion-smallest polar containing a.

We recall here a useful characterization of minimal prime subgroups.

Proposition 2.1.12 (cf. [49, Theorem 14.9])**.** *For any ℓ-group H and any $\mathfrak{p} \in \mathrm{Spec}\, H$, the following are equivalent:*

(1) *The prime $\mathfrak{p}$ is minimal.*
(2) $\mathfrak{p} = \bigcup\{a^{\perp} \mid a \notin \mathfrak{p}\}$.
(3) *For every $a \in \mathfrak{p}$, $a^{\perp} \nsubseteq \mathfrak{p}$.*

Remark 2.1.13. We mention here that the notion of a polar is purely lattice-theoretic, and that Proposition 2.1.12 can also be obtained from a study of the lattice $\mathcal{C}_p(H)$ of principal convex ℓ-subgroups (cf. [162]). This perspective will be treated in Chapter 3.

That minimal prime subgroups of an ℓ-group exist follows by applying Zorn's Lemma. Similarly, we can apply Zorn's Lemma to show that, in an ℓ-group H, there are 'enough' prime subgroups to separate every element $\mathrm{e} \neq a \in H$ from the identity.

Proposition 2.1.14 (cf. [49, Propositions 10.1 & 10.4])**.** *For any ℓ-group H, and any $\mathfrak{k} \in \mathcal{C}(H)$, if $a \in H \setminus \mathfrak{k}$, there exists a convex ℓ-subgroup $\mathfrak{p} \in \mathcal{C}(H)$ such that $\mathfrak{k} \subseteq \mathfrak{p}$, $a \notin \mathfrak{p}$, and $\mathfrak{p}$ is maximal with respect to not containing a. Every such maximal $\mathfrak{p}$ is completely meet-irreducible and hence, prime.*

It is a consequence of Proposition 2.1.14 that if $a \in H \setminus \{\mathrm{e}\}$, there exists a convex ℓ-subgroup of H which is maximal with respect to not containing a, and is prime. A convex ℓ-subgroup with the property just described is called a *value* of a. The set of values for all $a \in H \setminus \{\mathrm{e}\}$ coincides precisely with the set of completely meet-irreducible elements of the lattice $\mathcal{C}(H)$ ([49, Proposition 10.2]).

Remark 2.1.15. By Proposition 2.1.14 it follows that, for any $\mathrm{e} \neq a \in H$, there exists a prime subgroup $\mathfrak{p}_a \in H$ such that $a \notin \mathfrak{p}_a$.

This observation plays a key role in proving a 'Cayley-like representation theorem' for ℓ-groups.

For any ℓ-group H and any prime subgroup $\mathfrak{p}$ of H, easy calculations show that the map

$$\begin{aligned} H &\xrightarrow{R_{\mathfrak{p}}} \mathrm{Aut}(H/\mathfrak{p}) \\ a &\longmapsto R_{\mathfrak{p}}(a) \colon \mathfrak{p}b \mapsto \mathfrak{p}ba \end{aligned} \tag{2.4}$$

is an ℓ-group homomorphism (see, e.g., [49, Proposition 29.1]).

Remark 2.1.16. The kernel of the ℓ-group homomorphism $R_{\mathfrak{p}}\colon H \to \mathrm{Aut}(H/\mathfrak{p})$ is

$$\bigcap_{a\in H} a^{-1}\mathfrak{p}a = \bigcap_{a\in H}\{a^{-1}ta \mid t \in \mathfrak{p}\}, \tag{2.5}$$

and is the greatest ℓ-ideal contained in the prime subgroup $\mathfrak{p}$.

For any chain Ω, an ℓ-subgroup H of $\mathrm{Aut}(\Omega)$ is *transitive* (on Ω)—equivalently, *H acts transitively on* Ω—if for all $r, s \in \Omega$ there exists $f \in H$ such that $f(r) = s$. It was mentioned at the beginning of the section that ℓ-groups acting transitively on a chain play a key role in the theory of ℓ-groups, by acting as 'building blocks' (see Theorem 2.1.19). Generally, transitive ℓ-groups are structurally much easier to deal with than arbitrary ℓ-groups. For example:

Proposition 2.1.17 (cf. [112, Theorem 9.3.5])**.** *Every transitive representable ℓ-group of order-preserving permutations of some chain is totally ordered.*

There are several different ways to show that 'every ℓ-group is an ℓ-subgroup of $\mathrm{Aut}(\Omega)$ for some chain Ω' (see Remark 2.1.21). As we want to put some emphasis on Holland's original proof of this result (for reasons that will be clear in Chapter 4), in the reminder of this section we refer to his original paper [88].

The next key property is easy to check.

Proposition 2.1.18 ([88, Lemma 7])**.** *For any ℓ-group H and* $\mathfrak{p} \in \mathrm{Spec}\,H$*, the action of H on $H/\mathfrak{p}$ is transitive.*

If $\mathfrak{p}_a \in \mathrm{Spec}\,H$ is a value of $a \in H\setminus\{\mathrm{e}\}$, we write R_a for $R_{\mathfrak{p}_a}$. Consider the ℓ-group homomorphism

$$\begin{aligned} H &\xrightarrow{\beta} \prod_{a\in H\setminus\{\mathrm{e}\}} \mathrm{Aut}(H/\mathfrak{p}_a) \\ b &\longmapsto \langle R_a(b) \mid a \in H\setminus\{\mathrm{e}\}\rangle\,. \end{aligned} \tag{2.6}$$

The following results were first proved in [88, Theorem 1; Theorem 2] by Holland. They rely on Remark 2.1.15, as the fact that prime subgroups are 'enough' entails that β has trivial kernel.

Theorem 2.1.19. *For any ℓ-group H, the ℓ-group homomorphism β defined in* (2.6) *is a subdirect embedding of H into the product of transitive ℓ-subgroups $R_a[H]$ of* $\mathrm{Aut}(H/\mathfrak{p}_a)$*, for $a \in H\setminus\{\mathrm{e}\}$.*

Consider a well-order $\preceq$ on $H\setminus\{\mathrm{e}\}$, and define the following relation on $\bigsqcup_{a\in H\setminus\{\mathrm{e}\}} H/\mathfrak{p}_a$, where we use the symbol '$\bigsqcup$' to denote 'disjoint union':

$$\begin{aligned} b \le c \quad \Longleftrightarrow \quad & \text{there is } a \in H\setminus\{\mathrm{e}\} \text{ s.t. } b, c \in H/\mathfrak{p}_a \text{ and } b \le c \text{ in } H/\mathfrak{p}_a, \text{ or} \\ & \text{there are } a_1, a_2 \in H\setminus\{\mathrm{e}\} \text{ s.t. } b \in H/\mathfrak{p}_{a_1} \text{ and } c \in H/\mathfrak{p}_{a_2} \text{ and } a_1 \prec a_2. \end{aligned}$$

We write Ω_H for the resulting chain. For $f = \langle f_a \mid a \in H \setminus \{e\} \rangle$, the map

$$\prod_{a \in H\setminus\{e\}} \mathrm{Aut}(H/\mathfrak{p}_a) \xrightarrow{\gamma} \mathrm{Aut}(\Omega_H)$$

$$f \longmapsto \beta(f) \colon b \mapsto f_a(b) \quad \text{for } b \in H/\mathfrak{p}_a$$

is an ℓ-group homomorphism such that $\gamma(\beta(c))(b) = R_a(c)(b)$.

Theorem 2.1.20. *Every ℓ-group H is (isomorphic to) an ℓ-subgroup of* $\mathrm{Aut}(\Omega_H)$, *and the isomorphism is given by the map $\gamma \circ \beta$.*

Remark 2.1.21. The arguments sketched here for Theorem 2.1.19 and Theorem 2.1.20 are the original arguments from Holland's [88]. It is worth noticing that the statement 'every ℓ-group H is an ℓ-subgroup of $\mathrm{Aut}(\Omega)$ for some chain Ω' can be proved similarly using different sets of subsets of H. For example, by Proposition 2.1.11 and Remark 2.1.15, it is immediate that for each $a \in H \setminus \{e\}$, there exists a minimal prime subgroup $\mathfrak{m} \in \mathrm{Min}\, H$ such that $a \notin \mathfrak{m}$ (i.e., $\bigcap_{\mathfrak{m} \in \mathrm{Min}\, H} \mathfrak{m} = \{e\}$). Therefore, the proofs of Theorems 2.1.19 and 2.1.20 can be adapted to use the collection of all (minimal) prime subgroups of H, instead of the values of H. For similar reasons, prime lattice ideals of (the lattice reduct of) H can also be used, as they are 'enough' in a sense similar to (minimal) prime subgroups. (See Chapter 4.)

2.2 REVISITING CHAPTER 1: AN ALGEBRAIC PERSPECTIVE

In this section, we revisit the main results from Chapter 1, and explain how Theorems 1.3.1 and 1.4.1 can be derived employing the theory of ℓ-groups developed in Section 2.1. We push this further in the next section (Section 2.3), where we show how similar results can be obtained for equational classes of ℓ-groups that do not seem to be covered (at the present stage) by the approach from Chapter 1. Since the main scope of the current section is to clarify some aspects of Chapter 1 which might have been hidden by the syntactic approach, we leave out most of the arguments, and rely on the available literature.

Our primary goal here is to identify which properties of the interplay between ℓ-groups and right-orderable groups allow Theorem 1.3.1 to hold, and similarly for representable ℓ-groups and orderable groups. As we want to emphasize the distinction between properties that are particular to the theory of ℓ-groups and properties that hold in a general algebraic context, we talk generally about congruence relations, and do not exploit the correspondence between congruences and normal subgroups (resp., ℓ-ideals) available for groups (resp., ℓ-groups). We assume basic knowledge of the algebraic theory of quasivarieties, and we refer to [77, 47] for the necessary background on this topic. For the specific background on ℓ-groups and orderable groups used here and in the next section, we provide references to the original sources.

Let us recall some general terminology. If $\mathcal{L}$ is a signature and $\mathcal{L}' \subseteq \mathcal{L}$, an $\mathcal{L}'$-algebra A is said to be an $\mathcal{L}'$-subreduct of an $\mathcal{L}$-algebra B if A is a subalgebra of the

$\mathcal{L}'$-reduct of B. For simplicity, when $\mathcal{L}'$ is the monoid (resp., group, lattice or semilattice) signature, we sometimes say that A is a submonoid (resp., subgroup, sublattice or subsemilattice) of the $\mathcal{L}$-algebra B.

LATTICE-ORDERED GROUPS

Since right-orderable groups are precisely the subgroups (=group subreducts) of the groups $\mathrm{Aut}(\Omega)$ of order-preserving bijections of a chain Ω (see Proposition 1.1.3), by Theorem 2.1.20 we get the following result, originally due to Hollister.

Proposition 2.2.1 ([91, Theorem 5-31])**.** *A group is right-orderable if and only if it is a subgroup of an ℓ-group. In fact, the lattice order of any ℓ-group is the intersection of all the right orders on the underlying group that extend it.*

The following result arises from Conrad's study of free ℓ-groups.

Proposition 2.2.2 ([40])**.** *For any X, the subgroup generated by X in the free ℓ-group $F^{\ell}(X)$ is the free group $F(X)$.*

Remark 2.2.3. In fact, Proposition 2.2.2 can be alternatively proved using the following result, of which we sketch an argument here (cf. [175, Theorem 5.2]):

> Let V be a variety of algebras of type $\mathcal{L}$, and C the class of $\mathcal{L}'$-subreducts (=subalgebras of $\mathcal{L}'$-reducts) of V of type $\mathcal{L}' \subseteq \mathcal{L}$. For any set X, the free object $F_{\mathsf{C}}^{\mathcal{L}'}(X)$ relative to C is the $\mathcal{L}'$-algebra generated by X in the free object $F_{\mathsf{V}}^{\mathcal{L}}(X)$ relative to V.

Consider an assignment $h\colon X \to A$, where A is a subreduct of B of type $\mathcal{L}'$ for some $B \in \mathsf{V}$, and write D for the $\mathcal{L}'$-algebra generated by X in the free object $F_{\mathsf{V}}^{\mathcal{L}}(X)$. Then, the map h extends to a unique $\mathcal{L}$-homomorphism

$$k\colon F_{\mathsf{V}}^{\mathcal{L}}(X) \longrightarrow B,$$

whose restriction $\overline{h}\colon D \to A$ extends h uniquely. Hence, the universal property holds for D, which is thereby isomorphic to $F_{\mathsf{C}}^{\mathcal{L}'}(X)$. Therefore:

- The subgroup generated by X in the free ℓ-group $F^{\ell}(X)$ is the free group over X relative to the quasivariety of right-orderable groups (by Proposition 2.2.1).
- The free group relative to the quasivariety of right-orderable groups is the free group $F(X)$—as the latter is right-orderable (see Proposition 1.1.6).

Recall that, for any $R \subseteq F(X)$, we consider $G = \langle X \mid R \rangle$, and write $\alpha\colon F(X) \twoheadrightarrow G$ for the natural quotient map.

Lemma 2.2.4. *For any X, and $R \subseteq F(X)$ such that $G = \langle X \mid R\rangle$ is right-orderable, the group congruence $\Theta(\Sigma)$ generated by $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$ in the free group $F(X)$ over X coincides with the restriction $\Theta^{\ell}(\Sigma) \cap F(X)^2$ of the ℓ-group congruence $\Theta^{\ell}(\Sigma)$ generated by Σ in the free ℓ-group $F^{\ell}(X)$ over X.*

Proof. We show $\Theta^{\ell}(\Sigma) \cap F(X)^2 \subseteq \Theta(\Sigma)$, as the converse inclusion is immediate. For this, assume

$$(t_1, t_2) \in F(X)^2 \setminus \Theta(\Sigma).$$

Equivalently, assume that $\alpha(t_1 t_2{}^{-1}) \neq \mathrm{e}$. Let $\leq$ be a right order on G, and consider the unique valuation $\varphi\colon T^{\ell}(X) \to \mathrm{Aut}\,(G, \leq)$ extending the assignment

$$x \longmapsto \varphi(x)\colon \alpha(s) \mapsto \alpha(sx).$$

For any $r \in R$, it holds that $\varphi(r) = \mathrm{id}_G$, as

$$\begin{aligned}\varphi(r)(\alpha(s)) &= \alpha(sr)\\ &= \alpha(s)\alpha(r)\\ &= \alpha(s)\mathrm{e}\\ &= \alpha(s).\end{aligned}$$

On the other hand, the pair (t_1, t_2) is not an element of $\ker(\varphi)$, since

$$\begin{aligned}\varphi(t_1 t_2{}^{-1})(\mathrm{e}) &= \alpha(\mathrm{e}(t_1 t_2{}^{-1}))\\ &= \alpha(t_1 t_2{}^{-1})\\ &\neq \mathrm{e}.\end{aligned}$$

Therefore, this argument provides an ℓ-group—namely $\mathrm{Aut}\,(G, \leq)$—and a valuation φ, such that $\Sigma \subseteq \ker(\varphi)$, while $(t_1, t_2) \not\subseteq \ker(\varphi)$. Equivalently, (t_1, t_2) is not included in $\Theta^{\ell}(\Sigma)$. □

Remark 2.2.5. In the current setting, Lemma 2.2.4 is equivalent to the following: the quasiequational theory of ℓ-groups is a conservative extension of the quasiequational theory of right-orderable groups, i.e., a group quasiequation holds in the variety of ℓ-groups if and only if it holds in the quasivariety of right-orderable groups. This result can be obtained as an easy consequence of Proposition 2.2.1, as it is readily seen that:

> The quasiequational theory of a variety V of algebras of type $\mathcal{L}$ is a conservative extension of the quasiequational theory of the class C of $\mathcal{L}'$-subreducts of V of type $\mathcal{L}' \subseteq \mathcal{L}$.

Lemma 2.2.4 can be equivalently formulated as follows.

Lemma 2.2.6. *For any X, and $R \subseteq F(X)$ such that $G = \langle X \mid R\rangle$ is right-orderable, set p to be the natural ℓ-group quotient map $p\colon F^{\ell}(X) \to F^{\ell}(X)/\Theta^{\ell}(\Sigma)$, where $\Theta^{\ell}(\Sigma)$ is the ℓ-group congruence generated by $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$ in the free ℓ-group $F^{\ell}(X)$.*

$$\begin{array}{ccc} F(X) & \xhookrightarrow{\ \subseteq\ } & F^{\ell}(X) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ G & \xrightarrow{\ h\ } & F^{\ell}(X)/\Theta^{\ell}(\Sigma) \end{array} \tag{2.7}$$

Then, the unique group homomorphism h making the diagram (2.7) *commute is an embedding.*

Proof. First, observe that $\alpha(t) = \alpha(s)$ implies $p(t) = p(s)$, for all $s, t \in F(X)$. Hence, the map h obtained by assigning, for any $t \in F(X)$,

$$\alpha(t) \longmapsto p(t),$$

is a well-defined group homomorphism $h\colon G \to F^{\ell}(X)/\Theta^{\ell}(\Sigma)$ which makes the diagram commute. Uniqueness is trivial. Let $h(\alpha(s)) = h(\alpha(t))$ for $s, t \in F(X)$. But then, $p(s) = p(t)$ or equivalently, $(s, t) \in \Theta^{\ell}(\Sigma)$. Thus, by Lemma 2.2.4, also $(s, t) \in \Theta(\Sigma)$, where $\Theta(\Sigma)$ is the group congruence generated by Σ in the free group $F(X)$; that is, $\alpha(t) = \alpha(s)$. □

We thereby identify the group embedding h with the inclusion map.

We have now all the ingredients to provide an algebraic proof of Theorem 1.3.1.

Lemma 2.2.7. *For any right-orderable group G with presentation $\langle X \mid R\rangle$, and any finite set of group terms $t_1,\dots,t_n \in T(X)$, if the set $\{\alpha(t_1),\dots,\alpha(t_n)\} \subseteq G$ does not extend to a right order on G, then $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \cdots \vee t_n$, where $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$.*

Proof. We proceed by contraposition. Suppose $\Sigma \not\models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \cdots \vee t_n$ or, equivalently,

$$(\mathrm{e} \wedge (t_1 \vee \cdots \vee t_n), \mathrm{e}) \in F^{\ell}(X)^2 \setminus \Theta^{\ell}(\Sigma).$$

This means that $p(t_1 \vee \cdots \vee t_n) \not\ge \mathrm{e}$ in $F^{\ell}(X)/\Theta^{\ell}(\Sigma)$. Observe that $p(t_i) \le p(t_1 \vee \cdots \vee t_n)$ for every $1 \le i \le n$, and hence, $\mathrm{e} \le p((t_1 \vee \cdots \vee t_n)t_i{}^{-1})$ for every $1 \le i \le n$. By Proposition 2.2.1, there exists a right order $\preceq$ extending the lattice order of $F^{\ell}(X)/\Theta^{\ell}(\Sigma)$ such that $p(t_1 \vee \cdots \vee t_n) \prec \mathrm{e}$, and $p(t_i) \prec \mathrm{e}$ for every $1 \le i \le n$. But then, the elements $p(t_i)$ are all positive in the right order $\preceq^{\partial}$ on $F^{\ell}(X)/\Theta^{\ell}(\Sigma)$. Now, by Lemma 2.2.6, we have $p(t_i) = \alpha(t_i)$ for every $1 \le i \le n$, as $t_1,\dots,t_n$ are all group terms (and hence, elements of $F(X)$). Therefore, the restriction of $\preceq^{\partial}$ to G is a right order on G where the subset $\{\alpha(t_1),\dots,\alpha(t_n)\}$ is positive. □

Remark 2.2.8. In the argument above, and in (2) ⇒ (1) of Theorem 1.3.1 (see Section 1.3) we make use of the fact mentioned in Chapter 1 that the class of right orders is closed under the 'dual' operation; namely, the dual order (in the sense of (A.42)) of a right order is a right order.

Therefore, the following is now immediate.

Theorem 1.3.1. *For any right-orderable group G with presentation $\langle X \mid R\rangle$, and any finite set of group terms $t_1,\dots,t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{\alpha(t_1),\dots,\alpha(t_n)\} \subseteq G$ does not extend to a right order on G.*
(2) $\Sigma \models_{\mathsf{LG}} \mathrm{e} \le t_1 \vee \cdots \vee t_n$*, where $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$.*

Remarks 2.2.3, 2.2.5, and 2.2.8 are meant to highlight the main reasons why Theorem 1.3.1 holds. Thus, observing that results analogous to Remarks 2.2.3, 2.2.5, and 2.2.8 hold for a given variety of ℓ-groups, allows us to conclude a result analogous to Theorem 1.3.1.

REPRESENTABLE LATTICE-ORDERED GROUPS

The second part of this section deals with orders on groups, and validity in (varieties of) representable ℓ-groups. The analogues of Propositions 2.2.1 and 2.2.2 hold for representable ℓ-groups and orderable groups, and hence, so does the analogue of Theorem 1.3.1. Furthermore, since representable ℓ-groups are particularly well-behaved, we obtain a result about varieties of representable ℓ-groups defined by group equations (Theorem 1.4.1).

Proposition 2.2.9 ([91, Proposition 4-53]; cf. [49, Theorem 47.17])**.** *A group is orderable if and only if it is a subgroup of a representable ℓ-group. In fact, the lattice order of any representable ℓ-group is the intersection of all the orders on the underlying group that extend it.*

Observe the following:

- (Cf. Remark 2.2.3) As the free group $F(X)$ admits an order for any X (see Proposition 1.1.6), the free group over X relative to the class of orderable groups is $F(X)$. Thus, the subgroup of the free representable ℓ-group $F^{\ell}_{\mathsf{Rep}}(X)$ generated by the set X is the free group $F(X)$. (This was first proved by Conrad in [40]—cf. [110, Lemma 1].)
- (Cf. Remark 2.2.5) Proposition 2.2.9 entails that the quasiequational theory of representable ℓ-groups is a conservative extension of the quasiequational theory of orderable groups.
- (Cf. Remark 2.2.8) It is easy to see that the class of orders is closed under the 'dual' operation: the dual order (in the sense of (A.42)) of an order is still an order.

Now, we have all the ingredients to conclude that the analogues of Lemmas 2.2.4 and 2.2.6 hold in the setting of representable ℓ-groups. We therefore obtain the following result.

Theorem 2.2.10. *For any orderable group G with presentation $\langle X \mid R\rangle$, denote by $\alpha\colon F(X) \twoheadrightarrow G$ for the natural quotient map. For any finite set $\{t_1,\dots,t_n\} \subseteq T(X)$, the following are equivalent:*

(1) *The set $\{\alpha(t_1),\dots,\alpha(t_n)\} \subseteq G$ does not extend to an order on G.*
(2) $\Sigma \models_{\mathsf{Rep}} \mathrm{e} \le t_1 \vee \cdots \vee t_n$, *where* $\Sigma = \{r \approx \mathrm{e} \mid r \in R\}$.

Suppose that instead we consider the variety Rep_Σ axiomatized relative to Rep by the set Σ of group equations, and set G_Σ to be the variety of groups defined by Σ. We again write $F_\Sigma(X)$ for the free group over some set X relative to G_Σ, and $F^{\ell}_{\Sigma}(X)$ for the free ℓ-group relative to Rep_Σ. We identify a group term $t \in T(X)$ with its reduced form in $F_\Sigma(X)$.

We now consider the following observations. It follows from Proposition 2.2.9 that a group G in the variety G_Σ is orderable if and only if it is a subgroup of an ℓ-group H from Rep_Σ. Therefore, the quasivariety of subgroups of members of Rep_Σ is the class of orderable groups from G_Σ. If the free group $F_\Sigma(X)$ over a set X relative to G_Σ is orderable, we can conclude that the subgroup of $F_\Sigma^\ell(X)$ generated by X is the relatively free group $F_\Sigma(X)$. (See Remark 2.2.3.) Since the class of orders is closed under the 'dual' operation, it is immediate that if $(G,\leq)$ is a member of Rep_Σ, so is $(G,\leq^\partial)$. (See Remark 2.2.8.)

Lemma 2.2.11. *For any set Σ of group equations such that the relatively free group $F_\Sigma(X)$ is orderable, and for any finite set of group terms $t_1,\dots,t_n \in T(X)$, if the set $\{t_1,\dots,t_n\} \subseteq F_\Sigma(X)$ does not extend to an order on $F_\Sigma(X)$, then $\mathsf{Rep}_\Sigma \models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$.*

Proof. We proceed by contraposition. Suppose $\mathsf{Rep}_\Sigma \not\models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$. Hence, there exists an order $\preceq$ extending the lattice order of $F_\Sigma^\ell(X)$ such that $t = (t_1 \vee \cdots \vee t_n) \prec \mathrm{e}$, and thus $t_i \prec \mathrm{e}$ for every $1 \leq i \leq n$. But then, the elements t_i are all positive in the order $\preceq^\partial$ on $F_\Sigma^\ell(X)$. Thus, the restriction of $\preceq^\partial$ to $F_\Sigma(X)$ is an order on $F_\Sigma(X)$ where the subset $\{t_1,\dots,t_n\}$ is positive. □

Therefore, we get an algebraic proof of the following.

Theorem 1.4.1. *For any set Σ of group equations such that the relatively free group $F_\Sigma(X)$ is orderable, and any finite set of group terms $t_1,\dots,t_n \in T(X)$, the following are equivalent:*

(1) *The set $\{t_1,\dots,t_n\} \subseteq F_\Sigma(X)$ does not extend to an order on $F_\Sigma(X)$.*
(2) $\mathsf{Rep}_\Sigma \models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$.

We identify in the following observation the key difference between the ℓ-group case, and the case of representable ℓ-groups. The right regular representation of a group G in the variety defined by Σ equipped with a right order need not satisfy Σ (cf. [50]). The situation differs for orders, as the right regular representation of a totally ordered group is isomorphic (as an ℓ-group) to the totally ordered group at hand. In fact, *the quasivariety of group subreducts of representable ℓ-groups coincides with the class of group reducts of representable ℓ-groups.*[1]

2.3 NORMAL-VALUED AND WEAKLY ABELIAN VARIETIES

First, we saw in Chapter 1 that Theorems 1.3.1 and 1.4.1 can be applied to the varieties of ℓ-groups and representable ℓ-groups, respectively. Moreover, Thoerem 1.4.1 provides a correspondence result for a broad class of varieties, including the varieties of Abelian ℓ-groups and nilpotent ℓ-groups (of class $c \in \mathbb{N}^+$). In particular, in these two cases the result can be simplified as a consequence of Propositions 1.1.8 and 1.1.9 (see Example 1.4.3).

In this section, we illustrate how the arguments sketched in Section 2.2 can be adapted to other relevant classes of ℓ-groups: the variety of normal-valued ℓ-groups, and the variety of weakly Abelian ℓ-groups.

[1]Note that the class of groups that can be equipped with a compatible lattice order is not even elementary ([168])—that is, axiomatizable in first-order logic.

NORMAL-VALUED LATTICE-ORDERED GROUPS

We illustrate how the arguments sketched above can be adapted to another relevant variety, namely the variety of normal-valued ℓ-groups.

For any ℓ-group H, it is easy to show that every value $\mathfrak{p}$ of H has a cover $\hat{\mathfrak{p}}$ in $\mathcal{C}(H)$, that is, a smallest convex ℓ-subgroup of H properly extending $\mathfrak{p}$. We say that a value $\mathfrak{p}$ of an ℓ-group H is normal in its cover if for every $a \in \mathfrak{p}$ and $b \in \hat{\mathfrak{p}}$, it holds that $b^{-1}ab \in \mathfrak{p}$. An ℓ-group H is said to be *normal-valued* if every value of H is normal in its cover. The class of normal-valued ℓ-groups can be defined relative to the variety of ℓ-groups by the equation

$$|x||y| \wedge |x|^2|y|^2 \approx |x||y|,$$

as was first shown by Wolfenstein in [174]. The variety N of normal-valued ℓ-groups is of great importance, since it is the largest proper subvariety of ℓ-groups ([89]).

Seemingly unrelated, the notion of Conradian right order takes its name from Conrad who first introduced it in [34]. For any group G, a right order C on G is called *Conradian* if for all $a, b \in C$, there exists an $n \in \mathbb{N}^+$ such that $a^n ba^{-1} \in C$ (that is, $a <_C a^n ba^{-1}$).

Proposition 2.3.1 ([110, Theorem 4])**.** *A group admits a Conradian right order if and only if it is a subgroup of a normal-valued ℓ-group. The positive cone of any normal-valued ℓ-group is the intersection of all Conradian right orders on the underlying group extending its lattice order.*

It is possible to prove directly that the class of those groups that admit a Conradian right order is a quasivariety. Nonetheless, this also follows from the first part of Proposition 2.3.1, since it is a result of Mal'cev ([125]) that:

> For any $\mathcal{L}' \subseteq \mathcal{L}$, any variety V_1 of type $\mathcal{L}'$, and any variety V_2 of type $\mathcal{L}$, the subclass of V_1 consisting of all $\mathcal{L}'$-subreducts of algebras in V_2 is itself a quasivariety.

Therefore:

- (Cf. Remark 2.2.3) Since every order is a Conradian right order, the free group over X relative to the quasivariety of Conradian right-orderable groups is the free group $F(X)$. Thus, the subgroup of the free normal-valued ℓ-group $F^{\ell}_{\mathsf{N}}(X)$ generated by X is $F(X)$.
- (Cf. Remark 2.2.8) Further, it is easy to see that the class of Conradian right orders is closed under the 'dual' operation: the dual order (in the sense of (A.42)) of a Conradian right order still is a Conradian right order (see, e.g., [152, Exercise 1.1]).

We can now prove the following analogue of Corollary 1.3.2.

Theorem 2.3.2. *For any set $t_1, \dots, t_n \in T(X)$, the following are equivalent:*

(1) *The set* $\{t_1,\dots,t_n\} \subseteq F(X)$ *does not extend to a Conradian right order on* $F(X)$.
(2) $\mathsf{N} \models \mathrm{e} \le t_1 \vee \cdots \vee t_n$.

Proof. (1) ⇒ (2). We proceed by contraposition. Suppose $\mathsf{N} \not\models \mathrm{e} \le t_1 \vee \cdots \vee t_n$ or, equivalently, $\mathrm{e} \not\le t_1 \vee \cdots \vee t_n$ in $F^\ell_{\mathsf{N}}(X)$. Now, by Proposition 2.3.1 there exists a Conradian right order $\preceq$ extending the lattice order of $F^\ell_{\mathsf{N}}(X)$ such that $t_1 \vee \cdots \vee t_n \prec \mathrm{e}$. Observe that $\mathrm{e} \le (t_1 \vee \cdots \vee t_n) t_i^{-1}$ for every $1 \le i \le n$, and hence, $\mathrm{e} \preceq (t_1 \vee \cdots \vee t_n) t_i^{-1}$ for every $1 \le i \le n$, which entails $t_i \prec \mathrm{e}$ for every $1 \le i \le n$. But then, the elements t_i are all positive in the right order $\preceq^\partial$ on $F^\ell_{\mathsf{N}}(X)$. Therefore, the restriction of $\preceq^\partial$ to $F(X)$ is a Conradian right order on $F(X)$ where the subset $\{t_1,\dots,t_n\}$ is positive.

(2) ⇒ (1). We proceed by contraposition. Let C be the positive cone of a Conradian right order on $F(X)$ such that $t_1,\dots,t_n \in C$. Consider the dual order C^∂, which we know to be Conradian. Clearly, t_i is strictly negative for each $1 \le i \le n$. Consider the ℓ-subgroup H_{C^∂} of the ℓ-group $\mathrm{Aut}(G, \le_\partial)$ generated by the right regular representation of G defined in (1.1); consider also the valuation $\varphi\colon T^\ell(X) \to H_{C^\partial}$, obtained by extending the assignment

$$x \longmapsto \varphi(x)\colon s \mapsto sx, \quad \text{for } s \in F(X).$$

Every $t \in T(X) \subseteq T^\ell(X)$ is sent to the order-automorphism $s \mapsto st$, for $s \in F(X)$. Thus, we have

$$\begin{aligned} \varphi(t_1 \vee \cdots \vee t_n)(\mathrm{e}) &= (\varphi(t_1) \vee \cdots \vee \varphi(t_n))(\mathrm{e}) \\ &= \max_\partial\{\varphi(t_1)(\mathrm{e}),\dots,\varphi(t_n)(\mathrm{e})\} \\ &= \max_\partial\{t_1,\dots,t_n\}, \end{aligned}$$

and $\max_\partial\{t_1,\dots,t_n\}, <_\partial \mathrm{e}$, as t_i is strictly negative for each $1 \le i \le n$. Therefore, since the order on $\mathrm{Aut}(G, \le_\partial)$, and hence on H_{C^∂}, is defined pointwise, we conclude

$$\mathrm{id}_G = \varphi(\mathrm{e}) \not\le \varphi(t_1 \vee \cdots \vee t_n);$$

equivalently, $\mathsf{N} \not\models \mathrm{e} \le t_1 \vee \cdots \vee t_n$, as was to be shown. □

It was proved by McCleary ([127]) that the equational theory of normal-valued ℓ-groups is decidable. Hence, Theorem 2.3.2 yields the following result.

Corollary 2.3.3. *For any* $k \in \mathbb{N}$, *the problem of deciding if a finite subset of* $F(k)$ *extends to a Conradian right order is decidable.*

WEAKLY ABELIAN LATTICE-ORDERED GROUPS

An ℓ-group H is *weakly Abelian* if for all $\mathrm{e} < a \in H$ and for all $b \in H$, it holds that $a^2 > b^{-1}ab$. The class of weakly Abelian ℓ-groups is a variety, and can be defined relative to the variety of ℓ-groups by the equation

$$(x \wedge \mathrm{e})^2 \le y^{-1}(x \wedge \mathrm{e})y. \tag{2.8}$$

Weakly Abelian ℓ-groups were introduced by Martinez[2] in [126], with the purpose of providing an example of a variety between Abelian ℓ-groups and representable ℓ-groups. Uncountably many varieties are now known to exist between the varieties of Abelian ℓ-groups and representable ℓ-groups. Nonetheless, the variety W of weakly Abelian ℓ-groups remains one of the most well-studied varieties, in view of its interesting properties. For instance, every weakly Abelian ℓ-group H is *Hamiltonian*, i.e., every convex ℓ-subgroup of H is normal ([126, 3.2]); moreover, the variety of weakly Abelian ℓ-groups is the largest variety of Hamiltonian ℓ-groups ([148, Corollary 2.3]).

An order $\leq$ on a group G is called *weakly Abelian* if the resulting totally ordered group is weakly Abelian or, equivalently, if for all $\mathrm{e} < a \in G$ and $b \in G$, we have $b^{-1}ab \leq a^2$. We include a proof of the following, as we could not locate a convenient reference.

Proposition 2.3.4. *A group admits a weakly Abelian order if and only if it is the subgroup of a weakly Abelian ℓ-group; further, the positive cone of any weakly Abelian ℓ-group is the intersection of all weakly Abelian orders on the underlying group extending its lattice order.*

Proof. The first part of the statement is clear, since weakly Abelian ℓ-groups are representable. Let H be a weakly Abelian ℓ-group and, by representability, we can think of H as embedded into a direct product $\prod_{i\in I} H_i$ of weakly Abelian totally ordered groups. It is clear that the positive cone of H is included in the intersection of all weakly Abelian orders on the underlying group extending its lattice order. Suppose now $a \in H \setminus H^+$. This means that $a_k < \mathrm{e}$ for some component $k \in I$, by definition of the direct product order. We well-order the index set I in such a way that k is the least element, and consider the lexicographic order obtained by declaring positive $b \in \prod_{i\in I} H_i$ such that $\min\{i \in I \mid b_i \neq \mathrm{e}\}$ is positive. Clearly, a is negative. Further, the order is easily proved to be weakly Abelian, and to extend the lattice order of H. □

Further, the free group $F(X)$ over a set X admits a weakly Abelian order ([126, Proposition 3.7]), and is therefore free in the quasivariety of subgroups of W.

Lemma 2.3.5. *For any group G, if $\leq$ is a weakly Abelian order on G, then also $\leq^\partial$ is a weakly Abelian order on G.*

Proof. Let G be a group, and $\leq$ be a weakly Abelian order on G, that is, for all $\mathrm{e} < a \in G$, $b \in G$,

$$a^2 \leq b^{-1}ab.$$

Suppose now $\mathrm{e} <^\partial a$. Then, $\mathrm{e} < a^{-1}$ in the original order, and hence, $(a^{-1})^2 \leq b^{-1}a^{-1}b$ for all $b \in G$. Equivalently, $\mathrm{e} < b^{-1}a^{-1}ba^2$ for all $b \in G$. Thus, in the dual order we have $\mathrm{e} <^\partial (a^{-1})^2 b^{-1}ab$ and, since $\leq$ is an order (both right- and left-invariant), we can conclude $a^2 \leq^\partial b^{-1}ab$. □

[2]Kopytov and Medvedev originally referred to weakly Abelian ℓ-groups as *rigidly ordered ℓ-groups* ([111]).

Therefore, we have now all the ingredients to conclude:

Theorem 2.3.6. *For any set* $t_1,\dots,t_n \in T(X)$*, the following are equivalent:*

(1) *The set* $\{t_1,\dots,t_n\} \subseteq F(X)$ *does not extend to a weakly Abelian order on* $F(X)$*.*
(2) $\mathsf{W} \models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$.

Proof. (1) ⇒ (2) We proceed by contraposition. Suppose $\mathsf{W} \not\models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$. Hence, there exists a weakly Abelian order $\preceq$ extending the lattice order of the free weakly Abelian ℓ-group $F^{\ell}_{\mathsf{W}}(X)$ over X such that $t = (t_1 \vee \cdots \vee t_n) \prec \mathrm{e}$, and thus $t_i \prec \mathrm{e}$ for every $1 \leq i \leq n$. But then, the elements t_i are all positive in the order $\preceq^{\partial}$ on $F^{\ell}_{\mathsf{W}}(X)$. Thus, the restriction of $\preceq^{\partial}$ to $F(X)$ is an order on $F(X)$ where the subset $\{t_1,\dots,t_n\}$ is positive.

(2) ⇒ (1). We proceed by contraposition. Let C be the positive cone of a weakly Abelian order on $F(X)$ such that $t_1,\dots,t_n \in C$, and consider its dual order C^{∂} on $F(X)$. Clearly, the totally ordered group $(F(X), \leq_{\partial})$ is an ℓ-group in W. Consider the valuation $\varphi\colon T^{\ell}(X) \to (F(X), \leq_{\partial})$, obtained by extending the identity map $x \mapsto x$. But then, as t_i is strictly negative for each $1 \leq i \leq n$,

$$\varphi(t_1 \vee \cdots \vee t_n) = \max\nolimits_{\partial}\{\overline{t}_1,\dots,\overline{t}_n\} <_{\partial} \mathrm{e}.$$

Hence, we conclude $\varphi(\mathrm{e}) \not\leq \varphi(t_1 \vee \cdots \vee t_n)$, and $\mathsf{W} \not\models \mathrm{e} \leq t_1 \vee \cdots \vee t_n$ as was to be shown. □

A systematic account of this correspondence between varieties of ℓ-groups and classes of right-ordered groups will be carried out in the next chapter, where the relationship between (relatively) free groups and (relatively) free ℓ-groups mentioned here will be studied in a more general setting, via associated topological spaces.

2.4 CONCLUDING REMARKS

This chapter is intended as a bridge between the ideas developed in Chapter 1 and the rest of the thesis, providing an algebraic account of the results obtained so far. More precisely, the correspondence between validity of equations in (representable) ℓ-groups and subsets of free groups that extend to (right) orders on the group was proved here using general algebra, together with some central results proper to the theory of ℓ-groups.

This chapter provides an algebraic explanation of the results in Chapter 1. As remarked at the beginning of this chapter, at the present stage, the algebraic approach seems to have a broader scope than the syntactic approach, as the results obtained in Chapter 1 can be extended to other varieties of ℓ-groups, e.g., normal-valued and weakly Abelian ℓ-groups (Theorems 2.3.2 and 2.3.6). This allows us to draw some immediate conclusions, including the decidability of the problem whether a finite subset of a finitely generated free group extends to a Conradian right order. The following problem is related to Problem 1 in Section 1.5.

Problem 4. Use the syntactic approach (cf. Chapter 1) to obtain proofs of Theorems 2.3.2 and 2.3.6, thereby broadening the scope of those techniques so to encompass also, e.g., weakly Abelian and normal-valued ℓ-groups and, more generally, other varieties of ℓ-groups.

A systematic approach in this direction seems difficult, and would be related to some of the problems discussed in Section 1.5 (e.g., obtaining a framework suitable for a systematic account of the proof theory for ℓ-groups). However, it is reasonable to conjecture that something more may be said in specific well-behaved cases, as we already do for varieties of representable ℓ-groups defined by group equations. In fact, the main idea behind the proofs of Theorems 1.2.3 and 1.2.13 illustrated in Chapter 1 is that the ordering conditions described in Lemmas 1.1.10 and 1.1.11 can be 'translated' into inductive conditions, thereby providing a description of the sets of those subsets of a group that do not extend to right orders and orders, respectively. Consider now the following result for Conradian right orders.

> A subset $S \subseteq G$ of a group G extends to a Conradian right order on G if and only if for any finite set $\{a_1, \ldots, a_n\} \subseteq G \setminus \{e\}$ there exist $\delta_1, \ldots, \delta_n \in \{-1, 1\}$ such that e is not contained in the smallest subsemigroup C of G containing $(S \setminus \{e\}) \cup \{a_1^{\delta_1}, \ldots, a_n^{\delta_n}\}$ and such that, for all $a, b \in C$, the element a^2ba^{-1} also belongs to C. ([139, Proposition 3.10])

(Analogous result for weakly Abelian orders is due to Kopytov and Medvedev's [111].) The fact that results analogous to Lemmas 1.1.10 and 1.1.11 are available for Conradian right orders and weakly Abelian orders gives us hope to extend the scope of the syntactic techniques to these settings. Nonetheless, it is not clear at the present stage that the conditions that characterize Conradian right orders and weakly Abelian orders are suitable to be treated algorithmically.

The following problem is in line with the systematic approach that will be carried out in the next chapter.

Problem 5. Show that Corollaries 1.3.2 and 1.4.2, Theorems 2.3.2 and 2.3.6 are instances of a more general property, which can be formulated in terms of varieties of ℓ-groups and classes of group subreducts (with suitable (pre)orders).

Some results available in the literature seem to be pointing towards a positive solution to this problem. In particular, we mention the work by Valerii Kopytov, who showed in [110] that the free ℓ-group relative to any variety of ℓ-groups can be represented by considering collections of right orders on suitable relatively free groups.

CHAPTER 3

ORDERS ON GROUPS THROUGH SPECTRAL SPACES

That a topological space can be associated to any ℓ-group, by considering its spectral space (briefly, ℓ-spectrum) is a notable fact from the theory of ℓ-groups. The spectral space of an Abelian ℓ-group was introduced by Klaus Keimel in his doctoral dissertation (1971), as the set of its prime ℓ-ideals with the hull-kernel topology. The notion of ℓ-spectrum is not limited to the commutative setting, and can also be defined for an arbitrary ℓ-group (see, e.g., [43]), by considering the collection of its prime subgroups. (We write 'spectrum' when it is clear from the context that we mean 'ℓ-spectrum'.)

In 2004, Adam Sikora topologized the set of right orders on a group, and studied the resulting topological space ([160]). His paper 'Topology on the spaces of orderings of groups' pioneered a different perspective on the study of the interplay between topology and ordered groups, that has led to applications to both orderable groups and algebraic topology. The basic construction is the definition of a topology on the set of right orders on a given right-orderable group[1], which is then proved to be compact, Hausdorff, and zero-dimensional.

The theory of ℓ-groups and the theory of right-orderable groups are known to be deeply related, and this interdependence was largely the focus of Chapters 1 and 2. For this reason, it is natural to ask if there exists some relationship between the topological space of right orders on a right-orderable group, and the spectral space of some ℓ-group. In this chapter we provide a positive answer to this question, by showing, *inter alia*, that Sikora's space for a group G arises naturally from the study of the ℓ-group freely generated by the group G, as the subspace of minimal elements of its spectrum. More generally, by replacing right orders with right preorders—preorders that are invariant under group multiplication on the right—we provide a systematic, structural account of the relationship between right preorders on a group G and prime subgroups of the ℓ-group $F^{\ell}(G)$ freely generated by the group G (Theorem 3.3.6).

[1] More precisely, Sikora introduces a natural topology on the set of left orders on an arbitrary semigroup.

It is often relevant to restrict attention to special classes of right preorders on a group (e.g., Conradian right orders, orders) and, in such cases, studying the free ℓ-group over G is not enough. For example, to study orders on a group G, the free ℓ-group $F^{\ell}(G)$ needs to be replaced with the free representable ℓ-group generated by G. The study of specific varieties is the focus of Section 3.5, where minimal spectra of free ℓ-groups relative to specific varieties are related to spaces of right orders. To this end, Section 3.4 is concerned with the minimal spectrum, paying special attention to cases where it is compact.

This chapter is based on the paper [30]. The theory and terminology from topology, Stone duality, and category theory used here are reviewed in Appendix A.1 and Appendix A.3.

3.1 TOPOLOGICAL SPACES OF RIGHT ORDERS

For any group G, we write 2^G to denote the powerset of G, in view of the standard bijection mapping a subset $S \subseteq G$ to its characteristic function $\chi_S \colon G \to 2$. Now, consider

$$\mathbb{X}(a) = \{S \subseteq G \mid a \in S\} \quad \text{and} \quad \mathbb{X}^c(a) = \{S \subseteq G \mid a \notin S\}, \qquad \text{for } a \in G.$$

We endow 2^G with the smallest topology containing all sets $\mathbb{X}(a)$ and $\mathbb{X}^c(a)$ (this is the usual product topology). With this topology, the space 2^G is easily shown to be a Boolean space, i.e., compact, Hausdorff, and zero-dimensional (see [26, p. 6]). Recall that the set of all right orders on a right-orderable group G can be identified with the set $\mathcal{R}(G)$ of all positive cones on G (see Remark 1.1.1). The set $\mathcal{R}(G)$ carries a natural topology, first studied in [160] by Sikora. The topology is obtained by regarding the set $\mathcal{R}(G)$ as a closed subset of 2^G ([26, Problem 1.38]) with the subspace topology. The resulting topological space $\mathcal{R}(G)$ is a Boolean space, as it is a closed subspace of a Boolean space and, as such, is compact, Hausdorff, and zero-dimensional. The topology has a subbase (of clopens) consisting of

$$\{C \in \mathcal{R}(G) \mid a \in C\} \quad \text{and} \quad \{C \in \mathcal{R}(G) \mid a \notin C\}, \qquad \text{for } a \in G. \tag{3.1}$$

For each $a \in G \setminus \{\mathrm{e}\}$ and $C \in \mathcal{R}(G)$, $a \notin C$ if and only if $a^{-1} \in C$. Hence, the basic open sets are of the form $\{C \in \mathcal{R}(G) \mid a_1, \dots, a_n \in C\}$, for $n \in \mathbb{N}^+$ and $a_1, \dots, a_n \in G$. Finally, as the set of orders on G is in bijection with the subset $\mathcal{O}(G)$ of normal positive cones $C \in \mathcal{R}(G)$, we may also equip $\mathcal{O}(G)$ with the subspace topology inherited from $\mathcal{R}(G)$.

Remark 3.1.1. Recall that an *isolated point* in a topological space (X, τ) is an element $x \in X$ such that $\{x\}$ is open in the topology τ. Therefore, for any right-orderable group G, a right order $C \in \mathcal{R}(G)$ is an isolated point in the topology described by (3.1) if and only if C is the unique right order containing $\{a_1, \dots, a_n\}$, for some finite subset $\{a_1, \dots, a_n\}$ of the group G.

Example 3.1.2. For $k \geq 2$, the space $\mathcal{R}(\mathbb{Z}^k) = \mathcal{O}(\mathbb{Z}^k)$ has no isolated points, and hence is homeomorphic to the Cantor space. The idea behind the proof can be illustrated as follows. Pick $a_1,\ldots,a_4 \in \mathbb{Z}^2$, and consider an order C on $\mathbb{Z}^2$ making $a_1,\ldots,a_4$ positive (Figure 3.1).

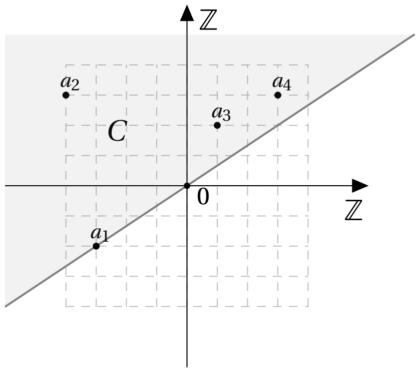

Figure 3.1: Right order C containing $\{a_1,\ldots,a_4\}$

It is now possible to perturbate the order C slightly, thereby obtaining a different order D on $\mathbb{Z}^2$ which also makes the points $a_1,\ldots,a_4$ positive. By this observation, and Example 1.1.7, it is clear that a similar perturbation can be applied to any order C on $\mathbb{Z}^2$, and any finitely many positive points $\{a_1,\ldots,a_n\} \subseteq \mathbb{Z}^2$ (Figure 3.2).

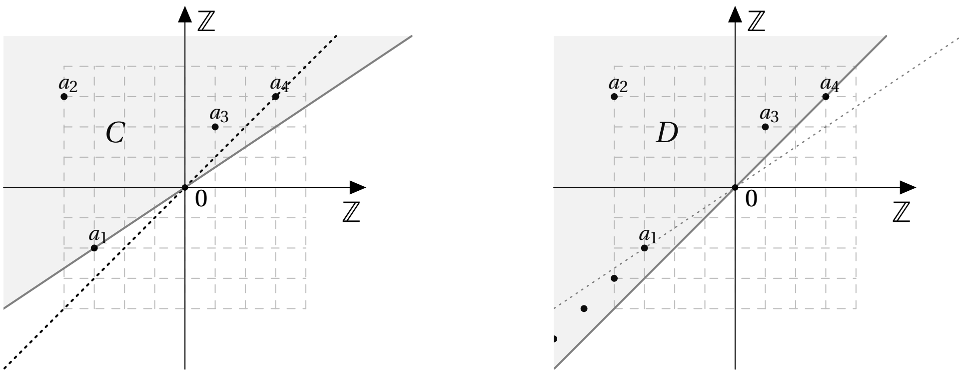

Figure 3.2: Right order $D \neq C$ containing $\{a_1,\ldots,a_4\}$

The absence of isolated points holds in greater generality in the Abelian setting. For any torsion-free Abelian group G with rank greater than one, the space $\mathcal{R}(G) = \mathcal{O}(G)$ has no isolated points ([4]).

Example 3.1.3. For any free group $F(k)$ with $k \geq 2$, the space $\mathcal{R}(F(k))$ has no isolated points, and hence is homeomorphic to the Cantor space ([129, Corollary 4]; see also [139, Theorem A], [25, Corollary 6], [153, Theorem B]).

We mention that groups admitting only finitely many right orders exist, and we have already seen one such example in Chapter 1.

Example 3.1.4. It is not hard to see that the space $\mathcal{R}(K)$ of right orders of the fundamental group K of the Klein bottle is finite ($|\mathcal{R}(K)| = 4$; e.g., [140]), and hence all its points are isolated.

We include here, for the sake of completeness, a family of examples of groups whose space of right orders is uncountable and has isolated points.

Example 3.1.5. Braid groups have played a key role in the study of interplay between right orders on groups, and topology (see, e.g., [53]). For all $n \geq 1$, the space $\mathcal{R}(B_n)$ of right orders on the braid group B_n is uncountable and has isolated points ([59]; cf. [26, Theorem 10.24]).

In the remainder of this section, we replace the notion of right order with that of right preorder, and the notion of group with the (more general) notion of partially ordered group, for the purpose of paving the way for a systematic, structural account of the relationship between right preorders on a (partially ordered) group, and the prime subgroups of a suitable ℓ-group. By a *partially ordered group* we mean a group G equipped with a partial order $\leq$ compatible with the group operation, that is, from $a \leq b$ we can conclude $cad \leq cbd$, for all $a, b, c, d \in G$. The positive cone of a partially ordered group G is $G^+ = \{a \in G \mid a \geq \mathrm{e}\}$. We use the notation '$G$' for both the notion of 'group' and that of 'partially ordered group', as the former can be seen as an instance of the latter. It will be clear from the context (or explicitly stated) whether a specific instance of 'G' refers to a group or to a partially ordered group.

For a partially ordered group G, a *(right) order on* G is a (right) order on the group G extending (the partial order with positive cone) G^+. We call (right-)orderable a partially ordered group G that can be equipped with a (right) order. Similarly to the case of partial orders, a (always total) preorder $\preceq$ on a group G is right-invariant (resp., left-invariant) if for all $a, b, c \in G$, whenever $a \preceq b$ then $ac \preceq bc$ (resp., $ca \preceq cb$). For a partially ordered group G, a *right preorder on* G is a proper (i.e., $\preceq \neq G \times G$) right-invariant preorder on G that extends the partial order on G; a *preorder on* G is a left-invariant right preorder on G. We set

$$\mathcal{P}(G) = \{C \subseteq G \mid C \text{ is a submonoid of } G,\ G^+ \subseteq C \text{ and } G = C \cup C^{-1}\}.$$

For a partially ordered group G, the set of right preorders on G is a poset under inclusion, and similarly, the set $\mathcal{P}(G)$ is partially ordered by inclusion. It is easy to see that these two posets are isomorphic via the map that associates to $C \in \mathcal{P}(G)$ the relation: $a \preceq_C b$ if and only if $ba^{-1} \in C$. The inverse of this bijection sends a right preorder $\preceq$ to its positive cone $C = \{a \in G \mid \mathrm{e} \preceq a\}$. This isomorphism restricts to an isomorphism between preorders on G and those elements of $\mathcal{P}(G)$ that are normal; we denote the subposet of such elements by $\mathcal{B}(G)$. It is convenient here to identify a right preorder $\preceq$ on G with its associated positive cone C. It follows from the definition that if $\preceq$ is a right preorder on a group G, and C is its positive cone, it is in general only true that $C \cap C^{-1} \supseteq \{\mathrm{e}\}$. It is clear that the subset $\mathcal{R}(G)$ of $\mathcal{P}(G)$ consisting of those $C \in \mathcal{P}(G)$ such that $C \cap C^{-1} = \{\mathrm{e}\}$ is in bijection with the set of right orders on G, and that this correspondence restricts to a bijection between the set $\mathcal{O}(G)$ of those elements $C \in \mathcal{R}(G)$ that are also normal, and the set of orders on G.

Remark 3.1.6. We bring to the reader's attention that we could have chosen to work with strict positive cones ($\{a \in G \mid \mathrm{e} \prec a\}$) instead of positive cones, as they too are in one-to-one correspondence with right preorders. Both choices have advantages and disadvantages. We have decided to work with positive cones, since the map that sends a right preorder to its strict positive cone reverses the inclusion order, in the sense that 'minimal right preorders correspond to maximal strict positive cones'.

Proposition 3.1.7. *A group G admits a right preorder if and only if G acts (non-trivially) on a chain by order-preserving bijections (see Proposition 1.1.3). Equivalently, a group G admits a right preorder if and only if it has a (non-trivial) right-orderable quotient.*

Example 3.1.8. Consider again the free Abelian group $\mathbb{Z}^2$ over two generators. We have seen in Example 1.1.7 that every line $y = qx$, where $q \in \mathbb{Q}$, determines four orders on $\mathbb{Z}^2$, obtained by choosing different half-planes and different half-lines to be positive. Similarly, every such line $y = qx$ determines two preorders on $\mathbb{Z}^2$ which are *not* orders. More precisely, a preorder on $\mathbb{Z}^2$ can be typically obtained by considering a line $y = qx$, where $q \in \mathbb{Q}$, and choosing which half-plane to make positive, therefore regarding every pair $(x, y) \in \mathbb{Z}^2$ on $y = qx$ as 'equivalent to 0' (i.e., we obtain a preorder C for which $C \cap C^{-1} = \{(x, y) \in \mathbb{Z}^2 \mid y = qx\}$). To the best of our knowledge, a full classification of preorders on $\mathbb{Z}^k$, for $k \geq 2$, is not present in the literature.

The following definition provides the natural extension to right preorders of the 'right regular representation' defined in (1.1), which is standard, *mutatis mutandis*, for right orders and groups. We write G_C for the partially ordered group G equipped with the right preorder $C \supseteq G^+$. Then C induces an equivalence relation $\equiv_C$ on G defined by: $a \equiv_C b$ if and only if $a \preceq_C b$ and $b \preceq_C a$. We write $[a]$ for the equivalence class of $a \in G$, where C is understood from the context. The quotient set of G modulo $\equiv_C$ (often called the 'poset reflection' of G_C), which we denote by Ω_C, is totally ordered by: $[a] \leq_C [b]$ if and only if $a \preceq_C b$. It is easy to see that the map

$$\begin{aligned} G &\xrightarrow{R_C} \mathrm{Aut}(\Omega_C) \\ a &\longmapsto R_C(a)\colon [b] \mapsto [ba] \end{aligned} \tag{3.2}$$

is a positive group homomorphism. However, its image $R_C[G]$ is in general not an ℓ-subgroup of $\mathrm{Aut}(\Omega_C)$. We denote by H_C the ℓ-subgroup of $\mathrm{Aut}(\Omega_C)$ generated by $R_C[G]$, and call the map R_C the *right regular representation* of G_C. We consistently use the notation H_C throughout this chapter as in the previous definition.

We recall that an ℓ-group H of order-preserving permutations of a chain Ω is transitive if for all $r, s \in \Omega$ there exists $f \in H$ such that $f(r) = s$.

Proposition 3.1.9. *For any partially ordered group G and any right preorder C on G, the ℓ-group H_C is transitive on the chain Ω_C.*

Proof. For $a, b \in G$, the equivalence class $[a]$ is sent to $[b]$ by $R_C(a^{-1}b)$ as defined in (3.2). □

By (3.2), we can classify any right preorder C on a partially ordered group G based on the equational properties of the ℓ-group H_C.

Definition 3.1.10. For any partially ordered group G and any variety V of ℓ-groups, we write $\mathcal{P}_{\mathsf{V}}(G)$ for the set of right preorders $C \in \mathcal{P}(G)$ such that $H_C \in \mathsf{V}$. Further, we write $\mathcal{B}_{\mathsf{V}}(G)$ for the subset of $\mathcal{P}_{\mathsf{V}}(G)$ consisting of preorders on G.

Thus, clearly $\mathcal{P}_{\mathsf{LG}}(G) = \mathcal{P}(G)$. Observe that $\mathcal{P}_{\mathsf{V}}(G)$ may well be empty even when G is non-trivial (see Example 3.1.7). For any partially ordered group G, we call representable a right preorder $C \in \mathcal{P}_{\mathsf{Rep}}(G)$, and Abelian a right preorder $C \in \mathcal{P}_{\mathsf{Ab}}(G)$. Note that a partially ordered group equipped with an Abelian right preorder does not need to be Abelian as a group.

For any variety V of ℓ-groups, we set

$$\mathbb{P}(a) = \{C \in \mathcal{P}_{\mathsf{V}}(G) \mid a \in C \text{ and } a^{-1} \notin C\}, \quad \text{for } a \in G,$$

and endow $\mathcal{P}_{\mathsf{V}}(G)$ with the topology generated by $\mathbb{P}(a)$ for all $a \in G$, and $\mathcal{B}_{\mathsf{V}}(G)$ with the subspace topology. For any group G, Sikora's space of right orders $\mathcal{R}(G)$ is homeomorphic to the subspace of $\mathcal{P}(G)$ consisting of all those $C \in \mathcal{P}(G)$ such that $C \cap C^{-1} = \{\mathrm{e}\}$. Later, it will be shown that, if nonempty, the space $\mathcal{R}(G)$ consists of the minimal elements of $\mathcal{P}(G)$ (Corollary 3.5.8).

3.2 SPECTRAL SPACES OF LATTICE-ORDERED GROUPS

First appearing in the work of Stone ([164]), the importance of spectral spaces became clear with Grothendieck's work in the field of algebraic geometry (see, e.g., [80]). The name 'spectral spaces' was coined by Hochster ([86]), and denotes those topological spaces that are sober, compact, and whose compact open subsets form a base closed under finite intersections.

In the next example and in the rest of the thesis, we are using the notion of 'specialization order' recalled in Appendix A.3, following the definition adopted in [56]; however, note that the dual relation is very often used in the literature.

Example 3.2.1. Examples of spectral spaces can be easily obtained by considering finite sets. Pick a finite set $n = \{0, \dots, n-1\}$ for $n \in \mathbb{N}^+$, with the natural total order. The latter yields a topology with opens described by $\{0, \dots, k-1\}$ for any $0 \le k \le n$. The resulting topological space is a spectral space, whose specialization order is the natural total order. Note that this construction generalizes to a spectral space for every finite poset P with its downset topology, i.e., the topology whose opens consist precisely of the downward closed subsets of P.

In the work of Stone, the spectrum of a bounded distributive lattice was defined as the set of its prime ideals with a suitable topology. What we nowadays call 'spectral spaces' were shown by Stone to be precisely the spectra of bounded distributive lattices—extending the most celebrated correspondence between Boolean spaces and Boolean algebras. As it turns out, the construction of spectra of distributive lattices can be adapted, *mutatis mutandis*, to associate spectral spaces with other kinds of mathematical structures.

The notion of the spectral space associated to an ℓ-group was introduced by Keimel ([106]; cf. [10, Chapitre 10]) in the Abelian setting, and later extended to the non-commutative setting (see, e.g., [43]; also, [169, 170]). We define here the notion of spectral space for an ℓ-group, and illustrate it with some examples; we then exhibit the spectrum of an ℓ-group H via a purely lattice-theoretic construction, as the Stone dual of the lattice of principal convex ℓ-subgroups of H (see Theorem 3.2.6). Most of the results in this section do not appear explicitly in the literature, even though they definitely are known within the community. For this reason, we often include (sketches of) proofs.

Recall that if H is any ℓ-group, $\operatorname{Spec} H$ is the set of prime subgroups of H; we also set $\operatorname{Spec}^* H$ to be the subset of $\operatorname{Spec} H$ consisting of prime ℓ-ideals. We topologize $\operatorname{Spec} H$ using the *spectral* (or *hull-kernel*, or *Stone*, or *Zariski*) topology whose open sets are those of the form

$$\mathbb{S}(A) = \{\mathfrak{p} \in \operatorname{Spec} H \mid A \not\subseteq \mathfrak{p}\} = \bigcap_{a \in A} \mathbb{S}(a), \quad \text{for } A \subseteq H,$$

where $\mathbb{S}(a)$ stands for $\mathbb{S}(\{a\})$ (see, e.g., [49, Proposition 49.6]). The closed sets are those of the form

$$\mathbb{V}(A) = \{\mathfrak{p} \in \operatorname{Spec} H \mid A \subseteq \mathfrak{p}\}, \quad \text{for } A \subseteq H.$$

We call $\operatorname{Spec} H$ with the spectral topology the *ℓ-spectrum* (and sometimes just 'spectrum') of H. We also topologize $\operatorname{Spec}^* H$ by the subspace topology, with opens $\mathbb{S}^*(A)$ for $A \subseteq H$. For the time being, we set aside the study of $\operatorname{Spec}^* H$; more will be said about its properties in Section 3.5.

Example 3.2.2. For any ℓ-group H isomorphic to a finite (of size $n \in \mathbb{N}^+$) lexicographic product of Archimedean totally ordered groups, the spectrum $\operatorname{Spec} H$ is a finite (of size n) totally ordered set. More generally, for any finite root system P, it is possible to find an (Abelian) ℓ-group whose spectral space is homeomorphic to P with the downset topology ([35]).

In full generality, the spectral space of an ℓ-group H need not be compact—we say that it is generalized spectral, as it is a sober space whose compact open subsets form a base closed under finite intersections (see Corollary 3.2.7).

Example 3.2.3. The spectrum $\operatorname{Spec} H$ of the ℓ-group $H := \mathrm{C}(\mathbb{R})$ of continuous functions over $\mathbb{R}$ is not compact. Also, the spectrum $\operatorname{Spec} H$ of the ℓ-group $H := \prod_{n\in\omega} \mathbb{Z}$ is not compact, showing that compactness of the spectral space is not preserved by direct products (as the spectral space of the ℓ-group $\mathbb{Z}$ is compact).

Throughout, we write $\mathbb{S}(a)$ in place of $\mathbb{S}(\{a\})$ for $a \in H$, and similarly for $\mathbb{V}(\{a\})$.

Proposition 3.2.4 (cf. [49, Proposition 49.7])**.** *For any ℓ-group H, the set $\{\mathbb{S}(a) \mid a \in H\}$ is a base for the topology of* $\operatorname{Spec} H$.

Proposition 3.2.5. *For any ℓ-group H, and for any $a, b \in H^+$, $c \in H$:*

(a) $\mathbb{S}(c) = \mathbb{S}(|c|)$.

(b) $\mathbb{S}(a \wedge b) = \mathbb{S}(a) \cap \mathbb{S}(b)$ *and* $\mathbb{S}(a \vee b) = \mathbb{S}(a) \cup \mathbb{S}(b)$.

Proof. (a) and (b) are immediate consequences of Proposition 2.1.5. □

Recall that $\mathcal{C}_p(H)$ denotes the sublattice of $\mathcal{C}(H)$ consisting of the principal convex ℓ-subgroups of H; see Proposition 2.1.5. The lattice $\mathcal{C}_p(H)$ has a minimum ($\mathfrak{C}(\mathrm{e})$), but not necessarily a maximum. We prove in this section that Spec H is the Stone dual of the distributive lattice $\mathcal{C}_p(H)$, thereby concluding that it is a generalized spectral space. Recall that the Stone dual of a distributive lattice D with minimum is obtained by considering on the set $X(D)$ of prime ideals of D a topology with subbase $\widehat{a} = \{I \in X(D) \mid a \notin I\}$, for $a \in D$ (see Appendix A.3).

Theorem 3.2.6. *For any ℓ-group H, the map*

$$\begin{aligned} X(\mathcal{C}_p(H)) &\xrightarrow{\mu} \mathcal{C}(H) \\ I &\longmapsto \bigvee\{\mathfrak{C}(a) \mid \mathfrak{C}(a) \in I\} \end{aligned} \tag{3.3}$$

restricts to a homeomorphism between $X(\mathcal{C}_p(H))$ and Spec H*. The compact open sets of* Spec H *are precisely those of the form $\mathbb{S}(a)$, for $a \in H$.*

Proof. We first show that μ from (3.3) is a bijection onto the set Spec H of prime subgroups.

Consider $I \in X(\mathcal{C}_p(H))$. For $a \in \mu(I)$, it follows from Proposition 2.1.6 that there are finitely many $\mathfrak{C}(a_1), \ldots, \mathfrak{C}(a_n) \in I$ such that $\mathfrak{C}(a) \subseteq \mathfrak{C}(a_1) \vee \cdots \vee \mathfrak{C}(a_n)$. Since I is closed under finite joins and downward closed, we conclude that $\mathfrak{C}(a) \in I$. Thus, $\mathfrak{C}(a) \in I$ if and only if $a \in \mu(I)$, and injectivity of μ is now obvious.

To prove primeness of $\mu(I)$ and surjectivity of μ, we make repeated use of Proposition 2.1.5.(b). For primeness, if $a \wedge b \in \mu(I)$, then $\mathfrak{C}(a) \cap \mathfrak{C}(b) = \mathfrak{C}(a \wedge b) \in I$. Since I is prime, either $\mathfrak{C}(a) \in I$ or $\mathfrak{C}(b) \in I$, that is, either $a \in \mu(I)$ or $b \in \mu(I)$.

For surjectivity, we pick a prime subgroup $\mathfrak{p}$ of H and consider the set $I_{\mathfrak{p}} = \{\mathfrak{C}(a) \mid a \in \mathfrak{p}\}$. Clearly, $I_{\mathfrak{p}}$ is downward closed and closed under finite joins. Now, $\mathfrak{C}(a) \cap \mathfrak{C}(b) \in I_{\mathfrak{p}}$ is equivalent to $\mathfrak{C}(a \wedge b) \in I_{\mathfrak{p}}$, and the latter is equivalent to $a \wedge b \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, and hence, either $\mathfrak{C}(a) \in I_{\mathfrak{p}}$ or $\mathfrak{C}(b) \in I_{\mathfrak{p}}$. This shows that $I_{\mathfrak{p}}$ is a prime ideal of $\mathcal{C}_p(H)$. Since, evidently, $\mathfrak{p} = \bigvee\{\mathfrak{C}(a) \mid a \in \mathfrak{p}\}$, we have $\mu(I_{\mathfrak{p}}) = \mathfrak{p}$.

Regarding now μ as a bijection $\mu\colon X(\mathcal{C}_p(H)) \to \operatorname{Spec} H$, we show that μ is a homeomorphism. First, since for $a \in H$, we have $\mathfrak{C}(a) \in I$ if and only if $a \in \mu(I)$, we may infer

$$\mu[\widehat{\mathfrak{C}(a)}] = \{\mathfrak{p} \in \operatorname{Spec} H \mid a \notin \mathfrak{p}\} = \mathbb{S}(a). \tag{3.4}$$

Since μ preserves arbitrary unions and intersections, this shows that μ is an open map. By Proposition 3.2.4, it also shows that μ is continuous, and hence a homeomorphism. Finally, it is a classical result that the compact open sets of $X(\mathcal{C}_p(H))$ are precisely those of the form $\widehat{\mathfrak{C}(a)}$ (see, e.g., [104]), and hence by (3.4) we have established that the compact open sets of Spec H are precisely those of the form $\mathbb{S}(a)$, for $a \in H$. □

Various versions of Theorem 3.2.6 have circulated as folklore amongst researchers in the field. We have included a full proof because we are not aware of a reference at this level of generality. For related work on Abelian ℓ-groups with a strong unit, *alias* MV-algebras, see [68, and references therein]. The construction in Theorem 3.2.6 is the exact analogue for ℓ-groups of Simmons' well-known reticulation of a ring ([161]).

An element u of an ℓ-group H is a *strong (order) unit* if for all $a \in H$ there is an $n \in \mathbb{N}^+$ such that $a \leq u^n$; equivalently, by Proposition 2.1.5.(a), if $\mathfrak{C}(u) = H$.

Corollary 3.2.7. *For any ℓ-group H, the space* $\operatorname{Spec} H$ *is a generalized spectral space. It is spectral if and only if the ℓ-group H has a strong unit.*

Proof. We here use the classical result that the Stone dual space of a distributive lattice D with minimum is a generalized spectral space that is compact if and only if D has a maximum (see Appendix A.3). Suppose now that $u \in H$ is a strong unit. Then, we have $\mathfrak{C}(u) = H$. Therefore, the lattice $\mathcal{C}_p(H)$ has a maximum, and its dual space $\operatorname{Spec} H$ is compact. Conversely, if $\operatorname{Spec} H$ is compact, then $\operatorname{Spec} H = \mathbb{S}(u)$ for some $u \in H$. But then, by the definition of $\mathbb{S}(u)$, every prime subgroup of H omits u. A standard Zorn's Lemma argument then shows that every proper convex ℓ-subgroup of H omits u. Hence, u is a strong unit. □

Proposition 3.2.8. *For any ℓ-group H, the poset* $\operatorname{Spec} H$ *is a root system, and the specialization order of the generalized spectral space* $\operatorname{Spec} H$ *coincides with the inclusion order.*

Proof. The first part of the statement follows from Proposition 2.1.11. For the second statement, first note that, for any $\mathfrak{p} \in \operatorname{Spec} H$, $\mathfrak{p} \in \mathbb{V}(\mathfrak{p})$ and every closed set $\mathbb{V}(A)$ that contains $\mathfrak{p}$ also contains $\mathbb{V}(\mathfrak{p})$. Thus, $\mathbb{V}(\mathfrak{p})$ is the closure of $\mathfrak{p}$. Further, for $\mathfrak{q} \in \operatorname{Spec} H$, if $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathfrak{q} \in \mathbb{V}(\mathfrak{p})$, that is, $\mathfrak{p} \leq \mathfrak{q}$ in the specialization order. Conversely, if the latter holds, then $\mathfrak{q} \in \mathbb{V}(\mathfrak{p})$, so $\mathfrak{p} \subseteq \mathfrak{q}$. □

It is not necessary for an ℓ-spectrum to have maximal elements. However, the existence of a strong unit suffices for the maximal elements to exist, and to be well-behaved.

Proposition 3.2.9 (cf. [10, Théorème 10.2.2])**.** *For any ℓ-group H, if H has a strong unit, then every prime subgroup* $\mathfrak{p} \in \operatorname{Spec} H$ *is extended by a (unique!) maximal prime subgroup.*

In this case, we write $\operatorname{Max} H$ for the set of maximal prime subgroups of H endowed with the subspace topology inherited from $\operatorname{Spec} H$.

Remark 3.2.10. Every finitely generated ℓ-group H has a strong unit (if $\{a_1, \ldots, a_n\}$ generates H, then $|a_1| \vee \cdots \vee |a_n|$ is a strong unit). Therefore, every prime subgroup $\mathfrak{p} \in \operatorname{Spec} H$ is extended by a unique maximal prime subgroup.

3.3 ORDER-PRESERVING HOMEOMORPHISMS

We describe now the construction that allows us to state—and later, prove—the main result of the chapter. The construction shows that to each variety of ℓ-groups there is associated a class of right preorders on groups. We refer to Appendix A.1 for the notions and terminology from category theory used in the next paragraph.

We write P for the category of partially ordered groups and their positive (equivalently, order-preserving) group homomorphisms. We identify here any variety V of ℓ-groups with the full subcategory of the category of ℓ-groups whose objects are the ℓ-groups in V. Let us write $P\colon \mathsf{V} \to \mathsf{P}$ for the inclusion functor that takes an ℓ-group H in V to H itself regarded as a partially ordered group in P. The functor P has a left adjoint $F^{\ell}_{\mathsf{V}}\colon \mathsf{P} \to \mathsf{V}$, in symbols, $F^{\ell}_{\mathsf{V}} \dashv P$. To show this, it suffices to exhibit an ℓ-group and a universal arrow η_G for any object G in P.[2]

Proposition 3.3.1. *The functor P has a left adjoint $F^{\ell}_{\mathsf{V}}\colon \mathsf{P} \to \mathsf{V}$.*

Proof. For G a partially ordered group, set $F^{\ell}_{\mathsf{V}}(X)$ to be the ℓ-group freely generated by the underlying set X of G relative to V. We consider now the smallest ℓ-group congruence θ_G on $F^{\ell}_{\mathsf{V}}(X)$ that contains $(x, a \cdot b)$ whenever $ab = x$ in G, (y, a^{-1}) provided that $a^{-1} = y$ in G, (z, e) where $z = \mathrm{e}$ in G, and finally, $(a \wedge b, a)$ whenever $a \leq b$ in G. Let $\alpha\colon F^{\ell}_{\mathsf{V}}(X) \twoheadrightarrow F^{\ell}_{\mathsf{V}}(X)/\theta_G$ be the natural quotient map. Let $\eta_G\colon G \to F^{\ell}_{\mathsf{V}}(X)/\theta_G$ be defined as

$$\eta_G(a) = \alpha(a).$$

By the construction of θ_G, it follows readily that η_G is a positive group homomorphism. Further, suppose that $p\colon G \to H$ is a positive group homomorphism from G to an ℓ-group H in V. Then, we consider the partial map h defined by $h(\eta_G(a)) = p(a)$. That this map extends to a (unique!) ℓ-group homomorphism $h\colon F^{\ell}_{\mathsf{V}}(X)/\theta_G \to H$ follows from the fact that the relations satisfied in the ℓ-group $F^{\ell}_{\mathsf{V}}(X)/\theta_G$ by the generating set $\eta_G[G]$ are preserved by h. This is a routine verification, given the construction of the objects involved. □

We denote $F^{\ell}_{\mathsf{V}}(X)/\theta_G$ by $F^{\ell}_{\mathsf{V}}(G)$, and call it the ℓ-group *free over the partially ordered group G in* V, or *freely generated by G in* V. We use F^{ℓ}_{V} to denote free ℓ-groups over sets and possibly, as in this chapter, over other algebraic structures. We favour this slight ambiguity over a heavier notation, and trust that context clarifies details.

By Proposition 3.3.1, the component at G of the unit of the adjunction $F^{\ell}_{\mathsf{V}} \dashv P$, written[3]

$$G \xrightarrow{\eta} F^{\ell}_{\mathsf{V}}(G), \tag{3.5}$$

[2] Here and elsewhere we adopt the style common in algebra of omitting forgetful functors—P, for the case in point—unless clarity requires otherwise.

[3] We write η in place of η_G for the component of the unit, G being understood.

is therefore characterized by the following universal property: *For each positive group homomorphism $p\colon G \to H$, with H an ℓ-group in V, there is exactly one ℓ-group homomorphism $h\colon F^{\ell}_{\mathsf{V}}(G) \to H$ such that $h \circ \eta = p$, i.e., such that the following diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \eta\ } & F^{\ell}_{\mathsf{V}}(G) \\ & {}_{p}\searrow & \Big\downarrow {\scriptstyle !\,h} \\ & & H \end{array} \tag{3.6}$$

commutes. We write $F^{\ell}(G)$ for $F^{\ell}_{\mathsf{LG}}(G)$.

Remark 3.3.2. The notion defined above is what Bigard, Keimel, Wolfenstein ([10, Appendice A.2]), and similarly, Conrad ([40]), call a ‘universal ℓ-group over a partially ordered group G’. In their terminology, the ‘free ℓ-group over a partially ordered group G’ has the further property that the universal arrow η is also an order-embedding. We do not follow their distinction, and speak of free objects in all cases. The construction sketched in Proposition 3.3.1 provides a generalization of the result on the existence of universal ℓ-groups over a partially ordered group given by Bigard, Keimel, Wolfenstein in [10, Théorème A.2.2].

Proposition 3.3.3. *For any partially ordered group G and any variety V of ℓ-groups, the image $\eta[G] \subseteq F^{\ell}_{\mathsf{V}}(G)$ of G under η generates $F^{\ell}_{\mathsf{V}}(G)$ as a lattice.*

Proof. Write $\widehat{G}$ for the ℓ-subgroup of $F^{\ell}_{\mathsf{V}}(G)$ generated by $\eta[G]$. Then the positive group homomorphism $G \to \widehat{G}$ that agrees with η on G enjoys the universal property of η because any ℓ-group homomorphism with domain $\widehat{G}$ is uniquely determined by its action on any generating set of $\widehat{G}$. It follows by a standard argument on the uniqueness of universal arrows (see Appendix A.1) that $\widehat{G} = F^{\ell}_{\mathsf{V}}(G)$. Since η is a group homomorphism, $\eta[G]$ is a subgroup of $\widehat{G}$ and, as such, is already closed under group operations; therefore, by the distributivity properties of ℓ-groups—recall that in any ℓ-group the lattice is distributive and the group operation distributes over meets and joins—$\eta[G]$ must generate $F^{\ell}_{\mathsf{V}}(G)$ as a lattice (see Remark 1.3.3). □

Remark 3.3.4. It was shown in [40, 1.1] that, for any ℓ-group H, if A is an Abelian subgroup of H, then the ℓ-subgroup of H obtained as the distributive lattice generated by A is also Abelian. Therefore, by Proposition 3.3.3, $F^{\ell}_{\mathsf{V}}(G)$ is Abelian if and only if $\eta[G]$ is Abelian, for every partially ordered group G, and any variety V of ℓ-groups.

For any partially ordered group G, and any right preorder in $\mathcal{P}_{\mathsf{V}}(G)$, we exhibit a corresponding prime subgroup of $F^{\ell}_{\mathsf{V}}(G)$ as follows. Given a right preorder $C \in \mathcal{P}_{\mathsf{V}}(G)$, write

$$h_C\colon F^{\ell}_{\mathsf{V}}(G) \longrightarrow H_C$$

for the ℓ-group homomorphism such that $h_C \circ \eta = R_C$. Let $H_C[\mathrm{e}]$ be the *stabilizer of* $[\mathrm{e}]$, i.e.,

$$H_C[\mathrm{e}] := \{f \in H_C \mid f([\mathrm{e}]) = [\mathrm{e}]\}. \tag{3.7}$$

Note that $H_C[\mathrm{e}] \cap R_C[G]$ coincides with the image under R_C of the equivalence class $[\mathrm{e}]$. It is immediate that $H_C[\mathrm{e}]$ is a subgroup of H_C, as for all $f, g \in H_C[\mathrm{e}]$, clearly $f(g([\mathrm{e}])) = [\mathrm{e}]$; similarly, $H_C([\mathrm{e}])$ is also a sublattice of H_C, since for all $f, g \in H_C[\mathrm{e}]$,

$$(f \wedge g)([\mathrm{e}]) = \min\{f([\mathrm{e}]), g([\mathrm{e}])\} = [\mathrm{e}] \quad \text{and} \quad (f \vee g)([\mathrm{e}]) = \max\{f([\mathrm{e}]), g([\mathrm{e}])\} = [\mathrm{e}].$$

Further, if $f, g \in H_C[\mathrm{e}]$ and $h \in H_C$ is such that $f \le h \le g$ in H_C, then

$$f([\mathrm{e}]) = [\mathrm{e}] \le h([\mathrm{e}]) \le [\mathrm{e}] = g([\mathrm{e}]),$$

which means $h([\mathrm{e}]) = [\mathrm{e}]$, making $H_C[\mathrm{e}]$ into a convex ℓ-subgroup of H_C. In fact, $H_C[\mathrm{e}]$ is a prime subgroup of H_C by Proposition 2.1.10: if $f, g \in H_C$ are such that $(f \wedge g)([\mathrm{e}]) = [\mathrm{e}]$, then $\min\{f([\mathrm{e}]), g([\mathrm{e}])\} = [\mathrm{e}]$ and hence, either $f([\mathrm{e}]) = [\mathrm{e}]$ or $g([\mathrm{e}]) = [\mathrm{e}]$ (cf. [72, Section 1.5]). Now, since it is readily seen that the preimage of a prime subgroup under an ℓ-group homomorphism still is a prime subgroup, also ${h_C}^{-1}(H_C[\mathrm{e}])$ is a prime subgroup of $F^{\ell}_{\mathsf{V}}(G)$. Hence, we may define a map

$$\kappa\colon \mathcal{P}_{\mathsf{V}}(G) \longrightarrow \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G) \tag{3.8}$$

by setting $\kappa(C) := {h_C}^{-1}(H_C[\mathrm{e}])$.

We now show how to associate to any prime subgroup of $F^{\ell}_{\mathsf{V}}(G)$ a right preorder in $\mathcal{P}_{\mathsf{V}}(G)$. For this, we need some preliminary observations. First, we recall the construction described in (2.4). If H is an ℓ-group and $\mathfrak{p} \in \operatorname{Spec} H$, the map

$$\begin{aligned} H &\xrightarrow{R_{\mathfrak{p}}} \operatorname{Aut}(H/\mathfrak{p}) \\ a &\longmapsto R_{\mathfrak{p}}(a)\colon \mathfrak{p}b \mapsto \mathfrak{p}ba \end{aligned}$$

is an ℓ-group homomorphism. Note that $H/\mathfrak{p}$ is naturally a totally ordered group if and only if $\mathfrak{p} \in \operatorname{Spec}^* H$—and in this case, $R_{\mathfrak{p}}[H]$ is isomorphic as an ℓ-group to $H/\mathfrak{p}$ (cf. [49, Theorem 8.4]). If $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$, we write $\Omega_{\mathfrak{p}}$ for the chain $F^{\ell}_{\mathsf{V}}(G)/\mathfrak{p}$.

Given a prime subgroup $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$, we define the relation $\preceq_{\mathfrak{p}}$ on G by

$$a \preceq_{\mathfrak{p}} b \quad \Longleftrightarrow \quad \mathfrak{p}\eta(a) \le \mathfrak{p}\eta(b). \tag{3.9}$$

We write $C_{\mathfrak{p}}$ for the set $\{a \in G \mid \mathrm{e} \preceq_{\mathfrak{p}} a\}$. Note that $\mathrm{e} \preceq_{\mathfrak{p}} a$ if and only if $\mathfrak{p}\mathrm{e} \le \mathfrak{p}\eta(a)$, which by (2.1) means $\mathrm{e} \le t\eta(a)$ for some $t \in \mathfrak{p}$. Since $\mathfrak{p}$ is a subgroup, and the order is preserved by left multiplication, $\mathrm{e} \le t\eta(a)$ for some $t \in \mathfrak{p}$ if and only if $t' \le \eta(a)$ for some $t' \in \mathfrak{p}$. Therefore, the set $C_{\mathfrak{p}}$ is $\eta^{-1}[\uparrow\mathfrak{p}]$, where $\uparrow\mathfrak{p}$ is, as customary, the set $\{b \in F^{\ell}_{\mathsf{V}}(G) \mid t \le b \text{ for some } t \in \mathfrak{p}\}$.

Lemma 3.3.5. *For any partially ordered group G and any variety V of ℓ-groups the map*

$$\pi\colon \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G) \longrightarrow \mathcal{P}_{\mathsf{V}}(G) \tag{3.10}$$

defined by $\pi(\mathfrak{p}) := C_{\mathfrak{p}}$ is a well-defined function.

Proof. It is easy to check that π is a function from $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$ into $\mathcal{P}(G)$, and it remains to show that $H_{C_{\mathfrak{p}}} \in \mathsf{V}$ for any $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$. We now show that the ℓ-group $H_{C_{\mathfrak{p}}}$ generated into $\operatorname{Aut}(\Omega_{C_{\mathfrak{p}}})$ by the image of G under the right regular representation $R_{C_{\mathfrak{p}}}$ (see (3.2)) is isomorphic to the ℓ-group $R_{\mathfrak{p}}[F^{\ell}_{\mathsf{V}}(G)]$ defined as in (2.4). This allows us to conclude that $C_{\mathfrak{p}} \in \mathcal{P}_{\mathsf{V}}(G)$ or, equivalently, $H_{C_{\mathfrak{p}}}$ is a member of V, as it is (isomorphic to) a homomorphic image of $F^{\ell}_{\mathsf{V}}(G)$ (namely, $R_{\mathfrak{p}}[F^{\ell}_{\mathsf{V}}(G)]$).

Claim 1. *For any $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$ and $a \in F^{\ell}_{\mathsf{V}}(G)$, it holds that $\mathfrak{p}a = \mathfrak{p}\eta(g)$, for some $g \in G$.*

Proof. By Proposition 3.3.3, each $a \in F^{\ell}_{\mathsf{V}}(G)$ is of the form $\bigwedge_I \bigvee_{J_i} \eta(g_{ij})$, for $g_{ij} \in G$ and $i \in I, j \in J_i$, where I and J_i are finite index sets. Since $\Omega_{\mathfrak{p}} := F^{\ell}_{\mathsf{V}}(G)/\mathfrak{p}$ is such that, for any $b, c \in F^{\ell}_{\mathsf{V}}(G)$, $\mathfrak{p}b \wedge \mathfrak{p}c = \mathfrak{p}(b \wedge c)$ and $\mathfrak{p}b \vee \mathfrak{p}c = \mathfrak{p}(b \vee c)$ (see (2.2)), we get $\mathfrak{p}a = \bigwedge_I \bigvee_{J_i} \mathfrak{p}\eta(g_{ij})$. But then, because $\Omega_{\mathfrak{p}}$ is in fact a chain and I, J_i are finite, $\mathfrak{p}a = \mathfrak{p}\eta(g_{ij})$, for some $g_{ij} \in G$ and some $i \in I, j \in J_i$. □

Claim 2. *For any $\mathfrak{p} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$, the images $R_{\mathfrak{p}}[\eta[G]]$ and $R_{C_{\mathfrak{p}}}[G]$ are isomorphic as groups.*

Proof. Note that for all $g, h \in G$, $[g] \leq_{C_{\mathfrak{p}}} [h]$ holds in the poset reflection $\Omega_{C_{\mathfrak{p}}}$ of $G_{C_{\mathfrak{p}}}$ if and only if $g \preceq_{\mathfrak{p}} h$, which is in turn equivalent to $\mathfrak{p}\eta(g) \leq \mathfrak{p}\eta(h)$, by (3.9). Therefore, the map $\tau\colon \Omega_{C_{\mathfrak{p}}} \to \Omega_{\mathfrak{p}}$ defined by $[g] \mapsto \mathfrak{p}\eta(g)$ is an order-isomorphism between the chains $\Omega_{\mathfrak{p}}$ and $\Omega_{C_{\mathfrak{p}}}$, since it is an order-embedding by construction, and it is onto by Claim 1. Therefore, the ℓ-groups $\operatorname{Aut}(\Omega_{\mathfrak{p}})$ and $\operatorname{Aut}(\Omega_{C_{\mathfrak{p}}})$ are isomorphic, where the required ℓ-group isomorphism is defined by

$$\operatorname{Aut}(\Omega_{C_{\mathfrak{p}}}) \xrightarrow{\hat{\tau}} \operatorname{Aut}(\Omega_{\mathfrak{p}}) \tag{3.11}$$

sending $f \in \operatorname{Aut}(\Omega_{C_{\mathfrak{p}}})$ to the order-automorphism $\hat{\tau}(f)\colon \mathfrak{p}\eta(g) \mapsto \tau(f([g]))$. Finally, the ℓ-group isomorphism $\hat{\tau}$ restricts to a bijection between (the subgroups) $R_{\mathfrak{p}}[\eta[G]]$ and $R_{C_{\mathfrak{p}}}[G]$, as for $h \in G$,

$$\hat{\tau}(R_{C_{\mathfrak{p}}}(h))\colon \mathfrak{p}\eta(g) \longmapsto \mathfrak{p}\eta(gh),$$

that is, $\hat{\tau}(R_{C_{\mathfrak{p}}}(h)) = R_{\mathfrak{p}}(\eta(h))$. □

By the preceding claim, and by the facts that the ℓ-group $R_{\mathfrak{p}}[F^{\ell}_{\mathsf{V}}(G)]$ is generated by $R_{\mathfrak{p}}[\eta[G]]$, and similarly $H_{C_{\mathfrak{p}}}$ is generated by $R_{C_{\mathfrak{p}}}[G]$, we infer that $R_{\mathfrak{p}}[F^{\ell}_{\mathsf{V}}(G)]$ and $H_{C_{\mathfrak{p}}}$ are isomorphic. Since $R_{\mathfrak{p}}[F^{\ell}_{\mathsf{V}}(G)]$ is a member of V, so is $H_{C_{\mathfrak{p}}}$. Therefore, $C_{\mathfrak{p}} \in \mathcal{P}_{\mathsf{V}}(G)$ as desired. □

Now we can state the main theorem of the chapter.

Theorem 3.3.6. *For any partially ordered group G and any variety V of ℓ-groups, the maps $\kappa\colon \mathcal{P}_{\mathsf{V}}(G) \to \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$ and $\pi\colon \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G) \to \mathcal{P}_{\mathsf{V}}(G)$ in (3.8) and (3.10) are mutually inverse, inclusion-preserving homeomorphisms restricting to maps between $\mathcal{B}_{\mathsf{V}}(G)$ and $\operatorname{Spec}^* F^{\ell}_{\mathsf{V}}(G)$.*

Definition 3.1.10 associates a class of right preorders on groups to any given variety V of ℓ-groups, namely, $\mathcal{P}_{\mathsf{V}}(G)$ as G ranges over all groups; Theorem 3.3.6 establishes a non-trivial property of this association. We do not address here the question of how to obtain a syntactic characterization of the class of right preorders associated in this manner to a variety V. For a more precise formulation of this problem, cf. Remark 3.5.3 below. The construction leading to the statement of Theorem 3.3.6 makes it clear that the correspondence can also be inverted: a class of right preorders on (a class of) groups uniquely determines a variety V of ℓ-groups.

We begin by showing the first two properties, namely that the functions κ and π are mutually inverse, inclusion-preserving maps between $\mathcal{P}_{\mathsf{V}}(G)$ and $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$.

Lemma 3.3.7. *For any object G in P and any variety V of ℓ-groups, the maps κ and π are mutually inverse.*

Proof. Let $C \in \mathcal{P}_{\mathsf{V}}(G)$, and let $\mathfrak{p} := {h_C}^{-1}(H_C[\mathrm{e}])$. We show that $\pi \circ \kappa$ is the identity on $\mathcal{P}_{\mathsf{V}}(G)$, that is $C_{\mathfrak{p}} = C$. (Recall from (3.9) the definition of the preorder associated to $C_{\mathfrak{p}}$.)

If $g \in C$, then

$$h_C(\eta(g) \wedge \mathrm{e})([\mathrm{e}]) = R_C(g)([\mathrm{e}]) \wedge [\mathrm{e}] = [\mathrm{e}].$$

Therefore, $(\eta(g) \wedge \mathrm{e}) \in \mathfrak{p}$, and hence, $\mathfrak{p}\eta(g) \geq \mathfrak{p}\mathrm{e}$. This shows $C \subseteq C_{\mathfrak{p}}$. Conversely, pick $g \in C_{\mathfrak{p}}$, that is, a is such that $\mathfrak{p}\mathrm{e} \leq \mathfrak{p}\eta(g)$ in $\Omega_{\mathfrak{p}}$. This means that $\mathrm{e} \leq t\eta(g)$, for some $t \in \mathfrak{p}$. Hence,

$$h_C(t\eta(g) \wedge \mathrm{e}) = h_C(\mathrm{e}).$$

Therefore, the element $h_C(t\eta(g) \wedge \mathrm{e})$ is in the stabilizer of $[\mathrm{e}]$, which entails

$$h_C(t\eta(g) \wedge \mathrm{e})([\mathrm{e}]) = (h_C(t)h_C(\eta(g)) \wedge h_C(\mathrm{e}))([\mathrm{e}]) = [\mathrm{e}]. \tag{3.12}$$

Since $h_C \circ \eta = R_C$, from (3.12) we obtain

$$R_C(g)(h_C(t)([\mathrm{e}])) \wedge h_C(\mathrm{e})([\mathrm{e}]) = [\mathrm{e}]. \tag{3.13}$$

But $t \in \mathfrak{p}$, and thus $h_C(t)([\mathrm{e}]) = [\mathrm{e}]$; so, from (3.13) we infer $R_C(g)([\mathrm{e}]) \wedge [\mathrm{e}] = [\mathrm{e}]$, i.e., $g \in C$.

To show that $\kappa \circ \pi$ is the identity on $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$, we prove $\kappa(C_{\mathfrak{p}}) = \mathfrak{p}$ for a prime $\mathfrak{p}$ of $F^{\ell}_{\mathsf{V}}(G)$. By definition, $a \in \kappa(C_{\mathfrak{p}})$ if and only if $h_{C_{\mathfrak{p}}}(a)([\mathrm{e}]) = [\mathrm{e}]$. By applying the map $\widehat{\tau}$ defined in (3.11), this is equivalent to $R_{\mathfrak{p}}(a)(\mathfrak{p}\mathrm{e}) = \mathfrak{p}\mathrm{e}$, that is, $a \in \mathfrak{p}$. □

In order to show that κ is order-preserving, we begin by making an easy observation.

Proposition 3.3.8. *Let H be an ℓ-group generated by a subgroup $S \subseteq H$, and let $a \in H^+$. Then a lies in the sublattice of H generated by $\{s \vee e \mid s \in S\}$.*

Proof. There are finite index sets I and J_i and elements $s_{ij} \in S$, $i \in I$ and $j \in J_i$, such that $a = \bigwedge_I \bigvee_{J_i} s_{ij}$. Since $a \geq \mathrm{e}$ we have $a \vee \mathrm{e} = a$, so we obtain $a = \left(\bigwedge_I \bigvee_{J_i} s_{ij}\right) \vee \mathrm{e}$. By distributivity, $a = \bigwedge_I \left(\bigvee_{J_i} s_{ij} \vee \mathrm{e}\right)$, so $a = \bigwedge_I \bigvee_{J_i} (s_{ij} \vee \mathrm{e})$. □

Lemma 3.3.9. *For any object G in* P *and any variety* V *of ℓ-groups, the map κ is inclusion preserving.*

Proof. Let $C, D \in \mathcal{P}_{\mathsf{V}}(G)$ be such that $C \subseteq D$, and pick $a = \bigwedge_I \bigvee_{J_i}(\eta(g_{ij}) \vee \mathrm{e}) \in F^{\ell}_{\mathsf{V}}(G)^+$ such that $a \in \kappa(C)$, i.e., $h_C(a)([\mathrm{e}]) = [\mathrm{e}]$. This means

$$\begin{aligned} h_C(a) &= h_C\left(\bigwedge_I \bigvee_{J_i}(\eta(g_{ij}) \vee \eta(\mathrm{e}))\right) \\ &= \bigwedge_I \bigvee_{J_i} h_C(\eta(g_{ij}) \vee \eta(\mathrm{e})) \\ &= \bigwedge_I \bigvee_{J_i}(R_C(g_{ij}) \vee R_C(\mathrm{e})). \end{aligned}$$

Hence, $h_C(a)([\mathrm{e}]) = [\mathrm{e}]$ if and only if

$$\bigwedge_I \bigvee_{J_i}([g_{ij}] \vee [\mathrm{e}]) = [\mathrm{e}] \text{ in } \Omega_C.$$

Observe that $\bigvee_{J_i}([g_{ij}] \vee [\mathrm{e}]) \geq_C [\mathrm{e}]$ for every $i \in I$ and hence, $h_C(a)([\mathrm{e}]) = [\mathrm{e}]$ if and only if

$$\bigvee_{J_{i^*}}([g_{i^*j}] \vee [\mathrm{e}]) = [\mathrm{e}]$$

for some $i^* \in I$. Writing $J_{i^*} = \{1, \dots, n\}$, and reindexing if necessary, we have

$$[g_{i^*1}] \leq_C [g_{i^*2}] \leq_C \cdots \leq_C [g_{i^*n}] \leq_C [\mathrm{e}] \text{ in } \Omega_C,$$

and hence,

$$[g_{i^*1}] \leq_D [g_{i^*2}] \leq_D \cdots \leq_D [g_{i^*n}] \leq_D [\mathrm{e}] \text{ in } \Omega_D.$$

Therefore,

$$\bigwedge_I \bigvee_{J_i}([g_{ij}] \vee [\mathrm{e}]) = [\mathrm{e}] \text{ in } \Omega_D,$$

which is equivalent to $h_D(a)([\mathrm{e}]) = [\mathrm{e}]$. □

Theorem 3.3.10. *For any partially ordered group G and any variety* V *of ℓ-groups, the maps κ and π are mutually inverse, inclusion-preserving bijections.*

Proof. By Lemma 3.3.7, the maps κ and π are mutually inverse. By Lemma 3.3.9, κ is inclusion preserving. If $\mathfrak{p} \subseteq \mathfrak{q} \in \operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$, then $g \in C_{\mathfrak{p}}$ if and only if $\mathfrak{p}\mathrm{e} \leq \mathfrak{p}\eta(g)$. The latter is equivalent to $\mathrm{e} \leq t\eta(g)$, for some $t \in \mathfrak{p}$. Hence, $\mathrm{e} \leq t\eta(g)$, for some $t \in \mathfrak{p} \subseteq \mathfrak{q}$, and therefore, $g \in C_{\mathfrak{q}}$. Thus, π is inclusion preserving. □

The next two results illustrate the behaviour of κ and π restricted to, respectively, the subspace $\mathcal{B}_{\mathsf{V}}(G)$ of preorders and the subspace $\operatorname{Spec}^* F^{\ell}_{\mathsf{V}}(G)$ of prime ℓ-ideals.

Proposition 3.3.11. *For any partially ordered group G and any right preorder C on G, the quotient Ω_C is a totally ordered group with group operation $[g][h] = [gh]$ if and only if $C \in \mathcal{B}(G)$. In that case, H_C is isomorphic to Ω_C.*

Proof. For a preorder $C \in \mathcal{B}(G)$, it is immediate that $\equiv_C$ is a group congruence, and Ω_C is a totally ordered group. Conversely, if Ω_C is a group with operation $[g][h] = [gh]$ totally ordered by $\leq_C$, we have that $g \preceq_C h$ implies $[sgt] \leq_C [sht]$ for all $g, h, t, s \in G$. That is, $sgt \preceq_C sht$.

Now, the map $q\colon \Omega_C \to H_C$ defined by $[g] \mapsto R_C(g)$ is a group homomorphism. Moreover, $[g] <_C [h]$ if and only if $[tg] <_C [th]$ for every $t \in G$. Hence, q is also an order-isomorphism onto $R_C[G]$, and since the ℓ-group H_C generated by the totally ordered group $R_C[G]$ is $R_C[G]$ itself, the proof is complete. □

Note that $q([g]) \in H_C[\mathrm{e}]$ if and only if $R_C(g)([\mathrm{e}]) = [\mathrm{e}]$, that is, $[g] = [\mathrm{e}]$.

Theorem 3.3.12. *For any partially ordered group G and any variety V of ℓ-groups, if $C \in \mathcal{B}_{\mathsf{V}}(G)$, then $\kappa(C)$ is a prime ℓ-ideal of $F^{\ell}_{\mathsf{V}}(G)$. Further, if $\mathfrak{p} \in \mathrm{Spec}^* F^{\ell}_{\mathsf{V}}(G)$, then $\pi(\mathfrak{p})$ is a preorder on G.*

Proof. For $C \in \mathcal{B}_{\mathsf{V}}(G)$, suppose $a \in \kappa(C)$. We show $b^{-1}ab \in \kappa(C)$, for every $b \in F^{\ell}_{\mathsf{V}}(G)$. By Proposition 3.3.11, we identify H_C with Ω_C, and have $h_C(a) = [\mathrm{e}]$. Similarly, given $b \in F^{\ell}_{\mathsf{V}}(G)$, we have $h_C(b) = [g]$ for some $g \in G$. Therefore,

$$h_C(b^{-1}ab) = h_C(b^{-1})h_C(a)h_C(b) = [g^{-1}][\mathrm{e}][g] = [\mathrm{e}].$$

If $\mathfrak{p} \in \mathrm{Spec}\, F^{\ell}_{\mathsf{V}}(G)$, and $g, h \in G$, we have $g \preceq_{\pi(\mathfrak{p})} h$ if and only if $\eta(g)\eta(h^{-1}) \leq a$ for some $a \in \mathfrak{p}$. Therefore, if $\mathfrak{p}$ is normal, we also have

$$\eta(s)\eta(g)\eta(t)\eta(t^{-1})\eta(h^{-1})\eta(s^{-1}) \leq \eta(s)a\eta(s^{-1}) \in \mathfrak{p},$$

which implies $sgt \preceq_{\pi(\mathfrak{p})} sht$, for all $s, t \in G$. □

Finally, we conclude with a proof that κ and π are homeomorphisms. Since κ and π are mutually inverse bijections by Theorem 3.3.10, it suffices to show that they both are open maps.

Theorem 3.3.13. *For any partially ordered group G and any variety V of ℓ-groups, the maps κ and π are homeomorphisms.*

Proof. We first show that

$$\kappa[\mathbb{P}(g)] = \mathbb{S}(\eta(g) \vee \mathrm{e}), \quad \text{for } g \in G. \tag{3.14}$$

Let $C \in \mathbb{P}(g)$. This means $R_C(g)([\mathrm{e}]) >_C [\mathrm{e}]$ in Ω_C, that is, $h_C(\eta(g))([\mathrm{e}]) >_C [\mathrm{e}]$. Therefore,

$$h_C(\eta(g) \vee \mathrm{e})([\mathrm{e}]) = [g] \vee [\mathrm{e}] >_C [\mathrm{e}],$$

and hence, ${h_C}^{-1}(H_C[\mathrm{e}]) \in \mathbb{S}(\eta(g) \vee \mathrm{e})$. Similarly, for ${h_C}^{-1}(H_C[e]) \in \mathbb{S}(\eta(g) \vee \mathrm{e})$, we prove $C \in \mathbb{P}(g)$. The assumption entails $h_C(\eta(g) \vee \mathrm{e})([\mathrm{e}]) = [g] \vee [\mathrm{e}] >_C [\mathrm{e}]$. Since Ω_C is a chain, this can only happen if $[g] >_C [\mathrm{e}]$. Therefore, $g \in C$ and $g^{-1} \notin C$.

Since $\{\mathbb{P}(g) \mid g \in G\}$ is a subbase, and κ, being a bijection, preserves arbitrary intersections and unions, it follows that κ is open. To show π is open, by Propositions 3.2.4 and 3.2.5.(a), together with the fact that π is a bijection, it suffices to prove $\pi[\mathbb{S}(a)]$ is open, for $a \in F^{\ell}_{\mathsf{V}}(G)^+$. By Proposition 3.3.8,

$$\mathbb{S}(a) = \mathbb{S}(\bigwedge_{I}\bigvee_{J_i}(\eta(g_{ij}) \vee \mathrm{e}))$$

for some finite sets I and J_i, and elements $g_{ij} \in G$. By the second item of Proposition 3.2.5,

$$\mathbb{S}(a) = \bigcap_{I}\bigcup_{J_i}\mathbb{S}(\eta(g_{ij}) \vee \mathrm{e}).$$

Since π is a bijection,

$$\pi[\mathbb{S}(a)] = \bigcap_{I}\bigcup_{J_i}\pi[\mathbb{S}(\eta(g_{ij}) \vee \mathrm{e})].$$

By (3.14), $\mathbb{P}(g_{ij}) = \pi[\mathbb{S}(\eta(g_{ij}) \vee \mathrm{e})]$. Therefore, $\pi[\mathbb{S}(a)] = \bigcap_I \bigcup_{J_i} \mathbb{P}(g_{ij})$ is open. □

Proof of Theorem 3.3.6. Combine Theorems 3.3.10, 3.3.12, and 3.3.13. □

The following is now immediate.

Corollary 3.3.14. *For any partially ordered group G and any variety* V *of* ℓ*-groups, the space* $\mathcal{P}_{\mathsf{V}}(G)$ *is a completely normal generalized spectral space whose specialization order coincides with the inclusion order.*

By Theorem 3.3.6 and Proposition 2.1.11, the space $\mathcal{P}_{\mathsf{V}}(G)$ has minimal elements, and every right preorder in $\mathcal{P}_{\mathsf{V}}(G)$ must extend a right preorder that is inclusion-minimal in $\mathcal{P}_{\mathsf{V}}(G)$. In the rest of the chapter, we refer to the set of minimal elements of $\mathcal{P}_{\mathsf{V}}(G)$ endowed with the subspace topology as the 'minimal layer' of $\mathcal{P}_{\mathsf{V}}(G)$.

3.4 MINIMAL AND QUASI-MINIMAL SPECTRA

We revisit here the much-studied minimal spectrum of an ℓ-group, paying special attention to the property of compactness, obtaining a general algebraic compactness criterion. It is known that, when the minimal spectrum is compact, it coincides with the Stone dual of the Boolean algebra of principal polars of the ℓ-group (cf. [43, Theorem 2.2]; [7, Lemma 3.2]). We broaden this perspective, and study the Stone dual of the algebra of principal polars in full generality—even when the latter is not a Boolean algebra, but only a distributive lattice with minimum. Its dual space turns out to be a generalization of the minimal spectrum, in a sense that will be clarified by Theorem 3.4.9.

In Section 3.5, we will then show that in the varieties of ℓ-groups, representable ℓ-groups, and Abelian ℓ-groups, any ℓ-group freely generated by a partially ordered group has a compact minimal spectrum (see Theorem 3.5.17). This is related to the spaces of right orders and orders on groups, and the space of orders on Abelian groups, as they correspond, through the bijection of Theorem 3.3.6, precisely to minimal spectra of the free ℓ-group over the given group relative to the appropriate variety.

Recall that we write $\operatorname{Min} H$ for the set of minimal prime subgroups of any ℓ-group H, and we topologize it with the subspace topology from $\operatorname{Spec} H$. We write $\mathbb{S}_{\mathrm{m}}(A)$ (resp., $\mathbb{V}_{\mathrm{m}}(A)$) for open subsets (resp., closed subsets) of $\operatorname{Min} H$ with A ranging over all subsets of H; more precisely, $\mathbb{S}_{\mathrm{m}}(A)$ denotes the open set $\{\mathfrak{m} \in \operatorname{Min} H \mid A \nsubseteq \mathfrak{m}\}$, and $\mathbb{V}_{\mathrm{m}}(A)$ denotes its complement closed set $\{\mathfrak{m} \in \operatorname{Min} H \mid A \subseteq \mathfrak{m}\}$, for any $A \subseteq H$.

Lemma 3.4.1. *For any $a \in H$ and $\mathfrak{m} \in \operatorname{Min} H$:*

(a) *For any $a \in \mathfrak{m}$, $a^{\perp\perp} \subseteq \mathfrak{m}$.*
(b) $\mathbb{S}_{\mathrm{m}}(a) = \mathbb{V}_{\mathrm{m}}(a^{\perp})$.

Proof. For (a), observe that by Proposition 2.1.12, if $a \in \mathfrak{m}$, then $a \in b^{\perp}$ for some $b \notin \mathfrak{m}$. Therefore, $a^{\perp\perp} \subseteq b^{\perp} \subseteq \mathfrak{m}$. Item (b) is an easy consequence of Proposition 2.1.12, as follows. If $a \notin \mathfrak{m}$ (i.e., $\mathfrak{m} \in \mathbb{S}_{\mathrm{m}}(a)$), it follows that $a^{\perp} \subseteq \mathfrak{m}$ or equivalently, $\mathfrak{m} \in \mathbb{V}_{\mathrm{m}}(a^{\perp})$. Conversely, if $a \in \mathfrak{m}$ (i.e., $\mathfrak{m} \notin \mathbb{S}_{\mathrm{m}}(a)$), then $a^{\perp} \nsubseteq \mathfrak{m}$ and hence, $\mathfrak{m} \notin \mathbb{V}_{\mathrm{m}}(a^{\perp})$. □

For the definition of the notion of a (principal) polar, we refer to Section 2.1. We write $\operatorname{Pol} H$ for the set of polars of H. Under the inclusion order, $\operatorname{Pol} H$ is a complete distributive lattice with $H = \mathrm{e}^{\perp}$ as its maximum, $\{\mathrm{e}\} = H^{\perp} = \mathrm{e}^{\perp\perp}$ as its minimum, meets given by intersection, and joins given by $\bigvee T_i = (\bigcup T_i)^{\perp\perp}$. It can be shown that $\operatorname{Pol} H$ is a complete Boolean algebra, with complementation given by the map $T \mapsto T^{\perp}$. We also write $\operatorname{Pol_p} H$ for the set of principal polars of H; it is a sublattice of $\operatorname{Pol} H$ because of the identitities

$$(a \wedge b)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}, \tag{3.15}$$

$$(a \vee b)^{\perp\perp} = a^{\perp\perp} \vee b^{\perp\perp}, \tag{3.16}$$

which hold for all $a, b \in H^{+}$. The minimum $\mathrm{e}^{\perp\perp}$ of $\operatorname{Pol} H$ lies in $\operatorname{Pol_p} H$, while the maximum $H = \mathrm{e}^{\perp}$ is principal if and only if H has a *weak (order) unit*—an element $w \in H^{+}$ such that for each $a \in H$, $w \wedge |a| = \mathrm{e}$ implies $a = \mathrm{e}$. In that case, $w^{\perp\perp} = H$.

Remark 3.4.2. By Proposition 2.1.12, an element $w \in H^{+}$ is a weak unit if and only if w misses every minimal prime.

Note that the existence of a weak unit is not sufficient for $\operatorname{Pol_p} H$ to be a Boolean subalgebra of $\operatorname{Pol} H$, because the complement of a principal polar need not be principal (see Theorem 3.4.9).

Lemma 3.4.3. *The map*

$$\mathcal{C}_p(H) \xrightarrow{f} \operatorname{Pol_p} H \tag{3.17}$$

defined by $\mathfrak{C}(a) \mapsto a^{\perp\perp}$ is an onto lattice homomorphism preserving minimum.

Proof. The map f is well defined, since $S^{\perp} = \mathfrak{C}(S)^{\perp}$ for $S \subseteq H$ ([10, 3.2.5]) and hence,

$$\mathfrak{C}(a) = \mathfrak{C}(b) \quad \Longrightarrow \quad a^{\perp\perp} = \mathfrak{C}(a)^{\perp\perp} = \mathfrak{C}(b)^{\perp\perp} = b^{\perp\perp};$$

also, f is clearly onto. Moreover, it is a lattice homomorphism by Proposition 2.1.5 and (3.15)–(3.16), and preserves the minimum since $\{\mathrm{e}\} \mapsto \mathrm{e}^{\perp\perp}$. □

Recall the notation $X(D)$ for the Stone dual of a distributive lattice D with minimum, and set for the rest of this section $D = \mathrm{Pol_p}\, H$. In light of Theorem 3.2.6, we identify the Stone dual of $\mathcal{C}_p(H)$ with $\mathrm{Spec}\, H$. From the map f in (3.17), we define

$$\begin{aligned} X(D) &\xrightarrow{f^*} \mathrm{Spec}\, H \\ I &\longmapsto \bigvee\{\mathfrak{C}(a) \mid a^{\perp\perp} \in I\}. \end{aligned} \tag{3.18}$$

Our next aim is to characterize the range of f^*. To this end, we first introduce the following new notion.

Definition 3.4.4. For any ℓ-group H, a prime subgroup $\mathfrak{p} \in \mathrm{Spec}\, H$ is *quasi-minimal* if

$$\mathfrak{p} = \bigcup\{a^{\perp\perp} \mid a \in \mathfrak{p}\}. \tag{3.19}$$

The quasi-minimal spectrum $\mathrm{Qin}\, H$ of H is the subset of quasi-minimal prime subgroups equipped with the subspace topology inherited from $\mathrm{Spec}\, H$.

We write $\{\mathbb{S}_\mathfrak{q}(a)\}_{a\in H}$ for the open base induced by $\{\mathbb{S}(a)\}_{a\in H}$ on $\mathrm{Qin}\, H$ by restriction, that is,

$$\mathbb{S}_\mathfrak{q}(a) = \{\mathfrak{p} \in \mathrm{Qin}\, H \mid a \notin \mathfrak{p}\}, \quad \text{for every } a \in H.$$

Remark 3.4.5. Observe that $S \subseteq \bigcup\{a^{\perp\perp} \mid a \in S\}$ holds for every $S \subseteq H$, as $a \in a^{\perp\perp}$ for each $a \in H$. We mention here the notion of 'z-subgroup', introduced by Bigard, and treated by Darnel in [49, Definition 15.1]. A convex ℓ-subgroup $\mathfrak{k}$ is a z-subgroup when $a \in \mathfrak{k}$ implies $a^{\perp\perp} \subseteq \mathfrak{k}$. This notion is clearly related to the notion of 'quasi-minimal prime subgroup'. More precisely, a prime subgroup of an ℓ-group H is quasi-minimal if and only if it is a z-subgroup of H.

Lemma 3.4.6. *For any ℓ-group H, the quasi-minimal spectrum extends the minimal spectrum, i.e.,* $\mathrm{Min}\, H \subseteq \mathrm{Qin}\, H$.

Proof. Immediate by Lemma 3.4.1.(a). □

Theorem 3.4.7. *For any ℓ-group H, $f^*[X(D)]$ coincides with the quasi-minimal spectrum* $\mathrm{Qin}\, H$, *and f^* (as defined in* (3.18) *above) is a homeomorphism onto its range.*

Proof. We establish the following equivalent description of $f^*(I)$, for any $I \in X(D)$:[4]

$$f^*(I) = \{a \in H \mid a^{\perp\perp} \in I\}. \tag{3.20}$$

First, observe that $b \in f^*(I)$ implies $\mathfrak{C}(b) \subseteq \mathfrak{C}(a_1) \vee \cdots \vee \mathfrak{C}(a_n)$ for some $a_1^{\perp\perp}, \ldots, a_n^{\perp\perp} \in I$ by Proposition 2.1.6. Further, from Proposition 2.1.5 it follows that $\mathfrak{C}(b) \subseteq \mathfrak{C}(a)$ for some $a^{\perp\perp} \in I$, since I is closed under finite joins. Thus, we obtain

$$a^\perp = \mathfrak{C}(a)^\perp \subseteq \mathfrak{C}(b)^\perp = b^\perp,$$

[4]Compare with Darnel's construction in [49, Proposition 49.18].

and hence,

$$b^{\perp\perp} = \mathfrak{C}(b)^{\perp\perp} \subseteq \mathfrak{C}(a)^{\perp\perp} = a^{\perp\perp},$$

which allows us to conclude that $b^{\perp\perp} \in I$ by downward closure of I. Conversely, if $b^{\perp\perp} \in I$, then $\mathfrak{C}(b) \subseteq f^*(I)$ by (3.18), and hence $b \in f^*(I)$.

That $f^*(I) \in \operatorname{Qin} H$ is immediate now, as $b \in \bigcup\{a^{\perp\perp} \mid a \in f^*(I)\}$ entails that $b^{\perp\perp} \subseteq a^{\perp\perp} \in I$, and hence $b^{\perp\perp} \in I$ by downward closure of I, that is, $b \in f^*(I)$. It remains to show that $\operatorname{Qin} H \subseteq f^*[X(D)]$. For this, suppose $\mathfrak{p} \in \operatorname{Qin} H$. We prove that

$$I_{\mathfrak{p}} = \{a^{\perp\perp} \mid a \in \mathfrak{p}\}$$

is a prime ideal of D, and hence, clearly, $f^*(I_{\mathfrak{p}}) = \mathfrak{p}$. It is easy to see that $I_{\mathfrak{p}}$ is an ideal of D using (3.15)–(3.16). Now, suppose $a^{\perp\perp} \cap b^{\perp\perp} \in I_{\mathfrak{p}}$; by (3.15), this is equivalent to $(a \wedge b)^{\perp\perp} \in I_{\mathfrak{p}}$. By definition of $I_{\mathfrak{p}}$, since $\mathfrak{p}$ is quasi-minimal, we get $a \wedge b \in \mathfrak{p}$. But $\mathfrak{p}$ is prime, and hence either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, from which we conclude that either $a^{\perp\perp} \in I_{\mathfrak{p}}$ or $b^{\perp\perp} \in I_{\mathfrak{p}}$.

Note that injectivity of f^* is now immediate from (3.20). Finally, to show that f^* is a homeomorphism onto its range it suffices to observe that (3.20) entails $f^*[\widehat{(a^{\perp\perp})}] = \mathbb{S}_{\mathfrak{q}}(a)$, where a ranges over H, and $\widehat{(a^{\perp\perp})}$ is the set of all prime ideals of D not containing $a^{\perp\perp}$.[5] □

We record a consequence that provides for ℓ-groups the spectral equivalent of the existence of a weak unit.

Corollary 3.4.8. *For any ℓ-group H, there is a weak unit $w \in H$ if and only if* $\operatorname{Qin} H$ *is compact.*

Proof. This follows immediately from Theorem 3.4.7 along with standard Stone duality. Suppose now that $w \in H$ is a weak unit. Then, we have $w^{\perp\perp} = H$. Therefore, the lattice $\operatorname{Pol}_{\mathrm{p}} H$ has a maximum, and its dual space $\operatorname{Qin} H$ is compact. Conversely, if $\operatorname{Qin} H$ is compact, then $\operatorname{Qin} H = \mathbb{S}_{\mathfrak{q}}(w)$ for some $w \in H$. But then, every quasi-minimal prime subgroup of H omits w, and so does every minimal prime (by Lemma 3.4.6). By Remark 3.4.2, w is a weak unit. □

We now use the notion of $\operatorname{Qin} H$ to clarify the relationship between $\operatorname{Min} H$ and $\operatorname{Pol}_{\mathrm{p}} H$. The equivalence of items (1)–(3) is essentially proved in [43, Theorem 2.2], and in [7, Lemma 3.2] in a commutative setting.

Theorem 3.4.9. *For any ℓ-group H, the following are equivalent:*

(1) $\operatorname{Pol}_{\mathrm{p}} H$ *is a Boolean subalgebra of* $\operatorname{Pol} H$.
(2) $\operatorname{Min} H$ *is compact.*

[5] That the map f^* is an embedding of topological spaces could be proved by observing that it is the dual of a surjective lattice homomorphism (namely, $f\colon \mathcal{C}_p(H) \to \operatorname{Pol}_{\mathrm{p}} H$). However, we are not aware of a suitable reference for a categorical duality for distributive lattices with minimum, and for this reason we choose to include an explicit argument.

(3) H^+ *is* complemented*: for every* $a \in H^+$ *there is* $b \in H^+$ *such that* $a \wedge b = \mathrm{e}$ *and* $a \vee b$ *is a weak unit.*

If any one of the equivalent conditions (1)–(3) holds, then $\operatorname{Min} H = \operatorname{Qin} H$.

Proof. (1) ⇒ (2). Observe that if $\operatorname{Pol_p} H$ is a Boolean algebra, then for every $a \in H$, we have $a^\perp = b^{\perp\perp}$ for some $b \in H$. Thus, a quasi-minimal prime $\mathfrak{p}$ is

$$\mathfrak{p} = \bigcup\{a^{\perp\perp} \mid a \in \mathfrak{p}\} = \bigcup\{b^\perp \mid b^\perp = a^{\perp\perp} \text{ for some } a \in \mathfrak{p}\}.$$

Now, we use Proposition 2.1.12 to show that $\mathfrak{p}$ is in fact minimal. Suppose for a contradiction that $b \in \mathfrak{p}$, for one of those b such that $b^\perp = a^{\perp\perp}$ for some $a \in \mathfrak{p}$. Then $b^{\perp\perp} \subseteq \mathfrak{p}$ and hence,

$$a^{\perp\perp} \vee b^{\perp\perp} = (a \vee b)^{\perp\perp} = H \subseteq \mathfrak{p},$$

which is a contradiction. Conversely, if $b \notin \mathfrak{p}$, then $b \notin \mathfrak{m}$ for every minimal prime $\mathfrak{m} \subseteq \mathfrak{p}$, that is, $b^\perp \subseteq \mathfrak{m}$ for every minimal prime $\mathfrak{m} \subseteq \mathfrak{p}$. Hence, $b^\perp \subseteq \mathfrak{p}$. Therefore, every quasi-minimal prime $\mathfrak{p} \in \operatorname{Qin} H$ is in fact minimal. Since $\operatorname{Qin} H$ is the Stone dual space of a Boolean algebra, it is compact, and hence so is $\operatorname{Min} H$.

(2) ⇒ (3). First recall that, by Lemma 3.4.1, $\mathbb{S}_\mathrm{m}(a) = \mathbb{V}_\mathrm{m}(a^\perp)$ for any $a \in H$, which is clearly equivalent to $\mathbb{V}_\mathrm{m}(a) = \mathbb{S}_\mathrm{m}(a^\perp)$. By Proposition 3.2.5, we can assume without loss of generality that $a \in H^+$. Hence, there are $b_1, \dots, b_n \in H^+$ such that $b_1, \dots, b_n \in a^\perp$ and

$$\mathbb{V}_\mathrm{m}(a) = \mathbb{S}_\mathrm{m}(a^\perp) = \bigcup_{c \in a^\perp} \mathbb{S}_\mathrm{m}(c) = \mathbb{S}_\mathrm{m}(b_1 \vee \cdots \vee b_n),$$

where the last equality follows from $\mathbb{V}_\mathrm{m}(a)$ being closed in a compact space, and from Proposition 3.2.5. Let $b = b_1 \vee \cdots \vee b_n$. We show $a \wedge b = \mathrm{e}$, and $\mathbb{S}_\mathrm{m}(a \vee b) = \operatorname{Min} H$. In fact,

$$\mathbb{S}_\mathrm{m}(a \vee b) = \mathbb{S}_\mathrm{m}(a) \cup \mathbb{S}_\mathrm{m}(b) = \mathbb{S}_\mathrm{m}(a) \cup \mathbb{V}_\mathrm{m}(a) = \operatorname{Min} H,$$

that is, $|a| \vee |b|$ is a weak unit. Further,

$$\mathbb{S}_\mathrm{m}(a \wedge b) = \mathbb{S}_\mathrm{m}(a) \cap \mathbb{S}_\mathrm{m}(b) = \mathbb{S}_\mathrm{m}(a) \cap \mathbb{V}_\mathrm{m}(a) = \emptyset,$$

or equivalently, $a \wedge b \in \mathfrak{m}$ for every $\mathfrak{m} \in \operatorname{Min} H$. Therefore, $a \wedge b = \mathrm{e}$ since $\bigcap_{\mathfrak{m} \in \operatorname{Min} H} \mathfrak{m} = \{\mathrm{e}\}$ (see Remark 2.1.21).

(3) ⇒ (1). First, we have that $a^{\perp\perp} = c^{\perp\perp}$ for some $c \in H^+$, as $a^\perp = |a|^\perp$ for any $a \in H$. Thus, it suffices to show that for each $a \in H^+$ there exists $b \in H^+$ such that $a^{\perp\perp} \cap b^{\perp\perp} = \{\mathrm{e}\}$ and $a^{\perp\perp} \vee b^{\perp\perp} = H$. This is immediate, since two positive elements $a, b \in H^+$ are orthogonal if and only if $a^{\perp\perp} \cap b^{\perp\perp} = \{\mathrm{e}\}$ by (3.15) and, similarly by (3.16), $a \vee b$ is a weak unit if and only if $a^{\perp\perp} \vee b^{\perp\perp} = H$.

Finally, if $\operatorname{Min} H$ is compact, then $\operatorname{Min} H = \operatorname{Qin} H$. In fact, assume $\mathfrak{p} \in \operatorname{Qin} H \setminus \operatorname{Min} H$. Then for every $\mathfrak{m} \in \operatorname{Min} H$, there is $a \in \mathfrak{p}$ such that $a \notin \mathfrak{m}$. Hence,

$$\operatorname{Min} H \subseteq \bigcup\{\mathbb{S}(a) \mid a \in \mathfrak{p}\}.$$

Assuming compactness of $\operatorname{Min} H$, we have $\operatorname{Min} H \subseteq \mathbb{S}(a_1 \vee \cdots \vee a_n)$ for some $w = a_1 \vee \cdots \vee a_n \in \mathfrak{p}$. Hence, the prime $\mathfrak{p}$ contains the weak unit w, which is a contradiction since $w^{\perp\perp} = H$. □

The following example shows that the converse to the last implication stated in Theorem 3.4.9 does not hold. That is, we exhibit an ℓ-group H for which $\operatorname{Min} H = \operatorname{Qin} H$ is not compact.

Example 3.4.10. Let H be the ℓ-group

$$H = \{f : \mathbb{N} \to \mathbb{Z} \mid \operatorname{supp}(f) \text{ is finite}\},$$

where $\operatorname{supp}(f) := \{n \in \mathbb{N} \mid f(n) \neq 0\}$, with pointwise operations and the map $\overline{0}$ constantly equal to 0 as the group identity. We show $\operatorname{Qin} H = \operatorname{Min} H = (\mathbb{N}, \tau_d)$, where τ_d is the discrete topology, and hence $\operatorname{Qin} H = \operatorname{Min} H$ is not compact.

Claim 3. *The distributive lattice* $\mathcal{C}_p(H)$ *is* $\operatorname{Pol_p} H$, *and it is isomorphic to* $(\mathbb{N}^*, \cap, \cup)$, *where*

$$\mathbb{N}^* = \{S \subseteq \mathbb{N} \mid S \textit{ is finite}\}.$$

Proof. We start from the latter, and observe that for any $f \in H^+$,

$$f \wedge g = \overline{0} \quad \text{if and only if} \quad \operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset.$$

Therefore, $h \wedge g = \overline{0}$ for every $g \in f^{\perp}$ precisely when $\operatorname{supp}(h) \subseteq \operatorname{supp}(f)$, from which we conclude

$$f^{\perp\perp} = \{h \in H \mid \operatorname{supp}(h) \subseteq \operatorname{supp}(f)\}.$$

Thus, by (3.15)–(3.16), the map $f^{\perp\perp} \mapsto \operatorname{supp}(f)$ is a lattice isomorphism $\operatorname{Pol_p} H \cong (\mathbb{N}^*, \cap, \cup)$. Moreover,

$$\mathfrak{C}(f) = \{g \in H \mid |g| \leq f^n \text{ for some } n \in \mathbb{N}\}.$$

Hence, every positive element of $\mathfrak{C}(f)$ has support included in $\operatorname{supp}(f)$. Conversely, if $g \in H^+$ and $\operatorname{supp}(g) \subseteq \operatorname{supp}(f)$, then $g(n) \geq 0$ implies $f(n) \geq 0$. Now, since the functions have finite support, it is possible to find $m \in \mathbb{N}$ so that $g(n) \leq mf(n)$, for every $n \in \mathbb{N}$. Hence, $g \in \mathfrak{C}(f)$. □

From Theorem 3.4.7 and Theorem 3.2.6, we conclude $\operatorname{Spec} H = \operatorname{Qin} H$.

Claim 4. *The Stone dual space of* $(\mathbb{N}^*, \cap, \cup)$ *is* $(\mathbb{N}, \tau_d)$.

Proof. It is straightforward that $(\mathbb{N}, \tau_d)$ is a generalized spectral space whose compact opens are precisely the finite subsets of $\mathbb{N}$. The result now follows from [164, Theorem 15]. □

Therefore, $\operatorname{Spec} H = \operatorname{Qin} H = (\mathbb{N}, \tau_d)$ is $\operatorname{Min} H$, since the specialization order of $(\mathbb{N}, \tau_d)$ is trivial. This completes Example 3.4.10.

The equivalence of (1), (2), and (3) in Theorem 3.4.9 is a well-known result, of which we have provided a streamlined proof for the reader's convenience. There is a substantial literature concerned with the compactness of minimal spectra of various structures, and we cannot do justice to it here. In connection with Theorem 3.4.9

we ought to at least mention Speed's paper [162] for distributive lattices (see Remark 2.1.13), and Conrad's and Martinez' paper [43] for ℓ-groups. Let us also mention that, in the Archimedean case, compactifications of minimal spectra of ℓ-groups were recently shown to be inextricably related to the construction of projectable hulls (see [7, 81]).

Corollary 3.4.11. *For any partially ordered group G and any variety V of ℓ-groups, the minimal layer of $\mathcal{P}_{\mathsf{V}}(G)$ is compact if and only if any one of the equivalent conditions of Theorem 3.4.9 holds for $F^{\ell}_{\mathsf{V}}(G)$.*

Proof. Combine Theorem 3.3.6 and Theorem 3.4.9. □

3.5 SPECIALIZING THE CORRESPONDENCE TO SPECIFIC VARIETIES

We begin this section by showing that the spaces of right orders and orders on groups, and the space of orders on Abelian groups are recovered in our framework as the subspaces of inclusion-minimal right preorders of the appropriate class in each case. Further, in view of Theorem 3.4.9, the spaces of right orders and orders on a partially ordered group G are shown to be the dual Boolean spaces of Boolean algebras of substructures of the free ℓ-group over G relative to the appropriate variety (see Theorem 3.5.18).

The second part of the section is devoted to orders on groups, and the variety of representable ℓ-groups.

Proposition 3.5.1. *For any partially ordered group G and any right preorder C on G, the following are equivalent:*

(1) *The right preorder C is in $\mathcal{P}_{\mathsf{Rep}}(G)$.*
(2) *H_C is a totally ordered group.*
(3) *For every $a \in G$, either $bab^{-1} \in C$ for every $b \in G$, or $bab^{-1} \in C^{-1}$ for every $b \in G$.*

Proof. (1) ⇔ (2). Recall that, by Proposition 3.1.9, H_C is always transitive. Hence, whenever H_C is representable (equivalently, C is in $\mathcal{P}_{\mathsf{Rep}}(G)$), then H_C is totally ordered by Proposition 2.1.17. The converse direction is a consequence of the fact that any totally ordered group is representable.

(2) ⇔ (3). Note that H_C is a chain if and only if for every $a \in G$, either $\mathrm{id}_{\Omega_C} \leq R_C(a)$ or $R_C(a) \leq \mathrm{id}_{\Omega_C}$. This means that, for each $a \in G$, either $[t] \preceq_C [ta]$ for all $t \in G$, or $[ta] \preceq_C [t]$ for all $t \in G$. Equivalently, for every $a \in G$, either $t \preceq_C ta$ for all $t \in G$, or $ta \preceq_C t$ for all $t \in G$, that is, either $\mathrm{e} \preceq_C tat^{-1}$ for all $t \in G$, or $tat^{-1} \preceq_C \mathrm{e}$ for all $t \in G$. □

Proposition 3.5.2. *For any partially ordered group G and any right preorder C on G, the following are equivalent:*

(1) *The right preorder C is in $\mathcal{P}_{\mathsf{Ab}}(G)$.*

(2) *H_C is a totally ordered Abelian group.*

(3) *For all $a, b \in G$, we have $a^{-1}b^{-1}ab \in C \cap C^{-1}$.*

Proof. (1) ⇒ (2). Follows by Proposition 3.5.1.

(2) ⇒ (3). Since H_C is totally ordered, by (the proof of) Proposition 3.3.11, it is isomorphic to $R_C[G]$; hence, H_C is Abelian if and only if $R_C[G]$ is Abelian. Thus, H_C is Abelian if and only if for all $a, b \in G$, $R_C(ab) = R_C(ba)$ or, equivalently, for all $a, b, t \in G$, $[tab] = [tba]$ in Ω_C. The latter entails $[ab] = [ba]$ for all $a, b \in G$, that is, $[aba^{-1}b^{-1}] = [\mathrm{e}]$, for all $a, b \in G$. Therefore, for all $a, b \in G$, we have $a^{-1}b^{-1}ab \in C \cap C^{-1}$.

(3) ⇒ (1). Pick a right preorder C on G satisfying $a^{-1}b^{-1}ab \in C \cap C^{-1}$, for all $a, b \in G$; that is, $[a^{-1}b^{-1}ab] = [\mathrm{e}]$, for all $a, b \in G$. We show that $[tab] = [tba]$ in Ω_C for all $a, b, t \in G$. Since $[ab] = [ba]$ and C is right-invariant, also $[abt] = [bat]$, for every $t \in G$. By using the assumption (3) again, $[bat] = [tba]$ and $[abt] = [tab]$. Thus, $[tab] = [tba]$ for all $a, b, t \in G$, which means $R_C(ab) = R_C(ba)$. The conclusion follows from Remark 3.3.4, since H_C is generated by $R_C[G]$, and the latter is Abelian. □

Remark 3.5.3. The class of groups equipped with a right preorder is elementary in the language of partially ordered groups. By Propositions 3.5.1 and 3.5.2, so are the classes of groups equipped with a representable (resp., Abelian) right preorder. More generally, we might ask for which varieties of ℓ-groups, the corresponding class of right preorders on groups as provided by Definition 3.1.10 is elementary in the language of partially ordered groups. The reader will find a few partial answers discussed in Section 3.6.

For any partially ordered group G and any variety V of ℓ-groups, consider the factorization of the universal map $\eta\colon G \to F^{\ell}_{\mathsf{V}}(G)$ given by

$$G \xrightarrow{\zeta} \eta[G] \xrightarrow{\xi} F^{\ell}_{\mathsf{V}}(G), \tag{3.21}$$

where $\eta[G]$ is the group image of G under η partially ordered by the restriction of the order on $F^{\ell}_{\mathsf{V}}(G)$, and ξ is the inclusion map.

Proposition 3.5.4. *The positive group homomorphism $\xi\colon \eta[G] \to F^{\ell}_{\mathsf{V}}(G)$ from* (3.21) *is an order-embedding satisfying the universal property, that is, for every positive group homomorphism $p\colon \eta[G] \to H$, with H an ℓ-group in V, there is exactly one ℓ-group homomorphism $h\colon F^{\ell}_{\mathsf{V}}(G) \to H$ such that $h \circ \xi = p$.*

Proof. It is evident by construction that ξ is an order-embedding. Consider the following commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\eta} & F^{\ell}_{\mathsf{V}}(G) \\ & {\scriptstyle \eta^*\circ\zeta}\searrow & \big\downarrow{\scriptstyle !\,h} \\ & & F^{\ell}_{\mathsf{V}}(\eta[G]) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \eta[G] & \xrightarrow{\eta^*} & F^{\ell}_{\mathsf{V}}(\eta[G]) \\ & {\scriptstyle \xi}\searrow & \big\downarrow{\scriptstyle !\,k} \\ & & F^{\ell}_{\mathsf{V}}(G) \end{array}$$

where $\eta^*\colon \eta[G] \to F^{\ell}_{\mathsf{V}}(\eta[G])$ satisfies the universal property from (3.6). We prove that $h \circ k$ and $k \circ h$ are, respectively, the identity map on $F^{\ell}_{\mathsf{V}}(\eta[G])$ and the identity map on $F^{\ell}_{\mathsf{V}}(G)$. First, since

$$h \circ k \circ \eta^* \circ \zeta = h \circ \xi \circ \zeta = h \circ \eta = \eta^* \circ \zeta,$$

and ζ is an epimorphism, we get $h \circ k \circ \eta^* = \eta^*$. Similarly,

$$k \circ h \circ \eta = k \circ h \circ \xi \circ \zeta = k \circ \eta^* \circ \zeta = \xi \circ \zeta = \eta.$$

By the universal property of η and η^*, the result follows. □

Remark 3.5.5. By Proposition 3.5.4, and by a standard argument on the uniqueness of universal arrows (see Appendix A.1), $F^{\ell}_{\mathsf{V}}(G)$ and $F^{\ell}_{\mathsf{V}}(\eta[G])$ are isomorphic and hence, their spectral spaces $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$ and $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(\eta[G])$ are homeomorphic. Thus, the space $\mathcal{P}_{\mathsf{V}}(\eta[G])$ is homeomorphic to $\operatorname{Spec} F^{\ell}_{\mathsf{V}}(G)$ and hence, by Theorem 3.3.6, to the space $\mathcal{P}_{\mathsf{V}}(G)$.

We say that a partially ordered group G is *isolated*[6] if $a^n \in G^+$ for some $n \in \mathbb{N}^+$ implies $a \in G^+$, for any $a \in G$.

Proposition 3.5.6. *For any partially ordered group G:*

(a) *The universal map $\eta\colon G \to F^{\ell}(G)$ is an order-embedding if and only if the positive cone G^+ is the intersection of the right orders on G.*
(b) *The universal map $\eta\colon G \to F^{\ell}_{\mathsf{Rep}}(G)$ is an order-embedding if and only if the positive cone G^+ is the intersection of the orders on G.*
(c) *The universal map $\eta\colon G \to F^{\ell}_{\mathsf{Ab}}(G)$ is an order-embedding if and only if G is an isolated partially ordered Abelian group.*

We refer to the literature for the (non-trivial) proofs. For the variety LG of all ℓ-groups, see [10, Théorème A.2.2]; for the variety Rep of representable ℓ-groups, see [10, Note de l'appendice]; for the variety Ab of Abelian ℓ-groups, it suffices to observe that the free Abelian ℓ-group $F^{\ell}_{\mathsf{Ab}}(G)$ over G is the free ℓ-group $F^{\ell}(G)$ over G whenever G is Abelian ([40, 1.2]).

Remark 3.5.7. For a partially ordered Abelian group G, being isolated is equivalent to G^+ being the intersection of the (right) orders that extend it ([10, Corollaire A.2.6]).

Corollary 3.5.8. *For any group G, the universal map $\eta\colon G \to F^{\ell}(G)$ is injective (resp., the universal map $\eta\colon G \to F^{\ell}_{\mathsf{Rep}}(G)$ is injective) if and only if G is right-orderable (resp., orderable). Further, the universal map $\eta\colon G \to F^{\ell}_{\mathsf{Ab}}(G)$ is injective if and only if G is torsion-free Abelian.*

Proof. It follows from Proposition 3.5.6 that the positive cone G^+ of any group G (with the trivial order) is $\{e\}$, and hence that G^+ is the intersection of right orders (resp., orders) if and only if the group G is right-orderable (resp., orderable). For the same reason, an Abelian group (with the trivial order) is isolated if and only if it is torsion-free. □

[6]Partially ordered Abelian groups with this property are also called *unperforated*; cf. [76, p. 19].

Remark 3.5.9. For any variety V of ℓ-groups, if a right preorder $C \in \mathcal{P}_{\mathsf{V}}(G)$ is a right order, then C must be inclusion-minimal in $\mathcal{P}_{\mathsf{V}}(G)$, since any proper subset $D \subset C$ would fail the condition $G = D \cup D^{-1}$. Conversely, suppose that $C \in \mathcal{P}_{\mathsf{V}}(G)$ is minimal. Remark 3.5.5 provides a natural way to associate to C a minimal element of $\mathcal{P}_{\mathsf{V}}(\eta[G])$. However, to the best of our knowledge, the latter need not be a right order. We will see that this is the case for the varieties of all ℓ-groups, representable ℓ-groups, and Abelian ℓ-groups. However, it is an open problem to characterize the varieties V of ℓ-groups such that, for all partially ordered groups G, the minimal elements of $\mathcal{P}_{\mathsf{V}}(G)$ are right orders on $\eta[G]$. Further discussion on this can be found in Remark 3.5.15.

We recall here a characterization of representable ℓ-groups that will be useful in the rest of the section.

Proposition 3.5.10 (cf. [49, Proposition 47.1])**.** *For any ℓ-group H, the following are equivalent:*

1. *H is representable.*
2. *Each minimal prime subgroup is an ℓ-ideal.*
3. *Each polar is normal.*

We now show that the spaces of right orders and orders on groups, and the space of orders on Abelian groups are recovered in our framework as the subspaces of inclusion-minimal right preorders of the appropriate class.

Theorem 3.5.11. *For any partially ordered group G:*

(a) *The minimal layer of $\mathcal{P}(G)$ is homeomorphic to the space $\mathcal{R}(\eta[G])$ of right orders on $\eta[G]$, where $\eta\colon G \to F^{\ell}(G)$ is the universal map.*
(b) *The minimal layer of $\mathcal{P}_{\mathsf{Rep}}(G)$ is homeomorphic to the space $\mathcal{O}(\eta[G])$ of orders on $\eta[G]$, where $\eta\colon G \to F^{\ell}_{\mathsf{Rep}}(G)$ is the universal map.*
(c) *The minimal layer of $\mathcal{P}_{\mathsf{Ab}}(G)$ is homeomorphic to the space $\mathcal{R}(\eta[G]) = \mathcal{O}(\eta[G])$ of orders on $\eta[G]$, where $\eta\colon G \to F^{\ell}_{\mathsf{Ab}}(G)$ is the universal map.*

Proof. First, we sketch the idea of the proof. For (a), we use the fact that $\eta[G]$ can be endowed with a right order $\leq$ (by Proposition 3.5.4 and Proposition 3.5.6). For any right preorder C on $\eta[G]$, we consider the restriction of a right order to $C \cap C^{-1}$. The relation obtained by lexicographically combining the original right preorder with such a right order on $C \cap C^{-1}$ is itself a right order on $\eta[G]$, which is a refinement of the right preorder C. This shows that every right preorder on $\eta[G]$ contains a right order on $\eta[G]$ and hence, right orders must be minimal elements of $\mathcal{P}(\eta[G])$. The result then follows from Remark 3.5.5, since $\mathcal{P}(\eta[G])$ is homeomorphic to $\mathcal{P}(G)$. The second claim is proved analogously, by showing that the resulting right order is an element of $\mathcal{P}_{\mathsf{Rep}}(\eta[G])$ (i.e., an order).

For $\eta\colon G \to F^{\ell}(G)$, the space of right orders on $\eta[G]$ is nonempty by Proposition 3.5.6. By Remark 3.5.9, the space of right orders on $\eta[G]$ consists of minimal elements of $\mathcal{P}(\eta[G])$. We now show that every $C \in \mathcal{P}(\eta[G])$ extends a right order. Let

P be a right order on $\eta[G]$, and $P(C)$ be its restriction $P \cap (C \cap C^{-1})$. Consider the binary relation on $\eta[G]$ defined by

$$a \leq b \iff [a] < [b] \text{ or } ([a] = [b] \text{ and } e \leq_{P(C)} ba^{-1}), \text{ for } a, b \in G. \tag{3.22}$$

The relation $\leq$ is a right order on $\eta[G]$ that extends $\eta[G]^+$, and $a \leq b$ implies $a \preceq_C b$. It is clear that $\leq$ is a total order. Suppose now that $a \leq b$ because $[a] < [b]$. Then, $a \prec_C b$, and hence, $at \prec_C bt$, which means $[at] < [bt]$. On the other hand, if $[a] = [b]$ and $\mathrm{e} \leq_{P(C)} ba^{-1}$, then $[ac] = [bc]$ and $\mathrm{e} \leq_{P(C)} bcc^{-1}a^{-1}$. Finally, it is clear that if $a \leq b$, then $a \preceq_C b$.

For $\eta\colon G \to F^{\ell}_{\mathsf{Rep}}(G)$, Proposition 3.5.6 entails that the space of orders on $\eta[G]$ is nonempty, and by Remark 3.5.9, the space of orders on $\eta[G]$ consists of minimal elements of $\mathcal{P}_{\mathsf{Rep}}(\eta[G])$. We pick an order P on $\eta[G]$ and its restriction $P(C)$, and show that if $C \in \mathcal{B}_{\mathsf{Rep}}(\eta[G])$, the binary relation $\leq$ defined in (3.22) is an order on $\eta[G]$ included in C. We can then conclude that every minimal element C of $\mathcal{P}_{\mathsf{Rep}}(\eta[G])$ is a preorder, using Proposition 3.5.10 and Theorem 3.3.12. Hence, we only need to prove that the right order $\leq$ that we obtain is also left-invariant. For this, suppose that $a \leq b$ because $[a] < [b]$. This means that $a \prec_C b$, and hence, $\mathrm{e} \prec_C ba^{-1}$. Now, by Proposition 3.5.1, $\mathrm{e} \preceq_C cba^{-1}c^{-1}$. If we had also $cba^{-1}c^{-1} \preceq_C \mathrm{e}$, we would get a contradiction with $ba^{-1} \not\preceq_C \mathrm{e}$ since C is a preorder. Therefore, $ca \prec_C cb$, and hence, $[ca] < [cb]$. Assume now that $[a] = [b]$ and $\mathrm{e} \leq_{P(C)} ba^{-1}$. If $ba^{-1} \in P(C)$, also $cba^{-1}c^{-1} \in P(C)$. The latter entails $[ca] = [cb]$ and $e \leq_{P(C)} cba^{-1}c^{-1}$. Therefore, if $a \leq b$, also $ca \leq cb$, and the right order $\leq$ is in fact an order on $\eta[G]$.

For $\eta\colon G \to F^{\ell}_{\mathsf{Ab}}(G)$, note that $\eta[G]$ is an isolated partially ordered Abelian group and hence, from Remark 3.5.9, $\eta[G]^+$ is the intersection of the orders that extend it. Moreover, the free ℓ-group $F^{\ell}(\eta[G])$ over $\eta[G]$ is the free Abelian ℓ-group $F^{\ell}_{\mathsf{Ab}}(\eta[G])$ over $\eta[G]$ [40, 1.2]. Thus, $\mathcal{P}_{\mathsf{Ab}}(\eta[G])$ is $\mathcal{P}(\eta[G])$ and hence, it follows by (a) that the minimal layer of $\mathcal{P}_{\mathsf{Ab}}(\eta[G])$ is the space of (right) orders on $\eta[G]$. □

We can now combine Corollary 3.3.14 and Theorem 3.5.11 to obtain:

Corollary 3.5.12. *For any partially ordered group G:*

(a) *The minimal spectrum* $\mathrm{Min}\, F^{\ell}(G)$ *is homeomorphic to the space* $\mathcal{R}(\eta[G])$ *of right orders on* $\eta[G]$*, where* $\eta\colon G \to F^{\ell}(G)$ *is the universal map.*

(b) *The minimal spectrum* $\mathrm{Min}\, F^{\ell}_{\mathsf{Rep}}(G)$ *is homeomorphic to the space* $\mathcal{O}(\eta[G])$ *of orders on* $\eta[G]$*, where* $\eta\colon G \to F^{\ell}_{\mathsf{Rep}}(G)$ *is the universal map.*

(c) *The minimal spectrum* $\mathrm{Min}\, F^{\ell}_{\mathsf{Ab}}(G)$ *is homeomorphic to the space* $\mathcal{R}(\eta[G]) = \mathcal{O}(\eta[G])$ *of orders on* $\eta[G]$*, where* $\eta\colon G \to F^{\ell}_{\mathsf{Ab}}(G)$ *is the universal map.*

The following examples illustrate how Corollary 3.5.12 can be used to derive some easy consequences.

Example 3.5.13. The topological space studied in Example 3.1.2 is homeomorphic to the minimal spectrum $\mathrm{Min}\, F^{\ell}_{\mathsf{Ab}}(2)$ of the free Abelian ℓ-group over two generators; thus, the space $\mathrm{Min}\, F^{\ell}_{\mathsf{Ab}}(2)$ is Cantor, and the same can be concluded for $\mathrm{Min}\, F^{\ell}_{\mathsf{Ab}}(k)$, for all $k \geq 2$.

Example 3.5.14. It was mentioned in Example 3.1.4, that the space of right orders of the fundamental group K of the Klein bottle is finite, having only 4 elements. Thus, by Corollary 3.5.12, $\operatorname{Min} F^{\ell}(K)$ has only finitely many minimal primes (namely, 4). Moreover, since it is finitely generated (as the Klein bottle group is), it has a strong unit (see Remark 3.2.10) and hence, every prime subgroup is contained in exactly one maximal prime. Therefore, $\operatorname{Spec} F^{\ell}(K)$ contains *at most* 4 maximal elements. We also mention that every right order on K is Conradian and hence, the free ℓ-group $F^{\ell}(K)$ is normal-valued.

Remark 3.5.15. We do not know at this stage whether a general characterization of the minimal elements of $\mathcal{P}_{\mathsf{V}}(G)$ along the lines of Theorem 3.5.11 is feasible, even in the case of well-studied varieties of ℓ-groups. For example, recall that N is the variety of normal-valued ℓ-groups (see Section 2.3). Suppose further that G is a group admitting a Conradian right order. Then it can be proved that each Conradian right order on G is a minimal element of $\mathcal{P}_{\mathsf{N}}(G)$. However, it is unclear to us at present whether each minimal member of $\mathcal{P}_{\mathsf{N}}(G)$ is a Conradian right order on G. Further discussion on this topic can be found in Section 3.6.

Lemma 3.5.16. *For any partially ordered group G, the minimal layers of the spaces $\mathcal{P}(G)$, $\mathcal{P}_{\mathsf{Rep}}(G)$, and $\mathcal{P}_{\mathsf{Ab}}(G)$ are compact.*

Proof. By Theorem 3.5.11, if V is the variety of all ℓ-groups (resp., representable or Abelian ℓ-groups), the minimal layer of $\mathcal{P}_{\mathsf{V}}(G)$ is the space $\mathcal{R}(\eta[G])$ of right orders (resp., orders $\mathcal{O}(\eta[G])$) on $\eta[G]$. We can now conclude that such a minimal layer is compact, since $\mathcal{R}(\eta[G])$ (resp., $\mathcal{O}(\eta[G])$) is a closed subspace of $2^{\eta[G]}$. □

By combining Theorem 3.3.6 and Lemma 3.5.16, we now get:

Theorem 3.5.17. *Suppose that V is the variety of all ℓ-groups (resp., representable or Abelian ℓ-groups), and G is any partially ordered group. Then, $\operatorname{Min} F^{\ell}_{\mathsf{V}}(G)$ is compact.*

Therefore, in view of Corollary 3.4.11 and Lemma 3.5.16, the spaces of right orders and orders on a partially ordered group G are the dual Boolean spaces of the Boolean algebras of principal polars of the free ℓ-group over G relative to the appropriate variety.

Theorem 3.5.18. *Suppose that V is the variety of all ℓ-groups (resp., representable or Abelian ℓ-groups). For any partially ordered group G, the minimal layer of $\mathcal{P}_{\mathsf{V}}(G)$ is a Boolean space with dual Boolean algebra $\operatorname{Pol_p} F^{\ell}_{\mathsf{V}}(G)$.*

Theorem 3.5.17 provides a criterion to determine whether a given ℓ-group is freely generated over some (partially ordered) group.

Example 3.5.19. Let L be the Archimedean ℓ-group of continuous and piecewise linear functions $f\colon [0,1]^2 \to \mathbb{R}$ with integer coefficients, equipped with pointwise operations. Consider

$$S = \{(x,y) \in [0,1]^2 \mid y = x^2\} \cup \{(x,y) \in [0,1]^2 \mid y = 0\},$$

and define H to be the ℓ-group obtained by restricting each element of L to S. It can be shown that $\operatorname{Min} H$ is not compact (see [7, Example 6.1]). Therefore, by Theorem 3.5.17, the ℓ-group H is not freely generated by any partially ordered group.

We now focus on orders on groups and representable ℓ-groups. We show that, in studying orders on a partially ordered group G, it is preferable to look at prime ℓ-ideals of the free representable ℓ-group generated by G, as opposed to all its prime subgroups. This amounts to saying, algebraically, that in varieties of representable ℓ-groups the notion of a prime subgroup should be replaced by the notion of a prime ℓ-ideal (see Theorem 3.5.27).

Remark 3.5.20. Theorem 3.4.7 and Proposition 3.5.10 ensure that $\operatorname{Qin} H \subseteq \operatorname{Spec}^* H$ for every representable ℓ-group H.

We saw in Proposition 2.1.8 that the poset $\mathcal{NC}_p(H)$ of all principal ℓ-ideals of any ℓ-group H is a $\vee$-semilattice. In fact, $\mathcal{NC}_p(H)$ need not be a lattice. More precisely, it is proved in [157, Theorem 6.3] that every distributive at most countable $\vee$-semilattice with minimum is isomorphic to $\mathcal{NC}_p(H)$ for some ℓ-group H. The next result shows that $\mathcal{NC}_p(H)$ is a lattice in all representable ℓ-groups.

Lemma 3.5.21. *For any ℓ-group H, the following are equivalent:*

(1) *H is representable.*
(2) *For all $a, b \in H^+$, $\mathrm{I}(a) \cap \mathrm{I}(b) \subseteq \mathrm{I}(a \wedge b)$.*

Proof. (1) $\Rightarrow$ (2). We show that $\mathrm{I}(a) \cap \mathrm{I}(b) \subseteq \mathrm{I}(a \wedge b)$ for any a, b in the positive cone of a representable ℓ-group H. For this, let H be a subdirect product of $\prod_{t\in T} C_t$, for C_t totally ordered groups. If $|c| \in \mathrm{I}(a) \cap \mathrm{I}(b)$, Proposition 2.1.7 entails

$$|c| \leq \prod_I u_i^{-1} a u_i \quad \text{and} \quad |c| \leq \prod_J v_j^{-1} b v_j.$$

Thus, in a given factor C_t of the product $\prod_{t\in T} C_t$,

$$(|c|)_t \leq (\prod_I u_i^{-1} a u_i)_t \quad \text{and} \quad (|c|)_t \leq (\prod_J v_j^{-1} b v_j)_t.$$

Since the group operation is also defined coordinate-wise, we obtain

$$(|c|)_t \leq \prod_I (u_i^{-1})_t (a)_t (u_i)_t \quad \text{and} \quad (|c|)_t \leq \prod_J (v_j^{-1})_t (b)_t (v_j)_t.$$

Without loss of generality, we can assume $(a)_t \leq (b)_t$, and hence

$$(|c|)_t \leq \prod_I (u_i^{-1})_t (a)_t (u_i)_t = \prod_I (u_i^{-1})_t (a \wedge b)_t (u_i)_t.$$

Since $t \in T$ was arbitrary, $|c| \leq \prod_I w_i^{-1} (a \wedge b) w_i$, for some $w_i \in H$, I finite. Thus, $c \in \mathrm{I}(a \wedge b)$.

(2) ⇒ (1). We use the characterizing property of representable ℓ-groups stated in Proposition 3.5.10: we show that $\mathfrak{I}(a) \cap \mathfrak{I}(b) \subseteq \mathfrak{I}(a \wedge b)$ entails that $a^{\perp}$ is normal, for every $a \in H$. Assume $a \wedge b = \mathrm{e}$, for some $a, b \in H^{+}$, that is, assume $b \in a^{\perp}$ for $a, b \in H^{+}$. Thus,

$$\mathfrak{I}(a) \cap \mathfrak{I}(b) = \mathfrak{I}(a \wedge b) = \{\mathrm{e}\}. \tag{3.23}$$

Since $\mathrm{e} \leq (a \wedge c^{-1}bc) \leq a \in \mathfrak{I}(a)$ and $\mathrm{e} \leq (a \wedge c^{-1}bc) \leq c^{-1}bc \in \mathfrak{I}(b)$ for any $c \in H$, by convexity, $(a \wedge c^{-1}bc) \in \mathfrak{I}(a) \cap \mathfrak{I}(b) = \{\mathrm{e}\}$ for any $c \in H$. Thus, $a \wedge c^{-1}bc = \mathrm{e}$ for any $c \in H$. □

Theorem 3.5.22. *For any ℓ-group H, the map*

$$\mathcal{C}_p(H) \xrightarrow{g} \mathcal{NC}(H)$$

defined by $\mathfrak{C}(a) \mapsto \mathfrak{I}(a)$ is a $\vee$-semilattice homomorphism preserving minimum such that

$$g[\mathcal{C}_p(H)] = \mathcal{NC}_p(H).$$

Moreover, it is a lattice homomorphism if and only if H is representable.

Proof. First, observe that from $\mathfrak{C}(a) = \mathfrak{C}(b)$, it follows that $a \in \mathfrak{C}(b) \subseteq \mathfrak{I}(b)$ and $b \in \mathfrak{C}(a) \subseteq \mathfrak{I}(a)$. Thus, if $\mathfrak{C}(a) = \mathfrak{C}(b)$, then $\mathfrak{I}(a) = \mathfrak{I}(b)$. Hence, g is well-defined. Further, it is clear that $g[\mathcal{C}_p(H)] = \mathcal{NC}_p(H)$. By Proposition 2.1.5.(b) and Proposition 2.1.8.(b),

$$g(\mathfrak{C}(a) \vee \mathfrak{C}(b)) = g(\mathfrak{C}(a \vee b)) = \mathfrak{I}(a \vee b) = \mathfrak{I}(a) \vee \mathfrak{I}(b).$$

Finally, $\mathfrak{C}(\mathrm{e}) = \{\mathrm{e}\} = \mathfrak{I}(\mathrm{e})$.

Since $g(\mathfrak{C}(a) \cap \mathfrak{C}(b)) = \mathfrak{I}(a) \wedge \mathfrak{I}(b)$ if and only if $\mathfrak{I}(a \wedge b) = \mathfrak{I}(a) \cap \mathfrak{I}(b)$, the second statement follows from Lemma 3.5.21, and Propositions 2.1.5.(b) and 2.1.8.(b). □

Corollary 3.5.23. *For any representable ℓ-group H,* Spec^*H *is homeomorphic to the dual space of the distributive lattice with minimum $\mathcal{NC}_p(H)$. Hence,* Spec^*H *is generalized spectral.*

Proof. For any representable ℓ-group H, set $D = \mathcal{NC}_p(H)$. Then, the map

$$\begin{aligned} X(D) &\longrightarrow \mathcal{NC}(H) \\ I &\longmapsto \bigvee\{\mathfrak{I}(a) \mid \mathfrak{I}(a) \in I\} \end{aligned}$$

restricts to a homeomorphism between $X(D)$ and Spec^*H. The proof proceeds along the same lines as the proof of Theorem 3.2.6, and is based on Propositions 3.2.4, 2.1.8, and 2.1.9, and Theorem 3.5.22 □

Remark 3.5.24. In this chapter, we make use of lattice-theoretic tools to study and compare the several different algebraic structures that arise from (the structure of) ℓ-groups. The application of lattice theory to the study of ℓ-groups is not the main goal of the chapter. Therefore, our treatment in this direction is somewhat initial and not developed to its full potential. The literature contains much more; e.g., besides the already mentioned [162], Cornish's well-known papers [44, 45] are clearly inspired by ℓ-groups. We also mention 'Conrad Program', whose goal was to study ℓ-groups by investigating their lattices of convex ℓ-subgroups; see also the remarks in the introduction to Chapter 5. We summarize the mutual relationship between the lattices treated here at the end of the chapter (see Table 3.1 and Figure 3.3).

For any ℓ-group H, we consider the function[7]

$$\begin{aligned} \mathcal{C}(H) &\xrightarrow{\nu} \mathcal{C}(H) \\ \mathfrak{k} &\longmapsto \bigcap_{a\in H} a^{-1}\mathfrak{k}a, \end{aligned} \tag{3.24}$$

that maps any convex ℓ-subgroup $\mathfrak{k}$ of H to the largest ℓ-ideal contained in $\mathfrak{k}$ (see Remark 2.1.16). Recall that an endofunction $\iota\colon S \to S$ on a partially ordered set S is an interior operator if it is contracting ($\iota(a) \leq a$), monotone ($a \leq b$ entails $\iota(a) \leq \iota(b)$), and idempotent ($\iota\circ\iota$ coincides with ι on S); see Appendix A.2. The map ν is an interior operator on $\mathcal{C}(H)$, and $\mathcal{NC}(H)$ consists precisely of the open elements of ν.

Lemma 3.5.25. *For any ℓ-group H, the map ν restricts to an interior operator ν on* $\operatorname{Spec} H$ *such that* $\operatorname{Spec}^* H$ *consists precisely of the open elements of ν if and only if H is representable. In this case,* $\nu\colon \operatorname{Spec} H \to \operatorname{Spec}^* H$ *is a continuous retraction.*

Proof. First, if ν is an interior operator onto $\operatorname{Spec}^* H$, since $\nu(\mathfrak{m}) \subseteq \mathfrak{m}$ for every $\mathfrak{m} \in \operatorname{Min} H$, we can conclude $\nu(\mathfrak{m}) = \mathfrak{m}$ and hence, every minimal prime is an ℓ-ideal. Thus, by Proposition 3.5.10, H is representable. Conversely, suppose that H is representable, and take $a, b \in H$ such that $a \wedge b = \mathrm{e}$. We show that either $a \in \nu(\mathfrak{p})$ or $b \in \nu(\mathfrak{p})$, for every $\mathfrak{p} \in \operatorname{Spec} H$. If $a \notin \nu(\mathfrak{p})$, there exists a $c \in H$ such that $c^{-1}ac \notin \mathfrak{p}$. Now, since $a \wedge b = \mathrm{e}$ and H is representable, by Proposition 3.5.10 also $c^{-1}ac \wedge d^{-1}bd = \mathrm{e}$ for every $d \in H$. Therefore, since $\mathfrak{p}$ is prime, $d^{-1}bd \in \mathfrak{p}$ for every $d \in H$, that is, $b \in \nu(\mathfrak{p})$. The fact that ν restricts to an interior operator $\nu\colon \operatorname{Spec} H \to \operatorname{Spec}^* H$ is immediate. Also, $\nu(\mathfrak{p}) = \mathfrak{p}$ if and only if $\mathfrak{p} \in \operatorname{Spec}^* H$. Finally, observe that

$$\nu^{-1}[\mathbb{S}^*(a)] = \bigcup_{b\in H} \mathbb{S}(b^{-1}ab)$$

and hence, ν is continuous. □

Under an appropriate duality theorem for distributive lattices with minimum, the retraction $\nu\colon \operatorname{Spec} H \to \operatorname{Spec}^* H$ will correspond to a suitable arrow $\mathcal{NC}_p(H) \to \mathcal{C}_p(H)$. At the time of writing it is not clear to the author what this arrow is.

Remark 3.5.26. The map $\nu\colon \operatorname{Spec} H \to \operatorname{Spec}^* H$ sends a prime subgroup $\mathfrak{p}$ to the kernel $\ker R_{\mathfrak{p}}$ of the map $R_{\mathfrak{p}}\colon H \to \operatorname{Aut}(H/\mathfrak{p})$ defined in (2.4).

[7] For a related use of this map, compare the characterization of representable ℓ-groups in [163, Theorem 2.4.4.(d)].

For any partially ordered group G and any variety V of representable ℓ-groups, consider

$$\begin{aligned} \mathcal{P}_{\mathsf{V}}(G) &\xrightarrow{\beta} \mathcal{P}_{\mathsf{V}}(G) \\ C &\longmapsto \bigcap_{t\in G} t^{-1}Ct. \end{aligned} \tag{3.25}$$

The following result concludes the section, showing that the space of preorders on any partially ordered group G is a retract of the space of representable right preorders on G.

Theorem 3.5.27. *For any partially ordered group G and any variety V of representable ℓ-groups:*

(a) *The set $\mathcal{B}_{\mathsf{V}}(G)$ with the subspace topology induced from $\mathcal{P}_{\mathsf{V}}(G)$ is generalized spectral.*

(b) *The map β is an interior operator on $\mathcal{P}_{\mathsf{V}}(G)$ such that $\mathcal{B}_{\mathsf{V}}(G)$ consists precisely of the open elements of β. Moreover, $\beta\colon \mathcal{P}_{\mathsf{V}}(G) \to \mathcal{B}_{\mathsf{V}}(G)$ is a continuous retraction.*

Proof. (a) follows immediately from Theorem 3.3.6 and Corollary 3.5.23. For (b), observe that for a right preorder C, the set $\beta(C)$ is clearly a normal submonoid of G. Moreover, if $a \notin \beta(C)$, there is a conjugate $t^{-1}at \notin C$ for $t \in G$; hence, by Proposition 3.5.1, $s^{-1}as \in C^{-1}$ for each $s \in G$, that is, $a^{-1} \in \beta(C)$. Therefore, $\beta(C)\cup\beta(C)^{-1} = G$, and β is a well-defined function onto $\mathcal{B}_{\mathsf{V}}(G)$. Further, β sends a right preorder C in $\mathcal{P}_{\mathsf{V}}(G)$ to the largest preorder contained in C, and $\beta(C) = \pi(\nu(\kappa(C)))$. Applying Theorem 3.3.6 and Lemma 3.5.25 completes the proof. □

3.6 CONCLUDING REMARKS

In this chapter we provided a systematic, structural account of the relationship between right preorders on a group G and prime subgroups of the ℓ-group $F^{\ell}(G)$ freely generated by the group G. More concretely, we showed that for any partially ordered group G, the space of right preorders on G is homeomorphic to the spectral space of the ℓ-group $F^{\ell}(G)$ freely generated by the partially ordered group G; this correspondence can be specialized to specific varieties of ℓ-groups, by considering the subspace of right preorders on G whose right regular representation is in the considered variety. The connection we exhibited and studied here was previously identified in its basic form by Stephen McCleary in his paper on representations of free ℓ-groups by ordered permutation groups, cf. [128, Lemma 16]. There, McCleary considers a free group $F(X)$ and constructs a bijection between right orders on $F(X)$ and minimal prime subgroups of $F^{\ell}(X)$. This chapter may be viewed as a generalization and extension of McCleary's result. Let us also mention [25], where the author acknowledges McCleary's work as a source for his own correspondence between closures of orbits (under the natural action of G) in the space of right orders on a group G, and kernels of certain maps from $F^{\ell}(G)$.

It follows from the correspondence established here that results formulated in the setting of (partially ordered) groups can be now translated to the setting of ℓ-groups. We have already seen examples of this. For instance, compactness of the minimal ℓ-spectrum can be formulated, for free ℓ-groups over a group, in terms of compactness of the space of suitable preorders on the group. Another consequence that can be drawn, and was not mentioned in the main body of the chapter, is that free representable ℓ-groups over a group can be represented as sections of a Hausdorff sheaf over a suitable space of orders (cf. [49, Proposition 49.9]). The following problem can also be immediately reformulated in terms of free representable ℓ-groups.

Problem 6. Determine whether the space of orders on the finitely generated free group $F(k)$ ($k \geq 2$) has isolated points; equivalently, determine whether the Boolean algebra of principal polars of any finitely generated free representable ℓ-group is atomless.

It was again McCleary, in his work with Ashok Arora ([3]), who first raised this problem. The analogous question was answered positively for the space of right orders by McCleary (cf. Example 3.1.3), and later by several other authors independently (e.g., [139]; [25]); in particular, we mention a geometric/combinatorial proof, which consists of two steps:

- An immediate consequence of the results contained in [27] is that (the positive cone of) any isolated point in the space of right orders on the free group must be finitely generated as a semigroup (cf. [55, Theorem 2.2.33]).
- It follows from a result in [107] that no finitely generated positive cone on the free group exists.

The problem for orders is still open. (We mention the recent preprint [137] where a solution to this problem is claimed.)

The notion of a right preorder associated to a given variety can be described in more details for certain specific varieties. This is the content of Propositions 3.5.1 and 3.5.2, where a description of representable and Abelian right-preordered groups is given.

Problem 7. Determine for which varieties of ℓ-groups, the corresponding class of right preorders on groups as provided by Definition 3.1.10 is elementary in the language $\{\cdot, ^{-1}, \mathrm{e}, \preceq\}$ of groups with a binary relation.

This problem, already mentioned in Remark 3.5.3, can be stated more precisely considering the following setting. Let G_C be a right-preordered group. It is possible to associate to G_C the variety of ℓ-groups generated by H_C (more precisely, the set of those equations that are valid in H_C). This can obviously be extended to a class of right-preordered groups, by taking the variety generated by H_C for any right-preordered group G_C in the class at hand.

Conversely, we can associate a class of right-preordered groups to any variety of ℓ-groups as follows. Let H be an ℓ-group and $\mathfrak{p} \in \operatorname{Spec} H$. We call a group G *big* (with respect to H and $\mathfrak{p}$) if there exists a group homomorphism $f\colon G \to H$ such that $\Omega_{\mathfrak{p}} = \{\mathfrak{p}a \mid a \in f[G]\}$, that is, the composition of the quotient induced by $\mathfrak{p}$ after f is onto. For any such G, we define

$$a \preceq_{\mathfrak{p}} b \iff \mathfrak{p}f(a) \leq \mathfrak{p}f(b),$$

and obtain the right-preordered group $G_{C_{\mathfrak{p}}}$, where $C_{\mathfrak{p}}$ is the set $\{a \in G \mid \mathrm{e} \preceq_{\mathfrak{p}} a\}$. By repeating this process for any prime $\mathfrak{p} \in \operatorname{Spec} H$ and any big group G, we obtain a class of right-preordered groups associated to H. Note that for any $\mathfrak{p} \in \operatorname{Spec} H$ and any big group G, the poset reflection $\Omega_{C_{\mathfrak{p}}}$ of $G_{C_{\mathfrak{p}}}$ is isomorphic to the chain $\Omega_{\mathfrak{p}}$.

The maps that we have just described, associating a class of right-preordered groups to any variety of ℓ-groups, and a variety of ℓ-groups to any class of right-preordered groups, induce a contravariant Galois connection between:

- classes of right-preordered groups ordered by inclusion;
- sets of ℓ-group equations ordered by inclusion.

The closed elements on the ℓ-group side correspond to varieties of ℓ-groups, namely those sets of equations that are closed under consequence. Conversely, a class of right-preordered groups is closed if and only if it contains all and only those right-preordered groups G_C such that H_C lies in a prescribed variety of ℓ-groups. Is every elementary class of right-preordered groups a closed element in the considered Galois connection? The answer to this question is negative. For instance, the elementary class of preordered Abelian groups corresponds to the variety of Abelian ℓ-groups; however, the corresponding closed element is the class of groups equipped with Abelian right preorders (see Definition 3.1.10). Conversely, are the closed elements all elementary? Is it possible to syntactically characterize varieties of ℓ-groups corresponding to elementary classes of right preordered groups? Further, if an equational axiomatization for a variety of ℓ-groups is given, can we recover an elementary axiomatisation for the corresponding class of right-preordered groups (similarly to what we did for the Abelian case and the representable case)? The development of a sufficiently systematic way to connect sets of ℓ-group equations to first-order theories in the language of groups with a binary relation would result in a 'Sahlqvist-type' correspondence theory for ℓ-groups (cf. [158]).

The last question can be answered easily in some well-behaved settings as, e.g., those varieties defined by group equations relative to the variety of representable ℓ-groups. (It was discussed already in Chapter 2 that these varieties are particularly suitable to work with—see also Theorem 1.4.1.) Every variety of ℓ-groups defined relative to Rep by a group equation

$$t(z_1,\dots,z_n) \approx \mathrm{e} \tag{3.26}$$

corresponds to an elementary class of right-preordered groups, axiomatized (relative to right-preordered groups) by

$$(\forall x)\big((\forall y)(y^{-1}xy \preceq \mathrm{e}) \vee (\forall z)(\mathrm{e} \preceq z^{-1}xz)\big)$$
$$(\forall x)(\forall z_1)\dots(\forall z_n)\big((xt(z_1,\dots,z_n) \preceq x) \wedge (x \preceq xt(z_1,\dots,z_n))\big).$$

Indeed, the first axiom ensures that the ℓ-group H_C obtained from any model G_C is totally ordered or, equivalently, representable (see Proposition 2.1.17). Since H_C is totally ordered, its group reduct is isomorphic to $R_C[G]$; therefore, it suffices to make sure that $R_C[G]$ satisfies (3.26). This is ensured by the second axiom. Note that we used the fact that transitive ℓ-groups have a simpler description and, in particular, are totally ordered in the representable case. Therefore, it would be probably hard to make this approach systematic.

As we mentioned at the beginning of the chapter, this work was motivated by the question whether there exists some relationship between the topological space of right orders on a group, and the spectral space of some ℓ-group. The framework developed here answers this question for any partially ordered group. More precisely, in Corollary 3.5.12 we showed that the spaces of right orders and orders on groups, and the space of orders on Abelian groups are recovered in our framework as the subspaces of inclusion-minimal right preorders of the appropriate class in each case. This result has interesting immediate consequences. For instance, that the space of (right) orders on a partially ordered group is compact is very easy to check, given Adam Sikora's definition. Therefore, we obtain a way to check whether an ℓ-group is freely generated by a (partially ordered) group (see Example 3.5.19). As was already mentioned in Remark 3.5.15, being able to fully describe the kind of right preorders that correspond to minimal ℓ-spectra for all varieties of ℓ-groups would be of interest and possibly lead to applications along the same lines of those described in Examples 3.5.13, 3.5.14, and 3.5.19

Problem 8. Extend Corollary 3.5.12 to other varieties of ℓ-groups and collections of right preorders.

We briefly discuss a related example. It is known that the free normal-valued ℓ-group $F^{\ell}_{\mathsf{N}}(X)$ can be represented as a subdirect product of all H_C such that C is a Conradian right order on the free group $F(X)$ ([110]). This fact shows that the subspace of (Conradian) right orders is *dense* in the minimal layer of $\mathcal{P}_{\mathsf{N}}(F(X))$, although at this stage we cannot conclude that every minimal element of $\mathcal{P}_{\mathsf{N}}(F(X))$ (equivalently, any minimal prime of $F^{\ell}_{\mathsf{N}}(X)$) corresponds to a Conradian right order. Moreover, a study of normal-valued ℓ-groups in terms of Theorem 3.5.11 is not possible at this stage, as we lack a suitable first-order description of those right preorders corresponding to the variety of normal-valued ℓ-groups. A deeper study of the arguments used by Valerii Kopytov in [110] would lead to a better understanding of these issues, and would possibly provide answers for several other varieties as well.

ALGEBRA	STRUCTURE	DUAL SPACE
Principal convex ℓ-subgroups $\mathcal{C}_p(H)$	Distributive lattice with minimum	Spec H
Principal polars $\mathrm{Pol_p}\, H$	Distributive lattice with minimum (when Boolean Algebra)	Qin H (Min H)
Principal ℓ-ideals $\mathcal{NC}_p(H)$	Distributive $(\vee, 0)$-semilattice (when H representable, distributive lattice with minimum)	 (Spec* H)

Table 3.1: Some of the distributive lattices arising from an ℓ-group H

Figure 3.3: The picture illustrates how the lattices and semilattices discussed above are related to each other. Some of the maps presented in the picture were explicitly discussed in the chapter. Note that, in general, the function g is only a $\vee$-semilattice homomorphism (preserving the minimum). However, when H is representable, g becomes a lattice homomorphism (see Theorem 3.5.22), and there exists a further lattice homomorphism that makes the diagram commute.

CHAPTER 4

DISTRIBUTIVE LATTICE-ORDERED MONOIDS

Cayley's theorem for groups can be generalized to the context of semigroups and monoids in an obvious way: every monoid is isomorphic to a monoid of transformations of some set. An analogous generalization in the setting of ℓ-groups leads us to consider order-preserving endomorphisms on chains. The monoid of all order-preserving endomorphisms on a chain ordered pointwise is a distributive lattice-ordered monoid (briefly, distributive ℓ-monoid), in the sense that the monoid operation distributes over both meet and join, and the lattice reduct is distributive. In 1984, Marlow Anderson and Constance Edwards showed, extending Holland's theorem for ℓ-groups, that any distributive ℓ-monoid is an ℓ-monoid of order-preserving endomorphisms on a chain.

It has long been known that the variety of those distributive ℓ-monoids that are commutative strictly contains the variety generated by the 'inverse-free' reduct of $\mathbb{Z}$ ([149]; [16]). Equivalently, there exists an ℓ-monoid equation that holds in all Abelian ℓ-groups and does not hold in some commutative distributive ℓ-monoid. In analogy with the commutative case, we prove in Section 4.3 that the variety of representable distributive ℓ-monoids—the variety of distributive ℓ-monoids generated by totally ordered monoids—is *not* the variety generated by 'inverse-free' reducts of representable ℓ-groups (Theorem 4.3.6). We also provide an axiomatization for the variety of representable distributive ℓ-monoids (Theorem 4.3.2).

Our primary goal is to show that the variety of distributive ℓ-monoids is the variety generated by the 'inverse-free' reducts of ℓ-groups (Theorem 4.4.3). This result, which is the focus of Section 4.4, is at first extremely surprising, especially in view of the fact that distributive ℓ-monoids satisfy the finite model property (Corollary 4.2.4). However, it should perhaps not come as a surprise for a reader familiar with the proof of Charles Holland and Stephen McCleary that the equational theory of ℓ-groups is decidable ([127]). It is clear from their argument that the validity of an equation in all ℓ-groups is witnessed by a finite set of points, which is determined by the initial subterms of the group terms involved. Our results also lead to a further proof of the decidability of the equational theory of ℓ-groups, based on Theorem 4.4.11 that the validity of an ℓ-group equation in LG corresponds to the validity of an algorithmically constructible finite set of ℓ-monoid equations. Finally, Corollary 4.4.7 establishes a relationship between right orders on free monoids and right orders on free groups, showing that finitely many inequalities extend to a right order on the free monoid if and only if they extend to a right order on the free group.

This chapter is based on joint ongoing work with Nikolaos Galatos and George Metcalfe. The theory and terminology from order theory and category theory used in this chapter is reviewed in Appendix A.1 and Appendix A.2, respectively.

4.1 HOLLAND-TYPE REPRESENTATION THEOREM

This section contains some examples and preliminaries, providing sufficient background to sketch a proof of the 'Holland-type' theorem for distributive ℓ-monoids (Theorem 2.1.20). This result was first proved by Anderson and Edwards ([2]; cf. [144]). Later, we use this representation theorem to show that the variety of distributive ℓ-monoids has the finite model property.

A *distributive ℓ-monoid* is an algebraic structure M, with operations $\cdot, \wedge, \vee, \mathrm{e}$ such that $(M, \cdot, \mathrm{e})$ is a monoid, $(M, \wedge, \vee)$ is a distributive lattice, and the monoid operation distributes over the lattice operations, i.e., the following equations hold:

$$z(x \wedge y)w \approx zxw \wedge zyw \tag{4.1}$$

$$z(x \vee y)w \approx zxw \vee zyw. \tag{4.2}$$

The set $M^+ = \{a \in M \mid a \geq \mathrm{e}\}$ is the *positive cone* of M. Clearly, distributive ℓ-monoids form a variety denoted by DLM. We call a distributive ℓ-monoid *commutative* if its monoid reduct is commutative.

Remark 4.1.1. We bring to the reader's attention the following two facts. A lattice order on (the carrier of) a group is preserved by multiplication on the left and right if and only if the multiplication distributes over meet and join; this is not the case for monoids, where only the right-to-left implication holds. Also, the distributivity of the lattice reduct of an ℓ-group follows from the distributivity of multiplication over meet and join, while this is not true for ℓ-monoids.

Example 4.1.2. The additive monoids $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$, when equipped with lattice operations min and max, are (commutative) distributive ℓ-monoids. More generally, any $\{\cdot, \wedge, \vee, \mathrm{e}\}$-subreduct of an ℓ-group is a distributive ℓ-monoid; in this sense, distributive ℓ-monoids are a generalization of ℓ-groups.

Example 4.1.3. For any topological space X with a preorder, the set of bounded continuous monotone functions from X to $\mathbb{R}$ with monoid and lattice operations defined pointwise is a commutative distributive ℓ-monoid.

We call any $\{\cdot, \wedge, \vee, \mathrm{e}\}$-(sub)reduct of an ℓ-group an *inverse-free* (sub)reduct. Subalgebras of any distributive ℓ-monoid M, namely sublattice submonoids of M, are called ℓ-submonoids.

Similarly to the ℓ-group case, the total order on a set naturally lifts to a lattice order on its monoid of order-preserving maps.

Example 4.1.4. The monoid of all order-preserving endomorphisms $\mathrm{End}(\Omega)$ of any chain Ω (with operation $f \cdot g$ defined as $g \circ f$) ordered pointwise is a distributive ℓ-monoid. We already know that its ℓ-submonoid $\mathrm{Aut}(\Omega)$ of order-preserving bijections forms an ℓ-group.

It was mentioned in Section 2.1 that the lattice congruence identified by any convex ℓ-subgroup is also a right (group) congruence. In [57], Dubreil extended the notion of a right congruence on a group to the setting of semigroups (and hence, monoids) in two ways, one of which is reviewed here. In any group G, if K is a subgroup of G, we usually define $a, b \in G$ to be related if $Ka = Kb$; the resulting equivalence relation is in fact a right congruence. Easy calculations show that $Ka = Kb$ if and only if, for all $c \in G$, $ac \in K$ if and only if $bc \in K$. For any monoid M and any subset $S \subseteq M$, define

$$a\backslash S := \{m \in M \mid am \in S\}, \quad \text{for } a \in M.$$

It is straightforward to prove that the relation $\rho_S \subseteq M \times M$ defined by

$$a\rho_S b \iff a\backslash S = b\backslash S \tag{4.3}$$

is a right (monoid) congruence—that is, compatible with multiplication on the right. Two-sided analogues of this relation have also been considered by several authors in semigroup theory (e.g., [145, 146, 46]), and can be defined as follows. Let

$$\frac{S}{a} := \{(m, n) \in M \times M \mid man \in S\}, \quad \text{for } a \in M.$$

For any monoid M and any subset $S \subseteq M$, the relation $\beta_S \subseteq M \times M$ defined by

$$a\beta_S b \iff \frac{S}{a} = \frac{S}{b} \tag{4.4}$$

is readily seen to be a (monoid) congruence contained in ρ_S. It will soon be clear what roles are played by ρ_S and β_S in the study of distributive ℓ-monoids.

Note that since any distributive ℓ-monoid M has a distributive lattice reduct, prime ideals of its lattice reduct exist. Let I be a prime lattice ideal of (the lattice reduct of) a distributive ℓ-monoid M. We consider the set $\{a\backslash I \mid a \in M\}$, with lattice operations defined by

$$a\backslash I \wedge b\backslash I = a\backslash I \cup b\backslash I = (a \wedge b)\backslash I \quad \text{and} \quad a\backslash I \vee b\backslash I = a\backslash I \cap b\backslash I = (a \vee b)\backslash I,$$

and denote it by M/I. That $\wedge$ and $\vee$ are well-defined can be easily checked, and we refer to [144, Lemma 19] for a detailed proof. Note that $a\backslash I \leq b\backslash I$ in the lattice order if and only if $b\backslash I \subseteq a\backslash I$.

Lemma 4.1.5 ([2, Lemma 1])**.** *For any distributive ℓ-monoid M and any prime lattice ideal I of M, M/I is a totally ordered lattice quotient of M.*

Consider now the set $\{\frac{I}{a} \mid a \in M\}$ partially ordered by $\frac{I}{a} \leq \frac{I}{b}$ if and only if $\frac{I}{b} \subseteq \frac{I}{a}$; routine verifications show that this is in fact a lattice order, with operations

$$\frac{I}{a} \wedge \frac{I}{b} = \frac{I}{a} \cup \frac{I}{b} = \frac{I}{(a \wedge b)} \quad \text{and} \quad \frac{I}{a} \vee \frac{I}{b} = \frac{I}{a} \cap \frac{I}{b} = \frac{I}{(a \vee b)}.$$

Further, the binary operation

$$\frac{I}{a} \cdot \frac{I}{b} = \frac{I}{a \cdot b},$$

is a well-defined monoid operation on $\{\frac{I}{a} \mid a \in M\}$, and the resulting algebraic structure, denoted by M/β_I, is a (distributive) ℓ-monoid quotient of M.

Proposition 4.1.6 ([130])**.** *For any distributive ℓ-monoid M and any prime lattice ideal I of M, M/β_I is an ℓ-monoid quotient of M; equivalently, β_I is an ℓ-monoid congruence.*

Lemma 4.1.7. *For any distributive ℓ-monoid M and any prime lattice ideal I of M, the map*

$$\begin{aligned} M &\xrightarrow{R_I} \mathrm{End}\,(M/I) \\ a &\longmapsto R_I(a)\colon\ m\backslash I \mapsto (ma)\backslash I \end{aligned} \tag{4.5}$$

is an ℓ-monoid homomorphism, and its kernel $\{(a,b) \in M \times M \mid R_I(a) = R_I(b)\}$ is β_I. Therefore, the distributive ℓ-monoid $R_I[M]$ is isomorphic to the quotient M/β_I.

Proof. The first part of the statement is well-known, and can already be found in Anderson and Edwards' original article [2]. For the second part, it suffices to observe that

$$\begin{aligned} R_I(a) = R_I(b) \quad &\Longleftrightarrow \quad \text{for all } m \in M\colon\ ma\backslash I = mb\backslash I \\ &\Longleftrightarrow \quad \text{for all } m,n \in M\colon\ man \in I \Leftrightarrow mbn \in I \\ &\Longleftrightarrow \quad \tfrac{I}{a} = \tfrac{I}{b} \end{aligned}$$

for all $a, b \in M$. □

Remark 4.1.8. Note that the relation β_I is the largest ℓ-monoid congruence contained in ρ_I, and this is the exact analogue of the situation for ℓ-groups, as described in Remark 2.1.16.

For any distributive ℓ-monoid M, we write $X(M)$ for the set of its prime lattice ideals. Consider the ℓ-monoid homomorphism

$$\begin{aligned} M &\xrightarrow{\beta} \prod_{I \in X(M)} \mathrm{End}\,(M/I) \\ b &\longmapsto \langle R_I(b) \mid I \in X(M) \rangle\,. \end{aligned} \tag{4.6}$$

The following result relies on the fact that prime lattice ideals are 'enough', and hence β has trivial kernel.

Theorem 4.1.9 ([2]). *For any distributive ℓ-monoid M, the ℓ-monoid homomorphism β defined in* (4.6) *is a subdirect embedding of M into the product $\prod_{I\in X(M)} R_I[M]$.*

Consider a well-order $\preceq$ on $X(M)$, and define the following relation on $\bigsqcup_{I\in X(M)} M/I$:

$$b \le c \quad\Longleftrightarrow\quad \begin{array}{l}\text{there is } I \in X(M) \text{ s.t. } b, c \in M/I \text{ and } b \le c \text{ in } M/I, \text{ or}\\ \text{there are } I_1, I_2 \in X(M) \text{ s.t. } b \in M/I_1 \text{ and } c \in M/I_2 \text{ and } I_1 \prec I_2.\end{array}$$

We write Ω_M for the resulting chain. For $f = \langle f_I \mid M/I\rangle$, the map

$$\prod_{M/I} \mathrm{End}\,(M/I) \xrightarrow{\ \gamma\ } \mathrm{End}\,(\Omega_M)$$
$$f \longmapsto \beta(f)\colon b \mapsto f_I(b) \quad \text{for } b \in M/I$$

is an ℓ-monoid embedding such that $\gamma(\beta(c))(b) = R_I(c)(b)$.

Theorem 4.1.10 ([2]). *Every distributive ℓ-monoid M is an ℓ-submonoid of the distributive ℓ-monoid* $\mathrm{End}\,(\Omega_M)$*, and the isomorphism is given by the map $\gamma\circ\beta$.*

By Theorem 4.1.10, DLM is generated by the class of endomorphism ℓ-monoids of chains. In what follows, we will show that in fact it suffices to consider those chains that are finite. For any finite chain Φ, the distributive ℓ-monoid $\mathrm{End}\,(\Phi)$ is clearly finite, and hence we get the finite model property for DLM. Note that an analogous result is clearly not possible in the context of ℓ-groups, as the group reduct of any ℓ-group is necessarily torsion-free (and hence, infinite).

4.2 THE FINITE MODEL PROPERTY

In this section, we obtain the finite model property for the variety of distributive ℓ-monoids. More precisely, we prove that the variety of distributive ℓ-monoids is generated by the class of (finite) ℓ-monoids of order-preserving endomorphisms of finite chains. We begin with a result on the interplay between $\mathrm{End}\,(\Omega)$ and $\mathrm{End}\,(\Phi)$ for any chain Ω and any finite $\Phi \subseteq \Omega$. We show that the latter is (isomorphic to) a substructure of the former.

First, we prove the following general fact.

Proposition 4.2.1. *Let Ω and Π be chains, and $\sigma\colon \Pi \hookrightarrow \Omega$ and $\nu\colon \Omega \twoheadrightarrow \Pi$ are order-preserving maps, and form a section-retraction pair between Ω and Π. Then, the map*

$$\mathrm{End}\,(\Pi) \longrightarrow \mathrm{End}\,(\Omega), \quad g \longmapsto \sigma\circ g\circ \nu, \tag{4.7}$$

is an injective ℓ-monoid homomorphism.

Proof. It is immediate that the map defined in (4.7) is well-defined. We now show that it is an ℓ-monoid homomorphism. Let $f, g \in \mathrm{End}\,(\Pi)$, and $w \in \Omega$. Then,

$$\begin{aligned}(\sigma\circ(f\wedge g)\circ\nu)(w) &= \sigma(\min\{f(\nu(w)), g(\nu(w))\})\\ &= \min\{\sigma(f(\nu(w))), \sigma(g(\nu(w)))\}\\ &= ((\sigma\circ f\circ\nu)\wedge(\sigma\circ g\circ\nu))(w),\end{aligned} \tag{4.8}$$

where (4.8) is a consequence of the fact that σ is order-preserving. Similarly,

$$\begin{aligned}(\sigma \circ (f \vee g) \circ \nu)(w) &= \sigma(\max\{f(\nu(w)), g(\nu(w))\}) \\ &= \max\{\sigma(f(\nu(w))), \sigma(g(\nu(w)))\} \\ &= ((\sigma \circ f \circ \nu) \vee (\sigma \circ g \circ \nu))(w).\end{aligned}$$

That the map defined in (4.7) is a monoid homomorphism follows from the observation that, for any $f, g \in \mathrm{End}(\Pi)$,

$$\begin{aligned}\sigma \circ (g \circ f) \circ \nu &= \sigma \circ (g \circ (\nu \circ \sigma) \circ f) \circ \nu \qquad (4.9)\\ &= (\sigma \circ g \circ \nu) \circ (\sigma \circ f \circ \nu),\end{aligned}$$

where (4.9) is a consequence of the fact that ν and σ form a section-retraction pair.

It remains to show that the map (4.7) is injective. For this, we observe that the map

$$\mathrm{End}(\Omega) \longrightarrow \mathrm{End}(\Pi), \quad f \longmapsto \nu \circ f \circ \sigma, \qquad (4.10)$$

is a left inverse for the map defined in (4.7). Namely,

$$\nu \circ (\sigma \circ f \circ \nu) \circ \sigma = (\nu \circ \sigma) \circ f \circ (\nu \circ \sigma) = f,$$

where the last equality follows from the fact that ν and σ form a section-retraction pair. □

We remark here that in general, the map defined in (4.10) need not be a monoid homomorphism, even though it is in fact a surjective lattice homomorphism.

For any chain Ω and any finite chain $\Phi \subseteq \Omega$, consider the map $\sigma_* : \Phi \hookrightarrow \Omega$ defined as the set-theoretic inclusion, and the map $\nu_* : \Omega \twoheadrightarrow \Phi$ defined by

$$\nu_*(u) = \begin{cases} \max \Phi, & \text{if } u > \max \Phi \\ \min\{v \in \Phi \mid u \le v\}, & \text{otherwise.} \end{cases}$$

It is easy to see that the maps ν_* and σ_* form a section-retraction pair. In what follows, we identify σ_* with the identity on Φ, and consider the resulting maps

$$\mathrm{End}(\Omega) \xrightarrow{(-)^*} \mathrm{End}(\Phi), \quad f \longmapsto \nu_* \circ f,$$

$$\mathrm{End}(\Phi) \xrightarrow{\overline{(-)}} \mathrm{End}(\Omega), \quad g \longmapsto g \circ \nu_*, \qquad (4.11)$$

where $f \in \mathrm{End}(\Omega)$ and $g \in \mathrm{End}(\Phi)$. We use the map $(-)^*$ to show that the failure of an equation in a distributive ℓ-monoid $\mathrm{End}(\Omega)$ can be translated into the failure of the same equation in $\mathrm{End}(\Phi)$, for some finite $\Phi \subseteq \Omega$. It then follows from Theorem 4.1.10 that the variety of distributive ℓ-monoids satisfies the finite model property and hence, has a decidable equational theory.

Let $T^m(X)$ and $T^{\ell m}(X)$ be the term algebras over a set X for the languages of monoids and ℓ-monoids, respectively. We refer to elements of $T^{\ell m}(X)$ as ℓ-monoid terms. Analogously to the ℓ-group case, it is not hard to show that in any distributive ℓ-monoid, any ℓ-monoid term is equivalent to a term of the form $\bigwedge_{i\in I}\bigvee_{j\in J_i} t_{ij}$ and to a term of the form $\bigvee_{i\in I}\bigwedge_{j\in J_i} t'_{ij}$ where each t_{ij}, t'_{ij} is a monoid term (cf. Remark 1.3.3). Therefore, the validity of an equation $s \approx t$ in the variety DLM is equivalent to the validity of two equations $s \leq t$ and $t \leq s$, whose left-hand side is (equivalent to) a finite join of finite meets and whose right-hand side is (equivalent to) a finite meet of finite joins. Hence, the validity of $s \approx t$ is equivalent to the validity of finitely many equations of the form

$$s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n,$$

where s_i, t_i are monoid terms for all $1 \leq i \leq n$. (By allowing repetition of terms, we can assume that the number of terms in the meet on the left is the same as the number of terms in the join on the right.)

Theorem 4.2.2. *For any chain Ω and ℓ-monoid terms $s, t \in T^{\ell m}(X)$, if the equation $s \leq t$ fails in* End(Ω)*, then it fails in* End(Φ) *for some finite chain Φ.*

Proof. Let Ω be a chain such that $s \leq t$ fails in End(Ω), i.e., there exists a valuation $\varphi\colon T^{\ell m}(X) \to \mathrm{End}(\Omega)$ and an element $w \in \Omega$ such that $\varphi(s)(w) > \varphi(t)(w)$. By the above reasoning, we can assume without loss of generality that $s \leq t$ is of the form

$$s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n,$$

where s_i, t_i are monoid terms in $T^{\ell m}(X)$ for all $1 \leq i \leq n$. That $\varphi(s)(w) > \varphi(t)(w)$ means that

$$\varphi(s_i)(w) > \varphi(t_j)(w) \quad \text{for all } 1 \leq i, j \leq n.$$

For a monoid term u, we write $\mathsf{sub}(u)$ for the set of its subterms (including the trivial word e). Consider the set

$$\Phi = \{\varphi(u)(w) \mid u \in \mathsf{sub}(s_i) \cup \mathsf{sub}(t_j) \text{ for some } 1 \leq i, j \leq n\}.$$

Note that Φ is finite, and $w \in \Phi$ (since $\varphi(\mathrm{e})(w) = w$). Consider the finite distributive ℓ-monoid End(Φ), and the valuation $\psi\colon T^{\ell m}(X) \to \mathrm{End}(\Phi)$ defined by

$$\psi(x) = (\varphi(x))^*,$$

for every $x \in X$. We show that $\psi(u)(w) = \varphi(u)(w)$ for every $u = x_1 \cdots x_k \in \mathsf{sub}(s_i) \cup \mathsf{sub}(t_j)$, for some $1 \leq i, j \leq n$, by induction on $k \in \mathbb{N}$. For $k = 0$, we have $u = \mathrm{e}$ and hence,

$$\psi(\mathrm{e})(w) = (\varphi(\mathrm{e}))^*(w) = w,$$

since $\varphi(\mathrm{e})(w) = w \in \Phi$. Consider now $u = x_1 \cdots x_k x_{k+1} \in \mathsf{sub}(s_i) \cup \mathsf{sub}(t_j)$, for some $1 \leq i, j \leq n$. Then,

$$\begin{aligned} \psi(x_1 \cdots x_k x_{k+1})(w) &= (\psi(x_{k+1}) \circ \psi(x_1 \cdots x_k))(w) \\ &= (\varphi(x_{k+1}))^*(\varphi(x_1 \cdots x_k)(w)) && (4.12) \\ &= \varphi(x_{k+1})(\varphi(x_1 \cdots x_k)(w)), && (4.13) \end{aligned}$$

where (4.12) follows by the induction hypothesis, and (4.13) by the definition of $(-)^*$ and Φ, as $\varphi(x_1 \cdots x_k)(w) \in \Phi$ and $\varphi(x_{k+1})(\varphi(x_1 \cdots x_k)(w)) \in \Phi$ by construction. Thus, $\psi(s_i)(w) = \varphi(s_i)(w)$ and $\psi(t_i)(w) = \varphi(t_i)(w)$, for any $1 \leq i \leq n$ and hence,

$$\psi(s_i)(w) > \psi(t_j)(w) \quad \text{for all } 1 \leq i, j \leq n.$$

Therefore, the equation $s(x_1, \ldots, x_n) \leq t(x_1, \ldots, x_n)$ fails in $\mathrm{End}(\Phi)$, where $\Phi \subseteq \Omega$ is finite. □

What Theorem 4.2.2 shows, in view of Proposition 4.2.1, is that when the equation $s \leq t$ fails in $\mathrm{End}(\Omega)$, then it must fail in a *finite substructure* of $\mathrm{End}(\Omega)$, since the map $\overline{(-)}$ defined in (4.11) is in fact an ℓ-monoid embedding of $\mathrm{End}(\Phi)$ into $\mathrm{End}(\Omega)$ by Proposition 4.2.1. Further, note that it does not matter which retraction we choose between Ω and Φ, since the only thing we actually use is that σ^* is the inclusion, and ν^* and σ^* form a section-retraction pair.

Corollary 4.2.3. *The variety of distributive ℓ-monoids is generated by the class of all ℓ-monoids of endomorphisms of finite chains.*

Proof. Immediate from Theorem 4.1.10 and Theorem 4.2.2. □

Corollary 4.2.4. *The variety of distributive ℓ-monoids has the finite model property; hence, the equational theory of distributive ℓ-monoids is decidable.*

4.3 REPRESENTABLE DISTRIBUTIVE LATTICE-ORDERED MONOIDS

For any monoid M, we say that $\leq \subseteq M \times M$ is a partial order on the monoid M if it is a partial order on its underlying set and, for all $a, b, c, d \in M$, whenever $a \leq b$, also $cad \leq cbd$; if the order $\leq$ is total, we call it a (total) *order on M*, and $(M, \leq)$ is called a *totally ordered monoid*. This section is concerned with representable distributive ℓ-monoids, by which we mean those ℓ-monoids that are subdirect products of totally ordered monoids. We provide a structural characterization of representable distributive ℓ-monoids, and use it to obtain an axiomatization of the variety of representable distributive ℓ-monoids. We conclude the section showing that the variety generated by the inverse-free reducts of representable ℓ-groups form a proper subvariety of the variety of representable distributive ℓ-monoids.

We have seen in Lemma 4.1.5 that, for any prime lattice ideal I of M, the lattice quotient M/I is totally ordered. The next result characterizes those prime lattice ideals I of M for which the ℓ-monoid quotient M/β_I is a chain.

Lemma 4.3.1. *For any distributive ℓ-monoid M and any prime lattice ideal I of M, the quotient M/β_I is totally ordered if and only if $m_1 a m_2 \in I$ and $n_1 b n_2 \in I$ entails $m_1 b m_2 \in I$ or $n_1 a n_2 \in I$, for all $a, b, m_1, m_2, n_1, n_2 \in M$.*

Proof. From right to left, we proceed by contraposition. Pick $a, b \in M$ and suppose

$$\frac{I}{a} \not\subseteq \frac{I}{b} \text{ and } \frac{I}{b} \not\subseteq \frac{I}{a}.$$

But then, by the definition of $\frac{I}{a}, \frac{I}{b}$, there are $m_1, m_2, n_1, n_2 \in M$ such that $m_1 a m_2 \in I$ and $n_1 b n_2 \in I$, but $m_1 b m_2 \notin I$ and $n_1 a n_2 \notin I$. Conversely, if for any $a, b \in M$,

$$\frac{I}{a} \subseteq \frac{I}{b} \quad \text{or} \quad \frac{I}{b} \subseteq \frac{I}{a},$$

then $m_1 a m_2 \in I$ (i.e., $(m_1, m_2) \in \frac{I}{a}$) and $n_1 b n_2 \in I$ (i.e., $(n_1, n_2) \in \frac{I}{b}$) must entail

$$m_1 b m_2 \in I \ \text{ (i.e., } (m_1, m_2) \in \frac{I}{b}) \quad \text{or} \quad n_1 a n_2 \in I \ \text{ (i.e., } (n_1, n_2) \in \frac{I}{a}).$$

This concludes the proof. □

The following result provides an axiomatization of the variety of representable distributive ℓ-monoids, and a characterization of the members of this variety in terms of their prime lattice ideals.

Theorem 4.3.2. *For any distributive ℓ-monoid M, the following are equivalent:*

(1) *M is representable.*

(2) *M satisfies the quasiequation:*

$$u \le v \vee z_1 x z_2 \ \text{ and } \ u \le v \vee w_1 y w_2 \quad \Longrightarrow \quad u \le v \vee z_1 y z_2 \vee w_1 x w_2. \tag{4.14}$$

(3) *M satisfies the equation:*

$$z_1 x z_2 \wedge w_1 y w_2 \le z_1 y z_2 \vee w_1 x w_2. \tag{4.15}$$

(4) *For any prime lattice ideal I of M, the quotient M/β_I is totally ordered.*

Proof. (1) ⇒ (2). As quasiequations are preserved by taking direct products and subalgebras, we may consider without loss of generality totally ordered monoids. Let M be a totally ordered monoid such that $u \le v \vee m_1 a m_2$ and $u \le v \vee n_1 b n_2$ hold, for some $a, b, m_1, m_2, n_1, n_2, u, v \in M$. We show that

$$u \le v \vee m_1 b m_2 \vee n_1 a n_2.$$

Since M is totally ordered, we can assume $a \le b$. Hence, we have that $m_1 a m_2 \le m_1 b m_2$ and, similarly, $n_1 a n_2 \le n_1 b n_2$, and therefore

$$\begin{aligned} u &\le \max\{v, m_1 a m_2\} \\ &\le \max\{v, m_1 b m_2\} \\ &\le \max\{v, m_1 b m_2, n_1 a n_2\}. \end{aligned}$$

Similarly, if we assume $b \le a$, we get

$$\begin{aligned} u &\le \max\{v, n_1 b n_2\} \\ &\le \max\{v, n_1 a n_2\} \\ &\le \max\{v, m_1 b m_2, n_1 a n_2\}. \end{aligned}$$

Therefore, $u \le v \vee m_1 b m_2 \vee n_1 a n_2$.

(2) ⇒ (3). Suppose that M satisfies the quasiequation (4.14), and write for simplicity

$$s_1 = z_1 x z_2, \quad s_2 = w_1 y w_2, \quad t_1 = z_1 y z_2, \text{ and } t_2 = w_1 x w_2.$$

Then, by assumption M satisfies

$$u \le v \vee s_1 \text{ and } u \le v \vee s_2 \quad \Longrightarrow \quad u \le v \vee t_1 \vee t_2,$$

and we want to show that it also satisfies the equation

$$s_1 \wedge s_2 \le t_1 \vee t_2.$$

This is immediate, as from $s_1 \wedge s_2 \le t_1 \vee t_2 \vee s_1$ and $s_1 \wedge s_2 \le t_1 \vee t_2 \vee s_2$, we can conclude

$$s_1 \wedge s_2 \le t_1 \vee t_2 \vee t_1 \vee t_2 = t_1 \vee t_2,$$

which is the desired conclusion.

(3) ⇒ (4). Suppose that $m_1 a m_2 \in I$ and $n_1 b n_2 \in I$ for some $a, b, m_1, m_2, n_1, n_1 \in M$. Hence, $m_1 a m_2 \vee n_1 b n_2 \in I$ since I is a lattice ideal, and also, $m_1 b m_2 \wedge n_1 a n_2 \in I$ by (4.15) and downwards closure of I. Then, as I is prime, it must be the case that either $m_1 b m_2 \in I$ or $n_1 a n_2 \in I$.

(4) ⇒ (1). By Lemma 4.1.7 and Theorem 4.1.9, every distributive ℓ-monoid is a subdirect product of $\prod_{I \in X(M)} M/\beta_I$. Hence, under the assumption that (4) holds, M is representable. □

Theorem 4.3.2 provides an inverse-free axiomatization of representable ℓ-groups, as the following example shows.

Example 4.3.3. Every representable ℓ-group is representable as an ℓ-monoid and hence, it satisfies (4.15). Suppose now that H is an ℓ-group that satisfies (4.15). Then, it is immediate that

$$\mathrm{e} = (yx^{-1})xy^{-1} \wedge \mathrm{e} \le x \vee yx^{-1}y^{-1}.$$

Therefore, H is representable, as $\mathrm{e} \le x \vee yx^{-1}y^{-1}$ axiomatizes representable ℓ-groups relative to LG (cf. Section 1.4).

Example 4.3.4. It was shown in [150, Lemma 1.4] (cf. [130, Corollary 2]) that commutative distributive ℓ-monoids are representable. This follows immediately from

Theorem 4.3.2. To see this, consider $a, b, m_1, m_2, n_1, n_2 \in M$, where M is a commutative distributive ℓ-monoid. Then,

$$\begin{aligned}
m_1 a m_2 \wedge n_1 b n_2 &= m_1 m_2 a \wedge n_1 n_2 b \\
&\leq (m_1 m_2 \vee n_1 n_2) a \wedge (m_1 m_2 \vee n_1 n_2) b \\
&= (m_1 m_2 \vee n_1 n_2)(a \wedge b) \\
&= m_1 m_2 (a \wedge b) \vee n_1 n_2 (a \wedge b) \\
&\leq m_1 m_2 b \vee n_1 n_2 a \\
&= m_1 b m_2 \vee n_1 a n_2,
\end{aligned}$$

which allows us to conclude that commutative distributive ℓ-monoids are representable. Also, the same conclusion could be obtained by observing that in commutative distributive ℓ-monoids $\rho_I = \beta_I$ for any I, and hence Theorem 4.3.2.(4) holds.

It is known ([149]) that the variety of commutative distributive ℓ-monoids properly contains the variety generated by the inverse-free reducts of Abelian ℓ-groups. We recall this result here, and then use it to conclude the same for representable distributive ℓ-monoids.

Proposition 4.3.5 ([149])**.** *The variety generated by all inverse-free reducts of Abelian ℓ-groups is a proper subvariety of the variety of commutative distributive ℓ-monoids.*

Proof Sketch. The argument relies on showing that the equation

$$x_1 x_2 x_3 \wedge x_4 x_5 x_6 \wedge x_7 x_8 x_9 \leq x_1 x_4 x_7 \vee x_2 x_5 x_8 \vee x_3 x_6 x_9 \tag{4.16}$$

holds in all Abelian ℓ-groups, and does not hold in some commutative totally ordered monoid. First, observe that cancellativity is what makes the difference. In fact, suppose (4.16) fails in a totally ordered Abelian group H. But then, we can find $a_1, \ldots, a_9 \in H$ such that

$$a_1 a_4 a_7,\ a_2 a_5 a_8,\ a_3 a_6 a_9 \ <\ a_1 a_2 a_3,\ a_4 a_5 a_6,\ a_7 a_8 a_9.$$

But then, by cancellativity,

$$a_1 a_4 a_7 a_2 a_5 a_8 a_3 a_6 a_9 < a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9,$$

which clearly is a contradiction as H is commutative. For the remaining part of the proof, we refer to [149, Lemma 7]. (By taking $k = 1$ and $n = 3$, [149, Lemma 7] provides a commutative totally ordered monoid where (4.16) does not hold.) □

What was proved in [149] is that the variety generated by inverse-free reducts of Abelian ℓ-groups (equivalently, by $\mathbb{Z}$) is not finitely based; it can be axiomatized by

$$\{s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n \mid s_i, t_i \in T^m(X) \text{ and } s_1 \cdots s_n \approx t_1 \cdots t_n \text{ in commutative monoids; } n \in \mathbb{N}^+\}.$$

To show the next result, we adapt (4.16) to a non-commutative setting, and provide an equation that holds in all totally ordered groups, but fails in some totally ordered monoid.

Theorem 4.3.6. *The variety generated by inverse-free reducts of representable ℓ-groups is a proper subvariety of the variety of representable distributive ℓ-monoids.*

Proof. Consider the following terms:

$$t_1 = x_1x_2x_3 \wedge x_5x_4x_6 \wedge x_9x_7x_8; \quad t_2 = x_1x_3x_2 \wedge x_5x_6x_4 \wedge x_9x_8x_7;$$

$$s_1 = x_1x_4x_7 \vee x_5x_2x_8 \vee x_9x_6x_3; \quad s_2 = x_1x_7x_4 \vee x_5x_8x_2 \vee x_9x_3x_6.$$

Clearly, in a commutative setting $t_1 = t_2$ and $s_1 = s_2$; therefore, the final part of the proof of Proposition 4.3.5 provides an example of a totally ordered (commutative) monoid where the equation

$$t_1 \wedge t_2 \leq s_1 \vee s_2 \tag{4.17}$$

fails. It remains to show that (4.17) holds in any totally ordered group. It suffices to show that

$$\mathrm{e} \leq ({t_1}^{-1} \vee {t_2}^{-1})(s_1 \vee s_2)$$

holds in any totally ordered group. First, recall that the quasiequation

$$\mathrm{e} \leq xy \vee z \quad \Longrightarrow \quad \mathrm{e} \leq x \vee y \vee z \tag{4.18}$$

holds in all ℓ-groups; see [67, Lemma 3.3]. Since the identity $\mathrm{e} \leq x \vee x^{-1}$ also holds, we get that

$$\mathrm{e} \leq u \vee {x_3}^{-1}x_8x_3{x_8}^{-1} \vee x_8{x_3}^{-1}{x_8}^{-1}x_3$$

is valid in any ℓ-group. Further,

$$\mathrm{e} \leq u \vee {x_3}^{-1}x_8{x_6}^{-1}x_7 \vee {x_7}^{-1}x_6x_3{x_8}^{-1} \vee x_8{x_3}^{-1}x_7{x_6}^{-1} \vee x_6{x_7}^{-1}{x_8}^{-1}x_3$$

also holds, by applying twice (4.18). Now, in any totally ordered group, we have that

$$\mathrm{e} \leq xy \vee z \quad \Longrightarrow \quad \mathrm{e} \leq yx \vee z; \tag{4.19}$$

hence,

$$\mathrm{e} \leq u \vee x_7{x_3}^{-1}x_8{x_6}^{-1} \vee {x_7}^{-1}x_6x_3{x_8}^{-1} \vee {x_3}^{-1}x_7{x_6}^{-1}x_8 \vee x_6{x_7}^{-1}{x_8}^{-1}x_3$$

holds in every totally ordered group. Using (4.18) again, we get

$$\begin{aligned}\mathrm{e} \leq u \vee x_7{x_3}^{-1}{x_2}^{-1}x_4 \vee {x_4}^{-1}x_2x_8{x_6}^{-1} \vee {x_7}^{-1}x_6x_3{x_8}^{-1} \vee \\ \vee {x_3}^{-1}x_7x_4{x_2}^{-1} \vee x_2{x_4}^{-1}{x_6}^{-1}x_8 \vee x_6{x_7}^{-1}{x_8}^{-1}x_3.\end{aligned}$$

We now conclude by applying (4.19) six times, thereby obtaining

$$\begin{aligned}\mathrm{e} \leq u \vee {x_3}^{-1}{x_2}^{-1}x_4x_7 \vee {x_6}^{-1}{x_4}^{-1}x_2x_8 \vee {x_8}^{-1}{x_7}^{-1}x_6x_3 \vee \\ \vee {x_2}^{-1}{x_3}^{-1}x_7x_4 \vee {x_4}^{-1}{x_6}^{-1}x_8x_2 \vee {x_7}^{-1}{x_8}^{-1}x_3x_6.\end{aligned}$$

Finally, observe that each of these disjuncts appears as a disjunct in $(t_1{}^{-1} \vee t_2{}^{-1})(s_1 \vee s_2)$ as follows:

$$\begin{aligned}
x_3{}^{-1}x_2{}^{-1}x_4x_7 &= x_3{}^{-1}x_2{}^{-1}x_1{}^{-1} \cdot x_1x_4x_7\\
x_6{}^{-1}x_4{}^{-1}x_2x_8 &= x_6{}^{-1}x_4{}^{-1}x_5{}^{-1} \cdot x_5x_2x_8\\
x_8{}^{-1}x_7{}^{-1}x_6x_3 &= x_8{}^{-1}x_7{}^{-1}x_9{}^{-1} \cdot x_9x_6x_3\\
x_2{}^{-1}x_3{}^{-1}x_7x_4 &= x_2{}^{-1}x_3{}^{-1}x_1{}^{-1} \cdot x_1x_7x_4\\
x_4{}^{-1}x_6{}^{-1}x_8x_2 &= x_4{}^{-1}x_6{}^{-1}x_5{}^{-1} \cdot x_5x_8x_2\\
x_7{}^{-1}x_8{}^{-1}x_3x_6 &= x_7{}^{-1}x_8{}^{-1}x_9{}^{-1} \cdot x_9x_3x_6;
\end{aligned}$$

therefore, we get $\mathrm{e} \leq (t_1{}^{-1} \vee t_2{}^{-1})(s_1 \vee s_2)$, by taking u to be, e.g., $x_3{}^{-1}x_2{}^{-1}x_1{}^{-1}x_5x_2x_8$. □

4.4 THE SUBREDUCTS OF LATTICE-ORDERED GROUPS

The main result of this section is Theorem 4.4.3, where we show that distributive ℓ-monoids are in fact the variety generated by the inverse-free reducts of ℓ-groups. Indeed, the failure of an ℓ-monoid equation $s \leq t$ in $\mathrm{End}(\Omega)$ for a chain Ω induces its failure in $\mathrm{Aut}(\mathbb{Q})$. This allows for some interesting consequences, that we discuss.

We begin with the following technical result (Lemma 4.4.1). The reader can find an example immediately after Lemma 4.4.1; the example is meant to illustrate how to implement the procedure described by Lemma 4.4.1 in a specific case, with the intent of clarifying the content of its proof. Recall that $T^{\ell}(X)$ and $T^{\ell m}(X)$ denote the term algebras over a set X for the languages of ℓ-groups and ℓ-monoids, respectively.

Lemma 4.4.1. *Let Ω be a chain, and $s_1 \leq t_1, \ldots, s_n \leq t_n$ be ℓ-monoid equations. Then, for any valuation $\varphi\colon T^{\ell m}(X) \to \mathrm{End}(\Omega)$ and any $w \in \Omega$ such that*

$$\varphi(s_1)(w) > \varphi(t_1)(w), \ldots, \varphi(s_n)(w) > \varphi(t_n)(w),$$

there exist a valuation $\psi\colon T^{\ell}(X) \to \mathrm{Aut}(\mathbb{Q})$ and an element $q \in \mathbb{Q}$, such that

$$\psi(s_1)(q) > \psi(t_1)(q), \ldots, \psi(s_n)(q) > \psi(t_n)(q).$$

Proof. We can assume that $s_1 \leq t_1, \ldots, s_n \leq t_n$ are ℓ-monoid equations such that

$$s_i = s_{i1} \wedge \cdots \wedge s_{im} \quad \text{and} \quad t_i = t_{i1} \vee \cdots \vee t_{im},$$

and s_{ij}, t_{ij} are monoid terms in $T^{\ell m}(X)$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Let Ω be a chain, and $\varphi\colon T^{\ell m}(X) \to \mathrm{End}(\Omega)$ a valuation satisfying the assumptions in the statement for some fixed $w \in \Omega$. Then, for any $1 \leq i \leq n$,

$$\varphi(s_{ij})(w) > \varphi(t_{ik})(w) \quad \text{for all } 1 \leq j, k \leq m.$$

For any monoid term $u = x_1 \cdots x_k$ where $x_1, \ldots, x_k \in X$, we write $\mathsf{is}(u)$ for the set of all initial subterms (including e) of u, and also define

$$\omega_u = (w_0, \ldots, w_k), \text{ where } w_0 = w \text{ and } w_{i+1} = \varphi(x_{i+1})(w_i) \text{ for any } 0 \le i \le k-1.$$

Then, set $\Phi_{s_{ij}} = \{\omega_u \mid u \in \mathsf{is}(s_{ij})\}$ and $\Phi_{t_{ij}} = \{\omega_v \mid v \in \mathsf{is}(t_{ij})\}$, for all possible indices $1 \le i \le n, 1 \le j \le m$. We define a total order on the (finite) set

$$\Phi = \bigcup_{1 \le i \le n, 1 \le j \le m} (\Phi_{s_{ij}} \cup \Phi_{t_{ij}}),$$

by

$$(p_{k_1}, \ldots, p_1) \preceq (q_{k_2}, \ldots, q_1) \iff \begin{array}{l} \text{there exists } j \in \mathbb{N} \text{ s.t. } p_i = q_i \text{ for all } i < j \text{ and} \\ \text{either } p_j < q_j \\ \text{or } j = k_1 + 1 \leqslant k_2. \end{array} \tag{4.20}$$

For any $x \in X$, we define the relation

$$f_x = \{((w_0, \ldots, w_k), (w_0, \ldots, w_k, w_{k+1})) \in \Phi \times \Phi \mid w_{k+1} = \varphi(x)(w_k)\}.$$

Then f_x is the graph of a partial function on Φ. Moreover, if

$$(w, a_1, \ldots, a_{k_1}) \prec (w, b_1, \ldots, b_{k_2})$$

in the domain of f_x, then

$$\begin{aligned} f_x(w, a_1, \ldots, a_{k_1}) &= (w, a_1, \ldots, a_{k_1}, \varphi(x)(a_{k_1})) \\ &\prec (w, b_1, \ldots, b_{k_2}, \varphi(x)(b_{k_2})) \\ &= f_x(w, b_1, \ldots, b_{k_2}), \end{aligned}$$

since either $\varphi(x)(a_{k_1}) < \varphi(x)(b_{k_2})$ (in which case, $a_{k_1} < b_{k_2}$), or $\varphi(x)(a_{k_1}) = \varphi(x)(b_{k_2})$; in this second case, since $(w, a_1, \ldots, a_{k_1}) \prec (w, b_1, \ldots, b_{k_2})$, also

$$(w, a_1, \ldots, a_{k_1}, \varphi(x)(a_{k_1})) \prec (w, b_1, \ldots, b_{k_2}, \varphi(x)(b_{k_2})),$$

by the definition of the reverse lexicographic order. We now consider an order-embedding $\Phi \hookrightarrow \mathbb{Q}$ (and identify it with the identity map, for simplicity), and define (partial) maps $\psi(x) \colon p \mapsto f_x(p)$, for any $p \in \Phi$, and $x \in X$. Such partial order-embeddings can be extended to order-preserving bijections on $\mathbb{Q}$, for any $x \in X$. Therefore, we now have a valuation

$$\psi \colon T^\ell(X) \longrightarrow \mathrm{Aut}(\mathbb{Q}),$$

and, by writing w for $\omega_{\mathrm{e}} = (w_0) = (w)$, we show that

$$\psi(s_1)(w) \succ \psi(t_1)(w), \ldots, \psi(s_n)(w) \succ \psi(t_n)(w).$$

For any $u = x_0 \cdots x_k \in \mathsf{is}(s_{ij}) \cup \mathsf{is}(t_{ij})$, we prove that

$$\psi(u)(w) = (w, \varphi(x_1)(w), \ldots, \varphi(x_1 \cdots x_k)(w)),$$

by induction on k. If $k = 0$, clearly $\psi(\mathrm{e})(w) = (w) = (\varphi(\mathrm{e})(w))$. For the induction step,

$$\begin{aligned}\psi(x_1 \cdots x_k x_{k+1}) &= \psi(x_{k+1})(w, \ldots, \varphi(x_1 \cdots x_k)(w)) \\ &= (w, \ldots, \varphi(x_1 \cdots x_k)(w), \varphi(x_{k+1})(\varphi(x_1 \cdots x_k)(w))) \\ &= (w, \ldots, \varphi(x_1 \cdots x_k)(w), (\varphi(x_1 \cdots x_k x_{k+1})(w))).\end{aligned}$$

Therefore, by the definition of the lexicographic order, for any $1 \le i \le n$,

$$\psi(s_{ij})(w) \succ \psi(t_{ik})(w) \quad \text{for all } 1 \le j, k \le m$$

follows from the fact that, for any $1 \le i \le n$, by assumption

$$\varphi(s_{ij})(w) > \varphi(t_{ik})(w) \quad \text{for all } 1 \le j, k \le m.$$

Thus, $\psi(s_{i1} \wedge \cdots \wedge s_{im})(w) \succ \psi(t_{i1} \vee \cdots \vee t_{im})(w)$ for any $1 \le i \le n$, that is,

$$\psi(s_1)(q) \succ \psi(t_1)(q), \ldots, \psi(s_n)(q) \succ \psi(t_n)(q)$$

in $\mathrm{Aut}(\mathbb{Q})$, as desired. □

Example 4.4.2. Let End(3) be the distributive ℓ-monoid of order-preserving endomorphisms of the 3-element chain $3 = \{0, 1, 2\}$, where $0 < 1 < 2$. We write $\langle w_0, w_1, w_2 \rangle$ for the order-preserving endomorphism of 3 such that $0 \mapsto w_0$, $1 \mapsto w_1$, and $2 \mapsto w_2$. The equation (4.15) fails in End(3), as the latter is not a representable distributive ℓ-monoid. To see this, consider

$$\varphi\colon T^{\ell m}(x, y, z_1, z_2, w_1, w_2) \longrightarrow \mathrm{End}(3),$$

defined by extending the assignment $x \mapsto \varphi(x) = \langle 0, 2, 2 \rangle$, $y \mapsto \varphi(y) = \langle 1, 1, 1 \rangle$, $z_1 \mapsto \varphi(z_1) = \langle 0, 1, 1 \rangle$, $z_2 \mapsto \varphi(z_2) = \langle 1, 1, 2 \rangle$, $w_1 \mapsto \varphi(w_1) = \langle 0, 0, 0 \rangle$, $w_2 \mapsto \varphi(w_2) = \langle 1, 2, 2 \rangle$. Observe that

$$\varphi(z_1 x z_2 \wedge w_1 y w_2)(1) > \varphi(z_1 y z_2 \vee w_1 x w_2)(1),$$

and this follows by calculating

$$\begin{aligned}&\varphi(z_2)(\varphi(x)(\varphi(z_1)(1))) = 2, & &\varphi(w_2)(\varphi(y)(\varphi(w_1)(1))) = 2, \\ &\varphi(z_2)(\varphi(y)(\varphi(z_1)(1))) = 1, & &\varphi(w_2)(\varphi(x)(\varphi(w_1)(1))) = 1.\end{aligned}$$

It is helpful to see each monoid term as a path. E.g., consider $z_1 x z_2$ and $z_1 y z_2$; then, $\varphi(z_1 x z_2)(1)$ and $\varphi(z_1 y z_2)(1)$ can be seen as the following two paths:

Figure 4.1: Paths identifying $\varphi(z_1 x z_2)(1)$ and $\varphi(z_1 y z_2)(1)$

where the dotted lines indicate those values of the considered order-preserving endomorphisms that are not relevant for the monoid term valuation. Our aim is to add points to the chain in such a way that each order-preserving valuation be made partially injective on the relevant points. For this, we consider all initial subterms. If an element is the endpoint of a 'path' of one such initial subterm, then we identify it with the 'path' itself, to remember the behaviour of the valuations in the original chain. The initial subterms to be considered are

$$\{\mathrm{e}, z_1, z_1x, z_1xz_2, z_1y, z_1yz_2, w_1, w_1x, w_1xw_2, w_1y, w_1yw_2\},$$

which results in the following 11-element chain Φ:

$$(1,0,0) < (1,0) < (1) < (1,0,0,1) < (1,0,1) < (1,1) < \\ < (1,1,1) < (1,1,1,1) < (1,0,1,2) < (1,1,2) < (1,1,2,2).$$

At this point, set

$$\begin{aligned} f_x &= \{\langle(1,0),(1,0,0)\rangle, \langle(1,0,1),(1,0,1,2)\rangle, \langle(1,1),(1,1,2)\rangle, \langle(1,1,2),(1,1,2,2)\rangle\} \\ f_y &= \{\langle(1,0,0),(1,0,0,1)\rangle, \langle(1,0),(1,0,1)\rangle, \langle(1),(1,1)\rangle, \\ &\qquad \langle(1,1),(1,1,1)\rangle, \langle(1,1,1),(1,1,1,1)\rangle\} \\ f_{z_1} &= \{\langle(1),(1,1)\rangle, \langle(1,1),(1,1,1)\rangle, \langle(1,1,1),(1,1,1,1)\rangle\} \\ f_{z_2} &= \{\langle(1,0,0),(1,0,0,1)\rangle, \langle(1,0),(1,0,1)\rangle, \langle(1),(1,1)\rangle, \\ &\qquad \langle(1,1),(1,1,1)\rangle, \langle(1,1,1),(1,1,1,1)\rangle, \langle(1,1,2),(1,1,2,2)\rangle\} \\ f_{w_1} &= \{\langle(1,0),(1,0,0)\rangle, \langle(1),(1,0)\rangle\} \\ f_{w_2} &= \{\langle(1,0,0),(1,0,0,1)\rangle, \langle(1,0),(1,0,1)\rangle, \langle(1,0,1),(1,0,1,2)\rangle, \\ &\qquad \langle(1,1),(1,1,2)\rangle, \langle(1,1,2),(1,1,2,2)\rangle\} \end{aligned}$$

and identify Φ with the chain $[0,10] \cap \mathbb{Z} \subseteq \mathbb{Q}$ via the order-embedding $\alpha\colon \Phi \to \mathbb{Q}$; observe that $\alpha(1) = 2$. The (partial) maps

$$\psi(z)\colon \alpha(p) \longmapsto \alpha(f_z(p)),$$

for any $p \in \Phi$, and $z \in \{x, y, z_1, z_2, w_1, w_2\}$ are partial order-embeddings. Moreover,

$$\begin{aligned} \varphi(z_2)(\varphi(x)(\varphi(z_1)(\alpha(1)))) &= \alpha(1,1,2,2) = 10, \\ \varphi(w_2)(\varphi(y)(\varphi(w_1)(\alpha(1)))) &= \alpha(1,0,1,2) = 8, \\ \varphi(z_2)(\varphi(y)(\varphi(z_1)(\alpha(1)))) &= \alpha(1,1,1,1) = 7, \\ \varphi(w_2)(\varphi(x)(\varphi(w_1)(\alpha(1)))) &= \alpha(1,0,0,1) = 3. \end{aligned}$$

Figure 4.2 shows how the paths identifying z_1xz_2 and z_1yz_2 have been modified. It is readily seen that the resulting partial maps are in fact order-embeddings.

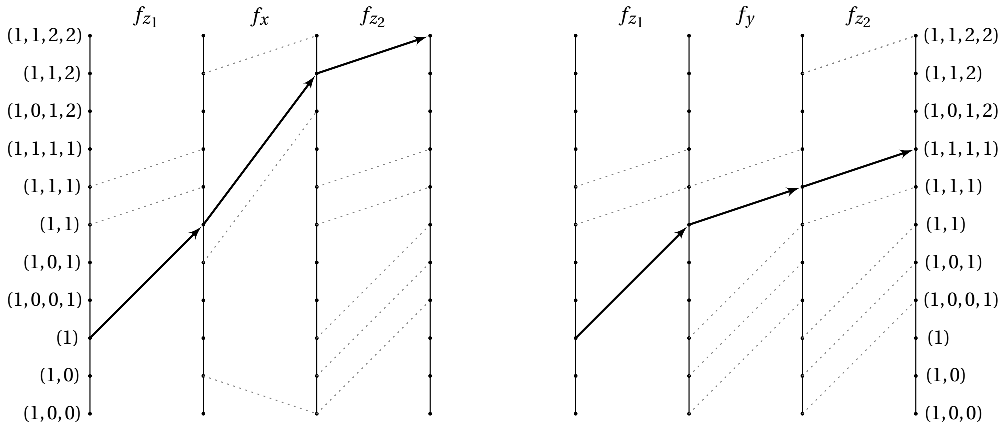

Figure 4.2: New paths identifying $\psi(z_1 x z_2)(\alpha(1))$ and $\psi(z_1 y z_2)(\alpha(1))$

Theorem 4.4.3. *An inverse-free equation holds in the variety of ℓ-groups if and only if it holds in the variety of distributive ℓ-monoids.*

Proof. Let Ω be a chain such that $s \le t$ fails in $\mathrm{End}(\Omega)$. That $s \le t$ fails in $\mathrm{End}(\Omega)$ means that we have a valuation $\varphi\colon T^{\ell m}(X) \to \mathrm{End}(\Omega)$ and an element $w \in \Omega$ such that $\varphi(s)(w) > \varphi(t)(w)$. By Lemma 4.4.1, we have a valuation

$$\psi\colon T^{\ell}(X) \longrightarrow \mathrm{Aut}(\mathbb{Q}),$$

such that $\psi(s)(w) > \psi(t)(w)$, that is,

$$s_1 \wedge \cdots \wedge s_n \le t_1 \vee \cdots \vee t_n,$$

fails in $\mathrm{Aut}(\mathbb{Q})$ (and hence, in the variety LG of ℓ-groups). □

Therefore, the equational theory of ℓ-groups is a conservative extension of the equational theory of distributive ℓ-monoids.

Corollary 4.4.4. *The variety generated by the inverse-free reducts of ℓ-groups is the variety of distributive ℓ-monoids.*

The following results provide suitable extensions of Corollary 1.3.2 to the context of distributive ℓ-monoids. For any monoid M, a total order $\le \subseteq M \times M$ is a right order on M if, for all $a, b, c \in M$, $a \le b$ implies $ac \le bc$. Since Theorem 4.4.3 establishes a correspondence between equations holding in distributive ℓ-monoids and (ℓ-monoid) equations holding in ℓ-groups, we combine these results and obtain the following first equivalence.

Theorem 4.4.5. *For any set $s_1, t_1 \ldots, s_n, t_n \in T^m(X)$, the following are equivalent:*

(1) $\mathsf{DLM} \models s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n$.

(2) *There exists no right order on the free monoid $F^m(X)$ such that $s_i < t_j$ for all $1 \leq i, j \leq n$.*

Proof. (1) ⇒ (2). We proceed by contraposition. Let ≤ be a right order on $F^m(X)$ such that $s_i < t_j$ for all $1 \leq i, j \leq n$. Consider the dual order $\leq^\partial$. Clearly, $t_j < s_i$ holds for all $1 \leq i, j \leq n$. Consider the distributive ℓ-monoid $\mathrm{End}(F^m(X), \leq)$, and the valuation $\varphi\colon T^{\ell m}(X) \to \mathrm{End}(F^m(X), \leq)$, obtained by extending the assignment

$$x \longmapsto \varphi(x)\colon s \mapsto sx, \quad \text{for } s \in F^m(X).$$

Clearly, $\varphi(t_j)(\mathrm{e}) < \varphi(s_i)(\mathrm{e})$ for any $1 \leq i, j \leq n$, and hence

$$\begin{aligned} \varphi(s_1 \wedge \cdots \wedge s_n)(\mathrm{e}) &= \min\{\varphi(s_i)(\mathrm{e}) \mid 1 \leq i \leq n\} \\ &> \max\{\varphi(t_i)(\mathrm{e}) \mid 1 \leq i \leq n\} \\ &= \varphi(t_1 \vee \cdots \vee t_n)(\mathrm{e}), \end{aligned}$$

which means that the equation $s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n$ fails in $\mathrm{End}(F^m(X), \leq)$ (hence, in DLM).

(2) ⇒ (1). We proceed by contraposition. Suppose that $\mathsf{DLM} \not\models s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n$. Then, by Theorem 4.4.3,

$$\mathsf{LG} \not\models s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n,$$

This is equivalent to

$$\mathsf{LG} \not\models \mathrm{e} \leq (t_1 \vee \cdots \vee t_n)(s_1{}^{-1} \vee \cdots \vee s_n{}^{-1}),$$

which simply means

$$\mathsf{LG} \not\models \mathrm{e} \leq \bigvee_{1 \leq i, j \leq n} t_i s_j{}^{-1}.$$

Therefore, by Corollary 1.3.2, there exists a right order on the free group $F(X)$ that makes $t_i s_j{}^{-1}$ positive for all $1 \leq i, j \leq n$ or, equivalently, a right order ≤ such that $s_i < t_j$ for all $1 \leq i, j \leq n$. Therefore, since $t_1, s_1, \ldots, t_n, s_n \in F^m(X)$, the restriction of this right order ≤ to $F^m(X)$ is a right order on the free monoid $F^m(X)$ such that $s_i < t_j$ for all $1 \leq i, j \leq n$. □

Theorem 4.4.5 provides a way to generalize Corollary 1.3.2; however, as the set of inequalities in Theorem 4.4.5.(1) has a specific form, it does not allow us to answer the question whether, for any finite set of monoid terms $s_1, t_1 \ldots, s_n, t_n$ such that $s_1 < t_1, \ldots, s_n < t_n$ in some right order on the free monoid, the inequalities $s_1 < t_1, \ldots, s_n < t_n$ hold in some right order on the free group. For this reason, we consider the following alternative generalization of Corollary 1.3.2.

Theorem 4.4.6. *For any set $s_1, t_1 \ldots, s_n, t_n \in T^m(X)$, the following are equivalent:*

(1) *There exists a right order on the free monoid $F^m(X)$ such that $s_1 < t_1, \ldots, s_n < t_n$.*

(2) $\mathsf{DLM} \not\models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n$, *where $y_1, \ldots, y_n$ are variables not occurring in $s_1, t_1, \ldots, s_n, t_n$.*

Proof. (1) ⇒ (2). We first remark that if $\mathsf{DLM} \models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n$ holds, where $y_1, \ldots, y_n$ are variables not occurring in $s_1, t_1, \ldots, s_n, t_n$, then also $\mathsf{LG} \models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n$ and, in particular, the following holds in all ℓ-groups:

$$s_1 {s_1}^{-1} \wedge \cdots \wedge s_n {s_n}^{-1} = \mathrm{e} \leq t_1 {s_1}^{-1} \vee \cdots \vee t_n {s_n}^{-1}. \tag{4.21}$$

It thus suffices to show that (4.21) fails in LG. Let $\leq$ be a right order on $F^m(X)$ such that $s_1 < t_1, \ldots, s_n < t_n$. Consider the dual order $\leq^\partial$. Clearly, $t_1 <^\partial s_1, \ldots, t_n <^\partial s_n$ holds. Consider the distributive ℓ-monoid $\mathrm{End}\,(F^m(X), \leq)$, and the valuation $\varphi\colon T^{\ell m}(X) \to \mathrm{End}\,(F^m(X), \leq)$, obtained by extending the assignment

$$x \longmapsto \varphi(x)\colon s \mapsto sx, \quad \text{for } s \in F^m(X).$$

Clearly, $\varphi(t_i)(\mathrm{e}) < \varphi(s_i)(\mathrm{e})$ for every $1 \leq i \leq n$, and hence, the equations

$$s_1 \leq t_1, \ldots, s_n \leq t_n$$

fail under the evaluation φ, at the point $\mathrm{e} \in F^m(X)$. By Lemma 4.4.1, we obtain a valuation $\psi\colon T^\ell(X) \to \mathrm{Aut}\,(\mathbb{Q})$ and a point $q \in \mathbb{Q}$ such that

$$\psi(t_1)(q) < \psi(s_1)(q), \ldots, \psi(t_n)(q) < \psi(s_n)(q).$$

Therefore,

$$\psi({s_1}^{-1})\psi(t_1)(q) < q, \ldots, \psi({s_1}^{-1})\psi(t_1)(q) < q,$$

which means that the equation $\mathrm{e} \leq t_1 {s_1}^{-1} \vee \cdots \vee t_n {s_n}^{-1}$ fails in $\mathrm{Aut}\,(\mathbb{Q})$ (hence, in LG).

(2) ⇒ (1). Suppose that $\mathsf{DLM} \not\models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n$, for some $y_1, \ldots, y_n$ not occurring in $s_1, t_1, \ldots, s_n, t_n$. Then, by Theorem 4.4.3,

$$\mathsf{LG} \not\models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n.$$

This is equivalent to

$$\mathsf{LG} \not\models \mathrm{e} \leq (t_1 y_1 \vee \cdots \vee t_n y_n)({y_1}^{-1} {s_1}^{-1} \vee \cdots \vee {y_n}^{-1} {s_n}^{-1}),$$

which entails

$$\mathsf{LG} \not\models \mathrm{e} \leq t_1 {s_1}^{-1} \vee \cdots \vee t_n {s_n}^{-1},$$

since

$$t_1 {s_1}^{-1} \vee \cdots \vee t_n {s_n}^{-1} \leq (t_1 y_1 \vee \cdots \vee t_n y_n)({y_1}^{-1} {s_1}^{-1} \vee \cdots \vee {y_n}^{-1} {s_n}^{-1}).$$

Therefore, by Corollary 1.3.2, there exists a right order on the free group $F(X)$ that makes $t_1 {s_1}^{-1}, \ldots, t_n {s_n}^{-1}$ positive or, equivalently, a right order $\preceq$ on $F(X)$ such that $s_1 \prec t_1, \ldots, s_n \prec t_n$ hold. Therefore, since $t_1, s_1, \ldots, t_n, s_n \in F^m(X)$, the restriction of this right order to $F^m(X)$ is a right order $\leq$ on the free monoid $F^m(X)$ such that $s_1 < t_1, \ldots, s_n < t_n$. □

Theorem 4.4.6 allows us to conclude the following striking result.

Corollary 4.4.7. *Every right order on the free monoid $F^m(X)$ extends to a right order on the free group $F(X)$.*

Proof. Let $\leq$ be a right order on $F^m(X)$. Since $F^m(X)$ is a submonoid of the free group $F(X)$, we can consider the set $S \subseteq F(X)$ defined as

$$S = \{ts^{-1} \mid s, t \in F^m(X) \text{ and } s < t\}.$$

Then, $\leq$ extends to a right order on $F(X)$ if and only if S extends to a right order on $F(X)$ in the sense defined in Chapter 1. By Proposition 1.1.10, it suffices to check that every finite subset of S extends to a right order on $F(X)$. Thus, suppose that $s_1, t_1 \ldots, s_n, t_n \in T^m(X)$ are such that $s_1 < t_1, \ldots, s_n < t_n$. By Theorem 4.4.6,

$$\mathsf{DLM} \not\models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n,$$

where $y_1, \ldots, y_n$ are variables not occurring in $s_1, t_1, \ldots, s_n, t_n$, which entails

$$\mathsf{LG} \not\models s_1 y_1 \wedge \cdots \wedge s_n y_n \leq t_1 y_1 \vee \cdots \vee t_n y_n,$$

by Theorem 4.4.3. This implies that

$$\mathsf{LG} \not\models \mathrm{e} \leq t_1 s_1{}^{-1} \vee \cdots \vee t_n s_n{}^{-1},$$

since $t_1 s_1{}^{-1} \vee \cdots \vee t_n s_n{}^{-1} \leq (t_1 y_1 \vee \cdots \vee t_n y_n)(y_1{}^{-1} s_1{}^{-1} \vee \cdots \vee y_n{}^{-1} s_n{}^{-1})$. Therefore, by Corollary 1.3.2, we get a right order $\preceq$ on $F(X)$ such that $s_1 \prec t_1, \ldots, s_n \prec t_n$. □

It is not clear at the present stage what role right orders (or left orders) on monoids play in the structure theory of distributive ℓ-monoids. It was claimed in [2] that:

> "From Holland's representation theorem it follows easily that the class of right-orderable groups coincides with the class of subgroups of ℓ-groups. The same proof and [Theorem 4.1.9] allows us to conclude that the class of right-orderable monoids coincides with the class of submonoids of distributive ℓ-monoids."

It can be observed by looking at End(3) that such a claim is in fact false.

Example 4.4.8. (This counterexample is based on an example provided by Simon Santschi while working on his BSc thesis at the University of Bern.) Recall that, for any monoid M, a total order $\leq \subseteq M \times M$ is a right order on M if, for all $a, b, c \in M$, $a \leq b$ implies $ac \leq bc$. Therefore, by contraposition, $ac < bc$ must entail $a < b$. Suppose there exists a right order $\leq$ on (the monoid reduct of) End(3), and assume

$$\langle 1,1,2 \rangle < \langle 1,2,2 \rangle.$$

Note that $\langle 1,1,2 \rangle = \langle 1,1,2 \rangle \circ \langle 0,1,2 \rangle$, and $\langle 1,2,2 \rangle = \langle 1,1,2 \rangle \circ \langle 1,2,2 \rangle$, which entails

$$\langle 0,1,2 \rangle < \langle 1,2,2 \rangle.$$

But then, by applying right multiplication,

$$\langle 0,2,2\rangle \circ \langle 0,1,2\rangle = \langle 0,2,2\rangle < \langle 2,2,2\rangle = \langle 0,2,2\rangle \circ \langle 1,2,2\rangle. \tag{4.22}$$

Similarly, since $\langle 1,1,2\rangle = \langle 1,1,2\rangle \circ \langle 1,1,2\rangle$, and $\langle 1,2,2\rangle = \langle 1,1,2\rangle \circ \langle 0,2,2\rangle$, we also get

$$\langle 1,1,2\rangle < \langle 0,2,2\rangle.$$

Therefore, again by applying right multiplication,

$$\langle 0,2,2\rangle \circ \langle 1,1,2\rangle = \langle 2,2,2\rangle < \langle 0,2,2\rangle = \langle 0,2,2\rangle \circ \langle 0,2,2\rangle. \tag{4.23}$$

Clearly, (4.22) and (4.23) lead to a contradiction. Therefore, if $\leq$ is a right order on $\mathrm{End}(3)$, it must be the case that

$$\langle 1,2,2\rangle < \langle 1,1,2\rangle.$$

Thus, for the same reason as before, $\langle 1,2,2\rangle < \langle 0,1,2\rangle$, and again,

$$\langle 0,2,2\rangle \circ \langle 1,2,2\rangle = \langle 2,2,2\rangle < \langle 0,2,2\rangle = \langle 0,2,2\rangle \circ \langle 0,1,2\rangle. \tag{4.24}$$

But then, $\langle 1,2,2\rangle < \langle 1,1,2\rangle$ entails $\langle 0,2,2\rangle < \langle 1,1,2\rangle$, and by multiplying on the right again, we conclude

$$\langle 0,2,2\rangle \circ \langle 0,2,2\rangle = \langle 0,2,2\rangle < \langle 2,2,2\rangle = \langle 0,2,2\rangle \circ \langle 1,1,2\rangle, \tag{4.25}$$

thereby obtaining a contradiction.

It is obvious what a left order on a monoid M should be: a total order $\leq \subseteq M \times M$ such that $a \leq b$ implies $ca \leq cb$, for all $a, b, c \in M$. The relation between right- and left-orderability in distributive ℓ-monoids is not as neat as it is in the context of ℓ-groups. In particular, it does not follow from Example 4.4.8 that $\mathrm{End}(3)$ does not admit a left order. However, by proceeding as in Example 4.4.8, it would be possible to show that in fact this is the case, and $\mathrm{End}(3)$ does not admit any left order.

We conclude the section by showing that the validity of any ℓ-group equation in all ℓ-groups is equivalent to the validity of finitely many ℓ-monoid equations. This result, together with Corollary 4.2.4, leads to a new proof of the decidability of the equational theory of free ℓ-groups.

Lemma 4.4.9. *The equation* $\mathrm{e} \leq v \vee sx \vee x^{-1}t$ *holds in all* ℓ*-groups for any variable* x *not occurring in the* ℓ*-group terms* s, t, v *if and only if the equation* $\mathrm{e} \leq v \vee st$ *holds in all* ℓ*-groups.*

Proof. The right-to-left direction is immediate, as $\mathrm{e} \leq xy \vee z$ implies $\mathrm{e} \leq x \vee y \vee z$ in any ℓ-group ([67, Lemma 3.3]). For the remaining direction, we proceed by contraposition. Suppose $\mathrm{e} \leq v \vee st$ fails in some ℓ-group and, in particular, we can assume that $\mathrm{e} \leq v \vee st$ fails in $\mathrm{Aut}(\mathbb{Q})$. This means that there exists a valuation $\varphi\colon T^{\ell}(X) \to$

$\mathrm{Aut}(\mathbb{Q})$ and an element $q \in \mathbb{Q}$ such that $q > \max\{\varphi(v)(q), \varphi(st)(q)\}$, that is, such that $\varphi(v)(q), \varphi(st)(q) < q$. We consider $X \cup \{x\}$, and extend φ to a valuation

$$\widehat{\varphi}\colon\ T^{\ell}(X \cup \{x\}) \longrightarrow \mathrm{Aut}(\mathbb{Q})$$

such that

$$q > \max\{\widehat{\varphi}(v)(q), \widehat{\varphi}(sx)(q), \widehat{\varphi}(x^{-1}t)(q)\}.$$

Note that from $\widehat{\varphi}(st)(q) < q$, we get $\widehat{\varphi}(s)(q) < \widehat{\varphi}(t^{-1})(q)$. We now want

$$\widehat{\varphi}(x)\widehat{\varphi}(s)(q) < q \quad \text{and} \quad q < \widehat{\varphi}(x)\widehat{\varphi}(t^{-1})q.$$

Therefore, we define a partial order-embedding on $\mathbb{Q}$ that maps:

$$\varphi(s)(q) \longmapsto p_1 < q \quad \text{and} \quad \varphi(t^{-1})(q) \longmapsto p_2 > q,$$

and extend it to an order-preserving bijection $\widehat{\varphi}(x) \in \mathrm{Aut}(\mathbb{Q})$. Clearly, the assignment $\widehat{\varphi}$ extends φ, and falsifies $\mathrm{e} \le v \vee sx \vee x^{-1}t$, as desired. □

Lemma 4.4.10. *For any set of monoid terms s, t, u, v, w, the following are equivalent:*

(1) $\mathsf{LG} \models u \le v \vee sw^{-1}t$.

(2) *For any variable x not occurring in s, t, u, v, w,*

$$\mathsf{LG} \models wxu \le wxv \vee wxsxu \vee t.$$

Proof. (1) ⇒ (2). Suppose that $u \le v \vee sw^{-1}t$, that is, $\mathrm{e} \le vu^{-1} \vee sw^{-1}tu^{-1}$ holds in all ℓ-groups. By Lemma 4.4.9,

$$\mathrm{e} \le vu^{-1} \vee sx \vee x^{-1}w^{-1}tu^{-1},$$

for some variable x not occurring in s, t, u, v, w. Thus, we get

$$wxu \le wxv \vee wxsxu \vee t.$$

(2) ⇒ (1). Suppose now that $wxu \le wxv \vee wxsxu \vee t$ holds in all ℓ-groups, for some variable x not occurring in s, t, u, v, w. Then,

$$u \le v \vee sxu \vee x^{-1}w^{-1}t$$

also holds, and so does $\mathrm{e} \le vu^{-1} \vee sx \vee x^{-1}w^{-1}tu^{-1}$. By Lemma 4.4.9,

$$\mathrm{e} \le vu^{-1} \vee sw^{-1}tu^{-1},$$

and therefore, $u \le v \vee sw^{-1}t$. □

Theorem 4.4.11. *For any monoid term t_0 and all group terms $t_1, \dots, t_n$, the validity of the ℓ-group equation $t_0 \le t_1 \vee \dots \vee t_n$ in LG is equivalent to the validity of an equation of the form*

$$s_0 \le s_1 \vee \dots \vee s_m,$$

where $s_0, s_1, \dots, s_m$ are monoid terms.

Proof. We proceed by induction on the number of inverses x^{-1} contained in $t_1, \dots, t_n$, for any variable x. Suppose there is only one variable inverse x^{-1}. If

$$\mathrm{e} \leq t_1 \vee \cdots \vee x^{-1}u_i \vee \cdots \vee t_n,$$

where $t_i = x^{-1}u_i$, then we simply multiply on the left by x and thereby obtain $x \leq xt_1 \vee \cdots \vee u_i \vee \cdots \vee xt_n$. Similarly, we multiply by x on the right if the equation is of the form

$$\mathrm{e} \leq t_1 \vee \cdots \vee u_i x^{-1} \vee \cdots \vee t_n.$$

On the other hand, if the equation is of the form $\mathrm{e} \leq t_1 \vee \cdots \vee u_i x^{-1} v_i \vee \cdots \vee t_n$, we apply Lemma 4.4.10 and obtain the following equivalent inverse-free equation

$$xy \leq xyt_1 \vee \cdots \vee xyu_i y \vee v_i \cdots \vee xyt_n.$$

Suppose now that we start from $\mathrm{e} \leq t_1 \vee \cdots \vee t_n$ containing ${x_1}^{-1}, \dots, {x_n}^{-1}, {x_{n+1}}^{-1}$, for some variables $x_1, \dots, x_{n+1}$. By applying either one of the three procedures described above to the inverse ${x_{n+1}}^{-1}$, we obtain an equivalent equation

$$u_0 \leq u_1 \vee \cdots \vee u_k$$

containing ${x_1}^{-1}, \dots, {x_n}^{-1}$. Therefore, by the induction hypothesis, we conclude. □

By Theorem 4.4.11 and Corollary 4.2.4, an ℓ-group equation fails in an ℓ-group if and only if finitely many ℓ-monoid equations fail in finitely many End(Φ), for some finite chains Φ. This, and Corollary 4.2.4, provide a new proof of the decidability of the equational theory of ℓ-groups.

4.5 BACK TO THE STRUCTURE OF LATTICE-ORDERED GROUPS

It should be clear at this point that most (if not all) of the results presented in this thesis have the theory of ℓ-groups as their conceptual and technical starting point. We conclude this chapter by making explicit how the structure theory of distributive ℓ-monoids (as described in Section 4.1) relates to the well-known structure theory of ℓ-groups. More precisely, we answer here the following question:

> If the distributive ℓ-monoid M is (the inverse-free reduct of) an ℓ-group, how does Theorem 4.1.9 relate exactly to Theorem 2.1.19?

For any distributive ℓ-monoid M, we call a submonoid K of M an *ideal submonoid* if K is a lattice ideal of M. Clearly, any ideal submonoid is nonempty, as it must include $M^- = \{a \in M \mid a \geq \mathrm{e}\}$ of M. It is easy to see that the intersection of a set of ideal submonoids is again an ideal submonoid. Thus, the set $\mathcal{S}(M)$ of ideal submonoids partially ordered by inclusion is a complete lattice, where $K_1 \wedge K_2$ is the intersection $K_1 \cap K_2$, and $K_1 \vee K_2$ is the ideal submonoid generated by the union, i.e.,

$$K_1 \vee K_2 = \bigcap \{K \in \mathcal{S}(M) \mid K_1 \cup K_2 \subseteq K\}.$$

In fact, $\mathcal{S}(M)$ is an algebraic lattice. This is because $\mathcal{S}(M)$ is an 'algebraic subset'—i.e., a subset of a complete lattice that is closed under finite meets and closed under joins of chains—of the algebraic lattice of all subsets of M (cf. [77, Lemma 1.3.3]).

We will also show that $\mathcal{S}(M)$ is distributive. For this, it is useful to provide a description of the ideal submonoid generated by a set.

Proposition 4.5.1. *For any distributive ℓ-monoid M, the ideal submonoid $\mathfrak{S}(S)$ generated by $S \subseteq M$ is*

$$\{a \in M \mid a \leq s_1 \vee \cdots \vee s_n \textit{ for some } s_i \in \langle S \rangle_{\mathrm{e}}\}, \tag{4.26}$$

where $\langle S \rangle_{\mathrm{e}}$ is the submonoid generated by S.

Proof. First, we show that the set

$$T = \{a \in M \mid a \leq s_1 \vee \cdots \vee s_n \text{ for some } s_i \in \langle S \rangle_{\mathrm{e}} \text{ for each } 1 \leq i \leq n\}$$

is an ideal submonoid extending S. That T contains S, and is downward closed, is immediate. Moreover, if $a \leq s_1 \vee \cdots \vee s_n$ and $b \leq t_1 \vee \cdots \vee t_m$, where $s_i, t_j \in \langle S \rangle_{\mathrm{e}}$, then also

$$a \vee b \leq s_1 \vee \cdots \vee s_n \vee t_1 \vee \cdots \vee t_m.$$

Therefore, T is a lattice ideal. Finally, we also have that

$$ab \leq (s_1 \vee \cdots \vee s_n)b \leq (s_1 \vee \cdots \vee s_n)(t_1 \vee \cdots \vee t_m) = \bigvee_{1 \leq i \leq n; 1 \leq j \leq m} s_i t_j,$$

and since $s_i t_j \in \langle S \rangle_{\mathrm{e}}$, we can conclude $ab \in T$ (that is, T is a submonoid). It remains to show that T is the least ideal submonoid containing S. For this, pick an ideal submonoid K such that $S \subseteq K$. Since K is a submonoid, $\langle S \rangle_{\mathrm{e}} \subseteq K$ and moreover, since it is a lattice ideal, $s_1 \vee \cdots \vee s_n \in K$ for all $s_1, \ldots, s_n \in \langle S \rangle_{\mathrm{e}}$, and $a \in K$ for every $a \in T$. This concludes the proof. □

The following result is the analogue of the 'Riesz Decomposition Property' for ℓ-groups (see Proposition 2.1.3), and as in the case of ℓ-groups, will be a key ingredient in showing that the lattice $\mathcal{S}(M)$, for a distributive ℓ-monoid M, is distributive.

Lemma 4.5.2. *For any distributive ℓ-monoid M, and any $a, b_1, \ldots, b_n \in M^+$, it holds that*

$$(b_1 \cdots b_n) \wedge a \leq (b_1 \wedge a) \cdots (b_n \wedge a).$$

Proof. We prove the statement by induction on $1 < n \in \mathbb{N}^+$, as for $n = 1$, the result is immediate. For $n = 2$, observe that

$$\begin{aligned} b_1 b_2 \wedge a &= b_1 b_2 \wedge a \wedge a \wedge a \\ &\leq b_1 b_2 \wedge b_1 a \wedge a b_2 \wedge a^2 \\ &= (b_1 \wedge a)(b_2 \wedge a), \end{aligned}$$

since a, b_1, b_2 are elements of the positive cone M^+. For $n = m+1$, observe that

$$\begin{aligned} b_1 \cdots b_n b_{n+1} \wedge a &= b_1 \cdots b_n b_{n+1} \wedge a \wedge a \wedge a \\ &\leq b_1 \cdots b_n b_{n+1} \wedge (b_1 \cdots b_n) a \wedge a b_{n+1} \wedge a^2 \\ &= ((b_1 \cdots b_n) \wedge a)(b_{n+1} \wedge a) \\ &\leq (b_1 \wedge a) \cdots (b_n \wedge a)(b_{n+1} \wedge a), \end{aligned}$$

where the first inequality follows from $a, b_1, \ldots, b_n$ being elements of the positive cone, and the last inequality follows by the induction hypothesis. □

For any distributive ℓ-monoid M, and any element $a \in M$, we write a^+ for the positive part of a, namely $a \vee \mathrm{e}$ (similarly, a^- denotes $a \wedge \mathrm{e}$).

Theorem 4.5.3. *For any distributive ℓ-monoid M, the poset $\mathcal{S}(M)$ of ideal submonoids is a distributive lattice.*

Proof. For any $H, J, K \in \mathcal{S}(M)$, we show that

$$K \cap (H \vee J) \subseteq (K \cap H) \vee (K \cap J).$$

Let $a \in K \cap (H \vee J)$ which, by Proposition 4.5.1, implies

$$a \leq (s_{11} \cdots s_{1m}) \vee \cdots \vee (s_{n1} \cdots s_{nm}), \tag{4.27}$$

for $s_{ij} \in H \cup J$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Notice that, by (4.27), we get

$$a \leq (s_{11} \cdots s_{1m}) \vee \cdots \vee (s_{n1} \cdots s_{nm}) \leq (s_{11}^+ \cdots s_{1m}^+) \vee \cdots \vee (s_{n1}^+ \cdots s_{nm}^+),$$

and, also,

$$a^+ = a \vee \mathrm{e} \leq (s_{11}^+ \cdots s_{1m}^+) \vee \cdots \vee (s_{n1}^+ \cdots s_{nm}^+) \vee \mathrm{e} = (s_{11}^+ \cdots s_{1m}^+) \vee \cdots \vee (s_{n1}^+ \cdots s_{nm}^+).$$

Now, by Lemma 4.5.2, we have

$$(s_{i1}^+ \cdots s_{im}^+) \wedge a^+ \leq (s_{i1}^+ \wedge a^+) \cdots (s_{im}^+ \wedge a^+), \tag{4.28}$$

for all $1 \leq i \leq n$; from this, we also get

$$\begin{aligned} a^+ &= \big((s_{11}^+ \cdots s_{1m}^+) \vee \cdots \vee (s_{n1}^+ \cdots s_{nm}^+)\big) \wedge a^+ \\ &= \big((s_{11}^+ \cdots s_{1m}^+) \wedge a^+\big) \vee \cdots \vee \big((s_{n1}^+ \cdots s_{nm}^+) \wedge a^+\big) \\ &\leq (s_{11}^+ \wedge a^+) \cdots (s_{1m}^+ \wedge a^+) \vee \cdots \vee (s_{n1}^+ \wedge a^+) \cdots (s_{nm}^+ \wedge a^+), \end{aligned}$$

where the last inequality follows from (4.28). Observe now that, if $s_{ij} \in H$, then $(s^+_{ij} \wedge a^+) \in K \cap H$, and whenever $s_{ij} \in J$, then $(s^+_{ij} \wedge a^+) \in K \cap J$, for any $1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore,

$$a^+ \in \mathfrak{S}((K \cap H) \cup (K \cap J)),$$

that is, $a \leq a^+ \in (K \cap H) \vee (K \cap J)$, as was to be shown. □

In the remainder of this section, we show that each convex ℓ-subgroup of an ℓ-group H can be identified with a unique ideal submonoid of its positive cone H^+. We use this to compare the representation theorem by Holland with the representation theorem by Anderson and Edwards, whenever the distributive ℓ-monoid at hand is (the inverse-free reduct of) an ℓ-group.

For any convex ℓ-subgroup $\mathfrak{k}$ of any ℓ-group H, $a \in \mathfrak{k}$ if and only if $a \vee a^{-1} = |a| \in \mathfrak{k}$ (this follows easily by definition, and from Proposition 2.1.4). Hence, $\mathfrak{k} \cap H^+ = |\mathfrak{k}|$.

Lemma 4.5.4. *For any ℓ-group H, the map*

$$\mathcal{C}(H) \xrightarrow{g} \mathcal{S}(H^+), \qquad \mathfrak{k} \longmapsto \mathfrak{k} \cap H^+ \tag{4.29}$$

is a lattice isomorphism, with inverse

$$\mathcal{S}(H^+) \xrightarrow{h} \mathcal{C}(H), \qquad K \longmapsto \mathfrak{C}(K) \tag{4.30}$$

Further, this correspondence restricts to a bijection between prime subgroups of H, and those ideal submonoids of H^+ that are prime (as lattice ideals).

Proof. Observe that the maps g, h are all well-defined. Moreover, g, h are inverse to each other. For this, it suffices to observe that $\mathfrak{k} \cap H^+ = \langle |\mathfrak{k}| \rangle_{\mathrm{e}}$. Therefore,

$$\begin{aligned} h(g(\mathfrak{k})) &= h(\mathfrak{k} \cap H^+) \\ &= \mathfrak{C}(\mathfrak{k} \cap H^+) \\ &= \mathfrak{C}(\langle |\mathfrak{k}| \rangle_{\mathrm{e}}) \\ &= \mathfrak{k}. \end{aligned}$$

Similarly,

$$g(h(K)) = g(\mathfrak{C}(K)) = \mathfrak{C}(K) \cap H^+ = K,$$

where the last equality follows because $a \in \mathfrak{C}(K) \cap H^+$ entails $\mathrm{e} \leq a \leq k$, for some $k \in \langle |K| \rangle_{\mathrm{e}} = K$; and hence, $a \in K$. It remains to show that h is a lattice homomorphism. For this, we assume $\mathfrak{k}_1, \mathfrak{k}_2 \in \mathcal{C}(H)$. It is immediate that

$$g(\mathfrak{k}_1 \cap \mathfrak{k}_2) = (\mathfrak{k}_1 \cap \mathfrak{k}_2) \cap H^+ = (\mathfrak{k}_1 \cap H^+) \cap (\mathfrak{k}_2 \cap H^+) = g(\mathfrak{k}_1) \cap g(\mathfrak{k}_2).$$

Moreover,

$$g(\mathfrak{k}_1 \vee \mathfrak{k}_2) = \{a \in H^+ \mid a \leq s_1 \cdots s_n \text{ for some } s_1, \ldots, s_n \in |\mathfrak{k}_1 \cup \mathfrak{k}_2|\}.$$

Now, it is immediate that

$$|\mathfrak{k}_1 \cup \mathfrak{k}_2| = |\mathfrak{k}_1| \cup |\mathfrak{k}_2| = (\mathfrak{k}_1 \cap H^+) \cup (\mathfrak{k}_2 \cap H^+);$$

thus, if $a \le s_1 \cdots s_n$ for some $s_1, \ldots, s_n \in (\mathfrak{k}_1 \cap H^+) \cup (\mathfrak{k}_2 \cap H^+)$, then

$$a \in \mathfrak{S}((\mathfrak{k}_1 \cap H^+) \cup (\mathfrak{k}_2 \cap H^+)) = (\mathfrak{k}_1 \cap H^+) \vee (\mathfrak{k}_2 \cap H^+) = g(\mathfrak{k}_1) \vee g(\mathfrak{k}_2).$$

Also, if $a \in g(\mathfrak{k}_1) \vee g(\mathfrak{k}_2) = (\mathfrak{k}_1 \cap H^+) \vee (\mathfrak{k}_2 \cap H^+)$, then

$$\mathrm{e} \le a \le s_1 \vee \cdots \vee s_n \quad \text{for some } s_i \in \langle (\mathfrak{k}_1 \cap H^+) \cup (\mathfrak{k}_2 \cap H^+) \rangle_{\mathrm{e}}.$$

Thus,

$$\mathrm{e} \le a \le s_1 \vee \cdots \vee s_n \quad \text{for some } s_i \in \langle |\mathfrak{k}_1| \cup |\mathfrak{k}_2| \rangle_{\mathrm{e}},$$

and hence

$$\mathrm{e} \le a \le s_1 \vee \cdots \vee s_n \quad \text{for some } s_i \in \langle |\mathfrak{k}_1 \cup \mathfrak{k}_2| \rangle_{\mathrm{e}}.$$

But then, $\mathrm{e} \le a \in \mathfrak{k}_1 \vee \mathfrak{k}_2$, that is, $a \in g(\mathfrak{k}_1 \vee \mathfrak{k}_2)$.

For the second part of the statement, assume that $\mathfrak{k}$ is a prime subgroup of H, and that $a \wedge b \in \mathfrak{k} \cap H^+$. Then, since $a, b \in H^+$ and $\mathfrak{k}$ is prime, either $a \in \mathfrak{k} \cap H^+$, or $b \in \mathfrak{k} \cap H^+$. Conversely, consider an ideal submonoid K which is prime as a lattice ideal of H^+, and let $a \wedge b \in \mathfrak{C}(K)$. Then,

$$|a \wedge b| = (a \wedge b) \vee (a^{-1} \vee b^{-1}) \le s \text{ for some } s \in \langle |K| \rangle_{\mathrm{e}} = K.$$

Now, by distributivity, and the fact that K is downward closed, we get

$$\begin{aligned} |a \wedge b| &= (a \wedge b) \vee (a^{-1} \vee b^{-1}) \\ &= (a \vee a^{-1} \vee b^{-1}) \wedge (b \vee a^{-1} \vee b^{-1}) \in K. \end{aligned}$$

Since K is prime, either $a \vee a^{-1} \vee b^{-1} \in K$ or $b \vee a^{-1} \vee b^{-1} \in K$, which entails (by downward closure) that either $|a| \in K$ or $|b| \in K$. Hence, either $a \in \mathfrak{C}(K)$ or $b \in \mathfrak{C}(K)$. □

For any ℓ-group H, every ideal submonoid $K \in \mathcal{S}(H^+)$ naturally determines a unique ideal submonoid of H, by considering $\downarrow K$. Therefore, if $\mathfrak{k}$ is a convex ℓ-subgroup of H, we associate to it a unique ideal submonoid of H, namely $\downarrow \mathfrak{k}$.

Lemma 4.5.5. *For any ℓ-group H, and any convex ℓ-subgroup $\mathfrak{k}$ of H, the ideal submonoid $\downarrow \mathfrak{k}$ is prime (as a lattice ideal) if and only if $\mathfrak{k}$ is a prime subgroup of H.*

Proof. Suppose $a \wedge b \in \downarrow \mathfrak{k}$. Then, $a \wedge b \leqslant k$, for some $\in \mathfrak{k} \cap H^+$. But then,

$$(a \vee k) \wedge (b \vee k) = (a \wedge b) \vee k = k \in \mathfrak{k} \cap H^+,$$

and hence, either $k_1 = (a \vee k) \in \mathfrak{k} \cap H^+$ or $k_2 = (b \vee k) \in \mathfrak{k} \cap H^+$. Therefore, either $a \le k_1 \in \mathfrak{k} \cap H^+$ or $b \le k_2 \in \mathfrak{k} \cap H^+$. Conversely, suppose $\downarrow \mathfrak{k}$ is prime, and $a \wedge b \in \mathfrak{k} \cap H^+$. Then, since $\mathfrak{k} \cap H^+ \subseteq \downarrow \mathfrak{k}$, we immediately have either $a \in (\downarrow \mathfrak{k}) \cap H^+ = \mathfrak{k} \cap H^+$ or $b \in (\downarrow \mathfrak{k}) \cap H^+ = \mathfrak{k} \cap H^+$. The result now follows from Lemma 4.5.4. □

It was mentioned in Chapter 2 that, for any ℓ-group H, and any of its convex ℓ-subgroups $\mathfrak{k}$, two (generally different) equivalence relations can be defined. The first one is the one induced by the right cosets, that is,

$$a\theta_{\mathfrak{k}} b \quad\Longleftrightarrow\quad ba^{-1} \in \mathfrak{k}; \tag{4.31}$$

the second one was implicitly considered in Remark 2.1.16, and can be explicitly defined by

$$a\phi_{\mathfrak{k}} b \quad\Longleftrightarrow\quad (c^{-1}ba^{-1}c) \in \mathfrak{k}, \quad \text{for any } c \in H. \tag{4.32}$$

The first equivalence relation is in fact a right (group) congruence, and a lattice congruence. The second equivalence relation is an ℓ-group congruence.

We conclude the section by showing that, for any ℓ-group, the quotients induced by its convex ℓ-subgroups naturally correspond to quotients induced by some of its lattice ideals.

Theorem 4.5.6. *For any ℓ-group H, any convex ℓ-subgroup $\mathfrak{k}$, and any $a, b \in H$:*

(a) *$a\theta_{\mathfrak{k}} b$ if and only if $a\rho_{\downarrow\mathfrak{k}} b$, where $\rho_{\downarrow\mathfrak{k}}$ is defined as in* (4.3).

(b) *$a\phi_{\mathfrak{k}} b$ if and only if $a\beta_{\downarrow\mathfrak{k}} b$, where $\beta_{\downarrow\mathfrak{k}}$ is defined as in* (4.4).

Proof. For (a), suppose $ba^{-1} \in \mathfrak{k}$, and $ac \in \downarrow\mathfrak{k}$ for some $c \in H$. Now, since $ba^{-1} \in \mathfrak{k}$, also its absolute value $ba^{-1} \vee ab^{-1} \in \mathfrak{k}^+ \subseteq \downarrow\mathfrak{k}$. Therefore, by downward closure, we get $ba^{-1} \in \downarrow\mathfrak{k}$ and $ab^{-1} \in \downarrow\mathfrak{k}$. Hence, also $ba^{-1}ac = bc \in \downarrow\mathfrak{k}$. (Analogously, if we start from $bc \in \downarrow\mathfrak{k}$, we reach the conclusion $ac \in \downarrow\mathfrak{k}$ by using $ab^{-1} \in \downarrow\mathfrak{k}$.) Conversely, suppose $a\rho_{\downarrow\mathfrak{k}} b$. Then, since $aa^{-1} = \mathrm{e} \in \downarrow\mathfrak{k}$, also $ba^{-1} \in \downarrow\mathfrak{k}$. Similarly, since $bb^{-1} = \mathrm{e} \in \downarrow\mathfrak{k}$, we conclude $ab^{-1} \in \downarrow\mathfrak{k}$. Therefore,

$$ba^{-1} \vee ab^{-1} = |ba^{-1}| \in \downarrow\mathfrak{k} \cap H^+ \subseteq \mathfrak{k},$$

and hence, $ba^{-1} \in \mathfrak{k}$. For (b), assume $c^{-1}ba^{-1}c \in \mathfrak{k}$ for every $c \in H$. But then, we also have

$$c^{-1}b^{-1}(ba)^{-1}bc = c^{-1}a^{-1}bc,\ c^{-1}ab^{-1}c,\ c^{-1}b^{-1}ac \in \mathfrak{k}, \tag{4.33}$$

for any $c \in H$. Suppose $sat \in \downarrow\mathfrak{k}$, for $s, t \in H$. Then, also $t^{-1}a^{-1}bt \in \downarrow\mathfrak{k}$ by (4.33) and hence

$$(sat)(t^{-1}a^{-1}bt) = sbt \in \downarrow\mathfrak{k},$$

as desired. Conversely, assume $a\beta_{\downarrow\mathfrak{k}} b$. Then, for every $c \in$ H, also

$$c^{-1}a^{-1}(a)c,\ c^{-1}b^{-1}(b)c \in \downarrow\mathfrak{k}. \tag{4.34}$$

Therefore, by the assumption, $c^{-1}a^{-1}bc, c^{-1}b^{-1}ac \in \downarrow\mathfrak{k}$ as well. Hence,

$$c^{-1}a^{-1}bc \vee c^{-1}b^{-1}ac = |c^{-1}a^{-1}bc| \in \downarrow\mathfrak{k} \cap H^+ \subseteq \mathfrak{k},$$

which implies $c^{-1}ba^{-1}c \in \mathfrak{k}$. □

Provided that a distributive ℓ-monoid is (the inverse-free reduct of) an ℓ-group, the factors of the product described by Theorem 2.1.19 are among the factors of the product obtained by Theorem 4.1.9; however, in general, they are strictly fewer. It is worth mentioning that more should be understood about the relationship between (sets of) prime lattice ideals, and prime subgroups. For instance, it is known that the collection of lattice ideals of a distributive ℓ-monoid that are maximal with respect to not containing a given element (sometimes called 'relative maximal') can still be used to show that 'every distributive ℓ-monoid is (isomorphic to) an ℓ-submonoid of End (Ω) for some chain Ω'. Hence, it is reasonable to study the relationship between values of an ℓ-group (originally used in Holland's proof), and those lattice ideals of the ℓ-group which are relative maximal, making it explicit along the lines of Theorem 4.5.6.

4.6 CONCLUDING REMARKS

The main contribution of this chapter, Theorem 4.4.3, is the result that the inverse-free reducts of ℓ-groups satisfy all and only the equations satisfied by distributive ℓ-monoids. The analogue of this result was already known to fail for Abelian ℓ-groups and commutative distributive ℓ-monoids; we showed here that it also fails for (inverse-free reducts of) representable ℓ-groups and representable distributive ℓ-monoids. Furthermore, it was proved in [149] that the variety generated by inverse-free reducts of Abelian ℓ-groups is not finitely based, although recursively axiomatizable.

Problem 9. Find a recursive axiomatization for the variety generated by the inverse-free reducts of representable ℓ-groups.

To do this, it will probably be necessary to adapt the axiomatization and the proof available for the commutative case.

Some interesting conclusions can be drawn from Theorem 4.4.3. For instance, we used this result to generalize the correspondence between validity of equations in ℓ-groups and subsets of free groups that extend to right orders on the group to the setting of distributive ℓ-monoids. This allowed us to establish, *inter alia*, a relationship between right orders on free groups and right orders on free monoids. Related to this is the space of right orders on monoids that, as already mentioned in Chapter 3, was defined by Adam Sikora in [160]. In analogy with Example 3.1.3 and Problem 6, it is reasonable to raise the following problem.

Problem 10. Determine whether the space of (right) orders on the finitely generated free monoid $F^m(k)$ ($k \geq 2$) has isolated points, and is hence homeomorphic to the Cantor space.

The proof given here for Corollary 4.4.7 does not provide an answer to this question. It would be helpful to obtain a proof that provides a construction of a right order on the free group from a right order on the free monoid. We mention, related to this, the following interesting references: [171, 172]; in these articles, the author constructs explicit right orders on the free group that extend certain right orders on free monoid. We also point out that a similar analysis for orders would be of interest.

Problem 11. Establish a relationship between orders on the free group and orders on the free monoid, similar to Corollary 4.4.7.

Theorem 4.4.3 is even more powerful when paired with the surprising fact, established in Section 4.4, that any ℓ-group equation can be reduced to finitely many ℓ-monoid equations. As a consequence, validity of ℓ-group equations in the variety of ℓ-groups amounts to validity of ℓ-monoid equations in the variety of distributive ℓ-monoids. Since the variety of distributive ℓ-monoids is generated by its finite members (Corollay 4.2.4), this provides a new proof of decidability for the equational theory of ℓ-groups. The situation differs for commutative and representable distributive ℓ-monoids, where decidability of the equational theory is still an open problem.

Problem 12. Provide an algorithm to decide the equational theory of commutative distributive ℓ-monoids and representable distributive ℓ-monoids. Similarly, provide an algorithm to decide the equational theory of inverse-free reducts of representable ℓ-groups.

The problem is particularly relevant for the variety of commutative distributive ℓ-monoids. In this direction, a first reasonable question is whether this variety satisfies the finite model property.

As already discussed in Section 1.5, the problem of obtaining a calculus for ℓ-groups that admits an algebraic proof of cut elimination is still open. It was shown in a recent paper by Nikolaos Galatos and Peter Jipsen ([65]) that a framework similar to that in [21] can be developed for varieties of distributive ℓ-monoids, by suitably adapting the notion of residuated frame (cf. [64]). The connection between ℓ-group equations and ℓ-monoid equations established in Theorem 4.4.11 might lead not only to obtain a first calculus for ℓ-groups admitting an algebraic proof of cut elimination, by importing tools and techniques developed in [65], but also to develop a framework for a more systematic study of the proof theory for ℓ-groups.

The last part of the chapter was concerned with the relationship between the structure theory of distributive ℓ-monoids, and the more well-established structure theory of ℓ-groups. At the present stage, the structure of distributive ℓ-monoids is not as well-understood as the structure of ℓ-groups. Even though convex subalgebras are central in the structure theory of ℓ-groups, it is not yet clear what role convex subalgebras play in the theory of distributive ℓ-monoids, if any at all. In what follows, we include some preliminary steps towards a better understanding of convex subalgebras in the context of distributive ℓ-monoids.

In Section 4.1, it was mentioned that there are two standard ways to transfer the notion of a right congruence from groups to the monoid setting. However, stepping from ℓ-groups to the setting of distributive ℓ-monoids, other possibilities are available to generalize the notion of right congruence, exploiting the presence of the (lattice) order. If we go back to Chapter 2, and focus on (2.1), we can see that for any

ℓ-group H and any convex ℓ-subgroup $\Bbbk$ of H,

$$\begin{aligned} \Bbbk a = \Bbbk b \quad &\Longleftrightarrow \quad \Bbbk a \leq^* \Bbbk b \text{ and } \Bbbk b \leq^* \Bbbk b \\ &\Longleftrightarrow \quad \text{there exist } s, t \in \Bbbk \text{ such that } a \leq sb \text{ and } b \leq ta, \end{aligned}$$

where $\leq^*$ denotes the order of the resulting lattice quotient. This notion, which was considered in [101], might allow us to generalize the structure theory of ℓ-groups to distributive ℓ-monoids, employing (order-)convex ℓ-submonoids instead of lattice ideals. More precisely, we should consider convex ℓ-submonoids with the following property: for any distributive ℓ-monoid M and any convex ℓ-submonoid $\Bbbk$ of M,

$$\text{if } m \in M,\, a, b \in \Bbbk \text{ and } m^+ a^- \leq m^- b^+, \text{ then } m \in \Bbbk \tag{4.35}$$

(similarly, if $a^- m^+ \leq b^+ m^-$). Indeed, a convex ℓ-submonoid of an ℓ-group is a convex ℓ-subgroup if and only if it satisfies (4.35). This notion was again considered in [101], in a more general setting (i.e., distributivity of the lattice reduct was not assumed).

Problem 13. Study the notion of convex ℓ-submonoid (with additional properties; cf. (4.35)), aiming at a uniform treatment of the structure theory of distributive ℓ-monoids and ℓ-groups.

Even though the notions discussed above are probably too general for studying all distributive ℓ-monoids, they might be suited for a smaller, more well-behaved, class (containing ℓ-groups).

The results in this chapter outlined the striking relationship between distributive ℓ-monoids and inverse-free reducts of ℓ-groups. It would be of interest to broaden the scope of this investigation, studying classes of distributive ℓ-monoids that are related to ℓ-groups in different ways.

Problem 14. Study the relationship between inverse-free (sub)reducts of negative cones of ℓ-groups, and integral distributive ℓ-monoids, i.e., those where the identity e is the greatest element.

From the point of view of the structure of such distributive ℓ-monoids, there seem to be some immediate simplification. For instance, the correct notion to study in this case would be the notion of a filter submonoid (defined analogously to the notion of ideal submonoid), as in an integral setting it coincides with the notion of convex ℓ-submonoid.

CHAPTER 5

HAMILTONIAN AND NILPOTENT CANCELLATIVE RESIDUATED LATTICES

The present chapter studies nilpotent and Hamiltonian cancellative residuated lattices and their relationship with nilpotent and Hamiltonian ℓ-groups. In particular, certain results about ℓ-groups are extended to the domain of residuated lattices. The two key ingredients behind the considerations of this chapter are the categorical equivalence of [135], which provides a new framework for the study of various classes of cancellative residuated lattices by viewing these structures as ℓ-groups with a suitable modal operator; and Malcev's description of nilpotent groups of a given nilpotency class by a semigroup equation ([124]; see also [142]).

The term 'Conrad Program' traditionally refers to Paul Conrad's approach to the study of ℓ-groups, which analyzes the structure of individual ℓ-groups, or classes of ℓ-groups, investigating the lattice-theoretic properties of their lattices of convex ℓ-subgroups. In the 1960s, Conrad's articles [35, 36, 37, 38, 39, 41] pioneered this approach and demonstrated its usefulness. Large parts of the Conrad Program can be profitably extended to the much wider class of e-cyclic residuated lattices (see, e.g., [115, 70, 15, 71, 116]), i.e., those satisfying the equation $x\backslash \mathrm{e} \approx \mathrm{e}/x$, which includes all residuated lattices that are cancellative, divisible, or commutative.

The notion of a Hamiltonian algebra arises as a generalization of the concept of a Hamiltonian group ([60]). Hamiltonian ℓ-groups were first introduced implicitly in [126], and later studied extensively (see, e.g., [42, 148, 74, 13]), as those ℓ-groups whose convex ℓ-subgroups are normal. While they do not form a variety ([42, Proposition 1.4]), a largest variety of Hamiltonian ℓ-groups does exist and was identified in [148]. Nilpotent ℓ-groups are Hamiltonian, and share other important properties with Abelian ℓ-groups, including representability ([109]; see also [92] and [148], respectively).

The present work builds on the aforementioned research. First, we dispatch some preliminaries on residuated lattices and their convex subalgebras. Section 5.2, and more precisely Theorem 5.2.3, provides a bridge for connecting nilpotent cancellative residuated lattices and nilpotent ℓ-groups. The focus of Section 5.3 is the prelinearity property and some of its equivalent formulations. Hamiltonian and nilpotent prelinear cancellative residuated lattices are the focus of Sections 5.4 and 5.5, respectively. The final section discusses varieties of representable cancellative residuated lattices. We show, *inter alia*, that any variety of representable cancellative integral residuated lattices defined by monoid equations is generated by those residuated chains which are finitely generated as monoids. To help the reader disentangle the several classes of residuated lattices that we study here, we refer to Table 5.1 and Figure 5.1 at the end of the chapter.

This chapter is based on the manuscript [33]. The theory and terminology from order theory and category theory used in this chapter is reviewed in Appendix A.1 and Appendix A.2.

5.1 RESIDUATED LATTICES AND THEIR STRUCTURE

In this section we briefly recall some basic facts about residuated lattices and their structure; we refer to [12], [102], [66], and [134] for further details.

A *residuated lattice* is an algebraic structure L with operations $\cdot, \wedge, \vee, \backslash, /, \mathrm{e}$ such that $(L, \cdot, \mathrm{e})$ is a monoid, $(L, \wedge, \vee)$ is a lattice, and $\backslash, /$ are binary operations with the following 'residuation property': for all $a, b, c \in L$,

$$ab \leq c \iff a \leq c/b \iff b \leq a\backslash c, \tag{5.1}$$

where $\leq$ is the lattice order. The operations $\backslash$ and $/$ are referred to as the left residual and the right residual of $\cdot$, respectively[1] (cf. Appendix A.2). We refer to a as the denominator of $a\backslash b$ (resp., b/a), and to b as the numerator of $a\backslash b$ (resp., b/a). Condition (5.1) is equivalent to $\cdot$ being order-preserving in each argument and, for all $a, b \in L$, the sets

$$\{c \in L \mid a \cdot c \leq b\} \quad \text{and} \quad \{c \in L \mid c \cdot a \leq b\} \tag{5.2}$$

containing greatest elements $a\backslash b$ and b/a, respectively. Residuated lattices form a variety denoted by RL. Residuated lattices with a commutative monoid reduct are called *commutative* residuated lattices, and form a subvariety of RL.

Example 5.1.1. For any ring R with unit, the set $\mathrm{Id}(R)$ of (two-sided) ideals of R partially ordered by inclusion forms a complete lattice, where meet is the intersection and join is ideal addition (i.e., $I + J = \{a + b \mid a \in I, b \in J\}$). The lattice $\mathrm{Id}(R)$ can be naturally equipped with monoid and residual operations, as follows. The monoid operation is given by ideal multiplication (i.e., $IJ = \{\sum_{k=1}^{n} a_k b_k \mid a_k \in I, b_k \in J; n \geq 1\}$), and the element R of $\mathrm{Id}(R)$ acts as the monoid identity. For all $I, J \in \mathrm{Id}(R)$, the residuals are given by

$$I\backslash J := \{a \in R \mid Ia \subseteq J\} \quad \text{and} \quad J/I := \{a \in R \mid aI \subseteq J\}.$$

This structure was one of the original motivations behind the study of residuated lattices ([173]).

[1] When such residuals exist, we say that '$\cdot$ is residuated with respect to $\leq$'.

We recall here some relevant standard facts.

Proposition 5.1.2 (cf. [66, Lemma 2.6])**.** *The monoid operation $\cdot$ of any residuated lattice preserves all existing joins in each argument. The residuals $\backslash$ and $/$ preserve all existing meets in the numerator, and convert existing joins in the denominator into meets. Consequently, residuals preserve order in the numerator, and reverse order in the denominator.*

We write xz/yw for $(xz)/(yw)$ and $yw\backslash xz$ for $(yw)\backslash(xz)$.

Proposition 5.1.3 (cf. [66, Lemma 2.6])**.** *Every residuated lattice satisfies the equations*

$$x\backslash(y/z) \approx (x\backslash y)/z, \quad x/yz \approx (x/z)/y, \quad xy\backslash z \approx y\backslash(x\backslash z).$$

We call a residuated lattice *cancellative* if its monoid reduct is cancellative as a monoid. Surprisingly, the class of cancellative residuated lattices is a variety (cf. [6, Lemma 2.5]) defined relative to RL by the equations

$$xy/y \approx x \approx y\backslash yx. \tag{5.3}$$

Proposition 5.1.4. *The equations $x/x \approx \mathrm{e} \approx x\backslash x$ hold in any cancellative residuated lattice.*

Proof. Follows immediately by substituting e for x and x for y in the equations (5.3). □

Example 5.1.5. The variety of ℓ-groups LG is term-equivalent to the subvariety of RL defined by the equations

$$x(x\backslash \mathrm{e}) \approx \mathrm{e} \approx (\mathrm{e}/x)x. \tag{5.4}$$

Every ℓ-group can be seen as a residuated lattice, where $a\backslash b := a^{-1}b$, and $b/a := ba^{-1}$; conversely, any residuated lattice satisfying (5.4) is an ℓ-group, where $a^{-1} := a\backslash \mathrm{e} = \mathrm{e}/a$. Clearly, ℓ-groups are cancellative residuated lattices.

For any residuated lattice L, an element $a \in L$ for which

$$a(a\backslash \mathrm{e}) = \mathrm{e} = (\mathrm{e}/a)a,$$

is said to be invertible. The variety of ℓ-groups is identified with the class of all those residuated lattices for which every element is invertible.

Example 5.1.6. For any ℓ-group G, the set $G^- = \{a \in L \mid a \le \mathrm{e}\}$ of its negative elements (including the monoid identity e) is its *negative cone*. It is a submonoid and a sublattice of G, and it can be made into a residuated lattice, by defining $\backslash_{G^-}$ and $/_{G^-}$ as

$$a\backslash_{G^-} b := a^{-1}b \wedge \mathrm{e}$$
$$a/_{G^-} b := ab^{-1} \wedge \mathrm{e},$$

for $a, b \in G^-$. More generally, the negative cone L^- of any residuated lattice L is itself a residuated lattice, with $a\backslash_{L^-} b := a\backslash b \wedge \mathrm{e}$ and $a/_{L^-} b := a/b \wedge \mathrm{e}$.

Remark 5.1.7. It is customary in the study of residuated lattices to focus on negative cones. Even though in the context of ℓ-groups positive cone and negative cone behave symmetrically, this is not the case in the setting of residuated lattices.

Residuated lattices satisfying $x \wedge \mathrm{e} \approx x$ are called *integral*. The class of integral residuated lattices can be equivalently defined relative to RL by the equation

$$x\backslash \mathrm{e} \approx \mathrm{e} \quad (\text{equivalently, } \mathrm{e}/x \approx \mathrm{e}). \tag{5.5}$$

A residuated lattice is said to be e*-cyclic* if it satisfies the equation $x\backslash \mathrm{e} \approx \mathrm{e}/x$. It is immediate that commutative residuated lattices are e-cyclic; also, ℓ-groups are standard examples of e-cyclic residuated lattices.

Proposition 5.1.8. *Every cancellative residuated lattice is* e*-cyclic.*

Proof. For any residuated lattice L and $a \in L$, we have $a\backslash(a/a) = (a\backslash a)/a$ by Proposition 5.1.3. Thus, by Proposition 5.1.4, if L is cancellative, $a\backslash \mathrm{e} = \mathrm{e}/a$ for every $a \in L$. □

For any e-cyclic residuated lattice L, the set $\mathcal{C}(L)$ of (order-)convex subalgebras of L partially ordered by inclusion is a distributive algebraic lattice (see, e.g., [15, Theorem 3.8]), where again meet is the intersection and join is the convex subalgebra generated by the union. We write $\mathfrak{C}(S)$ for the convex subalgebra generated by $S \subseteq L$. If $a \in L$, we write $\mathfrak{C}(a)$ for $\mathfrak{C}(\{a\})$, and call $\mathfrak{C}(a)$ the principal convex subalgebra generated by a. If L is a residuated lattice and $a \in L$, the *absolute value* $|a| \in L^-$ of a is defined as $a \wedge (\mathrm{e}/a) \wedge \mathrm{e}$.

The following results are established in [15, Lemma 3.2, Corollary 3.3, Lemma 3.6].

Lemma 5.1.9. *For any* e*-cyclic residuated lattice L:*

(a) *For any $S \subseteq L$, the convex subalgebra generated by S is*

$$\begin{aligned}\mathfrak{C}(S) = \mathfrak{C}(|S|) &= \{c \in L \mid t \le c \le t\backslash \mathrm{e}, \textit{for some } t \in \langle |S| \rangle_{\mathrm{e}}\} \\ &= \{c \in L \mid t \le |c|, \textit{for some } t \in \langle |S| \rangle_{\mathrm{e}}\},\end{aligned}$$

where $|S| = \{|s| \mid s \in S\}$, and $\langle T \rangle_{\mathrm{e}}$ is the monoid generated by a subset T of L.

(b) *For any $a \in L$, the convex subalgebra generated by a is*

$$\begin{aligned}\mathfrak{C}(a) = \mathfrak{C}(|a|) &= \{c \in L \mid |a|^n \le c \le |a|^n\backslash \mathrm{e}, \textit{for some } n \in \mathbb{N}^+\} \\ &= \{c \in L \mid |a|^n \le |c|, \textit{for some } n \in \mathbb{N}^+\}.\end{aligned}$$

(c) *For any $a, b \in L^-$, $\mathfrak{C}(a \vee b) = \mathfrak{C}(a) \wedge \mathfrak{C}(b) = \mathfrak{C}(a) \cap \mathfrak{C}(b)$ and $\mathfrak{C}(a \wedge b) = \mathfrak{C}(a) \vee \mathfrak{C}(b)$.*

If L is a residuated lattice, and $a, b \in L$, we define

$$\lambda_b(a) := (b\backslash ab) \wedge \mathrm{e} \;\text{ and }\; \rho_b(a) := (ba/b) \wedge \mathrm{e}, \tag{5.6}$$

and refer to λ_b and ρ_b respectively as left and right conjugation by b. For any residuated lattice L, a convex subalgebra $\Bbbk \in \mathcal{C}(L)$ is said to be *normal* if for any $a \in \Bbbk$ and any $b \in L$, it holds that $\lambda_b(a) \in \Bbbk$ and $\rho_b(a) \in \Bbbk$. It was proved in [12, Theorem 4.12] that the lattice $\mathcal{NC}(L)$ of normal convex subalgebras of any residuated lattice L is isomorphic to its congruence lattice $\mathrm{Con}\,L$. Note that if L is an ℓ-group, (normal) convex subalgebras of L in the sense defined here coincide with (normal) convex ℓ-subgroups of L.

5.2 SUBMONOIDS OF NILPOTENT LATTICE-ORDERED GROUPS

Our primary focus in this section is the quasivariety of all submonoids of nilpotent ℓ-groups. The main result of this section, Theorem 5.2.5, provides a characterization of submonoids of nilpotent ℓ-groups and, equivalently, of submonoids of nilpotent cancellative residuated lattices. In particular, a nilpotent monoid is a submonoid of a nilpotent ℓ-group if and only if it is cancellative and torsion-free (in the sense to be defined below).

Recall that nilpotent groups of class $c \in \mathbb{N}^+$ are those groups with a central series of length at most c; they form a variety defined by the equation

$$[[[x_1, x_2], \ldots, x_c], x_{c+1}] \approx \mathrm{e}.$$

It is possible to axiomatize nilpotent groups of class c using only a semigroup equation. Consider the equation L_c: $q_c(x, y, \bar{z}) \approx q_c(y, x, \bar{z})$, where $\bar{z}$ abbreviates a sequence of variables $z_1, z_2, \ldots$, and $q_c(x, y, \bar{z})$ is defined as follows, for $c \in \mathbb{N}^+$,

$$q_1(x, y, \bar{z}) = xy$$
$$q_{c+1}(x, y, \bar{z}) = q_c(x, y, \bar{z}) z_c q_c(y, x, \bar{z}).$$

Note that $q_{c+1}(x, y, \bar{z})$ contains $x, y, z_1, \ldots, z_c$. The next result was first proved by Malcev in [124].

Proposition 5.2.1 ([142, Corollary 1])**.** *For any $c \in \mathbb{N}^+$, a group is nilpotent of class c if and only if it satisfies the equation L_c.*

We call a monoid M right-reversible if $Ma \cap Mb \neq \emptyset$, for all $a, b \in M$. A *group of (left) quotients* for a monoid M is a group G that has M as a submonoid, and such that every $c \in G$ is of the form $c = a^{-1}b$ for some $a, b \in M$. By a classical result due to Ore (see, e.g., [28, Section 1.10], [58]), a cancellative monoid M has a group of quotients (unique up to isomorphism) if and only if M is right-reversible.

We call a right-reversible cancellative monoid Ore, and write $G(M)$ for its group of quotients. Further, we call *Ore* a residuated lattice whose monoid reduct is Ore.

Proposition 5.2.2 ([142, Theorem 1])**.** *For any $c \in \mathbb{N}^+$, a cancellative monoid M embeds into a nilpotent group of class c if and only if it satisfies the equation L_c; in this case, M is Ore and its group of quotients is nilpotent of class c.*

We call a monoid *nilpotent of class c* if it satisfies L_c, and call a residuated lattice *nilpotent of class c* if its monoid reduct is nilpotent of class c. Commutative residuated lattices coincide with nilpotent residuated lattices of class 1. The preceding result implies in particular that all nilpotent cancellative residuated lattices are Ore. The categorical equivalence in [135] provides a bridge between nilpotent cancellative residuated lattices and nilpotent ℓ-groups.

An interior operator σ on a partially ordered monoid is said to be a *conucleus* if $\sigma(\mathrm{e}) = \mathrm{e}$ and $\sigma(a)\sigma(b) \le \sigma(ab)$. If L is a residuated lattice and σ a conucleus on L, then the image $L_\sigma = \sigma[L]$ is a $\vee$-subsemilattice and a submonoid of L. It can be made into a residuated lattice by defining

$$a \wedge_\sigma b := \sigma(a \wedge b), \qquad a\backslash_\sigma b := \sigma(a\backslash b), \qquad a/_\sigma b := \sigma(a/b),$$

for any $a, b \in L_\sigma$ (see [135, Lemma 3.1]).

Let $\mathsf{LG}_{\mathrm{cn}}$ be the category with objects (G,σ) consisting of an ℓ-group G augmented with a conucleus σ such that the underlying group of the ℓ-group G is the group of quotients of the monoid reduct of $\sigma[G]$, and with morphisms given by ℓ-groups homomorphisms commuting with the conuclei. The category ORL of Ore residuated lattices and residuated lattice homomorphisms was shown to be equivalent to $\mathsf{LG}_{\mathrm{cn}}$ [135, Theorem 4.9]. The results collected here suffice to provide a specialization of that equivalence to the category $\mathsf{N}^c\mathsf{CanRL}$ of nilpotent cancellative residuated lattices of class $c \in \mathbb{N}^+$ and residuated lattice homomorphisms, and the full subcategory $\mathsf{N}^c\mathsf{LG}_{\mathrm{cn}}$ of $\mathsf{LG}_{\mathrm{cn}}$ consisting of objects whose first component is a nilpotent ℓ-group of class c.

We put aside the full categorical equivalence, and keep in mind the following key idea. Every nilpotent cancellative residuated lattice L (of class c) 'sits' inside a uniquely determined nilpotent ℓ-group $G(L)$ (of class c) as a submonoid and as a $\vee$-subsemilattice. Further, L can be seen as the image of $G(L)$ relative to a suitable conucleus.

Theorem 5.2.3 ([135] Lemmas 4.2–4.4)**.** *For any $c \in \mathbb{N}^+$, and any nilpotent cancellative residuated lattice L of class c, the relation $\preceq\, \subseteq G(L) \times G(L)$ defined, for $a, b, c, d \in L$, by*

$$a^{-1}b \preceq c^{-1}d \quad \Longleftrightarrow \quad \text{there exist } m, n \in L \text{ such that } mb \le nd \text{ and } ma = nc,$$

is the unique partial order on $G(L)$ that extends the lattice order $\le$ of L. Then, $G(L)$ with the resulting partial order is a nilpotent ℓ-group of class c, and the map

$$\sigma_L \colon G(L) \longrightarrow G(L); \qquad \sigma_L(a^{-1}b) = a\backslash b, \text{ for all } a, b \in L,$$

is a conucleus on $G(L)$ and $L = G(L)_{\sigma_L}$.

The main result of this section is Theorem 5.2.5, which characterizes monoids that embed into nilpotent ℓ-groups, and into nilpotent cancellative residuated lattices. We recall here a few relevant properties of nilpotent groups. In what follows, a monoid M is said to have *unique roots* if, whenever $a, b \in M$, and $a^n = b^n$ for some $n \in \mathbb{N}^+$, then $a = b$.

Lemma 5.2.4 (cf. [105, Theorems 16.2.3, 16.2.7 & 16.2.8])**.** *For any nilpotent group G:*

(a) *Every non-trivial normal subgroup of G intersects the center of G non-trivially.*
(b) *The set of torsion elements of G is a (normal) subgroup of G.*
(c) *If G is torsion-free, it has unique roots.*

These properties are useful to obtain the following result.

Theorem 5.2.5. *For any $c \in \mathbb{N}^+$ and any monoid M, the following are equivalent:*

(1) *M is a submonoid of a nilpotent ℓ-group of class c.*
(2) *M is nilpotent of class c, cancellative, and has unique roots.*
(3) *M has a group of quotients $G(M)$, that is nilpotent of class c and torsion-free.*
(4) *M is a submonoid of a totally ordered nilpotent group of class c.*
(5) *M is a submonoid of a nilpotent cancellative residuated lattice of class c.*

Proof. (1) ⇒ (2). Assume that M is a submonoid of a nilpotent ℓ-group G of class c. That M is nilpotent of class c is immediate by Proposition 5.2.1. It remains to show that M has unique roots. To this end, suppose that $a^n = b^n$ for some $n \in \mathbb{N}^+$, and $a, b \in M$. Then, $a^n = b^n$ in G. Now, since G is an ℓ-group, it is torsion-free, and by Lemma 5.2.4(c), $a = b$.

(2) ⇒ (3). Observe that $G(M)$ exists and is nilpotent of class c by Proposition 5.2.2. Suppose now that $(a^{-1}b)^n = \mathrm{e}$, for some $a \neq b \in M$, and $n \in \mathbb{N}^+$. Then, $a^{-1}b$ is in the torsion subgroup of $G(M)$, which is normal by Lemma 5.2.4(b). By Lemma 5.2.4(a), its intersection with the center of $G(M)$ is non-trivial, and hence, there exists a central element $c^{-1}d \in G(M)$ such that $c \neq d \in M$, and $(c^{-1}d)^m = \mathrm{e}$ for some $m \in \mathbb{N}^+$. As $c^{-1}d$ is a central element of $G(M)$, $c(c^{-1}d) = (c^{-1}d)c$ or, equivalently, $dc^{-1} = c^{-1}d$. Thus, an easy induction on $m \in \mathbb{N}^+$ shows that

$$(c^{-1}d)^m = (c^{-1})^m d^m = \mathrm{e}.$$

This implies $c^m = d^m$, which contradicts the assumption that M has unique roots, since c and d are assumed to be distinct.

(3) ⇒ (4). It suffices to observe that $G(M)$ admits a total order, as it is torsion-free and nilpotent (see [14, Theorem 2.2.4]).

(4) ⇒ (5). This implication is trivial, as any totally ordered nilpotent group of class c is a nilpotent cancellative residuated lattice of class c.

(5) ⇒ (1). By assumption M is a submonoid of a nilpotent cancellative residuated lattice L of class c. Let $G(L)$ be the ℓ-group of quotients of L, as defined in Theorem 5.2.3. Since L is a submonoid of $G(L)$, the result follows. □

Observe that the class described by Theorem 5.2.5 is a quasivariety.

5.3 PRELINEARITY AND ITS IMPLICATIONS

The remainder of this chapter will be concerned with classes of prelinear residuated lattices. A residuated lattice is said to be *prelinear* if it satisfies the following equations:

$$\text{(LPL)}\quad (x\backslash y \wedge \mathrm{e}) \vee (y\backslash x \wedge \mathrm{e}) \approx \mathrm{e} \quad \text{and} \quad \text{(RPL)}\quad (x/y \wedge \mathrm{e}) \vee (y/x \wedge \mathrm{e}) \approx \mathrm{e}.$$

This section is devoted to exploring prelinearity, focussing on some of its implications and equivalent formulations. More precisely, Theorem 5.3.3 shows that residuals in a prelinear residuated lattice preserve finite joins in the numerator, and convert finite meets to joins in the denominator.

We call a residuated lattice L *representable* if L is a subdirect product of totally ordered residuated lattices. A variety V of residuated lattices is representable[2] if each subdirectly irreducible member of V is totally ordered.

Example 5.3.1. Prelinearity plays a central role in the study of 'algebras of logic'. Boolean algebras, Gödel algebras, and MV-algebras are prelinear and, furthermore, they are representable. As a matter of fact, a commutative residuated lattice is representable if and only if it is prelinear ([83]).

While prelinearity implies representability in the presence of commutativity, this is no longer the case in non-commutative settings. For example, any ℓ-group is prelinear; however, it is not true that all ℓ-groups are representable. Theorem 5.3.3 shows that prelinear cancellative residuated lattices have distributive lattice reducts, thereby providing an alternative proof that ℓ-groups have distributive lattice reducts.

We show that validity of the equations

$$\text{(LPL2)}\quad (y \wedge z)\backslash x \approx (y\backslash x) \vee (z\backslash x) \quad \text{and} \quad \text{(LPL3)}\quad x\backslash(y \vee z) \approx (x\backslash y) \vee (x\backslash z)$$
$$\text{(RPL2)}\quad x/(y \wedge z) \approx (x/y) \vee (x/z) \quad \text{and} \quad \text{(RPL3)}\quad (y \vee z)/x \approx (y/x) \vee (z/x)$$

is necessary in prelinear residuated lattices, and sufficient to obtain prelinearity in residuated lattices with a distributive lattice reduct. The following results can be partially found in [12, Proposition 6.10], and [6, Corollary 4.2]. We begin with a preliminary lemma.

Lemma 5.3.2 (cf. [6, Proposition 4.1])**.** *For any lattice L, the following are equivalent:*

(1) *L is distributive.*
(2) *For all $a, b \in L$ with $a \leq b$, there exists a $\vee$-endomorphism $f\colon L \to L$ such that $f(b) = a$ and $f(x) \leq x$, for all $x \in L$.*

Theorem 5.3.3.

[2] It is common to call *semilinear* those residuated lattices that are representable; hence, the variety of representable residuated lattices is denoted by SemRL.

(a) *Any prelinear residuated lattice satisfies* (LPL2) *and* (LPL3).

(b) *In any residuated lattice that satisfies* $\mathrm{e} \wedge (y \vee z) \approx (\mathrm{e} \wedge y) \vee (\mathrm{e} \wedge z)$, *if* (LPL2) *or* (LPL3) *holds, then* (LPL) *holds.*

(c) *Any prelinear cancellative residuated lattice has a distributive lattice reduct.*

Remark 5.3.4. Even though Theorem 5.3.3 is presented here only for (LPL), (LPL2), (LPL3), the dual arguments establish the analogous results for the equations (RPL), (RPL2), (RPL3). More precisely, the equations (RPL), (RPL2), (RPL3) are equivalent under the hypothesis of Theorem 5.3.3(b). Further, (RPL3) and $x/x \approx \mathrm{e}$ entail distributivity of the lattice reduct.

Proof of Theorem 5.3.3. For (a), we consider any residuated lattice L satisfying (LPL). For any $a, b, c \in L$,

$$(b \wedge c)\backslash a \geq (b\backslash a) \vee (c\backslash a).$$

To obtain the reverse inequality, and hence conclude (LPL2), it suffices to show

$$\mathrm{e} \leq [(b\backslash a) \vee (c\backslash a)]/[(b \wedge c)\backslash a].$$

Let $u = (b\backslash a) \vee (c\backslash a)$. Then, we have

$$\begin{aligned} u/[(b \wedge c)\backslash a] &\geq (b\backslash a)/[(b \wedge c)\backslash a] &(5.7)\\ &= b\backslash[a/[(b \wedge c)\backslash a]] &(5.8)\\ &\geq b\backslash(b \wedge c) &(5.9)\\ &= (b\backslash c) \wedge (c\backslash c) &(5.10)\\ &\geq (b\backslash c) \wedge \mathrm{e}, &(5.11)\end{aligned}$$

where (5.7), (5.9), (5.10), and (5.11) hold in view of (5.1) – (5.2) and Proposition 5.1.2, and (5.8) follows from Proposition 5.1.3. Likewise,

$$u/[(b \wedge c)\backslash a] \geq (c\backslash b) \wedge \mathrm{e}.$$

Hence,

$$u/[(b \wedge c)\backslash a] \geq [(b\backslash c) \wedge \mathrm{e}] \vee [(c\backslash b) \wedge \mathrm{e}] = \mathrm{e},$$

as was to be shown.

For (LPL3), observe that it is always the case that

$$(a\backslash b) \vee (a\backslash c) \leq a\backslash(b \vee c).$$

To establish the reverse inequality, we show that

$$[a\backslash(b \vee c)]\backslash[(a\backslash b) \vee (a\backslash c)] \geq \mathrm{e}.$$

Let $u = (a\backslash b) \vee (a\backslash c)$. We have

$$\begin{aligned}
[a\backslash(b\vee c)]\backslash u &\geq [a\backslash(b\vee c)]\backslash(a\backslash b) && (5.12)\\
&= [a(a\backslash(b\vee c))]\backslash b && (5.13)\\
&\geq (b\vee c)\backslash b && (5.14)\\
&= (b\backslash b)\wedge(c\backslash b) && (5.15)\\
&\geq (c\backslash b)\wedge \mathrm{e} && (5.16)
\end{aligned}$$

where (5.12), (5.14), (5.15), and (5.16) follow from (5.1) – (5.2), and from Proposition 5.1.2, and (5.13) follows from Proposition 5.1.3. Likewise,

$$[a\backslash(b\vee c)]\backslash u \geq (b\backslash c)\wedge \mathrm{e}.$$

Consequently,

$$[a\backslash(b\vee c)]\backslash u \geq [(c\backslash b)\wedge \mathrm{e}]\vee[(b\backslash c)\wedge \mathrm{e}] = \mathrm{e},$$

and the conclusion follows.

For (b), assume L satisfies (LPL2), and let $a, b, c \in L$. Then

$$\begin{aligned}
[(a\backslash b)\wedge \mathrm{e}]\vee[(b\backslash a)\wedge \mathrm{e}] &= [a\backslash(a\wedge b)\wedge \mathrm{e}]\vee[b\backslash(a\wedge b)\wedge \mathrm{e}] && (5.17)\\
&= [(a\backslash(a\wedge b))\vee(b\backslash(a\wedge b))]\wedge \mathrm{e} && (5.18)\\
&= [(a\wedge b)\backslash(a\wedge b)]\wedge \mathrm{e} && (5.19)\\
&\geq \mathrm{e}\wedge\mathrm{e} = \mathrm{e}, && (5.20)
\end{aligned}$$

where (5.17) and (5.20) follow from (5.1) – (5.2), and from Proposition 5.1.2, the equality (5.18) follows from the assumption, and (5.19) is a consequence of (LPL2).

Finally, assume L satisfies (LPL3), and let $a, b, c \in L$. Then

$$\begin{aligned}
[(a\backslash b)\wedge \mathrm{e}]\vee[(b\backslash a)\wedge \mathrm{e}] &= [(a\vee b)\backslash b)\wedge \mathrm{e}]\vee[(a\vee b)\backslash a)\wedge \mathrm{e}] && (5.21)\\
&= [((a\vee b)\backslash b)\vee((a\vee b)\backslash a)]\wedge \mathrm{e} && (5.22)\\
&= [(a\vee b)\backslash(a\vee b)]\wedge \mathrm{e} && (5.23)\\
&\geq \mathrm{e}\wedge\mathrm{e} = \mathrm{e}, && (5.24)
\end{aligned}$$

where (5.21) and (5.24) follow from (5.1) – (5.2), and from Proposition 5.1.2, the equality (5.22) follows from the assumption, and (5.23) is a consequence of (LPL3).

For (c), we show a stronger result than the one stated above, as it suffices to assume that (LPL3) and $x\backslash x \approx \mathrm{e}$ hold in L to obtain the conclusion. For any $a \leq b \in L$, define

$$f\colon L \longrightarrow L, \qquad f(x) = a(b\backslash x).$$

The fact that f is a $\vee$-endomorphism follows from

$$\begin{aligned}
a(b\backslash(x\vee y)) &= a((b\backslash x)\vee(b\backslash y)) && (5.25)\\
&= a(b\backslash x)\vee a(b\backslash y), && (5.26)
\end{aligned}$$

where (5.25) follows from (LPL3), and (5.26) by Proposition 5.1.2. Further, we have

$$f(b) = a(b\backslash b) = a$$

by assumption, and $f(x) \leq x$ since

$$a \leq b \quad \Longrightarrow^{(1)} \quad b\backslash x \leq a\backslash x \quad \Longrightarrow^{(2)} \quad a(b\backslash x) \leq x,$$

where we get (1) by Proposition 5.1.2, and (2) by (5.1). The conclusion follows from Lemma 5.3.2. □

Theorem 5.3.3(c) provides an alternative proof that ℓ-groups have distributive lattice reducts.

Following the proof of Theorem 5.3.3(c), it is easy to see that every prelinear integral residuated lattice has a distributive lattice reduct, as it satisfies (LPL3) and $x\backslash x \approx \mathrm{e}$. Finally, in the case of cancellative (resp., integral) residuated lattices, prelinearity is equivalent to (LPL3) and (RPL3). The left-to-right direction is immediate from Theorem 5.3.3(a). For the converse, observe that (LPL3) and cancellativity (resp., integrality) together entail distributivity of the lattice reduct. Therefore, by Theorem 5.3.3(b), (LPL) must hold.

5.4 CANCELLATIVITY AND PRELINEARITY: HAMILTONIAN VARIETIES

This section is devoted to Hamiltonian residuated lattices. A residuated lattice is said to be *Hamiltonian* if its convex subalgebras are normal, i.e., every convex subalgebra is closed under left and right conjugation as defined in (5.6). A variety V of residuated lattices is Hamiltonian if every member of V is Hamiltonian.

The fact that Hamiltonian ℓ-groups are representable is extended here to prelinear e-cyclic residuated lattices. More precisely, Theorem 5.4.2 shows that (LPL) and (RPL) provide an axiomatization for representability relative to any variety of Hamiltonian e-cyclic residuated lattices. Later, this is used to show that a largest variety of Hamiltonian prelinear cancellative residuated lattices exists, thereby extending the analogous result for ℓ-groups.

The following result generalizes an analogous result for representable ℓ-groups (cf. (1.3)).

Proposition 5.4.1 ([15, Theorem 5.6]). *For any residuated lattice L, the following are equivalent:*

(1) *L is representable.*

(2) *L is prelinear, and it satisfies the quasiequation:*

$$x \vee y \approx \mathrm{e} \quad \Longrightarrow \quad \lambda_u(x) \vee \rho_v(y) \approx \mathrm{e}. \tag{5.27}$$

The laws (LPL) and (RPL) hold in all totally ordered residuated lattices and hence in all representable residuated lattices. In the Hamiltonian e-cyclic case, the converse also holds.

Theorem 5.4.2. *Any Hamiltonian prelinear* e*-cyclic residuated lattice is representable.*

Proof. Let L be a Hamiltonian e-cyclic residuated lattice satisfying the prelinearity laws, and suppose that $a \vee b = \mathrm{e}$, for $a, b \in L$. Then,

$$\mathrm{e} = \mathfrak{C}(a \vee b) = \mathfrak{C}(a) \cap \mathfrak{C}(b)$$

by Lemma 5.1.9. As L is Hamiltonian, for any $c, d \in L$, we have $\lambda_c(a) \in \mathfrak{C}(a)$, and $\rho_d(b) \in \mathfrak{C}(b)$. Therefore, again by Lemma 5.1.9,

$$\begin{aligned}\mathfrak{C}(\lambda_c(a) \vee \rho_d(b)) &= \mathfrak{C}(\lambda_c(a)) \cap \mathfrak{C}(\rho_d(b)) \\ &\subseteq \mathfrak{C}(a) \cap \mathfrak{C}(b) \\ &= \mathrm{e},\end{aligned}$$

and hence, $\lambda_c(a) \vee \rho_d(b) = \mathrm{e}$. □

By Theorem 5.4.2, prelinear commutative residuated lattices are representable [83].

Corollary 5.4.3. *Every Hamiltonian prelinear cancellative residuated lattice is representable.*

Proof. The conclusion follows from Proposition 5.1.8 and Theorem 5.4.2. □

It was mentioned in Chapter 2 that the variety of weakly Abelian ℓ-groups is the largest variety of Hamiltonian ℓ-groups. We extend this result to the context of prelinear cancellative residuated lattices, providing an axiomatization for the largest Hamiltonian variety that generalizes the equation (2.8) defining weakly Abelian ℓ-groups. It was shown that the analogous result fails for e-cyclic residuated lattices (cf. [15, Theorem 6.3]).

Theorem 5.4.4. *There exists a largest variety of Hamiltonian prelinear cancellative residuated lattices. More precisely, a variety* V *of prelinear cancellative residuated lattices is Hamiltonian if and only if* V *satisfies the equations*

$$(x \wedge \mathrm{e})^2 \leq \lambda_y(x) \quad \textit{and} \quad (x \wedge \mathrm{e})^2 \leq \rho_y(x), \tag{5.28}$$

where λ_y *and* ρ_y *are defined as in (5.6).*

Proof. Given $L \in \mathsf{V}$, and a convex subalgebra $H \in \mathcal{C}(L)$, $a \in H$ implies $(a \wedge \mathrm{e})^2 \in H$. Hence, by convexity, since

$$(a \wedge \mathrm{e})^2 \leq (b \backslash ab) \wedge \mathrm{e} \leq \mathrm{e} \quad \text{and} \quad (a \wedge \mathrm{e})^2 \leq (ba / b) \wedge \mathrm{e} \leq \mathrm{e}$$

we get $\lambda_b(a), \rho_b(a) \in H$, for any $b \in L$. Thus, H is normal.

For the remaining direction, we seek a contradiction. Suppose that there exists a variety V of prelinear cancellative residuated lattices that does not satisfy either one of the equations in (5.28). Then, by Corollary 5.4.3, there exists a chain $T \in \mathsf{V}$ and an element $a \in T^-$ such that $a^2 \not\leq b\backslash ab \wedge \mathrm{e}$ for some $b \in T$ or, by cancellativity,

$$ab < ba^2 \quad \text{for some } b \in T. \tag{5.29}$$

Condition (5.29) can be used to construct a non-Hamiltonian member $L \in \mathsf{V}$, contradicting the assumption. Consider

$$L = \prod_{i \in \mathbb{N}} L_i,$$

where L_i is a copy of T for every $i \in \mathbb{N}$. Let $\bar{a}, \bar{b} \in L$ be the elements $\bar{a}(i) = a$ and $\bar{b}(i) = b^i$. We want to conclude that the element $\lambda_{\bar{b}}(\bar{a}) = b^i\backslash ab^i$ witnesses the failure of the Hamiltonian property for L, that is, $\lambda_{\bar{b}}(\bar{a}) \notin \mathfrak{C}(\bar{a})$. To this aim, by Lemma 5.1.9, it suffices to show that $(\bar{a})^n \not\leq \bar{b}$ for any $n \in \mathbb{N}^+$. First, we show:

Claim 5. *For any $n \in \mathbb{N}^+$, $a^n b < ba^{2n}$.*

Proof. We proceed by induction on $n \in \mathbb{N}^+$. The base case follows from (5.29). For the induction step, observe that

$$\begin{aligned} a^{n+1}b &= aa^n b \\ &< aba^{2n} \qquad (5.30) \\ &< ba^2a^{2n} \qquad (5.31) \\ &= ba^{2(n+1)}, \end{aligned}$$

where (5.30) follows from the induction hypothesis, and (5.31) from (5.29). □

Claim 6. *For any $n \in \mathbb{N}^+$, $ab^n < b^n a^{2^n}$.*

Proof. We proceed by induction on $n \in \mathbb{N}^+$. The base case follows from (5.29). For the induction step, observe that

$$\begin{aligned} ab^{n+1} &= ab^n b \\ &< b^n a^{2^n} b \qquad (5.32) \\ &< b^n b a^{2 \cdot 2^n} \qquad (5.33) \\ &= b^{n+1} a^{2^{(n+1)}}, \end{aligned}$$

where (5.32) follows by the induction hypothesis, and (5.33) by Claim 5. □

Therefore, for any $n \in \mathbb{N}^+$ there exists $m \in \mathbb{N}^+$ such that $b^m\backslash ab^m < a^n$, namely,

$$ab^n < b^n a^{2^n} < b^n a^n,$$

since $a \in T^-$, and T is cancellative. □

We use the term *weakly Abelian* for a residuated lattice that satisfies (5.28).

We conclude this section with an easy consequence of Corollary 5.4.3. Recall that a class K of algebras is said to have the *amalgamation property* if for all $A, B, C \in \mathsf{K}$, and any embeddings $i\colon A \hookrightarrow B$ and $j\colon A \hookrightarrow C$, there exist $D \in \mathsf{K}$, and embeddings $h\colon B \hookrightarrow D$ and $k\colon C \hookrightarrow D$ making the following diagram

$$\begin{array}{ccc} B & \overset{h}{\dashrightarrow} & D \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle k} \\ A & \underset{j}{\longrightarrow} & C \end{array} \tag{5.34}$$

commute. The algebra D (sometimes, the triple (D, h, k)) is called an amalgam for A, B, C. We use the fact that the variety of weakly Abelian ℓ-groups fails the amalgamation property [75] to show that the same happens for the variety of weakly Abelian prelinear cancellative residuated lattices. The argument is based on ideas from [70] (cf. Lemma 4.2, and Theorem 4.3), and we sketch it here for the interested reader.

Lemma 5.4.5. *For any representable residuated lattice L, the set of invertible elements of L, i.e.,*

$$\mathrm{Inv}(L) = \{a \in L \mid \text{ there exists a unique } b \in L \text{ such that } ab = \mathrm{e} = ba\},$$

is a subalgebra of L which is an ℓ-group.

Proof. First, note that it is immediate to check that the identities

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \quad\text{and}\quad z(x \wedge y)w \approx zxw \wedge zyw \tag{5.35}$$

hold in a totally ordered residuated lattice and hence, in a representable one. The set $\mathrm{Inv}(L)$ is obviously a submonoid. Further, if $a, b \in \mathrm{Inv}(L)$ and $c \in L$, it is immediate that $a(a^{-1}b) = b \leq b$, and that from $ac \leq b$ it follows $c \leq a^{-1}b$. Therefore, the submonoid $\mathrm{Inv}(L)$ is closed under residuals. Moreover, for all $a, b \in \mathrm{Inv}(L)$, the residual $a\backslash b$ coincides with $a^{-1}b$, and b/a coincides with ba^{-1}. It remains to show that $\mathrm{Inv}(L)$ is a sublattice. For this, observe that

$$a \wedge a^{-1} \leq \mathrm{e},$$

for any $a \in \mathrm{Inv}(L)$. In fact, since $a \wedge a^{-1} \leq a, a^{-1}$, also $(a \wedge a^{-1})^2 \leq aa^{-1} = \mathrm{e}$. Moreover, for every element $b \in L$, if $b^2 \leq \mathrm{e}$, also

$$(b \wedge \mathrm{e})^2 = b^2 \wedge b \wedge \mathrm{e} = b \wedge \mathrm{e},$$

where the first equality follows using (5.35). Hence, $b \wedge \mathrm{e} = \mathrm{e}$ by cancellativity. Thus, $a \wedge a^{-1} \leq \mathrm{e}$. Similarly, and dually, we can show that

$$\mathrm{e} \leq a \vee a^{-1}.$$

Therefore, for $a, b \in \mathrm{Inv}(L)$, we obtain the following:

$$\begin{aligned}(a \wedge b)(a^{-1} \vee b^{-1}) &= (aa^{-1} \wedge ab^{-1}) \vee (ba^{-1} \wedge bb^{-1}) \\ &= (\mathrm{e} \wedge ab^{-1}) \vee (ba^{-1} \wedge \mathrm{e}) \\ &= \mathrm{e} \wedge (ab^{-1} \vee ba^{-1}) \\ &= \mathrm{e}.\end{aligned}$$

Analogously, $(a^{-1} \vee b^{-1})(a \wedge b) = \mathrm{e}$. Hence, $\mathrm{Inv}(L)$ is a subalgebra of L which is an ℓ-group. □

Suppose that the variety of weakly Abelian prelinear cancellative residuated lattices satisfies the amalgamation property. Then, since weakly Abelian ℓ-groups are weakly Abelian prelinear cancellative residuated lattices, a pair of weakly Abelian ℓ-groups B, C with a common ℓ-subgroup A must have an amalgam D in the variety of weakly Abelian prelinear cancellative residuated lattices. Let $h\colon B \hookrightarrow D$ and $k\colon C \hookrightarrow D$ be the resulting embeddings as described in (5.34). The images $h[B]$ and $k[C]$ are subalgebras of D that are also ℓ-groups. By Corollary 5.4.3 and Lemma 5.4.5, $h[B]$ and $k[C]$ are in fact subalgebras of $\mathrm{Inv}(D)$, which is therefore an amalgam for A, B, C in the variety of weakly Abelian ℓ-groups. Hence:

Corollary 5.4.6. *The variety of weakly Abelian prelinear cancellative residuated lattices does not have the amalgamation property.*

5.5 NILPOTENT PRELINEAR CANCELLATIVE RESIDUATED LATTICES

The preceding section demonstrates that Hamiltonian prelinear cancellative residuated lattices bear striking similarities to Hamiltonian ℓ-groups. We now move on to the study of nilpotent prelinear cancellative residuated lattices. It is known that nilpotent ℓ-groups are representable ([109]; cf. [92, Theorem 4]), and Hamiltonian ([148, Theorem 2.4]; cf. [109, Corollary 2]). The main result of this section is Theorem 5.5.1, where nilpotent cancellative residuated lattices are in fact proved to be Hamiltonian. As a consequence, we get that nilpotent prelinear cancellative residuated lattices are representable.

Theorem 5.5.1. *Every nilpotent cancellative residuated lattice is Hamiltonian.*

Proof. Let L be a nilpotent cancellative residuated lattice. By Theorem 5.4.4, it suffices to show that L satisfies the equations (5.28). For this, pick $c, d \in L$, with $c \leq \mathrm{e}$. Then, both $dc^2 \preceq cd$ and $c^2d \preceq dc$ hold in $G(L)$, since the latter is a nilpotent, and hence Hamiltonian, ℓ-group. Since L is a submonoid of $G(L)$ (see Theorem 5.2.3), and the restriction of the order $\preceq$ to L is the order $\leq$ of L, it follows that $dc^2 \leq cd$

and $c^2 d \leq dc$ hold in L. Therefore, using the equations (5.3) we can conclude that L satisfies $c^2 \leq d\backslash cd$ and $c^2 \leq dc/d$, for $c, d \in L$ with $c \leq \mathrm{e}$. Thus, for all $a, b \in L$,

$$(a \wedge \mathrm{e})^2 \leq b\backslash(a \wedge \mathrm{e})b \leq (b\backslash ab) \wedge \mathrm{e} \quad \text{and} \quad (a \wedge \mathrm{e})^2 \leq b(a \wedge \mathrm{e})/b \leq (ba/b) \wedge \mathrm{e},$$

as was to be shown. □

Remark 5.5.2. For the variety $\mathsf{N}^2\mathsf{CanRL}$ of nilpotent cancellative residuated lattices of class 2, we also provide a direct argument, without going through Theorem 5.2.3. Pick any residuated lattice $L \in \mathsf{N}^2\mathsf{CanRL}$. Then, for $a, b \in L$,

$$b(a \wedge \mathrm{e})\mathrm{e}(a \wedge \mathrm{e})b = (a \wedge \mathrm{e})b\mathrm{e}b(a \wedge \mathrm{e}) \tag{5.36}$$
$$\leq b^2(a \wedge \mathrm{e}) \tag{5.37}$$
$$\leq b^2 a, \tag{5.38}$$

where (5.36) follows from the equation L_2, and (5.37) and (5.38) follow from $(a \wedge \mathrm{e}) \leq a, \mathrm{e}$. Thus,

$$(a \wedge \mathrm{e})^2 b = b\backslash b(a \wedge \mathrm{e})^2 b \tag{5.39}$$
$$\leq b\backslash b^2 a \tag{5.40}$$
$$= ba, \tag{5.41}$$

where (5.39) and (5.41) follow from (5.3), and (5.40) follows from what we showed above, together with Proposition 5.1.2. The other equation can be proved similarly.

From Theorem 5.4.2 and Theorem 5.5.1, we can conclude that nilpotent prelinear cancellative residuated lattices are representable. We also give an alternative argument, based on Theorem 5.2.3.

Theorem 5.5.3. *Nilpotent prelinear cancellative residuated lattices are representable.*

Proof. Let L be a nilpotent prelinear cancellative residuated lattice and let $G(L)$ be its ℓ-group of quotients. We show that L satisfies (5.27) of Proposition 5.4.1. Let $a, b, c \in L$, and assume $a \vee b = \mathrm{e}$. By Theorem 5.2.3, $a \vee b = \mathrm{e}$ holds in the nilpotent ℓ-group $G(L)$. This implies that $c^{-1}ac \vee b = \mathrm{e}$ by Proposition 5.4.1, as nilpotent ℓ-groups are representable. Hence, also $ac \vee cb = c$ in L, and therefore,

$$c\backslash(ac \vee cb) = c\backslash c = \mathrm{e}.$$

Now, by Theorem 5.3.3(a), we get $c\backslash ac \vee c\backslash cb = \mathrm{e}$, that is, $\lambda_c(a) \vee b = \mathrm{e}$. Similarly, and by Remark 5.3.4, we can conclude that $\lambda_c(a) \vee \rho_d(b) = \mathrm{e}$. □

Given that the variety of nilpotent ℓ-groups of class c does not have the amalgamation property [166, Theorem 2.2], we can proceed as in Section 5.4 and conclude that amalgamation also fails for the variety of nilpotent prelinear cancellative residuated lattices of class c.

Corollary 5.5.4. *For any $c \in \mathbb{N}^+$, the variety of nilpotent prelinear cancellative residuated lattices of class c does not have the amalgamation property.*

5.6 ORDERING INTEGRAL RESIDUATED LATTICES

The results of the preceding sections provide strong evidence of the importance of the notion of representability in the study of Hamiltonian and nilpotent prelinear cancellative varieties. The present section is concerned with the variety $\mathsf{SemCanIRL}$ of representable cancellative integral residuated lattices. In particular, we consider *monoid-subvarieties* of $\mathsf{SemCanIRL}$, i.e., those classes defined relative to $\mathsf{SemCanIRL}$ by monoid equations. Since the setting is fixed throughout the section, we often refer to a monoid-subvariety V of $\mathsf{SemCanIRL}$ simply as a 'monoid-variety'. For instance, commutative representable cancellative integral residuated lattices form a monoid-variety; clearly, also nilpotent representable cancellative integral residuated lattices of class c form a monoid-variety.

It is known that the subgroups of nilpotent ℓ-groups are precisely the torsion-free nilpotent groups or, equivalently, nilpotent groups that admit a total order. In view of Theorem 5.2.5, it is natural to ask whether every nilpotent cancellative monoid with unique roots admits a (residuated) total order. We provide a partial answer to this question, and show that any finitely generated free monoid relative to the quasivariety of nilpotent cancellative monoids with unique roots admits an integral (residuated) total order (Lemma 5.6.8). This leads to a concrete description of the generating algebras in the variety of nilpotent representable cancellative residuated lattices of class c.

We begin by defining a few relevant notions. For any monoid M, we say that M admits a residuated total order if it admits a total order $\leq$ that is residuated (i.e., $\cdot$ is residuated with respect to $\leq$), and sometimes write $(M,\leq)$ for the resulting residuated lattice. It is immediate that any total order on a group is a residuated total order. Finally, a residuated lattice admits a (residuated) total order if its underlying monoid admits a (residuated) total order that extends its lattice order. We call a total order $\leq$ on a monoid (not necessarily residuated) *integral* if the monoid identity is the greatest element with respect to $\leq$. We say that a poset P satisfies the *ascending chain condition* (ACC) if P does not contain any infinite (strictly) ascending chain. By (5.2), if a total order on a monoid M satisfies the ACC, then it is a residuated total order.

Lemma 5.6.1. *Every integral total order on a finitely generated monoid is residuated.*

Proof. Let M be a monoid generated by n elements, and set $\leq$ to be an integral total order on M. Then, there exists a surjective monoid homomorphism φ from the free monoid $M(x_1,\dots,x_n) = M(n)$ over n generators to M. We show that $(M,\leq)$ satisfies the ACC. Suppose that

$$m_0 < m_1 < m_2 < \cdots < m_i < \cdots,$$

is an infinite ascending chain in $(M,\leq)$. As φ is onto, $\varphi^{-1}[\{m_i\}] \neq \emptyset$ for all $i \in \mathbb{N}$. Consider

$$\{t_i = f(\varphi^{-1}[\{m_i\}]) \mid i \in \mathbb{N}\},$$

where $f\colon \mathbb{N} \to \bigcup_{i\in\mathbb{N}} \varphi^{-1}[\{m_i\}]$ is a choice function. Then, $\{t_i\}$ is an infinite sequence of words over the finite alphabet $\{x_1,\dots,x_n\}$. By Higman's Lemma [85], there must be indices $i < j$ such that t_i can be obtained from t_j by deleting some symbols: e.g.,

$$t_i = x_{i_1} \cdots x_{i_k} \text{ and } t_j = s_{j_0} x_{i_1} s_{j_1} \cdots x_{i_k} s_{j_k},$$

where $s_{j_0}, s_{j_1}, \ldots, s_{j_k}$ are arbitrary words in $M(n)$. Then, $\varphi(t_i) = m_i < m_j = \varphi(t_j)$, which entails

$$\varphi(x_{i_1}) \cdots \varphi(x_{i_k}) < \varphi(s_{j_0})\varphi(x_{i_1})\varphi(s_{j_1}) \cdots \varphi(x_{i_k})\varphi(s_{j_k}).$$

This is a contradiction, since for all $a, b \in M$, $ab \le a, b$ due to the integrality of the order $\le$. Therefore, $(M, \le)$ satisfies the ACC, and hence it is residuated. □

Let V be any monoid-subvariety of SemCanIRL, and write $\mathcal{M}(\mathsf{V})$ for the class of monoid subreducts of V, that is, those monoids that are submonoids of (the monoid reduct of) a residuated lattice from V. Observe that, as the variety V is defined relative to SemCanIRL by a set Σ of monoid equations, every member of the quasivariety $\mathcal{M}(\mathsf{V})$ satisfies Σ.

Lemma 5.6.2. *For any monoid-subvariety* V *of* SemCanIRL*, every finitely generated monoid in the quasivariety* $\mathcal{M}(\mathsf{V})$ *is the monoid reduct of a totally ordered member of* V.

Proof. Let M be a finitely generated member of $\mathcal{M}(\mathsf{V})$ and a submonoid of a member L of V. Since V is representable, L is the subdirect product of cancellative integral residuated chains L_i, $i \in I$. Let $\le$ be a well-order on I, and for $a = (a_i)_{i \in I}$, $b = (b_i)_{i \in I} \in L$, set

$$a \trianglelefteq b \iff a = b \text{ or } (a_j < b_j, \text{ where } j = \min\{i \in I \mid a_i \ne b_i\}).$$

We claim that $\trianglelefteq$ is an integral total order on L extending its lattice order. Indeed, let a, b, c be elements of L such that $a \triangleleft b$. Then $a_j < b_j$ for $j = \min\{i \in I \mid a_i \ne b_i\}$. By cancellativity, $a_j c_j < b_j c_j$ (resp., $c_j a_j < c_j b_j$), and hence, $ac \triangleleft bc$ (resp., $ca \triangleleft cb$). The restriction of the total order $\trianglelefteq$ to the finitely generated monoid M is residuated by Lemma 5.6.1. Moreover, as V is a monoid-subvariety of SemCanIRL and M is a submonoid of L, $(M, \le)$ is a member of V. □

Let $T^r(X)$ be the term algebra over a set X for the language of residuated lattices. Every variety of representable residuated lattices is generated by its finitely generated totally ordered members. In the case of monoid-varieties, we have the following stronger result.

Lemma 5.6.3. *Every monoid-subvariety of* SemCanIRL *is generated by the class of residuated chains whose monoid reduct is a finitely generated monoid.*

Proof. For any monoid-subvariety V of SemCanIRL, we show that an equation $t_1 \approx t_2$ that fails in V necessarily fails in a V-chain whose monoid reduct is a finitely generated monoid. Let $t_1(x_1, \ldots, x_m)$ and $t_2(x_1, \ldots, x_m)$ be two residuated lattice terms such that

$$\varphi(t_1) = t_1(\varphi(x_1), \ldots, \varphi(x_m)) \ne t_2(\varphi(x_1), \ldots, \varphi(x_m)) = \varphi(t_2),$$

under the valuation $\varphi\colon T^r(X) \to C$. Let $\mathsf{sub}(t_1)$ and $\mathsf{sub}(t_2)$ denote, respectively, the set of all subterms of t_1 and the set of all subterms of t_2. Let M be the submonoid of C generated by the finite set

$$\{\varphi(u) \mid u \in \mathsf{sub}(t_1) \cup \mathsf{sub}(t_2)\},$$

and consider the restriction of the order $\leq$ from C to M. The resulting integral residuated lattice $(M, \leq\restriction_M)$ is a submonoid, and a sublattice, of C—although it need not be a substructure, since residuals might not be preserved. Consider the valuation $\psi\colon T^r(X) \to M \subseteq C$, defined by $\psi(x_i) = \varphi(x_i)$ for any $1 \leq i \leq m$. We show that $\psi(u) = \varphi(u)$, for every $u \in \mathsf{sub}(t_1) \cup \mathsf{sub}(t_2)$, by induction on the structure of u. The base case is trivial, since it follows from the definition of ψ. The cases involving monoid operation ($u = u_1 \cdot u_2$), and the lattice operations ($u = u_1 \wedge u_2$ or $u = u_1 \vee u_2$) follows from the fact that M is a submonoid and a sublattice of C. Suppose that $u = u_1 \backslash u_2$. It suffices to show

$$\psi(u_1) \backslash_M \psi(u_2) = \psi(u_1) \backslash_C \psi(u_2).$$

By induction hypothesis, $\psi(u_1) = \varphi(u_1)$ and $\mu(u_2) = \varphi(u_2)$. Therefore,

$$\psi(u_1) \backslash_C \psi(u_2) = \varphi(u_1) \backslash_C \varphi(u_2) = \varphi(u_1 \backslash u_2) \in M.$$

Hence, we can conclude that

$$\psi(u) = \psi(u_1) \backslash_M \psi(u_2) = \psi(u_1) \backslash_C \psi(u_2) = \varphi(u_1) \backslash_C \varphi(u_2) = \varphi(u),$$

as was to be shown. Therefore, $\psi(t_1) \neq \psi(t_2)$ in M, and $t_1 \approx t_2$ fails in $(M, \leq\restriction_M)$. $\square$

In what follows, we write $M_{\mathsf{V}}(X)$ to denote the free monoid over a set X relative to the quasivariety $\mathcal{M}(\mathsf{V})$. The next result shows that it suffices to consider residuated chains over finitely generated free monoids relative to $\mathcal{M}(\mathsf{V})$.

Lemma 5.6.4. *For any monoid-subvariety* V *of* SemCanIRL*, every integral residuated chain whose monoid reduct is finitely generated is a homomorphic image of a residuated chain whose monoid reduct is a finitely generated free monoid relative to* $\mathcal{M}(\mathcal{V})$.

Proof. Let C be a chain member of V whose monoid reduct is finitely generated by $\{a_1, \dots, a_n\}$, and set $\varphi\colon M_{\mathsf{V}}(n) \twoheadrightarrow C$ to be the monoid homomorphism extending $x_i \mapsto a_i$, for $i \in \{1, \dots, n\}$. Since $M_{\mathsf{V}}(n)$ is a member of $\mathcal{M}(\mathsf{V})$, we can consider by Lemma 5.6.2 a residuated total order $\leq$ on $M_{\mathsf{V}}(n)$ such that the resulting algebra is in V. Let $\preceq$ be the total order on C. We now modify the order $\leq$, making φ into an order-preserving map relative to the modified order. Define

$$s \leq^* t \quad \Longleftrightarrow \quad \varphi(s) \prec \varphi(t) \text{ or } (\varphi(s) = \varphi(t) \text{ and } s \leq t),$$

for $s, t \in M_{\mathsf{V}}(n)$. The binary relation $\leq^*$ is an integral residuated total order on $M_{\mathsf{V}}(n)$ and hence, $(M_{\mathsf{V}}(n), \leq^*) \in \mathsf{V}$. Further, the map φ is order-preserving, and hence can be lifted to a residuated lattice homomorphism from $(M_{\mathsf{V}}(n), \leq^*)$ onto C. $\square$

Theorem 5.6.5. *Every monoid-subvariety* V *of* $\mathsf{SemCanIRL}$ *is generated by the class of residuated chains whose monoid reducts are finitely generated free monoids in* $\mathcal{M}(\mathcal{V})$.

Proof. Immediate by Lemma 5.6.3 and Lemma 5.6.4. □

We next specialize the preceding results to the nilpotent case. Nilpotent representable cancellative integral residuated lattices of class c form a monoid-subvariety of $\mathsf{SemCanIRL}$, which we denote by $\mathsf{N}^c\mathsf{SemCanIRL}$. We write $M_c(X)$ for the free cancellative nilpotent monoid of class c over X.

Lemma 5.6.6. *For any* $c \in \mathbb{N}^+$ *and any set* X, *the monoid* $\langle X\rangle_{\mathrm{e}}$ *generated by* X *in the free nilpotent group* $F_c(X)$ *of class* c *is the free cancellative nilpotent monoid* $M_c(X)$ *of class* c *over* X. *Further,* $F_c(X)$ *is the group of quotients of* $M_c(X)$.

Proof. First, observe that $\langle X\rangle_{\mathrm{e}}$ is nilpotent of class c, cancellative and, in view of Theorem 5.2.5, has unique roots. Therefore, the unique monoid homomorphism

$$\gamma\colon M_c(X) \longrightarrow \langle X\rangle_{\mathrm{e}}$$

extending the identity map on X exists by the universal property of $M_c(X)$, and is clearly onto, since $\langle X\rangle_{\mathrm{e}}$ is generated by X as a monoid. Further, observe that $M_c(X)$ is Ore and hence, it sits as a submonoid inside a nilpotent group H of class c. Thus, there exists a unique group homomorphism

$$\delta\colon F_c(X) \longrightarrow H$$

extending the identity map on X. This map restricts to a surjective monoid homomorphism

$$\hat{\delta}\colon \langle X\rangle_{\mathrm{e}} \longrightarrow M_c(X),$$

since $M_c(X)$ is generated by X. Thus, γ and δ are inverses to each other. Finally, the second part of the statement follows from the fact that the group of quotients of the monoid $\langle X\rangle_{\mathrm{e}}$ exists and is the free nilpotent group $F_c(X)$ of class c. This is because any group generated by an Ore monoid M is a group of quotients of M (see, e.g., [28, Section 1.10]). □

Note that Lemma 5.6.6 entails that the free cancellative nilpotent monoid $M_c(X)$ of class c over X has unique roots. Let us remark that Lemma 5.6.6 could also be obtained as a consequence of a more general result that can be found in [175, §5].

Remark 5.6.7. We briefly consider here the problem of characterizing submonoids of nilpotent cancellative integral residuated lattices of class $c \in \mathbb{N}^+$. By Theorem 5.2.3, the problem can be rephrased as follows: characterize the monoids that can be embedded into the negative cone of a nilpotent ℓ-group of class c. For the commutative case, the following holds: a monoid M can be embedded into the negative cone (equivalently, positive cone) of an Abelian ℓ-group if and only if M is commutative, cancellative, has unique roots, and (*) does not contain any (non-trivial) invertible element. It is clear that (*) is necessary. To see that it suffices, let M be a submonoid of a torsion-free Abelian group G such that $M \cap M^{-1} = \{\mathrm{e}\}$. Then, M is the negative cone of a partial order on G. Since every partial order on a torsion-free Abelian group G extends to a total order (cf. Proposition 1.1.8), M can be extended to the negative cone of a total order on G, and hence can be embedded into the negative cone of an Abelian totally ordered group. For non-commutative ℓ-groups the condition (*) does not suffice. A submonoid of a torsion-free nilpotent group G of class $c \geq 2$ satisfying (*) is, in general, the negative cone of a partial right order on G, and it is not true that any partial right order can be extended to a total order on G [51].

Lemma 5.6.8. *Every finitely generated free cancellative nilpotent monoid of class c admits an integral residuated total order.*

Proof. Let $F_c(n)$ be the free nilpotent group of class c generated by $X = \{x_1, \dots, x_n\}$. We consider a total order $\leq$ on $F_c(n)$. This is possible since $F_c(n)$ is torsion-free. Let $\langle X^\delta\rangle_{\mathrm{e}}$ be the submonoid of $F_c(n)$ generated by $X^\delta = \{x_1^{\delta_1}, \dots, x_n^{\delta_n}\}$, with $\delta_i \in \{-1, 1\}$ and $x_i^{\delta_i} < \mathrm{e}$ for each $i \in \{1, \dots, n\}$. The restriction of the total order $\leq$ to $\langle X^\delta\rangle_{\mathrm{e}}$ induces an integral residuated total order on $\langle X^\delta\rangle_{\mathrm{e}}$, by Lemma 5.6.1. Now, we conclude by observing that $\langle X^\delta\rangle_{\mathrm{e}}$ is isomorphic to $\langle X\rangle_{\mathrm{e}}$, and hence to $M_c(n)$ by Lemma 5.6.6. For this, it suffices to consider the unique group homomorphism

$$\alpha\colon F_c(n) \longrightarrow F_c(n),$$

extending the map $x_i \mapsto x_i^{\delta_i}$. This is a group automorphism of $F_c(n)$, whose restriction to $\langle X\rangle_{\mathrm{e}}$ is a monoid isomorphism onto $\langle X^\delta\rangle_{\mathrm{e}}$. □

We now give an example of the construction described in Lemma 5.6.8. The resulting residuated lattice is a non-commutative nilpotent cancellative integral residuated chain, which is neither an ℓ-group nor the negative cone of an ℓ-group.

Example 5.6.9. Consider the monoid $\langle\{x, y\}\rangle_{\mathrm{e}}$ generated by the variables in the free nilpotent group $F_2(x, y)$ of class 2. As was already mentioned in Example 1.4.4, the group $F_2(x, y)$ is isomorphic to the group $\mathrm{UT}_3(\mathbb{Z})$ of unitriangular matrices; the isomorphism in Example 1.4.4, obtained by extending a variable assignment, can be equivalently described on each element of $F_2(x, y)$ by

$$x^\alpha y^\beta [y, x]^\gamma \longmapsto \begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$$

(For instance, the variable x is represented by $\alpha = 1, \beta = \gamma = 0$, as described in Example 1.4.4.) Further, $\langle\{x, y\}\rangle_{\mathrm{e}}$ is isomorphic to the submonoid of $\mathrm{UT}_3(\mathbb{Z})$ whose underlying set is

$$\{A \in \mathrm{UT}_3(\mathbb{Z}) \mid \alpha, \beta, \gamma \in \mathbb{N} \text{ and } \gamma \leq \alpha\beta\}.$$

We consider the total order on $\langle\{x, y\}\rangle_{\mathrm{e}}$ induced by the (anti-)lexicographic order on the triples (α, β, γ). That is, if we identify a with $(\alpha_1, \beta_1, \gamma_1)$ and b with $(\alpha_2, \beta_2, \gamma_2)$, define

$$a \leq^* b \iff (\alpha_1, \beta_1, \gamma_1) \geq_{lex} (\alpha_2, \beta_2, \gamma_2).$$

The monoid equipped with the considered order is a nilpotent cancellative residuated chain that is neither an ℓ-group nor the negative cone of an ℓ-group—even though it is integral.

By Lemma 5.6.8, the free monoid in the quasivariety $\mathcal{M}(\mathsf{N^cSemCanIRL})$ over a finite set X coincides with the free cancellative nilpotent monoid $M_c(X)$ of class c. Hence:

Theorem 5.6.10. *For any* $c \in \mathbb{N}^+$*, the variety* $\mathsf{N^cSemCanIRL}$ *is generated by the class of residuated chains with monoid reducts* $\langle X \rangle_e$*, where* $\langle X \rangle_e$ *is the submonoid of the free nilpotent group* $F_c(X)$ *over* X *and* X *is an arbitrary finite set.*

Proof. Immediate by Theorem 5.6.5, Lemma 5.6.6, and Lemma 5.6.8. □

Theorem 5.6.10 provides, in particular, a generation result for the variety of commutative representable cancellative integral residuated lattices in terms of integral total orders on $\mathbb{N}^n$. Several generation and decidability results for commutative representable cancellative integral residuated lattices are available in the literature (see, e.g., [93, 94, 95, 96, 97]), and it might be worth studying how Theorem 5.6.10 relates to the available results in the commutative case.

5.7 CONCLUDING REMARKS

In this chapter, we studied nilpotent and Hamiltonian cancellative residuated lattices. The results obtained in this chapter are of interest from two different perspectives. First, Hamiltonian and nilpotent cancellative residuated lattices, in view of the results presented here, retain some of the key properties of commutative (cancellative) residuated lattices. For instance, nilpotent cancellative residuated lattices are Hamiltonian and, moreover, representable Hamiltonian cancellative residuated lattices are axiomatized by the equations for prelinearity. Second, it is reasonable to regard prelinear cancellative residuated lattices as a suitable generalization of ℓ-groups, in the sense that many of the relevant properties that are known to hold for ℓ-groups, are shown to hold for prelinear cancellative residuated lattices. This is not surprising, as prelinearity and cancellativity play a key role in the theory of ℓ-groups (e.g., as shown in Theorem 5.3.3, they are the reason why ℓ-groups are distributive as lattices). From this point of view, the results presented here contribute to the extension of the Conrad Program to the domain of (e-cyclic) residuated lattices.

The study of nilpotent cancellative residuated lattices carried out in this chapter is largely based on [124] and [142], where the notion of nilpotent semigroup and its relation with nilpotent groups is studied. In particular, in order to get a useful categorical equivalence (Theorem 5.2.3), it is of key importance that the group of quotients of a nilpotent semigroup always is nilpotent as a group. It is worth mentioning that Bernhard Neumann and Tekla Taylor prove another surprising result in [142]: any cancellative semigroup that satisfies a non-trivial equation is Ore, and therefore has a group of quotients. An immediate question is the following. Suppose that M is a monoid that satisfies the equation $s \approx t$, where s, t are monoid terms; determine what properties $s \approx t$ should satisfy for it to still be valid in $G(M)$ (this property is referred to in (semi)group theory as 'transferability'). This question was originally asked by George Bergman ([8]), and for this reason the problem is sometimes called the 'GB-Problem'; some answers are available in the literature (see, e.g., [114, 17, 120, 99, 122, 121]). More generally, given the equational properties of M, what can we say about the equational properties of $G(M)$? It would be interesting to study the available literature on this topic to solve the following problem.

Problem 15. Extend Theorem 5.2.3 to other varieties of cancellative residuated lattices and other varieties of ℓ-groups (defined by group equations) with a conucleus, which are defined by transferable equations.

Essentially all the results in this chapter generalize analogous results from the theory of ℓ-groups in a natural way; e.g., nilpotent cancellative residuated lattices are Hamiltonian (nilpotent ℓ-groups are Hamiltonian); nilpotent prelinear cancellative residuated lattices are representable (nilpotent ℓ-groups are representable). The proofs of these results rely on the analogous results for ℓ-groups. On the one hand, this allows us to make use of the categorical equivalence, thereby adding value to the beautiful correspondence between cancellative residuated lattices and ℓ-groups with a conucleus first established by Franco Montagna and Constantine Tsinakis in [135]. On the other hand, it is not possible to claim that our results *extend* the results for ℓ-groups, in the cases where the latter are used to conclude the former.

Problem 16. Obtain proofs of, e.g., Theorems 5.5.1 and 5.5.3, that do not rely on the analogous results for ℓ-groups.

The final part of the chapter is concerned with varieties of representable cancellative (integral) residuated lattices. We prove generation results in terms of free objects in the quasivarieties of monoid subreducts, along the same lines as analogous generation results for ℓ-groups (in the case of ℓ-groups, we would consider quasivarieties of group subreducts). These results are powerful tools, as they allow us to 'split' the underlying monoid structure and the residuated ordered structure. A relatively free monoid endowed with a total order is a much simpler object than a free residuated lattice.

The main question that motivated the results in Section 5.6 is the study of the free objects in the considered varieties. Given that residuated lattices are very general structures, it is often hard to provide neat descriptions of free objects (for some recent work on free objects see, e.g., [22, 23, 19, 20, 24, 48, 1]). In the context of ℓ-groups, free objects can be uniformly described in terms of quasivarieties of group subreducts (cf. [110]).

Problem 17. Obtain a representation theorem for, e.g., nilpotent representable cancellative (integral) residuated lattices, in terms of suitable residuated chains over the free objects in the quasivarieties of monoid subreducts.

By Theorem 5.6.10, the variety of commutative representable cancellative integral residuated lattices is generated by the class of integral chains whose monoid reducts are $\mathbb{N}^n$. In relation to this, a deeper study of the relationship between Theorem 5.6.10 and Rostislav Horčík's work ([93, 94, 95, 96, 97]) should be considered. As a matter of fact, some of the results included here generalize results contained in Horčík's papers. We briefly compare Theorem 5.6.10 to one of the main results in [93], namely Corollary 4.8. There, the author considers the class of finitely generated submonoids of the negative cone of the lexicographic product of n copies of $\mathbb{Z}$—which he denotes by $\mathbb{Z}^n_{\text{lex}}$—with the (integral) total order inherited from the whole group $\mathbb{Z}^n_{\text{lex}}$; such totally ordered monoids are residuated chains (by Lemma 5.6.1; cf. [93, Lemma 3.1]), and the author shows that the class of these residuated chains generates the variety of commutative representable cancellative integral residuated lattices. This generating class has nonempty intersection with the generating class that we describe in Theorem 5.6.10, although it is hard at this stage to say anything more.

PROPERTY	DEFINITION	EQUATION(S)
e-cyclic	—	$x\backslash e \approx e/x$
Cancellative Can RL	$xz \approx yz \Rightarrow x \approx y$ $zx \approx zy \Rightarrow x \approx y$	$xy/y \approx x$ $y\backslash yx \approx x$
Prelinear PreRL	—	$(x\backslash y \wedge e) \vee (y\backslash x \wedge e) \approx e$ $(x/y \wedge e) \vee (y/x \wedge e) \approx e$
Integral IRL	e is the greatest element	$x\backslash e \approx e$
Semilinear SemRL	subdirect product of totally ordered RLs	$\lambda_u((x \vee y)\backslash x) \vee \rho_v((x \vee y)\backslash y) \approx e$ $\lambda_u(x/(x \vee y)) \vee \rho_v(y/(x \vee y)) \approx e$
Hamiltonian HamRL	convex subalgebras are normal	—
Weakly Abelian WRL	—	$(x \wedge e)^2 \leq \lambda_y(x)$ $(x \wedge e)^2 \leq \rho_y(x)$
Nilpotent class c $\mathsf{N}^c\mathsf{RL}$	—	L_c
Commutative CRL	—	$xy \approx yx$
ℓ-group LG	every element is invertible	$x(x\backslash e) \approx e$ $(e/x)x \approx e$

Table 5.1: Relevant properties of residuated lattices

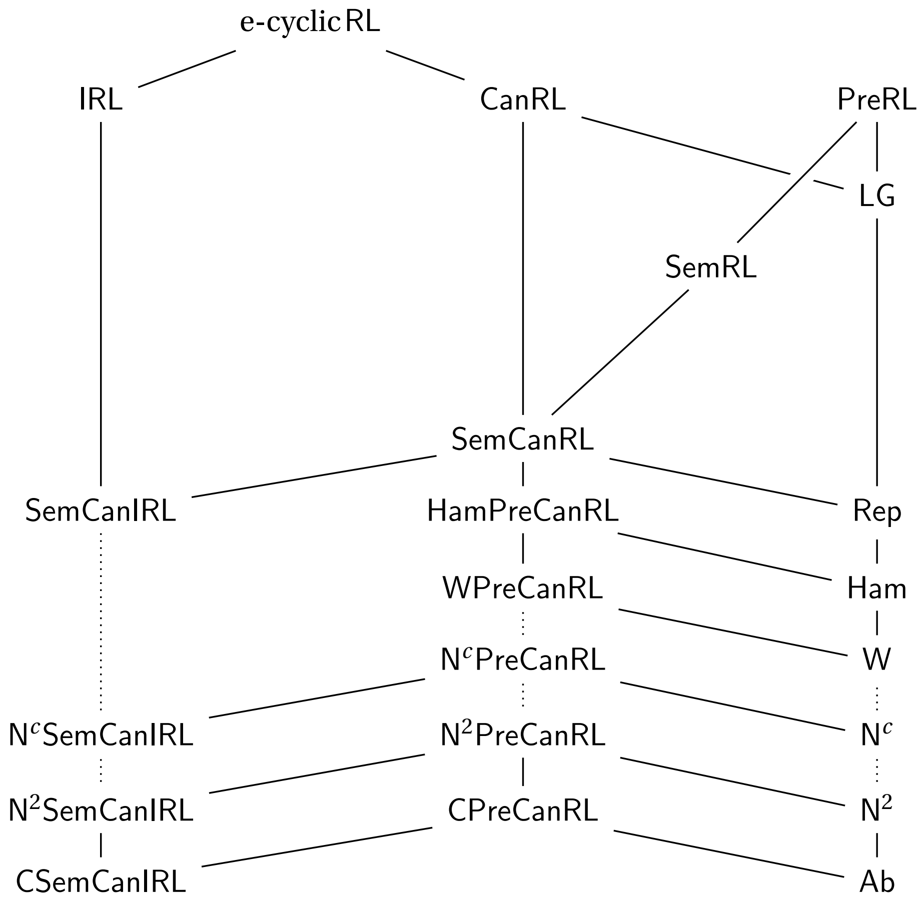

Figure 5.1: The picture illustrates mutual inclusions between relevant classes of residuated lattices. Note that HamPreCanRL is in fact HamSemCanRL; similarly N^cPreCanRL coincides with N^cSemCanRL, and so on. However, we choose this notation to emphasize that semilinearity in these contexts is actually axiomatized by prelinearity (cf. Corollary 5.4.3 and Theorem 5.5.3).

APPENDIX

The aim of this appendix is to provide a brief account of some of the concepts used in this thesis. The notions and results presented here are well-known, and we choose to introduce them at the level of generality needed for this thesis. We gather here notions that play a role in specific parts of the thesis. More precisely, we introduce the concepts from category theory used in Chapters 3 and 5. Further, we define notions from order theory that are mentioned specifically in Chapter 3 (e.g., interior operator), Chapter 4 (e.g., section-retraction pair), and Chapter 5 (e.g., residuation). Finally, we sketch the basic duality-theoretic results used in Chapter 3.

A.1 CATEGORY THEORY

For the material covered in this section, we refer to [119].

CATEGORIES AND FUNCTORS

A *category* C consists of a class $\mathrm{ob}(\mathsf{C})$ of objects and a class $\mathrm{hom}(\mathsf{C})$ of morphisms (or arrows) between objects. For any object A in C, there exists an identity arrow id_A. For all objects A, B, C in C and all morphisms $f\colon A \to B$, $g\colon B \to C$ in C, the composition $g \circ f\colon A \to C$ is a morphism in C; composition is associative, $\mathrm{id}_B \circ f = f$ and $g \circ \mathrm{id}_B = g$. We write $\mathrm{Hom}_{\mathsf{C}}(A, B)$ to denote the class of all morphisms from objects A to B in C. For any category C, the *opposite category* C^{op} is the category with the same objects as C and such that, for any $A, B \in \mathrm{ob}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A)$.

In what follows, C and D denote two categories.

If $\mathrm{ob}(\mathsf{D}) \subseteq \mathrm{ob}(\mathsf{C})$ and $\mathrm{hom}(\mathsf{D}) \subseteq \mathrm{hom}(\mathsf{C})$, the category D is said to be a *subcategory* of C. The subcategory is *full* provided that $\mathrm{Hom}_{\mathsf{C}}(A, B) = \mathrm{Hom}_{\mathsf{D}}(A, B)$, for any pair of objects A, B in D.

Let A, B be objects in C. A morphism $f\colon A \to B$ is

- a *monomorphism* if, for any $g, h\colon C \to A$ such that $f \circ g = f \circ h$, we have $g = h$;
- an *epimorphism* if, for any $g, h\colon B \to C$ such that $g \circ f = h \circ f$, we have $g = h$;
- an *isomorphism* provided that there exists $g\colon B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ between categories is a pair of assignments

$$\mathrm{ob}(\mathsf{C}) \longrightarrow \mathrm{ob}(\mathsf{D}) \quad \text{and} \quad \mathrm{hom}(\mathsf{C}) \longrightarrow \mathrm{hom}(\mathsf{D})$$

such that, for all morphisms f, g in C and all objects A, B in C, if $f\colon A \to B$, then $Ff\colon FA \to FB$, and also, $F\,\mathrm{id}_A = \mathrm{id}_{FA}$ and $F(f \circ g) = Ff \circ Fg$, whenever $f \circ g$ exists. A functor $F\colon \mathsf{C} \to \mathsf{D}^{\mathrm{op}}$ is called a *contravariant functor* between C and D. The notion of composition between functors is defined in the obvious way: if $F\colon \mathsf{C}_1 \to \mathsf{C}_2$ and $G\colon \mathsf{C}_2 \to \mathsf{C}_3$ are functors, then so is their composition $G \circ F\colon \mathsf{C}_1 \to \mathsf{C}_3$; we denote $G \circ F$ by GF. We also write I_{C} for the functor $\mathsf{C} \to \mathsf{C}$ that is the identity on both objects and morphisms.

ADJUNCTIONS

Consider two functors $F, G\colon \mathsf{C} \rightrightarrows \mathsf{D}$. We say that $\tau\colon F \to G$ is a *natural transformation* if it assigns to each object A in C a morphism $\tau_A\colon FA \to GA$ of D in such a way that, for any morphism $f\colon A \to B$ in C, the following diagram

$$\begin{array}{ccc} FA & \xrightarrow{\tau_A} & GA \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\tau_B} & GB \end{array}$$

commutes. The morphism τ_A is called the *component of* τ *at* A. We say that τ is a (natural) *isomorphism* if τ_A is an isomorphism for any object A in C.

Consider two functors $F\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon G$. We say that (F, G) is an *adjuction* if there exist natural transformations $\eta\colon \mathrm{I}_{\mathsf{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{I}_{\mathsf{D}}$ such that, for any object A in C and any object B in D, the following diagrams (known as 'triangle identities') commute:

$$\begin{array}{ccc} GB & \xrightarrow{\eta_{GB}} & GFGB \\ & {\scriptstyle \mathrm{id}_{GB}}\searrow & \big\downarrow{\scriptstyle G\varepsilon_B} \\ & & GB \end{array} \qquad \begin{array}{ccc} FA & \xrightarrow{F\eta_A} & FGFA \\ & {\scriptstyle \mathrm{id}_{FA}}\searrow & \big\downarrow{\scriptstyle \varepsilon_{FA}} \\ & & FA \end{array}$$

The transformations η and ε are called *unit* and *counit* of the adjunction, respectively.

We also include an equivalent definition of adjunction that does not rely on the notion of natural transformation, and is most easily specialized to the setting of preordered sets (see Appendix A.2). Consider two functors $F\colon \mathsf{C} \rightleftarrows \mathsf{D} \colon G$. We say that (F, G) is an *adjuction* if there exists a function φ that assigns to each pair of objects A in C and B in D a bijection of sets

$$\varphi_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(FA, B) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, GB)$$

which is natural, i.e., for all morphisms $k\colon C \to A$ in C and $h\colon B \to D$ in D,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}}(FA, B) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathsf{C}}(A, GB) \\ {\scriptstyle \overline{h}}\big\downarrow & & {\scriptstyle \overline{Gh}}\big\downarrow \\ \mathrm{Hom}_{\mathsf{D}}(FA, D) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathsf{C}}(A, GD) \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}_{\mathsf{D}}(FA, B) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathsf{C}}(A, GB) \\ {\scriptstyle \overline{Fk}}\big\downarrow & & {\scriptstyle \overline{k}}\big\downarrow \\ \mathrm{Hom}_{\mathsf{D}}(FC, B) & \xrightarrow{\varphi} & \mathrm{Hom}_{\mathsf{C}}(C, GB) \end{array}$$

commute, where $\overline{h}(f) = h \circ f \colon FA \to D$ and $\overline{Fk}(f) = f \circ Fk \colon FC \to B$ for any $f \in \mathrm{Hom}_{\mathsf{D}}(FA, B)$ and similarly, $\overline{Gh}(g) = Gh \circ g \colon A \to GD$ and $\overline{k}(g) = g \circ k \colon A \to GB$ for any $g \in \mathrm{Hom}_{\mathsf{C}}(A, GB)$. For the equivalence of the two definitions provided here, see [119, Chapter IV].

A *contravariant adjunction* between C and D is an adjunction between C and D^{op}. When (F, G) is an adjunction, we write $F \dashv G$. In this case, F and G are called the *left* (or *lower*) and *right* (or *upper*) *adjoint*, respectively.

Let $F \colon \mathsf{C} \to \mathsf{D}$ be a functor, and B be an object in D. A *universal arrow* from B to F is a morphism $u \colon B \to FA$ together with an object A in C such that, for any object C in C and arrow $f \colon B \to FC$ in D, there exists a unique morphism $g \colon A \to C$ in C such that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & FA \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle Fg} \\ & & FC \end{array}$$

commutes. The following uniqueness property is a straightforward consequence of the definition of universal arrow: if $u \colon B \to FA_1, w \colon B \to FA_2$ are universal arrows from B to F, then there is exactly one morphism $h \colon A_1 \to A_2$ in C such that the following diagram

$$\begin{array}{ccc} B & \xrightarrow{u} & FA_2 \\ & {\scriptstyle w}\searrow & \downarrow{\scriptstyle Fh} \\ & & FA_2 \end{array}$$

commutes, and Fh is an isomorphism. For any functor $G \colon \mathsf{D} \to \mathsf{C}$, a left adjoint $F \colon \mathsf{C} \to \mathsf{D}$ exists if, for each object A in C, there exists an object B in D and a universal arrow $\eta_A \colon A \to GB$ (from A to G). When this is the case, the morphism η_A is the component at A of the unit of the adjunction $F \dashv G$, and B is FA.

Consider two functors $F \colon \mathsf{C} \rightleftarrows \mathsf{D} \colon G$. We say that F, G define a (*categorical*) *equivalence* between C and D if (F, G) is an adjunction such that the unit η and the counit ε are isomorphisms. A dual equivalence (or duality) between C and D is an equivalence between C and D^{op}.

A.2 ORDER AND RESIDUATION

For the material covered in this section, we refer to [54], [78].

PREORDERS AND PARTIAL ORDERS

The set of positive natural numbers is $\mathbb{N}^+ := \{1, 2, \dots\}$, and $\mathbb{N}$ is the set $\mathbb{N}^+ \cup \{0\}$.

A binary relation $\preceq \subseteq P \times P$ on a set P is a *preorder* if it is

- reflexive: for every $a \in P$, $a \preceq a$;
- transitive: for all $a, b, c \in P$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$;

- total: for all $a, b \in P$, either $a \preceq b$ or $b \preceq a$.

Also, a reflexive and transitive binary relation $\preceq \subseteq P \times P$ is a *partial order* if it is

- antisymmetric: for all $a, b \in P$, $a \preceq b$ and $b \preceq a$ implies $a = b$.

We write $a \prec b$ to mean $a \preceq b$ and $b \npreceq a$. A *total order* is a partial order which is also total; equivalently, a total order is an antisymmetric preorder. We write $(P, \preceq)$ for the set P endowed with the preorder (resp., partial order) $\preceq$, and call $(P, \preceq)$ a *preordered set* (resp., a partially ordered set or poset). We often call a set endowed with a total order, a *totally ordered set* (or a *chain*). For any preordered set (resp., poset) $(P, \preceq)$, the binary relation $\preceq^\partial$ defined by

$$a \preceq^\partial b \iff b \preceq a \tag{A.42}$$

is a preorder (resp., a partial order) on P, called the *dual (pre)order* of $\preceq$.

Remark A.2.1. Note that any preordered set $(P, \preceq)$ can be seen as a category P, with objects the elements of P, and with arrows $a \to b$ if and only if $a \preceq b$. In this sense, considering the dual order $(P, \preceq^\partial)$ simply amounts to taking the suitable opposite category P^{op}.

ORDER-PRESERVING FUNCTIONS

Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be preordered sets. A map $f : P \to Q$ is an *order-embedding* if it is both

- order-preserving: if for all $a, b \in P$, $a \preceq_P b$ implies $f(a) \preceq_Q f(b)$;
- order-reflecting: if for all $a, b \in P$, $f(a) \preceq_Q f(b)$ implies $a \preceq_P b$.

Any order-embedding is clearly injective. A function $f : P \to Q$ is an *order-isomorphism* if it is an onto order-embedding; in this case, it has an order-preserving inverse. An order-isomorphism $f : P \to P$ is sometimes called *order-automorphism*.

Let $(P, \preceq_P)$ and $(Q, \preceq_Q)$ be two preordered sets. In view of Remark A.2.1, a *(contravariant) Galois connection* between $(P, \preceq_P)$ and $(Q, \preceq_Q)$ consists of a (contravariant) adjunction between the categories P and Q. Since the setting of preordered sets is simpler than arbitrary categories, we also recall the following definition. The pair (f, g) of order-preserving functions $f : P \to Q$ and $g : Q \to P$ is a Galois connection (or a *residuated pair*) if, for any $a \in P$ and $b \in Q$,

$$f(a) \preceq_Q b \iff a \preceq_P g(b).$$

When a map f has a right adjoint, we say that f is *residuated*; conversely, when a map g has a left adjoint, we say that g is a *residual*. A contravariant Galois connection is a Galois connection between $(P, \preceq_P)$ and $(Q, \preceq_Q^\partial)$.

A pair (f, g) of order-preserving functions $f : P \to Q$ and $g : Q \to P$ is called a *section-retraction pair* if f is an order-embedding, g is onto, and $g \circ f : P \to P$ is the identity. In this case, f and g are called a *section* (of g) and *retraction* (of f), respectively.

We say that a function $\iota : P \to P$ on a poset $(P, \preceq)$ is an *interior operator* if it is

- contracting: $\iota(a) \preceq a$,
- order-preserving,
- idempotent: $\iota \circ \iota$ coincides with ι on P.

Analogously, an order-preserving map $\gamma\colon P \to P$ is a *closure operator* if it is enlarging ($a \preceq \gamma(a)$) and idempotent. Any element in the image $\iota[P]$ is called an open element, and any element in $\gamma[P]$ is called a closed element.

Remark A.2.2. Note that any Galois connection (f,g) between $(P,\preceq_P)$ and $(Q,\preceq_Q)$ gives rise to an interior operator $f \circ g\colon Q \to Q$, and to a closure operator $g \circ f\colon P \to P$. In the case of contravariant Galois connections, we obtain two closure operators and, for this reason, we only talk about 'closed elements' (see Section 3.6).

LATTICES

For any preordered set $(P,\preceq)$, a subset $S \subseteq P$ is *downward closed* (resp., *upward closed*) provided that $b \preceq a$ implies $b \in S$ (resp., $a \preceq b$ implies $b \in S$) for any $a \in S$. In this case, we call S a *downset* (resp., an *upset*).

For any preordered set $(P,\preceq)$, and $S \subseteq P$, an element $a \in P$ is a *lower bound* (resp., *upper bound*) of S if $a \preceq b$ (resp., $b \preceq a$) for all $b \in S$. For any $a \in P$, we denote by $\downarrow a$ (resp., $\uparrow a$) the set $\{b \in P \mid b \preceq a\}$ (resp., $\{b \in P \mid a \preceq b\}$).

A poset $(P,\preceq)$ is a *root system* if for all $a,b,c \in P$, if $a \preceq b$ and $a \preceq c$, then $b \preceq c$ or $c \preceq b$; namely, the upset $\uparrow a$ with the restricted order is a chain.

Let $(P,\preceq)$ be a poset. If for all $a,b \in P$, the set of lower bounds of $\{a,b\}$ has a greatest element, written $a \wedge b$, then $(P,\preceq)$ is a $\wedge$-*semilattice* and $a \wedge b$ is the *infimum* (or *meet*) of a,b. Similarly, if for all $a,b \in P$, the set of upper bounds of $\{a,b\}$ has a least element, denoted by $a \vee b$, then $(P,\preceq)$ is called a $\vee$-*semilattice* and $a \vee b$ is the *supremum* (or *join*) of a,b. The poset $(P,\preceq)$ is a *lattice* if it is both a $\vee$-semilattice and a $\wedge$-semilattice.

We most often think of a $\wedge$-semilattice (resp., $\vee$-semilattice) L as an algebraic structure with a binary operation $\wedge$ (resp., $\vee$), which is associative, commutative, and idempotent; similarly, we think of a lattice L as an algebra with two semilattice operations $\wedge$ and $\vee$, and satisfying the following absorption laws:

$$a \wedge (a \vee b) \approx a \quad \text{and} \quad a \vee (a \wedge b) \approx a.$$

If L is a lattice, the corresponding poset $(L,\preceq)$ is obtained by $a \preceq b$ if and only if $a \wedge b = a$ (equivalently, $a \vee b = b$). We recall here some relevant properties. A lattice L is said to be

- *distributive*, provided that for all $a,b,c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad \text{and} \quad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c);$$

- *bounded*, if the poset $(L,\preceq)$ has a greatest element, called the *top* element (or *maximum*), and a least element, called the *bottom* element (or *minimum*); we often denote the maximum by $\top$ and the minimum by $\bot$;

- *complete*, provided that for any subset $S \subseteq L$, the greatest lower bound of S, denoted by $\bigwedge S$, exists in L; equivalently, a lattice is complete if any subset $S \subseteq L$ has a least upper bound in L, denoted by $\bigvee S$.

An element a of a lattice L is *compact* if, whenever $a \preceq \bigvee S$ for some $S \subseteq L$, then $a \preceq \bigvee T$ for some finite subset $T \subseteq S$. A lattice L is said to be

- *algebraic*, if it is complete and, for any $a \in L$,

$$a = \bigvee \{b \in L \mid b \preceq a \text{ and } b \text{ is compact}\}.$$

An element a of a bounded lattice L has a *complement* $b \in L$ provided that $a \vee b = \top$ and $a \wedge b = \bot$. A *Boolean algebra* is a distributive lattice in which every element has a complement.

Let L be a lattice. An *ideal*[3] of L is a nonempty downset $I \subseteq L$ such that, for any $a, b \in I$, also $a \vee b \in I$. An ideal I is called proper if $I \neq L$; a proper ideal is called *prime* if, for any $a, b \in L$, whenever $a \wedge b \in I$, either $a \in I$ or $b \in I$. The notion of a (prime) filter is defined dually.

If a, b are elements of a distributive lattice L such that $b \npreceq a$, then there exists a prime ideal I such that $a \in I$ and $b \notin I$. This result follows by an application of Zorn's Lemma, and is known as the 'Prime Ideal Separation Theorem'; its content is often summarized by saying that 'in a distributive lattice, there are *enough* prime ideals to separate distinct elements'. We point out some relevant original references: [84, 79, 108].

A.3 TOPOLOGY AND DUALITY

A comprehensive account of the topics touched on here can be found in [104], [56].

SPECTRAL SPACES

Recall that a topological space is a pair (X, τ) where X is a set and τ is a set of subsets of X closed under finite intersections and arbitrary unions, and containing X and $\emptyset$. The elements of τ are called open sets while their complements are said to be closed sets; a subset that is both open and closed is called clopen. Whenever τ is the smallest topology containing a set of subsets $S \subseteq \tau$, then S is a subbase for τ; in this case, every open set can be written as a union of finite intersections of elements from S, and τ is said to be generated by S. A base for the topology τ on X is a collection $B \subseteq \tau$ such that every open in τ can be obtained as the union of opens from B.

In what follows, X and Y denote topological spaces (X, τ_X) and (Y, τ_Y).

For any subset S of X, the smallest closed subset of X that contains S is called the closure of S. A subset S of a topological space X is said to be *dense* in X if X is the closure of S.

[3] In this thesis, we sometimes use the terminology 'lattice ideal' to make explicit the distinction between the notion of ideal in this context, and the notion of ideal in the context of, e.g., ℓ-groups.

A function $f\colon X \to Y$ is

- *continuous*, provided that $f^{-1}[V]$ is open for any open $V \subseteq Y$;
- *open*, provided that $f[U]$ is open for any open $U \subseteq X$;
- a *homeomorphism*, if f is a continuous bijection with continuous inverse f^{-1}.

We say that a function $v\colon X \to X$ is a *continuous retraction* if it is continuous, and is a retraction of the inclusion map $v[X] \hookrightarrow X$.

A subset $K \subseteq X$ is *compact* provided every open cover of K contains a finite subcover. We say that X is

- *Kolmogorov* (or T_0) if, for all $x, y \in X$ such that $x \neq y$ there exist an open $U \subseteq X$ that contains exactly one of x and y;
- *Hausdorff* (or T_2) if, for all $x, y \in X$ such that $x \neq y$ there exist opens $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$;
- *totally disconnected* if, for all $x, y \in X$ such that $x \neq y$ there exist a clopen $U \subseteq X$ such that $x \in U$, $y \notin U$;
- *zero-dimensional* if its clopen subsets form a base;
- a *Boolean* (or *Stone*) *space* provided that it is compact and totally disconnected; equivalently, if it is compact, Hausdorff and zero-dimensional.

An element $x \in X$ is said to be an *isolated point* if $\{x\}$ is an open subset of X. A topological space X is (homeomorphic to) the *Cantor space* if and only if it is a Boolean space with countably many clopen sets, and does not have isolated points.

The *specialization (pre)order* of a topological space X is the relation defined on X by: $x \preceq y$ if and only if y is in the closure of $\{x\}$; when the space is T_0, it becomes a partial order.

A closed set $\emptyset \neq Z \subseteq X$ is *irreducible* if it is not the union of two proper closed subsets of itself. We say that X is

- *sober*, if every irreducible closed set is the closure of a unique point;
- *generalized spectral*, if X is sober, and its compact open subsets $\mathring{\mathcal{K}}(X)$ form a base closed under finite intersections;
- *spectral*, provided that X is a generalized spectral space, and it is compact.

Every generalized spectral space is T_0. A generalized spectral space X is *completely normal* if for any $x, y \in X$ in the closure of a singleton $\{z\}$, either x is in the closure of $\{y\}$, or y is in the closure of $\{x\}$. Therefore, a generalized spectral space is completely normal if and only if its specialization order forms a root system.

DUALITY FOR DISTRIBUTIVE LATTICES

We limit the treatment of duality to what is actually needed in this thesis. More precisely, we describe the behaviour of the duality at the level of the objects in the relevant categories, and refrain from discussing morphisms and adjoint functors. We focus on *Stone duality*, as *Priestley duality* does not play a role in this thesis. For a broader and more detailed treatment of topological dualities for distributive lattices, we refer to [164], [147]; cf. [104, 52]. The facts included here essentially appear in Stone's original article [164].

Let D be a distributive lattice with a minimum—but not necessarily with a maximum. We endow the set $X(D)$ of prime ideals of D with a topology, by declaring that the sets

$$\widehat{a} = \{I \in X(D) \mid a \notin I\}, \quad \text{for } a \in D$$

form a subbase. The generated topology $\tau^{\downarrow}$ is known as the *spectral* (or *hull-kernel*, or *Stone*, or *Zariski*) topology on $X(D)$. If D is a distributive lattice with minimum, the space $(X(D), \tau^{\downarrow})$ is a generalized spectral space. The set $\{\widehat{a} \mid a \in D\}$ is exactly the set of compact open subsets of $(X(D), \tau^{\downarrow})$, and its specialization order coincides with the inclusion order between the prime ideals. For any generalized spectral space X, the set $\mathring{\mathcal{K}}(X)$ of its compact open subsets partially ordered by inclusion is a distributive lattice with minimum. These constructions are inverse to each other. More precisely, any distributive lattice D with minimum is isomorphic to the lattice $\mathring{\mathcal{K}}(X(D))$ and, conversely, every generalized spectral space Y is homeomorphic to the space of prime ideals of its own lattice $\mathring{\mathcal{K}}(Y)$ of compact open sets. These isomorphisms are natural, in the mathematical sense defined above.

If D is a bounded distributive lattice, the space $X(D)$ is a spectral space. Further, for any spectral space X, the set of its compact open subsets partially ordered by inclusion is a bounded distributive lattice. In particular, for any distributive lattice D with minimum, the space $X(D)$ is compact if and only if D has a maximum. If D is a Boolean algebra, the space $X(D)$ is a Boolean space; also, the set of clopen subsets of any Boolean space X partially ordered by inclusion is a Boolean algebra. In this case, the specialization order is trivial, since prime ideals are maximal ideals in Boolean algebra and, as such, they are mutually incomparable. We conclude by mentioning also that, in this case, $X(D)$ is a Cantor space if and only if the Boolean algebra D is countable and atomless, that is, it has no non-trivial minimal elements.

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INDEX OF SYMBOLS

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Erklärung

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