Strength and Noetherianity
for infinite Tensors

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von
Michel Arthur Bik
von den Niederlanden

Leiter der Arbeit:
Prof. dr. ir. Jan Draisma
Mathematisches Institut
der Universität Bern

Strength and Noetherianity
for infinite Tensors

Inauguraldissertation
der Philosophisch-naturwissenschaftlichen Fakultät
der Universität Bern

vorgelegt von

Michel Arthur Bik
von den Niederlanden

Leiter der Arbeit:
Prof. dr. ir. Jan Draisma
Mathematisches Institut
der Universität Bern

Von der Philosophisch-naturwissenschaftlichen Fakultät angenommen.

Bern, 29. Juni 2020

Der Dekan:
Prof. Dr. Zoltán Balogh

Preface

This thesis comprises the results I have obtained during my PhD studies under the supervision of Jan Draisma at the Mathematical Institute of the University of Bern.
During my PhD studies I have been supported by Jan Draisma's Vici Grant 639.033.514 entitled "Stabilization in Algebra and Geometry" from the Netherlands Organisation for Scientific Research.

Some of the content of this thesis is available in published form and/or on Arxiv: Chapter 2 is based on [4], Chapter 3 is based on joint work [7] with Jan Draisma and Rob Eggermont except for the second section, which is based on joint work [10] with Alessandro Oneto, and Chapter 4 is based on joint work [8] with Jan Draisma, Rob Eggermont and Andrew Snowden. During my time in Bern, I also published [6] together with Jan Draisma and [5] together with Adam Czapliński and Markus Wageringel. These results are not included in this thesis.

I would like to thank Jan Draisma and Emanuele Ventura for their helpful comments on the early version of my thesis. I would also like to thank JM Landsberg and Mateusz Michałek for refereeing my thesis and for their insightful questions.

Acknowledgements

There are many people I want to thank for the four years I spent in Bern. Let me begin before the beginning. Jan, thank you for giving me the chance to be your PhD student. I moved to Switzerland solely because you did the same, and the decision to do so has turned out to be one of the best ones I have ever made. Also, more importantly, thank you for being the greatest advisor I could wish for. I will be forever grateful that you gave me the space needed to find myself as a Mathematician while, at the same time, always being available to talk when I had a question and many times bringing up interesting problems you were thinking about. If not for you, I would never have discovered the most beautiful objects in all of Mathematics.[1]

There are, of course, a lot more people that deserve thanks for my mathematical development, starting with my officemate during the first half of my time here. Pierre-Marie, you were the best officemate I can imagine, willing to listen with interest every time I got stuck on a problem. Of all the courses I assisted with, I enjoyed the Lineare Algebra courses that you taught the most. One of my few regrets regarding my time in Bern is that, in the end, I never did beat you in a game of shogi. I am also very grateful to my fellow members of our Bern/Eindhoven research group. Jan, Michał, Alejandro, Rob, Chi Ho, Alessandro, Azhar and Emanuele, over the years I learned a lot together with you that I would not know now otherwise, and my interactions with you and the relaxing yet engaging atmosphere you helped create made this possible. Thirdly, I want to thank the mathematicians I have written articles with. Jan, Rob, Alessandro, Emanuele, Adam, Markus and Andrew, I could not have done this without you, literally, and working together with you has taught me a lot. And, at the end of this paragraph, let me use this opportunity to thank everyone else with whom I discussed Math with during the past four years.

Those who know me know that I enjoy playing a game with others from time to time. Alejandro, Lucas, Pierre-Marie, Olim, Rob, Guus, Jan-Willem, Olivier, Ruben, Sebastian, Lukas, Carlos, Frank, Tim, Chi Ho, Timo, Livio, Aline and the many others that I have forgotten to mention, the key to success in game playing, as in all things, is to do it with good people. And my last few years have been very successful indeed. Thank you. I also want to thank everyone from the mathematical institute for the warm work environment in general and for all the times we lunched together in particular. And let me thank Alejandro, Lucas and Aline most of all for making sure that I did things other than thinking about Mathematics and playing games all the time.

[1]It goes without saying that the objects I refer to are closed subsets of polynomial functors of finite degree over an algebraically closed field of characteristic zero.

Tot slot will ik mijn familie bedanken. Mam, Pap, Robbert, Dwarrel, Anna, Opa, Thierry en Jorunn, zonder jullie hulp was ik nooit gekomen tot waar ik nu ben. Bedankt voor alles.

Contents

Notation

Numbers, sets and maps

$\mathbb{N}$	the positive natural numbers (not including 0)
$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	the natural, rational, real and complex numbers
δ_{ij}	the constant 1 when $i = j$ and the constant 0 otherwise
$[n]$	the set $\{k \in \mathbb{N} \mid k \leq n\}$ for $n \in \mathbb{N} \cup \{\infty\}$
$S \twoheadrightarrow T$	a surjective morphism
$S \hookrightarrow T$	an injective morphism
$\overline{S}$	the closure of S
$\lambda \vdash d$	a partition of d for $d \in \mathbb{Z}_{\geq 0}$
$G_\infty = \varinjlim_n G_n$	the direct limit of a sequence of groups $(G_n)_{n \in \mathbb{N}}$
$V_\infty = \varprojlim_n V_n$	the inverse limit of a sequence of spaces $(V_n)_{n \in \mathbb{N}}$
pr_n	the projection map $V_m \twoheadrightarrow V_n$ for $n \leq m \leq \infty$

Vector spaces and matrices

K	an infinite field
K^S	the vector space of maps $S \to K$ for a set S
$K^{n_1 \times \cdots \times n_k}$	the vector space $K^{[n_k] \times \cdots \times [n_k]}$
V^*	the vector space of linear maps $V \to K$
$\mathrm{Diag}(A_1, \ldots, A_k)$	the block-diagonal matrix with blocks $A_1, \ldots, A_k$
GL_n	the group $\mathrm{GL}(K^n)$ for $n \in \mathbb{N}$
GL_∞	the group $\bigcup_{n \in \mathbb{N}} \mathrm{GL}_n$

Categories

Set	the category of sets
Vec	the category of finite-dimensional vector spaces over K
Top	the category of topological spaces
Vec^μ	the category of μ-tuples of finite-dimensional vector spaces over K for $\mu \in \mathbb{N}$

The category Vec^μ

V, W	objects of Vec^μ
ℓ, ℓ'	morphisms of Vec^μ
π_V	the projection map $V \oplus W \twoheadrightarrow V$
ι_V	the inclusion map $V \hookrightarrow V \oplus W$
$\mathrm{Hom}(V, W)$	the vector space of morphisms $V \to W$
$\mathrm{End}(V)$	the vector space $\mathrm{Hom}(V, V)$
$\ell \oplus \ell'$	the morphism $V \oplus V' \to W \oplus W'$ obtained from $\ell\colon V \to W$ and $\ell'\colon V' \to W'$
$\mathrm{GL}(V)$	the group of invertible morphisms $V \to V$

Polynomial functors over K

$\mathrm{GL}(P)$	the group of linear automorphisms of a polynomial functor P
$\mathrm{Aut}(P)$	the group of polynomial automorphisms of a polynomial functor P
$\mathrm{Mor}(Q, P)$	the vector space of polynomial transformations $Q \to P$

Introduction

This thesis mainly revolves around spaces of infinite tensors and expressing tensors using simpler objects. In the following pages, we will introduce these concepts by considering infinite matrices over the complex numbers $\mathbb{C}$. Here, the objects that are simpler than matrices are vectors.

Let $n, m \in \mathbb{N}$ be integers. A vector $v \in \mathbb{C}^n$ is is a map $[n] \to \mathbb{C}$ sending $i \mapsto v_i$ and a matrix $A \in \mathbb{C}^{n\times m}$ is a map $[n] \times [m] \to \mathbb{C}$ sending $(i, j) \mapsto A_{ij}$. We also write

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix}.$$

We define infinite vectors and matrices by replacing the sets $[n]$ and $[m]$ by $\mathbb{N}$. So an infinite vector $v \in \mathbb{C}^\infty$ is a map $\mathbb{N} \to \mathbb{C}$ sending $i \mapsto v_i$ and an infinite matrix $A \in \mathbb{C}^{\infty\times\infty}$ is a map $\mathbb{N}\times\mathbb{N} \to \mathbb{C}$ sending $(i, j) \mapsto A_{ij}$. Now we also write

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_{11} & A_{12} & \dots \\ A_{21} & A_{22} & \dots \\ \vdots & \vdots & \end{pmatrix}.$$

For finite matrices we have to following equivalence.

Proposition. Let $A \in \mathbb{C}^{n\times m}$ be a matrix and $k \geq 0$ an integer. Then the following conditions are equivalent:

(1) The determinant of each $(k + 1) \times (k + 1)$ submatrix of A is 0.

(2) We have $A = v_1 w_1^T + \dots + v_k w_k^T$ for some vectors $v_1, \dots, v_k \in \mathbb{C}^n$ and $w_1, \dots, w_k \in \mathbb{C}^m$.

When these equivalent conditions are satisfied, we say that A has rank $\leq k$. □

Again, the same statement is true when we replace n and m by ∞. Proving that (2) implies (1) in the infinite case is easy. To see that (1) implies (2), let $A \in \mathbb{C}^{\infty\times\infty}$ be an infinite matrix and assume that the determinant of each $(k + 1) \times (k + 1)$ submatrix of A is 0 for some minimal integer $k \geq 0$. Since k is minimal, there must be k columns $v_1, \dots, v_k$ of A that are linearly independent. To show that there exists $w_1, \dots, w_k \in \mathbb{C}^\infty$ such that

$$A = v_1 w_1^T + \dots + v_k w_k^T$$

it suffices to show that every column v of A is a linear combination of $v_1, \ldots, v_k$. If v is among the vectors $v_1, \ldots, v_k$ this is true. Otherwise, we know that every $(k+1) \times (k+1)$ submatrix of the matrix $B := (v_1\ v_2\ \cdots\ v_k\ v)$ is also a submatrix of A and hence has determinant 0. For every $n \in \mathbb{N}$, let $B_n \in \mathbb{C}^{n \times (k+1)}$ be the matrix consisting of the first n rows of B. Then B has rank $\leq k$. Let $V_n \subseteq \mathbb{C}^{k+1}$ be the kernel of B_n. Then we have a descending chain

$$\mathbb{C}^{k+1} \supseteq V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4 \supseteq \cdots$$

of nonzero subspaces. It follows that

$$n \geq \dim V_1 \geq \dim V_2 \geq \dim V_3 \geq \dim V_4 \geq \cdots$$

is a descending chain of postive integers. This second chain stabilizes, i.e., there exists an $m \in \mathbb{N}$ such that $\dim V_n = m$ for all $n \gg 0$. It follows that the space of linear dependencies of $v_1, \ldots, v_k, v$

$$V := \bigcap_{n=1}^{\infty} V_n$$

has dimension $m > 0$. So $v_1, \ldots, v_k, v$ must be linearly dependent. Since $v_1, \ldots, v_k$ are linearly independent, this means that v is a linear combination of $v_1, \ldots, v_k$. This shows that the proposition also holds in the infinite case and we say that the infinite matrix $A \in \mathbb{C}^{\infty \times \infty}$ has rank $\leq k$ when the equivalent conditions from the proposition hold. When $\mathrm{rk}(A) \leq k$ does not hold for any $k < \infty$, then we say that the rank of A is infinite.

Using the proposition, it is easy to write down infinite matrices with low rank. One example of a matrix with infinite rank is the infinite identity matrix

$$I_\infty = \begin{pmatrix} 1 & 0 & \ldots \\ 0 & 1 & \ddots \\ \vdots & \ddots & \ddots \end{pmatrix} \in \mathbb{C}^{\infty \times \infty}.$$

The multiplication of infinite matrices is defined using the usual formulas

$$A \cdot B = C, \quad C_{ij} = \sum_{k=1}^{\infty} A_{ik} \cdot B_{kj}.$$

However the product of two infinite matrices is not well-defined in general. If A or B is an element of the subset

$$\mathrm{GL}_\infty := \{\mathrm{Diag}(g, I_\infty) \mid n \in \mathbb{N}, g \in \mathrm{GL}_n\} \subseteq \mathbb{C}^{\infty \times \infty},$$

then the product is always well-defined. This turns the subset GL_∞ into a group called the infinite general linear group, which acts on $\mathbb{C}^{\infty \times \infty}$ by multiplication on the left and on the right. Matrices in the same GL_∞-orbit have the same rank.

Next we define the Zarisky topology on the space of infinite matrices. A polynomial function on $\mathbb{C}^{\infty \times \infty}$ is a function

$$f\colon \mathbb{C}^{\infty \times \infty} \to \mathbb{C}$$

that sends an infinite matrix A to a finite polynomial expression in its entries A_{ij}. For integers $i, j \in \mathbb{N}$, let x_{ij} be the polynomial function sending $A \mapsto A_{ij}$. Then $\mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$ is the ring of polynomial functions on $\mathbb{C}^{\infty\times\infty}$. A closed subset of $\mathbb{C}^{\infty\times\infty}$ is any subset of the form

$$Z(S) = \{A \in \mathbb{C}^{\infty\times\infty} \mid \forall f \in S : f(A) = 0\}$$

where S is a subset of $\mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$.

Since the determinants of finite submatrices are polynomial functions, the subset

$$\{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

is closed for each integer $k \geq 0$. Note that these subsets are also stable under the action of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$. In fact, these subsets are the only $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subsets of $\mathbb{C}^{\infty\times\infty}$ apart from $\emptyset$ and $\mathbb{C}^{\infty\times\infty}$ itself.

Theorem. The only nonempty proper $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subsets of $\mathbb{C}^{\infty\times\infty}$ are

$$\{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

for integers $k \geq 0$.

Before we prove this theorem, we discuss its consequences. First, the theorem shows that the space $\mathbb{C}^{\infty\times\infty}$ is Noetherian up to the action of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$. This means that any descending chain of $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subsets of $\mathbb{C}^{\infty\times\infty}$

$$\mathbb{C}^{\infty\times\infty} \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \cdots$$

stabilizes. Equivalently, this means that any $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subset of $\mathbb{C}^{\infty\times\infty}$ is defined using finitely many orbits of equations. As an example, the subset

$$\{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

is defined by the vanishing of the determinants of all $(k + 1) \times (k + 1)$ submatrices, which form one orbit under the action of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$. When considering a topological space with an action of a group, whether Noetherianity up to the action of that group holds is the first question one wants to answer. This is because it tells you that group-stable properties of points in the space given by closed conditions are always given by finitely many orbits of closed conditions, making checking whether such a property holds theoretically possible. We will generalize the statement that $\mathbb{C}^{\infty\times\infty}$ is Noetherian up to the action of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$ in Chapter 2.

Second, the theorem shows that the following statement holds: let $A \in \mathbb{C}^{\infty\times\infty}$ be an infinite matrix. Then either its orbit of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$ is Zariski-dense in $\mathbb{C}^{\infty\times\infty}$ or the rank of A is finite. In the latter case, the (infinite) matrix A can be expressed using finitely many (infinite) vectors. So either an infinite matrix is GL_∞-generic or it can be expressed using simpler object. We will prove more general versions of this statement in Chapters 3 and 4.

Finally, let $\mathcal{P}$ be a property of matrices given by the vanishing of polynomial functions such that PAQ has property $\mathcal{P}$ for all matrices $A \in \mathbb{C}^{n\times m}$ with property $\mathcal{P}$, $P \in \mathbb{C}^{n'\times n}$ and $Q \in \mathbb{C}^{m\times m'}$. Then the set

$$\left\{A \in \mathbb{C}^{\infty\times\infty} \,\middle|\, \forall n, m \in \mathbb{N} : (A_{ij})_{i,j=1}^{n,m} \text{ has property } \mathcal{P}\right\}$$

is a $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subset of $\mathbb{C}^{\infty\times\infty}$. So since the space $\mathbb{C}^{\infty\times\infty}$ is Noetherian up to the action of $\mathrm{GL}_\infty \times \mathrm{GL}_\infty$, the property $\mathcal{P}$ can be checked using finitely many orbits of polynomial functions. Moreover, since the only nonempty proper $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subsets of $\mathbb{C}^{\infty\times\infty}$ are

$$\{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

for integers $k \geq 0$, it follows that $\mathcal{P}$ must be either trivial or the property that a matrix has rank $\leq k$ for some integer $k \geq 0$.

We will end this introduction with a proof of the theorem. We will use the following result.

Lemma. Let $n, m \in \mathbb{N}$ be integers and let $A \in \mathbb{C}^{n\times m}$ be a matrix of rank $k \leq \min(n, m)$. Then the $(\mathrm{GL}_n \times \mathrm{GL}_m)$-orbit of A consists of all matrices in $\mathbb{C}^{n\times m}$ of rank k and the closure of this orbit consists of all matrices in $\mathbb{C}^{n\times m}$ of rank $\leq k$. □

Proof of the theorem. Let X be a nonempty proper $(\mathrm{GL}_\infty \times \mathrm{GL}_\infty)$-stable closed subset of the space $\mathbb{C}^{\infty\times\infty}$. We first will prove

$$X \subseteq \{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

for some integer $k \geq 0$. Since X is a proper closed subset of $\mathbb{C}^{\infty\times\infty}$, there is a polynomial $f \in \mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$ such that $f(A) = 0$ for all $A \in X$. Only finitely many variables occur in the polynomial f. Let $n, m \in \mathbb{N}$ be such that only x_{ij} with $i \leq n$ and $j \leq m$ occur in f. Then we see that $f(B) = 0$ for all $n \times m$ matrices B in the $(\mathrm{GL}_n \times \mathrm{GL}_m)$-stable set

$$Y = \left\{(A_{ij})_{i,j=1}^{n,m} \,\middle|\, A \in X\right\}$$

and hence it follows from the lemma that $\mathrm{rk}(B) < \min(n, m)$ for all matrices $B \in Y$. Let $A \in X$ be an infinite matrix. Then $gAh \in X$ for all $(g, h) \in \mathrm{GL}_\infty \times \mathrm{GL}_\infty$. So in particular, every matrix obtained from A by finitely many row and column permutations is contained in X. It follows that every $n \times m$ submatrix of A is contained in Y. Hence $\mathrm{rk}(A) \leq k$ for $k = \min(n, m) - 1$.

Now, let $k \geq 0$ be minimal with the property that

$$X \subseteq \{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

holds. Then there is an $A \in X$ such that $\mathrm{rk}(A) = k$. Let $f \in \mathbb{C}[x_{ij} \mid i, j \in \mathbb{N}]$ such that $f(A) = 0$ for all $A \in X$ and let $n, m \in \mathbb{N}$ be such that only x_{ij} with $i \leq n$ and $j \leq m$ occur in f. Then it follows from the lemma that $f(B) = 0$ for all matrices $B \in \mathbb{C}^{n\times m}$ of rank $\leq k$. It follows that $f(A) = 0$ for all $A \in \mathbb{C}^{\infty\times\infty}$ of rank $\leq k$. Hence

$$X \supseteq \{A \in \mathbb{C}^{\infty\times\infty} \mid \mathrm{rk}(A) \leq k\}$$

and we conclude that these sets must be equal. □

Chapter 1

Preliminaries

This thesis is centered around two concepts; the first being Noetherianity up to the action of a group and the second being rank functions that try to measure the complexity of an object by how that object can be expressed using simpler objects. In this chapter, we collect some basic definitions and results related to these concepts that we will need in the later chapters:

- We define what it means for a topological space equipped with an action of a group to be Noetherian up to the action of that group.
- We discuss inverse limits of sequences of finite-dimensional vector spaces, which are a source of many interesting examples of group-Noetherian spaces.
- We introduce polynomial functors, which we view as a generalization of finite-dimensional affine spaces equipped with the trivial group action.
- We list several examples of the kind of rank functions that are related to expressing objects using simpler objects.

1.1 Noetherianity up to the action of a group

Let us start with the definition of Noetherianity up to the action of a group together with a surprisingly useful proposition. Let X be a topological space and let G be a group acting on X.

Definition 1.1.1. We say that X is Noetherian up to the action of G (or G-Noetherian) when every descending chain

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

of G-stable closed subsets of X, i.e., closed subsets $Y \subseteq X$ such that $gY \subseteq Y$ for all $g \in G$, stabilizes. This means that there exists an $i \in \mathbb{N}$ such that $X_j = X_i$ for all $j \geq i$.

Proposition 1.1.2. *The space X is G-Noetherian if and only if every proper G-stable closed subset $Y \subsetneq X$ is G-Noetherian.*

Proof. If X is G-Noetherian and Y is a G-stable closed subset of X, then every descending chain of G-stable closed subsets of Y is also a descending chain of G-stable closed subsets of X and hence stabilizes. Conversely, if every proper G-stable closed subset $Y \subsetneq X$ is G-Noetherian and

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

is a descending chain of G-stable closed subsets of X, then either $X_i = X$ for all $i \in \mathbb{N}$ and the chain stabilizes or $X_i \subsetneq X$ is G-Noetherian for some $i \in \mathbb{N}$. In the latter case, the chain also stabilizes since X_i is G-Noetherian. Hence X must itself be G-Noetherian. □

Our first example of group-Noetherian spaces are finite-dimensional vector spaces equipped with the Zariski topology. Let V be a vector space that is the dual of a vector space with countable basis $\mathcal{X}$.

Definition 1.1.3. We define the coordinate ring $K[V]$ of V to be the polynomial ring over K in $\mathcal{X}$, i.e., the elements of $\mathcal{X}$ are independent variables and the elements of $K[V]$ are finite polynomial expressions in the elements of $\mathcal{X}$.

We call the elements of $\mathcal{X}$ variables and we call elements of $K[V]$ polynomials on V. Every variable $x \in \mathcal{X}$ induces a linear function $(v \mapsto v(x)) \in V^*$. Using these linear functions, every polynomial on V induces a function $V \to K$. We call functions that arise in this manner polynomial functions on V. Here is an example of a linear combination of the variables of V that is not a polynomial on V.

Nonexample 1.1.4. The series $f = \sum_{x \in \mathcal{X}} x$ is not an element of $K[V]$. This corresponds to the fact that it does not induce a map $V \to K$. ♣

Definition 1.1.5. For a subset $S \subseteq K[V]$, we define its zero set $Z(S) \subseteq V$ to be the subset of V consisting of all points v such that $f(v) = 0$ for all $f \in S$. The zero sets inside V form the closed subsets of a topology on V. We call this topology the Zariski topology.

We now have our first example of spaces that are Noetherian up to the action of a group. Let V be a finite-dimensional vector space equipped with the Zariski topology and the action of the trivial group $\{*\}$.

Theorem 1.1.6 (Hilbert's basis theorem)**.** *The space V is $\{*\}$-Noetherian.* □

We also get our first important nonexample.

Nonexample 1.1.7. Let V be the dual of a countably infinite-dimensional vector space with basis $\mathcal{X} = \{x_1, x_2, \dots\}$ and equip V with the Zariski topology and the action of the trivial group $\{*\}$. Then

$$V \supsetneq Z(x_1) \supsetneq Z(x_1, x_2) \supsetneq Z(x_1, x_2, x_3) \supsetneq Z(x_1, x_2, x_3, x_4) \supsetneq \dots$$

is an infinite descending chain of $\{*\}$-stable closed subsets that does not stabilize. ♣

The following proposition tells us that G-Noetherianity is preserved when we make the group G bigger.

Proposition 1.1.8. *Let X be a topological space equipped with the actions of two groups H, G. Suppose that X is H-Noetherian and and that every G-stable closed subset of X is also H-stable. Then X is also G-Noetherian.*

Proof. Every descending chain of G-stable closed subsets of X is also a chain of H-stable closed subsets of X and hence stabilizes. □

As a consequence, we see that a finite-dimensional vector space is Noetherian up to any action of any group. The following example shows that any vector space becomes G-Noetherian when we make the group G big enough.

Example 1.1.9. Let V be a vector space equipped with the Zariski topology and let the group $\mathrm{GL}(V)$ act on V by left-multiplication. Then the orbits of V are $\{0\}$ and $V \setminus \{0\}$. So the $\mathrm{GL}(V)$-stable closed subsets of V are $\emptyset$, $\{0\}$ and V. As there are only finitely many such subsets, every descending chain of them must stabilize. So the space V is $\mathrm{GL}(V)$-Noetherian. ♣

Noetherianity up to the action of a group is also preserved when we take quotients.

Proposition 1.1.10. *Let V be a G-Noetherian vector space and let W be a G-stable subspace of V. Then V/W is also G-Noetherian.*

Proof. Any descending chain of G-stable closed subsets of V/W can be pulled back along the projection map $V \twoheadrightarrow V/W$ to get a descending chain of G-stable closed subsets of V. This chain must stabilize and, since the map $V \twoheadrightarrow V/W$ is surjective, so must the chain of subsets of V/W as well. □

Theorem 1.1.6 and Proposition 1.1.8 tell us that interesting examples of vector spaces that are Noetherian up to the action of some group are infinite-dimensional. Nonexample 1.1.7 shows that in this case the group cannot act trivially. The first result that gave us a space with these properties is the following theorem.

Theorem 1.1.11 (Cohen [14], Hillar-Sullivant [23, Theorem 1.1])**.** *Fix an integer $k \in \mathbb{N}$ and let $\mathrm{Sym}(\mathbb{N})$ act on the polynomial ring*

$$R = K\left[x_{ij} \,\middle|\, i \in \mathbb{N}, j \in [k]\right]$$

by permuting the first index of the variables. Then the ring R is $\mathrm{Sym}(\mathbb{N})$-Noetherian, i.e., every ascending chain of $\mathrm{Sym}(\mathbb{N})$-stable ideals of R stabilizes. □

Corollary 1.1.12. *The space $K^{\mathbb{N}}$ is $\mathrm{Sym}(\mathbb{N})$-Noetherian.*

Proof. Let

$$K^{\mathbb{N}} = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

be a descending chain of $\mathrm{Sym}(\mathbb{N})$-stable closed subsets of $K^{\mathbb{N}}$. Then there is an ideal $\mathcal{I}_k \subseteq R$ for every $k \in \mathbb{N}$ such that $X_k = Z(\mathcal{I}_k)$. Note that we also have

$$X_k = Z(\mathrm{Sym}(\mathbb{N})\,\mathcal{I}_1 \cup \dots \cup \mathrm{Sym}(\mathbb{N})\,\mathcal{I}_k)$$

for each $k \in \mathbb{N}$. So we may replace each $\mathcal{I}_k$ by the ideal generated by $\mathrm{Sym}(\mathbb{N})\,\mathcal{I}_1 \cup \dots \cup \mathrm{Sym}(\mathbb{N})\,\mathcal{I}_k$ and hence we may assume that the ideals $\mathcal{I}_k$ form an ascending chain of $\mathrm{Sym}(\mathbb{N})$-stable ideals. By the theorem, this chain must stabilize and hence the chain of closed subsets of $K^{\mathbb{N}}$ must stabilize as well. □

1.2 Limits of spaces and groups

Many examples of vector spaces that are Noetherian up to the action of some group arise as the inverse limit of a sequence of finite-dimensional vector spaces. In these examples, the group acting on the inverse limit is a direct limit of groups acting on the finite-dimensional spaces. Throughout this section, let

$$V_1 \twoheadleftarrow V_2 \twoheadleftarrow V_3 \twoheadleftarrow \dots$$

be a sequence of finite-dimensional vector spaces connected by surjective linear maps. For each integer $n \in \mathbb{N}$, let G_n be a group acting on V_n and assume that G_n is a subgroup of G_{n+1} in some natural way. This gives us an action of G_n on V_{n+1}. We also assume that the map $V_{n+1} \twoheadrightarrow V_n$ is G_n-equivariant for each $n \in \mathbb{N}$. Now, we define the inverse limit

$$V_\infty := \varprojlim_n V_n = \{(v_n)_n \in \textstyle\prod_{n\in\mathbb{N}} V_n \,\big|\, v_{n+1} \text{ maps to } v_n \text{ for all } n \in \mathbb{N}\}$$

and the direct limit $G_\infty := \varinjlim_n G_n = \bigcup_{n\in\mathbb{N}} G_n$. Note that, for an integer $m \in \mathbb{N}$, an element $(v_n)_n \in \prod_{n\geq m} V_n$ such that v_{n+1} maps to v_n for all $n \geq m$ defines a unique element of V_∞ by letting v_n be the image of v_{n+1} in V_n for all $n < m$. Every element of V_∞ can be represented in this way and the group G_m acts on V_∞ by $g(v_n)_n = (gv_n)_n$ for all such elements. The fact that the maps $V_{n+1} \twoheadrightarrow V_n$ are G_n-equivariant ensures that $(gv_n)_n$ again defines an element of V_∞. As these actions are compatible with the inclusions $G_n \subseteq G_{n+1}$, we get an action of the whole group G_∞ on V_∞. For each $m \in \mathbb{N}$, we have projection maps $V_\infty \twoheadrightarrow V_m$ and $V_n \twoheadrightarrow V_m$ for $n \geq m$. We denote all these maps by pr_m. Each of the spaces V_n is equipped with the Zariski topology. We use these topologies to define a topology on V_∞.

Definition 1.2.1. We say that a subset X_∞ of V_∞ is closed when it is the inverse limit of a sequence of closed subsets, i.e., when

$$X_\infty = \{v \in V_\infty \mid \forall n \in \mathbb{N} : \mathrm{pr}_n(v) \in X_n\}$$

for closed subsets $X_n \subseteq V_n$. The closed subsets of V_∞ form a topology. We call this topology the Zariski topology.

Example 1.2.2. Let V_∞ be the dual of a countably infinite-dimensional vector space with basis $x_1, x_2, \dots$ and let V_n be the dual of the vector space with basis $x_1, \dots, x_n$ for each $n \in \mathbb{N}$. Then V_∞ is the inverse limit of the vector spaces V_n where the maps $V_{n+1} \to V_n$ are given by precomposition with the inclusion map

$$\mathrm{span}(x_1, \dots, x_n) \hookrightarrow \mathrm{span}(x_1, \dots, x_{n+1}).$$

The topologies on V_∞ from Definitions 1.1.5 and 1.2.1 are the same. ♣

Let X_∞ be a closed subset of V_∞ and take $X_n = \overline{\mathrm{pr}_n(X_\infty)}$ for each $n \in \mathbb{N}$. Then the set X_∞ is the inverse limit of the sets X_n and the maps $X_{n+1} \to X_n$ are dominant. So every closed subset of V_∞ is an inverse limit of closed subsets that map dominantly into each other. The following proposition relates the irreducibility of X_∞ with that of the X_n.

Proposition 1.2.3. *The following statements are equivalent:*

(1) X_∞ is irreducible.

(2) X_n *is irreducible for all* $n \in \mathbb{N}$.

(3) X_n *is irreducible for all* $n \gg 0$.

Proof. Suppose that X_n is reducible for some $n \in \mathbb{N}$. Then $X_n = Y \cup Z$ for some closed subsets $Y, Z \subsetneq X_n$. In this case, we see that

$$X_\infty = (\mathrm{pr}_n^{-1}(Y) \cap X_\infty) \cup (\mathrm{pr}_n^{-1}(Z) \cap X_\infty), \quad \mathrm{pr}_n^{-1}(Y) \cap X_\infty, \mathrm{pr}_n^{-1}(Z) \cap X_\infty \subsetneq X_\infty$$

and so X_∞ is reducible. This establishes (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial. So next, suppose that $X_\infty = Y_\infty \cup Z_\infty$ for some closed subsets $Y_\infty, Z_\infty \subseteq X_\infty$ with closures Y_n, Z_n in V_n. Then $X_n = Y_n \cup Z_n$ for all $n \in \mathbb{N}$. If $Y_\infty \subsetneq X_\infty$, then $Y_n \subsetneq X_n$ for some (and then also all bigger) $n \in \mathbb{N}$. The same holds for Z_∞. So we see that if X_∞ is reducible, then X_n is reducible for all $n \gg 0$. This shows (3) $\Rightarrow$ (1). □

Remark 1.2.4. In general, we cannot expect an open subset of V_∞ to be the inverse limit of a sequence of open subsets of the V_n. ♠

1.3 Polynomial functors

In this section, we give an introduction to polynomial functors. We start with univariate polynomial functors, which should be compared with univariate polynomials.

Definition 1.3.1. A univariate polynomial functor P assigns to every vector space $V \in \mathrm{Vec}$ a vector space $P(V) \in \mathrm{Vec}$ and to every linear map $\ell\colon V \to W$ a linear map $P(\ell)\colon P(V) \to P(W)$ such that $P(\mathrm{id}_V) = \mathrm{id}_{(P(V)}$ for all $V \in \mathrm{Vec}$, $P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2)$ for all linear maps $\ell_1\colon V \to W$ and $\ell_2\colon U \to V$ and the map

$$\begin{array}{rcl} \mathrm{Hom}_K(V, W) & \to & \mathrm{Hom}_K(P(V), P(W)) \\ \ell & \mapsto & P(\ell) \end{array}$$

is a polynomial map for all $V, W \in \mathrm{Vec}$.

Example 1.3.2. Let $U \in \mathrm{Vec}$ be a fixed finite-dimensional vector space. Then the constant functor $C_U\colon \mathrm{Vec} \to \mathrm{Vec}$ assigning U to every vector space and assigning id_U to every linear map is a polynomial functor. ♣

Example 1.3.3. The functor $T\colon \mathrm{Vec} \to \mathrm{Vec}$ assigning all vector spaces and linear maps to themselves is a polynomial functor. ♣

Definition 1.3.4. A functor Q is a subfunctor of a polynomial functor P when $Q(V)$ is a subspace of $P(V)$ for all $V \in \mathrm{Vec}$ and $Q(\ell)\colon Q(V) \to Q(W)$ is the restriction of $P(\ell)\colon P(V) \to P(W)$ for all linear maps $\ell\colon V \to W$.

Definition 1.3.5. Let Q be a subfunctor of a polynomial functor P. Then we define the quotient P/Q as the functor $\mathrm{Vec} \to \mathrm{Vec}$ that assigns to a vector space $V \in \mathrm{Vec}$ the quotient space $P(V)/Q(V)$ and assigns to a linear map $\ell\colon V \to W$ the linear map $P(V)/Q(V) \to P(W)/Q(W)$ induced by $P(\ell)$.

Remark 1.3.6. Subfunctors and quotients of a polynomial functor are themselves polynomial functors. ♠

Like the set of univariate polynomials, the set of univariate polynomial functors has an addition and multiplication. Let P, Q be polynomial functors.

Definition 1.3.7. We define the direct sum $P \oplus Q$ of P and Q as the functor $\mathrm{Vec} \to \mathrm{Vec}$ that assigns to a vector space $V \in \mathrm{Vec}$ the space $P(V) \oplus Q(V)$ and assigns to a linear map $\ell \colon V \to W$ the linear map $P(V) \oplus Q(V) \to P(W) \oplus Q(W)$ sending $(v_1, v_2) \mapsto (P(\ell)(v_1), Q(\ell)(v_2))$.

Definition 1.3.8. We define the tensor product $P \otimes Q$ of P and Q as the functor $\mathrm{Vec} \to \mathrm{Vec}$ that assigns to a vector space $V \in \mathrm{Vec}$ the space $P(V) \otimes Q(V)$ and assigns to a linear map $\ell \colon V \to W$ the linear map $P(V) \otimes Q(V) \to P(W) \otimes Q(W)$ sending $v_1 \otimes v_2 \mapsto P(\ell)(v_1) \otimes Q(\ell)(v_2)$.

Remark 1.3.9. Direct sums and tensor products of polynomial functors are themselves polynomial functors. ♠

Using the direct sum as addition and the tensor product as multiplication, the set of univariate polynomial functors gets the structure of a semiring. The constant functors serve a role similar to that of the constants in a polynomial ring and the functor T serves a role similar to that of the variable.

Example 1.3.10. Let $d \in \mathbb{N}$ be an integer. Then we get the polynomial functor

$$\begin{array}{rccc} T^{\otimes d} \colon & \mathrm{Vec} & \to & \mathrm{Vec} \\ & V & \mapsto & V^{\otimes d} \\ & \ell & \mapsto & \ell^{\otimes d} \end{array}$$

by taking the tensor product of d copies of T. We get the polynomial functor S^d by taking the subspace of $T^{\otimes d} V = V^{\otimes d}$ consisting of all symmetric tensors for all $V \in \mathrm{Vec}$. ♣

Like the univariate polynomial ring, the semiring of polynomial functors is graded.

Definition 1.3.11. Let P be a polynomial functor and let $d \geq 0$ be an integer.

(1) We say that P is homogeneous of degree d when $P(t \cdot \mathrm{id}_V) = t^d \cdot \mathrm{id}_V$ for every vector space $V \in \mathrm{Vec}$ and scalar $t \in K$.

(2) We define the degree-d part $P_{(d)}$ of P to be the subfunctor of P with

$$P_{(d)}(V) = \left\{ v \in P(V) \,\middle|\, P(t \cdot \mathrm{id}_V)(v) = t^d \cdot v \text{ for all } t \in K \right\}$$

for all $V \in \mathrm{Vec}$.

Proposition 1.3.12. *Let P be a polynomial functor. Then $P = \bigoplus_{d \geq 0} P_{(d)}$.* □

Just like polynomials, polynomial functors are the sum of their homogeneous parts. However, for polynomial functors this sum need not be finite.

Example 1.3.13. The functor $\mathrm{Vec} \to \mathrm{Vec}$ assigning $\bigoplus_{d \geq 0} \bigwedge^d V$ to V for every $V \in \mathrm{Vec}$ is a polynomial functor. For each integer $d \geq 0$, its degree-d part is the d-th alternating power functor $\bigwedge^d$. In particular, all its homogeneous parts are nonzero. ♣

Definition 1.3.14. Let P be a polynomial functor and let $d \geq 0$ be an integer. We say that P has degree d when $P_{(d)} \neq 0$ and $P_{(e)} = 0$ for all $e > d$. When $P = 0$, we say that P has degree -1.

In this thesis, we will only consider polynomial functors of finite degree.

Proposition 1.3.15. *Let P, Q be polynomial functors.*

1. *(1) The constant functor C_U from Example 1.3.2 is homogeneous of degree 0 for every vector space $U \in \mathrm{Vec}$ of positive dimension.*
2. *(2) The polynomial functor T from Example 1.3.3 is homogeneous of degree 1.*
3. *(3) Nonzero subfunctors and quotients of a homogeneous polynomial functor of degree d are again homogeneous of degree d.*
4. *(4) The degree of the direct sum of P and Q is the maximum of the degrees of P and Q.*
5. *(5) The degree of the tensor product of P and Q is the sum of the degrees of P and Q.* □

A univariate polynomial ring is generated by its constants and its variable. When $\mathrm{char}(K) = 0$, it is similarly true that every polynomial functor can be obtained from the constant functors and the functor T by taking direct sums, tensor products, subfunctors and quotients. This follows from the next lemma together with the theory of polynomial representations of general linear groups. In order to state it, we first need to define what it means to be a morphism between polynomial functors.

Definition 1.3.16. Let P, Q be polynomial functors.

(1) A natural transformation $\alpha = (\alpha_V)_V : Q \to P$ consists of a map $\alpha_V : Q(V) \to P(V)$ for every vector space $V \in \mathrm{Vec}$ such that the diagram

$$\begin{tikzcd} Q(V) \arrow[r, "\alpha_V"] \arrow[d, "Q(\ell)"'] & P(V) \arrow[d, "P(\ell)"] \\ Q(W) \arrow[r, "\alpha_W"'] & P(W) \end{tikzcd}$$

commutes for each linear map $\ell : V \to W$.

(2) A linear transformation $\alpha : Q \to P$ is a natural transformation such that α_V is a linear map for each $V \in \mathrm{Vec}$.

(3) A polynomial transformation $\alpha : Q \to P$ is a natural transformation such that α_V is a polynomial map for each $V \in \mathrm{Vec}$.

Let P be a polynomial functor and $V \in \mathrm{Vec}$ a vector space. Then the map

$$\begin{aligned} \mathrm{GL}(V) &\to \mathrm{GL}(P(V)) \\ \ell &\mapsto P(\ell) \end{aligned}$$

is a homomorphism. This gives $P(V)$ the structure of a polynomial representation of $\mathrm{GL}(V)$. When the functor P is homogeneous of degree d, then the representation is also homogeneous of degree d.

Lemma 1.3.17 (Friedlander-Suslin [22, Lemma 3.4])**.** *For any integer $d \geq 0$ and vector space $V \in \mathrm{Vec}$ with $\dim(V) \geq d$, the functor sending*

$$\begin{aligned} P &\mapsto P(V) \\ \alpha &\mapsto \alpha_V \end{aligned}$$

is an equivalence of categories between the category of homogeneous polynomial functors of degree d whose morphisms are linear transformations and the category of homogeneous polynomial representations of $\mathrm{GL}(V)$ *of degree d.* $\square$

When $\mathrm{char}(K) = 0$, every homogeneous polynomial representation of $\mathrm{GL}(V)$ of degree d is a direct sum of Schur representations $S_\lambda(V)$ where $\lambda \vdash d$. In this case, it follows that every homogeneous polynomial functor of degree d is a direct sum of Schur functors S_λ. Since the Schur functor S_λ is a subfunctor of $T^{\otimes d}$ for every partition $\lambda \vdash d$, this means that every polynomial functor can be obtained from the constant functors together with T.

The closed subsets of a polynomial functor

Polynomial functors of degree 0 are the same as finite-dimensional vector spaces. In algebraic geometry, we give such spaces the structure of a topological space. Here, we do the same with polynomial functors of arbitrary (finite) degree.

Definition 1.3.18. A closed subset X of a polynomial functor P assigns to each vector space $V \in \mathrm{Vec}$ a Zariski-closed subset $X(V)$ of $P(V)$ such that $P(\ell)$ maps $X(V)$ into $X(W)$ for each linear map $\ell \colon V \to W$.

Remark 1.3.19. A closed subset X of a polynomial functor P is the same as a subfunctor of the functor $\mathrm{Vec} \to \mathrm{Top}$ obtained by composing P with the functor $\mathrm{Vec} \to \mathrm{Top}$ that equips a vector space in Vec with the Zariski topology. In particular, the set $X(V)$ naturally comes with an action of the group $\mathrm{GL}(V)$ for every $V \in \mathrm{Vec}$. ♠

As expected of objects called closed subsets, intersections and finite unions of closed subsets of a polynomial functor are again closed subsets. For polynomial functors of degree 0, closed subsets coincide with the usual notion of a closed subset of a finite-dimensional affine space. So Theorem 1.1.6 tells us that polynomial functors of degree 0 are Noetherian. This is in fact true in general.

Theorem 1.3.20 (Draisma [17]). *Let P be a polynomial functor of finite degree. Then every descending chain of closed subsets*

$$P = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

stabilizes. $\square$

We want to view closed subsets of polynomial functors as generalizations of embedded affine varieties. In order to do so, we need to define what the maps between them are. Let $X \subseteq P$ and $Y \subseteq Q$ be closed subsets of polynomial functors.

Definition 1.3.21. A regular transformation $\beta \colon Y \to X$ is a natural transformation such that the map $\beta_V \colon Y(V) \to X(V)$ is a regular map for each $V \in \mathrm{Vec}$.

Proposition 1.3.22. *Suppose that* $\mathrm{char}(K) = 0$*. Then every regular tranformation $\beta \colon Y \to X$ is the restriction of a polynomial transformation $\alpha \colon Q \to P$, i.e., we have $\beta_V = \alpha_V|_{Y(V)}$ for each $V \in$* Vec.

Remark 1.3.23. The proposition is not valid in positive characteristic. To see this, let K be an algebraically closed field of characteristic $p > 0$ and consider the image P of the polynomial transformation $S^1 \to S^p$ sending $v \mapsto v^p$. The functor P is both a closed subfunctor of S^p and a polynomial functor itself. However, the identity transformation $P \to P$ does not extend to a polynomial transformation $S^p \to P$. Indeed, any such extension would need to be linear and would hence imply that $P(V)$ is a direct summand of $S^p(V)$ for all $V \in \mathrm{Vec}$, which is not the case. ♠

In order to prove Proposition 1.3.22, we need a better understanding of the space $\mathrm{Mor}(Q, P)$ of polynomial transformations $Q \to P$. Given two finite-dimensional vector spaces V and W, the set of polynomial maps $W \to V$ equals $K[W] \otimes V$. So it is a finitely generated $K[W]$-module and additive in V. Similar statements hold for $\mathrm{Mor}(Q, P)$. Write $P_{(0)} = C_V$, $Q_{(0)} = C_W$ for vector spaces $V, W \in \mathrm{Vec}$. Take $Q' = \bigoplus_{e \geq 1} Q_{(e)}$ and

$$R_d = \bigoplus_{\substack{(e_1,\dots,e_d) \in \mathbb{Z}_{\geq 0}^d \\ 1 \cdot e_1 + \dots + d \cdot e_d = d}}^{d} S^{e_1} Q_{(1)} \otimes \cdots \otimes S^{e_d} Q_{(d)}.$$

for $d \geq 1$.

Lemma 1.3.24. *Suppose that $d! \neq 0$ in K. Then every polynomial transformation $Q' \to P$ factors uniquely as the composition of the polynomial transformation*

$$\begin{array}{rcl} \gamma \colon Q' & \to & R_d \\ (q_1, \dots, q_d, \dots) & \mapsto & \displaystyle\sum_{\substack{(e_1,\dots,e_d) \in \mathbb{Z}_{\geq 0}^d \\ 1 \cdot e_1 + \dots + d \cdot e_d = d}}^{d} q_1^{e_1} \otimes \cdots \otimes q_d^{e_d} \end{array}$$

and a linear transformation $R_d \to P$.

Proof. Let $\alpha \colon Q' \to P$ be a polynomial transformation. Then α_V factors uniquely as the composition of γ_V and a $\mathrm{GL}(V)$-equivariant linear map $\beta_V \colon R_d(V) \to P(V)$ for every $V \in \mathrm{Vec}$. It is easy to check that $\beta = (\beta_V)_V$ is a linear transformation $R_d \to P$ such that $\alpha = \beta \circ \gamma$ and that β is unique with this property. □

Proposition 1.3.25. *The following statements hold:*

(1) The set $\mathrm{Mor}(Q, P)$ *is the direct sum of* $\mathrm{Mor}(Q, P_{(d)})$ *over all* $d \geq 0$*.*

(2) The set $\mathrm{Mor}(Q, P)$ *is the tensor product of* $K[W]$ *and* $\mathrm{Mor}(Q', P)$*.*

(2) We have $\mathrm{Mor}(Q', P_{(0)}) = V$*.*

(3) If $d! \neq 0$*, then we have* $\mathrm{Mor}(Q', P_{(d)}) = \mathrm{Hom}(R_d, P_{(d)})$*.*

In particular, the set $\mathrm{Mor}(Q, P)$ *is a free* $K[W]$*-module of finite rank.* □

Example 1.3.26. The set of polynomial transformations $K^2 \oplus S^1 \oplus S^2 \to K \oplus S^3$ consists of all natural transformations α given by maps of the form

$$\begin{array}{rcl} \alpha_V \colon K^2 \oplus V \oplus S^2 V & \to & K \oplus S^3 V \\ (\lambda_1, \lambda_2, v, w) & \mapsto & \big(f_1(\lambda_1, \lambda_2), f_2(\lambda_1, \lambda_2) \cdot v^3 + f_3(\lambda_1, \lambda_2) \cdot v \cdot w\big) \end{array}$$

where $f_1, f_2, f_3 \in K[x_1, x_2]$ are polynomials that do not depend on V. ♣

Proof of Proposition 1.3.22. Let $V \in \mathrm{Vec}$ be a vector space and let $\alpha_V \colon Q(V) \to P(V)$ be any polynomial map extending β_V. We consider α_V as an element of the space

$$P(V) \otimes K[Q(V)]$$

and note that $\alpha_V \in P(V) \otimes K[Q(V)]_{\leq k}$ for some $k \in \mathbb{N}$. If $\iota \colon U \to V$ and $\pi \colon V \to U$ are morphisms such that $\pi \circ \iota = \mathrm{id}_U$, then we see that $\alpha_U = P(\pi) \circ \alpha_V \circ Q(\iota)$ is a polynomial map $Q(U) \to P(U)$ extending β_U. So if $P(V) \otimes K[Q(V)]_{\leq k}$ contains a polynomial map extending β_V, then $P(U) \otimes K[Q(U)]_{\leq k}$ contains a polynomial map extending β_U for all $U \in \mathrm{Vec}$ with $\dim(U) \leq \dim(V)$.

Now let $W \in \mathrm{Vec}$ and assume that $\dim(W) \geq \dim(V) \geq \deg(P)$. Then we claim that

$$\bigcup_{L \colon W \twoheadrightarrow V} P(L)^* P(V)^*$$

contains a basis of $P(W)^*$. To see that this is true, first consider the case where $P = S_\lambda$ for some partition λ. Choose a basis of W. Then we also get a basis of $S_\lambda(W)$. One can check that its dual basis is contained in

$$\bigcup_{L \colon W \twoheadrightarrow V} P(L)^* P(V)^*$$

where we may even restrict to the linear maps $L \colon W \twoheadrightarrow V$ that send $\dim(W) - \dim(V)$ elements of the basis of W to zero. This proves the claim for $P = S_\lambda$. One can check that if the claim holds for polynomial functors P_1 and P_2, then it also holds for $P_1 \oplus P_2$. So, since every polynomial functor is a direct sum of Schur functors, it follows that the claim holds for all polynomial functors.

To find a polynomial map $\alpha_W \colon Q(W) \to P(W)$ extending β_W, we need to find a homomorphism

$$\alpha_W^* \colon K[P(W)] \to K[Q(W)]$$

of K-algebras such that the diagram

$$\begin{tikzcd} K[P(W)] \arrow[r, "\alpha_W^*"] \arrow[dr, "\beta_W^*"'] & K[Q(W)] \arrow[d] \\ & K[X(W)] \end{tikzcd}$$

commutes. To find such a map, it suffices to find images $\alpha_W^*(x) \in K[Q(W)]$ for elements x of some basis $\mathcal{B}$ of $P(W)^*$ such that $\alpha_W^*(x)$ maps to $\beta_W^*(x)$ in $K[X(W)]$ for all $x \in \mathcal{B}$. We consider a basis $\mathcal{B}$ that is contained in

$$\bigcup_{L \colon W \twoheadrightarrow V} P(L)^* P(V)^*$$

and let $x \in \mathcal{B}$ be some element. Let $L \colon W \twoheadrightarrow V$ be a linear map such that $P(L)^*(y) = x$ for some $y \in P(V)^*$. Then the diagrams

$$\begin{tikzcd} K[P(V)] \arrow[r, "P(L)^*"] \arrow[d, "\beta_V^*"'] & K[P(W)] \arrow[d, "\beta_W^*"] \\ K[X(V)] \arrow[r, "X(L)^*"'] & K[X(W)] \end{tikzcd}$$

and

$$\begin{tikzcd}
K[Q(V)] \arrow[r, "Q(L)^*"] \arrow[d] & K[Q(W)] \arrow[d] \\
K[X(V)] \arrow[r, "X(L)^*"'] & K[X(W)]
\end{tikzcd}$$

commute. So if $z \in K[Q(V)]$ is an element mapping to $\beta_V^*(y)$ in $K[X(V)]$, then $Q(L)^*(z) \in K[Q(W)]$ maps to $\beta_W^*(x)$. This way we can construct a polynomial map

$$\alpha_W \colon Q(W) \to P(W)$$

extending β_W from a polynomial map $\alpha_V \colon Q(V) \to P(V)$ extending β_V. Note here that if α_V is contained in $P(V) \otimes K[Q(V)]_{\leq k}$, then the map α_W we constructed is contained in $P(W) \otimes K[Q(W)]_{\leq k}$. This shows that there exists a $k \in \mathbb{N}$ such that we can extend β_V to a polynomial map $\alpha_V \in P(V) \otimes K[Q(V)]_{\leq k}$ for each $V \in \mathrm{Vec}$.

Next, let $\alpha_V \in P(V) \otimes K[Q(V)]_{\leq k}$ be a polynomial map extending β_V. Consider the projection map

$$P(V) \otimes K[Q(V)]_{\leq k} \twoheadrightarrow (P(V) \otimes K[Q(V)]_{\leq k})^{\mathrm{GL}(V)}$$

and let $\hat{\alpha}_V$ be the image of α_V under this map. Since $X(V)$ is a $\mathrm{GL}(V)$-stable Zariski-closed subset of $Q(V)$ and β_V is $\mathrm{GL}(V)$-equivariant, we see that $\hat{\alpha}_V$ also extends β_V. So for each $V \in \mathrm{Vec}$ there exists a $\mathrm{GL}(V)$-equivariant polynomial map $Q(V) \to P(V)$ contained in $P(V) \otimes K[Q(V)]_{\leq k}$ extending β_V. Next we will show that these maps can be chosen in such a way that they form a polynomial transformation $Q \to P$ extending the regular transformation β.

Let $m \gg 0$ be an integer. For each integer $n \geq m$, let Y_n be the set of all polynomial transformations $\alpha \colon Q \to P$ such that α_{K^n} is contained in

$$P(K^n) \otimes K[Q(K^n)]_{\leq k}$$

and extends β_{K^n}. We have shown that $Y_{n+1} \subseteq Y_n$ and $Y_n \neq \emptyset$ for all $n \geq m$. By Proposition 1.3.25, the set of polynomial transformations $\alpha \colon Q \to P$ such that α_{K^m} is contained in

$$P(K^m) \otimes K[Q(K^m)]_{\leq k}$$

is a finite-dimensional vector space. So it is in particular a Noetherian topological space and Y_n is a Zariski-closed subset of this space for all $n \geq m$. Hence $\bigcap_{n \geq m} Y_n$ contains some polynomial transformation $\alpha \colon Q \to P$ and this polynomial transformation extends β. □

The limit of a polynomial functor

Let P be a polynomial functor. Then we get the vector space $P_n := P(K^n)$ for $n \in \mathbb{N}$. Let $\mathrm{pr}_n \colon K^{n+1} \twoheadrightarrow K^n$ be the projection map on the first n coordinates. Then we get the map $P(\mathrm{pr}_n) \colon P_{n+1} \twoheadrightarrow P_n$. Since pr_n is surjective, so is $P(\mathrm{pr}_n)$. Every element $g \in \mathrm{GL}_n$ is a linear map $K^n \to K^n$ and hence induces a linear map $P(g) \colon P_n \to P_n$. Since P is a functor, the map $\mathrm{GL}_n \to \mathrm{GL}(P_n)$ sending $g \mapsto P(g)$ is a homomorphism and hence gives P_n the structure of a representation of GL_n. We view GL_n as a subgroup of GL_{n+1} via the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1}$ sending $g \mapsto \mathrm{Diag}(g, 1)$. Now one can check that the map $P(\mathrm{pr}_n)$ is GL_n-equivariant for every $n \in \mathbb{N}$. So, following the construction from the previous section, we get an inverse limit $P_\infty = \varprojlim_n P_n$ acted on by the group $\mathrm{GL}_\infty = \varinjlim_n \mathrm{GL}_n$ and equipped with the Zariski topology. We also get a map $\mathrm{pr}_n\colon P_\infty \twoheadrightarrow P_n$ for every $n \in \mathbb{N}$.

Given a closed subset X of P, we get a GL_∞-stable closed subset X_∞ of P_∞ by taking the inverse limit of the GL_n-stable closed subsets $X(K^n)$ of $P(K^n)$. In the other direction, given a GL_∞-stable closed subset X_∞ of P_∞, we get a closed subset X of P given by taking

$$X(V) = \{v \in P(V) \mid f(P(\ell)(v)) = 0 \text{ for all } f \in \mathcal{I}(\mathrm{pr}_n(X_\infty)) \text{ and linear maps } \ell\colon V \to K^n\}$$

for every $V \in \mathrm{Vec}$.

Definition 1.3.27. We call X_∞ the affine GL_∞-variety corresponding to X.

Proposition 1.3.28. *The map $X \mapsto X_\infty$ is a one-to-one correspondence between the closed subsets of P and the GL_∞-subvarieties of P_∞. Furthermore, we have $X_n = \mathrm{pr}_n(X_\infty)$ for all closed subsets X of P and for all $n \in \mathbb{N}$.*

Proof. Let X_∞ be the inverse limit of a closed subset X of P. Then we have $\mathrm{pr}_n(X_\infty) \subseteq X_n$ for every $n \in \mathbb{N}$. Let $v_n \in X_n$ be a point and take $v_{m+1} = P(i_m)(v_m) \in X_{m+1}$ for $m \geq n$ where $i_m\colon K^m \to K^{m+1}$ is a section of pr_m. Then $(v_m)_{m\geq n} \in X_\infty$ and hence $v_n = \mathrm{pr}_n(v_m)_{m\geq n} \in \mathrm{pr}_n(X_\infty)$. So $\mathrm{pr}_n(X_\infty) = X_n$. It now follows easily that

$$X(V) = \{v \in P(V) \mid f(P(\ell)(v)) = 0 \text{ for all } f \in \mathcal{I}(\mathrm{pr}_n(X_\infty)) \text{ and linear maps } \ell\colon V \to K^n\}.$$

Next, let X_∞ be a GL_∞-subvariety of P_∞ and let X be the associated closed subset of P. Then the inverse limit of X is contained in X_∞. Take $v \in X_\infty$ and let $\ell\colon K^m \to K^n$ be a linear map. Then $P(\ell)(v_m)$ is a limit of elements of the GL_n-orbit of v_n. We have $f(w) = 0$ for all $f \in \mathcal{I}(\mathrm{pr}_n(X_\infty))$ and all w in this orbit. Hence $v_m \in X(k^m)$. It follows that X_∞ is the inverse limit of X. $\square$

Corollary 1.3.29. *The space P_∞ is GL_∞-Noetherian, i.e. every chain of GL_∞-subvarieties of P_∞ stabilizes.*

Proof. This follows directly from the proposition together with Theorem 1.3.20. $\square$

A regular transformation $\alpha\colon Y \to X$ induces a map

$$\begin{aligned} \alpha_\infty\colon Y_\infty &\to X_\infty \\ (y_n)_n &\mapsto (\alpha_{K^n}(y_n))_n. \end{aligned}$$

Definition 1.3.30. A morphism $\alpha_\infty\colon X_\infty \to Y_\infty$ of affine GL_∞-varieties is a map that arises from a regular transformation $\alpha\colon X \to Y$ in this manner.

Multivariate polynomial functors

Fix an integer $\mu \in \mathbb{N}$. We finish this section by defining multivariate polynomial functors. Let Vec^μ be the category whose objects are tuples $V = (V_1, \ldots, V_\mu)$ of finite-dimensional vector spaces and in which a morphism $\ell\colon V \to W$ is a tuple $(\ell_1, \ldots, \ell_\mu)$ where each $\ell_i\colon V_i \to W_i$ is a linear map.

Definition 1.3.31. A μ-variate polynomial functor P assigns to every $V \in \mathrm{Vec}^\mu$ a vector space $P(V) \in \mathrm{Vec}$ and to every morphism $\ell\colon V \to W$ a linear map $P(\ell)\colon P(V) \to P(W)$ such that $P(\mathrm{id}_V) = \mathrm{id}_{(P(V)}$ for all $V \in \mathrm{Vec}^\mu$, $P(\ell_1 \circ \ell_2) = P(\ell_1) \circ P(\ell_2)$ for all morphisms $\ell_1\colon V \to W$ and $\ell_2\colon U \to V$ and the map

$$\begin{array}{rcl} \mathrm{Hom}(V,W) & \to & \mathrm{Hom}(P(V),P(W)) \\ \ell & \mapsto & P(\ell) \end{array}$$

is a polynomial map for all $V, W \in \mathrm{Vec}^\mu$.

Just like univariate polynomial functors, multivariate polynomial functors have subfunctors, quotients, direct sums and tensor products. And, just like μ-variate polynomial rings, the semiring of μ-variate polynomial functors is $\mathbb{Z}^\mu_{\geq 0}$-graded.

Example 1.3.32. Let $U \in \mathrm{Vec}$ be a fixed finite-dimensional vector space. Then the functor $C_U\colon \mathrm{Vec}^\mu \to \mathrm{Vec}$ assigning U to every $V \in \mathrm{Vec}^\mu$ and assigning id_U to every morphism is a μ-variate polynomial functor. When the space U has positive dimension, the functor C_U has degree 0. ♣

Example 1.3.33. Take $i \in [\mu]$. Then the functor $T_i\colon \mathrm{Vec}^\mu \to \mathrm{Vec}$ assigning V_i to $V \in \mathrm{Vec}^\mu$ and ℓ_i to a morphism ℓ is a homogeneous μ-variate polynomial functor of degree e_i. ♣

Example 1.3.34. Let $d = (d_1, \dots, d_\mu) \in \mathbb{N}^\mu$ be a tuple of integers and let $\lambda = (\lambda_1, \dots, \lambda_\mu)$ be a tuple of partitions such that $\lambda_i \vdash d_i$. Then we define the multivariate Schur functor S_λ to be the tensor product of the functors $S_{\lambda_1} \circ T_1, \dots, S_{\lambda_\mu} \circ T_\mu$, i.e., we have $S_\lambda V = S_{\lambda_1} V_1 \otimes \cdots \otimes S_{\lambda_\mu} V_\mu$ for each $V \in \mathrm{Vec}^\mu$. The functor S_λ is a homogeneous μ-variate polynomial functor of degree d. ♣

The definitions of linear and polynomial transformations between multivariate polynomial functors generalize as one expects and a multigraded version of Proposition 1.3.25 holds. Given a μ-variate polynomial functor P and a tuple $V \in \mathrm{Vec}^\mu$ of vector spaces, the vector space $P(V)$ naturally has the structure of a polynomial representation of $\mathrm{GL}(V) := \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_\mu)$. When P is homogeneous of degree $d \in \mathbb{Z}^\mu_{\geq 0}$, then so is the representation $P(V)$. We again have the following lemma.

Lemma 1.3.35 (Touze [34, Théorème 7.2]). *For any tuples $d \in \mathbb{Z}^\mu_{\geq 0}$ and $V \in \mathrm{Vec}^\mu$ such that $\dim(V_i) \geq d_i$ for all $i \in [\mu]$, the functor sending*

$$\begin{array}{rcl} P & \mapsto & P(V) \\ \alpha & \mapsto & \alpha_V \end{array}$$

is an equivalence of categories between the category of homogeneous μ-variate polynomial functors of degree d whose morphisms are linear transformations and the category of homogeneous polynomial representations of $\mathrm{GL}(V)$ of degree d. □

As a consequence of the lemma, we find that, when $\mathrm{char}(K) = 0$, every μ-variate polynomial functor is a direct sum of multivariate Schur functors. So in particular, every μ-variate polynomial functor can be obtained from the constant functors C_U and the functors $T_1, \dots, T_\mu$ by taking subfunctors, quotients, direct sums and tensor products in this case.

The definition of a closed subset of a polynomial functor also generalizes. A closed subset of a μ-variate polynomial functor is itself a functor from Vec^μ to Top. The Noetherianity of multivariate polynomial functors easily follows from the univariate case. Let P be a μ-variate polynomial functor of finite degree and let $\Delta\colon \mathrm{Vec} \to \mathrm{Vec}^\mu$ be the functor assigning $(V,\dots,V)$ to $V \in \mathrm{Vec}$ and $(\ell,\dots,\ell)$ to a linear map ℓ.

Lemma 1.3.36. *Let $X, Y \subseteq P$ be closed subsets such that $X \circ \Delta = Y \circ \Delta$. Then $X = Y$.*

Proof. Let $V \in \mathrm{Vec}^\mu$ and $W \in \mathrm{Vec}$ be such that $\dim(V_i) \leq \dim(W)$ for each $i \in [\mu]$. Also, let $\iota_i\colon V_i \hookrightarrow W$ and $\pi_i\colon W \twoheadrightarrow V_i$ be linear maps such that $\pi_i \circ \iota_i = \mathrm{id}_{V_i}$ for each $i \in [\mu]$. Then we have

$$X(V) = P(\pi_1,\dots,\pi_\mu)(P(\iota_1,\dots,\iota_\mu)(X(V))) \subseteq P(\pi_1,\dots,\pi_\mu)(X(W,\dots,W)) \subseteq X(V)$$

since X is a closed subset of P. Hence $X(V) = P(\pi_1,\dots,\pi_\mu)(X(W,\dots,W))$. Similarly, we have $Y(V) = P(\pi_1,\dots,\pi_\mu)(Y(W,\dots,W))$. Since $(W,\dots,W) = \Delta(W)$, it follows that

$$X(V) = P(\pi_1,\dots,\pi_\mu)((X \circ \Delta)(W)) = P(\pi_1,\dots,\pi_\mu)((Y \circ \Delta)(W)) = Y(V)$$

for each $V \in \mathrm{Vec}$. Hence $X = Y$. □

Theorem 1.3.37. *Every descending chain of closed subsets of P stabilizes.*

Proof. Let

$$P = X_0 \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

be a descending chain of closed subsets of P. Then we get a descending chain

$$P \circ \Delta = X_0 \circ \Delta \supseteq X_1 \circ \Delta \supseteq X_2 \circ \Delta \supseteq X_3 \circ \Delta \supseteq X_4 \circ \Delta \supseteq \dots$$

of closed subsets of the univariate polynomial functor $P \circ \Delta$. This chain must stabilize by Theorem 1.3.20. Hence the original chain must also stabilize by the previous lemma. □

A natural transformation between closed subsets of multivariate polynomial functors is regular when each of the maps it consists of is regular and every regular transformation is the restriction of a polynomial transformation when $\mathrm{char}(K) = 0$.

Finally, we construct the limits of multivariate polynomial functors. Let P be a μ-variate polynomial functor. Then we take $P_n := P(K^n,\dots,K^n)$ for every integer $n \in \mathbb{N}$. Let $\mathrm{pr}_n\colon K^{n+1} \twoheadrightarrow K^n$ be the projection map on the first n coordinates. Then we get the maps $P(\mathrm{pr}_n,\dots,\mathrm{pr}_n)\colon P_{n+1} \twoheadrightarrow P_n$ and we define $P_\infty := \varprojlim_n P_n$. The map

$$\begin{array}{rcl} \mathrm{GL}_n^\mu & \to & \mathrm{GL}(P_n) \\ (\ell_1,\dots,\ell_\mu) & \mapsto & P(\ell_1,\dots,\ell_\mu) \end{array}$$

gives P_n the structure of a representation of GL_n^μ. We view GL_n^μ as a subgroup of GL_{n+1}^μ via the map sending $(g_1,\dots,g_\mu) \mapsto (\mathrm{Diag}(g_1,1),\dots,\mathrm{Diag}(g_\mu,1))$. This makes map $P(\mathrm{pr}_n,\dots,\mathrm{pr}_n)$ into a GL_n^μ-equivariant map. This gives us an action of the direct limit $\mathrm{GL}_\infty^\mu = \varinjlim_n \mathrm{GL}_n^\mu$ on P_∞.

Remark 1.3.38. Alternatively, we could define P_∞ as the inverse limit of the maps $P(\mathrm{pr}_{n_1}, \dots, \mathrm{pr}_{n_\mu})$ where $n_1, \dots, n_\mu \in \mathbb{N}$ can be chosen independent from each other. As every space $P(K^{n_1}, \dots, K^{n_\mu})$ in this inverse system has a space of the form $P(K^n, \dots, K^n)$ above it, the inverse limit is naturally isomorphic to the space P_∞ we use. ♠

We can move between closed subsets X of P and GL_∞^μ-subvarieties X_∞ of P_∞ as expected.

Proposition 1.3.39. *The map $X \mapsto X_\infty$ is a one-to-one correspondence between the closed subsets of P and the GL_∞^μ-subvarieties of P_∞. Furthermore, we have $X_n = \mathrm{pr}_n(X_\infty)$ for all closed subsets X of P and for all $n \in \mathbb{N}$.*

Proof. This follows by combining Lemma 1.3.36 with the arguments from Proposition 1.3.28. □

Corollary 1.3.40. *The space P_∞ is GL_∞^μ-Noetherian.*

Proof. This follows directly from the proposition together with Theorem 1.3.37. □

A regular transformation $\alpha\colon Y \to X$ induces a map $\alpha_\infty\colon Y_\infty \to X_\infty$ and we call the maps that arise in this way the morphisms $Y \to X$.

1.4 Rank functions

Many rank functions can be used to define closed subsets of polynomial functors. And, many of these rank functions can be defined using polynomial transformations from smaller polynomial functors. This is no coincidence: see Theorem 4.2.5. In this section, we list several examples of such rank functions in order to give some intuition for this theorem.

Example 1.4.1. Let $k \in \mathbb{Z}_{\geq 0}$ be an integer and let $P = T^{\oplus k+1}$ be the univariate polynomial functor sending $V \in \mathrm{Vec}$ to $V^{\oplus k+1}$. The elements of P are $(k+1)$-tuples of vectors from the same vector space. Consider the closed subset $X \subseteq P$ defined by

$$X(V) = \{(v_1, \dots, v_{k+1}) \in V^{\oplus k+1} \mid v_1, \dots, v_{k+1} \text{ are linearly dependent}\}$$

for all $V \in \mathrm{Vec}$. This is a closed subset of P, because the condition

$$\dim \operatorname{span}\{v_1, \dots, v_{k+1}\} \leq k$$

is closed and functorial in $(v_1, \dots, v_{k+1})$. For vectors $v_1, \dots, v_m$ from the same vector space $V \in \mathrm{Vec}$, note that

$$\dim \operatorname{span}\{v_1, \dots, v_m\}$$

is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that $(v_1, \dots, v_m)$ is contained in the image of the polynomial transformation $\alpha\colon K^{\oplus k \times m} \oplus T^{\oplus k} \to T^{\oplus m}$ given by the maps

$$\begin{aligned} \alpha_V\colon K^{k \times m} \oplus V^{\oplus k} &\to V^{\oplus m} \\ (A, v_1, \dots, v_k) &\mapsto (v_1, \dots, v_k)A \end{aligned}$$

for $V \in \mathrm{Vec}$. Here $(v_1, \dots, v_k)$ is treated as a $1 \times k$ matrix. ♣

Example 1.4.2. Let $P = T_1 \otimes T_2$ be the 2-variate polynomial functor sending a pair $(V, W) \in \mathrm{Vec}^2$ to their tensor product $V \otimes W$. The elements of P are matrices. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V, W) = \{A \in V \otimes W \mid \mathrm{rk}(A) \leq k\}$$

for all $(V, W) \in \mathrm{Vec}^2$. These are closed subsets of P, because the condition

$$\mathrm{rk}(A) \leq k$$

is closed and functorial in A. For a finite-by-finite matrix A, note that $\mathrm{rk}(A)$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that A is contained in the image of the polynomial transformation $\alpha\colon (T_1 \oplus T_2)^{\oplus k} \to P$ given by the maps

$$\begin{array}{rcl} \alpha_{(V,W)}\colon (V \oplus W)^{\oplus k} & \to & V \otimes W \\ (v_1, w_1, \dots, v_k, w_k) & \mapsto & v_1 \otimes w_1 + \dots + v_k \otimes w_k \end{array}$$

for $(V, W) \in \mathrm{Vec}^2$. ♣

Fix an integer $m \in \mathbb{Z}_{\geq 2}$.

Example 1.4.3. Define the rank of a tuple $(A_1, \dots, A_m)$ of matrices of the same size as

$$\mathrm{rk}(A_1, \dots, A_m) := \min\left\{\mathrm{rk}(\mu_1 A_1 + \dots + \mu_m A_m) \,\middle|\, (\mu_1 : \dots : \mu_m) \in \mathbb{P}^{m-1}\right\}.$$

This rank was first defined in [18]. Let $P = (T_1 \otimes T_2)^{\oplus m}$ be the 2-variate polynomial functor sending a pair $(V, W) \in \mathrm{Vec}^2$ to $(V \otimes W)^{\oplus k}$. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V, W) = \{(A_1, \dots, A_m) \in (V \otimes W)^{\oplus m} \mid \mathrm{rk}(A_1, \dots, A_m) \leq k\}$$

for all $(V, W) \in \mathrm{Vec}^2$. These are closed subsets of P, because the condition

$$\mathrm{rk}(A_1, \dots, A_m) \leq k$$

is closed and functorial in $(A_1, \dots, A_m)$. For finite-by-finite matrices $A_1, \dots, A_m$ of the same size, note that $\mathrm{rk}(A_1, \dots, A_m)$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that $(A_1, \dots, A_m)$ is contained in the image of the polynomial transformation

$$\alpha\colon K^{m \times m} \oplus (T_1 \oplus T_2)^{\oplus k} \oplus (T_1 \otimes T_2)^{\oplus m-1} \to P$$

given by the maps

$$\begin{array}{rcl} \alpha_{(V,W)}\colon K^{m \times m} \oplus (V \oplus W)^{\oplus k} \oplus (V \otimes W)^{\oplus m-1} & \to & (V \otimes W)^{\oplus m} \\ (B, v_1, w_1, \dots, v_k, w_k, A_1, \dots, A_{m-1}) & \mapsto & (A_1, \dots, A_{m-1}, v_1 \otimes w_1 + \dots + v_k \otimes w_k)B \end{array}$$

for $(V, W) \in \mathrm{Vec}^2$. Here $(A_1, \dots, A_{m-1}, v_1 \otimes w_1 + \dots + v_k \otimes w_k)$ is treated as a $1 \times m$ matrix. ♣

Example 1.4.4. Let $P = T_1 \otimes \dots \otimes T_m$ be the m-variate polynomial functors sending a tuple $(V_1, \dots, V_m) \in \mathrm{Vec}^m$ to $V_1 \otimes \dots \otimes V_m$. The elements of P are m-way tensors. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V_1, \dots, V_m) = \overline{\{t \in V_1 \otimes \dots \otimes V_m \mid \mathrm{rk}(t) \leq k\}}$$

for all $(V_1, \ldots, V_m) \in \mathrm{Vec}^m$. These are closed subsets of P, because the condition

$$\mathrm{rk}(t) \leq k$$

is functorial in t. For a tensor t, note that its tensor rank $\mathrm{rk}(t)$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that t is contained in the image of the polynomial transformation

$$\alpha\colon (T_1 \oplus \cdots \oplus T_m)^{\oplus k} \to P$$

given by the maps

$$\begin{array}{rcl} \alpha_{(V_1,\ldots,V_m)}\colon (V_1 \oplus \cdots \oplus V_m)^{\oplus k} & \to & V_1 \otimes \cdots \otimes V_m \\ (v_{11}, \ldots, v_{m1}, \ldots, v_{1k}, \ldots, v_{mk}) & \mapsto & v_{11} \otimes \cdots \otimes v_{m1} + \cdots + v_{1k} \otimes \cdots \otimes v_{mk} \end{array}$$

for $(V_1, \ldots, V_m) \in \mathrm{Vec}^m$. ♣

Example 1.4.5. Again take $P = T_1 \otimes \cdots \otimes T_m$. A nonzero tensor $t \in V_1 \otimes \cdots \otimes V_m$ has slice rank 1 when it is of the form $t' \otimes v_i$ for some $i \in [m]$, $t' \in V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_m$ and $v_i \in V_i$. The slice rank $\mathrm{slrk}(t)$ of a tensor t is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that t is a sum of k tensors with slice rank 1. The slice rank of a tensor was first defined in [32]. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V_1, \ldots, V_m) = \{t \in V_1 \otimes \cdots \otimes V_m \mid \mathrm{slrk}(t) \leq k\}$$

for all $(V_1, \ldots, V_m) \in \mathrm{Vec}^m$. These are closed subsets of P, because the condition

$$\mathrm{slrk}(t) \leq k$$

is closed and functorial in t. For a tensor t, note that its slice rank is the minimal sum $k_1 + \cdots + k_m$ of integers $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ such that t is contained in the image of the polynomial transformation

$$\alpha\colon \bigoplus_{i=1}^{m} \Big((T_1 \otimes \cdots \otimes \hat{T}_i \otimes \cdots \otimes T_m) \oplus T_i\Big)^{\oplus k_i} \to P$$

given by the maps

$$\begin{array}{rcl} \alpha_{(V_1,\ldots,V_m)}\colon \bigoplus_{i=1}^{m} \Big((V_1 \otimes \cdots \otimes \hat{V}_i \otimes \cdots \otimes V_m) \oplus V_i\Big)^{\oplus k_i} & \to & V_1 \otimes \cdots \otimes V_m \\ ((t_{ij}, v_{ij}))_{i,j} & \mapsto & \sum_{i=1}^{m} \sum_{j=1}^{k_i} t_{ij} \otimes v_{ij} \end{array}$$

for $(V_1, \ldots, V_m) \in \mathrm{Vec}^m$. ♣

Fix an integer $d \in \mathbb{Z}_{\geq 2}$.

Example 1.4.6. Let $P = S^d$ be the univariate polynomial functors sending a $V \in \mathrm{Vec}$ to its dth symmetric power. The elements of P are homogeneous polynomials of degree d. The Waring rank of a polynomial $f \in S^d(V)$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that

$$f = \ell_1^d + \cdots + \ell_k^d$$

for some linear forms $\ell_1, \dots, \ell_k \in V$. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V) = \overline{\{f \in S^d(V) \mid \mathrm{wrk}(f) \leq k\}}$$

for all $V \in \mathrm{Vec}$. These are closed subsets of P, because the condition

$$\mathrm{wrk}(t) \leq k$$

is functorial in f. For a homogeneous polynomial f of degree d, note that its Waring rank is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that f is contained in the image of the polynomial transformation $\alpha \colon T^{\oplus k} \to P$ given by the maps

$$\begin{array}{rcl} \alpha_V \colon V^{\oplus k} & \to & S^d(V) \\ (\ell_1, \dots, \ell_k) & \mapsto & \ell_1^d + \dots + \ell_k^d \end{array}$$

for $V \in \mathrm{Vec}$. ♣

Example 1.4.7. Again take $P = S^d$. The strength of a polynomial $f \in S^d(V)$ is the minimal $k \in \mathbb{Z}_{\geq 0}$ such that

$$f = g_1 \cdot h_1 + \dots + g_k \cdot h_k$$

for some homogeneous polynomials $g_1, h_1, \dots, g_k, h_k$ of degree $< d$. The strength of a polynomial was first defined in [2]. For $k \in \mathbb{Z}_{\geq 0}$, consider the closed subset $X \subseteq P$ defined by

$$X(V) = \overline{\{f \in S^d(V) \mid \mathrm{str}(f) \leq k\}}$$

for all $V \in \mathrm{Vec}$. These are closed subsets of P, because the condition

$$\mathrm{str}(t) \leq k$$

is functorial in f. For a homogeneous polynomial f of degree d, note that its strength is the minimal sum $k_1 + \dots + k_{\lfloor d/2 \rfloor}$ of integer $k_e \in \mathbb{Z}_{\geq 0}$ such that f is contained in the image of the polynomial transformation

$$\alpha \colon \bigoplus_{e=1}^{\lfloor d/2 \rfloor} (S^e \otimes S^{d-e})^{\oplus k_e} \to P$$

given by the maps

$$\begin{array}{rcl} \alpha_V \colon \displaystyle\bigoplus_{e=1}^{\lfloor d/2 \rfloor} (S^e(V) \otimes S^{d-e}(V))^{\oplus k_e} & \to & S^d(V) \\ ((g_{ej}, h_{ej}))_{e,j} & \mapsto & \sum_{e=1}^{\lfloor d/2 \rfloor} \sum_{j=1}^{k_e} g_{ej} \cdot h_{ej} \end{array}$$

for $V \in \mathrm{Vec}$. ♣

Chapter 2

Inverse limits of locally diagonal sequences

In this chapter, the field K is assumed to be infinite.

2.1 Introduction

Consider a sequence of embeddings

$$G_1 \xhookrightarrow{\iota_1} G_2 \xhookrightarrow{\iota_2} G_3 \xhookrightarrow{\iota_3} \dots$$

built up out of homomorphisms between the following classical algebraic groups

$$\begin{aligned}
\mathrm{A}_{n-1}\colon\ \mathrm{SL}_n &= \{A \in \mathrm{GL}_n \mid \det(A) = 1\}\\
\mathrm{B}_n\ :\ \mathrm{O}_{2n+1} &= \left\{A \in \mathrm{GL}_{2n+1} \,\middle|\, A\begin{pmatrix} & & I_n\\ & 1 & \\ I_n & & \end{pmatrix}A^T = \begin{pmatrix} & & I_n\\ & 1 & \\ I_n & & \end{pmatrix}\right\}\\
\mathrm{C}_n\ :\ \mathrm{Sp}_{2n} &= \left\{A \in \mathrm{GL}_{2n} \,\middle|\, A\begin{pmatrix} & I_n\\ -I_n & \end{pmatrix}A^T = \begin{pmatrix} & I_n\\ -I_n & \end{pmatrix}\right\}\\
\mathrm{D}_n\ :\ \mathrm{O}_{2n} &= \left\{A \in \mathrm{GL}_{2n} \,\middle|\, A\begin{pmatrix} & I_n\\ I_n & \end{pmatrix}A^T = \begin{pmatrix} & I_n\\ I_n & \end{pmatrix}\right\}
\end{aligned}$$

each of which we view as embedded subgroups of GL_n for some $n \in \mathbb{N}$. Let G, H be such groups, let V, W be their standard representations and consider K as the trivial representation of G. In [3], an embedding $G \hookrightarrow H$ is called diagonal if

$$W \cong V^{\oplus l} \oplus (V^*)^{\oplus r} \oplus K^{\oplus z}$$

as representations of G for some $l, r, z \in \mathbb{Z}_{\geq 0}$ with $l + r \geq 1$. The triple (l, r, z) is called the signature of the embedding. If G is of type B, C or D, then the representation V is isomorphic to V^*. In this case, we will always assume that $r = 0$, which makes the pair (l, z) unique, and we also denote the signature by (l, z). For more on diagonal embeddings, see the previous chapter.

Example 2.1.1. For all $n \in \mathbb{N}$ and $l, r, z \in \mathbb{Z}_{\geq 0}$ with $l + r \geq 1$, the map

$$\begin{array}{rcl} \mathrm{SL}_n & \to & \mathrm{SL}_{(l+r)n+z} \\ A & \mapsto & \mathrm{Diag}(\underbrace{A, \dots, A}_{l}, \underbrace{A^{-T}, \dots, A^{-T}}_{r}, I_z) \end{array}$$

is a diagonal embedding with signature (l, r, z). ♣

We will assume that the sequence

$$G_1 \xhookrightarrow{\iota_1} G_2 \xhookrightarrow{\iota_2} G_3 \xhookrightarrow{\iota_3} \dots$$

consists of diagonal embeddings and we let G be its direct limit. We have an associated sequence of linear maps

$$\mathfrak{g}_1 \hookrightarrow \mathfrak{g}_2 \hookrightarrow \mathfrak{g}_3 \hookrightarrow \dots$$

where $\mathfrak{g}_i$ is the Lie algebra of G_i. Let V be the inverse limit of the sequence

$$\mathfrak{g}_1^* \twoheadleftarrow \mathfrak{g}_2^* \twoheadleftarrow \mathfrak{g}_3^* \twoheadleftarrow \dots$$

obtained by dualizing the previous sequence. Then V has a natural action of the group G. The goal of this chapter is to prove the following theorem.

Theorem 2.1.2. *Assume that one of the following conditions holds:*

(a) The group G_i has type A *for infinitely many $i \in \mathbb{N}$.*

(b) The characteristic of K does not equal 2.

Then the space V is G-Noetherian.

Remark 2.1.3. We would like to point out that the G-Noetherianity of V also follows from [20, Theorem 1.2] when all groups G_i have the same type and all signatures are of the form $(1, 0, z)$. The same is true for Theorem 2.1.4 below. ♠

Note that the conjugation-actions of GL_n and SL_n on $\mathfrak{gl}_n$ have the same orbits. This observation might make one hope that one can prove case (a) of the theorem by considering sequences of homomorphisms between general linear groups instead of special linear groups. This turns out to indeed be the case. Consider sequences of the form

$$\mathrm{GL}_{n_1} \xhookrightarrow{\iota_1} \mathrm{GL}_{n_2} \xhookrightarrow{\iota_2} \mathrm{GL}_{n_n} \xhookrightarrow{\iota_3} \dots$$

consisting of embeddings of the form

$$\begin{array}{rrcl} \iota_i\colon & \mathrm{GL}_{n_i} & \to & \mathrm{GL}_{n_{i+1}} \\ & A & \mapsto & \mathrm{Diag}(\underbrace{A, \dots, A}_{l_i}, \underbrace{A^{-T}, \dots, A^{-T}}_{r_i}, I_{z_i}) \end{array}$$

with $l_i, r_i, z_i \in \mathbb{Z}_{\geq 0}$ such that $l_i + r_i \geq 1$ and $n_{i+1} = (l_i + r_i)n_i + z_i$. Let G be the direct limit of this sequence. Then, similarly to before, the group G acts naturally on the inverse limit V of the sequence

$$\mathfrak{gl}_{n_1} \twoheadleftarrow \mathfrak{gl}_{n_2} \twoheadleftarrow \mathfrak{gl}_{n_3} \twoheadleftarrow \dots$$

consisting of the maps

$$\begin{array}{ccc} & \mathfrak{gl}_{n_{i+1}} & \to & \mathfrak{gl}_{n_i} \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i1} & \dots & P_{l_il_i} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r_i} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r_i1} & \dots & Q_{r_ir_i} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix} & & \mapsto & \displaystyle\sum_{k=1}^{l_i} P_{kk} - \sum_{\ell=1}^{r_i} Q_{\ell\ell}^T. \end{array}$$

Here each $\bullet$ represents some matrix of the appropriate size. Take

$$\alpha = \#\{i \mid l_i > 1\},\ \beta = \#\{i \mid r_i > 0\},\ \gamma = \#\{i \mid z_i > 0\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}.$$

We assume that $\alpha + \beta + \gamma = \infty$ since V is finite-dimensional otherwise. Based on α, β, γ we distinguish the following cases:

(1) $\alpha + \beta < \infty$;

(2) $\alpha + \beta = \gamma = \infty$;

(3a) $\beta = \infty, \gamma < \infty$ and $\mathrm{char}(K) \neq 2$;

(3b) $\beta = \infty, \gamma < \infty$ and $\mathrm{char}(K) = 2$; and

(4) $\beta + \gamma < \infty$.

Note here that if $\gamma < \infty$, then $n_i | n_{i+1}$ for all $i \gg 0$. Denote the element of V representated by the sequence of zero matrices by 0. The following theorem completely classifies the G-stable closed subsets of V.

Theorem 2.1.4. *The space V is G-Noetherian. Any G-stable closed subset of V is a finite union of irreducible G-stable closed subsets. The irreducible G-stable closed subsets of V are* $\{0\}$ *and V together with*

$$\{(P_i)_i \in V \mid \forall i \gg 0\colon\ \mathrm{rk}(P_i, I_{n_i}) \leq k\},\ \{(P_i)_i \in V \mid \forall i \gg 0\colon\ \mathrm{rk}(P_i - \lambda I_{n_i}) \leq k\}$$

for $\lambda \in K$ and $k \in \mathbb{Z}_{\geq 0}$ in case (1) *and together with*

$$\{(P_i)_i \in V \mid \forall i \gg 0\colon\ \mathrm{tr}(P_i) = \mu\}$$

for $\mu \in K$ in cases (3b) *and* (4).

When proving case (a) of Theorem 2.1.2, we may assume that each group G_i is of type A. And we will show that, when this is the case, the space from Theorem 2.1.2 is a quotient of the space from Theorem 2.1.4 if we choose the tuples (l_i, r_i, z_i) to be the signatures from our orginal sequence of diagional embeddings. This allows us to prove case (a) of Theorem 2.1.2.

Outline of this chapter. There are many useful ways in which we can change the sequence of groups

$$G_1 \xhookrightarrow{\iota_1} G_2 \xhookrightarrow{\iota_2} G_3 \xhookrightarrow{\iota_3} \dots$$

without changing its direct limit G or the inverse limit V of the associated sequence

$$\mathfrak{g}_1^* \twoheadleftarrow \mathfrak{g}_2^* \twoheadleftarrow \mathfrak{g}_3^* \twoheadleftarrow \dots .$$

We may in particular assume that all groups G_i are of the same type and we will prove Theorem 2.1.2 for each type seperately. These proofs nevertheless share the same overall structure. The first section of this chapter in devoted to these sequence changes and the shared structure of the proofs. After this, we prove Theorem 2.1.4 and Theorem 2.1.2 for groups of type A, C, D and B in that order in five more sections.

2.2 Structure of the proofs

In this section, we reduce Theorem 2.1.2 to a number of cases and we outline the structure that the proofs of each of those cases and of Theorem 2.1.4 share.

Reduction to standard diagonal embeddings

When the vector space V is finite-dimensional over K, Theorem 2.1.2 becomes trivial. So we will only consider the cases where V is infinite-dimensional. For all $i \in \mathbb{N}$, let (l_i, r_i, z_i) be the signature of the embedding $\iota_i \colon G_i \hookrightarrow G_{i+1}$. When G_i is of type B, C or D, we will assume that $r_i = 0$. The following lemma tells us that we can assume that $l_i \geq r_i$ for all $i \in \mathbb{N}$.

Lemma 2.2.1. *For all $i \in \mathbb{N}$, let $\sigma_i \colon G_i \to G_i$ be the automorphism sending $A \mapsto A^{-T}$ and take $k_i \in \mathbb{Z}/2\mathbb{Z}$. Then the bottom row of the commutative diagram*

$$\begin{tikzcd}
G_1 \arrow[r, hook, "\iota_1"] \arrow[d, "\sigma_1^{k_1}"] & G_2 \arrow[r, hook, "\iota_2"] \arrow[d, "\sigma_2^{k_2}"] & G_3 \arrow[r, hook, "\iota_3"] \arrow[d, "\sigma_3^{k_3}"] & \dots \\
G_1 \arrow[r, hook] & G_2 \arrow[r, hook] & G_3 \arrow[r, hook] & \dots
\end{tikzcd}$$

is a sequence of diagonal embeddings with signatures $\sigma^{k_i+k_{i+1}}(l_i, r_i, z_i)$ where σ acts by permuting the first two entries. □

The lemma follows from the fact that the automorphism $G_i \to G_i, A \mapsto A^{-T}$ is diagonal and its own inverse. We can choose the k_i recursively so that $l_i \geq r_i$ for all $i \in \mathbb{N}$ in the bottom sequence. Since the vertical maps are isomorphisms and the diagram commutes, the bottom sequence gives rise to isomorphic G and V. This allows us to indeed assume that $l_i \geq r_i$.

Let G be a classical group of type A, B, C or D. Let $l, r, z \in \mathbb{Z}_{\geq 0}$ be integers with $r = 0$ if G is not of type A. Assume that β_1, β_2 are nondegenerate G-invariant bilinear forms on $V^{\oplus l} \oplus (V^*)^{\oplus r} \oplus K^{\oplus z}$.

Lemma 2.2.2. *Assume that $K = \overline{K}$ and that one of the following conditions holds:*

(a) β_1 and β_2 are both skew-symmetric.

(b) β_1 and β_2 are both symmetric and $\mathrm{char}(K) \neq 2$.

Then there exists a G-equivariant automorphism φ of $V^{\oplus l} \oplus (V^*)^{\oplus r} \oplus K^{\oplus z}$ such that

$$\beta_2(\varphi(v), \varphi(w)) = \beta_1(v, w)$$

for all $v, w \in V^{\oplus l} \oplus (V^*)^{\oplus r} \oplus K^{\oplus z}$.

Proof. First suppose that $l = r = 0$. In this case, the lemma reduces to the well-known statement that the matrices corresponding to β_1 and β_2 are congruent. In general, Schur's Lemma splits the lemma into the cases $z = 0$ and $l = r = 0$. Suppose that $z = 0$. If G is of type B, C or D, then Schur's Lemma also shows the matrices corresponding to β_1 and β_2 are Kronecker products of $l \times l$ matrices with the identity matrix. If G is of type A, then Schur's Lemma shows that $l = r$ and that the matrices corresponding to β_1 and β_2 are Kronecker products of $l \times l$ matrices with the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

Here we order the copies of V and V^* alternatingly. This reduces the case $z = 0$ to the the case $l = r = 0$. □

Let $f, g \colon G \to H \subseteq \mathrm{GL}_n$ be two diagonal embeddings with signature (l, r, z).

Lemma 2.2.3. *If the type of H is* B, C *or* D, *assume that $K = \overline{K}$. If the type of H is* B *or* D, *assume in addition that* $\mathrm{char}(K) \neq 2$. *Then there is a $P \in H$ such that the isomorphism $\pi \colon H \to H, A \mapsto PAP^{-1}$ makes the diagram*

$$\begin{tikzcd} G \arrow[r, hook, "f"] \arrow[d, "\mathrm{id}"] & H \arrow[d, "\pi"] \\ G \arrow[r, hook, "g"] & H \end{tikzcd}$$

commute.

Proof. The maps f and g both induce an isomorphism

$$K^n \cong V^{\oplus l} \oplus (V^*)^{\oplus r} \oplus K^{\oplus z}$$

of representations of G. This means that there are matrices Q, R such that

$$Qf(A)Q^{-1} = Rg(A)R^{-1} = \mathrm{Diag}(A, \ldots, A, A^{-T}, \ldots, A^{-T}, I_z)$$

for all $A \in G$ where the block-diagonal matrix has l blocks A and r blocks A^{-T}. If H is of type A, then we take $P = \lambda R^{-1}Q$ for some $\lambda \in K$ such that $P \in \mathrm{SL}_n$ and see that the isomorphism $\pi \colon H \to H, A \mapsto PAP^{-1}$ makes the diagram commute.

Assume that H is not of type A. Then $H = \{g \in \mathrm{GL}_n \mid g^T B g = B\}$ for some matrix $B \in \mathrm{GL}_n$. Let β_1 and β_2 be the G-invariant bilinear forms on K^n defined by $Q^{-T}BQ^{-1}$ and $R^{-T}BR^{-1}$. By the previous lemma, there exists a G-equivariant automorphism φ of K^n such that

$$\beta_2(\varphi(v), \varphi(w)) = \beta_1(v, w)$$

for all $v, w \in K^n$. Let S be the matrix corresponding to φ. Then

$$S^T Q^{-T} B Q^{-1} S = R^{-T} B R^{-1}$$

and

$$S\,\mathrm{Diag}(A,\dots,A,A^{-T},\dots,A^{-T},I_z) = \mathrm{Diag}(A,\dots,A,A^{-T},\dots,A^{-T},I_z)S$$

for all $A \in G$. Take $P = R^{-1}S^{-1}Q$. Then $P^{-1} \in H$ and therefore $P \in H$. The isomorphism $\pi\colon H \to H, A \mapsto PAP^{-1}$ makes the diagram commute. □

Proposition 2.2.4. *For every $i \in \mathbb{N}$, let $\iota_i'\colon G_i \hookrightarrow G_{i+1}$ be a diagonal embedding with the same signature (l_i, r_i, z_i) as ι_i. If the type of G_i is* B, C *or* D *for any $i \in \mathbb{N}$, assume that $K = \overline{K}$. If the type of G_i is* B *or* D *for any $i \in \mathbb{N}$, assume in addition that* $\mathrm{char}(K) \neq 2$. *Then there exist isomorphisms $\varphi_i\colon G_i \to G_i$ making the diagram*

$$\begin{tikzcd}
G_1 \arrow[r, hook, "\iota_1"] \arrow[d, "\mathrm{id}"] & G_2 \arrow[r, hook, "\iota_2"] \arrow[d, "\varphi_2"] & G_3 \arrow[r, hook, "\iota_3"] \arrow[d, "\varphi_3"] & \dots \\
G_1 \arrow[r, hook, "\iota_1'"] & G_2 \arrow[r, hook, "\iota_2'"] & G_3 \arrow[r, hook, "\iota_3'"] & \dots
\end{tikzcd}$$

commute.

Proof. We construct the isomorphisms φ_i recursively in such a way that the φ_i are also diagonal embeddings with signature $(1, 0, 0)$. Write $\varphi_1 = \mathrm{id}$, let $i \geq 2$ and assume that φ_{i-1} has already been constructed. Then $\iota_{i-1}' \circ \varphi_{i-1}$ has the same signature as ι_{i-1}. So by the previous lemma, there exists an isomorphism φ_i making the diagram

$$\begin{tikzcd}
G_{i-1} \arrow[r, hook, "\iota_{i-1}"] \arrow[d, "\mathrm{id}"] & G_i \arrow[d, "\varphi_i"] \\
G_{i-1} \arrow[r, hook, "\iota_{i-1}' \circ \varphi_{i-1}"] & G_i
\end{tikzcd}$$

commute that also has signature $(1, 0, 0)$ as a diagonal embedding. □

Recall that, when we replace

$$G_1 \xhookrightarrow{\iota_1} G_2 \xhookrightarrow{\iota_2} G_3 \xhookrightarrow{\iota_3} \dots$$

by supersequences or infinite subsequences, we do not change G or V. Therefore we may assume that each group G_i has the same type and we will prove Theorem 2.1.2 for sequences of groups of type A, B, C and D separately. The proposition tells us that, if we replace K by its algebraic closure, the limits G and V only depend on the signatures of the diagonal embeddings. Since G-Noetherianity of V over $\overline{K}$ implies G-Noetherianity of V over the original field K, we only have to consider one diagonal embedding per possible signature.

Identifying V with the inverse limit of a sequence of quotients/subspaces of matrix spaces

We encounter the following Lie algebras:

$$\begin{aligned}
\mathrm{A}_{n-1}: \ \mathfrak{sl}_n \ &= \{P \in \mathfrak{gl}_n \mid \mathrm{tr}(P) = 0\} \\
\mathrm{B}_n \ : \ \mathfrak{o}_{2n+1} &= \left\{ P \in \mathfrak{gl}_{2n+1} \ \middle| \ P \begin{pmatrix} & & I_n \\ & 1 & \\ I_n & & \end{pmatrix} + \begin{pmatrix} & & I_n \\ & 1 & \\ I_n & & \end{pmatrix} P^T = 0 \right\} \\
\mathrm{C}_n \ : \ \mathfrak{sp}_{2n} &= \left\{ P \in \mathfrak{gl}_{2n} \ \middle| \ P \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} + \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} P^T = 0 \right\} \\
\mathrm{D}_n \ : \ \mathfrak{o}_{2n} &= \left\{ P \in \mathfrak{gl}_{2n} \ \middle| \ P \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} + \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} P^T = 0 \right\}
\end{aligned}$$

These are all subspaces of $\mathfrak{gl}_m$ for some $m \in \mathbb{N}$. Consider the symmetric bilinear form $\mathfrak{gl}_m \times \mathfrak{gl}_m \to K, (P, Q) \mapsto \mathrm{tr}(PQ)$. This map is nondegenerate and therefore the map $\mathfrak{gl}_m \to \mathfrak{gl}_m^*, P \mapsto (Q \mapsto \mathrm{tr}(PQ))$ is an isomorphism. By composing this map with the restriction map $\mathfrak{gl}_m^* \to \mathfrak{sl}_m^*$ and factoring out the kernel, we find that

$$\begin{aligned}
\mathfrak{gl}_m \,/\, \mathrm{span}(I_m) \ &\to \ \mathfrak{sl}_m^* \\
P \bmod I_m \ &\mapsto \ (Q \mapsto \mathrm{tr}(PQ))
\end{aligned}$$

is an isomorphism. When $\mathrm{char}(K) \neq 2$ and $\mathfrak{g} \subseteq \mathfrak{gl}_m$ is a Lie algebra of type B, C or D, the restriction of the bilinear map to $\mathfrak{g} \times \mathfrak{g}$ is nondegenerate. So the map

$$\begin{aligned}
\mathfrak{g} \ &\to \ \mathfrak{g}^* \\
P \ &\mapsto \ (Q \mapsto \mathrm{tr}(PQ))
\end{aligned}$$

is an isomorphism. Since the map $\mathfrak{gl}_n \to \mathfrak{gl}_n^*$ is in fact GL_n-equivariant, the maps $\mathfrak{gl}_m \,/\, \mathrm{span}(I_m) \to \mathfrak{sl}_m^*$ and $\mathfrak{g} \to \mathfrak{g}^*$ are all isomorphisms of representations of the groups acting on them. Using these isomorphisms, we identify the duals $\mathfrak{g}_i^*$ of the Lie algebras of the groups G_i with quotients/subspaces of spaces of matrices. This in particular allows us to define the coordinate rings of the $\mathfrak{g}_i^*$ in terms of entries of matrices. For type A, we get

$$K[\mathfrak{gl}_n \,/\, \mathrm{span}(I_n)] = \{f \in K[\mathfrak{gl}_n] \mid \forall P \in \mathfrak{gl}_n \ \forall \lambda \in K \colon f(P + \lambda I_n) = f(P)\}$$

which is the graded subring

$$K[p_{k\ell} \mid k \neq \ell] \otimes_K K[p_{11} - p_{kk} \mid k \neq 1]$$

of $K[\mathfrak{gl}_n] = K[p_{k\ell} \mid 1 \leq k, \ell \leq n]$. For type B, assuming that $\mathrm{char}(K) \neq 2$, we have

$$\mathfrak{o}_{2n+1} = \left\{ \begin{pmatrix} P & v & Q \\ -w^T & 0 & -v^T \\ R & w & -P^T \end{pmatrix} \in \mathfrak{gl}_{2n+1} \ \middle| \ \begin{matrix} Q + Q^T = 0 \\ R + R^T = 0 \end{matrix} \right\}$$

and therefore we get

$$K[\mathfrak{o}_{2n+1}] = K[p_{k\ell}, q_{k\ell}, r_{k\ell}, v_k, w_k \mid 1 \leq k, \ell \leq n]/(q_{k\ell} + q_{\ell k}, r_{k\ell} + r_{\ell k}).$$

For type C, we have

$$\mathfrak{sp}_{2n} = \left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{gl}_{2n} \;\middle|\; \begin{array}{c} Q = Q^T \\ R = R^T \end{array} \right\}$$

and we get

$$K[\mathfrak{sp}_{2n}] = K[p_{k\ell}, q_{k\ell}, r_{k\ell} \mid 1 \leq k, \ell \leq n]/(q_{k\ell} - q_{\ell k}, r_{k\ell} - r_{\ell k}).$$

For type D, assuming that $\mathrm{char}(K) \neq 2$, we have

$$\mathfrak{o}_{2n} = \left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{gl}_{2n} \;\middle|\; \begin{array}{c} Q + Q^T = 0 \\ R + R^T = 0 \end{array} \right\}$$

and get

$$K[\mathfrak{o}_{2n}] = K[p_{k\ell}, q_{k\ell}, r_{k\ell} \mid 1 \leq k, \ell \leq n]/(q_{k\ell} + q_{\ell k}, r_{k\ell} + r_{\ell k}).$$

For Lie algebras $\mathfrak{g} \subseteq \mathfrak{gl}_m$ of type B, C or D, we will denote elements of $K[\mathfrak{g}]$ by their representatives in $K[\mathfrak{gl}_m]$. Define a grading on each of these coordinate rings by $\mathrm{grad}(r_{k\ell}) = \mathrm{grad}(w_k) = 0$, $\mathrm{grad}(p_{k\ell}) = \mathrm{grad}(v_k) = 1$ and $\mathrm{grad}(q_{k\ell}) = 2$ for all $k, \ell \in [n]$.

Moving equations around

Let $X \subsetneq V$ be a G-stable closed subset. For each $i \in \mathbb{N}$, let V_i be the vector space (we identified with) $\mathfrak{g}_i^*$ which is acted on by G_i by conjugation and let X_i be the closure of the projection from X to V_i. Then X_i is a G_i-stable closed subset of V_i for all $i \in \mathbb{N}$ and there exists an $i \in \mathbb{N}$ such that $X_i \neq V_i$. This means that the ideal $\mathcal{I}(X_i) \subseteq K[V_i]$ is nonzero. Let f be a nonzero element of $\mathcal{I}(X_i)$ and let d be its degree. The first step of the proof of Theorem 2.1.2 is to use this polynomial f to get elements f_j of $\mathcal{I}(X_j)$ such that $f_j \neq 0$, such that $\deg(f_j) \leq d$ and such that f_j is "off-diagonal" for all $j \gg i$. When the groups G_i are of type B, C or D, this last condition means that f_j is a polynomial in only the variables $r_{k\ell}$ and w_k. When the groups G_i are of type A, we similarly require that the f_j are polynomials in the variables $p_{k\ell}$ with $k \in \mathcal{K}$ and $\ell \in \mathcal{L}$ for some disjoint sets $\mathcal{K}, \mathcal{L}$.

The projection maps $\mathrm{pr}_i\colon V_{i+1} \to V_i$ induce maps $\mathrm{pr}_i^*\colon K[V_i] \to K[V_{i+1}]$ which are injective and degree-preserving. We will see that, for many of the maps pr_i we will encounter, the map pr_i^* is also grad-preserving. Since X_{i+1} projects into X_i, we have $\mathrm{pr}_i^*(\mathcal{I}(X_i)) \subseteq \mathcal{I}(X_{i+1})$. So f induces nonzero elements $g_j \in \mathcal{I}(X_j)$ of degree d for all $j > i$.

Let $A\colon K^k \to G_j$ be a polynomial map such that the map

$$\begin{array}{ccc} K^k & \to & G_j \\ \Lambda & \mapsto & A(\Lambda)^{-1} \end{array}$$

is polynomial as well. Then $A(\Lambda) \cdot g_j \in \mathcal{I}(X_j)$ for all $\Lambda \in K^k$ and therefore linear combinations of such elements also lie in $\mathcal{I}(X_j)$. Note that we can view $A(\Lambda) \cdot g_j$ as a polynomial in the entries of Λ whose coefficients are elements of $K[V_j]$. Let R be a K-algebra and $h \in R[x]$ a polynomial. Then, since the field K is infinite, one sees using a Vandermonde matrix that the coefficients of h are contained in the K-span of

$\{h(\lambda) \mid \lambda \in K\}$. Applying this fact k times, we see that all the coefficients of $A(\Lambda) \cdot g_j$ lie in $\operatorname{span}(A(\Lambda) \cdot g_j \mid \Lambda \in K^k) \subseteq \mathcal{I}(X_j)$.

We will let f_j be a certain one of these coefficients. We have $\deg(f_j) \leq d$ by construction and we will choose A in such a way that f_j is "off-diagonal". We will see that f_j is obtained from g_j by substituting variables into the top-graded part of g_j with respect to the right grading (in most cases deg or grad). Since the polynomial g_j is nonzero, so is its top-graded part with respect to any grading. So it then suffices to check that this top-graded part does not become zero after the substitution. In the cases where this is not obvious, it will follow from a lemma stating that a certain morphism is dominant.

Using knowledge about stable closed subsets of the "off-diagonal" part

The space V_j consists of matrices. When we have an "off-diagonal" polynomial which is contained in $\mathcal{I}(X_j)$, we know that the projection Y of X_j onto some off-diagonal submatrix cannot form a dense subset of the projection W of the whole space V_j. We then give W the structure of a representation such that Y is stable and use the fact that we know that the ideal of Y contains a nonzero polynomial of degree at most d to find conditions that hold for all elements of Y. These in turn give conditions that must hold for all elements of X_j, which will be enough to prove that X is G-Noetherian.

2.3 Limits of general linear groups

In this section, we let G be the direct limit of a sequence

$$\mathrm{GL}_{n_1} \xhookrightarrow{\iota_1} \mathrm{GL}_{n_2} \xhookrightarrow{\iota_2} \mathrm{GL}_{n_3} \xhookrightarrow{\iota_3} \dots$$

of embeddings given by

$$\begin{array}{rccl} \iota_i\colon & \mathrm{GL}_{n_i} & \hookrightarrow & \mathrm{GL}_{n_{i+1}} \\ & A & \mapsto & \mathrm{Diag}(\underbrace{A, \dots, A}_{l_i}, \underbrace{A^{-T}, \dots, A^{-T}}_{r_i}, I_{z_i}) \end{array}$$

for some $l_i \in \mathbb{N}$ and $r_i, z_i \in \mathbb{Z}_{\geq 0}$ with $l_i \geq r_i$. We let V be the inverse limit of the sequence

$$\mathfrak{gl}_{n_1} \twoheadleftarrow \mathfrak{gl}_{n_2} \twoheadleftarrow \mathfrak{gl}_{n_3} \twoheadleftarrow \dots$$

where the maps are given by

$$\begin{array}{ccc} \mathfrak{gl}_{n_{i+1}} & \twoheadrightarrow & \mathfrak{gl}_{n_i} \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i1} & \dots & P_{l_il_i} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r_i} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r_i1} & \dots & Q_{r_ir_i} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix} & \mapsto & \displaystyle\sum_{k=1}^{l_i} P_{kk} - \sum_{\ell=1}^{r_i} Q_{\ell\ell}^T. \end{array}$$

Our goal is to prove Theorem 2.1.4. We start by proving some basic properties of the tuple rank of a matrix with the identity matrix.

Proposition 2.3.1. *Let $P, P_1, \ldots, P_k$ be elements of $\mathfrak{gl}_\infty$.*

1. *We have* $\mathrm{rk}(P_1, \ldots, P_k) = \sup\{\mathrm{rk}(\mathrm{pr}_n(P_1), \ldots, \mathrm{pr}_n(P_k)) \mid n \in \mathbb{N}\}$.
2. *If* $\mathrm{rk}(P, I_\infty) < \infty$*, then* $\mathrm{rk}(P - \lambda I_\infty) < \infty$ *for some unique $\lambda \in K$.*

Proof. We have

$$\mathrm{rk}(\mathrm{pr}_n(P_1), \ldots, \mathrm{pr}_n(P_k)) \leq \mathrm{rk}(\mu_1 P_1 + \cdots + \mu_k P_k)$$

for all $n \in \mathbb{N}$ and $(\mu_1 : \cdots : \mu_k) \in \mathbb{P}^{k-1}$. So

$$r := \sup\{\mathrm{rk}(\mathrm{pr}_n(P_1), \ldots, \mathrm{pr}_n(P_k)) \mid n \in \mathbb{N}\} \leq \mathrm{rk}(P_1, \ldots, P_k)$$

with equality when $r = \infty$. Suppose that $r < \infty$ and consider the descending chain

$$Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq Y_4 \supseteq \ldots$$

of closed subsets of $\mathbb{P}^{k-1}$ defined by

$$Y_n = \left\{(\mu_1 : \cdots : \mu_k) \in \mathbb{P}^{k-1} \;\middle|\; \mathrm{rk}(\mu_1 \,\mathrm{pr}_n(P_1) + \cdots + \mu_k \,\mathrm{pr}_n(P_k)) \leq r\right\}.$$

By construction, each Y_n is nonempty. And by the Noetherianity of $\mathbb{P}^{k-1}$, the chain stabilizes. Let $(\mu_1 : \cdots : \mu_k) \in \mathbb{P}^{k-1}$ be an element contained in Y_n for all $n \in \mathbb{N}$. Then we see that $\mathrm{rk}(P_1, \ldots, P_k) \leq \mathrm{rk}(\mu_1 P_1 + \cdots + \mu_k P_k) \leq r$. This shows (1).

If $\mathrm{rk}(P, I_\infty) < \infty$, then $\mathrm{rk}(P - \lambda I_\infty) < \infty$ for some $\lambda \in K$. If this holds for distinct $\lambda, \lambda' \in K$, then

$$\infty = \mathrm{rk}((\lambda - \lambda') I_\infty) = \mathrm{rk}\left((P - \lambda' I_\infty) - (P - \lambda I_\infty)\right) \leq \mathrm{rk}(P - \lambda' I_\infty) + \mathrm{rk}(P - \lambda I_\infty) < \infty$$

and hence the $\lambda \in K$ such that $\mathrm{rk}(P - \lambda I_\infty) < \infty$ must be unique. This shows (2). □

The following proposition, which is due to Jan Draisma, connects the tuple rank of a matrix P with the identity matrix to the rank of off-diagonal submatrices of matrices similar to P.

Proposition 2.3.2. *Let $k, m, n \in \mathbb{Z}_{\geq 0}$ be such that $n \geq 2m \geq 2(k+1)$, let $\mathscr{K}, \mathscr{L}$ be disjoint subsets of $[n]$ of size m and let P be an $n \times n$ matrix. Then* $\mathrm{rk}(P, I_n) \leq k$ *if and only if the submatrix $Q_{\mathscr{K},\mathscr{L}}$ of Q has rank at most k for every $Q \sim P$.*

Proof. Suppose that $\mathrm{rk}(P, I_n) \leq k$. Let $Q \sim P$ be a similar matrix. Then $\mathrm{rk}(Q, I_n) \leq k$. So since $\mathscr{K} \cap \mathscr{L} = \emptyset$ and the off-diagonal entries of Q and $Q - \lambda I_n$ are equal for all $\lambda \in K$, we see that $\mathrm{rk}(Q_{\mathscr{K},\mathscr{L}}) \leq k$.

Suppose that the submatrix $Q_{\mathscr{K},\mathscr{L}}$ has rank at most k for every $Q \sim P$. Then this statement still holds when we replace $\mathscr{K}$ and $\mathscr{L}$ by subsets of themselves of size $k+1$. This reduces the proposition to the case $m = k + 1$. Now the statement we want to prove is implied by the following coordinate-free version:

(*) Let V be a vector space of dimension n and let $\varphi\colon V \to V$ be an endomorphism. If the induced map $\varphi\colon W \to V/W$ has a nontrivial kernel for all $(k+1)$-dimensional subspaces W of V, then φ has an eigenvalue of geometric multiplicity at least $n - k$.

Indeed, taking $\varphi\colon K^n \to K^n$ the endomorphism corresponding to P and $W \subseteq K^n$ a $(k+1)$-dimensional subspace, we can first replace P be a matrix $Q \sim P$ to get $W = K^{k+1} \times \{0\}$. Since Q is similar to all its conjugates by permutation matrices, we know that $\det(Q_{\mathcal{K},\mathcal{L}}) = 0$ for all disjoint subsets of $\mathcal{K}, \mathcal{L} \subseteq [n]$ of size m. Hence $Q_{[n]\setminus[k+1],[k+1]}$ has rank at most k. So the induced map $W \to V/W$ has a nontrivial kernel. We conclude from (*) that

$$\mathrm{rk}(P - \lambda I_n) = \mathrm{rk}(Q - \lambda I_n) \leq n - (n-k) = k$$

for some $\lambda \in K$. So $\mathrm{rk}(P, I_n) \leq k$.

To prove (*), consider the incidence variety

$$Z = \{(W, [v]) \in \mathrm{Gr}_{k+1}(V) \times \mathbb{P}(V) \mid v, \varphi(v) \in W\}$$

and let π_1, π_2 be the projections from Z to the Grassmannian $\mathrm{Gr}_{k+1}(V)$ and to $\mathbb{P}(V)$. By assumption π_1 is surjective. So we have

$$\dim Z \geq \dim(\mathrm{Gr}_{k+1}(V)) = (k+1)(n-k-1).$$

On the other hand, let $v \in V \setminus \{0\}$ be a non-eigenvector of φ. Then $\pi_1(\pi_2^{-1}([v]))$ consists of all $W \in \mathrm{Gr}_{k+1}(V)$ containing $\mathrm{span}(v, \varphi(v))$ and these form the Grassmannian $\mathrm{Gr}_{k-1}(V/\,\mathrm{span}(v, \varphi(v)))$ of dimension $(k-1)(n-k-1)$. Thus the union of the fibres $\pi_2^{-1}([v])$ for v not an eigenvector of φ has dimension at most

$$(k-1)(n-k-1) + \dim(\mathbb{P}(V)) = (k+1)(n-k-1) + 2k + 1 - n.$$

This dimension is strictly smaller than $\dim(Z)$. Let v be an eigenvector of φ. Then $\pi_1(\pi_2^{-1}([v]))$ consists of all $W \in \mathrm{Gr}_{k+1}(V)$ with $v \in W$ and these form the Grassmannian $\mathrm{Gr}_k(V/\,\mathrm{span}(v))$ of dimension $k(n-k-1)$. So we see that the union of the eigenspaces of φ must have dimension at least $\dim(Z) - k(n-k-1) + 1 \geq n-k$. Hence some eigenspace of φ must have dimension at least $n-k$. □

2.3.1 The case $\alpha + \beta < \infty$

By replacing

$$\mathrm{GL}_{n_1} \xhookrightarrow{\iota_1} \mathrm{GL}_{n_2} \xhookrightarrow{\iota_2} \mathrm{GL}_{n_3} \xhookrightarrow{\iota_3} \cdots$$

with some infinite subsequence, we may assume that $(l_i, r_i) = (1, 0)$ and $z_i > 0$ for all $i \in \mathbb{N}$. Then, by replacing the sequence by a supersequence, we may assume that $n_i = i$ and $z_i = 1$ for all $i \in \mathbb{N}$. So we consider the inverse limit $V = \mathfrak{gl}_\infty$ of the sequence

$$\mathfrak{gl}_1 \twoheadleftarrow \mathfrak{gl}_2 \twoheadleftarrow \mathfrak{gl}_3 \twoheadleftarrow \cdots$$

acted on by the group $G = \mathrm{GL}_\infty$.

Definition 2.3.3. For $n \in \mathbb{N}$, we call a polynomial $f \in K[\mathfrak{gl}_n]$ off-diagonal if

$$f \in K[p_{k\ell} \mid k \in \mathcal{K}, \ell \in \mathcal{L}]$$

for some disjoint subsets $\mathcal{K}, \mathcal{L} \subset [n]$ of size $m \leq n/2$.

Lemma 2.3.4. *Let $n \in \mathbb{N}$ be an integer, let Y be a GL_n-stable closed subset of $\mathfrak{gl}_n$ and suppose that $\mathcal{I}(Y)$ contains a nonzero off-diagonal polynomial f. Then $\mathrm{rk}(P, I_n) < \deg(f)$ for all $P \in Y$.*

Proof. Let $\mathcal{K}, \mathcal{L} \subset [n]$ be disjoint subsets of size $m \leq n/2$ and let

$$f \in K[p_{k\ell} \mid k \in \mathcal{K}, \ell \in \mathcal{L}] \cap \mathcal{I}(Y)$$

be a nonzero element. If $m = 0$, then f is constant and $Y = \emptyset$. So in particular, $\mathrm{rk}(P, I_n) < \deg(f)$ for all $P \in Y$. For $m > 0$, let Z be the closure of the set

$$\{(y_{k\ell})_{k \in \mathcal{K}, \ell \in \mathcal{L}} \mid (y_{k\ell})_{k,\ell} \in Y\}$$

in $\mathfrak{gl}_m$. Then $f \in \mathcal{I}(Z)$. By conjugating with ± 1 times a permutation matrix, we may assume that $\mathcal{K} = [m]$ and $\mathcal{L} = [2m] \setminus [m]$. Now consider the map

$$\begin{array}{rcl} \mathrm{GL}_m \times \mathrm{GL}_m & \to & \mathrm{GL}_n \\ (A, B) & \mapsto & \mathrm{Diag}(A, B, I_{n-2m}). \end{array}$$

Since Y is $\mathrm{GL}_m \times \mathrm{GL}_m$-stable, we see that Z is closed under $\mathrm{GL}_m \times \mathrm{GL}_m$ acting by left and right multiplication. So Z must consist of all matrices of rank at most ℓ for some $\ell \leq m$. Since $f \in \mathcal{I}(Z)$, we see that $\ell < \min(m, \deg(f))$. So by Proposition 2.3.2, we see that Y consists of matrices P such that $\mathrm{rk}(P, I_n) < \min(m, \deg(f)) \leq \deg(f)$. $\square$

Let X be a proper GL_∞-stable closed subset of $\mathfrak{gl}_\infty$. Denote the closure of the projection of X to $\mathfrak{gl}_n$ by X_n and let $\mathcal{I}(X_n) \subseteq K[\mathfrak{gl}_n]$ be its corresponding ideal.

Lemma 2.3.5. *Let m be a positive integer and suppose that $\mathcal{I}(X_m)$ contains a nonzero polynomial f. Then rk$(P, I_\infty) < \deg(f)$ for all $P \in X$.*

Proof. Note that the morphism $X_n \to X_m$ is dominant for all positive integers $m \leq n$. So it suffices to prove that $\mathrm{rk}(\mathrm{pr}_n(P), I_n) < \deg(f)$ for $n \gg 0$. Let $n \geq 2m$ be an integer. Then f induces the element

$$g = \left(\begin{pmatrix} P & Q & \bullet \\ R & S & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \mapsto f(P) \right)$$

of $\mathcal{I}(X_n)$ where $P, Q, R, S \in \mathfrak{gl}_m$. This allows us to assume that $\deg(f) < m$ without loss of generality. For $\lambda \in K$, consider the matrix

$$A(\lambda) = \begin{pmatrix} I_m & \lambda I_m & \\ & I_m & \\ & & I_{n-2m} \end{pmatrix} \in \mathrm{GL}_n.$$

We have

$$A(\lambda) \begin{pmatrix} P & Q & \bullet \\ R & S & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} A(\lambda)^{-1} = \begin{pmatrix} P + \lambda R & Q + \lambda(S - P) - \lambda^2 R & \bullet \\ R & S - \lambda R & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

for all $\lambda \in K$. So we see that if we let $A(\lambda)$ act on g, we obtain the element

$$h_\lambda = \left(\begin{pmatrix} P & Q & \bullet \\ R & S & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \mapsto f(P + \lambda R) \right)$$

of $\mathcal{I}(X_n)$. Let d be the degree of f and let f_d be the homogeneous part of f of degree d. Then the homogeneous part of h_λ of degree d in λ equals the polynomial $\lambda^d f_d(R)$. Since the field K is infinite, the polynomial $f_d(R)$ is a linear combination of the h_λ. Hence $f_d(R) \in \mathcal{I}(X_n)$. So $\mathrm{rk}(P, I_n) < \deg(f)$ for all $P \in X_n$ by Lemma 2.3.4 and therefore $\mathrm{rk}(P, I_\infty) < \deg(f)$ for all $P \in X$. □

Lemma 2.3.6. *Let $k < n$ be nonnegative integers and let $P \in \mathfrak{gl}_{2n}$ and $Q \in \mathfrak{gl}_n$ be matrices with $\mathrm{rk}(P) = k$ and $\mathrm{rk}(Q) \leq k$. Then P is similar to*

$$\begin{pmatrix} Q & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

for some $Q_{12}, Q_{21}, Q_{22} \in \mathfrak{gl}_n$.

Proof. First note that $\mathrm{rk}(P, I_{2n}) = 2n - \dim\ker(P) = k$, since 0 has the highest geometric multiplicity among all eigenvalues of P. Since $2(k+1) \leq 2n$, it follows by Proposition 2.3.2 that

$$P \sim \begin{pmatrix} \bullet & \bullet \\ R & \bullet \end{pmatrix}$$

for some matrix $R \in \mathfrak{gl}_n$ with $\mathrm{rk}(R) = k$. By conjugating the latter matrix with $\mathrm{Diag}(g, I_n)$ for some $g \in \mathrm{GL}_n$ such that $g\ker(R) \subseteq \ker(Q)$, we see that

$$\begin{pmatrix} \bullet & \bullet \\ R & \bullet \end{pmatrix} \sim \begin{pmatrix} \bullet & \bullet \\ R' & \bullet \end{pmatrix}$$

for some matrix $R' \in \mathfrak{gl}_n$ with $\mathrm{rk}(R') = k$ and $\ker(R') \subseteq \ker(Q)$. This means that $Q = SR'$ for some $S \in \mathfrak{gl}_n$. Since both R' and any matrix similar to P have rank k, we see that the matrix on the right must be of the form

$$\begin{pmatrix} TR' & \bullet \\ R' & \bullet \end{pmatrix}$$

for some $T \in \mathfrak{gl}_n$. Now note that the matrix

$$\begin{pmatrix} I_n & S-T \\ 0 & I_n \end{pmatrix}\begin{pmatrix} TR' & \bullet \\ R' & \bullet \end{pmatrix}\begin{pmatrix} I_n & T-S \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} SR' & \bullet \\ R' & \bullet \end{pmatrix} = \begin{pmatrix} Q & \bullet \\ R' & \bullet \end{pmatrix}$$

is similar to P and of the form we want. □

Proposition 2.3.7. *Let $P \in \mathfrak{gl}_\infty$ be an element. Then either the orbit of P is dense in $\mathfrak{gl}_\infty$ or $k = \mathrm{rk}(P - \lambda I_\infty) < \infty$ for some unique $\lambda \in K$. In the second case, the closure of the orbit of P equals the irreducible closed subset $\{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) \leq k\}$ of $\mathfrak{gl}_\infty$.*

Proof. Let X be the closure of the orbit of P. Then either $X = \mathfrak{gl}_\infty$ or $\mathrm{rk}(P, I_\infty) = k$ for some $k \in \mathbb{Z}_{\geq 0}$ by Lemma 2.3.5. In the second case, we see that $\mathrm{rk}(P - \lambda I_\infty) = k$ for some unique $\lambda \in K$ by (1) of Proposition 2.3.1. Our goal is to prove that $X = \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) \leq k\}$. Using the GL_∞-equivariant affine isomorphism

$$\begin{aligned} \mathfrak{gl}_\infty &\to \mathfrak{gl}_\infty \\ Q &\mapsto Q - \lambda I_\infty \end{aligned}$$

we may assume that $\lambda = 0$ and hence that $k = \mathrm{rk}(P)$ is finite. It suffices to prove that

$$\mathrm{pr}_n(\{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q) \leq k\}) = \{Q \in \mathfrak{gl}_n \mid \mathrm{rk}(Q) \leq k\} = \mathrm{pr}_n(\mathrm{GL}_\infty \cdot P)$$

for all $n \gg 0$ since the middle set is irreducible. See Proposition 1.2.3. The inclusions

$$\mathrm{pr}_n(\mathrm{GL}_\infty \cdot P) \subseteq \mathrm{pr}_n(\{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q) \leq k\}) \subseteq \{Q \in \mathfrak{gl}_n \mid \mathrm{rk}(Q) \leq k\}$$

are clear for all $n \in \mathbb{N}$. Let $n > k$ be an integer such that the rank of $\mathrm{pr}_{2n}(P)$ equals k. Then

$$\{Q \in \mathfrak{gl}_n \mid \mathrm{rk}(Q) \leq k\} \subseteq \mathrm{pr}_n(\mathrm{GL}_{2n} \cdot \mathrm{pr}_{2n}(P)) \subseteq \mathrm{pr}_n(\mathrm{GL}_\infty \cdot P)$$

by Lemma 2.3.6. So indeed $\mathrm{pr}_n(\{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q) \leq k\}) = \mathrm{pr}_n(\mathrm{GL}_\infty \cdot P)$ for all $n \gg 0$. □

Lemma 2.3.8. *Let m be a positive integer and suppose that $\mathcal{I}(X_m)$ contains a nonzero polynomial f with $\deg(f) < m$. Let $g(t) = f(tI_m) \in K[t]$ be the restriction of f to $\mathrm{span}(I_m)$. Then X is contained in*

$$\bigcup_\lambda \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) < \deg(f)\}$$

where $\lambda \in K$ ranges over the zeros of g.

Proof. Let P be an element of X. Since f is nonzero, we know that X is a proper GL_∞-stable closed subset of $\mathfrak{gl}_\infty$. Hence the orbit of P cannot be dense in $\mathfrak{gl}_\infty$. So $k = \mathrm{rk}(P - \lambda I_\infty) < \deg(f)$ for some $\lambda \in K$ by Lemma 2.3.5. This λ is unique and the closure of the orbit of P equals $\{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) \leq k\}$ by Proposition 2.3.7. So we see that λI_∞ is an element of X. So λI_m is an element of X_m and hence $g(\lambda) = f(\lambda I_m) = 0$. We see that for all $P \in X$ there is a $\lambda \in K$ with $g(y) = 0$ such that

$$P \in \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) < \deg(f)\}.$$ □

Proposition 2.3.9. *Either the GL_∞-stable closed subset $\mathrm{span}(I_\infty)$ of $\mathfrak{gl}_\infty$ is contained in X or there exist $\lambda_1, \dots, \lambda_\ell \in K$ and $k_1, \dots, k_\ell \in \mathbb{Z}_{\geq 0}$ such that*

$$X = \bigcup_{i=1}^{\ell} \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda_i I_\infty) \leq k_i\}.$$

Proof. Assume that $\mathrm{span}(I_\infty)$ is not contained in X. Then, for some $m \in \mathbb{N}$, X_m is a proper subset of $\mathfrak{gl}_m$ that does not contain $\mathrm{span}(I_m)$. The ideal $\mathcal{I}(X_m)$ must contain a nonzero polynomial f such that the polynomial $g(t) = f(tI_m) \in K[t]$ is nonzero. By Lemma 2.3.8, we see that X is contained in

$$\bigcup_\lambda \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) < \deg(f)\}$$

where $\lambda \in K$ ranges over the finitely many zeros of g. Take

$$\Lambda = \{\lambda \in K \mid g(\lambda) = 0, \exists P \in X\colon\ \mathrm{rk}(P - \lambda I_\infty) < \deg(f)\}$$

and take

$$k_\lambda = \max\{\mathrm{rk}(P - \lambda I_\infty) \mid P \in X, \mathrm{rk}(P - \lambda I_\infty) < \infty\}$$

for all $\lambda \in \Lambda$. Then we see that

$$X = \bigcup_{\lambda \in \Lambda} \{Q \in \mathfrak{gl}_\infty \mid \mathrm{rk}(Q - \lambda I_\infty) \leq k_\lambda\}$$

using Proposition 2.3.7. □

The proposition implies in particular that any descending chain of GL_∞-stable closed subsets of $\mathfrak{gl}_\infty$ stabilizes as long as one of these subsets does not contain $\mathrm{span}(I_\infty)$. Next we will classify the subsets that do contain $\mathrm{span}(I_\infty)$.

Proposition 2.3.10. *Let k be a nonnegative integer. Then the GL_∞-stable subset*

$$\{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\}$$

of $\mathfrak{gl}_\infty$ is closed and irreducible.

Proof. Using (1) of Proposition 2.3.1, we see that

$$\{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\}$$

is the inverse limit of its projections $\{P \in \mathfrak{gl}_n \mid \mathrm{rk}(P, I_n) \le k\}$ onto $\mathfrak{gl}_n$. So it suffices to show that this is a closed irreducible subset of $\mathfrak{gl}_n$ for all $n \in \mathbb{N}$. See Proposition 1.2.3. The subset $\{P \in \mathfrak{gl}_n \mid \mathrm{rk}(P, I_n) \le k\}$ is the inverse image of the subset

$$Y = \left\{(P, Q) \in \mathfrak{gl}_n^2 \;\middle|\; \mathrm{rk}(P, Q) \le k\right\}$$

under the map $\mathfrak{gl}_n \to \mathfrak{gl}_n^2, P \mapsto (P, I_n)$. The subset Y is closed in $\mathfrak{gl}_n^2$ since it is the image of the closed subset

$$\left\{((\mu_1 : \mu_2), P, Q) \in \mathbb{P}^1 \times \mathfrak{gl}_n^2 \;\middle|\; \mathrm{rk}(\mu_1 P + \mu_2 Q) \le k\right\}$$

under the projection map along the complete variety $\mathbb{P}^1$. So $\{P \in \mathfrak{gl}_n \mid \mathrm{rk}(P, I_n) \le k\}$ is a closed subset of $\mathfrak{gl}_n$. This subset is also the image of the map

$$\begin{array}{rcl} \{Q \in \mathfrak{gl}_n \mid \mathrm{rk}(Q) \le k\} \times K & \to & \mathfrak{gl}_n \\ (Q, \lambda) & \mapsto & Q + \lambda I_n \end{array}$$

and hence irreducible. □

Proposition 2.3.11. *Suppose that X contains $\mathrm{span}(I_\infty)$. Then*

$$X = \{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\} \cup Y$$

for some nonnegative integer k and some GL_∞-stable closed subset Y of $\mathfrak{gl}_\infty$ that does not contain $\mathrm{span}(I_\infty)$.

Proof. Since X is a proper subset of $\mathfrak{gl}_\infty$, we know that

$$X \subseteq \{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le \ell\}$$

for some $\ell \in \mathbb{Z}_{\ge 0}$ by Lemma 2.3.5. Let k be the maximal nonnegative integer such that

$$\{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\} \subseteq X.$$

We will prove the statement by induction on the difference between ℓ and k.

Suppose that $\ell = k$. Then $X = \{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\}$ and the statement holds. Now suppose that $\ell > k$ and let Y' be a GL_∞-stable closed subset of $\mathfrak{gl}_\infty$ that does not contain $\mathrm{span}(I_\infty)$ such that

$$X \cap \{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le \ell - 1\} = \{P \in \mathfrak{gl}_\infty \mid \mathrm{rk}(P, I_\infty) \le k\} \cup Y'.$$

Consider the set $Z = \{\lambda \in K \mid \exists P \in X\colon \operatorname{rk}(P - \lambda I_\infty) = \ell\}$ and fix an element $Q \in \mathfrak{gl}_\infty$ with $\operatorname{rk}(Q) = \ell$. By Proposition 2.3.7, we know for $\lambda \in K$ that $Q + \lambda I_\infty \in X$ if and only if $\lambda \in Z$. This shows that Z is a closed subset of K. So either $Z = K$ or Z is finite. If $Z = K$, then we see that X contains all $P \in \mathfrak{gl}_\infty$ with $\operatorname{rk}(P, I_\infty) \leq \ell$ by Proposition 2.3.7. Since $\ell > k$, this is not true and hence Z is finite. Take

$$Y = Y' \cup \bigcup_{\lambda \in Z} \{P \in \mathfrak{gl}_\infty \mid \operatorname{rk}(P - \lambda I_\infty) \leq \ell\}.$$

Then we see that $X = \{P \in \mathfrak{gl}_\infty \mid \operatorname{rk}(P, I_\infty) \leq k\} \cup Y$. □

Proof of Theorem 2.1.4 in case (1). Let S be the set of pairs (k, f) where $k \in \mathbb{Z}_{\geq -1}$ and where $f\colon K \to \mathbb{Z}_{\geq k}$ is a function such that $f^{-1}(\mathbb{Z}_{>k})$ is finite. Define a partial ordering on S by $(k, f) \leq (\ell, g)$ when $k \leq \ell$ and $f(\lambda) \leq g(\lambda)$ for all $\lambda \in K$. Then for all $(k, f) \in S$, the set $\{(k, g) \in S \mid (k, g) \leq (k, f)\}$ is finite. So any descending chain in S stabilizes. For a proper GL_∞-stable closed subset X of $\mathfrak{gl}_\infty$, let k_X be the maximal integer such that $\{P \in \mathfrak{gl}_\infty \mid \operatorname{rk}(P, I_\infty) \leq k_X\} \subseteq X$ and let $f_X\colon K \to \mathbb{Z}_{\geq k}$ be the function sending $\lambda \in K$ to the maximal k such that $\{P \in \mathfrak{gl}_\infty \mid \operatorname{rk}(P - \lambda I_\infty) \leq k\} \subseteq X$. Then, by Propositions 2.3.9 and 2.3.11, we see that

$$X = \{P \in \mathfrak{gl}_\infty \mid \operatorname{rk}(P, I_\infty) \leq k_X\} \cup \bigcup_{\lambda \in f_X^{-1}(\mathbb{Z}_{>k_X})} \left\{P \in \mathfrak{gl}_\infty \;\middle|\; \operatorname{rk}(P - \lambda I_\infty) \leq f_X(\lambda)\right\}$$

and that the map $X \mapsto (k_X, f_X)$ is an order preserving bijection between the set of proper GL_∞-stable closed subsets of $\mathfrak{gl}_\infty$ and S. Now consider a descending chain

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

of GL_∞-stable closed subsets of $\mathfrak{gl}_\infty$. We get a descending chain

$$(k_{X_1}, f_{X_1}) \geq (k_{X_2}, f_{X_2}) \geq (k_{X_3}, f_{X_3}) \geq (k_{X_4}, f_{X_4}) \geq \dots$$

in S which must stabilize. Therefore the original chain also stabilizes. Hence $\mathfrak{gl}_\infty$ is GL_∞-Noetherian. The irreducible GL_∞-stable closed subsets of $\mathfrak{gl}_\infty$ are as described in the theorem by Propositions 2.3.7, 2.3.9, 2.3.10 and 2.3.11. □

2.3.2 The proof of the other cases

Now, we turn our attention to cases (2)-(4) of Theorem 2.1.4. We start by proving some statements that are useful in multiple cases.

Lemma 2.3.12. *Let k, n be positive integers with $k \leq n$ and let $P \in \mathfrak{gl}_n$ be a matrix. Then* $\operatorname{rk}(P) < k$ *if and only if* $\det(Q_{[k],[k]}) = 0$ *for all $Q \sim P$.*

Proof. If $\operatorname{rk}(P) < k$, then $\det(Q_{[k],[k]}) = 0$ for all $Q \sim P$. Suppose that $\det(Q_{[k],[k]}) = 0$ for all $Q \sim P$. Note that $\operatorname{rk}(P) < k$ if and only if $\det(P_{\mathcal{K},\mathcal{L}}) = 0$ for all subsets $\mathcal{K}, \mathcal{L} \subset [n]$ of size k. One can prove this using reverse induction of the size of $\mathcal{K} \cap \mathcal{L}$. If $\mathcal{K} = \mathcal{L}$, then $P_{\mathcal{K},\mathcal{L}} = Q_{[k],[k]}$ for some matrix $Q \sim P$ obtained from P by conjugating with a permutation matrix. So $\det(P_{\mathcal{K},\mathcal{L}}) = 0$. For $|\mathcal{K} \cap \mathcal{L}| < k$, we take $i \in \mathcal{K} \setminus \mathcal{L}$, $j \in \mathcal{L} \setminus \mathcal{K}$ and $\mathcal{K}' = \{j\} \cup \mathcal{K} \setminus \{i\}$ and note that, since $|\mathcal{K}' \cap \mathcal{L}| > |\mathcal{K} \cap \mathcal{L}|$,

$$\det(P_{\mathcal{K},\mathcal{L}}) = \pm \det(P_{\mathcal{K}',\mathcal{L}}) \pm \det(Q_{\mathcal{K}',\mathcal{L}}) = 0$$

where $Q \sim P$ is the matrix obtained from P by adding row i to row j and substracting column j from column i. □

Lemma 2.3.13. *Let $k, \ell, n \in \mathbb{N}$ be integers with $n \geq 6k$ and $\ell \geq 2$.*

(1) Let $P_1, \dots, P_\ell \in \mathfrak{gl}_n$ be matrices of rank k. Then there exist $Q_1 \sim P_1, \dots, Q_\ell \sim P_\ell$ such that $k < \mathrm{rk}(Q_1 + \cdots + Q_\ell) \leq 3k$.

(2) Let $P_1, \dots, P_\ell \in \mathfrak{gl}_n$ be matrices with $\mathrm{rk}(P_1, I_n) = \cdots = \mathrm{rk}(P_\ell, I_n) = k$. Then there exist $Q_1 \sim P_1, \dots, Q_\ell \sim P_\ell$ such that $k < \mathrm{rk}(Q_1 + \cdots + Q_\ell, I_n) \leq 3k$.

Proof. Let $P, P' \in \mathfrak{gl}_n$ be matrices such that $\mathrm{rk}(P), \mathrm{rk}(P') \leq n/2$. We start with three claims.

(i) For all $Q \sim P$ and $Q' \sim P'$, we have $\mathrm{rk}(Q + Q') \geq |\mathrm{rk}(P) - \mathrm{rk}(P')|$.

(ii) There exist $Q \sim P$ and $Q' \sim P'$ with $\mathrm{rk}(Q + Q') = \mathrm{rk}(P) + \mathrm{rk}(P')$.

(iii) There exist $Q \sim P$ and $Q' \sim P'$ with $\mathrm{rk}(Q + Q') \leq \max(\mathrm{rk}(P), \mathrm{rk}(P'))$.

Claim (i) is obvious. For (ii) and (iii), take $m = \max(\mathrm{rk}(P), \mathrm{rk}(P'))$ and note that

$$P \sim \begin{pmatrix} TR & TRS \\ R & RS \end{pmatrix} \sim \begin{pmatrix} I_m & -S \\ & I_{n-m} \end{pmatrix}^{-1} \begin{pmatrix} TR & TRS \\ R & RS \end{pmatrix} \begin{pmatrix} I_m & -S \\ & I_{n-m} \end{pmatrix} = \begin{pmatrix} (S+T)R & 0 \\ R & 0 \end{pmatrix}$$

for some matrices R, S, T with R an $(n-m) \times m$ matrix of rank $\mathrm{rk}(P)$ by Proposition 2.3.2, because otherwise $\mathrm{rk}(P, I_n) < \mathrm{rk}(P)$ would hold. Similarly, we have

$$P' \sim \begin{pmatrix} \bullet & 0 \\ R' & 0 \end{pmatrix} \sim \begin{pmatrix} \bullet & R'' \\ 0 & 0 \end{pmatrix}$$

for some $(n-m) \times m$ matrix R' and $m \times (n-m)$ matrix R'' that both have the same rank as P'. Now (ii) follows from the fact that

$$\begin{pmatrix} \bullet & 0 \\ R & 0 \end{pmatrix} + \begin{pmatrix} \bullet & R'' \\ 0 & 0 \end{pmatrix}$$

has rank $\mathrm{rk}(P) + \mathrm{rk}(P')$ and (iii) follows from the fact that

$$\begin{pmatrix} \bullet & 0 \\ R & 0 \end{pmatrix} + \begin{pmatrix} \bullet & 0 \\ R' & 0 \end{pmatrix}$$

has rank at most m.

Let $P_1, \dots, P_\ell \in \mathfrak{gl}_n$ be matrices of rank k. To show (1), we use induction on ℓ. For $\ell = 2$, we see that (1) follows from (ii). Now suppose that $\ell > 2$ and that

$$k < \mathrm{rk}(Q_1 + \cdots + Q_{\ell-1}) \leq 3k$$

for some $Q_1 \sim P_1, \dots, Q_{\ell-1} \sim P_{\ell-1}$. Using (ii) if $\mathrm{rk}(Q_1 + \cdots + Q_{\ell-1}) \leq 2k$ and using (i) and (iii) otherwise, we see that

$$k < \mathrm{rk}\Big(g(Q_1 + \cdots + Q_{\ell-1})g^{-1} + Q_\ell\Big) \leq 3k$$

for some $g \in \mathrm{GL}_n$ and $Q_\ell \sim P_\ell$. Since $gQ_1g^{-1} \sim P_1, \dots, gQ_{\ell-1}g^{-1} \sim P_{\ell-1}$ and $Q_\ell \sim P_\ell$ this proves (1).

Next, let $P_1, \dots, P_\ell \in \mathfrak{gl}_n$ be matrices with $\mathrm{rk}(P_1, I_n) = \dots = \mathrm{rk}(P_\ell, I_n) = k$ and let $\lambda_1, \dots, \lambda_\ell \in K$ be such that $\mathrm{rk}(P_1 - \lambda_1 I_n) = \dots = \mathrm{rk}(P_\ell - \lambda_\ell I_n) = k$. Then (1) tells us that there exist $Q_1' \sim P_1 - \lambda_1 I_n, \dots, Q_\ell' \sim P_\ell - \lambda_\ell I_n$ such that $k < \mathrm{rk}(Q_1' + \dots + Q_\ell') \le 3k$. From this follows that

$$k < \mathrm{rk}(Q_1 + \dots + Q_\ell, I_n) \le 3k$$

for $Q_1 = Q_1' + \lambda_1 I_n \sim P_1, \dots, Q_\ell = Q_\ell' + \lambda_\ell I_n \sim P_\ell$. This shows (2). □

Let X be a G-stable closed subset of V and let X_i be the closure of the projection of X to $\mathfrak{gl}_{n_i}$.

Lemma 2.3.14. *Suppose that $l_i + r_i \ge 2$ for all $i \in \mathbb{N}$. If there exists a $k \in \mathbb{Z}_{\ge 0}$ such that X_i only contains elements P with $\mathrm{rk}(P, I_{n_i}) \le k$ for all $i \gg 0$, then $X \subseteq \{0\}$.*

Proof. The lemma follows by induction on k from the following statement.

(*) Let $k, i \in \mathbb{N}$ be integers such that $n_i \ge 6k$. If X_{i+1} contains an element P with $\mathrm{rk}(P, I_{n_{i+1}}) = k$, then X_i contains an element Q with $\mathrm{rk}(Q, I_{n_i}) > k$.

Let $k, i \in \mathbb{N}$ be integers such that $n_i \ge 6k$ and let P be an element of X_{i+1} with $\mathrm{rk}(P, I_{n_{i+1}}) = k$. By Lemma 2.3.12, we have

$$gPg^{-1} = \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i 1} & \dots & P_{l_i l_i} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r_i} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r_i 1} & \dots & Q_{r_i r_i} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix} + \lambda I_{n_{i+1}}$$

for some $g \in \mathrm{GL}_{n_{i+1}}$, $\lambda \in K$ and $P_{11}, \dots, P_{l_i l_i}, Q_{11}, \dots, Q_{r_i r_i} \in \mathfrak{gl}_{n_i}$ with $\mathrm{rk}(P_{11}) = k$. Since this is an open condition on g, the matrix gPg^{-1} is in fact of this form for sufficiently general $g \in \mathrm{GL}_{n_{i+1}}$. This allows us to assume that $\mathrm{rk}(P_{jj}) = k$ for all $j \in [l_i]$ and $\mathrm{rk}(-Q_{\ell\ell}^T) = \mathrm{rk}(Q_{\ell\ell}) = k$ for all $\ell \in [r_i]$. Lemma 2.3.13 now tell us that by replacing g by $\mathrm{Diag}(g_1, \dots, g_{l_i + r_i}, I_{z_i}) g$ for some $g_1, \dots, g_{l_i + r_i} \in \mathrm{GL}_{n_i}$, we may also assume that

$$Q = \sum_{j=1}^{l_i} P_{jj} - \sum_{\ell=1}^{r_i} Q_{\ell\ell}^T + \lambda(l_i - r_i) I_{n_i} \in X_i$$

satisfies $k < \mathrm{rk}\,(Q, I_{n_i})$ and this proves (*). □

Note that if $z_i = 0$ and in addition $\mathrm{char}(K) = 2$ or $r_i = 0$, then the map

$$\begin{array}{ccc} \mathfrak{gl}_{n_{i+1}} & \twoheadrightarrow & \mathfrak{gl}_{n_i} \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & \dots & \bullet \\ \vdots & & \vdots & \vdots & & \vdots \\ P_{l_i 1} & \dots & P_{l_i l_i} & \bullet & \dots & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r_i} \\ \vdots & & \vdots & \vdots & & \vdots \\ \bullet & \dots & \bullet & Q_{r_i 1} & \dots & Q_{r_i r_i} \end{pmatrix} & \mapsto & \displaystyle\sum_{k=1}^{l_i} P_{kk} - \sum_{\ell=1}^{r_i} Q_{\ell\ell}^T \end{array}$$

commutes with taking the trace.

Definition 2.3.15. When $z_i = 0$ for all $i \gg 0$ and in addition $\mathrm{char}(K) = 2$ or $r_i = 0$ for all $i \gg 0$, define the trace of an element $(P_i)_i \in V$ to be the $\mu \in K$ such that $\mathrm{tr}(P_i) = \mu$ for all $i \gg 0$. Otherwise, define the trace of any element of V to be zero.

Note that in all cases the trace of an element of V is G-invariant. For $\mu \in K$, denote the G-stable closed subset $\{P \in V \mid \mathrm{tr}(P) = \mu\}$ of V by Y_μ. Denote the closure of the projection of Y_μ to $\mathfrak{gl}_{n_i}$ by $Y_{\mu,i}$.

Theorem 2.3.16. *Assume that $l_i+r_i \geq 2$ for all $i \in \mathbb{N}$ and that $X \subsetneq Y_\mu$ for some $\mu \in K$. Suppose that for all $i \in \mathbb{N}$ such that $\mathcal{I}(Y_{\mu,i}) \subsetneq \mathcal{I}(X_i)$ and for all nonzero polynomials $f \in \mathcal{I}(X_i) \setminus \mathcal{I}(Y_{\mu,i})$ of minimal degree, the span of the $\mathrm{GL}_{n_{i+1}}$-orbit of the polynomial*

$$f(P_{11} + \cdots + P_{l_il_i} - Q_{11}^T - \cdots - Q_{r_ir_i}^T) \in \mathcal{I}(X_{i+1})$$

contains a nonzero off-diagonal polynomial. Then either $X = \emptyset$ or $X = \{0\}$.

Proof. Since X is strictly contained in Y_μ, there exists an integer $j \geq 2$ such that $\mathcal{I}(Y_{\mu,j}) \subsetneq \mathcal{I}(X_j)$. Note that $\mathcal{I}(Y_{\mu,i}) \subsetneq \mathcal{I}(X_i)$ for all integers $i \geq j$. For all $i \geq j$, let $f_i \in \mathcal{I}(X_i) \setminus \mathcal{I}(Y_{\mu,i})$ be an element of minimal degree d_i. Then $d_i \leq d_j$ for all $i \geq j$ and by choosing j large enough we may assume that $d_j \leq n_j$.

For $i \geq j$, let $g_i \in \mathcal{I}(X_{i+1})$ be a nonzero off-diagonal polynomial contained in the span of the $\mathrm{GL}_{n_{i+1}}$-orbit of $f_i(P_{11} + \cdots + P_{l_il_i} - Q_{11}^T - \cdots - Q_{r_ir_i}^T)$. Then we have $\deg(g) \leq d_i \leq d_j \leq n_j \leq n_{i+1}/2$ since $n_{i+1} = (l_i + r_i)n_i + z_i \geq 2n_i$. So by Lemmas 2.3.4 and 2.3.14, we see that $X \subseteq \{0\}$. □

Corollary 2.3.17. *Assume that $l_i + r_i \geq 2$ for all $i \in \mathbb{N}$. Suppose that for all $\mu \in K$, for all G-stable closed subsets $X \subsetneq Y_\mu$, for all $i \in \mathbb{N}$ such that $\mathcal{I}(Y_{\mu,i}) \subsetneq \mathcal{I}(X_i)$ and for all nonzero polynomials $f \in \mathcal{I}(X_i) \setminus \mathcal{I}(Y_{\mu,i})$ of minimal degree, the span of the $\mathrm{GL}_{n_{i+1}}$-orbit of the polynomial*

$$f(P_{11} + \cdots + P_{l_il_i} - Q_{11}^T - \cdots - Q_{r_ir_i}^T) \in \mathcal{I}(X_{i+1})$$

contains a nonzero off-diagonal polynomial. Then the irreducible G-stable closed subsets of V are the nonempty subsets among $\{0\}$, V and $\{P \in V \mid \mathrm{tr}(P) = \mu\}$ for $\mu \in K$ and every G-stable closed subset of V is a finite union of irreducible G-stable closed subsets.

Proof. Using Proposition 1.2.3, it is easy to check that the mentioned subsets are either irreducible or empty. If the trace map on V is zero, this corollary is just Theorem 2.3.16 applied with $\mu = 0$. Assume the trace map is nonzero. Then the linear map

$$\begin{aligned} \varphi\colon K &\to V \\ \mu &\mapsto ((\mu + 1)E_{11} - E_{22})_i \end{aligned}$$

has the property that $\mathrm{tr}(\varphi(\mu)) = \mu$ for all $\mu \in K$. Let X be a G-stable closed subset of V. Then

$$\varphi^{-1}(X) = \left\{\mu \in K \,\middle|\, Y_\mu \subseteq X\right\}$$

is a closed subset of K. So either $\varphi^{-1}(X)$ is finite or $\varphi^{-1}(X) = K$. By Theorem 2.3.16, the intersection of X with Y_0 is either $\emptyset$, $\{0\}$ or Y_0 and the intersection of X with Y_μ for $\mu \in K \setminus \{0\}$ is either $\emptyset$ or Y_μ. So either

$$X = \{0\} \cup \bigcup_{\mu \in \varphi^{-1}(X) \setminus \{0\}} Y_\mu \qquad \text{or} \qquad X = \bigcup_{\mu \in \varphi^{-1}(X)} Y_\mu$$

when $\varphi^{-1}(X)$ is finite and $X = V$ when $\varphi^{-1}(X) = K$. □

What remains is to reduce the cases (2)-(4) of Theorem 2.1.4 to sequences

$$\mathrm{GL}_{n_1} \xhookrightarrow{\iota_1} \mathrm{GL}_{n_2} \xhookrightarrow{\iota_2} \mathrm{GL}_{n_3} \xhookrightarrow{\iota_3} \dots$$

where the conditions of the corollary are satisfied.

Case (2): $\alpha + \beta = \gamma = \infty$

Since $\gamma = \infty$, we do not have $z_i = 0$ for all $i \gg 0$. So we get $Y_0 = V$ and $Y_\mu = \emptyset$ for all $\mu \in K \setminus \{0\}$. By restricting to an infinite subsequence we may assume that $l_i + r_i \geq 2$ and $z_i \geq n_i$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be such that $\mathcal{I}(X_i) \neq 0$ and let $f \in \mathcal{I}(X_i)$ be a nonzero polynomial of minimal degree. Take $l = l_i$, $r = r_i$, $z = z_i$, $m = n_i$ and $n = n_{i+1} = (l+r)m + z$. To prove that the conditions of Corollary 2.3.17 are satisfied, we need to check the following condition:

(*) The span of the GL_n-orbit of the polynomial

$$g := f(P_{11} + \dots + P_{ll} - Q_{11}^T - \dots - Q_{rr}^T)$$

contains a nonzero off-diagonal polynomial.

Consider the matrix

$$H = \begin{pmatrix} P_{11} & \dots & P_{1l} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l1} & \dots & P_{ll} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r1} & \dots & Q_{rr} & \bullet \\ R_1 & \dots & R_l & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix}$$

where $P_{k,\ell}, Q_{k,\ell}, R_k \in \mathfrak{gl}_m$. For $\lambda \in K$, consider the matrix

$$A(\lambda) = \begin{pmatrix} I_m & & & & & & \lambda I_m & \\ & \ddots & & & & & & \\ & & I_m & & & & & \\ & & & I_m & & & & \\ & & & & \ddots & & & \\ & & & & & I_m & & \\ & & & & & & I_m & \\ & & & & & & & I_{z-m} \end{pmatrix}.$$

For all $\lambda \in K$, we have

$$A(\lambda)HA(\lambda)^{-1} = \begin{pmatrix} P'_{11} & \dots & P'_{1l} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P'_{l1} & \dots & P'_{ll} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r1} & \dots & Q_{rr} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix}$$

where $P'_{11} = P_{11} + \lambda R_1$ and $P'_{jj} = P_{jj}$ for all $j \in \{2, \dots, l\}$. This means that if we let $A(\lambda)$ act on g, we obtain the polynomial $h(\lambda) = f(P_{11} + \dots + P_{ll} - Q_{11}^T - \dots - Q_{rr}^T + \lambda R_1)$. Let d be the degree of f and let $f_d = f_d(P)$ be the homogeneous part of f of degree d. Then $f_d(R_1)$ is a nonzero off-diagonal polynomial on $\mathfrak{gl}_n$ since $m \le n/2$. Since $f_d(R_1)$ is the coefficient of $h(\lambda)$ at λ^d, it is contained in the span of the $h(\lambda)$. So (*) holds. So we can apply Corollary 2.3.17 and this proves Theorem 2.1.4 in case (2).

Case (3a): $\beta = \infty$, $\gamma < \infty$ and $\mathrm{char}(K) \neq 2$

We do not have $\mathrm{char}(K) = 2$ or $r_i = 0$ for all $i \gg 0$. So we again get $Y_0 = V$ and $Y_\mu = \emptyset$ for all $\mu \in K \setminus \{0\}$. By restricting to an infinite subsequence we may assume that $r_i > 0$ and $z_i = 0$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be such that $\mathcal{I}(X_i) \neq 0$ and let $f \in \mathcal{I}(X_i)$ be a nonzero polynomial of minimal degree. Take $l = l_i$, $r = r_i$, $m = n_i$ and $n = n_{i+1} = (l + r)m$. To prove that the conditions of Corollary 2.3.17 are satisfied, we need to check the following condition:

(*) The span of the GL_n-orbit of the polynomial

$$g := f(P_{11} + \dots + P_{ll} - Q_{11}^T - \dots - Q_{rr}^T)$$

contains a nonzero off-diagonal polynomial.

Consider the matrix

$$H = \begin{pmatrix} P_{11} & \dots & P_{1l} & \bullet & \dots & \bullet \\ \vdots & & \vdots & \vdots & & \vdots \\ P_{l1} & \dots & P_{ll} & \bullet & \dots & \bullet \\ R_{11} & \dots & R_{1l} & Q_{11} & \dots & Q_{1r} \\ \vdots & & \vdots & \vdots & & \vdots \\ R_{r1} & \dots & R_{rl} & Q_{r1} & \dots & Q_{rr} \end{pmatrix}$$

where $P_{k,\ell}, Q_{k,\ell}, R_k \in \mathfrak{gl}_m$. Also consider the matrix

$$A(\Lambda) = \begin{pmatrix} I_m & & & \Lambda & & \\ & \ddots & & & & \\ & & I_m & & & \\ & & & I_m & & \\ & & & & \ddots & \\ & & & & & I_m \end{pmatrix}$$

for $\Lambda \in \mathfrak{gl}_m$. For all $\Lambda \in \mathfrak{gl}_m$, we have

$$A(\Lambda) H A(\Lambda)^{-1} = \begin{pmatrix} P'_{11} & \dots & P'_{1l} & \bullet & \dots & \bullet \\ \vdots & & \vdots & \vdots & & \vdots \\ P'_{l1} & \dots & P'_{ll} & \bullet & \dots & \bullet \\ \bullet & \dots & \bullet & Q'_{11} & \dots & Q'_{1r} \\ \vdots & & \vdots & \vdots & & \vdots \\ \bullet & \dots & \bullet & Q'_{r1} & \dots & Q'_{rr} \end{pmatrix}$$

where

$$\begin{array}{rcl} P'_{11} & = & P_{11} + \Lambda R_{11} \\ P'_{jj} & = & P_{jj} \text{ for } j \in \{2, \ldots, l\} \\ Q'_{11} & = & Q_{11} - R_{11}\Lambda \\ Q'_{jj} & = & Q_{\ell\ell} \text{ for } \ell \in \{2, \ldots, r\}. \end{array}$$

This means that if we let $A(\Lambda)$ act on the polynomial g, we obtain the polynomial $h(\Lambda) = f(P_{11} + \cdots + P_{ll} - Q_{11}^T - \cdots - Q_{rr}^T + \Lambda R_{11} + \Lambda^T R_{11}^T)$. Let d be the degree of f and let $f_d = f_d(P)$ be the homogeneous part of f of degree d. Then we see that the homogeneous part of $h(\Lambda)$ of degree d in the coordinates of Λ equals $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$. Since the polynomial f is nonzero, so is f_d. Using $\mathrm{char}(K) \neq 2$, we have

$$\mathfrak{gl}_n = \{PQ + P^T Q^T \mid P, Q \in \mathfrak{gl}_n\}$$

since every matrix is a product of two symmetric matrices by [33, (ii)]. So we see that the polynomial $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ is nonzero. Now view $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ as a polynomial in Λ whose coefficients are polynomials in the entries of R_{11}. Any of its nonzero coefficients is a nonzero off-diagonal polynomial on $\mathfrak{gl}_n$ which is contained in the span of the orbit of g. Here we use that $m \leq n/2$ since $r > 0$. So (*) holds. So we can apply Corollary 2.3.17 and this proves Theorem 2.1.4 in case (3a).

Case (3b): $\beta = \infty$, $\gamma < \infty$ and $\mathrm{char}(K) = 2$

Note that in this case the trace map on V is nonzero. By restricting to an infinite subsequence we may assume that $r_i > 0$ and $z_i = 0$ for all $i \in \mathbb{N}$. Let $\mu \in K$, suppose that $X \subsetneq Y_\mu$ and let $i \in \mathbb{N}$ be such that $\mathcal{I}(Y_{\mu,i}) \subsetneq \mathcal{I}(X_i)$. Let $f \in \mathcal{I}(X_i) \setminus \mathcal{I}(Y_{\mu,i})$ be a polynomial of minimal degree. Take $l = l_i$, $r = r_i$, $m = n_i$ and $n = n_{i+1} = (l + r)n$. To prove that the conditions of Corollary 2.3.17 are satisfied, we need to check the following condition:

(*) The span of the GL_n-orbit of the polynomial

$$g := f(P_{11} + \cdots + P_{ll} - Q_{11}^T - \cdots - Q_{rr}^T)$$

contains a nonzero off-diagonal polynomial.

As in case (3a), we find that all coefficients of $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ are off-diagonal polynomials on $\mathfrak{gl}_n$ which are contained in the span of the orbit of g. So it suffices to prove that $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ is not the zero polynomial. To do this, we will use reduction rules for graphs. See for example [12] for more on this. Let Γ be an undirected multigraph. Denote its vertex and edge sets by $V(\Gamma)$ and $E(\Gamma)$.

Definition 2.3.18. We consider the following three reduction rules:

(1) Remove an edge from Γ.

(2) Remove a vertex of Γ that has at least one loop.

(3) Pick a vertex v of Γ that has at least one loop. Replace an edge of Γ with endpoints $v \neq w$ by a loop at w.

We say that Γ reduces to a multigraph Γ' if Γ' can be obtained from Γ by applying a series of reductions.

Lemma 2.3.19. *If Γ reduces to the empty graph, then the linear map*

$$\begin{array}{rcl} \ell_\Gamma\colon K^{E(\Gamma)} & \to & K^{V(\Gamma)} \\ (x_e)_e & \mapsto & \left(\sum_{e\ni v} x_e\right)_v \end{array}$$

is surjective. Here entries corresponding to loops are only added once.

Proof. If Γ is the empty graph, then ℓ_Γ is surjective. So it suffices to check that ℓ_Γ is surjective whenever we have a reduction Γ' of Γ such that the similarly defined map $\ell_{\Gamma'}$ is surjective. When Γ' is obtained from Γ by applying reduction rule (1), this is easy. The other cases follow from the fact that x_e only appears in coordinate v when e is a loop with endpoint v. $\square$

Lemma 2.3.20. *If $\mathrm{char}(K) = 2$, then $\{PQ + P^TQ^T \mid P, Q \in \mathfrak{gl}_n\}$ is dense in $\mathfrak{sl}_n$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $\mathrm{char}(K) = 2$ and let $n \in \mathbb{N}$ be an integer. Then $PQ + P^TQ^T \in \mathfrak{sl}_n$ for all $P, Q \in \mathfrak{gl}_n$. Note that $\{PQ + P^TQ^T \mid P, Q \in \mathfrak{gl}_n\}$ is dense in $\mathfrak{sl}_n$ if and only if the morphism

$$\begin{array}{rcl} \varphi\colon \mathfrak{gl}_n \times \mathfrak{gl}_n & \to & \mathfrak{gl}_n \,/\, \mathrm{span}(E_{n,n}) \\ (P, Q) & \mapsto & PQ + P^TQ^T \bmod E_{n,n} \end{array}$$

is dominant. To show that φ is dominant, it suffices to show that its derivative

$$\begin{array}{rcl} \mathrm{d}_{(R,S)}\varphi\colon \mathfrak{gl}_n \oplus \mathfrak{gl}_n & \to & \mathfrak{gl}_n \,/\, \mathrm{span}(E_{n,n}) \\ (P, Q) & \mapsto & PS + P^TS^T + RQ + R^TQ^T \bmod E_{n,n} \end{array}$$

at the point

$$(R, S) = \left(\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}, \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & \iddots & & \\ 1 & & & \end{pmatrix} \right)$$

is surjective. Note that

$$\begin{array}{rcl} (\mathrm{d}_{(R,S)}\varphi)(E_{i,j}, 0) & = & E_{i,n+1-j} + E_{j,n+1-i} \\ (\mathrm{d}_{(R,S)}\varphi)(0, E_{k,\ell}) & = & (1 - \delta_{k1})E_{k-1,\ell} + (1 - \delta_{\ell n})E_{\ell+1,k} \end{array}$$

and hence $(\mathrm{d}_{(R,S)}\varphi)(0, E_{1,n}) = 0$ and $(\mathrm{d}_{(R,S)}\varphi)(E_{i,i}, 0) = 0$ for all $i \in [n]$, because $\mathrm{char}(K) = 2$. The other basis elements of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ all get sent to a sum of one or two basis elements of $\mathfrak{gl}_n \,/\, \mathrm{span}(E_{n,n})$. To prove that $\mathrm{d}_{(R,S)}\varphi$ is surjective, it suffices by the previous lemma to prove that the restriction of $\mathrm{d}_{(R,S)}\varphi$ to the span of these other basis vectors equals ℓ_Γ for some multigraph Γ that reduces to the empty graph.

Define the multigraph Γ as follows: We let $V(\Gamma)$ be the basis $\{E_{i,j} \mid (i, j) \neq (n, n)\}$ of $\mathfrak{gl}_n \,/\, \mathrm{span}(E_{n,n})$ and we let $E(\Gamma)$ be the set

$$\{(E_{i,j}, 0) \mid i \neq j\} \cup \{(0, E_{k,\ell}) \mid k, \ell \in [n]\} \setminus \{(0, E_{1,n})\}$$

of basis elements of $\mathfrak{gl}_n \oplus \mathfrak{gl}_n$ that are not mapped to 0. This allows us to define the set of endpoints of an edge in such a way that $(\mathrm{d}_{(R,S)}\varphi)|_{\mathrm{span}(E(\Gamma))} = \ell_\Gamma$. Next we check that Γ reduces to the empty graph. One can check that Γ has two loops at $E_{1,1}$, a loop at $E_{k,1}$ for all $k > 1$ and a loop at $E_{\ell,n}$ for all $\ell < n$. We also have:

(x) edges with endpoints $E_{i,j}$ and $E_{j+1,i+1}$ for all $i, j \in [n-1]$;

(y) edges with endpoints $E_{k,1}$ and $E_{n,n+1-k}$ for all $1 < k < n$; and

(z) edges with endpoints $E_{\ell,n}$ and $E_{1,n+1-\ell}$ for $1 < \ell < n$.

First, we remove all other edges from Γ using reduction rule (1). Next, we replace the edges (y) and (z) by loops at $E_{n,k}$ for $1 < k < n$ and $E_{1,\ell}$ for $1 < \ell < n$ using reduction rule (3). The graph Γ' obtained this way has the edges (x) together with loops at $E_{1,1}$ and $E_{1,i}, E_{n,i}, E_{i,1}, E_{i,n}$ for $1 < i < n$. Now consider the connected components of Γ'. One connected component consists of a path from $E_{1,1}$ to $E_{n,n}$ with a loop at $E_{1,1}$. All other components are paths with loops at both ends starting at a vertex of the form $E_{1,i}$ or $E_{i,1}$ and ending at a vertex of the form $E_{n,i}$ or $E_{i,n}$. Each of these components reduces to the empty graph by repeatedly using reduction rules (2) and (3). Therefore Γ' and Γ also reduce to the empty graph. Hence $\mathrm{d}_{(R,S)}\varphi$ is surjective and φ is dominant. □

Suppose that the polynomial $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ is the zero polynomial. Then $f_d(P) = 0$ for all $P \in \mathfrak{sl}_m$ by Lemma 2.3.20. So f_d is a multiple of the trace function on $\mathfrak{gl}_m$ and we can write $f_d = \mathrm{tr} \cdot h$ for some h. But then $f - (\mathrm{tr} - \mu)h \in \mathcal{I}(X_i) \setminus \mathcal{I}(Y_{\mu,i})$. This contradicts the minimality of the degree of f. So $f_d(\Lambda R_{11} + \Lambda^T R_{11}^T)$ cannot be the zero polynomial. So (*) again holds. So we can apply Corollary 2.3.17 and this proves Theorem 2.1.4 in case (3b).

Case (4): $\beta + \gamma < \infty$

Note that in this case the trace map on V is nonzero. By restricting to an infinite subsequence we may assume that $l_i > 2$ and $r_i = z_i = 0$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ be such that $\mathcal{I}(X_i) \neq 0$ and let $f \in \mathcal{I}(X_i)$ be a nonzero polynomial of minimal degree. Take $l = l_i$, $m = n_i$ and $n = n_{i+1} = lm$. Then $m \leq n/2$. To prove that the conditions of Corollary 2.3.17 are satisfied, we need to check the following condition:

(*) The span of the GL_n-orbit of the polynomial

$$g := f(P_{11} + \cdots + P_{ll})$$

contains a nonzero off-diagonal polynomial.

Consider the matrix

$$H = \begin{pmatrix} P_{11} & \dots & P_{1l} \\ \vdots & & \vdots \\ P_{l1} & \dots & P_{ll} \end{pmatrix}$$

where $P_{k,\ell} \in \mathfrak{gl}_m$. Also consider the matrix

$$A(\Lambda) = \begin{pmatrix} I_m & \Lambda & & & \\ & I_m & & & \\ & & I_m & & \\ & & & \ddots & \\ & & & & I_m \end{pmatrix}$$

for $\Lambda \in \mathfrak{gl}_m$. For all $\Lambda \in \mathfrak{gl}_m$, we have

$$A(\Lambda)HA(\Lambda)^{-1} = \begin{pmatrix} P'_{11} & \dots & P'_{1l} \\ \vdots & & \vdots \\ P'_{l1} & \dots & P'_{ll} \end{pmatrix}$$

where $P'_{11} = P_{11} + \Lambda P_{21}$, $P'_{22} = P_{22} - P_{21}\Lambda$ and $P'_{jj} = P_{jj}$ for $j \in \{3, \dots, l\}$. This means that if we let $A(\Lambda)$ act on g, we obtain the polynomial $h(\Lambda) = f(P_{11} + \dots + P_{ll} + [\Lambda, P_{21}])$ where $[-,-]$ is the commutator bracket. Let d be the degree of f and let $f_d = f_d(P)$ be the homogeneous part of f of degree d. Then we see that the homogeneous part of $h(\Lambda)$ of degree d in the coordinates of Λ equals $f_d([\Lambda, P_{21}])$. Since f is nonzero, so is f_d. By [30, Theorem 6.3], we know that every element of $\mathfrak{sl}_m$ is of the form $[X, Y]$ for some $X, Y \in \mathfrak{gl}_m$. So like the previous case, we see that $f_d([\Lambda, P_{21}])$ is not the zero polynomial. Any nonzero coefficient of $f_d([\Lambda, P_{21}])$ as a polynomial in Λ satisfies (*). So we can apply Corollary 2.3.17 and this proves Theorem 2.1.4 in case (4).

2.4 Limits of classical groups of type A

In this section, we let H be the direct limit of a sequence

$$\mathrm{SL}_{n_1} \xhookrightarrow{\iota_1} \mathrm{SL}_{n_2} \xhookrightarrow{\iota_2} \mathrm{SL}_{n_3} \xhookrightarrow{\iota_3} \dots$$

of diagonal embeddings given by

$$\begin{array}{rcl} \iota_i \colon \mathrm{SL}_{n_i} & \hookrightarrow & \mathrm{SL}_{n_{i+1}} \\ A & \mapsto & \mathrm{Diag}(\underbrace{A, \dots, A}_{l_i}, \underbrace{A^{-T}, \dots, A^{-T}}_{r_i}, I_{z_i}) \end{array}$$

for some $l_i \in \mathbb{N}$ and $r_i, z_i \in \mathbb{Z}_{\geq 0}$ with $l_i \geq r_i$. We let W be the inverse limit of the sequence

$$\mathfrak{gl}_{n_1} / \mathrm{span}(I_{n_1}) \twoheadleftarrow \mathfrak{gl}_{n_2} / \mathrm{span}(I_{n_2}) \twoheadleftarrow \mathfrak{gl}_{n_3} / \mathrm{span}(I_{n_3}) \twoheadleftarrow \dots$$

where the maps are given by

$$\begin{array}{ccc} \mathfrak{gl}_{n_{i+1}} / \mathrm{span}(I_{n_{i+1}}) & \twoheadrightarrow & \mathfrak{gl}_{n_i} / \mathrm{span}(I_{n_i}) \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & \dots & \bullet & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i1} & \dots & P_{l_il_i} & \bullet & \dots & \bullet & \bullet \\ \bullet & \dots & \bullet & Q_{11} & \dots & Q_{1r_i} & \bullet \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ \bullet & \dots & \bullet & Q_{r_i1} & \dots & Q_{r_ir_i} & \bullet \\ \bullet & \dots & \bullet & \bullet & \dots & \bullet & \bullet \end{pmatrix} \bmod I_{n_{i+1}} & \mapsto & \displaystyle\sum_{k=1}^{l_i} P_{kk} - \sum_{\ell=1}^{r_i} Q_{\ell\ell}^T \bmod I_{n_i}. \end{array}$$

Again take $\alpha = \#\{i \mid l_i > 1\}$, $\beta = \#\{i \mid r_i > 0\}$, $\gamma = \#\{i \mid z_i > 0\} \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then we have $\alpha + \beta + \gamma = \infty$, since H is assumed to be infinite-dimensional. Based on α, β, γ we distinguish the following cases:

(1) $\alpha + \beta < \infty$;

(2) $\alpha + \beta = \gamma = \infty$;

(3a) $\beta = \infty, \gamma < \infty$ and $\mathrm{char}(K) \neq 2$ or $2 \nmid n_i$ for all $i \gg 0$;

(3b) $\beta = \infty, \gamma < \infty$, $\mathrm{char}(K) = 2$ and $2 \mid n_i$ for all $i \gg 0$;

(4a) $\beta + \gamma < \infty$ and $\mathrm{char}(K) \nmid n_i$ for all $i \gg 0$; and

(4b) $\beta + \gamma < \infty$ and $\mathrm{char}(K) \mid n_i$ for all $i \gg 0$.

Theorem 2.1.4 has to following corollary.

Corollary 2.4.1. *The space W is H-Noetherian. Any H-stable closed subset of W is a finite union of irreducible H-stable closed subsets. The irreducible H-stable closed subsets of W are* $\{(0 \bmod I_{n_i})_i\}$ *and W together with*

$$\{(P_i \bmod I_{n_i})_i \in W \,\big|\, \forall i \gg 0\colon\ \mathrm{rk}(P_i, I_{n_i}) \leq k\}$$

for $k \in \mathbb{N}$ in case (1) *and together with*

$$\{(P_i \bmod I_{n_i})_i \in W \,\big|\, \forall i \gg 0\colon\ \mathrm{tr}(P_i) = \mu\}$$

for $\mu \in K$ in cases (3b) *and* (4b).

Proof. Let the tuples (l_i, r_i, z_i) from this section and the previous section be the same. Then H is a subgroup of G and the linear map

$$\begin{array}{rcl} \pi\colon V & \to & W \\ (P_i)_i & \mapsto & (P_i \bmod I_{n_i})_i \end{array}$$

is both H-equivariant and surjective. Furthermore, the orbits in V of H and G are the same. So the H-stable closed subsets of W are precisely the images of G-stable closed subsets X of V such that $X + \ker(\pi) = X$. This yields the corollary for the cases (1), (2), (3a) and (3b). For the cases (4a) and (4b), we note that

$$\{(P_i)_i \in V \mid \forall i \gg 0\colon\ \mathrm{tr}(P_i) = \mu\} + \ker(\pi) = \{(P_i)_i \in V \mid \forall i \gg 0\colon\ \mathrm{tr}(P_i) = \mu\}$$

if and only if $\mathrm{tr}(I_{n_i}) = n_i$ is zero modulo $\mathrm{char}(K)$ for all $i \gg 0$. Here we use that $(n_i^{-1} I_{n_i})_{i \geq j}$ is an element of $\ker(\pi)$ with trace 1 when $\mathrm{char}(K) \nmid n_i$ for all $i \geq j$. ◻

2.5 Limits of classical groups of type C

For the remainder of this chapter, we assume that $\mathrm{char}(K) \neq 2$. In this section, we let G be the direct limit of a sequence

$$\mathrm{Sp}_{2n_1} \xhookrightarrow{\iota_1} \mathrm{Sp}_{2n_2} \xhookrightarrow{\iota_2} \mathrm{Sp}_{2n_3} \xhookrightarrow{\iota_3} \dots$$

of diagonal embeddings given by

$$\begin{array}{rcl} \iota_i\colon \mathrm{Sp}_{2n_i} & \hookrightarrow & \mathrm{Sp}_{2n_{i+1}} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \mapsto & \begin{pmatrix} \mathrm{Diag}(A, \dots, A, I_{z_i}) & \mathrm{Diag}(B, \dots, B, 0) \\ \mathrm{Diag}(C, \dots, C, 0) & \mathrm{Diag}(D, \dots, D, I_{z_i}) \end{pmatrix} \end{array}$$

with l_i blocks $A, B, C, D \in \mathfrak{gl}_{n_i}$ for some $l_i \in \mathbb{N}$ and $z_i \in \mathbb{Z}_{\geq 0}$. We let V be the inverse limit of the sequence

$$\mathfrak{sp}_{2n_1} \twoheadleftarrow \mathfrak{sp}_{2n_2} \twoheadleftarrow \mathfrak{sp}_{2n_3} \twoheadleftarrow \dots$$

where the maps are given by

$$\begin{array}{ccc} \mathfrak{sp}_{2n_{i+1}} & \twoheadrightarrow & \mathfrak{sp}_{2n_i} \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & Q_{11} & \dots & Q_{1l_i} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i1} & \dots & P_{l_il_i} & \vdots & Q_{l_i1} & \dots & Q_{l_il_i} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \\ R_{11} & \dots & R_{1l_i} & \bullet & S_{11} & \dots & S_{1l_i} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{l_i1} & \dots & R_{l_il_i} & \vdots & S_{l_i1} & \dots & S_{l_il_i} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \end{pmatrix} & \mapsto & \begin{pmatrix} \sum_{k=1}^{l_i} P_{kk} & \sum_{k=1}^{l_i} Q_{kk} \\ \sum_{k=1}^{l_i} R_{kk} & \sum_{k=1}^{l_i} S_{kk} \end{pmatrix} \end{array}$$

with $P_{k\ell} = -S_{\ell k}^T, Q_{k\ell}, R_{k\ell} \in \mathfrak{gl}_{n_i}$ such that $Q_{k\ell} = Q_{\ell k}^T$ and $R_{k\ell} = R_{\ell k}^T$.

Theorem 2.5.1. *The space V is G-Noetherian.*

Let $X \subsetneq V$ be a G-stable closed subset. Let X_i be the closure of the projection of X to $\mathfrak{sp}_{2n_i}$ and let $\mathcal{I}(X_i) \subseteq K[\mathfrak{sp}_{2n_i}]$ be the ideal of X_i. If $\#\{i \mid l_i > 1\} < \infty$, then Theorem 2.5.1 follows from [20, Theorem 1.2].

Remark 2.5.2. Let $X \subsetneq V$ be a G-stable closed subset in the case where $\#\{i \mid l_i > 1\} < \infty$. Then V can be identified with a subspace of the space of $\mathbb{N} \times \mathbb{N}$ matrices and we can prove (using technique similar to the ones used in this paper) that X consists of matrices of bounded rank. The G-Noetherianity of V then follows from the $\mathrm{Sym}(\mathbb{N})$-Noetherianity of $K^{\mathbb{N} \times k}$ for $k \in \mathbb{N}$. Important to note here is that, for every $n \in \mathbb{N}$, the group Sp_{2n} contains all matrices corresponding to permutations $\pi \in S_{2n}$ such that $\pi(i + n) = \pi(i) + n$ for all $i \in [n]$. This allows us to define an action of $\mathrm{Sym}(\mathbb{N})$ on V, up to which the closed subset X is Noetherian. Similar statements hold for sequences of types B and D. ♠

We assume that $\#\{i \mid l_i > 1\} = \infty$. By restricting to an infinite subsequence, we may assume that $l_i \geq 3$ for all $i \in \mathbb{N}$.

Lemma 2.5.3. *Let $n \in \mathbb{N}$, let $Y \subsetneq \mathfrak{sp}_{2n}$ be an Sp_{2n}-stable closed subset and let Z be the closed subset*

$$\left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{sp}_{2n} \;\middle|\; P = P^T \right\}$$

of $\mathfrak{sp}_{2n}$. Then there is a nonzero polynomial $f \in \mathcal{I}(Y)$ whose top-graded part is not contained in the ideal of Z.

Proof. Since $Y \subsetneq \mathfrak{sp}_{2n}$, there is a nonzero polynomial $f \in \mathcal{I}(Y)$. Since f is nonzero, so is its top-graded part g. Let the group GL_n act on $\mathfrak{sp}_{2n}$ via the diagonal embedding $\mathrm{GL}_n \hookrightarrow \mathrm{Sp}_{2n}, A \mapsto \mathrm{Diag}(A, A^{-T})$. Then we get a action of GL_n on $K[\mathfrak{sp}_{2n}]$. Note that this action respects the grading on $K[\mathfrak{sp}_{2n}]$ and that the ideal $\mathcal{I}(Y)$ is GL_n-stable. So for all $A \in \mathrm{GL}_n$ we have $A \cdot f \in \mathcal{I}(Y)$ and the top-graded part of this polynomial is $A \cdot g$. Hence it suffices to prove that $A \cdot g \notin \mathcal{I}(Z)$ for some $A \in \mathrm{GL}_n$. Note that

$$\begin{aligned}
\mathrm{GL}_n \cdot Z &= \left\{ A \cdot \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \;\middle|\; \begin{matrix} P = P^T, A \in \mathrm{GL}_n \\ Q = Q^T, R = R^T \end{matrix} \right\} \\
&= \left\{ \begin{pmatrix} APA^{-1} & AQA^T \\ A^{-T}RA^{-1} & -A^{-T}P^TA^T \end{pmatrix} \;\middle|\; \begin{matrix} P = P^T, A \in \mathrm{GL}_n \\ Q = Q^T, R = R^T \end{matrix} \right\} \\
&= \left\{ \begin{pmatrix} APA^{-1} & Q \\ R & -(APA^{-1})^T \end{pmatrix} \;\middle|\; \begin{matrix} P = P^T, A \in \mathrm{GL}_n \\ Q = Q^T, R = R^T \end{matrix} \right\}
\end{aligned}$$

and that $\{APA^{-1} \mid P = P^T, A \in \mathrm{GL}_n\}$ is dense in $\mathfrak{gl}_n$ since K is infinite and diagonal matrices are symmetric. So $\mathrm{GL}_n \cdot Z$ is dense in $\mathfrak{sp}_{2n}$. So since the polynomial g is nonzero, there must be an $A \in \mathrm{GL}_n$ such that $A \cdot g \notin \mathcal{I}(Z)$. $\square$

Lemma 2.5.4. *Let $i \in \mathbb{N}$ and let $f = f(P, Q, R) \in \mathcal{I}(X_i)$ be a nonzero polynomial whose top-graded part g is not contained in the ideal of*

$$\left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{sp}_{2n_i} \;\middle|\; P = P^T \right\}.$$

Then $\mathcal{I}(X_{i+1}) \cap K[r_{k\ell} | 1 \leq k, \ell \leq n_{i+1}]/(r_{k\ell} - r_{\ell k})$ contains a nonzero polynomial with degree at most $\deg(f)$.

Proof. Take $m = n_i$, $l = l_i$, $z = z_i$ and $n = n_{i+1} = lm + z$. Consider the matrix

$$H = \begin{pmatrix}
P_{11} & \dots & P_{1l} & \bullet & Q_{11} & \dots & Q_{1l} & \bullet \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
P_{l1} & \dots & P_{ll} & \vdots & Q_{l1} & \dots & Q_{ll} & \vdots \\
\bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \\
R_{11} & \dots & R_{1l} & \bullet & S_{11} & \dots & S_{1l} & \bullet \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
R_{l1} & \dots & R_{ll} & \vdots & S_{l1} & \dots & S_{ll} & \vdots \\
\bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet
\end{pmatrix} \in \mathfrak{sp}_{2n}$$

and consider the matrix

$$A(\lambda) = \begin{pmatrix}
I_m & & & & \lambda I_m & \\
& I_m & & \lambda I_m & & \\
& & I_{n-2m} & & & \\
& & & I_m & & \\
& & & & I_m & \\
& & & & & I_{n-2m}
\end{pmatrix} \in \mathrm{Sp}_{2n}$$

for $\lambda \in K$. The polynomial $f = f(P, Q, R) \in \mathcal{I}(X_i)$ pulls back to the element

$$f\left(\sum_{k=1}^{l} P_{kk}, \sum_{k=1}^{l} Q_{kk}, \sum_{k=1}^{l} R_{kk} \right)$$

of $\mathcal{I}(X_{i+1})$. For $\lambda \in K$, we have

$$A(\lambda)HA(\lambda)^{-1} = \begin{pmatrix} P'_{11} & \dots & P'_{1l} & \bullet & Q'_{11} & \dots & Q'_{1l} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ P'_{l1} & \dots & P'_{ll} & \vdots & Q'_{l1} & \dots & Q'_{ll} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \\ R_{11} & \dots & R_{1l} & \bullet & S'_{11} & \dots & S'_{1l} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{l1} & \dots & R_{ll} & \vdots & S'_{l1} & \dots & S'_{ll} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \end{pmatrix}$$

where

$$\begin{aligned} P'_{11} &= P_{11} + \lambda R_{21} \\ P'_{22} &= P_{22} + \lambda R_{12} \\ P'_{kk} &= P_{kk} \text{ for } k = 3, \dots, l \\ Q'_{11} &= Q_{11} + \lambda(S_{21} - P_{12}) - \lambda^2 R_{22} \\ Q'_{22} &= Q_{22} + \lambda(S_{12} - P_{21}) - \lambda^2 R_{11} \\ Q'_{kk} &= Q_{kk} \text{ for } k = 3, \dots, l \end{aligned}$$

Let g be the top-graded part of f. Then we see that

$$g(R_{21} + R_{12}, -(R_{11} + R_{22}), \sum_{k=1}^{l} R_{kk})$$

is contained in the span of

$$A(\lambda) \cdot f\left(\sum_{k=1}^{l} P_{kk}, \sum_{k=1}^{l} Q_{kk}, \sum_{k=1}^{l} R_{kk}\right)$$

over all $\lambda \in K$. We have $g(P, Q, R) \neq 0$ for some symmetric matrices $P, Q, R \in \mathfrak{gl}_m$. Since $\mathrm{char}(K) \neq 2$, there are matrices R_{12}, R_{21} such that $R_{12} = R_{21}^T$ and $R_{21} + R_{12} = P$. And, since $l > 2$, there are symmetric matrices $R_{11}, \dots, R_{ll}$ such that $-(R_{11} + R_{22}) = Q$ and $\sum_{k=1}^{l} R_{kk} = R$. So we see that the polynomial

$$g\left(R_{21} + R_{12}, -(R_{11} + R_{22}), \sum_{k=1}^{l} R_{kk}\right) \in \mathcal{I}(X_{i+1})$$

is nonzero. □

Since $X \subsetneq V$, we know that $X_j \subsetneq \mathfrak{sp}_{2n_j}$ for some $j \in \mathbb{N}$. Using the previous lemma, we see that there is a $d \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{I}(X_i) \cap K[r_{k\ell} | 1 \leq k, \ell \leq n_i]/(r_{k\ell} - r_{\ell k})$ contains a nonzero polynomial of degree at most d for all $i > j$.

Lemma 2.5.5. *Let $n \in \mathbb{N}$, let $Y \subsetneq \mathfrak{sp}_{2n}$ be an Sp_{2n}-stable closed subset, let*

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in Y$$

be an element and suppose that

$$\mathcal{I}(Y) \cap K[r_{k\ell} | 1 \leq k, \ell \leq n]/(r_{k\ell} - r_{\ell k})$$

contains a nonzero polynomial of degree $m+1$*. Then* $\mathrm{rk}(M_{12}), \mathrm{rk}(M_{21}) \leq m$*. Furthermore, if* $n > 6m$*, then* $\mathrm{rk}(M_{11}) = \mathrm{rk}(M_{22}) \leq 3m/2$ *and* $\mathrm{rk}(M) \leq 5m$*.*

Proof. Let GL_n act on $\mathfrak{sp}_{2n}$ via the diagonal embedding

$$\begin{array}{rcl} \mathrm{GL}_n & \hookrightarrow & \mathrm{Sp}_{2n} \\ g & \mapsto & \mathrm{Diag}(g, g^{-T}) \end{array}$$

and on $\{R \in \mathfrak{gl}_n \mid R = R^T\}$ by $g \cdot R = g^{-T} R g^{-1}$. Then the projection map

$$\begin{array}{rcl} \pi\colon \mathfrak{sp}_{2n} & \to & \mathfrak{gl}_n \\ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} & \mapsto & R \end{array}$$

is GL_n-equivariant. Let Z be the closure of $\pi(Y)$ in $\{R \in \mathfrak{gl}_n \mid R = R^T\}$. Since Y is GL_n-stable, so are $\pi(Y)$ and Z. Since $\mathrm{char}(K) \neq 2$, the GL_n-orbits of $\{R \in \mathfrak{gl}_n \mid R = R^T\}$ consist of all symmetric matrices of equal rank. So Z must consist of all symmetric matrices of rank at most h for some $h \leq n$. Since $\mathcal{I}(Z)$ contains a nonzero polynomial of degree $m+1$, we see that $h \leq m$. See, for example, [31, §4]. So

$$Y \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{sp}_{2n} \;\middle|\; \mathrm{rk}(R) \leq m \right\}.$$

Let $A \in \mathfrak{gl}_n$ be a symmetric matrix. Then we have

$$\begin{pmatrix} 0 & I_n \\ -I_n & A \end{pmatrix} \in \mathrm{Sp}_{2n}$$

with inverse

$$\begin{pmatrix} A & -I_n \\ I_n & 0 \end{pmatrix}.$$

Let

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

be an element of Y. Then

$$\begin{pmatrix} 0 & I_n \\ -I_n & A \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & A \end{pmatrix}^{-1} = \begin{pmatrix} \bullet & \bullet \\ ARA + AS - PA - Q & \bullet \end{pmatrix} \in Y.$$

So we get $\mathrm{rk}(ARA + AS - PA - Q) \leq m$ for all symmetric matrices $A \in \mathfrak{gl}_n$. For $A = 0$, this gives us $\mathrm{rk}(Q) \leq m$ and so $\mathrm{rk}(M_{12}) \leq m$ in particular. For all A, we can write

$$PA + (PA)^T = (ARA + AS - PA - Q) - ARA + Q$$

since $S = -P^T$. We get

$$\mathrm{rk}(PA + (PA)^T) \leq \mathrm{rk}(ARA + AS - PA - Q) + \mathrm{rk}(ARA) + \mathrm{rk}(Q) \leq 3m.$$

Since we had no conditions on the element

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in Y,$$

we also get $\mathrm{rk}(P'A + (P'A)^T) \leq 3m$ for all

$$\begin{pmatrix} P' & \bullet \\ \bullet & \bullet \end{pmatrix} \in \mathrm{GL}_n \cdot \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \subseteq Y$$

and hence $\mathrm{rk}(P'A + (P'A)^T) \leq 3m$ for all $P' \sim P$. Now assume that $n > 6m$. Choose $A = \mathrm{Diag}(I_{2m+1}, 0)$ and write

$$P' = \begin{pmatrix} P'_{11} & P'_{12} \\ P'_{21} & P'_{22} \end{pmatrix} \sim P$$

with $P'_{21} \in \mathfrak{gl}_{2m+1}$. Then

$$P'A + (P'A)^T = \begin{pmatrix} \bullet & \bullet & P'^T_{21} \\ \bullet & & \\ P'_{21} & & \end{pmatrix}$$

and hence $\mathrm{rk}(P'_{21}) \leq 3m/2$. By Proposition 2.3.2, we see that $\mathrm{rk}(P, I_n) \leq 3m/2$ and hence $\mathrm{rk}(P + \lambda I_n) \leq 3m/2$ for some $\lambda \in K$. Next, choose $A = I_n$. Then we see that $\mathrm{rk}(P + P^T) \leq 3m$. So

$$\mathrm{rk}(2\lambda I_n) \leq \mathrm{rk}(P + P^T) + \mathrm{rk}(P + \lambda I_n) + \mathrm{rk}(P^T + \lambda I_n) \leq 6m < n$$

and hence $\lambda = 0$. So we in fact have $\mathrm{rk}(P) \leq 3m/2$. In particular, we see that $\mathrm{rk}(M_{11}) = \mathrm{rk}(M_{22}) \leq 3m/2$. Combining this with $\mathrm{rk}(M_{12}), \mathrm{rk}(M_{21}) \leq m$, we get $\mathrm{rk}(M) \leq 5m$. □

Using Lemma 2.5.5, we see that there is an $m \in \mathbb{Z}_{\geq 0}$ such that

$$X_i \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{sp}_{2n} \;\middle|\; \mathrm{rk}(P) \leq m \right\}$$

for all $i \gg 0$. As in the proof of Lemma 2.3.14, we see using Lemma 2.3.13 that this in fact holds for $m = 0$.

Lemma 2.5.6. *Let $n \in \mathbb{N}$ and let $Y \subsetneq \mathfrak{sp}_{2n}$ be an Sp_{2n}-stable closed subset of*

$$\left\{ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \;\middle|\; \begin{matrix} Q \in \mathfrak{gl}_n, Q = Q^T \\ R \in \mathfrak{gl}_n, R = R^T \end{matrix} \right\}.$$

Then $Y \subseteq \{0\}$.

Proof. Let

$$\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$$

be an element of Y. Then

$$\begin{pmatrix} 0 & I_n \\ -I_n & I_n \end{pmatrix} \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & I_n \end{pmatrix}^{-1} = \begin{pmatrix} R & \bullet \\ \bullet & \bullet \end{pmatrix} \in Y$$

since Y is Sp_{2n}-stable and therefore $R = 0$. By Lemma 2.5.5, we see that $Q = 0$. □

The lemma shows that $X \subseteq \{0\}$. So when $\#\{i \mid l_i > 1\} = \infty$, the only G-stable closed subsets of V are V, $\{0\}$ and $\emptyset$. This proves in particular that V is G-Noetherian.

2.6 Limits of classical groups of type D

Recall that we assume that $\mathrm{char}(K) \neq 2$. In this section, we let G be the direct limit of a sequence

$$\mathrm{O}_{2n_1} \xhookrightarrow{\iota_1} \mathrm{O}_{2n_2} \xhookrightarrow{\iota_2} \mathrm{O}_{2n_3} \xhookrightarrow{\iota_3} \dots$$

of diagonal embeddings given by

$$\begin{array}{rcl} \iota_i \colon \mathrm{O}_{2n_i} & \hookrightarrow & \mathrm{O}_{2n_{i+1}} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \mapsto & \begin{pmatrix} \mathrm{Diag}(A,\dots,A,I_{z_i}) & \mathrm{Diag}(B,\dots,B,0) \\ \mathrm{Diag}(C,\dots,C,0) & \mathrm{Diag}(D,\dots,D,I_{z_i}) \end{pmatrix} \end{array}$$

with l_i blocks $A,B,C,D \in \mathfrak{gl}_{n_i}$ for some $l_i \in \mathbb{N}$ and $z_i \in \mathbb{Z}_{\geq 0}$. We let V be the inverse limit of the sequence

$$\mathfrak{o}_{2n_1} \twoheadleftarrow \mathfrak{o}_{2n_2} \twoheadleftarrow \mathfrak{o}_{2n_3} \twoheadleftarrow \dots$$

where the maps are given by

$$\begin{array}{ccc} \mathfrak{o}_{2n_{i+1}} & \twoheadrightarrow & \mathfrak{o}_{2n_i} \\ \begin{pmatrix} P_{11} & \dots & P_{1l_i} & \bullet & Q_{11} & \dots & Q_{1l_i} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ P_{l_i1} & \dots & P_{l_il_i} & \vdots & Q_{l_i1} & \dots & Q_{l_il_i} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \\ R_{11} & \dots & R_{1l_i} & \bullet & S_{11} & \dots & S_{1l_i} & \bullet \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ R_{l_i1} & \dots & R_{l_il_i} & \vdots & S_{l_i1} & \dots & S_{l_il_i} & \vdots \\ \bullet & \dots & \dots & \bullet & \bullet & \dots & \dots & \bullet \end{pmatrix} & \mapsto & \begin{pmatrix} \sum_{k=1}^{l_i} P_{kk} & \sum_{k=1}^{l_i} Q_{kk} \\ \sum_{k=1}^{l_i} R_{kk} & \sum_{k=1}^{l_i} S_{kk} \end{pmatrix} \end{array}$$

with $P_{k\ell} = -S_{\ell k}^T, Q_{k\ell}, R_{k\ell} \in \mathfrak{gl}_{n_i}$ such that $Q_{k\ell} + Q_{\ell k}^T = R_{k\ell} + R_{\ell k}^T = 0$.

Theorem 2.6.1. *The space V is G-Noetherian.*

This proof of this theorem will have the same structure as the proof of Theorem 2.5.1. Let $X \subsetneq V$ be a G-stable closed subset. Let X_i be the closure of the projection of X to $\mathfrak{o}_{2n_i}$ and let $\mathcal{I}(X_i) \subseteq K[\mathfrak{o}_{2n_i}]$ be the ideal of X_i. If $\#\{i \mid l_i > 1\} < \infty$, then Theorem 2.6.1 follows from [20, Theorem 1.2]. So we assume that $\#\{i \mid l_i > 1\} = \infty$. By restricting to an infinite subsequence, we may assume that $l_i \geq 3$ for all $i \in \mathbb{N}$.

Lemma 2.6.2. *Let $n \in \mathbb{N}$, let $Y \subsetneq \mathfrak{o}_{2n}$ be an O_{2n}-stable closed subset and let Z be the closed subset*

$$\left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{sp}_{2n} \;\middle|\; P = P^T \right\}$$

of $\mathfrak{o}_{2n}$. Then there is a nonzero polynomial $f \in \mathcal{I}(Y)$ whose top-graded part is not contained in the ideal of Z.

Proof. The proof is analogous to the proof of Lemma 2.5.3. $\square$

Lemma 2.6.3. *Let $i \in \mathbb{N}$ and let $f = f(P, Q, R) \in \mathcal{I}(X_i)$ be a nonzero polynomial whose top-graded part g is not contained in the ideal of*

$$\left\{ \begin{pmatrix} P & Q \\ R & -P^T \end{pmatrix} \in \mathfrak{o}_{2n_i} \;\middle|\; P = P^T \right\}.$$

Then $\mathcal{I}(X_{i+1}) \cap K[r_{k\ell} | 1 \leq k, \ell \leq n_{i+1}]/(r_{k\ell} + r_{\ell k})$ contains a nonzero polynomial with degree at most $\deg(f)$.

Proof. The proof is analogous to the proof of Lemma 2.5.4, replacing $A(\lambda)$ by the matrix

$$\begin{pmatrix} I_m & & & & \lambda I_m & \\ & I_m & & -\lambda I_m & & \\ & & I_{n-2m} & & & \\ & & & I_m & & \\ & & & & I_m & \\ & & & & & I_{n-2m} \end{pmatrix} \in \mathrm{O}_{2n}\,.$$

□

Since $X \subsetneq V$, we know that $X_j \subsetneq \mathfrak{o}_{2n_j}$ for some $j \in \mathbb{N}$. Using the previous lemma, we see that there is a $d \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{I}(X_i) \cap K[r_{k\ell} \mid 1 \leq k, \ell \leq n_i]/(r_{k\ell} + r_{\ell k})$ contains a nonzero polynomial of degree at most d for all $i > j$.

Lemma 2.6.4. *Let $n \in \mathbb{N}$, let $Y \subsetneq \mathfrak{o}_{2n}$ be an O_{2n}-stable closed subset and suppose that*

$$\mathcal{I}(Y) \cap K[r_{k\ell} | 1 \leq k, \ell \leq n]/(r_{k\ell} + r_{\ell k})$$

contains a nonzero polynomial of degree $m + 1$. Then

$$Y \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{o}_{2n} \;\middle|\; \mathrm{rk}(Q), \mathrm{rk}(R) \leq 2m \right\}.$$

Furthermore, if $n \geq 20m + 2$, then $\mathrm{rk}(M) \leq 10m$ for all $M \in Y$.

Proof. Let Z be the closure of the subset

$$\left\{ R \;\middle|\; \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in Y \right\}$$

of $\{R \in \mathfrak{gl}_n \mid R + R^T = 0\}$. Let GL_n act on $\mathfrak{o}_{2n}$ via the diagonal embedding

$$\begin{array}{rcl} \mathrm{GL}_n & \hookrightarrow & \mathrm{O}_{2n} \\ g & \mapsto & \mathrm{Diag}(g, g^{-T}) \end{array}$$

and on $\{R \in \mathfrak{gl}_n \mid R + R^T = 0\}$ by $g \cdot R = gRg^T$. Then we see that Y is GL_n-stable and therefore Z is also GL_n-stable. So Z must consist of all skew-symmetric matrices of rank at most h for some even $h \leq n$. Since $\mathcal{I}(Z)$ contains a nonzero polynomial of degree $m + 1$, we see that $h \leq 2m$. See [1, §3]. So

$$Y \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{o}_{2n} \;\middle|\; \mathrm{rk}(R) \leq 2m \right\}.$$

Let $A \in \mathfrak{gl}_n$ be a skew-symmetric matrix and let

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

be an element of Y. Then we have

$$\begin{pmatrix} 0 & I_n \\ I_n & A \end{pmatrix} \in \mathrm{O}_{2n}$$

and hence

$$\begin{pmatrix} 0 & I_n \\ I_n & A \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix}\begin{pmatrix} 0 & I_n \\ I_n & A \end{pmatrix}^{-1} = \begin{pmatrix} \bullet & \bullet \\ Q + AS - PA - ARA & \bullet \end{pmatrix} \in Y.$$

So we get $\mathrm{rk}(Q + AS - PA - ARA) \leq 2m$. Choosing $A = 0$, we see that

$$Y \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{o}_{2n} \;\middle|\; \mathrm{rk}(Q) \leq 2m \right\}.$$

Assume that $n \geq 2(3m + 1)$. Since $S = -P^T$ and $A = -A^T$, we get

$$\mathrm{rk}(PA - (PA)^T) \leq \mathrm{rk}(Q + AS - PA - ARA) + \mathrm{rk}(ARA) + \mathrm{rk}(Q) \leq 6m.$$

Since Y is GL_n-stable, we have $\mathrm{rk}(P'A - (P'A)^T) \leq 6m$ for all $P' \sim P$. Choose

$$A = \begin{pmatrix} & & I_{3m+1} \\ & 0 & \\ -I_{3m+1} & & \end{pmatrix}$$

and write

$$P' = \begin{pmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{pmatrix}$$

with $P'_{11}, P'_{13}, P'_{31}, P'_{33} \in \mathfrak{gl}_{3m+1}$. Then

$$P'A - (P'A)^T = \begin{pmatrix} \bullet & P'^T_{23} & \bullet \\ -P'_{23} & 0 & P'_{21} \\ \bullet & -P'^T_{21} & \bullet \end{pmatrix}$$

has rank at most $6m$. Therefore the submatrix

$$\begin{pmatrix} 0 & P'_{21} \\ -P'^T_{21} & \bullet \end{pmatrix}$$

also has rank at most $6m$ and hence $\mathrm{rk}(P'_{21}) \leq 3m$. By Proposition 2.3.2, we see that $\mathrm{rk}(P, I_n) \leq 3m$. Hence

$$Y \subseteq \{M \in \mathfrak{o}_{2n} \mid \mathrm{rk}(M, \mathrm{Diag}(I_n, -I_n)) \leq 2 \cdot 2m + 2 \cdot 3m = 10m\}.$$

Assume that $n \geq 20m + 2$, let $M + \lambda\,\mathrm{Diag}(I_n, -I_n)$ be an element of Y with $\mathrm{rk}(M) \leq 10m$ and $\lambda \in K$ and let $B \in \mathfrak{gl}_n$ be a skew-symmetric matrix of rank at least $n - 1$. Then

$$\begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} \in \mathrm{O}_{2n}$$

and therefore

$$\begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} (M + \lambda \operatorname{Diag}(I_n, -I_n)) \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix}^{-1} \in Y.$$

So this element must be of the form $M' - \mu \operatorname{Diag}(I_n, -I_n)$ with $\operatorname{rk}(M) \leq 10m$ and $\mu \in K$. Now note that

$$\operatorname{rk}\left(\lambda \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} \operatorname{Diag}(I_n, -I_n) \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix}^{-1} + \mu \operatorname{Diag}(I_n, -I_n) \right)$$

is at most $\operatorname{rk}(M) + \operatorname{rk}(M') \leq 20m$. So since

$$\lambda \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix} \operatorname{Diag}(I_n, -I_n) \begin{pmatrix} I_n & B \\ & I_n \end{pmatrix}^{-1} + \mu \operatorname{Diag}(I_n, -I_n) = \begin{pmatrix} \bullet & -2\lambda B \\ \bullet & \bullet \end{pmatrix}$$

and $\operatorname{rk}(2B) \geq n - 1 > 20m$, we see that $\lambda = 0$. Hence Y consists of matrices of rank at most $10m$. ◻

Using Lemma 2.6.4, we see that there is an $m \in \mathbb{Z}_{\geq 0}$ such that

$$X_i \subseteq \left\{ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathfrak{o}_{2n} \;\middle|\; \operatorname{rk}(P) \leq m \right\}$$

for all $i \gg 0$. As in the proof of Lemma 2.3.14, we see using Lemma 2.3.13 that this in fact holds for $m = 0$.

Lemma 2.6.5. *Let $n \in \mathbb{N}$ and let $Y \subsetneq \mathfrak{o}_{2n}$ be an O_{2n}-stable closed subset of*

$$\left\{ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \;\middle|\; \begin{array}{c} Q \in \mathfrak{gl}_n, Q + Q^T = 0 \\ R \in \mathfrak{gl}_n, R + R^T = 0 \end{array} \right\}.$$

Then $Y \subseteq \{0\}$.

Proof. Let

$$\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix}$$

be an element of Y. Then

$$\begin{pmatrix} I_n & A \\ & I_n \end{pmatrix} \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \begin{pmatrix} I_n & A \\ & I_n \end{pmatrix}^{-1} = \begin{pmatrix} AR & \bullet \\ \bullet & \bullet \end{pmatrix} \in Y$$

for all $A \in \mathfrak{gl}_n$ with $A + A^T = 0$ since Y is O_{2n}-stable and therefore $R = 0$. By Lemma 2.6.4, we see that $Q = 0$. ◻

As in the previous section, the lemma shows that $X \subsetneq \{0\}$. So again, when $\#\{i \mid l_i > 1\} = \infty$, the only G-stable closed subsets of V are V, $\{0\}$ and $\emptyset$ and the space V is G-Noetherian.

2.7 Limits of classical groups of type B

In the last section of this chapter, we still assume that $\mathrm{char}(K) \neq 2$. Now, we let G be the direct limit of a sequence

$$\mathrm{O}_{2n_1+1} \xhookrightarrow{\iota_1} \mathrm{O}_{2n_2+1} \xhookrightarrow{\iota_2} \mathrm{O}_{2n_3+1} \xhookrightarrow{\iota_3} \dots$$

of diagonal embeddings. To prove that the corresponding inverse limit V is G-Noetherian, it suffices to consider the case where K is algebraically closed. The following proposition shows that, if $K = \overline{K}$ and ι_i has signature (l_i, z_i) with l_i even, then we can insert a group of type D into the sequence defining G.

Proposition 2.7.1. *Suppose that K is algebraically closed. Let $m, n \in \mathbb{Z}_{\geq 0}$ be integers and let $\iota\colon \mathrm{O}_{2m+1} \hookrightarrow \mathrm{O}_{2n+1}$ be a diagonal embedding with signature (l, z). If l is even, then ι is the composition of diagonal embeddings $\mathrm{O}_{2m+1} \hookrightarrow \mathrm{O}_{l(2m+1)}$ and $\mathrm{O}_{l(2m+1)} \hookrightarrow \mathrm{O}_{2n+1}$.*

Proof. By Lemma 2.2.3, it suffices to find one diagonal embedding $\iota\colon \mathrm{O}_{2m+1} \hookrightarrow \mathrm{O}_{2n+1}$ with signature (l, z) for which the proposition holds. For $k \in \mathbb{N}$, note that the group

$$H_k = \left\{ A \in \mathrm{GL}_k \;\middle|\; A \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} A^T = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \right\}$$

is conjugate to O_k in GL_k. The map

$$\begin{array}{ccc} H_{2m+1} & \hookrightarrow & H_{l(2m+1)} \\ A & \mapsto & \mathrm{Diag}(A, \dots, A) \end{array}$$

induces a diagonal embedding $\mathrm{O}_{2m+1} \hookrightarrow \mathrm{O}_{l(2m+1)}$ with signature $(l, 0)$. Note that $2n + 1 = l(2m + 1) + z$ and so z is odd. Write $z = 2k + 1$. Then the map

$$\begin{array}{ccc} \mathrm{O}_{l(2m+1)} & \hookrightarrow & \mathrm{O}_{2n+1} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \mapsto & \begin{pmatrix} A & & & B & \\ & I_k & & & \\ & & 1 & & \\ C & & & D & \\ & & & & I_k \end{pmatrix} \end{array}$$

is a diagonal embedding with signature $(1, z)$. Now, let ι be the composition of these two diagonal embeddings. Then ι is itself a diagonal embedding and has signature (l, z). □

Suppose that K is algebraically closed and that the diagonal embeddings ι_i have signatures (l_i, z_i) with l_i even for infinitely many $i \in \mathbb{N}$. Then the proposition shows that we can replace our sequence by a supersequence in which groups of type D appear infinitely many times. In this case V is G-Noetherian by the previous section. So, even if K is not algebraically closed, we only have to consider the case where this does not happen. And, by replacing our sequence by an infinite subsequence, we may assume that $l_i \in \mathbb{N}$ odd for every $i \in \mathbb{N}$. As both n_i and $n_{i+1} = l_i n_i + z_i$ are odd, this forces $z_i \in \mathbb{Z}_{\geq 0}$ to be even for all $i \in \mathbb{N}$. Our next task is to find diagonal embeddings with such signatures.

First, note that for $n \in \mathbb{N}$ and $z \in \mathbb{Z}_{\geq 0}$ the map

$$\begin{aligned} \iota_{1,2z}\colon \mathrm{O}_{2n+1} &\hookrightarrow \mathrm{O}_{2(n+z)+1} \\ \begin{pmatrix} A & \alpha & B \\ \beta & \mu & \gamma \\ C & \delta & D \end{pmatrix} &\mapsto \begin{pmatrix} A & & \alpha & B & \\ & I_z & & & \\ \beta & & \mu & \gamma & \\ C & & \delta & D & \\ & & & & I_z \end{pmatrix} \end{aligned}$$

is a diagonal embedding with signature $(1, 2z)$. Here $A, B, C, D \in \mathfrak{gl}_n$, $\alpha, \beta^T, \gamma^T, \delta \in K^n$ and $\mu \in K$. The associated map of Lie algebras is

$$\begin{aligned} \mathrm{pr}_{1,2z}\colon \mathfrak{o}_{2(n+z)+1} &\twoheadrightarrow \mathfrak{o}_{2n+1} \\ \begin{pmatrix} P & \bullet & v & Q & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ \phi & \bullet & 0 & \psi & \bullet \\ R & \bullet & w & S & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet \end{pmatrix} &\mapsto \begin{pmatrix} P & v & Q \\ \phi & 0 & \psi \\ R & w & S \end{pmatrix} \end{aligned}$$

with $P = -S^T, Q, R \in \mathfrak{gl}_n$ and $v = -\psi^T, w = -\phi^T \in K^n$ such that $Q + Q^T = R + R^T = 0$.

Next, we construct a diagonal embedding $\mathrm{O}_{2n+1} \hookrightarrow \mathrm{O}_{l(2n+1)}$ with signature $(l, 0)$ for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$ odd. Write

$$J_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_k$$

for $k \in \mathbb{N}$ and take

$$H_{2n+1,l} = \left\{ A \in \mathrm{GL}_{l(2n+1)} \;\middle|\; A \begin{pmatrix} & & I_{ln} \\ & J_l & \\ I_{ln} & & \end{pmatrix} A^T = \begin{pmatrix} & & I_{ln} \\ & J_l & \\ I_{ln} & & \end{pmatrix} \right\}$$

for all $n \in \mathbb{N}$ and $l \in \mathbb{N}$ odd. Then we have

$$P \begin{pmatrix} & & I_{ln} \\ & J_l & \\ I_{ln} & & \end{pmatrix} P^T = \begin{pmatrix} & & I_{ln+k} \\ & 1 & \\ I_{ln+k} & & \end{pmatrix}$$

where

$$P = \begin{pmatrix} I_{ln} & & & & \\ & I_k & & & \\ & & 1 & & \\ & & & & I_{ln} \\ & & & J_k & \end{pmatrix}$$

is a permutation matrix. So the map

$$\begin{aligned} H_{2n+1,l} &\to \mathrm{O}_{l(2n+1)} \\ A &\mapsto PAP^T \end{aligned}$$

is an isomorphism. Consider the map

$$\begin{array}{ccc} \mathrm{O}_{2n+1} & \hookrightarrow & H_{2n+1,l} \\ \begin{pmatrix} A & \alpha & B \\ \beta & \mu & \gamma \\ C & \delta & D \end{pmatrix} & \mapsto & \begin{pmatrix} A & & & \alpha & & & & & B \\ & \ddots & & & \ddots & & & \iddots & \\ & & A & & & \alpha & B & & \\ \beta & & & \mu & & & & & \gamma \\ & \ddots & & & \ddots & & & \iddots & \\ & & \beta & & & \mu & \gamma & & \\ & & C & & & \delta & D & & \\ & \iddots & & & \iddots & & & \ddots & \\ C & & & \delta & & & & & D \end{pmatrix} \end{array}$$

where $A, B, C, D \in \mathfrak{gl}_n$, $\alpha, \beta^T, \gamma^T, \delta \in K^n$ and $\mu \in K$ all occur l times on the right hand side. Write $l = 2k+1$. By taking the composition of these two maps, we get a diagonal embedding $\mathrm{O}_{2n+1} \hookrightarrow \mathrm{O}_{l(2n+1)}$ with signature $(l, 0)$.

Write $J = J_l$ and consider the Lie algebra

$$\begin{aligned} \mathfrak{h}_{2n+1,l} &= \left\{ P \in \mathfrak{gl}_{l(2n+1)} \;\middle|\; P \begin{pmatrix} & & I_{ln} \\ & J_l & \\ I_{ln} & & \end{pmatrix} + \begin{pmatrix} & & I_{ln} \\ & J_l & \\ I_{ln} & & \end{pmatrix} P^T = 0 \right\} \\ &= \left\{ \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \in \mathfrak{gl}_{l(2n+1)} \;\middle|\; \begin{array}{c} P + S^T = Q + Q^T = R + R^T = 0 \\ VJ + \Psi^T = WJ + \Phi^T = 0 \\ UJ + JU^T = 0 \end{array} \right\} \end{aligned}$$

of $H_{2n+1,l}$. The map $\mathrm{O}_{2n+1} \hookrightarrow H_{2n+1,l}$ corresponds to the map $\mathfrak{h}_{2n+1,l} \twoheadrightarrow \mathfrak{o}_{2n+1}$ sending

$$\begin{pmatrix} P_{11} & \dots & P_{1l} & V_{11} & \dots & V_{1l} & Q_{11} & \dots & Q_{1l} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ P_{l1} & \dots & P_{ll} & V_{l1} & \dots & V_{ll} & Q_{l1} & \dots & Q_{ll} \\ \Phi_{11} & \dots & \Phi_{1l} & U_{11} & \dots & U_{1l} & \Psi_{11} & \dots & \Psi_{1l} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \Phi_{l1} & \dots & \Phi_{ll} & U_{l1} & \dots & U_{ll} & \Psi_{l1} & \dots & \Psi_{ll} \\ R_{11} & \dots & R_{1l} & W_{11} & \dots & W_{1l} & S_{11} & \dots & S_{1l} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ R_{l1} & \dots & R_{ll} & W_{l1} & \dots & W_{ll} & S_{l1} & \dots & S_{ll} \end{pmatrix}$$

to

$$\begin{pmatrix} P_{11} + \dots + P_{ll} & V_{11} + \dots + V_{ll} & Q_{1l} + \dots + Q_{l1} \\ \Phi_{11} + \dots + \Phi_{ll} & U_{11} + \dots + U_{ll} & \Psi_{1l} + \dots + \Psi_{l1} \\ R_{1l} + \dots + R_{l1} & W_{1l} + \dots + W_{l1} & S_{11} + \dots + S_{ll} \end{pmatrix}.$$

Here, for each entry, we either sum along the diagonal or along the anti-diagonal in a manner consistent with the definition of the map $\mathrm{O}_{2n+1} \hookrightarrow H_{2n+1,l}$. The map $H_{2n+1,l} \to \mathrm{O}_{l(2n+1)}$ corresponds to the map $\mathfrak{o}_{l(2n+1)} \to \mathfrak{h}_{2n+1,l}$ sending Q to $P^T Q P^{-T}$.

We let the diagonal embeddings in the sequence

$$\mathrm{O}_{2n_1+1} \xhookrightarrow{\iota_1} \mathrm{O}_{2n_2+1} \xhookrightarrow{\iota_2} \mathrm{O}_{2n_3+1} \xhookrightarrow{\iota_3} \dots$$

be (compositions) of the forms above. As in the previous sections, if only finitely many embeddings have signature $(l_i, 2z_i)$ with $l_i > 1$, then Theorem 2.5.1 follows from [20, Theorem 1.2]. So we assume that $\#\{l_i \mid l_i > 1\} = \infty$. Now, by replacing our sequence by an infinite subsequence, we may assume that $l_i \in \mathbb{N}$ is odd and at least 3 for every $i \in \mathbb{N}$.

Lemma 2.7.2. *Let $Y \subsetneq \mathfrak{h}_{2n+1,l}$ be an $H_{2n+1,l}$-stable closed subset and let Z be the closed subset*

$$\left\{ \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \in \mathfrak{h}_{2n+1,l} \;\middle|\; P = P^T \right\}$$

of $\mathfrak{h}_{2n+1,l}$. Then there is a nonzero polynomial $f \in \mathcal{I}(Y)$ whose top-graded part is not contained in the ideal of Z.

Proof. The proof is analogous to the proof of Lemma 2.5.3. □

Lemma 2.7.3. *Let X be an $H_{2n+1,l}$-stable closed subset of $\mathfrak{h}_{2n+1,l}$ and let Y be the closure of its image in $\mathfrak{o}_{2n+1}$. Let $f \in \mathcal{I}(Y) \subseteq K[\mathfrak{o}_{2n+1}]$ be a nonzero polynomial whose top-graded part g is not contained in the ideal of*

$$\left\{ \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \in \mathfrak{h}_{2n+1,l} \;\middle|\; P = P^T \right\}.$$

Then $\mathcal{I}(X)$ contains a nonzero polynomial with degree at most $\deg(f)$ *that only depends on R and two columns of W.*

Proof. Consider the matrix

$$\begin{pmatrix}
P_{11} & \dots & P_{1l} & V_{11} & \dots & V_{1l} & Q_{11} & \dots & Q_{1l} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
P_{l1} & \dots & P_{ll} & V_{l1} & \dots & V_{ll} & Q_{l1} & \dots & Q_{ll} \\
\Phi_{11} & \dots & \Phi_{1l} & U_{11} & \dots & U_{1l} & \Psi_{11} & \dots & \Psi_{1l} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
\Phi_{l1} & \dots & \Phi_{ll} & U_{l1} & \dots & U_{ll} & \Psi_{l1} & \dots & \Psi_{ll} \\
R_{11} & \dots & R_{1l} & W_{11} & \dots & W_{1l} & S_{11} & \dots & S_{1l} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
R_{l1} & \dots & R_{ll} & W_{l1} & \dots & W_{ll} & S_{l1} & \dots & S_{ll}
\end{pmatrix} \in \mathfrak{h}_{2n+1,l}$$

and note that the polynomial $f = f(P, Q, R, v, w) \in \mathcal{I}(Y)$ induces the element

$$f(P_{11} + \dots + P_{ll}, Q_{1l} + \dots + Q_{l1}, R_{1l} + \dots + R_{l1}, V_{11} + \dots + V_{ll}, W_{1l} + \dots + W_{l1})$$

of $\mathcal{I}(X)$. Consider the matrix

$$A(\lambda) = \begin{pmatrix}
I_n & & & & & & -\lambda I_n \\
& \ddots & & & & & \\
& & I_n & & \lambda I_n & & \\
& & & I_l & & & \\
& & & & I_n & & \\
& & & & & \ddots & \\
& & & & & & I_n
\end{pmatrix} \in H_{2n+1,l}$$

for $\lambda \in K$. One can check that

$$g(R_{1l} - R_{l1}, -(R_{1l} + R_{l1}), R_{1l} + \cdots + R_{l1}, W_{1l} - W_{l1}, W_{1l} + \cdots + W_{l1})$$

is contained in the span of

$$A(\lambda) \cdot f(P_{11} + \cdots + P_{ll}, Q_{1l} + \cdots + Q_{l1}, R_{1l} + \cdots + R_{l1}, V_{11} + \cdots + V_{ll}, W_{1l} + \cdots + W_{l1})$$

over all $\lambda \in K$. So it is an element of $\mathcal{I}(X)$ and its degree is at most $\deg(f)$.

Next, consider the matrix

$$B(\mu) = \begin{pmatrix} I_{ln} & & & & & \\ & 1 & & & & \\ & \mu & \ddots & & & \\ & & & \ddots & & \\ & & & -\mu & 1 & \\ & & & & & I_{ln} \end{pmatrix} \in H_{2n+1,l}$$

for $\mu \in K$. Let $h(P, Q, R, v, w)$ be the top-graded part of g with respect to the grading where P, Q, R get grading 0 and v, w get grading 1. Then one can check that

$$h(R_{1l} - R_{l1}, -(R_{1l} + R_{l1}), R_{1l} + \cdots + R_{l1}, -W_{l-1,2}, W_{1l} + W_{l-1,2})$$

is contained in the span of

$$B(\mu) \cdot g(R_{1l} - R_{l1}, -(R_{1l} + R_{l1}), R_{1l} + \cdots + R_{l1}, W_{1l} - W_{l1}, W_{1l} + \cdots + W_{l1})$$

over all $\mu \in K$. This polynomial is contained in $\mathcal{I}(X)$ and has degree at most $\deg(f)$. □

The following proposition tells us how to use the equation we gain from Lemma 2.7.2. Let GL_n act on $\{Q \in \mathfrak{gl}_n \mid Q = -Q^T\}$ by $g \cdot Q = gQg^T$. Let $k \leq n$ be an integer and let GL_n act on $K^{n \times k}$ by left-multiplication.

Proposition 2.7.4. *Let $R \in \mathfrak{gl}_n$ be a skew-symmetric matrix and let $W \in K^{n\times k}$ be a matrix of rank k. Then the closure of the GL_n-orbit of (R, W) inside $\{Q \in \mathfrak{gl}_n \mid Q = -Q^T\} \oplus K^{n\times k}$ contains all tuples (Q, V) with $\mathrm{rk}(Q) \leq \mathrm{rk}(R) - 2k$.*

Proof. We will prove the proposition using induction on k. The case $k = 0$ is well-known. So assume that $0 < 2k \leq \mathrm{rk}(R)$. Let X be the closure of the GL_n-orbit of (R, W). Note that we may replace (R, W) with any element in its GL_n-orbit. Since $\mathrm{rk}(W) = k$, we may therefore assume that the last column of W equals e_n. Now, if we act with a matrix of the form

$$\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & \ddots & \\ a_1 & \dots & a_{n-1} & 1 \end{pmatrix},$$

then the last column of W stays equal to e_n. And, the last column of R becomes

$$\begin{pmatrix} a_1 r_1 + \cdots + a_{n-1} r_{n-1} + r_n \\ 0 \end{pmatrix}$$

if we write

$$R = \begin{pmatrix} r_1 & \dots & r_{n-1} & r_n \\ \bullet & \dots & \bullet & 0 \end{pmatrix}$$

with $r_1, \dots, r_n \in K^{n-1}$. As $\mathrm{rk}(R) > k = \mathrm{rk}(W)$ and e_n is contained in the image of W, we see that

$$\begin{pmatrix} a_1 r_1 + \dots + a_{n-1} r_{n-1} + r_n \\ 0 \end{pmatrix}$$

is not contained in the image of W for some $a_1, \dots, a_{n-1}$. So we may also assume that the last column of R is not contained in the image of W. Next, note that the last column of W stays e_n and the last column of R stays outside the image of W if we act with a matrix of the form $\mathrm{Diag}(g, 1)$ with $g \in \mathrm{GL}_{n-1}$. Since the last column of R is nonzero, we may therefore assume in addition that

$$R = \begin{pmatrix} R' & w & 0 \\ -w^T & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

for some $R' \in \mathfrak{gl}_{n-2}$ and $w \in K^{n-2}$. So the vector e_{n-1} is not contained in the image of W. Note that $\mathrm{rk}(R') \geq \mathrm{rk}(R) - 2$. Write

$$W = \begin{pmatrix} W' & 0 \\ v^T & 0 \\ u^T & 1 \end{pmatrix}$$

with $W' \in K^{(n-2)\times(k-1)}$ and $u, v \in K^{k-1}$. Since e_{n-1} is not contained in the image of W, the matrix $(W\ e_{n-1})$ has rank $k+1$ and hence $\mathrm{rk}(W') = k - 1$. The limit

$$\lim_{\lambda \to 0} \mathrm{Diag}(I_{n-2}, \lambda, 1) \cdot (R, W) = \left(\begin{pmatrix} R' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} W' & 0 \\ 0 & 0 \\ u^T & 1 \end{pmatrix} \right)$$

is an element of X. Using the induction hypothesis, we see that X contains

$$\left(\begin{pmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} V & 0 \\ 0 & 0 \\ u^T & 1 \end{pmatrix} \right)$$

for all skew-symmetric matrices $Q \in \mathfrak{gl}_{n-2}$ of rank at most $\mathrm{rk}(R) - 2k$ and all $V \in K^{(n-2)\times(k-1)}$. By acting with a permutation matrix, we see in particular that

$$\left(\begin{pmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ I_{k-1} & 0 \\ u^T & 1 \end{pmatrix} \right) \in X$$

for all skew-symmetrix matrices $Q \in \mathfrak{gl}_{n-k}$ of rank at most $\mathrm{rk}(R) - 2k$. Therefore $(\mathrm{Diag}(Q, 0), V) \in X$ since it equals

$$\lim_{\lambda \to 0} \left(\mathrm{Diag}(I_{n-k}, \lambda I_k) + \left(0\ V \begin{pmatrix} I_{k-1} & 0 \\ -u^T & 1 \end{pmatrix} \right) \right) \cdot \left(\begin{pmatrix} Q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ I_{k-1} & 0 \\ u^T & 1 \end{pmatrix} \right)$$

for all skew-symmetrix matrices $Q \in \mathfrak{gl}_{n-k}$ of rank at most $\mathrm{rk}(R) - 2k$ and all matrices $V \in K^{n\times k}$. So since X is GL_n-stable, we see that $(Q, V) \in X$ for all skew-symmetric matrices $Q \in \mathfrak{gl}_n$ of rank at most $\mathrm{rk}(R) - 2k$ and all matrices $V \in K^{n\times k}$. $\square$

Lemma 2.7.5. *There are integers $c_0, c_1, c_2 \in \mathbb{N}$ such that the following holds: let $m \in \mathbb{Z}_{\geq 0}$ be an integer with $c_2 m \leq n$ and let $M \in \mathfrak{h}_{2n+1,l}$ be an element such that for all matrices*

$$\begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \in H_{2n+1,l} \cdot M$$

it holds that $\mathrm{rk}(R) \leq m$ *or the first and last column of W are linearly dependent. Then we have* $\mathrm{rk}(M) \leq c_1 m + c_0$.

Proof. Let

$$\begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix}$$

be an element of the orbit of M. We assume that $c_2 m \leq n$ with c_2 high enough and we will prove a series of claims, which together imply that $\mathrm{rk}(M) \leq c_1 m + c_0$ for suitable $c_0, c_1 \in \mathbb{N}$.

(x) We have $\mathrm{rk}(R) \leq m + 4$.

Suppose that $\mathrm{rk}(R) > m$. Note that $\mathrm{Diag}(I_{ln}, g, I_{ln}) \in H_{2n+1,l}$ for all $g \in \mathrm{GL}_l$ with $gJg^T = J$. We have

$$\begin{pmatrix} I_{ln} & & \\ & g & \\ & & I_{ln} \end{pmatrix} \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \begin{pmatrix} I_{ln} & & \\ & g & \\ & & I_{ln} \end{pmatrix}^{-1} = \begin{pmatrix} P & Vg^{-1} & Q \\ g\Phi & gUg^{-1} & g\Psi \\ R & Wg^{-1} & S \end{pmatrix}$$

for all $g \in \mathrm{GL}_l$. So we see that the first and last column of Wg^{-1} are linearly dependent for all $g \in \mathrm{GL}_l$ with $gJg^T = J$. Using the fact that

$$g = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ \lambda & & \ddots & & \\ & & & \ddots & \\ & & -\lambda & & 1 \end{pmatrix}$$

satisfies $gJg^T = J$ as long as λ is not in the middle row together with $JJJ^T = J$, it is now easy to check that $\mathrm{rk}(W) \leq 2$. Next, note that

$$\begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix} \in H_{2n+1,l}$$

for all $A \in K^{ln \times l}$. For all $A \in K^{ln \times l}$, we have

$$\begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix}^{-1} \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ R & W + RA & \bullet \end{pmatrix}$$

and hence $\mathrm{rk}(W + RA) \leq 2$. So $\mathrm{rk}(RA) \leq 4$ and hence $\mathrm{rk}(R) \leq 4$.

(y) We have $\mathrm{rk}(Q) \leq m+4$ and $\mathrm{rk}(P) = \mathrm{rk}(S) \leq 3(m+4)/2$.

Repeat the proof of Lemma 2.6.4 and act with matrices

$$\begin{pmatrix} & & I_{ln} \\ & I_l & \\ I_{ln} & & A \end{pmatrix}, \begin{pmatrix} I_{ln} & & B \\ & I_l & \\ & & I_{ln} \end{pmatrix}$$

with $A = -A^T$ and $B = -B^T$.

(z) We have $\mathrm{rk}(W) = \mathrm{rk}(\Phi), \mathrm{rk}(V) = \mathrm{rk}(\Psi) \leq 4(m+4)$ and $\mathrm{rk}(U) \leq 22(m+4)$.

We have

$$\begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix}^{-1} \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & T \end{pmatrix}$$

with $T = -\frac{1}{2}RAJA^T - WJA^T + S$ for all $A \in K^{ln \times l}$. So $\mathrm{rk}(WJA^T) \leq 4(m+4)$ for all $A \in K^{ln \times l}$. So $\mathrm{rk}(W) = \mathrm{rk}(\Phi) \leq 4(m+4)$. By conjugating with

$$\begin{pmatrix} & & I_{ln} \\ & I_l & \\ I_{ln} & & \end{pmatrix}$$

we also see that $\mathrm{rk}(V) = \mathrm{rk}(\Psi) \leq 4(m+4)$. We have

$$\begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix}^{-1} \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \begin{pmatrix} I_{ln} & A & -\frac{1}{2}AJA^T \\ & I_l & -JA^T \\ & & I_{ln} \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & T \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}$$

with

$$T = \begin{pmatrix} I_{ln} & -A & -\frac{1}{2}AJA^T \end{pmatrix} \begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \begin{pmatrix} -\frac{1}{2}AJA^T \\ -JA^T \\ I_{ln} \end{pmatrix}.$$

Now, we know that $\mathrm{rk}(T) \leq m+4$. Also, the matrix T is a sum of nine matrices: the matrix $AUJA^T$ and eight other matrices for which we have found bounds on the rank. Adding all these bounds together, we find that

$$\mathrm{rk}(AUJA^T) \leq (1+1+1+3/2+3/2+4+4+4+4)(m+4) = 22(m+4)$$

for all $A \in K^{ln \times l}$. Hence $\mathrm{rk}(U) \leq 22(m+4)$.

Together (x), (y) and (z) show that

$$\mathrm{rk}\begin{pmatrix} P & V & Q \\ \Phi & U & \Psi \\ R & W & S \end{pmatrix} \leq c_1 m + c_0$$

for some $c_0, c_1 \in \mathbb{N}$. So this holds in particular if we let this matrix be M itself. □

We combine these results as in the previous section. Lemmas 2.7.2 and 2.7.3 play the roles of Lemmas 2.6.2 and 2.6.3 and give us off-diagonal polynomials. Then, Proposition 2.7.4 with $k = 2$ shows us the structure of the off-diagonal part of the matrix as a GL_n-representation with the Zariski topology. From this and the degree of the off-diagonal polynomial, we get bounds on ranks of some submatrices. Lemma 2.7.5 turns these bounds into a rank bound on the matrix itself. Finally, we find similarly to Lemma 2.3.14 that $X \subseteq \{0\}$ and this implies that V is G-Noetherian.

Chapter 3

Strength of polynomials and tensors

Section 3.2 of this chapter is based on ongoing work with Alessandro Oneto. Sections 3.3 to 3.6 are based on work [7] with Jan Draisma and Rob Eggermont. In this chapter, the field K is assumed to be infinite.

3.1 Introduction

Fix an integer $d \geq 2$ and let $V \in \mathrm{Vec}$ be a finite-dimensional vector space. This chapter concerns decompositions of polynomials $q \in S^d(V)$ of the form

$$q = r_1 s_1 + \cdots + r_k s_k$$

where $r_i \in S^{e_i}(V)$ and $s_i \in S^{d-e_i}(V)$ for suitable natural numbers $e_i \in \{1, \dots, d-1\}$. The minimal number of terms k among all such decompositions of q is called the strength $\mathrm{str}(q)$ of q.

Remark 3.1.1. This term was introduced in [2], except that we have taken the liberty of adding 1 to the strength defined there. ♠

We begin by listing some basic properties and examples of the strength of a polynomial.

Example 3.1.2. Let $f \in S^d(V)$ be a polynomial. Then $\mathrm{str}(f) = 0$ if and only if $f = 0$. And, we have $\mathrm{str}(f) = 1$ if and only if $f \neq 0$ and f is reducible. ♣

Proposition 3.1.3. *Let $f, g \in S^d(V)$ be polynomials.*

(1) We have $\mathrm{str}(f + g) \leq \mathrm{str}(f) + \mathrm{str}(g)$.

(2) We have $\mathrm{str}(f) \leq \dim V$. *If K is algebraically closed, then* $\mathrm{str}(f) \leq \dim(V) - 1$.

(3) Suppose that V is a subspace of a vector space W. Then the strengths of f viewed as an element of $S^d(V)$ and as an element of $S^d(W)$ coincide.

(4) Let $\ell\colon V \to W$ be a linear map and take $L := S^d(\ell)\colon S^d(V) \to S^d(W)$. Then we have

$$\mathrm{str}(f) - \dim\ker(\ell) \leq \mathrm{str}(L(f)) \leq \mathrm{str}(f).$$

Proof. (1) Suppose that

$$f = r_1 s_1 + \cdots + r_k s_k \text{ and } g = r'_1 s'_1 + \cdots + r'_{k'} s'_{k'}$$

for polynomials r_i, s_i, r'_j, s'_j of degree $< d$. Then we see that

$$f + g = r_1 s_1 + \cdots + r_k s_k + r'_1 s'_1 + \cdots + r'_{k'} s'_{k'}$$

and this shows that $\operatorname{str}(f + g) \leq \operatorname{str}(f) + \operatorname{str}(g)$.
(2) Let $x_1, x_2, \ldots, x_n$ be a basis of V where $n = \dim V$. Then we can write

$$f = x_1 g_1 + x_2 g_2 + \cdots + x_n g_n$$

where $g_1, g_2, \ldots, g_n \in S^{d-1}(V)$ and hence $\operatorname{str}(f) \leq n$. If the field K is algebraically closed, then we can write

$$f = h + x_3 g_3 + \cdots + x_n g_n$$

where $h := f(x_1, x_2, 0, \ldots, 0)$ and $g_3 \ldots, g_n \in S^{d-1}(V)$. Since K is algebraically closed, the binary form h is has a linear factor and therefore $\operatorname{str}(h) \leq 1$. Hence we have

$$\operatorname{str}(f) \leq \operatorname{str}(h) + n - 2 \leq n - 1.$$

(3) If

$$f = r_1 s_1 + \cdots + r_k s_k$$

for $r_i \in S^{e_i}(V)$ and $s_i \in S^{d-e_i}(V)$ with $1 \leq e_i \leq d - 1$, then the strength of f viewed as an element of $S^d(W)$ is at most k. Conversely, suppose that

$$f = r_1 s_1 + \cdots + r_k s_k$$

for $r_i \in S^{e_i}(W)$ and $s_i \in S^{d-e_i}(W)$ with $1 \leq e_i \leq d - 1$ and let $\ell \colon W \twoheadrightarrow V$ be a linear map restricting to the identity of V. Then we see that

$$f = S^d(\ell)(f) = S^{e_1}(\ell)(r_1) S^{d-e_1}(\ell)(s_1) + \cdots + S^{e_k}(\ell)(r_k) S^{d-e_k}(\ell)(s_k)$$

and hence the strength of f viewed as an element of $S^d(W)$ is at most k. So the strengths of f viewed as an element of $S^d(V)$ and as an element of $S^d(W)$ coincide.
(4) Write $f = r_1 s_1 + \cdots + r_k s_k$ for $k = \operatorname{str}(f)$ and polynomials r_i, s_i of degrees $e_i, d - e_i < d$ and take

$$L_e := S^e(\ell) \colon S^e(V) \to S^e(W)$$

for $e = 1, \ldots, d$. Then

$$L(f) = L_d(f) = L_{e_1}(r_1) L_{d-e_1}(s_1) + \cdots + L_{e_k}(r_k) L_{d-e_k}(s_k)$$

and hence $\operatorname{str}(L(f)) \leq k = \operatorname{str}(f)$. Let $x_1, \ldots, x_k$ be a basis of $\ker(\ell)$ and let $V' \subseteq V$ be a subspace such that $V = V' \oplus \operatorname{span}(x_1, \ldots, x_k)$. Then we can write

$$f = f' + x_1 g_1 + \cdots + x_k g_k$$

for some $f' \in S^d(V')$ and $g_1, \ldots, g_k \in S^{d-1}(V)$. The map $\ell|_{V'} \colon V' \to W$ is injective. So by (3) we see that $\operatorname{str}(f') = \operatorname{str}(L(f')) = \operatorname{str}(L(f))$. Hence $\operatorname{str}(f) \leq \operatorname{str}(f') + k = \operatorname{str}(L(f)) + \dim \ker(\ell)$. □

The following example shows that the strength of polynomials of degree 2 is completely understood when $\operatorname{char}(K) \neq 2$ and $x^2 = -1$ has a solution in K.

Example 3.1.4. Suppose that $K = \mathbb{C}$. Then symmetric $n \times n$ matrices correspond one-to-one to homogeneous polynomials of degree 2 in the variables $x_1, \dots, x_n$ via the GL_n-equivariant map

$$\begin{array}{rcl} \{A \in \mathbb{C}^{n\times n} \mid A = A^T\} & \to & S^2(\mathbb{C}^n) \\ A & \mapsto & (x_1 \ \cdots \ x_n)A(x_1 \ \cdots \ x_n)^T \end{array}$$

Consider the matrix

$$A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}$$

of rank k and the corresponding polynomial $f = x_1^2 + \cdots + x_k^2$. Note that every symmetric matrix in $\mathbb{C}^{n\times n}$ is congruent to such a matrix. We have

$$x^2 + y^2 = (x + iy)(x - iy)$$

and hence $\mathrm{str}(f) \leq \lceil k/2 \rceil$. On the other hand, suppose that

$$f = (x_1 \ \cdots \ x_n)v_1 \cdot (x_1 \ \cdots \ x_n)w_1 + \cdots + (x_1 \ \cdots \ x_n)v_\ell \cdot (x_1 \ \cdots \ x_n)w_\ell$$

for some vectors $v_1, w_1, \dots, v_\ell, w_\ell \in \mathbb{C}^n$. Then we get

$$A = \frac{1}{2}\left((v_1 w_1^T + w_1 v_1^T) + \cdots + (v_\ell w_\ell^T + w_\ell v_\ell^T)\right)$$

and hence $k = \mathrm{rk}(A) \leq 2\ell$. Hence $\mathrm{str}(f) = \lceil \mathrm{rk}(A)/2 \rceil$. ♣

We also have an example that shows that the strength of a polynomial may go down when you extend the base field.

Example 3.1.5. Consider the polynomial $f = x_1^2 + \cdots + x_n^2$ over $\mathbb{R}$. If

$$f = r_1 s_1 + \cdots + r_k s_k$$

for real linear forms $r_1, s_1, \dots, r_k, s_k$, then we see that $f(v) = 0$ for all vectors v in the subspace $\ker(r_1, \dots, r_k) \subseteq \mathbb{R}^n$ of codimension at most k. Since $f(v) = 0$ only holds for $v = 0$, we see that $\mathrm{str}(f) \geq n$ must hold. Since $\mathrm{str}(f) \leq n$ always holds, we get $\mathrm{str}(f) = n$. This is roughly twice as high as the strength of f over $\mathbb{C}$, which equals $\lceil n/2 \rceil$ by the previous example. ♣

The following example explicitly gives the strength of some homogeneous polynomials of degree 3 and thereby shows that the strength of such polynomials is unbounded.

Example 3.1.6. Let $n \in \mathbb{N}$ be an integer and $x_1, y_1, z_1, \dots, x_n, y_n, z_n$ be a basis of K^{3n}. Derksen, Eggermont and Snowden proved in [15] that the polynomial

$$x_1 y_1 z_1 + \cdots + x_n y_n z_n$$

has strength n. ♣

Polynomials of arbitrarily high strength in fact exist in every degree $d \geq 2$. To show this, we need the following lemma.

Lemma 3.1.7. *Let $d \geq 3$ and suppose that $f \in S^d(V)$ has strength $\leq k$. Then there exists an $\ell \leq k$ and an ℓ-dimensional subspace $W \subseteq V$ such that the image g of f in $S^d(V/W)$ satisfies*

$$\operatorname{str}\left(\frac{\partial g}{\partial x}\right) \leq 2(k-\ell)$$

for all $x \in (V/W)^*$.

Proof. Write

$$f = w_1 g_1 + \cdots + w_\ell g_\ell + r_1 s_1 + \cdots + r_{k-\ell} s_{k-\ell}$$

where $w_1, \ldots, w_\ell \in V$ are linear and $r_1, s_1, \ldots, r_{k-\ell}, s_{k-\ell}$ have degree ≥ 2 and let W be the span of $w_1, \ldots, w_\ell$. Then the image g of f in $S^d(V/W)$ satisfies

$$g = \overline{f} = \overline{r_1} \cdot \overline{s_1} + \cdots + \overline{r_{k-\ell}} \cdot \overline{s_{k-\ell}}$$

and hence we see that

$$\frac{\partial g}{\partial x} = \frac{\partial \overline{r_1}}{\partial x} \cdot \overline{s_1} + \overline{r_1} \cdot \frac{\partial \overline{s_1}}{\partial x} + \cdots + \frac{\partial \overline{r_{k-\ell}}}{\partial x} \cdot \overline{s_{k-\ell}} + \overline{r_{k-\ell}} \cdot \frac{\partial \overline{s_{k-\ell}}}{\partial x}$$

has strength $\leq 2(k-\ell)$ for all $x \in (V/W)^*$. □

Example 3.1.8. For $n \in \mathbb{N}$, write $f_{d,n} = x_1^d + \cdots + x_n^d$. Note that we obtain $f_{d,n-1}$ from $f_{d,n}$ by setting x_n to zero and hence $\operatorname{str}(f_{d,n-1}) \leq \operatorname{str}(f_{d,n})$. We will prove that $\operatorname{str}(f_{d,n}) \to \infty$ when $n \to \infty$ for all $d \geq 2$ using induction on d. We have $\operatorname{str}(f_{2,n}) = \lceil n/2 \rceil$ and hence the statement holds for $d = 2$. Now assume that the statement holds for $d-1$. Suppose that $\operatorname{str}(f_{d,n}) \leq k$. Let V be the vector space with basis $x_1, \ldots, x_n$ and let $\ell \leq k$ and $W \subseteq V$ be as in the lemma. We may assume that $\overline{x_1}, \ldots, \overline{x_{n-\ell}}$ form a basis of V/W. Take $x = \overline{x_1} + \cdots + \overline{x_{n-\ell}}$. Then we see that

$$\frac{\partial}{\partial x}\left(\overline{x_1}^d + \cdots + \overline{x_n}^d\right)$$

has strength at most $2(k-\ell)$. Therefore

$$d\left(\overline{x_1}^{d-1} + \cdots + \overline{x_{n-\ell}}^{d-1}\right) = \frac{\partial}{\partial x}\left(\overline{x_1}^d + \cdots + \overline{x_n}^d\right) - \sum_{j=1}^{\ell} \overline{x_{n-\ell+j}}^{d-1} \frac{\partial \overline{x_{n-\ell+1}}}{\partial x}$$

has strength at most $2(k-\ell) + \ell \leq 2k$. Its strength is also equal to

$$\operatorname{str}(f_{d-1,n-\ell}) \geq \operatorname{str}(f_{d-1,n-k})$$

and so we see that the strength of $f_{d,n}$ cannot be bounded by any $k < \infty$ as $n \to \infty$. ♣

The main result of this chapter is the following theorem.

Theorem 3.1.9. *Fix $d \in \mathbb{Z}_{\geq 2}$ and assume that K is a perfect and infinite field with* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$. *Then for any closed subset $X \subsetneq S^d$ there exists an $N \geq 0$ such that for all vector spaces $V \in$* Vec *the strength of all elements in $X(V)$ is at most N.*

Assume that K is a perfect and infinite field with $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ and let $\mathcal{P}$ be any property of polynomials such that the following condition holds:

(*) If a polynomial $f \in S^d(V)$ has property $\mathcal{P}$ and $\ell\colon V \to W$ is any linear map, then the polynomial $S^d(\ell)(f)$ has property $\mathcal{P}$ as well.

Then the theorem applied to the closure of $\mathcal{P}$ tells us that either the set of $f \in S^d(V)$ with property $\mathcal{P}$ is Zariski-dense for every $V \in \mathrm{Vec}$ or the strength of polynomials with property $\mathcal{P}$ is bounded independently of V.

Example 3.1.10. Let $k \in \mathbb{N}$ be an integer. Then the dimension of the subset

$$X(V) := \overline{\{f \in S^d(V) \mid \operatorname{str}(f) \le k\}}$$

of $S^d(V)$ is bounded by a polynomial in $\dim V$ of degree $d-1$. Hence $X(V)$ cannot be equal to $S^d(V)$ for $V \in \mathrm{Vec}$ with $\dim V \gg 0$. Using Proposition 3.1.3(4) and Theorem 3.1.9, it follows that there exists an $N \ge 0$ such that $\operatorname{str}(g) \le N$ for all $V \in \mathrm{Vec}$ and all polynomials $g \in X(V)$. ♣

Example 3.1.11. The paper [24] concerns polynomials all of whose directional derivatives have bounded strength. Let $x \in V^*$ be nonzero. Then the map that sends a polynomial to its directional derivative to x is surjective. So since the set of polynomials of bounded strength is not dense for $\dim V \gg 0$, the same is true for the set of polynomials all of whose directional derivatives have bounded strength. Hence the strength of such polynomials is bounded. This implies [24, Theorem 1.2]. ♣

Remark 3.1.12. See [25, Lemma 1.23] for a strengthening of this result, which shows that either all $f \in S^d(V)$ have property $\mathcal{P}$ for every $V \in \mathrm{Vec}$ or the strength of polynomials with property $\mathcal{P}$ is bounded independently of V. ♠

We define the strength $\operatorname{str}(f)$ of a series

$$f \in S^d_\infty = \varprojlim_n S^d(K^n)$$

as the supremum of the strengths of all its projections $\mathrm{pr}_n(f) \in S^d(K^n)$. Note that a strength decomposition of f also gives strength decompositions of all its projections with the same number of terms. So it is easy to write down series with low strength. The previous examples also allow us to write down some series whose strength is infinite.

Example 3.1.13. The series $\sum_{i=1}^\infty x_i y_i z_i$ has infinite strength over $\mathbb{C}$. ♣

Example 3.1.14. The series $\sum_{i=1}^\infty x_i^d$ has infinite strength over $\mathbb{C}$. ♣

Theorem 3.1.9 has the following consequence in this setting.

Corollary 3.1.15. *Fix $d \in \mathbb{Z}_{\ge 2}$ and assume that K is a perfect and infinite field with* $\operatorname{char} K = 0$ *or* $\operatorname{char} K > d$*. Then the* GL_∞*-orbit of any series in S^d_∞ of infinite strength is dense.*

Proof. Let X be the closed subset of S^d corresponding to the closure of the GL_∞-orbit of a series $f \in S^d_\infty$. If f is not dense in S^d_∞, then $X \ne S^d$ and hence the strength of polynomials in X is bounded. This would in particular imply that the strength of the projections $\mathrm{pr}_n(f) \in X(K^n)$ is bounded and that therefore that the strength of f is finite. So if the series f has infinite strength, then its GL_∞-orbit is dense. □

Remark 3.1.16. Alternatively, one could define the strength of a series $f \in S^d_\infty$ as the infimal number of terms of a strength decomposition of f. These definitions are in fact equivalent. See Remark 4.5.25 for a proof of this statement.

When studying rank-like measures of tensors, there are several natural questions one can ask. Let $d \geq 2$, $n \geq 1$ and $k \geq 0$ be integers. The first question to ask is whether having bounded strength is a closed condition.

Question 3.1.17. *Is the subset* $\{f \in \mathbb{C}[x_1, \dots, x_n]_{(d)} \mid \mathrm{str}(f) \leq k\}$ *closed?*

We know from Example 3.1.4 that the answer is yes for $d = 2$. It is also known [15, Proposition 2.2] that the answer is yes for $d = 3$. Since $\mathbb{C}$ is algebraically closed, we also know that the answer is yes when $k \leq 1$. So the first open case is $(d, k) = (4, 2)$.

Question 3.1.18. *Is the subset* $\{f \in \mathbb{C}[x_1, \dots, x_n]_{(4)} \mid \mathrm{str}(f) \leq 2\}$ *closed?*

Note that we have

$$\{f \in \mathbb{C}[x_1, \dots, x_{n-1}]_{(d)} \mid \mathrm{str}(f) \leq k\} = \{f \in \mathbb{C}[x_1, \dots, x_n]_{(d)} \mid \mathrm{str}(f) \leq k\} \cap \mathbb{C}[x_1, \dots, x_{n-1}]_{(d)}$$

for all $n \in \mathbb{N}$ by Proposition 3.1.3. So to answer Questions 3.1.17 and 3.1.18 it suffices to ask the question for $n \gg 0$.

Second, we want to know the strength of a generic polynomial and the maximal strength a polynomial can have given its degree and number of variables.

Question 3.1.19. *What is the strength of a generic polynomial in* $\mathbb{C}[x_1, \dots, x_n]_{(d)}$*?*

Question 3.1.20. *What is the maximal strength of polynomials in* $\mathbb{C}[x_1, \dots, x_n]_{(d)}$*?*

In the next section, we compare the strength of polynomials to their slice rank and conjecture that they are generically equal, which would in particular imply that the generic and maximum strengths are the same.

Third, like for the rank of tensors, we can ask how to compute the strength of a polynomial, low strength approximations and one can try to find families of polynomials with high strength.

Question 3.1.21. *Given a polynomial* $f \in \mathbb{C}[x_1, \dots, x_n]_{(d)}$*, can we compute its strength?*

Question 3.1.22. *Given a polynomial* $f \in \mathbb{C}[x_1, \dots, x_n]_{(d)}$*, can we compute its best low strength approximation?*

Question 3.1.23. *Can we explicitly write down families of polynomials with high strength?*

See [27] for a recent paper trying to answer this last question for tensors.

Last, let f, g be homogeneous polynomials of the same degree in distinct variables. Then we have $\mathrm{str}(f + g) \leq \mathrm{str}(f) + \mathrm{str}(g)$. One would hope that, since the polynomials use distinct variables, we in fact have $\mathrm{str}(f + g) = \mathrm{str}(f) + \mathrm{str}(g)$. This is however not the case: the polynomials x^d and $-y^d$ both have strength 1, but their sum is divisible by $x - y$ and hence also has strength 1. One can nevertheless ask whether it is possible to find interesting lower bounds for $\mathrm{str}(f + g)$ given $\mathrm{str}(f)$ and $\mathrm{str}(g)$.

Question 3.1.24. *Given polynomials f, g in different variables with known strengths, can we find a lower bound for the strength of their sum?*

Outline of this chapter. In the next section, we introduce the slice rank of a polynomial and compare it with its strength. The three sections after that are devoted to proving Theorem 3.1.9 and its analogues for alternating and ordinary tensors. We conclude with a section discussing versions of Theorem 3.1.9 and its analogues over $\mathbb{Z}$.

3.2 Slice rank and generic strength of polynomials

In this section, we compare the strength of polynomials with their slice rank in the case that $K = \mathbb{C}$. Let $d \geq 2$ be an integer and $V \in \mathrm{Vec}$ a vector space.

Definition 3.2.1. Let $f \in S^d(V)$ be a polynomial. The slice rank $\mathrm{slrk}(f)$ of f is the minimal number of terms k among all decompositions of f of the form

$$g = \ell_1 g_1 + \cdots + \ell_k g_k$$

where $\ell_1, \ldots, \ell_k \in V$ are linear and $g_1, \ldots, g_k \in S^{d-1}(V)$ have degree $d-1$.

We again start with listing some basic properties of the slice rank of polynomials.

Proposition 3.2.2. *Let $f, g \in S^d(V)$ be polynomials.*

(1) We have $\mathrm{slrk}(f+g) \leq \mathrm{slrk}(f) + \mathrm{slrk}(g)$.

(2) We have $\mathrm{slrk}(f) \leq \dim V$. *If K is algebraically closed, then* $\mathrm{slrk}(f) \leq \dim(V) - 1$.

(3) Suppose that V is a subspace of a vector space W. Then the slice ranks of f viewed as an element of $S^d(V)$ and as an element of $S^d(W)$ coincide.

(4) Let $\ell \colon V \to W$ be a linear map and take $L := S^d(\ell) \colon S^d(V) \to S^d(W)$. Then we have

$$\mathrm{slrk}(f) - \dim\ker(\ell) \leq \mathrm{slrk}(L(f)) \leq \mathrm{slrk}(f).$$

(5) We have $\mathrm{str}(f) \leq \mathrm{slrk}(f)$.

(6) We have $\mathrm{slrk}(f) = \min\{\mathrm{codim}\, W \mid W \subseteq V, f(W) = 0\}$.

(7) Let $k \geq 0$ be an integer. Then $\{h \in S^d(V) \mid \mathrm{slrk}(h) \leq k\}$ is a closed subset of $S^d(V)$.

Proof. Parts (1), (2), (3) and (4) have the same proof as in Proposition 3.1.3. Part (5) follows immediately from the definitions of strength and slice rank. Part (6) is proven in [15, Proposition 2.2] for $d = 3$ and this proof in fact works for any $d \geq 2$. Part (7) follows from part (6), which shows that the set

$$\{h \in S^d(V) \mid \mathrm{slrk}(h) \leq k\}$$

is a projection of a closed subset along the Grassmannian of codimension-k subspaces of V. □

By part (7), the generic and maximal slice rank of polynomials in $S^d(V)$ coincide. And in fact, their value is also know.

Theorem 3.2.3 ([11, Theorem 6.11]). *The generic slice rank of a polynomial in* $\mathbb{C}[x_1,\dots,x_n]_{(d)}$ *equals*

$$\min\left\{k\in\mathbb{Z}_{\geq n/2}\;\middle|\;\binom{d+n-k-1}{d}\leq k(n-k)\right\}.$$

□

We can calculate the slice rank of the polynomial $f=x_1^d+\cdots+x_n^d$ using the Fano varieties of the hypersurface defined by f.

Example 3.2.4. Let $d\geq 2$ and $n\geq 1$ be integers. Then the slice rank of the polynomial $f=x_1^d+\cdots+x_n^d$ is at most $\lceil n/2\rceil$ since it is the sum of at most that many binary forms. In order to prove that the slice rank of f is exactly $\lceil n/2\rceil$, we consider the Fano varieties

$$F_m(X_f):=\{W\in\mathbb{G}(m,\mathbb{C}^n)\mid\mathbb{P}(W)\subseteq X_f\}=\{W\in\mathbb{G}(r,\mathbb{C}^n)\mid f(W)=0\}$$

of the hypersurface $X_f:=\{f=0\}\subseteq\mathbb{P}^{n-1}$ for $m\in[n]$. The projective hypersurface X_f is smooth. Therefore $F_m(X_f)=\emptyset$ for all integers m with $2(m-1)\geq n-1$ by [29, Proposition 0.1]. This means that any subspace $W\subseteq\mathbb{C}^n$ with $f(W)=0$ has codimension at least $\lceil n/2\rceil$ and hence $\operatorname{slrk}(f)=\lceil n/2\rceil$ by Proposition 3.2.2(6). ♣

By Proposition 3.2.2(7), we see that the generic and maximal slice rank of polynomials in $S^d(V)$ coincide. Let

$$\operatorname{str}^\circ(d,n)\leq\operatorname{str}^{\max}(d,n)\leq\operatorname{slrk}^\circ(d,n)=\operatorname{slrk}^{\max}(d,n)$$

denote the generic strength, maximum strength, generic slice rank and maximal slice rank in $K[x_1,\dots,x_n]_{(d)}$ for integers $d\geq 2$ and $n\geq 1$. Then we have the following conjecture.

Conjecture 3.2.5. *We have*

$$\operatorname{str}^\circ(d,n)=\operatorname{str}^{\max}(d,n)=\operatorname{slrk}^\circ(d,n)=\operatorname{slrk}^{\max}(d,n)$$

for all integers $d\geq 2$ *and* $n\geq 1$.

Note that the conjecture holds for $d=2,3$ since in those cases the strength and slice rank of a polynomial coincide.

Remark 3.2.6. While we conjecture the strength and slice rank of a generic polynomial to be equal, we know that this does not hold for all polynomials. In fact, the difference between the slice rank and strength of a polynomial can be arbitrarily big. See Subsection 4.7.3 for details.

3.2.1 The proof of the conjecture for $d\leq 6$

Our goal for the remainder of this section is to prove this conjecture for $d\leq 6$. To do this, we have to show that the image of the map

$$\begin{aligned}\varphi\colon\bigoplus_{e=1}^{\lfloor d/2\rfloor}\left(\mathbb{C}[x_1,\dots,x_n]_{(e)}\times\mathbb{C}[x_1,\dots,x_n]_{(d-e)}\right)^{\oplus k_e}&\to\mathbb{C}[x_1,\dots,x_n]_{(d)}\\((g_{e,i},h_{e,i}))_{e,i}&\mapsto\sum_{e=1}^{\lfloor d/2\rfloor}\sum_{i=1}^{k_e}g_{e,i}\cdot h_{e,i}\end{aligned}$$

does not have full dimension whenever the sum of the k_e's is lower than the generic slice rank in $\mathbb{C}[x_1,\dots,x_n]_{(d)}$. Let the $g_{e,i}, h_{e,i}$ be generic and consider the derivative

$$\begin{aligned}\varphi\colon \bigoplus_{e=1}^{\lfloor d/2\rfloor}\left(\mathbb{C}[x_1,\dots,x_n]_{(e)}\times\mathbb{C}[x_1,\dots,x_n]_{(d-e)}\right)^{\oplus k_e} &\to \mathbb{C}[x_1,\dots,x_n]_{(d)}\\ ((G_{e,i},H_{e,i}))_{e,i} &\mapsto \sum_{e=1}^{\lfloor d/2\rfloor}\sum_{i=1}^{k_e}(g_{e,i}\cdot H_{e,i}+G_{e,i}\cdot h_{e,i})\end{aligned}$$

of φ at $((g_{e,i},h_{e,i}))_{e,i}$. It suffices to prove that the image of this map, which is precisely the degree-d part of the ideal generated by the $g_{e,i}, h_{e,i}$, is not the whole space.

Let $V\in\mathrm{Vec}$ be a vector space of dimension $n\geq 1$, let $k\leq n$ be an integer and take $\ell_1,\dots,\ell_k\in V$ and $f_1,\dots,f_k\in S^{d-1}(V)$ generic. Then $\ell_1,\dots,\ell_k$ are linearly independent. Let $\ell_{k+1},\dots,\ell_n\in V$ be such that $\ell_1,\dots,\ell_n$ form a basis of V. Then we see that

$$S^d(V)=\mathbb{C}[\ell_1,\dots,\ell_n]_{(d)}=(\ell_1,\dots,\ell_k)_{(d)}\oplus\mathbb{C}[\ell_{k+1},\dots,\ell_n]_{(d)}.$$

The codimension of $(\ell_1,\dots,\ell_k,f_1,\dots,f_k)_{(d)}$ in $S^d(V)$ is equal to the codimension of

$$(\overline{f_1},\dots,\overline{f_k})_{(d)}$$

in $\mathbb{C}[\ell_{k+1},\dots,\ell_n]_{(d)}=\mathbb{C}[\ell_1,\dots,\ell_n]_{(d)}/(\ell_1,\dots,\ell_k)_{(d)}$. The dimension of this subspace is at most $k\dim\mathbb{C}[\ell_{k+1},\dots,\ell_n]_{(1)}$. So we see that the codimension of $(\ell_1,\dots,\ell_k,f_1,\dots,f_k)_{(d)}$ in $S^d(V)$ is at least

$$\dim\mathbb{C}[\ell_{k+1},\dots,\ell_n]_{(d)}-k\dim\mathbb{C}[\ell_{k+1},\dots,\ell_n]_{(1)}=\binom{d+n-k-1}{d}-k(n-k).$$

Now back to the degree-d part of the ideal generated by the $g_{e,i},h_{e,i}$. We know that the generic slice rank is at most n. So in particular, we can assume that $k:=k_1\leq n$. This means that the codimension of the degree-d part of ideal generated by the $g_{e,i},h_{e,i}$ is at least

$$\binom{d+n-k-1}{d}-k(n-k)-\dim W$$

where W is the degree-d part of the ideal in the ring $\mathbb{C}[x_1,\dots,x_n]/(g_{1,1},\dots,g_{1,k})$ generated by the images of the $g_{e,i},h_{e,i}$ with $e\geq 2$. Note that

$$\mathbb{C}[x_1,\dots,x_n]/(g_{1,1},\dots,g_{1,k})\cong\mathbb{C}[y_1,\dots,y_m]$$

for $m=n-k$. So our next task is to find an upper bound for the dimension of the degree-d part of an ideal in $\mathbb{C}[y_1,\dots,y_m]$ generated by generic $g_{e,i},h_{e,i}$ where $e\in\{2,\dots,\lfloor d/2\rfloor\}$, $i\in[k_e]$, $\deg(g_{e,i})=e$, $\deg(h_{e,i})=d-e$. We do this for $d=4,5,6$ separately.

Lemma 3.2.7. *Let $\ell\geq 0$ and $m\geq\ell+2$ be integers and let $g_1,\dots,g_{2\ell}\in\mathbb{C}[y_1,\dots,y_m]_{(2)}$ be generic. Then*

$$2\ell\binom{m+1}{2}-\binom{2\ell}{2}$$

is an upper bound for the dimension of $(g_1,\dots,g_{2\ell})_{(4)}$.

Proof. Note that $(g_1,\dots,g_{2\ell})_{(4)}$ is the image of the linear map

$$\begin{array}{rcl}\mathbb{C}[y_1,\dots,y_m]_{(2)}^{\oplus 2\ell} & \to & \mathbb{C}[y_1,\dots,y_m]_{(4)}\\ \displaystyle\sum_{i=1}^{2\ell} H_i\cdot[g_i] & \mapsto & \displaystyle\sum_{i=1}^{2\ell} H_i g_i\end{array}$$

where $[g_1],\dots,[g_{2\ell}]$ are formal symbols. In the kernel of this map, we have:

(1) $g_j\cdot[g_i]-g_i\cdot[g_j]$ for $1\le i<j\le 2\ell$.

Since $\ell\le m-2$, we have $2\ell\le\binom{m+1}{2}=\dim\mathbb{C}[y_1,\dots,y_m]_{(2)}$. So since the g_i are generic, they are linearly independent. Let $W\subseteq\mathbb{C}[y_1,\dots,y_m]_{(2)}$ be the subspace they span. Then the $g_j\cdot[g_i]-g_i\cdot[g_j]$ are elements of $W^{\oplus 2\ell}$. Consider the basis of $W^{\oplus\ell}$ consisting of $g_j\cdot[g_i]$ for $i,j\in[2\ell]$ ordered first be the index of $[g_i]$ and then by the index of g_j. Then we see that each $g_j\cdot[g_i]-g_i\cdot[g_j]$ has a distinct leading term. Hence the $g_j\cdot[g_i]-g_i\cdot[g_j]$ must be linearly independent. The upper bound now follows. □

Lemma 3.2.8. *Let $\ell\ge 0$ and $m\ge\ell+2$ be integers and let $g_1,\dots,g_\ell\in\mathbb{C}[y_1,\dots,y_m]_{(2)}$ and $h_1,\dots,h_\ell\in\mathbb{C}[y_1,\dots,y_m]_{(3)}$ be generic. Then*

$$\ell\binom{m+1}{2}+\ell\binom{m+2}{3}-\binom{\ell}{2}m-\ell^2$$

is an upper bound for the dimension of $(g_1,\dots,g_\ell,h_1,\dots,h_\ell)_{(5)}$.

Proof. Note that $(g_1,\dots,g_\ell,h_1,\dots,h_\ell)_{(5)}$ is the image of the linear map

$$\begin{array}{rcl}\mathbb{C}[y_1,\dots,y_m]_{(3)}^{\oplus\ell}\oplus\mathbb{C}[y_1,\dots,y_m]_{(2)}^{\oplus\ell} & \to & \mathbb{C}[y_1,\dots,y_m]_{(5)}\\ \displaystyle\sum_{i=1}^{\ell} H_i\cdot[g_i]+\sum_{i=1}^{\ell} G_i\cdot[h_i] & \mapsto & \displaystyle\sum_{i=1}^{\ell} H_i g_i+\sum_{i=1}^{\ell} G_i h_i\end{array}$$

and that the kernel of this map contains:

(1) $h_j\cdot[g_i]-g_i\cdot[h_j]$ for $i,j\in[\ell]$ and

(2) $y_k\cdot(g_j\cdot[g_i]-g_i\cdot[g_j])$ for $1\le i<j\le\ell$ and $k\in[m]$.

It suffices to prove that these elements are linearly independent.

Consider the projection on $\mathbb{C}[y_1,\dots,y_m]_{(2)}^{\oplus\ell}$. The $h_j\cdot[g_i]-g_i\cdot[h_j]$ project onto the elements $-g_i\cdot[h_j]$ and the $y_k\cdot(g_j\cdot[g_i]-g_i\cdot[g_j])$ project to zero. Since the g_i are generic, they are linearly independent. So that means that the $-g_i\cdot[h_j]$ are independent as well. This takes care of the elements from (1). So it suffices to prove that the $y_k\cdot(g_j\cdot[g_i]-g_i\cdot[g_j])$ are linearly independent.

As in the previous lemma, it is enough to show that the $y_k\cdot g_j$ are linearly independent for generic g_j. Since this is an open condition, it is moreover enough to show that the $y_k\cdot g_j$ are linearly independent for some g_j. Take $g_j=y_j^2$. Then the $y_k\cdot g_j=y_k y_j^2$ are distinct monomials and hence linearly independent. This finishes the proof. □

Lemma 3.2.9. *Let $\ell, \ell' \geq 0$ and $m \geq \ell + \ell' + 2$ be integers and let $g_1, \dots, g_\ell \in \mathbb{C}[y_1, \dots, y_m]_{(2)}$, $h_1, \dots, h_\ell \in \mathbb{C}[y_1, \dots, y_m]_{(4)}$ and $a_1, \dots, a_{2\ell'} \in \mathbb{C}[y_1, \dots, y_m]_{(3)}$ be generic. Then*

$$\ell\binom{m+1}{2} + \ell\binom{m+3}{4} + 2\ell'\binom{m+2}{3} - \binom{\ell}{2}\binom{m+1}{2} + \binom{\ell}{3} - 2\ell\ell' - \ell^2 - \binom{2\ell'}{2}$$

is an upper bound for the dimension of $(g_1, \dots, g_\ell, h_1, \dots, h_\ell, a_1, \dots, a_{2\ell'})_{(4)}$.

Proof. Note that $(g_1, \dots, g_\ell, h_1, \dots, h_\ell, a_1, \dots, a_{2\ell'})_{(4)}$ is the image of the linear map

$$\begin{array}{rcl} \mathbb{C}[y_1, \dots, y_m]_{(4)}^{\oplus \ell} \oplus \mathbb{C}[y_1, \dots, y_m]_{(2)}^{\oplus \ell} \oplus \mathbb{C}[y_1, \dots, y_m]_{(3)}^{\oplus 2\ell'} & \to & \mathbb{C}[y_1, \dots, y_m]_{(6)} \\ \displaystyle\sum_{i=1}^{\ell} H_i \cdot [g_i] + \sum_{i=1}^{\ell} G_i \cdot [h_i] + \sum_{i=1}^{2\ell'} B_i \cdot [a_i] & \mapsto & \displaystyle\sum_{i=1}^{\ell} H_i g_i + \sum_{i=1}^{\ell} G_i h_i + \sum_{i=1}^{2\ell'} B_i a_i \end{array}$$

and the kernel of this map contains:

(1) $y_{k_1} y_{k_2} \cdot (g_j \cdot [g_i] - g_i \cdot [g_j])$ for $1 \leq k_1 \leq k_2 \leq m$ and $1 \leq i < j \leq \ell$,

(2) $y_k \cdot (a_j \cdot [g_i] - g_i \cdot [a_j])$ for $k \in [m]$, $i \in [\ell]$ and $j \in [2\ell']$,

(3) $h_j \cdot [g_i] - g_i \cdot [h_j]$ for $i, j \in [\ell]$ and

(4) $a_j \cdot [a_i] - a_i \cdot [a_j]$ for $1 \leq i < j \leq 2\ell'$.

We need to show that after leaving out $\binom{\ell}{3}$ elements from (1), we get a generically linearly independent set. We may assume that the g_i are linearly independent. So the projection on $\mathbb{C}[y_1, \dots, y_m]_{(2)}^{\oplus \ell}$ takes care of the elements from (3).

Next we take care of (2) and (4) by projecting on $\mathbb{C}[y_1, \dots, y_m]_{(3)}^{\oplus 2\ell'}$. When we project the elements from (2), we get $-y_k \cdot g_i \cdot [a_j]$. Take $g_i = y_i^2$. Then we see that the monomials $y_k \cdot g_i = y_k y_i^2$ are distinct from each other. We have ℓm such monomials. So since

$$\ell m + 2\ell' \leq (\ell + \ell')m \leq (m-2)m \leq \binom{m+2}{3}$$

we can choose the a_i to be monomials distinct from the monomials $y_k \cdot g_i$ and distinct from each other. It follows that the projection of the elements from (2) and (4) are generically linearly independent. This leaves the elements from (1).

Finally, we need to show that the elements from (1) are generically linearly independent after leaving out $\binom{\ell}{3}$ of them. We consider the elements $y_{k_1} y_{k_2} \cdot (g_j \cdot [g_i] - g_i \cdot [g_j])$ for $1 \leq k_1 \leq k_2 \leq m$ and $1 \leq i < j \leq \ell$ where the condition $i < k_1 = k_2 < j$ does not hold. This leaves out exactly $\binom{\ell}{3}$ elements. We need to show that the leading terms $y_{k_1} y_{k_2} \cdot g_j \cdot [g_i]$ are linearly independent for some choice of the g_j. Take $g_j = y_j^2$ and assume that

$$y_{k_1} y_{k_2} \cdot g_j \cdot [g_i] = y_{k_1'} y_{k_2'} \cdot g_{j'} \cdot [g_i]$$

for some $k_1, k_2, j, k_1', k_2', j', i$. Then either $j = j'$, which implies that $(k_1, k_2) = (k_1', k_2')$, or $j = k_1' = k_2'$ and $j' = k_1 = k_2$. The latter case implies that $i < k_1 = k_2 < j$ or $i < k_1' = k_2' < j'$. So among the elements from (1) that we consider, the leading terms are all distinct. Hence those elements from (1) are linearly independent. □

By combining the previous lemmas with the discussion before them, we now find lower bounds for the codimension of the image of the map φ for $d \in \{4,5,6\}$. We now need to check that these lower bounds are positive when the sum of the k_e's is lower than the generic slice rank in $\mathbb{C}[x_1,\dots,x_n]_{(d)}$. For fixed d,n, this can be checked by computer. So we now first focus on the case where $n \gg 0$.

Write $m = n-k$ and note that the codimension of the image of φ is at least

$$f_4(m,\ell) := \binom{m+3}{4} - (n-m)m - \left(2\ell\binom{m+1}{2} - \binom{2\ell}{2}\right)$$

for $(d,k_1,k_2) = (4,k,\ell)$, is at least

$$f_5(m,\ell) := \binom{m+4}{5} - (n-m)m - \left(\ell\binom{m+1}{2} + \ell\binom{m+2}{3} - \binom{\ell}{2}m - \ell^2\right)$$

for $(d,k_1,k_2) = (5,k,\ell)$ and is at least

$$f_6(m,\ell,\ell') := \binom{m+5}{6} - (n-m)m - \dim W$$

where

$$\dim W = \ell\binom{m+1}{2} + \ell\binom{m+3}{4} + 2\ell'\binom{m+2}{3} - \binom{\ell}{2}\binom{m+1}{2} + \binom{\ell}{3} - 2\ell\ell' - \ell^2 - \binom{2\ell'}{2}$$

for $(d,k_1,k_2,k_3) = (6,k,\ell,\ell')$. We need to show that $f_d(m,\ell) > 0$ for $d = 4,5$ and $m - \ell = n - \mathrm{slrk}^\circ(d,n) + 1$ and that $f_6(m,\ell,\ell') > 0$ for $m - \ell - \ell' = n - \mathrm{slrk}^\circ(6,n) + 1$. To proceed, we first need a lower bound for $n - \mathrm{slrk}^\circ(d,n)$.

Lemma 3.2.10. *Suppose that $d \geq 4$ and that*

$$n \geq \max\left(d^{d-1}/d!,\ \sqrt[d-3]{(d-1)!^{d-1}/d!^{d-3}},\ 2\sqrt[d-1]{2d!}\right)$$

holds. Take $p(x) = (x+d-1)\cdots(x+1) - d!(n-x)$. Then $p(x)$ has a unique positive root $a > 0$. We have

$$\sqrt[d-1]{d!n} - (d+1) < a < \sqrt[d-1]{d!n} - 1$$

and $n - \mathrm{slrk}^\circ(d,n) = \lfloor a \rfloor$.

Proof. Recall that

$$\mathrm{slrk}^\circ(d,n) = \min\left\{k \in \mathbb{Z}_{\geq n/2} \,\middle|\, \binom{d+n-k-1}{d} \leq k(n-k)\right\}.$$

Take $x = n-k$. Then we see that

$$\binom{d+n-k-1}{d} \leq k(n-k)$$

holds if and only if $p(x) \leq 0$. Note that $p(0) = (d-1)! - d!n < 0$ and that $p(x)$ is strictly increasing on $\mathbb{R}_{\geq 0}$. So $p(x)$ has a unique positive root $a > 0$. Take $x = \sqrt[d-1]{d!n} - 1$. Then

$$p(x) \geq d!n - d!(n-x) > 0$$

and so $a < \sqrt[d-1]{d!n} - 1$. Take $y = \sqrt[d-1]{d!n} \geq d$. Then

$$p(y-(d+1)) < y^{d-2}(y-d) - d!(n-y) = d!y - dy^{d-2} = dy((d-1)! - y^{d-3}) \leq 0$$

since $(d!n)^{d-3} \geq (d-1)!^{d-1}$ and hence $(d-1)! \leq y^{d-3}$. So $a > \sqrt[d-1]{d!n} - (d+1)$. Since $n \geq 2\sqrt[d-1]{2d!}$, we have $a \leq \sqrt[d-1]{d!n} \leq n/2$. So $\lfloor a \rfloor$ is the maximal integer $\leq n/2$ such that $p(x) \leq n$. So $\lfloor a \rfloor = n - \mathrm{slrk}^{\circ}(d,n)$. □

Remark 3.2.11. Note that the condition on n in the lemma is satisfied when $d \in \{4,5,6\}$ and $n \geq 11$. ♠

Note that $f_4(m,0) > 0$ for $m = n - \mathrm{slrk}^{\circ}(4,n) + 1$. So for $d = 4$ it suffices to prove that

$$f_4(m+1, \ell+1) \geq f_4(m,\ell)$$

whenever $m - \ell = n - \mathrm{slrk}^{\circ}(4,n) + 1$. We have

$$\begin{array}{rcl} f_4(m+1,\ell+1) - f_4(m,\ell) & = & \frac{1}{6}m^3 - 2m\ell + \frac{5}{6}m + 2\ell - n + 1 \\ & \geq & \frac{1}{6}m^3 - 2m(m-2) + \frac{5}{6}m - n + 1 \end{array}$$

since $0 \leq \ell \leq m-2$. Assume that $n \geq 11$. Then by the lemma, we see that

$$m \geq n - \mathrm{slrk}^{\circ}(4,n) + 1 \geq \left\lfloor \sqrt[3]{24n} - 5 \right\rfloor + 1 \geq \sqrt[3]{24n} - 5$$

and so $n \leq (m+5)^3/24$. So

$$\begin{array}{rcl} \frac{1}{6}m^3 - 2m(m-2) + \frac{5}{6}m - n + 1 & \geq & \frac{1}{6}m^3 - 2m(m-2) + \frac{5}{6}m - (m+5)^3/24 + 1 \\ & = & \frac{1}{8}m^3 - \frac{21}{8}m^2 + \frac{41}{24}m - \frac{101}{24}. \end{array}$$

One can check numerically that the latter is positive for $m \geq 41/2$. So we see that

$$f_4(m+1,\ell+1) \geq f_4(m,\ell) > 0$$

for all $m - \ell = n - \mathrm{slrk}^{\circ}(4,n) + 1$ when $n \geq 691 \geq (41/2+5)^3/24$. Note that one can check in a finite amount of time that $f_4(m,\ell) > 0$ for all $m - \ell = n - \mathrm{slrk}^{\circ}(4,n) + 1$ when $n < 691$. We checked using a computer that this is indeed the case.

Next, note that $f_5(m,0) > 0$ for $m = n - \mathrm{slrk}^{\circ}(5,n) + 1$. So for $d = 5$ it suffices to prove that

$$f_5(m+1,\ell+1) \geq f_5(m,\ell)$$

whenever $m - \ell = n - \mathrm{slrk}^{\circ}(5,n) + 1$. Similarly to before, we find that

$$f_5(m+1,\ell+1) - f_5(m,\ell) \geq \frac{1}{30}m^4 - \frac{9}{20}m^3 - \frac{281}{120}m^2 - \frac{69}{20}m - \frac{49}{5}.$$

One can check numerically that the latter is positive for $m \geq 18$. So we see that

$$f_5(m+1,\ell+1) \geq f_5(m,\ell) > 0$$

for all $m - \ell = n - \mathrm{slrk}^\circ(4, n) + 1$ when $n \geq 2765 \geq (18+6)^4/120$. We checked that $f_5(m, \ell) > 0$ for all $m - \ell = n - \mathrm{slrk}^\circ(5, n) + 1$ when $n < 2765$.

Finally, note that $f_6(m, 0, 0) > 0$ for $m = n - \mathrm{slrk}^\circ(6, n) + 1$. So for $d = 6$ it suffices to prove that

$$f_6(m, \ell, \ell') \geq f_6(m, \ell+1, \ell'-1) \quad \text{and} \quad f_6(m+1, \ell+1, 0) \geq f_6(m, \ell, 0)$$

whenever $m - \ell = n - \mathrm{slrk}^\circ(6, n) + 1$. Similarly to before, we find that

$$f_6(m, \ell, \ell') - f_6(m, \ell+1, \ell'-1) \geq \frac{1}{24}m^4 - \frac{7}{12}m^3 + \frac{11}{24}m^2 + \frac{37}{12}m - 6$$

and

$$f_6(m+1, \ell+1, 0) - f_6(m, \ell, 0) \geq \frac{1}{144}m^5 - \frac{19}{144}m^4 - \frac{5}{9}m^3 - \frac{409}{72}m^2 - \frac{8861}{720}m - \frac{16087}{720}.$$

The right hand sides of these expressions are positive when $m \geq 24$. So we see that

$$f_6(m, \ell, \ell') > 0$$

for all $m - \ell - \ell' = n - \mathrm{slrk}^\circ(4, n) + 1$ when $n \geq 39763 \geq (24+7)^5/720$. We checked that $f_6(m, \ell, \ell') > 0$ for all $m - \ell - \ell' = n - \mathrm{slrk}^\circ(6, n) + 1$ when $n < 39763$.

3.3 Bounded strength of polynomials

The goal of this section is to prove Theorem 3.1.9. By assumption, there exists a $U \in \mathrm{Vec}$ such that $X(U) \subsetneq S^d(U)$. We fix this U throughout the proof. The bound $N \geq 0$ that we will obtain depends only on d and $\dim U$. See Remark 3.3.8.

Irreducibility. The following lemma is a standard fact from representation theory. Recall that, since $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > d$, the representations $S^d(V^*)$ and $S^d(V)^*$ of the group $\mathrm{GL}(V)$ are isomorphic for any $V \in \mathrm{Vec}$.

Lemma 3.3.1. *For each $V \in \mathrm{Vec}$, the representation $S^d(V)$ of $\mathrm{GL}(V)$ is irreducible and linearly spanned by its subset $\{v^d \mid v \in V \setminus \{0\}\}$. Furthermore, any $\mathrm{GL}(V)$-equivariant polynomial map from V into a representation N of $\mathrm{GL}(V)$ on which $t\,\mathrm{id}_V$ acts via multiplication with t^d factors as $V \to S^d(V)$, $v \mapsto v^d$ and a unique $\mathrm{GL}(V)$-equivariant linear map $S^d(V) \to N$.* □

Homogeneity. We equip the coordinate ring $K[S^d(V)]$ with the grading in which the elements of $S^d(V)^*$ have degree 1. For any closed subset $X \subseteq S^d$ we find, from the fact that $X(V)$ is $\mathrm{GL}(V)$-stable, that the ideal $\mathcal{I}(X(V)) \subseteq K[S^d(V)]$ is $\mathrm{GL}(V)$-stable and in particular homogeneous. We define $\delta_X := \min\{\deg(f) \mid f \in \mathcal{I}(X(U)) \setminus \{0\}\}$.

Induction. If $\delta_X = 0$, then $\mathcal{I}(X(U))$ contains a nonzero constant and hence $X(U) = \emptyset$. In this case, for any $V \in \mathrm{Vec}$, the d-th symmetric power of the zero map $V \to U$ maps $X(V)$ into $X(U)$ and so all $X(V)$ are empty. Hence in Theorem 3.1.9 we may take $N = 0$. We proceed by induction, assuming that $\delta_X > 0$ and that the theorem holds for all closed subsets $Y \subseteq S^d$ with $Y(U) \subsetneq S^d(U)$ and $\delta_Y < \delta_X$.

Derivative. Let $f \in \mathcal{I}(X(U)) \setminus \{0\}$ be homogeneous of degree δ_X. By the minimality of δ_X and perfectness of K, there exists an $r \in S^d(U)$ such that the directional derivative

$$h := \frac{\partial f}{\partial r}$$

is not the zero polynomial. By Lemma 3.3.1, $S^d(U)$ is spanned by d-th powers, so we may further assume that $r = u^d$ for some $u \in U$.

We define the closed subset $Y \subsetneq S^d$ by

$$Y(V) := \left\{ q \in X(V) \,\middle|\, \forall \ell \in \mathrm{Hom}(V, U) : h(S^d(\ell)(q)) = 0 \right\}.$$

Now we have $\delta_Y \leq \deg(h) = \deg(f) - 1 < \deg(f)$. So, by the induction hypothesis, the theorem holds for Y. We define $Z(V) := X(V) \setminus Y(V)$ and set out to prove that all elements in $Z(V)$ have strength bounded independently of V.

Shifting. For $V \in \mathrm{Vec}$, we define

$$\begin{aligned}
P'(V) &:= S^d(U \oplus V) = \bigoplus_{i=0}^{d} S^{d-i}(U) \otimes S^i(V),\\
X'(V) &:= X(U \oplus V) \subseteq P'(V),\\
Z'(V) &:= \left\{ q \in X'(V) \,\middle|\, h(S^d(\pi_U)(q)) \neq 0 \right\}.
\end{aligned}$$

The notation is chosen compatible with [17]. We think of $P'(V), X'(V)$ as varieties over $S^d(U), X(U)$, respectively, via the linear map $S^d(\pi_U)$. Accordingly, by slight abuse of notation, we will write h for $h \circ S^d(\pi_U)$.

Lemma 3.3.2. *We have*

$$Z(U \oplus V) = \bigcup_{g \in \mathrm{GL}(U \oplus V)} gZ'(V).$$

In particular, $\sup_{q \in Z(U \oplus V)} \mathrm{str}(q) = \sup_{q \in Z'(V)} \mathrm{str}(q)$.

Proof. First, we have $Z(U \oplus V) \supseteq Z'(V)$, and since the left-hand side is $\mathrm{GL}(U \oplus V)$-stable, the inclusion $\supseteq$ follows. Conversely, if $q \in Z(U \oplus V)$, then there exists a linear map $\ell \colon U \oplus V \to U$ for which $h(S^d(\ell)(q)) \neq 0$. Since this is an open condition on ℓ, we may further assume that ℓ has full rank. Then for a suitable $g \in \mathrm{GL}(U \oplus V)$ we find that $\ell = \pi_U \circ g$. Accordingly,

$$h(g \cdot q) = (h \circ S^d(\pi_U))(S^d(g)(q)) = h(S^d(\ell)(q)) \neq 0$$

and hence $g \cdot q \in Z'(V)$. □

Lemma 3.3.3. *We have*

$$\sup_{\substack{V \in \mathrm{Vec} \\ q \in X(V)}} \mathrm{str}(q) = \sup_{V \in \mathrm{Vec}} \max \left\{ \sup_{q \in Y(V)} \mathrm{str}(q), \sup_{q \in Z'(V)} \mathrm{str}(q) \right\}.$$

Proof. The same statement with $Z'(V)$ replaced by $Z(V)$ is obvious given the fact that $X(V) = Y(V) \cup Z(V)$. Let $\iota\colon V \hookrightarrow U \oplus V$ be the map sending $v \mapsto (0, v)$. Then the map $S^d(\iota)$ maps $Z(V)$ into $Z(U \oplus V)$ and is easily seen to preserve the strength. By the previous lemma

$$\sup_{q \in Z'(V)} \operatorname{str}(q) = \sup_{q \in Z(U \oplus V)} \operatorname{str}(q)$$

and so the statement follows. $\square$

So it suffices to show that elements in $Z'(V)$ have bounded strength.

Chopping.

Lemma 3.3.4. *For $q \in P'(V)$ write $q = q_0 + \cdots + q_d$ with $q_i \in S^{d-i}(U) \otimes S^i(V)$. Then*

$$\operatorname{str}(q) \leq \dim U + \operatorname{str}(q_d).$$

Proof. Note that $q_0 + \ldots + q_{d-1}$ is in the image of the map

$$\begin{array}{rcl} U \otimes S^{d-1}(U \oplus V) & \to & S^d(U \oplus V) \\ r \otimes s & \mapsto & rs \end{array}$$

and hence has strength at most $\dim U$. Now, strength is subadditive, so

$$\operatorname{str}(q) \leq \operatorname{str}(q_0 + \ldots + q_{d-1}) + \operatorname{str}(q_d) \leq \dim U + \operatorname{str}(q_d). \qquad \square$$

So, as U is fixed, it suffices to prove that for V ranging through Vec and q ranging through $Z'(V)$ the component q_d has bounded strength.

Embedding. Define

$$Q'(V) := P'(V)/S^d(V) = \bigoplus_{i=0}^{d-1} S^{d-i}(U) \otimes S^i(V)$$

and write $\pi_{Q'(V)}\colon P'(V) \to Q'(V)$ for the natural projection. Take

$$B(V) := \{q \in Q'(V) \mid h(q) \neq 0\} = \{(q_0, \ldots, q_{d-1}) \in Q'(V) \mid h(q_0) \neq 0\}.$$

Then $\pi_{Q'(V)}$ maps $Z'(V)$ into $B(V)$. And, by [17, Lemma 7] and Lemma 3.3.1, the following lemma holds.

Lemma 3.3.5. *The map $\pi_{Q'(V)}$ restricts to a closed embedding $Z'(V) \to B(V)$.* $\square$

We will not actually use this lemma, but we will use its proof method.

An equivariant map back. We construct a suitable map opposite to the embedding of Lemma 3.3.5.

Lemma 3.3.6. *There exists a $\mathrm{GL}(V)$-equivariant polynomial map $\Psi\colon Q'(V) \to S^d(V)$ such that q_d is a scalar multiple of $\Psi(q_0, \ldots, q_{d-1})$ for all $q = (q_0, \ldots, q_d) \in Z'(V)$.*

Proof. For $x \in V^*$ and $t \in K$, let $\ell_x \colon V \to U$ and $\ell'_x(t) \colon U \oplus V \to U$ be the linear maps sending $v \mapsto x(v)u$ and $(u, v) \mapsto u + t\ell_x(v)$. Here u is the vector used in the definition of h. Note that $x \mapsto \ell_x$ is a $\mathrm{GL}(V)$-equivariant linear map $V^* \to \mathrm{Hom}(V, U)$. Now consider the linear map $\Phi_x(t) := S^d(\ell'_x(t)) \colon P'(V) \to S^d(U)$.

The restriction of $\Phi_x(t)$ to the summand $S^{d-i}(U) \otimes S^i(V)$ equals $t^i\Phi_{x,i}$ where $\Phi_{x,i}$ is the composition of $S^{d-i}(\mathrm{id}_U) \otimes S^i(\ell_x) \colon S^{d-i}(U) \otimes S^i(V) \to S^{d-i}(U) \otimes S^i(U)$ and the multiplication map into $S^d(U)$. In particular, $\Phi_{x,0}$ is the identity on $S^d(U)$ and $\Phi_{x,d} \colon S^d(V) \to S^d(U)$ is the linear map sending $q_d \mapsto x^d(q_d)u^d$. Note that the map

$$\begin{array}{rcl} V^* & \to & \mathrm{Hom}(S^{d-i}(U) \otimes S^i(V), S^d(U)) \\ x & \mapsto & \Phi_{x,i} \end{array}$$

is a $\mathrm{GL}(V)$-equivariant polynomial map of degree i.

The functoriality of X implies that $\Phi_x(t)(X'(V)) \subseteq X(U)$. In particular, the pull-back of f along $\Phi_x(t)$ to $P'(V)$ vanishes on $X'(V)$. Take $q = (q_0, \ldots, q_d) \in P'(V)$. Then

$$f(\Phi_x(t)(q_0 + q_1 + \cdots + q_{d-1} + q_d)) = f\left(q_0 + t\Phi_{x,1}(q_1) + \cdots + t^{d-1}\Phi_{x,d-1}(q_{d-1}) + t^d x^d(q_d)u^d\right)$$

vanishes for $q \in X'(V)$. In particular, the coefficient of t^d in the Taylor expansion of this expression vanishes for $q \in X'(V)$. This coefficient equals

$$x^d(q_d)\frac{\partial f}{\partial u^d}(q_0) + \Psi(x, q_0, \ldots, q_{d-1}) = x^d(q_d)h(q_0) + \Psi(x, q_0, \ldots, q_{d-1})$$

where the function $\Psi \colon V^* \times Q'(V) \mapsto K$ is $\mathrm{GL}(V)$-invariant and homogeneous of degree d in its first argument x.

For $q \in Z'(V)$ we have $h(q_0) \neq 0$ and hence

$$x^d(q_d) = -\frac{1}{h(q_0)}\Psi(x, q_0, \ldots, q_{d-1}).$$

By Lemma 3.3.1 the space $S^d(V)^*$ of coordinates on S^d is spanned by the $\{x^d \mid x \in V^*\}$ and this shows that $Z'(V) \to B(V)$ is a closed embedding. But it yields more: by Lemma 3.3.1, Ψ factors as

$$\begin{array}{rcl} V^* \times Q'(V) & \to & S^d V^* \times Q'(V) \\ (x, q') & \mapsto & (x^d, q') \end{array}$$

and a unique $\mathrm{GL}(V)$-invariant map $S^d V^* \times Q'(V) \to K$. We denote the latter map also by Ψ, which is now linear in its first argument. If we reinterpret Ψ as a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to S^d V$, then for $q \in Z'(V)$ we have

$$q_d = -\frac{1}{h(q_0)}\Psi(q_0, \ldots, q_{d-1}).$$

In particular, for $q \in Z'(V)$ we have $q_d \in \mathrm{im}\,\Psi$. $\square$

Covariants. A covariant of $Q'(V)$ (of order $S^d(V)$) is a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to S^d(V)$. So the map Ψ constructed in Lemma 3.3.6 is a covariant. For each integer $i \in [d-1]$, choose a basis $u_{i,1}, \dots, u_{i,n_i}$ of $S^{d-i}(U)$. Then the map

$$\begin{array}{rrcl} \Phi\colon & \bigoplus_{i=0}^{d-i} S^i(V)^{\oplus n_i} & \to & \bigoplus_{i=1}^{d-1} S^{d-i}(U) \otimes S^i(V) \\ & (w_{i,j})_{i,j} & \mapsto & \left(\sum_{j=1}^{n_i} u_{i,j} \otimes w_{i,j} \right)_{i=1}^{d-1} \end{array}$$

is a $\mathrm{GL}(V)$-equivariant isomorphism and the following lemma holds.

Lemma 3.3.7. *Let $\Psi\colon Q'(V) \to S^d(V)$ be a covariant. Then the composition*

$$\Psi \circ (\mathrm{id}_{S^d(U)}, \Phi)\colon S^d(U) \oplus \bigoplus_{i=1}^{d-1} S^i(V)^{\oplus n_i} \to S^d(V)$$

is given by

$$\Big(q, (w_{i,j})_{i,j}\Big) \mapsto \sum_{\substack{\alpha_{i,j} \in \mathbb{Z}_{\geq 0} \\ \sum_{i=1}^{d-1} \sum_{j=1}^{n_i} i \cdot \alpha_{i,j} = d}} p_\alpha(q) \cdot \prod_{i=1}^{d-1} \prod_{j=1}^{n_i} w_{i,j}^{\alpha_{i,j}}$$

for some polynomial functions $p_\alpha\colon S^d(U) \to K$.

Proof. Polynomial $\mathrm{GL}(V)$-equivariant maps

$$S^d(U) \oplus \bigoplus_{i=1}^{d-1} S^i(V)^{\oplus n_i} \to S^d(V)$$

correspond one-to-one to linear $\mathrm{GL}(V)$-equivariant maps

$$K[S^d(U)] \otimes \bigoplus_{\substack{\alpha_{i,j} \in \mathbb{Z}_{\geq 0} \\ \sum_{i=1}^{d-1} \sum_{j=1}^{n_i} i \cdot \alpha_{i,j} = d}} \bigotimes_{i=1}^{d-1} \bigotimes_{j=1}^{n_i} S^{\alpha_{i,j}}(S^i(V)) \to S^d(V)$$

by the universal properties of tensor products and symmetric powers. For each α, the vector space

$$\mathrm{Hom}_{\mathrm{GL}(V)}\left(\bigotimes_{i=1}^{d-1} \bigotimes_{j=1}^{n_i} S^{\alpha_{i,j}}(S^i(V)), S^d(V) \right)$$

is one-dimensional and consists of multiples of the homomorphism ℓ_α sending

$$\bigotimes_{i=1}^{d-1} \bigotimes_{j=1}^{n_i} w_{i,j,1} \cdot \dots \cdot w_{i,j,\alpha_{i,j}} \mapsto \prod_{i=1}^{d-1} \prod_{j=1}^{n_i} w_{i,j,1} \cdot \dots \cdot w_{i,j,\alpha_{i,j}}.$$

Hence the set of $\mathrm{GL}(V)$-equivariant linear maps

$$K[S^d(U)] \otimes \bigoplus_{\substack{\alpha_{i,j} \in \mathbb{Z}_{\geq 0} \\ \sum_{i=1}^{d-1} \sum_{j=1}^{n_i} i \cdot \alpha_{i,j} = d}} \bigotimes_{i=1}^{d-1} \bigotimes_{j=1}^{n_i} S^{\alpha_{i,j}}(S^i(V)) \to S^d(V)$$

is spanned as a $K[S^d(U)]$-module by the set of all ℓ_α. The corresponding statement for polynomial $\mathrm{GL}(V)$-equivariant maps is the statement of the lemma. $\square$

Conclusion of the proof

Proof of Theorem 3.1.9. By the induction hypothesis and Lemma 3.3.3, to bound the strength of elements of $X(V)$ for all $V \in \mathrm{Vec}$ it suffices to bound the strength of elements of $Z'(V)$ for all $V \in \mathrm{Vec}$. By Lemma 3.3.4, it suffices to bound the strength of q_d over all $q = (q_0, \dots, q_d) \in Z'(V)$. By Lemma 3.3.6, we know that such a q_d is contained in the image of a covariant. So using Lemma 3.3.7, we see that q_d is a linear combination of products of polynomials $w_{i,j} \in S^i(V)$ where i ranges over $[d-1]$ and j ranges over $[\dim S^{d-i}(U)]$. Since each of those products has degree d and since, for each pair (i, j), the polynomial $w_{i,j}$ has degree i, we see that each of the products is divisible by $w_{i,j}$ for some $i \leq d/2$. We find that the strength of q_d is at most

$$\#\{w_{i,j} \mid i \in \{1, \dots, \lfloor d/2 \rfloor\}, j \in [\dim S^{d-i}(U)]\} \leq \sum_{i=1}^{\lfloor d/2 \rfloor} \dim S^{d-i}(U)$$

and this bounds the strength of q_d independently of V. $\square$

Remark 3.3.8. It follows from the induction that N from Theorem 3.1.9 can be taken equal to

$$\dim U + \sum_{i=1}^{\lfloor d/2 \rfloor} \dim S^{d-i}(U).$$

♠

3.4 Bounded strength of alternating tensors

Let $V \in \mathrm{Vec}$. For alternating tensors $q \in \bigwedge^d(V)$, the strength $\mathrm{str}(q)$ is defined as the minimal number of terms k in any decomposition of the form

$$q = r_1 \wedge s_1 + \cdots + r_k \wedge s_k$$

where $r_i \in \bigwedge^{e_i}(V)$ and $s_i \in \bigwedge^{d-e_i}(V)$ for suitable natural numbers $e_i \in \{1, \dots, d-1\}$. By taking all e_i equal to 1 and using standard properties of the wedge product, we obtain the bound $\mathrm{str}(q) \leq \dim V - d + 1$.

The goal of this section is to adapt the statement of Theorem 3.1.9 and its proof from Section 3.3 to the polynomial functor $\bigwedge^d$. In order to state the theorem, we only need to replace S^d by $\bigwedge^d$.

Theorem 3.4.1. *Fix $d \in \mathbb{Z}_{\geq 2}$ and assume that K is a perfect and infinite field with* $\mathrm{char}\, K = 0$ *or* $\mathrm{char}\, K > d$. *Then for any closed subset $X \subsetneq \bigwedge^d$ there exists an $N \geq 0$ such that for all vector spaces $V \in$* Vec *the strength of all elements in $X(V)$ is at most N.*

Fix a vector space $U \in \mathrm{Vec}$ such that $X(U) \subsetneq \bigwedge^d(U)$. Note that $\dim U \geq d$ as $\bigwedge^d(U) \neq \emptyset$.

Irreducibility. Note that for any $V \in \mathrm{Vec}$ the $\mathrm{GL}(V)$-modules $\bigwedge^d(V^*)$ and $\bigwedge^d(V)^*$ are isomorphic. The analogue of Lemma 3.3.1 is as follows.

Lemma 3.4.2. *For each $V \in \mathrm{Vec}$, the $\mathrm{GL}(V)$-module $\bigwedge^d(V)$ is irreducible and linearly spanned by its subset $\{v_1 \wedge \cdots \wedge v_d \mid v_1, \ldots, v_d \in V$ linearly independent$\}$. Furthermore, any $\mathrm{GL}(V)$-equivariant multilinear and alternating map from V^d into a $\mathrm{GL}(V)$-module N on which $t\,\mathrm{id}_V$ acts via multiplication with t^d extends uniquely to a $\mathrm{GL}(V)$-equivariant linear map $\bigwedge^d(V) \to N$.* □

Homogeneity. We equip the coordinate ring $K[\bigwedge^d(V)]$ with the grading in which the elements of $\bigwedge^d(V)^*$ have degree 1. For any closed $X \subseteq \bigwedge^d$ we find, from the fact that $X(V)$ is stable under $\mathrm{GL}(V)$, that the ideal $\mathcal{I}(X(V)) \subseteq K[\bigwedge^d(V)]$ is $\mathrm{GL}(V)$-stable and in particular homogeneous. We define $\delta_X := \min\{\deg f \mid f \in \mathcal{I}(X(U)) \setminus \{0\}\}$.

Induction. If $\delta_X = 0$, then we find that $X(V) = \emptyset$ for all $V \in \mathrm{Vec}$. We may therefore assume that $\delta_X > 0$ and we proceed by induction, assuming that the theorem holds for all $Y \subseteq \bigwedge^d$ with $Y(U) \subsetneq \bigwedge^d(U)$ and $\delta_Y < \delta_X$.

Derivative. Let $f \in \mathcal{I}(X(U)) \setminus \{0\}$ be a homogeneous polynomial of degree δ_X. Then, there exists an $r \in \bigwedge^d(U)$ such that the directional derivative

$$h := \frac{\partial f}{\partial r}$$

is not the zero polynomial. By Lemma 3.4.2 we may assume that $r = u_1 \wedge \cdots \wedge u_d$ for some linearly independent $u_1, \ldots, u_d \in U$.

We define $Y \subsetneq \bigwedge^d$ by

$$Y(V) := \left\{ q \in X(V) \,\middle|\, \forall \ell \in \mathrm{Hom}(V, U) : h(\textstyle\bigwedge^d(\ell)(q)) = 0 \right\}$$

and note that, by the induction hypothesis, the theorem holds for Y. We define

$$Z(V) := X(V) \setminus Y(V)$$

and prove that all elements of $Z(V)$ have strength bounded independently of V.

Shifting. For $V \in \mathrm{Vec}$ we define

$$\begin{aligned} P'(V) &:= \textstyle\bigwedge^d(U \oplus V) = \displaystyle\bigoplus_{i=0}^{d} \textstyle\bigwedge^{d-i}(U) \otimes \bigwedge^i(V), \\ X'(V) &:= X(U \oplus V) \subseteq P'(V), \\ Z'(V) &:= \left\{ q \in X'(V) \,\middle|\, h(\textstyle\bigwedge^d(\pi_U)(q)) \neq 0 \right\}. \end{aligned}$$

We think of $P'(V), X'(V)$ as varieties over $\bigwedge^d(U), X(U)$, respectively, via the linear map $\bigwedge^d(\pi_U)$ and we will write h for $h \circ \bigwedge^d(\pi_U)$.

Lemma 3.4.3. *We have*

$$\sup_{\substack{V \in \mathrm{Vec} \\ q \in X(V)}} \mathrm{str}(q) = \sup_{V \in \mathrm{Vec}} \max \left\{ \sup_{q \in Y(V)} \mathrm{str}(q), \sup_{q \in Z'(V)} \mathrm{str}(q) \right\}.$$

□

Chopping.

Lemma 3.4.4. *For $q \in P'(V)$ write $q = q_0 + \ldots + q_d$ with $q_i \in \bigwedge^{d-i}(U) \otimes \bigwedge^{i}(V)$. Then*

$$\mathrm{str}(q) \leq \dim U + \mathrm{str}(q_d).$$

$\square$

Embedding. Define

$$Q'(V) := P'(V)/\bigwedge^{d}(V) = \bigoplus_{i=0}^{d-1} \bigwedge^{d-i}(U) \otimes \bigwedge^{i}(V)$$

and write $\pi_{Q'(V)} \colon P'(V) \to Q'(V)$ for the natural projection. Take

$$B(V) := \{q \in Q'(V) \mid h(t) \neq 0\} = \{(q_0, \ldots, q_{d-1}) \in Q'(V) \mid h(q_0) \neq 0\}.$$

Then $\pi_{Q'(V)}$ maps $Z'(V)$ into $B(V)$ (and this is a closed embedding by [17, Lemma 7] and Lemma 3.4.2).

An equivariant map back.

Lemma 3.4.5. *There exists a* $\mathrm{GL}(V)$*-equivariant polynomial map* $\Psi \colon Q'(V) \to \bigwedge^{d}(V)$ *such that* q_d *is a scalar multiple of* $\Psi(q_0, \ldots, q_{d-1})$ *for all* $q = (q_0, \ldots, q_d) \in Z'(V)$.

Proof. For $x = (x_1, \ldots, x_d) \in (V^*)^d$ and $t = (t_1, \ldots, t_d) \in K^d$, consider the linear map

$$\begin{array}{rcl} \ell'_x(t) \colon U \oplus V & \to & U \\ (u, v) & \mapsto & u + \sum_{j=1}^{d} t_j \ell_{x,j}(v) \end{array}$$

where $\ell_{x,j} \colon V \to U$ sends $v \mapsto x_j(v) u_j$ and $u_1, \ldots, u_d$ are the vectors used in the definition of h. Note that $x \mapsto \ell_{x,j}$ is a $\mathrm{GL}(V)$-equivariant linear map $(V^*)^d \to \mathrm{Hom}(V, U)$.

Now take $\Phi_x(t) := \bigwedge^{d}(\ell'_x(t)) \colon P'(V) \to \bigwedge^{d}(U)$ and denote the restriction of $\Phi_x(t)$ to the summand $\bigwedge^{d-i}(U) \otimes \bigwedge^{i}(V)$ by $\Phi_{x,i}$. Note that $\Phi_{x,0}$ is the identity on $\bigwedge^{d}(U)$ and $\Phi_{x,d}$ is the linear map

$$\begin{array}{rcl} \bigwedge^{d} V & \to & \bigwedge^{d} U \\ v_1 \wedge \cdots \wedge v_d & \mapsto & t_1 \cdots t_d \cdot (\sum_{j=1}^{d} \ell_{x,j}(v_1)) \wedge \cdots \wedge (\sum_{j=1}^{d} \ell_{x,j}(v_d)) \end{array}$$

where the latter is a multiple of $u_1 \wedge \ldots \wedge u_d$. Also note that $x \mapsto \Phi_{x,i}$ is a $\mathrm{GL}(V)$-equivariant polynomial map of degree i and that $x \mapsto \Phi_{x,d}$ is multilinear and alternating.

By functoriality of X, we have $\Phi_x(t)(X'(V)) \subseteq X(U)$, and for $q = (q_0, \ldots, q_d) \in P'(V)$ we find that

$$f(\Phi_x(t)(q_0 + \ldots + q_d)) = f\left(q_0 + \Phi_{x,1}(q_1) + \cdots + \Phi_{x,d-1}(q_{d-1}) + t_1 \cdots t_d \cdot \bigwedge^{d}(\textstyle\sum_{j=1}^{d} \ell_{x,j})(q_d)\right)$$

and this expression vanishes for $q \in X'(V)$. The coefficient of $t_1 \cdots t_d$ in the Taylor expansion of this expression equals

$$h(q_0) \cdot (x_1 \wedge \cdots \wedge x_d)(q_d) + \Psi(x, q_0, \ldots, q_{d-1})$$

where the function $\Psi\colon (V^*)^d \times Q'(V) \to K$ is $\mathrm{GL}(V)$-invariant and multilinear in $(V^*)^d$. We note that for $q \in Z'(V)$, we have $h(q_0) \neq 0$ by definition of $Z'(V)$, and therefore

$$(x_1 \wedge \cdots \wedge x_d)(q_d) = -\frac{1}{h(q_0)}\Psi(x, q_0, \ldots, q_{d-1}).$$

The map Ψ factors as

$$\begin{array}{rcl} (V^*)^d \times Q'(V) & \to & (V^*)^{\otimes d} \times Q'(V) \\ (x, q') & \mapsto & (x_1 \otimes \cdots \otimes x_d, q') \end{array}$$

and a unique $\mathrm{GL}(V)$-equivariant map $(V^*)^{\otimes d} \times Q'(V) \to K$. If we re-interpret Ψ as a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to V^{\otimes d}$ and compose the map with the projection $V^{\otimes d} \to \bigwedge^d V$, then we get a map $Q'(V) \to \bigwedge^d V$ which we also denote by Ψ. We see that

$$d! \cdot q_d = -\frac{1}{h(q_0)}\Psi(q_0, \ldots, q_{d-1}).$$

for all $q \in Z'(V)$, since

$$(x_1 \wedge \cdots \wedge x_d)(q_d) = (x_1 \otimes \cdots \otimes x_d)\iota(q_d)$$

where ι is the $\mathrm{GL}(V)$-equivariant map

$$\begin{array}{rrcl} \iota\colon & \bigwedge^d V & \to & V^{\otimes d} \\ & v_1 \wedge \cdots \wedge v_d & \mapsto & \displaystyle\sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) \cdot v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \end{array}$$

□

Covariants. A covariant of $Q'(V)$ of order $\bigwedge^d(V)$ is a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to \bigwedge^d V$. So the map Ψ constructed in Lemma 3.4.5 is a covariant. For each integer $i \in [d-1]$, choose a basis $u_{i,1}, \ldots, u_{i,n_i}$ of $\bigwedge^{d-i}(U)$. Then the map

$$\begin{array}{rrcl} \Phi\colon & \displaystyle\bigoplus_{i=0}^{d-i} \bigwedge^i(V)^{\oplus n_i} & \to & \displaystyle\bigoplus_{i=1}^{d-1} \bigwedge^{d-i}(U) \otimes \bigwedge^i(V) \\ & (w_{i,j})_{i,j} & \mapsto & \displaystyle\left(\sum_{j=1}^{n_i} u_{i,j} \otimes w_{i,j}\right)_{i=1}^{d-1} \end{array}$$

is a $\mathrm{GL}(V)$-equivariant isomorphism and the following lemma holds.

Lemma 3.4.6. *Let $\Psi\colon Q'(V) \to \bigwedge^d V$ be a covariant. Then the composition*

$$\Psi \circ (\mathrm{id}_{\bigwedge^d(U)}, \Phi)\colon \bigwedge^d(U) \oplus \bigoplus_{i=1}^{d-1} \bigwedge^i(V)^{\oplus n_i} \to \bigwedge^d(V)$$

is given by

$$\Big(q,(w_{i,j})_{i,j}\Big)\mapsto \sum_{\substack{\alpha_{i,j}\in\mathbb{Z}_{\geq 0}\\ \sum_{i=1}^{d-1}\sum_{j=1}^{n_i} i\cdot\alpha_{i,j}=d}} p_\alpha(q)\cdot \bigwedge_{i=1}^{d-1}\bigwedge_{j=1}^{n_i}\bigwedge_{\ell=1}^{\alpha_{i,j}} w_{i,j}$$

for some polynomial functions $p_\alpha\colon \bigwedge^d(U)\to K$. □

Conclusion of the proof.

Proof of Theorem 3.4.1. To bound the strength of elements of $X(V)$ independently of V, by the induction assumption applied to Y, it suffices to bound the strength of elements of $Z(V)$ independently of V. Lemma 3.4.3, which focusses the attention to Z', and Lemma 3.4.4 together reduce this problem further to bounding the strength of elements q_d for all $(q_0,\dots,q_d)\in Z'(V)$ independently of V. Lemma 3.4.5 shows that such a q_d is contained in the image of a covariant. So Lemma 3.4.6 implies that the strength of q_d is bounded by

$$\sum_{i=1}^{\lfloor d/2\rfloor}\dim \textstyle\bigwedge^{d-i}(U),$$

which completes the proof. □

Remark 3.4.7. It follows from the induction that N from Theorem 3.4.1 can be taken equal to

$$\dim U+\sum_{i=1}^{\lfloor d/2\rfloor}\dim \textstyle\bigwedge^{d-i}(U).$$

♠

3.5 Bounded strength of ordinary tensors

In this section, we consider the d-variate polynomial functor $P := T_1\otimes\cdots\otimes T_d$. Let $V=(V_1,\dots,V_d)\in \mathrm{Vec}^d$. Then $P(V)=V_1\otimes\cdots\otimes V_d$. For tensors $q\in P(V)$, the strength $\mathrm{str}(q)$ is defined as the minimal number of terms k in any decomposition of the form

$$q=r_1\otimes s_1+\cdots+r_k\otimes s_k$$

where $r_i\in\bigotimes_{j\in J_i}V_j$ and $s_i\in\bigotimes_{j\in[d]\setminus J_i}V_j$ for suitable nonempty subsets $J_i\subsetneq[d]$. By taking all J_i equal to $\{\ell\}$, we obtain the bound $\mathrm{str}(q)\leq\dim V_\ell$ for any $\ell\in[d]$.

The goal of this section is to adapt the statement of Theorem 3.1.9 and its proof from Section 3.3 to the polynomial functor P. This time, we do not need to assume anything about the characteristic of K.

Theorem 3.5.1. *Fix* $d\in\mathbb{Z}_{\geq 2}$ *and assume that* K *is a perfect and infinite field. Then for any closed subset* $X\subsetneq P$ *there exists an* $N\geq 0$ *such that for all* $V\in\mathrm{Vec}^d$ *the strength of all elements in* $X(V)$ *is at most* N.

Fix a tuple of vector spaces $U \in \mathrm{Vec}^d$ such that $X(U) \subsetneq P(U)$.

Homogeneity. We equip the coordinate ring $K[P(V)]$ with the grading in which the elements of $P(V)^*$ have degree 1. For any closed $X \subseteq P$ we find, from the fact that $X(V)$ is stable under $\mathrm{GL}(V)$, that the ideal $\mathcal{I}(X(V)) \subseteq K[P(V)]$ is stable under $\mathrm{GL}(V)$ and in particular homogeneous. We define $\delta_X := \min\{\deg f \mid f \in \mathcal{I}(X(U)) \setminus \{0\}\}$.

Induction. If $\delta_X = 0$, then we find that $X(V) = \emptyset$ for all $V \in \mathrm{Vec}^d$. We may therefore assume that $\delta_X > 0$ and we proceed by induction, assuming that the theorem holds for all $Y \subseteq P$ with $Y(U) \subsetneq P(U)$ and $\delta_Y < \delta_X$.

Derivative. Let $f \in \mathcal{I}(X(U)) \setminus \{0\}$ be a homogeneous polynomial of degree δ_X. Then, there exists an $r \in P(U)$ such that the directional derivative

$$h := \frac{\partial f}{\partial r}$$

is not the zero polynomial and we may assume that $r = u_1 \otimes \cdots \otimes u_d$ for some $u_i \in U_i$.

We define the closed subset $Y \subsetneq P$ by

$$Y(V) := \{q \in X(V) \,\big|\, \forall \ell \in \mathrm{Hom}(V, U) : h(P(\ell)(q)) = 0\}\,.$$

Note that, by the induction hypothesis, the theorem holds for Y. We define

$$Z(V) := X(V) \setminus Y(V)$$

and prove that all elements in $Z(V)$ have strength bounded independently of V.

Shifting. For $V \in \mathrm{Vec}^d$ we define

$$\begin{aligned} P'(V) &:= P(U \oplus V) = \bigoplus_{J \subseteq [d]} \left(\bigotimes_{j \in [d] \setminus J} U_j \otimes \bigotimes_{j \in J} V_j \right), \\ X'(V) &:= X(U \oplus V) \subseteq P'(V), \\ Z'(V) &:= \{q \in X'(V) \,\big|\, h(P(\pi_U)(q)) \neq 0\}\,. \end{aligned}$$

We think of $P'(V), X'(V)$ as varieties over $P(U), X(U)$, respectively, via the linear map $P(\pi_U)$. We will write h for $h \circ P(\pi_U)$.

Lemma 3.5.2. *We have*

$$\sup_{\substack{V \in \mathrm{Vec}^d \\ q \in X(V)}} \mathrm{str}(q) = \sup_{V \in \mathrm{Vec}^d} \max \left\{ \sup_{q \in Y(V)} \mathrm{str}(q), \sup_{q \in Z'(V)} \mathrm{str}(q) \right\}.$$

□

Chopping. We write $n_\ell := \dim U_\ell$ for $\ell = 1, \ldots, d$.

Lemma 3.5.3. *For $q \in P'(V)$ write $q = \sum_{J \subseteq [d]} q_J$ with $q_J \in \bigotimes_{j \in [d] \setminus J} U_j \otimes \bigotimes_{j \in J} V_j$. Then*

$$\mathrm{str}(q) \leq n_1 + \cdots + n_d + \mathrm{str}(q_{[d]}).$$

□

Embedding. Define

$$Q'(V) := P'(V)/P(V) = \bigoplus_{J \subsetneq [d]} \left(\bigotimes_{j \in [d] \setminus J} U_j \otimes \bigotimes_{j \in J} V_j \right)$$

and write $\pi_{Q'(V)}: P'(V) \to Q'(V)$ for the natural projection. Take

$$B(V) := \{q \in Q'(V) \mid h(q) \neq 0\} = \{(q_J)_{J \subsetneq [d]} \in Q'(V) \mid h(q_\emptyset) \neq 0\}.$$

Then $\pi(V)$ maps $Z'(V)$ into $B(V)$.

An equivariant map back.

Lemma 3.5.4. *There exists a* $\mathrm{GL}(V)$*-equivariant polynomial map* $\Psi: Q'(V) \to P(V)$ *such that* $q_{[d]}$ *is a scalar multiple of* $\Psi((q_J)_{J \subsetneq [d]})$ *for all* $q = (q_J)_{J \subseteq [d]} \in Z'(V)$.

Proof. For $x = (x_1, \dots, x_d) \in V_1^* \times \cdots \times V_d^*$ and $t = (t_1, \dots, t_d) \in K^d$, consider the linear map

$$\begin{array}{rcl} \ell'_x(t): U \oplus V & \to & U \\ ((u_i)_i, (v_i)_i) & \mapsto & (u_i + t_i \ell_{x,i}(v_i))_i \end{array}$$

where $\ell_{x,i}: V_i \to U_i$ sends $v_i \mapsto x_i(v_i) u_i$ and $u_1, \dots, u_d$ are the vectors used in the definition of h. Note that $x \mapsto \ell_{x,i}$ is a $\mathrm{GL}(V)$-equivariant linear map.

Now take $\Phi_x(t) := P(\ell'_x(t)): P'(V) \to P(U)$. The restriction of $\Phi_x(t)$ to the summand

$$\bigotimes_{j \in [d] \setminus J} U_j \otimes \bigotimes_{j \in J} V_j$$

equals $\prod_{i \in J} t_i \cdot \Phi_{x,J}$ where $\Phi_{x,J}$ is the map $\bigotimes_{j \in [d] \setminus J} \mathrm{id}_{U_j} \otimes \bigotimes_{j \in J} \ell_{x,j}$. Note that $x \mapsto \Phi_{x,J}$ is a $\mathrm{GL}(V)$-equivariant polynomial map of degree $|J|$ and that $x \mapsto \Phi_{x,[d]}$ is multilinear.

By functoriality of X, we have $\Phi_x(t)(X'(V)) \subseteq X(U)$, and for $q = (q_J)_{J \subseteq [d]} \in P'(V)$ we find that

$$f(\Phi_x(t)((q_J)_{J \subseteq [d]})) = f\left(\sum_{J \subsetneq [d]} \left(\prod_{i \in J} t_i \cdot \Phi_{x,J}(q_J) \right) + t_1 \cdots t_d \cdot P((\ell_{x,i})_i)(q_{[d]}) \right),$$

and this expression vanishes for $q \in X'(V)$. The coefficient of $t_1 \cdots t_d$ in the Taylor expansion of this expression equals

$$h(q_\emptyset) \cdot (x_1 \otimes \cdots \otimes x_d)(q_{[d]}) + \Psi(x, (q_J)_{J \subsetneq [d]})$$

where the function $\Psi: V_1^* \times \cdots \times V_d^* \times Q'(V) \to K$ is $\mathrm{GL}(V)$-invariant and multilinear in $V_1^* \times \cdots \times V_d^*$. We note that for $q \in Z'(V)$, we have $h(q_\emptyset) \neq 0$ by definition of $Z'(V)$, and therefore

$$(x_1 \otimes \cdots \otimes x_d)(q_{[d]}) = -\frac{1}{h(q_\emptyset)} \Psi(x, (q_J)_{J \subsetneq [d]}).$$

The map Ψ factors as the composition of

$$\begin{array}{rcl} V_1^* \times \cdots \times V_d^* \times Q'(V) & \to & (V_1^* \otimes \cdots \otimes V_d^*) \times Q'(V) \\ (x, q') & \mapsto & (x_1 \otimes \cdots \otimes x_d, q') \end{array}$$

and a unique $\mathrm{GL}(V)$-equivariant map $(V_1^* \otimes \cdots \otimes V_d^*) \times Q'(V) \to K$. We denote the latter map also by Ψ, which is now linear in its first argument. If we re-interpret Ψ as a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to T^d V$, then

$$q_{[d]} = -\frac{1}{h(q_\emptyset)} \Psi((q_J)_{J \subsetneq [d]}).$$

for all $q \in Z'(V)$. □

Covariants. A covariant of $Q'(V)$ of order $P(V)$ is a $\mathrm{GL}(V)$-equivariant polynomial map $Q'(V) \to P(V)$. So the map Ψ constructed in Lemma 3.5.4 is a covariant. For each nonempty subset $J \subsetneq [d]$, choose a basis $u_{J,1}, \ldots, u_{J,n_J}$ of $\bigotimes_{j \in [d] \setminus J} U_j$. Then the map

$$\begin{array}{rcl} \Phi \colon \bigoplus_{\substack{J \subsetneq [d] \\ J \neq \emptyset}} \left(\bigotimes_{j \in J} V_j \right)^{\oplus n_J} & \to & \bigoplus_{\substack{J \subsetneq [d] \\ J \neq \emptyset}} \left(\bigotimes_{j \in [d] \setminus J} U_j \otimes \bigotimes_{j \in J} V_j \right) \\ (w_{J,\ell})_{J,\ell} & \mapsto & \left(\sum_{\ell=1}^{\prod_{j \in [d] \setminus J} \dim U_j} u_{J,\ell} \otimes w_{J,\ell} \right)_J . \end{array}$$

is a $\mathrm{GL}(V)$-equivariant isomorphism and the following lemma holds.

Lemma 3.5.5. *Let $\Psi \colon Q'(V) \to P(V)$ be a covariant. Then the composition*

$$\Psi \circ (\mathrm{id}_{P(U)}, \Phi) \colon P(U) \oplus \bigoplus_{\substack{J \subsetneq [d] \\ J \neq \emptyset}} \left(\bigotimes_{j \in J} V_j \right)^{\oplus n_J} \to P(V)$$

is given by

$$\left(q, (w_{J,\ell})_{J,\ell}\right) \mapsto \sum_{\{J_1, \ldots, J_k\} \in \mathcal{J}} \sum_{\ell_1 \in [n_{J_1}]} \cdots \sum_{\ell_k \in [n_{J_k}]} p_{\{J_1, \ldots, J_k\}, \ell_1, \ldots, \ell_k}(q) \cdot \bigotimes_{i=1}^{k} w_{J_i, \ell_i}$$

for some polynomial functions

$$p_{\{J_1, \ldots, J_k\}, \ell_1, \ldots, \ell_k} \colon P(U) \to K,$$

where $\mathcal{J}$ consists of all unordered partitions of $[d]$ into nonempty sets, i.e. all collections $\{J_1, \ldots, J_k\}$ of nonempty subsets $J_i \subsetneq [d]$ with $J_i \cap J_{i'} = \emptyset$ if $i \neq i'$ and $\bigcup_{i=1}^{k} J_i = [d]$. □

Conclusion of the proof.

Proof of Theorem 3.5.1. To bound the strength of elements of $X(V)$ independently of V, by the induction assumption applied to Y, it suffices to bound the strength of elements of $Z(V)$ independently of V. Lemma 3.5.2 and Lemma 3.5.3 together reduce this problem further to bounding the strength of elements $q_{[d]}$ for all $(q_J)_{J\subseteq[d]} \in Z'(V)$ independently of V. Lemma 3.5.4 shows that such a $q_{[d]}$ is contained in the image of a covariant. So using Lemma 3.5.5, we see that $q_{[d]}$ is a linear combination of tensor products of elements $w_{J,\ell} \in \bigotimes_{j\in J} V_j$ where J ranges over nonempty subsets of $[d]$ and j ranges from 1 to $\prod_{j\in[d]\setminus J} \dim U_j$. Fix an integer $m \in [d]$. Then we note for each $\{J_1, \dots, J_k\} \in \mathcal{J}$ that $m \in J_i$ for some $i \in [k]$. So the strength of $q_{[d]}$ is at most

$$
\begin{aligned}
\#\left\{ w_{J,\ell} \,\middle|\, m \in J \subsetneq [d], \ell \in \left[\prod_{j\in[d]\setminus J} \dim U_j\right]\right\} &\leq \sum_{\substack{J\subsetneq[d]\\ J\ni m}} \prod_{j\in[d]\setminus J} \dim U_j \\
&= \sum_{\substack{J'\subsetneq[d]\setminus\{m\}\\ J'\neq\emptyset}} \prod_{j\in J'} \dim U_j \\
&= \prod_{j\in[d]\setminus\{m\}} (\dim U_j + 1) - 1
\end{aligned}
$$

and the latter expression is minimized over m when $\dim U_m$ is maximal. This bounds the strength of $q_{[d]}$ independently of V. □

Remark 3.5.6. It follows from the induction that N from Theorem 3.5.1 can be taken equal to

$$n_1 + \cdots + n_d + \prod_{j\in[d]\setminus\{m\}} (n_j + 1) - 1$$

where $n_\ell = \dim U_\ell$ and where $m \in [d]$ such that $n_m \geq n_\ell$ for all $\ell \in [d]$. ♠

Remark 3.5.7. The definitions of strength we have used have the following common generalisation: For integers $0 \leq m \leq n$, $d_1, \dots, d_n \in \mathbb{N}$ with $\sum_i d_i \geq 2$ and vector spaces $V_1, \dots, V_n \in \mathrm{Vec}$, the strength $\mathrm{str}(q)$ of a composite tensor

$$q \in S^{d_1}(V_1) \otimes \cdots \otimes S^{d_m}(V_m) \otimes \textstyle\bigwedge^{d_{m+1}}(V_{m+1}) \otimes \cdots \otimes \bigwedge^{d_n}(V_n)$$

is the minimal number of terms k in any composition of the form

$$q = r_1 s_1 + \cdots + r_k s_k$$

where

$$
\begin{aligned}
r_i &\in S^{e_1}(V_1) \otimes \cdots \otimes S^{e_m}(V_m) \otimes \textstyle\bigwedge^{e_{m+1}}(V_{m+1}) \otimes \cdots \otimes \bigwedge^{e_n} V_n \\
s_i &\in S^{d_1-e_1}(V_1) \otimes \cdots \otimes S^{d_m-e_m}(V_m) \otimes \textstyle\bigwedge^{d_{m+1}-e_{m+1}}(V_{m+1}) \otimes \cdots \otimes \bigwedge^{d_n-e_n} V_n
\end{aligned}
$$

for suitable $0 \leq e_i \leq d_i$ with $(e_1, \dots, e_n) \neq (0, \dots, 0), (d_1, \dots, d_n)$.

A version of Theorems 3.1.9, 3.4.1 and 3.5.1 for composite tensors generalising the three versions exists. A proof of this version can be obtained by modifying the proof in this section. The most important changes are:

(a) we must assume that $\mathrm{char}\, K = 0$ or $\mathrm{char}\, K > d_i$ for all $i \in \{1, \dots, n\}$;

(b) we take $h := \frac{\partial f}{\partial r}$ where $r = r_1 \otimes \cdots \otimes r_n$ with $r_i = u_i^{d_i}$, $u_i \in U_i$ for $i \leq m$ and $r_i = u_{i,1} \wedge \cdots \wedge u_{i,d_i}$, $u_{i,j} \in U_i$ for $i > m$; and

(c) for $i \le m$ we take $x_i \in V_i^*$ and $t_i \in K$, for $i > m$ we take $x_{i,j} \in V_i^*$ and $t_{i,j} \in K$ and we let $\Phi_x(t) = \Phi^{(1)}_{x_1}(t_1) \otimes \cdots \otimes \Phi^{(n)}_{x_n}(t_n)$ where $\Phi^{(i)}_{x_i}(t_i)$ is the map from the symmetric case for $i \le m$ and the map from the alternating case for $i > m$.

In addition, the bounds must be adjusted to (more complicated) expressions. ♠

3.6 Bounded strength over $\mathbb{Z}$

Theorems 3.1.9, 3.4.1 and 3.5.1 require that K be fixed in advance and allow for the closed subsets of $S^d, \bigwedge^d, T_1 \otimes \cdots \otimes T_d$ to be defined by equations specific to K. The price that we pay for this generality is that we need to require K to be perfect and infinite and that the values of N in these theorems depend on K.

Indeed, in the proofs, perfectness of the field is used to ensure that a squarefree nonzero polynomial has some nonzero directional derivative. And, infiniteness of the field is used to ensure that if some polynomial in t vanishes for all $t \in K$, then the coefficients of all monomials t^d vanish. We can get around both of these restrictions by working only with tensor properties defined over $\mathbb{Z}$ before specialising to K.

Let $\mathrm{Vec}_{\mathbb{Z}}$ be the category of finite-rank free $\mathbb{Z}$-modules with $\mathbb{Z}$-linear maps. Every object $V \in \mathrm{Vec}_{\mathbb{Z}}$ gives rise to an affine scheme, the spectrum of the symmetric algebra (over $\mathbb{Z}$) on the module dual to V. By abuse of notation, we write V for this scheme as well. The scheme of a product $V \times W$ is canonically isomorphic to the product of the schemes and an $\ell \in \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{Z}}}(V, W)$ determines a morphism of schemes $V \to W$.

A module $V \in \mathrm{Vec}_{\mathbb{Z}}$ has a symmetric power $S^d_{\mathbb{Z}}(V) \in \mathrm{Vec}_{\mathbb{Z}}$ characterised by the usual universal property. A closed subscheme of $S^d_{\mathbb{Z}}$ is a rule $X_{\mathbb{Z}}$ that assigns to each $V \in \mathrm{Vec}_{\mathbb{Z}}$ a closed subscheme of $S^d_{\mathbb{Z}}(V)$ in such a manner that for $V, W \in \mathrm{Vec}_{\mathbb{Z}}$ and $\ell \in \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{Z}}}(V, W)$ the morphism $S^d_{\mathbb{Z}}(\ell)$ maps $X_{\mathbb{Z}}(V)$ into $X_{\mathbb{Z}}(W)$. This is equivalent to the condition that the morphism of schemes determined by

$$\begin{array}{rcl} S^d_{\mathbb{Z}}(V) \times \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{Z}}}(V, W) & \to & S^d_{\mathbb{Z}}(W) \\ (v_1 \cdots v_d, \ell) & \mapsto & \ell(v_1) \cdots \ell(v_d) \end{array}$$

maps $X_{\mathbb{Z}}(V) \times \mathrm{Hom}_{\mathrm{Vec}_{\mathbb{Z}}}(V, W)$ into $X_{\mathbb{Z}}(W)$.

In terms of equations this means the following: Suppose that $V = \mathbb{Z}^m$ and $W = \mathbb{Z}^n$, let f be any polynomial in the $\binom{n-1+d}{d}$ standard coordinates on $S^d_{\mathbb{Z}}(W)$ with coefficients in $\mathbb{Z}$ and let ℓ be an $n \times m$ matrix whose entries ℓ_{ij} are variables. Then one can expand $f \circ S^d_{\mathbb{Z}}(\ell)$ as a polynomial $\sum_{\alpha \in \mathbb{Z}^{n \times m}_{\ge 0}} c_\alpha \ell^\alpha$ in the ℓ_{ij} whose coefficients c_α are polynomials in the $\binom{m-1+d}{d}$ standard coordinates on $S^d_{\mathbb{Z}}(V)$. The condition above says that if f is in the ideal of $X_{\mathbb{Z}}(W)$, then all the c_α lie in the ideal of $X_{\mathbb{Z}}(V)$.

If $X_{\mathbb{Z}}$ is a closed subscheme of $S^d_{\mathbb{Z}}$, then for each field K we obtain a closed subset X_K of $S^d = S^d_K$ as follows: for $V \in \mathrm{Vec} = \mathrm{Vec}_K$ choose any linear isomorphism $\ell \colon V \to K^n$ and let $X_K(V)$ be the preimage under $S^d(\ell)$ of the set of K-valued points of the scheme $X(\mathbb{Z}^n) \subseteq S^d(\mathbb{Z}^n)$.

Remark 3.6.1. We have

$$X_L(V \otimes_K L) \cap S^d(V) = X_K(V)$$

for all field extensions $K \subseteq L$ and all vector spaces $V \in \mathrm{Vec}_K$. ♠

Theorem 3.6.2. *Let $d \in \mathbb{Z}_{\geq 2}$ and let $X_{\mathbb{Z}}$ be a closed subscheme of $S^d_{\mathbb{Z}}$. Then there exists an $N \geq 0$ such that the following holds:*

(†) *Let K be any field with $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ such that $X_K \subsetneq S^d_K$. Then for all $V \in \mathrm{Vec}_K$ the strength of all elements in $X_K(V)$ is at most N.*

□

The $\mathbb{Z}$-constructions in this subsection have analogues for the polynomial functors $\bigwedge^d$ and $T_1 \otimes \cdots \otimes T_d$. And, the analogues of Theorems 3.4.1 and 3.5.1 also hold over $\mathbb{Z}$.

Theorem 3.6.3. *Let X be a closed subscheme of $\bigwedge^d_{\mathbb{Z}}$. Then there exists an $N \geq 0$ such that the following holds:*

(†) *Let K be any field with $\operatorname{char} K = 0$ or $\operatorname{char} K > d$ such that $X_K \subsetneq \bigwedge^d_K$. Then for all $V \in \mathrm{Vec}_K$ the strength of all elements in $X_K(V)$ is at most N.*

□

Theorem 3.6.4. *Let X be a closed subscheme of $T_{1,\mathbb{Z}} \otimes \cdots \otimes T_{d,\mathbb{Z}}$. Then there exists an $N \geq 0$ such that the following holds:*

(†) *Let K be any field such that $X_K \subsetneq T_{1,K} \otimes \cdots \otimes T_{d,K}$. Then for all $V \in \mathrm{Vec}^d_K$ the strength of all elements in $X_K(V)$ is at most N.*

□

Chapter 4

The geometry of polynomial functors

This chapter is based on work [8] with Jan Draisma, Rob Eggermont and Andrew Snowden. We let K be an algebraically closed field of characteristic zero, we let $\mu \in \mathbb{N}$ be an integer and we assume all polynomial functors to be μ-variate and of finite degree. From Section 4.5 onward, we restrict to the case where $\mu = 1$ holds.

4.1 Introduction

Let P be a polynomial functor. If the degree of P equals 0, then $P(V) = U$ for some fixed vector space $U \in \mathrm{Vec}$ and the closed subsets of P are the same as the Zariski-closed subsets of U. The goal of this chapter is to also study closed subsets of polynomial functors of positive degree and thereby extend the field of affine finite-dimensional algebraic geometry.

We start with the technical heart of this chapter, which is Theorem 4.2.5. When studying matrices/polynomials/tensors, one of the questions to ask is always: how can we express a matrix/polynomial/tensor using simpler objects? In a way, Theorem 4.2.5 proves that attempting to do this is a good idea. The theorem states that, given a closed subset X of a polynomial functor P, there are two cases:

(1) We have $X = P$.

(2) The subset X is covered by images from smaller polynomial functors.

So families of objects that share a common structure must have an uniform bound on the number of simpler objects needed to express them.

Since Theorem 4.2.5 has two mutually exclusive cases, we call it the Dichotomy Theorem. This theorem helps us to set up the theory of GL_∞-equivariant infinite-dimensional affine algebraic geometry in two ways: first, we can use the Dichotomy Theorem as a tool to do induction on polynomial functors. This allows us to extend theorems from the base case, which is finite-dimensional affine algebraic geometry, to all polynomial functors. As a first example, we get an easy proof of the following theorem.

Theorem 4.1.1 (Draisma [17, Theorem 1]). *Let P be a polynomial functor of finite degree. Then every descending chain of closed subsets of P stabilizes.*

Proof. Let

$$P \supseteq X_1 \supseteq X_2 \supseteq X_3 \supseteq X_4 \supseteq \dots$$

be a descending chain of closed subsets of P. There are two cases: either $X_n = P$ for all $n \in \mathbb{N}$ or $X_n \neq P$ for some $n \in \mathbb{N}$. In the first case, we are done. In the second case, we may assume that $n = 1$ by removing the first $n-1$ closed subsets from the chain. Now, we have $X_1 \neq P$ and hence X_1 is covered by images from smaller polynomial functors, i.e., there exist finitely many polynomial transformations $\alpha_i\colon Q_i \to P$ such that $X_1 \subseteq \bigcup_i \mathrm{im}(\alpha_i)$. Take $Y_{i,n} = \alpha_i^{-1}(X_n)$ for all i and $n \in \mathbb{N}$. Then we get a chain of closed subsets of Q_i

$$Q_i \supseteq Y_{i,1} \supseteq Y_{i,2} \supseteq Y_{i,3} \supseteq Y_{i,4} \supseteq \dots$$

for every i. As the Q_i are smaller than P, these chains stabilize by the induction hypothesis. Here the base case consists of polynomial functors of degree zero and is hence implied by Hilbert's Basis Theorem. As there are only finitely many i, the chains must stabilize at some common point. As $X_n = \bigcup_i \alpha_i(Y_{i,n})$ for each $n \in \mathbb{N}$, we see that therefore the original chain in P also stabilizes. □

As a second example of a result we can prove using the Dichotomy Theorem as a tool for induction, we prove a version of Chevalley's Theorem on the constructibility of images of constructible sets. See Theorem 4.6.7.

The second use of the Dichotomy Theorem is that it provides a basis for the study of the lattice of orbit closures in P_∞. When the degree of P is zero, then this lattice consists of infinitely many disconnected points. However, when P is pure and of positive degree, we will see that this lattice is more interesting. Let $p \in P_\infty$ be a point. Then its orbit closure $\overline{\mathrm{GL}_\infty \cdot p}$ corresponds to the smallest closed subset $X \subseteq P$ such that $\mathrm{pr}_n(p) \in X_n$ for all $n \in \mathbb{N}$. Now, we again have two cases: either $X = P$ or $X \neq P$. In the first case, we call the point p a GL_∞-generic point. In the second case, the subset X is covered by images from smaller polynomial functors and therefore $p = \alpha(q)$ for some polynomial functor $Q < P$ and some polynomial transformation $\alpha\colon Q \to P$ and some point $q \in Q_\infty$. As long as the point q is not GL_∞-generic, we continue to write it as the image of a point coming from a smaller polynomial functor. Since the ordering on polynomial functors is well-founded, this can only continue for finitely many steps. We conclude that every point p is the image of a GL_∞-generic point $q \in Q_\infty$. Studying the minimal Q for which this is the case leads to the definition of the type of p, which we will define in Section 4.5. Our main result here is that the lattices of orbit closures in P_∞ and of types of points in P_∞ are the same.

Outline of this chapter. In the next two sections, we state and prove the Dichotomy Theorem. Then, before we move on to the applications of the Dichotomy Theorem, we have a short intermission where we give more details on the structure of dominant polynomial transformations between two polynomial functors. The sections after that are about types of points and about a variant of Chevalley's Theorem for our setting. We conclude with a section filled with interesting examples and open questions.

4.2 Ordering polynomial functors

Definition 4.2.1. Let P be a polynomial functor. We define the magnitude of P to be the sequence $\mathrm{mag}(P) := (n_0, n_1, n_2, \ldots)$ where n_d is the number of irreducible components of the $\mathrm{GL}(V)$-module $P_{(d)}(V)$ for all $V \in \mathrm{Vec}^{\mu}$ with $\dim(V_i) \gg 0$ for all $i \in [\mu]$.

Remark 4.2.2. The magnitude of P is well-defined by Lemma 1.3.35. ♠

Let P, Q be polynomial functors with magnitudes $\mathrm{mag}(P) = (n_1, n_2, \ldots)$ and $\mathrm{mag}(Q) = (m_1, m_2, \ldots)$. We compare the magnitudes of P and Q lexicographically, i.e., we say that $\mathrm{mag}(Q) < \mathrm{mag}(P)$ when $\mathrm{mag}(Q) \neq \mathrm{mag}(P)$ and $m_d < n_d$ where $d \in \mathbb{N}$ is maximal with the property that $m_d \neq n_d$. Note that this ordering is well-founded.

Definition 4.2.3. A polynomial functor P is called pure when $P_{(0)} = P(0) = 0$.

Write $Q = Q_{(0)} \oplus Q'$, consider a polynomial transformation $\alpha\colon Q \to P$ and let

$$q = (q_0, q') \in Q_\infty = Q_{(0)} \oplus Q'_\infty$$

be a point with image $p = \alpha_\infty(q)$ in P_∞. Then the polynomial transformation $\beta\colon Q' \to P$ defined by the maps

$$\begin{array}{rcl} Q'(V) & \to & P(V) \\ x & \mapsto & \alpha_V(q_0, x) \end{array}$$

also maps the point q' to p. So if a point in P_∞ lies in the image of a polynomial transformation from Q, then it also lies in the image of a polynomial transformation from the pure polynomial functor Q' whose magnitude only differs from the magnitude of Q in its first entry.

Definition 4.2.4. Define the partial ordering $\leq$ on the set of isomorphism classes of polynomial functors by $Q < P$ when $Q \not\cong P$ and $Q_{(d)}$ is a quotient of $P_{(d)}$ where $d \in \mathbb{N}$ is maximal with the property that $Q_{(d)} \not\cong P_{(d)}$.

Note that when $Q < P$, we have also $\mathrm{mag}(Q) < \mathrm{mag}(P)$. So $\leq$ is well-founded.

Theorem 4.2.5 (Dichotomy Theorem)**.** *Assume that K is an algebraically closed field of characteristic 0 and let X be a closed subset of a μ-variate polynomial functor P of finite degree. Then either $X = P$ or there exist a finite number of finite-dimensional affine varieties A_i, pure μ-variate polynomial functors $Q_i < P$ and regular transformations*

$$\alpha_i\colon A_i \times Q_i \to P$$

such that $X(V) = \bigcup_i \mathrm{im}(\alpha_{i,V})$ for all $V \in \mathrm{Vec}^{\mu}$ and $X_\infty = \bigcup_i \mathrm{im}(\alpha_{i,\infty})$.

4.3 The proof of the Dichotomy Theorem

Similar to the proof of Draisma in [17], we will prove the Dichotomy Theorem using induction on P. Note that, when the degree of P is 0, any nonempty closed subset $X \subsetneq P$ is itself a finite-dimensional affine variety and so we can choose $Q = 0$. So we assume that P has positive degree. Write $P = Q \oplus R$ for some irreducible polynomial functor $R \subseteq P_{(d)}$ where $d = \deg(P)$. Then we have $Q < P$. If $X = Y \times R$ for some closed subset $Y \subseteq Q$, then there exist a finite number of finite-dimensional affine varieties A_i, pure polynomial functors $Q_i \leq Q$ and regular transformations

$$\alpha_i \colon A_i \times Q_i \to Q$$

such that $Y(V) = \bigcup_i \operatorname{im}(\alpha_{i,V})$ for all $V \in \mathrm{Vec}^\mu$ and $Y_\infty = \bigcup_i \operatorname{im}(\alpha_{i,\infty})$. In this case, we see that the regular transformations

$$\beta_i \colon A_i \times (Q_i \oplus R) \to P$$

defined by $\beta_{i,V}(a,q,r) = (\alpha_{i,V}(a,q),r)$ satisfy $X(V) = \bigcup_i \operatorname{im}(\beta_{i,V})$ for all $V \in \mathrm{Vec}^\mu$ and $X_\infty = \bigcup_i \operatorname{im}(\beta_{i,\infty})$. Here we take the identity transformation $Q \to Q$ if $Y = Q$. So we suppose that such a closed subset Y does not exists.

Fix a $U \in \mathrm{Vec}^\mu$ such that the ideal $\mathcal{I}(X(U)) \subseteq K[P(U)]$ is not generated by $\mathcal{I}(X(U)) \cap K[Q(U)]$ and let Y be a closed subset of P. Note that $\mathcal{I}(Y(U))$ is homogeneous with respect to the grading where nonzero elements of $P_{(e)}(U)^*$ have degree e. If the ideal $\mathcal{I}(Y(U)) \subseteq K[P(U)]$ is not generated by $\mathcal{I}(Y(U)) \cap K[Q(U)]$, then we define δ_Y as the minimal degree of an element of $\mathcal{I}(Y(U))$ that is not contained in the ideal generated by $\mathcal{I}(Y(U)) \cap K[Q(U)]$. Now, let $\preceq$ be the partial ordering on closed subsets Y of P defined by $Y_1 \prec Y_2$ if either the closure of the projection of Y_1 to Q is strictly contained in the closure of the projection of Y_2 or both closures are equal and $\delta_{Y_1} < \delta_{Y_2}$. The partial ordering $\preceq$ is well-founded since Q is Noetherian by the induction hypothesis and Theorem 4.1.1. We will also do induction on $\preceq$.

Let $f \in \mathcal{I}(X(U))$ be a polynomial of degree $\delta_X > 0$ that is not contained in the ideal generated by $\mathcal{I}(X(U)) \cap K[Q(U)]$. Then there is a vector $r \in R(U)$ such that

$$h = \frac{\partial f}{\partial r}$$

is not the zero polynomial. We let Y be the biggest closed subset of X such that $h \in \mathcal{I}(Y(U))$. Now either h is contained in the ideal generated by $\mathcal{I}(Y(U)) \cap K[Q(U)]$ or it is not. In both cases, we see that $Y \prec X$. So we can cover Y with images of regular transformations from smaller polynomial functors. We will next construct finitely many additional regular transformations that cover $Z(V) := X(V) \setminus Y(V)$ for all $V \in \mathrm{Vec}^\mu$ and also cover $Z_\infty := X_\infty \setminus Y_\infty$.

Take $P' = P \circ \mathrm{Sh}_U$ where $\mathrm{Sh}_U \colon \mathrm{Vec}^\mu \to \mathrm{Vec}^\mu$ is the functor assigning $U \oplus V$ to $V \in \mathrm{Vec}^\mu$ and $\mathrm{id}_U \oplus \ell$ to a morphism ℓ. Then P' is a polynomial functor of degree d and $P'_{(d)} = P_{(d)}$. Write $P' = Q' \oplus R$ and note that $Q' < P$. Take $Z'(V) := \{p' \in X(U \oplus V) \mid h(P(\pi_U)(p')) \neq 0\}$ for each $V \in \mathrm{Vec}^\mu$ and $Z'_\infty := \{(p'_n)_n \in P'_\infty \mid \forall n : p'_n \in Z'(K^n, \dots, K^n)\}$. We consider the polynomial transformation $\gamma \colon P' \to P$ given by the maps

$$\gamma_V := P(\pi_V) \colon P'(V) = P(U \oplus V) \to P(V).$$

First note that $\gamma_V(Z'(V)) \subseteq X(V)$ for all $V \in \mathrm{Vec}^\mu$. Second, note that for all $V \in \mathrm{Vec}^\mu$ and $p \in Z(V)$, there is a morphism $\ell \colon V \to U$ such that $h(P(\ell)(p)) \neq 0$ and the element

$$p' = P(\ell')(p) \in Z'(V), \quad \ell' \colon V \to U \oplus V, \quad \ell'_i(v_i) = (\ell(v_i), v_i) \text{ for } i \in [\mu]$$

is mapped to p by γ_V. So we have $X(V) \setminus Y(V) = Z(V) \subseteq \gamma_V(Z'(V)) \subseteq X(V)$ for all $V \in \mathrm{Vec}^\mu$. Next, for a point $p = (p_n)_n \in X_\infty \setminus Y_\infty$, there is an $m \in \mathbb{N}$ such that $p_m \in Z(K^m, \dots, K^m)$. Let $\ell\colon (K^m, \dots, K^m) \to U$ be a morphism such that $h(P(\ell)(p_m)) \neq 0$. Then the elements

$$p'_n = P(\ell')(p_n) \in Z'(K^n), \quad \ell'\colon (K^n)_i \to (U_i \oplus K^n)_i, \quad \ell'_i(v) = (\ell_i(\mathrm{pr}_m(v)), v) \text{ for } i \in [\mu]$$

map to the elements p_n and $P'(\mathrm{pr}_n, \dots, \mathrm{pr}_n)(p'_{n+1}) = p'_n$ for all $n \geq m$. So the sequence $(p'_n)_{n\geq m}$ defines a point in Z'_∞ that is mapped to p. We also see that $\gamma_\infty(p') \in X_\infty$ for all $p' \in Z'_\infty$. So we also have $X_\infty \setminus Y_\infty = Z_\infty \subseteq \gamma_\infty(Z'_\infty) \subseteq X_\infty$. Hence it suffices to contruct a finite number of finite-dimensional affine varieties B_j, pure polynomial functors $R_j < P$ and regular transformations

$$\beta_j\colon B_j \times R_j \to P'$$

such that $Z'(V) = \bigcup_j \mathrm{im}(\beta_{j,V})$ for all $V \in \mathrm{Vec}^\mu$ and $Z'_\infty = \bigcup_j \mathrm{im}(\beta_{j,\infty})$. Note that, for each $V \in \mathrm{Vec}^\mu$, the map $h \circ P(\pi_U)\colon P'(V) \to K$ is the composition of the projection map $\pi_{Q'(V)}\colon P'(V) \to Q'(V)$ with the map $h \circ P(\pi_U) \circ \iota_{Q'(V)\subseteq P'(V)}\colon Q'(V) \to K$. Take

$$Z''(V) := \left\{ q \in Q'(V) \,\middle|\, (h \circ P(\pi_U) \circ \iota_{Q'(V)\subseteq P'(V)})(q) \neq 0 \right\}$$

for each $V \in \mathrm{Vec}^\mu$. Then the restriction $\pi_{Q'(V)}|_{Z'(V)}\colon Z'(V) \to Z''(V)$ is a closed embedding by [17, Lemma 21]. We view the codomain $Z''(V)$ as a closed subset of $K \oplus Q'(V)$ and in this manner we obtain a closed subset Z'' of $K \oplus Q'$. The regular transformation $Z'' \to P'$ given by the maps $\pi_{Q'(V)}|^{-1}_{Z'(V)}$ extends to a polynomial transformation $\beta\colon K \oplus Q' \to P'$ such that $\beta_V(Z''(V)) = Z'(V)$ for all $V \in \mathrm{Vec}^\mu$ and $\beta_\infty(Z''_\infty) = Z'_\infty$. Since $K \oplus Q' < P$, we know that there are a finite number of finite-dimensonal affine varieties B_j, pure polynomial functors $R_j \leq K \oplus Q'$ and regular transformations

$$\beta'_j\colon B_j \times R_j \to K \oplus Q'$$

such that $Z''(V) = \bigcup_j \mathrm{im}(\beta_{j,V})$ for all $V \in \mathrm{Vec}^\mu$ and $Z''_\infty = \bigcup_j \mathrm{im}(\beta_{j,\infty})$. So the regular transformations

$$\beta_j := \beta \circ \beta'_j\colon B_j \times R_j \to P'$$

satisfy the desired properties. This concludes the proof.

4.4 The structure of dominant polynomial transformations

Let P, Q be pure polynomial functors. The goal of this section is to better understand the polynomial transformations $\alpha\colon Q \to P$ that are dominant, i.e., such that α_V is dominant for all $V \in \mathrm{Vec}^\mu$. In particular, we want to understand the structure of the group $\mathrm{Aut}(P)$ of polynomial automorphisms $P \to P$.

Consider a polynomial transformation

$$\alpha\colon Q = \bigoplus_\lambda S_\lambda^{\oplus m_\lambda} \to P = \bigoplus_\lambda S_\lambda^{\oplus n_\lambda}$$

and fix a λ. Then the composition

$$\pi_{S_\lambda^{\oplus n_\lambda}} \circ \alpha\colon Q \to S_\lambda^{\oplus n_\lambda}$$

only depends on

$$\bigoplus_{\lambda':|\lambda'|<|\lambda|} S_{\lambda'}^{\oplus m_{\lambda'}} \oplus S_{\lambda}^{\oplus m_{\lambda}}$$

and is the sum of a polynomial transformation

$$\beta_\lambda\colon \bigoplus_{\lambda':|\lambda'|<|\lambda|} S_{\lambda'}^{\oplus m_{\lambda'}} \to S_{\lambda}^{\oplus n_{\lambda}}$$

and a linear transformation $\ell_\lambda\colon S_\lambda^{\oplus m_\lambda} \to S_\lambda^{\oplus n_\lambda}$. This linear transformation ℓ_λ corresponds to a matrix $A_\lambda \in K^{n_\lambda \times m_\lambda}$. We start with the following lemma.

Lemma 4.4.1. *The following statements are equivalent:*

(1) The polynomial transformation $\pi_{S_\lambda^{\oplus n_\lambda}} \circ \alpha$ is dominant.

(2) The polynomial transformation $\pi_{S_\lambda^{\oplus n_\lambda}} \circ \alpha$ is surjective.

(3) The linear transformation ℓ_λ is surjective.

(4) The matrix A_λ has rank n_λ.

Proof. Clearly, we have (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). So it suffices to prove that (1) $\Rightarrow$ (3) holds. Suppose that (1) holds and that ℓ_λ is not surjective. Then there exists a surjective linear transformation $\ell'\colon S_\lambda^{n_\lambda} \to S_\lambda$ such that $\ell' \circ \ell_\lambda = 0$. So then

$$\ell' \circ \pi_{S_\lambda^{\oplus n_\lambda}} \circ \alpha = \ell' \circ \beta_\lambda\colon \bigoplus_{\lambda':|\lambda'|<|\lambda|} S_{\lambda'}^{\oplus m_{\lambda'}} \to S_\lambda$$

must be a dominant polynomial transformation. Note that for $V \in \mathrm{Vec}$, the dimensions of

$$\bigoplus_{\lambda':|\lambda'|<|\lambda|} S_{\lambda'}(V, \dots, V)^{\oplus m_{\lambda'}}$$

and of $S_\lambda(V, \dots, V)$ are polynomials in the dimension of V. As the former has a lower degree than the latter, we see that $(\ell' \circ \beta_\lambda)_{(V,\dots,V)}$ cannot be dominant when the dimension of V is big enough. Hence ℓ_λ must be surjective. $\square$

When $\alpha\colon Q \to P$ is dominant, it follows that

$$\pi_{S_\lambda^{\oplus n_\lambda}} \circ \alpha\colon Q \to S_\lambda^{\oplus n_\lambda}$$

is dominant for all λ. It follows that $\alpha = \beta \circ \ell$ where $\ell \in \mathrm{GL}(Q)$ is the linear automorphism made up out of the linear automorphisms $\ell_\lambda^{-1} \in \mathrm{GL}(S_\lambda^{m_\lambda})$ and $\beta\colon Q \to P$ is a polynomial transformation given by maps

$$\begin{aligned} \beta_V\colon \bigoplus_\lambda S_\lambda(V)^{\oplus m_\lambda} &\to \bigoplus_\lambda S_\lambda(V)^{\oplus n_\lambda} \\ q = ((q_{\lambda,1}, \dots, q_{\lambda,m_\lambda}))_\lambda &\mapsto ((q_{\lambda,1} + \beta_{\lambda,1,V}(\pi_{<\lambda}(q)), \dots, q_{\lambda,n_\lambda} + \beta_{\lambda,n_\lambda,V}(\pi_{<\lambda}(q))))_\lambda \end{aligned}$$

for $V \in \mathrm{Vec}^\mu$ where

$$\beta_{\lambda,i}\colon \bigoplus_{\lambda':|\lambda'|<|\lambda|} S_{\lambda'}^{\oplus m_{\lambda'}} \to S_\lambda$$

are polynomial transformations and where

$$\pi_{<\lambda}\colon Q \to \bigoplus_{\lambda' : |\lambda'|<|\lambda|} S_{\lambda'}^{\oplus m_{\lambda'}}$$

is the natural projection. In particular, we see that $m_\lambda \geq n_\lambda$ for all λ.

Remark 4.4.2. Suppose that $\alpha\colon Q \to P$ is a polynomial isomorphism. Then it follows that $m_\lambda = n_\lambda$ for all λ and hence that Q and P are also linearly isomorphic.

Definition 4.4.3. We call ℓ and β the linear and affine parts of α.

Definition 4.4.4. We call a polynomial transformation $\beta\colon Q \to P$ of the form above an affine polynomial transformation.

Definition 4.4.5. Let $\beta\colon Q \to P$ be an affine polynomial transformation and suppose that $\beta_{\lambda,i} = 0$ for all but one pair (λ, i). Then we call β an elementary affine polynomial transformation.

Example 4.4.6. Take $Q = S^1 \oplus (S^2)^{\oplus 2}$ and consider the polynomial transformation $\alpha \in \mathrm{Aut}(Q)$ given by the maps

$$\begin{array}{rcl} \alpha_V\colon V \oplus S^2(V) \oplus S^2(V) & \to & V \oplus S^2(V) \oplus S^2(V) \\ (v, A, B) & \mapsto & (2v, A + B + v^2, A - B - 3v^2) \end{array}$$

for $V \in \mathrm{Vec}$. We have $\alpha = \beta \circ \ell = \ell \circ \beta'$ where $\ell \in \mathrm{GL}(Q)$ and $\beta, \beta' \in \mathrm{Aut}(Q)$ are given by the maps

$$\begin{array}{rcl} \ell_V\colon V \oplus S^2(V) \oplus S^2(V) & \to & V \oplus S^2(V) \oplus S^2(V) \\ (v, A, B) & \mapsto & (2v, A + B, A - B), \end{array}$$

$$\begin{array}{rcl} \beta_V\colon V \oplus S^2(V) \oplus S^2(V) & \to & V \oplus S^2(V) \oplus S^2(V) \\ (v, A, B) & \mapsto & (v, A + v^2, B - 3v^2) \end{array}$$

and

$$\begin{array}{rcl} \beta'_V\colon V \oplus S^2(V) \oplus S^2(V) & \to & V \oplus S^2(V) \oplus S^2(V) \\ (v, A, B) & \mapsto & (v, A - v^2, B + 2v^2) \end{array}$$

for $V \in \mathrm{Vec}$. Here we note that that the matrices

$$(2), \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

are invertible and that the solution of the system of equations

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

is $x = (-1, 2)^T$. ♣

Example 4.4.7. Take $Q = S^1 \oplus S^2 \oplus S^3$ and consider the polynomial transformation $\beta \in \mathrm{Aut}(Q)$ given by the maps

$$\begin{array}{rcl}\beta_V\colon V \oplus S^2(V) \oplus S^3(V) & \to & V \oplus S^2(V) \oplus S^3(V)\\ (v,q,f) & \mapsto & (v, q+v^2, f+vq)\end{array}$$

for $V \in \mathrm{Vec}$. We have $\beta = \beta_2 \circ \beta_1$ where $\beta_1, \beta_2 \in \mathrm{Aut}(Q)$ are given by the maps

$$\begin{array}{rcl}\beta_{1,V}\colon V \oplus S^2(V) \oplus S^3(V) & \to & V \oplus S^2(V) \oplus S^3(V)\\ (v,q,f) & \mapsto & (v, q, f+vq)\end{array}$$

and

$$\begin{array}{rcl}\beta_{2,V}\colon V \oplus S^2(V) \oplus S^3(V) & \to & V \oplus S^2(V) \oplus S^3(V)\\ (v,q,f) & \mapsto & (v, q+v^2, f)\end{array}$$

for $V \in \mathrm{Vec}$. ♣

Definition 4.4.8. The affine polynomial automorphisms $\beta\colon Q \to Q$ form a subgroup of $\mathrm{Aut}(Q)$. We denote this subgroup by $\mathrm{Aff}(Q)$.

Lemma 4.4.9. *The subgroup* $\mathrm{Aff}(Q)$ *of* $\mathrm{Aut}(Q)$ *is normal.*

Proof. As $\mathrm{Aut}(Q)$ is generated by $\mathrm{GL}(Q)$ and $\mathrm{Aff}(Q)$ by Lemma 4.4.1, this follows from the fact that $\ell \circ \mathrm{Aff}(Q) \circ \ell^{-1} = \mathrm{Aff}(Q)$ for all $\ell \in \mathrm{GL}(Q)$. □

Lemma 4.4.10. *Every affine polynomial transformation $Q \to P$ is a composition of the polynomial transformation $\pi\colon Q \to P$ given by the maps*

$$\begin{array}{rcl}\pi_V\colon \bigoplus_\lambda S_\lambda(V)^{\oplus m_\lambda} & \to & \bigoplus_\lambda S_\lambda(V)^{\oplus n_\lambda}\\ ((q_{\lambda,1},\dots,q_{\lambda,m_\lambda}))_\lambda & \mapsto & ((q_{\lambda,1},\dots,q_{\lambda,n_\lambda}))_\lambda\end{array}$$

for $V \in \mathrm{Vec}^\mu$ and elementary affine polynomial transformations of Q. In particular, the group $\mathrm{Aff}(Q)$ *is generated by the set of all elementary affine polynomial transformations.*

Proof. It is easy to see that every affine polynomial transformation is a composition of elementary affine automorphisms

$$\pi \circ \prod_\lambda \prod_{i=1}^{n_\lambda} \gamma_{\lambda,i}$$

where the order of the composition is such that the $\gamma_{\lambda,i}$ with higher $|\lambda|$ are applied first. When $Q = P$, we get $\pi = \mathrm{id}_Q$ and hence $\mathrm{Aff}(Q)$ is generated by the set of all elementary affine polynomial transformations. □

Lemma 4.4.11. *Every $\beta \in \mathrm{Aff}(Q)$ is a unipotent element of* $\mathrm{Mor}(Q,Q)$.

Proof. Write $Q = Q_{(1)} \oplus \cdots \oplus Q_{(d)}$. Then one can check using induction on k that

$$\pi_{Q_{(1)}\oplus\cdots\oplus Q_{(k)}} \circ (\beta - \mathrm{id}_Q)^k = 0$$

for all $k \in \{1,\dots,d\}$. In particular, we have $(\beta - \mathrm{id}_Q)^d = 0$ and hence β is unipotent. □

The discussion above gives us the following result.

Proposition 4.4.12. *Any domimant polynomial transformation $\alpha\colon Q \to P$ is the composition of a linear automorphism $\ell \in \mathrm{GL}(Q)$ and an affine polynomial transformation $\beta\colon Q \to P$. The unipotent radical of the finite-dimensional algebraic group* $\mathrm{Aut}(Q)$ *is* $\mathrm{Aff}(Q)$ *and has* $\mathrm{GL}(Q)$ *as a Levi complement.* □

4.5 Theory of types

From now on, we take $\mu = 1$ and consider univariate polynomial functors. Fix a pure polynomial functor P. In this section, we consider the lattice of orbit closures

$$\overline{\mathrm{GL}_\infty \cdot p} \subseteq P_\infty$$

over all points $p \in P_\infty$ ordered by containment, i.e., we say that $\overline{\mathrm{GL}_\infty \cdot q} \leq \overline{\mathrm{GL}_\infty \cdot p}$ when the former is contained in the latter.

4.5.1 GL_∞-generic points

Definition 4.5.1. A point $p \in P_\infty$ is called GL_∞-generic if $\overline{\mathrm{GL}_\infty \cdot p} = P_\infty$. When the point p is not GL_∞-generic, we call p degenerate.

The GL_∞-generic points will play a role similar to generic points in finite-dimensional algebraic geometry.

Remark 4.5.2. Note that if the polynomial functor P is not pure, then there are no points in P_∞ whose GL_∞-orbit is dense. Hence the definition of GL_∞-generic points only makes sense when P is pure.

By the Dichotomy Theorem, the point p is GL_∞-generic if and only if it is not in the image of any polynomial transformation $Q \to P$ with $Q < P$. We start by relating the GL_∞-genericity of a point in P_∞ to that of its projections onto the irreducible components of P.

Lemma 4.5.3. *Write $P = P_{(1)} \oplus \cdots \oplus P_{(d)}$ and let*

$$p = (p_1, \ldots, p_d) \in P_\infty = P_{(1),\infty} \oplus \cdots \oplus P_{(d),\infty}$$

be a point. Then $p \in P_\infty$ is GL_∞-generic if and only if $p_1 \in P_{(1),\infty}, \ldots, p_d \in P_{(d),\infty}$ are.

Proof. It is clear that if one of $p_1, \ldots, p_d$ is degenerate, then so is p. Suppose that p is degenerate, but p_d is not. Then there exists a polynomial transformation $\alpha \colon Q \to P$ with $Q < P$ whose image contains p. Note that p_d is in the image of the composition $\pi_d \circ \alpha$ where $\pi_d \colon P \to P_{(d)}$ is the natural projection. So $Q \not\leq P_{(d)}$ since p_d is GL_∞-generic. This means that $Q_{(d)} \cong P_{(d)}$. So

$$Q_{(1)} \oplus \cdots \oplus Q_{(d-1)} < P_{(1)} \oplus \cdots \oplus P_{(d-1)}$$

and therefore $(p_1, \ldots, p_{d-1})$ is degenerate. It follows by induction on d that, if $p_1, \ldots, p_d$ are GL_∞-generic, then so is p. □

Lemma 4.5.4. *Suppose that P is homogeneous of degree d, write $P = \bigoplus_{\lambda \vdash d} S_\lambda^{n_\lambda}$ and let*

$$p = (p_\lambda)_\lambda \in P_\infty = \bigoplus_{\lambda \vdash d} S_{\lambda,\infty}^{n_\lambda}$$

be a point. Then $p \in P_\infty$ is GL_∞-generic if and only if $p_\lambda \in S_{\lambda,\infty}^{n_\lambda}$ is GL_∞-generic for all $\lambda \vdash d$.

Proof. If one of p_λ is degenerate, then so is p. Assume that p is degenerate. Then there exists a polynomial transformation $\alpha\colon Q \to P$ with $Q < P$ whose image contains p. Write

$$Q = R \oplus \bigoplus_{\lambda \vdash d} S_\lambda^{m_\lambda}$$

with R a polynomial functor whose degree is lower than the degree of P. Then $m_\lambda < n_\lambda$ for some λ since $Q < P$. This implies that p_λ is degenerate since it is in the image of the composition $\pi_{S_\lambda^{n_\lambda}} \circ \alpha$ which only depends on $Q = R \oplus S_\lambda^{m_\lambda} < S_\lambda^{n_\lambda}$. So if all p_λ are GL_∞-generic, then so is p. □

Let λ be a partition. Then the subset $D_\lambda \subseteq S_{\lambda,\infty}$ consisting of all points in $S_{\lambda,\infty}$ that are degenerate is a subspace of $S_{\lambda,\infty}$. To see this, let $p, q \in D_\lambda$ be points coming from smaller polynomial functors $Q, R < S_\lambda$. Then $\deg(Q), \deg(R) < |\lambda|$. So $\deg(Q \oplus R) < |\lambda|$ and hence $Q \oplus R < S_\lambda$. As $p + q$ comes from $Q \oplus R$, this means that $p + q \in D_\lambda$. So D_λ is closed under addition. For any polynomial functor Q, the set of points in $S_{\lambda,\infty}$ coming from Q is closed under scaling. So D_λ is indeed a subspace of $S_{\lambda,\infty}$.

Lemma 4.5.5. *Suppose that $P = S_\lambda^n$ and let*

$$p = (p_1, \ldots, p_n)_\lambda \in P_\infty = S_{\lambda,\infty}^n$$

be a point. Then $p \in P_\infty$ is GL_∞-generic if and only if the points $p_1, \ldots, p_n \in S_{\lambda,\infty}$ are linearly independent modulo D_λ.

Proof. This proof is left to the reader. □

Proposition 4.5.6. *Write $P = \bigoplus_\lambda S_\lambda^{n_\lambda}$ and let*

$$p = ((p_{\lambda,1}, \ldots, p_{\lambda,n_\lambda}))_\lambda \in P_\infty = \bigoplus_\lambda S_{\lambda,\infty}^{n_\lambda}$$

be a point. Then $p \in P_\infty$ is GL_∞-generic if and only if the points $p_{\lambda,1}, \ldots, p_{\lambda,n_\lambda} \in S_{\lambda,\infty}$ are linearly independent modulo D_λ for all λ.

Proof. This follows directly from the previous three lemmas. □

Our next task is to show that GL_∞-generic points exist. Recall that $T^{\otimes d}$ is the univariate polynomial functor sending $V \mapsto V^{\otimes d}$.

Lemma 4.5.7. *The point*

$$p = \left(\sum_{i=0}^{\infty} e_{i\cdot nd+1} \otimes \cdots \otimes e_{i\cdot nd+d}, \ldots, \sum_{i=0}^{\infty} e_{i\cdot nd+(n-1)d+1} \otimes \cdots \otimes e_{i\cdot nd+nd} \right) \in (T^{\otimes d})^{\oplus n}_\infty$$

is GL_∞-generic.

Proof. We need to prove that $(T^{\otimes d})^{\oplus n}$ is the smallest closed subset X of itself such that $p \in X_\infty$. So let $X \subseteq (T^{\otimes d})^{\oplus n}$ be a closed subset such that $p \in X_\infty$. Then we see that

$$\mathrm{pr}_{knd}(p) = \left(\sum_{i=0}^{k} e_{i\cdot nd+1} \otimes \cdots \otimes e_{i\cdot nd+d}, \ldots, \sum_{i=0}^{k} e_{i\cdot nd+(n-1)d+1} \otimes \cdots \otimes e_{i\cdot nd+nd} \right) \in X(K^{knd})$$

for all $k \in \mathbb{N}$. Let $\ell\colon K^{knd} \to V$ be any linear map and write $\ell(e_j) = v_j \in V$. Then we see that

$$P(\ell)(\mathrm{pr}_{knd}(p)) = \left(\sum_{i=0}^{k} v_{i\cdot nd+1} \otimes \cdots \otimes v_{i\cdot nd+d}, \dots, \sum_{i=0}^{k} v_{i\cdot nd+(n-1)d+1} \otimes \cdots \otimes v_{i\cdot nd+nd}\right) \in X(V)$$

for all $v_1, \dots, v_{knd} \in V$. For $k = n(\dim V)^d$, we see that every element of $(V^{\otimes d})^{\oplus n}$ is of this form. Hence $X = (T^{\otimes d})^{\oplus n}$ and so p is GL_∞-generic. □

Lemma 4.5.8. *For every partition λ and every $n \in \mathbb{N}$, the space $S_{\lambda,\infty}^{\oplus n}$ contains a GL_∞-generic point.*

Proof. Take $d = |\lambda|$. Then

$$(T^{\otimes d})^{\oplus n} = \bigoplus_{\lambda' \vdash d} S_{\lambda'}^{\oplus n_{\lambda'}}$$

with $n_{\lambda'} \geq n$ for all $\lambda \vdash d$. By the previous lemma and Proposition 4.5.6, it follows that $S_{\lambda,\infty}^{\oplus n_\lambda}$ has a GL_∞-generic point $(p_1, \dots, p_{n_\lambda})$. By Lemma 4.5.5, we see that $(p_1, \dots, p_n)$ is a GL_∞-generic point of $S_{\lambda,\infty}^{\oplus n}$. □

Proposition 4.5.9. *The space P_∞ contains a GL_∞-generic point.*

Proof. This follows from Proposition 4.5.6 and the previous lemma. □

Proposition 4.5.10. *For every partition λ, the quotient space $S_{\lambda,\infty}/D_\lambda$ is infinite-dimensional.*

Proof. The previous lemma shows that $S_{\lambda,\infty}/D_\lambda$ contains n linearly independent elements for all $n \in \mathbb{N}$. Hence $S_{\lambda,\infty}/D_\lambda$ must be infinite-dimensional. □

Before we define the type of a point, we state some easy but important observations.

Proposition 4.5.11. *Let $q \in Q_\infty$ be a GL_∞-generic point and let $\alpha, \beta\colon Q \to P$ be polynomial transformations such that $\alpha_\infty(q) = \beta_\infty(q)$. Then $\alpha = \beta$.*

Proof. Since $\alpha_\infty - \beta_\infty$ is GL_∞-equivariant, we see that $\alpha_\infty - \beta_\infty$ is zero on $\mathrm{GL}_\infty \cdot q$. As $\alpha_\infty - \beta_\infty$ is continuous, it follows that $\alpha_\infty - \beta_\infty$ is the zero map. So $\alpha_\infty = \beta_\infty$ and hence we get $\alpha = \beta$. □

Proposition 4.5.12. *Let $q \in Q_\infty$ be a GL_∞-generic point and let $\alpha\colon Q \to P$ be a polynomial transformation. Then $\overline{\mathrm{im}(\alpha_\infty)} = \overline{\mathrm{GL}_\infty \cdot \alpha_\infty(q)}$.*

Proof. We have $Q_\infty = \overline{\mathrm{GL}_\infty \cdot q}$ and hence

$$\overline{\mathrm{im}(\alpha_\infty)} = \overline{\alpha_\infty(Q_\infty)} = \overline{\alpha_\infty(\mathrm{GL}_\infty \cdot q)} = \overline{\mathrm{GL}_\infty \cdot \alpha_\infty(q)}$$

since α_∞ is GL_∞-equivariant. □

4.5.2 Types of points

Let $p \in P_\infty$ be a point.

Proposition 4.5.13. *There exist a polynomial functor $Q \leq P$, a GL_∞-generic point $q \in Q_\infty$ and a polynomial transformation $\alpha\colon Q \to P$ with $p = \alpha_\infty(q)$.*

Proof. We prove the proposition using induction on P. If p is GL_∞-generic, then we take $Q = P$, $q = p$ and $\alpha = \mathrm{id}_P$. Otherwise, let X be the closed subset of P with $\overline{\mathrm{GL}_\infty \cdot p} = X_\infty$. Then $X \neq P$ and hence X is covered by images from smaller polynomial functors by Theorem 4.2.5. It follows that there exist a polynomial functor $R < P$, a point $r \in R_\infty$ and a polynomial transformation $\beta\colon R \to P$ with $p = \beta_\infty(r)$. By induction, there also exist a polynomial functor $Q \leq R$, a GL_∞-generic point $q \in Q_\infty$ and a polynomial transformation $\gamma\colon Q \to R$ with $r = \alpha_\infty(q)$. We see that the conditions of the proposition hold with $\alpha = \beta \circ \gamma$. □

Roughly speaking, the type of the point p is the smallest polynomial functor Q (up to isomorphism) satisfying the conditions of the proposition. To prove that such a functor Q is well-defined, we need to understand the polynomial transformations whose image contain p better. Let $\alpha\colon Q \to P$ and $\beta\colon R \to P$ be polynomial transformations.

Definition 4.5.14. We write $\alpha \preceq \beta$ if there exists a polynomial transformation $\gamma\colon Q \to R$ such that $\alpha = \beta \circ \gamma$. We say that α and β are equivalent if both $\alpha \preceq \beta$ and $\beta \preceq \alpha$. The quasi-order $\preceq$ induces a partial order on equivalence classes $[\alpha]$ of polynomial transformations, also denoted $\preceq$.

Let $q \in Q_\infty$ be a GL_∞-generic point. Note that this implies that Q is pure. Also let $r \in R_\infty$ be an arbitrary point.

Lemma 4.5.15. *Assume that $R = S_{\lambda'}$ for some partition λ'. Then there exist a polynomial functor Q', a point $q' \in Q'_\infty$ and a polynomial transformation $\gamma\colon Q \oplus Q' \to R$ such that $(q, q') \in Q_\infty \oplus Q'_\infty$ is GL_∞-generic and $\gamma_\infty(q, q') = r$.*

Proof. Write

$$Q = \bigoplus_\lambda S_\lambda^{n_\lambda} \quad \text{and} \quad q = ((q_{\lambda,1}, \dots, q_{\lambda,n_\lambda}))_\lambda \in \bigoplus_\lambda S_{\lambda,\infty}^{n_\lambda}.$$

We are allowed to enlarge the n_λ as long as $q_{\lambda,1}, \dots, q_{\lambda,n_\lambda}$ remain linearly independent modulo D_λ and we need to prove that after doing so the point r lies in the image of a polynomial transformation from $\bigoplus_\lambda S_\lambda^{n_\lambda}$. Note that, after possibly enlarging $n_{\lambda'}$, we may assume that r lies in the span of $q_{\lambda',1}, \dots, q_{\lambda',n_{\lambda'}}$ modulo $D_{\lambda'}$. So we can write r as a linear combination of $q_{\lambda',1}, \dots, q_{\lambda',n_{\lambda'}}$ and a degenerate point in $S_{\lambda',\infty}$. We can write this degenerate point as the image of a point $(r'_1, \dots, r'_k) \in S_{\lambda_1,\infty} \oplus \dots \oplus S_{\lambda_k,\infty}$ where $\lambda_1, \dots, \lambda_k$ are partitions with $|\lambda_i| < |\lambda'|$. So, using induction on $|\lambda'|$, we can enlarge the n_λ finitely many times and write each of the r'_i as images from $\bigoplus_\lambda S_\lambda^{n_\lambda}$. This in turn allows us to write r as an image from $\bigoplus_\lambda S_\lambda^{n_\lambda}$. □

Now we drop the assumption that R is a Schur functor.

Lemma 4.5.16. *There exist a polynomial functor Q', a point $q' \in Q'_\infty$ and a polynomial transformation $\gamma\colon Q \oplus Q' \to R$ such that $(q, q') \in Q_\infty \oplus Q'_\infty$ is GL_∞-generic and $\gamma_\infty(q, q') = r$.*

Proof. It suffices to prove the lemma assuming that the polynomial functor R is pure. We prove the lemma using induction on the number of irreducible components of R. When $R = 0$, we simply take $Q' = 0$ and $\gamma = 0$. So assume that $R \neq 0$ and write

$$R = R' \oplus S_\lambda \quad \text{and} \quad r = (r', s) \in R'_\infty \oplus S_{\lambda,\infty}$$

for some partition λ. Using the induction hypothesis, there exist a polynomial functor Q', a point $q' \in Q'_\infty$ and a polynomial transformation $\gamma\colon Q \oplus Q' \to R'$ such that $(q, q') \in Q_\infty \oplus Q'_\infty$ is GL_∞-generic and $\gamma_\infty(q, q') = r'$. Using the previous lemma, there also exist a polynomial functor Q'', a point $q'' \in Q''_\infty$ and a polynomial transformation $\gamma'\colon Q \oplus Q' \oplus Q'' \to S_\lambda$ such that

$$(q, q', q'') \in Q_\infty \oplus Q'_\infty \oplus Q''_\infty$$

is GL_∞-generic and $\gamma'_\infty(q, q', q'') = s$. We are now done since $Q' \oplus Q''$, (q', q'') and (γ, γ') satisfy the conditions of the lemma. $\square$

Proposition 4.5.17. *If $\alpha_\infty(q) = \beta_\infty(r)$, then $\alpha \leq \beta$.*

Proof. By the previous lemma, there exist a polynomial functor Q', a point $q' \in Q'_\infty$ and a polynomial transformation $\gamma\colon Q \oplus Q' \to R$ such that $(q, q') \in Q_\infty \oplus Q'_\infty$ is GL_∞-generic and $\gamma_\infty(q, q') = r$. Note that $\alpha_\infty(\pi_{Q_\infty}(q, q')) = \alpha_\infty(q) = \beta_\infty(\gamma_\infty(q, q'))$. It follows that $\alpha \circ \pi_Q = \beta \circ \gamma$ as (q, q') is GL_∞-generic and hence $\alpha = \beta \circ \gamma(-, 0)$. $\square$

Corollary 4.5.18. *All polynomial transformations $\alpha\colon Q \to P$ for which there exists a GL_∞-generic point $q \in Q_\infty$ with $\alpha_\infty(q) = p$ are equivalent and have $\overline{\mathrm{im}(\alpha_\infty)} = \overline{\mathrm{GL}_\infty \cdot p}$.* $\square$

Corollary 4.5.19. *There is a unique polynomial functor whose magnitude is minimal among all polynomial functors Q for which there are a GL_∞-generic point $q \in Q_\infty$ and a polynomial transformation $\alpha\colon Q \to P$ with $p = \alpha_\infty(q)$.*

Proof. Let Q be any polynomial functor whose magnitude is minimal among all polynomial functors for which there are a GL_∞-generic point $q \in Q_\infty$ and a polynomial transformation $\alpha\colon Q \to P$ with $p = \alpha_\infty(q)$. Also let R be a polynomial functor, $r \in R_\infty$ a GL_∞-generic point and $\beta\colon R \to P$ a polynomial transformation with $p = \beta_\infty(r)$. We will prove that Q is a quotient of R, which implies that Q is unique up to isomorphism.

By the previous proposition, we see that $\beta \leq \alpha$. So there exists a polynomial transformation $\gamma\colon R \to Q$ such that $\beta = \alpha \circ \gamma$. Consider the closed subset $X = \overline{\mathrm{im}(\gamma)} \subseteq Q$ and the point $\gamma_\infty(r) \in X_\infty$. If $X \neq Q$, then by Theorem 4.2.5 there exist a polynomial functor $Q' < Q$, a polynomial transformation $\alpha'\colon Q' \to Q$ and a GL_∞-generic point $q' \in Q'_\infty$ with $\alpha'_\infty(q') = \gamma_\infty(r)$. But then

$$p = \beta_\infty(r) = \alpha_\infty(\gamma_\infty(r)) = \alpha_\infty(\alpha'_\infty(q')) = (\alpha \circ \alpha')_\infty(q'),$$

which contradicts the minimality of Q. Hence $X = Q$ and so the polynomial transformation $\gamma\colon R \to Q$ is dominant. We conclude using Lemma 4.4.1 that Q is a quotient of R and hence Q is unique. $\square$

Definition 4.5.20. We call the polynomial functor Q from the corollary the type of p.

From the proof of the previous corollary, we also get the following result.

Corollary 4.5.21. *Let $r \in R_\infty$ be a GL_∞-generic point, let $\alpha\colon R \to P$ be a polynomial transformation and let Q be the type of $\alpha_\infty(r)$. Then α factors through Q.* $\square$

4.5.3 A map between lattices

We conclude this section by proving the following theorem. Note that equivalent polynomial transformations have the same image.

Theorem 4.5.22. *The map*

$$\begin{array}{rcl}\{\textit{classes of pure polynomial transformations into } P\} & \to & \{\mathrm{GL}_\infty\textit{-orbit closures in } P_\infty\} \\ [\alpha] & \mapsto & \overline{\mathrm{im}(\alpha_\infty)}\end{array}$$

is an order-preserving bijection.

Here the former set is ordered by the partial order $\preceq$ from Definition 4.5.14 and the latter set is ordered by containment.

Remark 4.5.23. The inverse of this map is not order-preserving in general. See Subsection 4.7.2 for an example where this happens.

The fact that this map is order-preserving follows from directly from the definition of the partial order $\preceq$. Surjectivity of the map follows from Corollary 4.5.18. To prove injectivity, we need to following lemmas.

Lemma 4.5.24. *Let $\alpha\colon Q \to P$ be a polynomial transformation. If $\mathrm{pr}_n(p) \in \mathrm{im}(\alpha_n)$ for all $n \in \mathbb{N}$, then $p \in \mathrm{im}(\alpha_\infty)$.*

Proof. Before we prove the general case, we first consider the case where the field K is uncountable. Let $q \in Q_\infty$ be a point. Then the equality $\alpha_\infty(q) = p$ holds if and only if $\alpha_n(\mathrm{pr}_n(q)) = \mathrm{pr}_n(p_n)$ holds for all $n \in \mathbb{N}$. This translates $\alpha_\infty(q) = p$ into polynomial equations in countably many variables and the condition that $\mathrm{pr}_n(p) \in \mathrm{im}(\alpha_n)$ for all $n \in \mathbb{N}$ shows that any finite number of these equations has a solution. Hence, by Lang's theorem from [28] the entire system has a solution when K is uncountable.

Now for the general case, let $\beta\colon R \to P$ be a polynomial transformation and let $r \in R_\infty$ be a GL_∞-generic point such that $\beta_\infty(r) = p$. Choose an uncountable algebraically closed extension L/K. Then r is still GL_∞-generic in R_∞^L: indeed, for each $n \in \mathbb{N}$, there exists an $m \geq n$ such that $\mathrm{pr}_n(\mathrm{GL}_m \cdot \mathrm{pr}_m(r))$ is dense in R_n and then this set is also dense in R_n^L. Proposition 4.5.17 yields that $\beta^L \preceq \alpha^L$. But then also $\beta \preceq \alpha$, i.e., there exists a polynomial transformation $\gamma\colon R \to Q$ (defined over K) such that $\beta = \alpha \circ \gamma$, as the space $\mathrm{Mor}(R, Q)$ is finite-dimensional and the field K is algebraically closed. So $q = \gamma_\infty(r)$ satisfies $\alpha_\infty(q) = p$. □

Remark 4.5.25. The rank functions from Section 1.4 all can be extended to their respective infinite settings. There are two a priori different way of doing this. One possibility is to define the rank of a series p as the infimum of all its projections $\mathrm{pr}_n(p)$. Another way is to define the rank of a series using the description of the rank in terms of polynomial transformations. The previous lemma shows that these two definitions coincide.

Lemma 4.5.26. *Suppose that K has infinite transcendence degree over $\mathbb{Q}$. Let $p \in P_\infty$ have coordinates that are algebraically independent over $\mathbb{Q}$. Then p is GL_∞-generic.*

Proof. Let $Q < P$ be a pure polynomial functor. Then the polynomial transformation $\alpha\colon Q' = \mathrm{Mor}(Q, P) \oplus Q \to P$ defined by the maps

$$\alpha_V(\beta, q) = \beta_V(q)$$

for $V \in \mathrm{Vec}$ is defined over $\mathbb{Q}$ and for any polynomial transformation $\beta\colon Q \to P$ we have $\mathrm{im}(\beta_\infty) \subseteq \mathrm{im}(\alpha_\infty)$. We have $Q' < P$. So there is a d such that $Q'_{(e)} \cong P_{(e)}$ for all $e > d$ and $Q'_{(d)} \not\cong P_{(d)}$ is a quotient of $P_{(d)}$. We see that the coefficients of the polynomials $n \mapsto \dim Q'(K^n)$ and $n \mapsto \dim P(K^n)$ coincide in degrees $> d$ and that the coefficient of $n \mapsto \dim Q'(K^n)$ is lower in degree d. So $\dim Q'(K^n) < \dim P(K^n)$ for $n \in \mathbb{N}$ big enough. For such an integer $n \in \mathbb{N}$, we find that

$$\overline{\mathrm{im}(\alpha_n)}$$

is a strict closed subset of $P(K^n)$ defined over $\mathbb{Q}$. Hence it does not contain the point $\mathrm{pr}_n(p) \in P(K^n)$ since its coordinates are algebraically independent over $\mathbb{Q}$. □

Injectivity now follows from the following proposition.

Proposition 4.5.27. *Let $\alpha\colon Q \to P$ and $\beta\colon R \to P$ be polynomial transformations such that*

$$\overline{\mathrm{im}(\alpha_\infty)} = \overline{\mathrm{im}(\beta_\infty)}.$$

If Q is pure, then $\alpha \preceq \beta$. If in addition R is also pure, then α and β are equivalent.

Proof. The polynomial transformations α and β are defined over our fixed ground field K. Let L/K be an algebraically closed extension of infinite transcendence degree and let $q \in Q^L_\infty$ have coordinates that are algebraically independent over K. Now define $p_n = \alpha^L_n(\mathrm{pr}^L_n(q))$. This is a generic point of the K-variety $\overline{\mathrm{im}\,\alpha_n} = \overline{\mathrm{im}\,\beta_n}$. So in particular, it is contained in the image of the map $\beta^L_n\colon R^L(L^n) \to P^L(L^n)$. By Lemma 4.5.24 applied with L instead of K, it follows that $p := (p_n)_n \in P_\infty$ lies in the image of the map $\beta^L_\infty\colon R_\infty(L) \to P_\infty(L)$. Moreover, the point q is GL_∞-generic by Lemma 4.5.26. Here we use that Q is pure. So by Proposition 4.5.17 we find $\alpha^L \preceq \beta^L$. But the polynomial transformation $\gamma\colon Q^L \to R^L$ such that $\alpha^L = \beta^L \circ \gamma$ is a solution to a finite-dimensional system of polynomial equations with coefficients from K. Hence this system has a solution over the algebraically closed field K. So $\alpha \preceq \beta$. When R is also pure, the same argument shows that also $\beta \preceq \alpha$. □

Proof of Theorem 4.5.22. The proof of the theorem follows from Corollary 4.5.18 together with the previous proposition. □

4.6 A version of Chevalley's Theorem

Let Q, P be polynomial functors and let $\alpha\colon Q \to P$ be a polynomial transformation. The goal of this section is to use the Dichotomy Theorem to prove that α_∞ sends GL_∞-stable constructible subsets of Q_∞ to GL_∞-stable constructible subsets of P_∞. First, we define what it means for a GL_∞-stable subset of P_∞ to be constructable.

Definition 4.6.1. A GL_∞-stable subset $C \subseteq P_\infty$ is called constructible if it is a finite union of subsets of the form $X_\infty \cap U$ where X_∞ is a GL_∞-stable closed subset and U is a GL_∞-stable open subset of P_∞.

Remark 4.6.2. Note that GL_∞-stable open subsets of P_∞ do not in general correspond to functors. Hence we will not denote them and constructible subsets of P_∞ using a subscript ∞.

4.6.1 Constructibility of the whole image

As a first step, we prove that $\mathrm{im}(\alpha_\infty)$ is constructible. We use the following lemma.

Lemma 4.6.3. *Let A, B be finite-dimensional affine varieties and assume that A is irreducible. Let P, Q, R be polynomial functors and assume that Q and R are pure. Let $\alpha\colon A \times Q \to P$ and $\beta\colon B \times R \to P$ be regular transformations and assume that*

$$\overline{\mathrm{im}(\alpha_\infty)} = \overline{\mathrm{im}(\beta_\infty)}$$

holds. Then there exists an open dense subset $A' \subseteq A$ such that $\alpha_\infty(A' \times Q_\infty) \subseteq \mathrm{im}(\beta_\infty)$.

Proof. Let L be an algebraic closure of $K(A)$. For each $n \in \mathbb{N}$ we have a K-algebra homorphism $\alpha_n^*\colon K[P(K^n)] \to K[A] \otimes_K K[Q(K^n)]$. Compose this with the natural K-algebra homomorphism

$$K[A] \otimes_K K[Q(K^n)] \to L \otimes_K K[Q(K^n)] \cong L[Q^L(L^n)]$$

and extend the resulting map L-linearly to a map $L[P^L(L^n)] \to L[Q^L(L^n)]$. This is a homomorphism of L-algebras. The maps $Q^L(L^n) \to P^L(L^n)$ we obtain in this manner form a polynomial transformation $\alpha^L\colon Q^L \to P^L$. We have

$$\ker\left((\alpha_n^L)^*\right) = L \otimes_K \ker(\alpha_n^*)$$

and this implies that $\overline{\mathrm{im}(\alpha_\infty^L)}$ is defined by the same equations (with coefficients from K) as $\overline{\mathrm{im}(\alpha_\infty)}$. Similarly, the regular transformation β gives rise to a regular transformation

$$\beta^L\colon B^L \times R^L \to P^L$$

such that $\overline{\mathrm{im}(\beta_\infty^L)}$ is defined by the same equations as $\overline{\mathrm{im}(\beta_\infty)}$. We conclude that

$$\overline{\mathrm{im}\,\alpha_\infty^L} = \overline{\mathrm{im}\,\beta_\infty^L}$$

holds. By Proposition 4.5.27 applied with L instead of K, we find that $\beta^L = \alpha^L \circ \gamma$ for some polynomial transformation $\gamma\colon Q^L \to B^L \times R^L$. Thinking of γ as a (multi-valued) algebraic map from A to the finite-dimensional space $B \times \mathrm{Mor}(Q, R)$ of regular transformations $Q \to B \times R$, we find that there exists a finite-dimensional affine variety C over K together with a dominant morphism $a\colon C \to A$, a morphism $b\colon C \to B$ and a morphism $\psi\colon C \to \mathrm{Mor}(Q, R)$ such that $\alpha(a(c), -) = \beta(b(c), -) \circ \psi(c)$ as polynomial transformations $Q \to P$ for all $c \in C$. By Chevalley's theorem $a(C)$ is constructible, so it contains an open dense subset $A' \subseteq A$. This set has the desired property. □

Let $\alpha\colon Q \to P$ be a polynomial transformation.

Proposition 4.6.4. *The set $\mathrm{im}(\alpha_\infty) \subseteq P_\infty$ is constructible.*

Proof. We prove the proposition using induction on Q. Take $X_\infty = \overline{\mathrm{im}(\alpha_\infty)}$. By Theorem 4.2.5, we know that there exist a finite number of finite-dimensional affine varieties A_i, pure polynomial functors $Q_i \preceq P$ and regular transformations

$$\alpha_i\colon A_i \times Q_i \to P$$

such that $X_\infty = \bigcup_i \mathrm{im}(\alpha_{i,\infty})$. We may assume that $\alpha_1 = \alpha$ and that each variety A_i is irreducible. Using the previous lemma, we may also assume that $\overline{\mathrm{im}(\alpha_i)} \subsetneq \overline{\mathrm{im}(\alpha)}$ for all $i \neq 1$. Now take $Y_\infty = \bigcup_{i\neq 1} \mathrm{im}(\alpha_{i,\infty})$. Then we see that

$$\mathrm{im}(\alpha_\infty) = (X_\infty \setminus Y_\infty) \cup \alpha_\infty\left(\alpha_\infty^{-1}(Y_\infty)\right).$$

Note that $Z_\infty := \alpha_\infty^{-1}(Y_\infty)$ is a strict closed subset of Q_∞. So there exist a finite number of finite-dimensional affine varieties B_j, pure polynomial functors $R_j < Q$ and regular transformations

$$\beta_j \colon B_j \times R_j \to Q$$

such that $Z_\infty = \bigcup_j \mathrm{im}(\beta_{j,\infty})$. So

$$\mathrm{im}(\alpha_\infty) = (X_\infty \setminus Y_\infty) \cup \bigcup_j \mathrm{im}(\alpha_\infty \circ \beta_{j,\infty}).$$

is constructible by induction as the polynomial transformations $\alpha \circ \beta_j$ have as domains the polynomial functors $R_j < P$. $\square$

Corollary 4.6.5. *Let X be a closed subset of Q. Then $\alpha_\infty(X_\infty)$ is a constructible subset of P_∞.*

Proof. There exist a finite number of finite-dimensional affine varieties A_i, pure polynomial functors $Q_i \leq Q$ and regular transformations

$$\alpha_i \colon A_i \times Q_i \to Q$$

such that $X_\infty = \bigcup_i \mathrm{im}(\alpha_{i,\infty})$. So $\alpha_\infty(X_\infty) = \bigcup_i \mathrm{im}(\alpha_\infty \circ \alpha_{i,\infty})$ is constructible. $\square$

4.6.2 Constructablility of the image of a constructible subset

Let $n \in \mathbb{N}$ be an integer. Note that the set

$$\mathrm{Diag}(I_n, \mathrm{GL}_\infty) = \{\mathrm{Diag}(I_n, g) \mid g \in \mathrm{GL}_\infty\} = \bigcup_{m\in\mathbb{N}} \{\mathrm{Diag}(I_n, g, I_\infty) \mid g \in \mathrm{GL}_m\}$$

is a subgroup of GL_∞. Let P be a polynomial functor and let Z be a closed subset of the polynomial functor $P \circ \mathrm{Sh}_{K^n}$. Then we have a corresponding GL_∞-stable closed subset Z_∞ of $(P \circ \mathrm{Sh}_{K^n})_\infty$. We can identify

$$(P \circ \mathrm{Sh}_{K^n})_\infty = \varprojlim_{m\geq 1} P(K^{m+n}) = \varprojlim_{m\geq n} P(K^m) = \varprojlim_{m\geq 1} P(K^m) = P_\infty$$

as topological spaces. Note however that GL_∞-stable subsets of $(P \circ \mathrm{Sh}_{K^n})_\infty$ correspond under this identification with $\mathrm{Diag}(I_n, \mathrm{GL}_\infty)$-stable subsets of P_∞. Now, let $\alpha \colon Q \to P$ be a polynomial transformation. To prove the analogue of Chevalley's Theorem for our setting, we need one more lemma.

Lemma 4.6.6. *Let U be a nonempty GL_∞-stable open subset of Q_∞. Then $\alpha_\infty(U)$ contains a nonempty GL_∞-stable open subset of $\overline{\mathrm{im}(\alpha_\infty)}$.*

Proof. Take $X_\infty = \overline{\mathrm{im}(\alpha_\infty)}$ and $Y_\infty = Q_\infty \setminus U$. Then Y is a proper closed subset of Q. So there exist an $n \in \mathbb{N}$ and $f \in \mathcal{I}(Y(K^n))$ such that f is not the zero polynomial. Let $Z \subseteq K \oplus Q \circ \mathrm{Sh}_{K^n}$ be the closed subset defined by

$$Z(V) = \{(\lambda, q) \in K \oplus Q(K^n \oplus V) \mid \lambda f(Q(\pi_{K^n})(q)) = 1\}$$

and consider the polynomial transformation $\beta\colon K \oplus Q \circ \mathrm{Sh}_{K^n} \to P \circ \mathrm{Sh}_{K^n}$ defined by the maps

$$\begin{array}{rcl}\beta_V\colon K \oplus Q(K^n \oplus V) & \to & P(K^n \oplus V)\\ (\lambda, q) & \mapsto & \alpha_{K^n \oplus V}(q)\end{array}$$

for $V \in \mathrm{Vec}$. By Corollary 4.6.5, the set $\beta_\infty(Z_\infty)$ is constructible. So it contains a GL_∞-stable open dense subset of its closure. Note that β factors through the projection on $Q \circ \mathrm{Sh}_{K^n}$. Denote the projection of Z_∞ on $(Q \circ \mathrm{Sh}_{K^n})_\infty$ by U'. We see that the subset U' is GL_∞-stable, open and dense in $(Q \circ \mathrm{Sh}_{K^n})_\infty$ and that the image of U' in $(P \circ \mathrm{Sh}_{K^n})_\infty$ is constructible.

Now we identify $(Q \circ \mathrm{Sh}_{K^n})_\infty$ with Q_∞ and $(P \circ \mathrm{Sh}_{K^n})_\infty$ with P_∞. Then we know that $U' \subseteq U$ is a $\mathrm{Diag}(I_n, \mathrm{GL}_\infty)$-stable dense open subset of Q_∞ such that $\alpha_\infty(U')$ is constructible. So $\alpha_\infty(U')$ contains a $\mathrm{Diag}(I_n, \mathrm{GL}_\infty)$-stable dense open subset V of its closure, which is $X_\infty = \overline{\mathrm{im}(\alpha_\infty)}$ as U' is dense in Q_∞. So we see that $V \subseteq \alpha_\infty(U)$ and since the latter is GL_∞-stable, we find that the GL_∞-stable open dense subset $\bigcup_{g \in \mathrm{GL}_\infty} gV$ of X_∞ is also contained in $\alpha_\infty(U)$. □

Theorem 4.6.7. *Let C be a GL_∞-stable constructible subset of Q_∞. Then $\alpha_\infty(C)$ is a GL_∞-stable constructible subset of P_∞.*

Proof. If C is contained in a GL_∞-stable closed subset $X_\infty \subsetneq Q_\infty$, then we can cover the closed subset X of Q using finitely many images $\mathrm{im}(\beta_\infty)$ where the $\beta\colon R \to Q$ are polynomial transformations with $R < Q$ by Theorem 4.2.5 and pull C back along the maps β_∞. In this case, we see that $\alpha_\infty(C)$ is constructible using induction on Q. So we may assume that C is dense in Q_∞. This means in particular that C contains a nonempty GL_∞-stable open subset of Q_∞. By the previous lemma, we see that $\alpha_\infty(C)$ therefore contains a nonempty GL_∞-stable open subset V of $\overline{\mathrm{im}(\alpha_\infty)}$. Take $U = \alpha_\infty^{-1}(V)$ and $X_\infty = Q_\infty \setminus U$. Then $\alpha_\infty(U \cap C) = V_\infty$ is constructible and $\alpha_\infty(X_\infty \cap C)$ is also constructible using induction on Q. Hence their union $\alpha_\infty(C)$ is constructible as well. □

4.7 Examples and open questions

We conclude with some interesting examples and open questions.

4.7.1 Dimension functions

Let P be a polynomial functor of degree d. Then there is a polynomial $g \in \mathbb{Q}[x]$ of degree d such that $\dim P(V) = g(\dim V)$ for all $V \in \mathrm{Vec}$ with $\dim V \gg 0$. Let $X \subseteq P$ be a closed subset. Then a natural question to ask is whether is is also true that there exists a polynomial $g \in \mathbb{Q}[x]$ such that $\dim X(V) = f(\dim V)$ for all $V \in \mathrm{Vec}$ with $\dim V \gg 0$. By adjusting the proof of the Dichotomy Theorem, we can show that this is indeed the case.

Proposition 4.7.1. *There is a polynomial $g \in \mathbb{Q}[x]$ such that*

$$\dim X(V) = g(\dim V)$$

for all $V \in \mathrm{Vec}$ with $\dim V \gg 0$.

Proof. The closed subset X of P is irreducible if and only if the closed subset $X(V)$ of $P(V)$ is irreducible for every vector space $V \in \mathrm{Vec}$. And, the closed subset X of P can only have finitely many components since P is Noetherian. Suppose that the proposition holds for all irreducible closed subsets of P. Then we see that $\dim X(V)$ is a maximum of finitely many polynomials in $\dim V$ when $\dim V \gg 0$. This implies that $\dim X(V)$ is itself a polynomial in $\dim V$ for $\dim V \gg 0$. So it suffices to prove the proposition for irreducible closed subsets X of P. We will prove the proposition using induction of P. So we may assume that the proposition holds for all closed subsets of polynomial functors $Q < P$.

Assume that X is irreducible. If $\deg(P) = 0$, then $\dim X(V)$ is constant. Assume that the degree of P is positive. Let R be an irreducible subfunctor of $P_{(d)}$ and write $P = Q \oplus R$. If $X = Y \times R$ for some closed subset $Y \subseteq Q$, then $\dim X(V) = \dim Y(V) + \dim R(V)$ is a polynomial in $\dim V$ for $\dim V \gg 0$ since $Q < P$ and since $\dim R(V)$ is polynomial in $\dim V$ for $\dim V \gg 0$. So we assume that this is not the case. This means that for some $U \in \mathrm{Vec}$ the ideal $\mathcal{I}(X(U)) \subseteq K[P(U)]$ is not generated by $\mathcal{I}(X(U)) \cap K[Q(U)]$. Let

$$f \in \mathcal{I}(X(U)) \setminus (\mathcal{I}(X(U)) \cap K[Q(U)])$$

be a polynomial of minimal degree. Then

$$h := \frac{\partial f}{\partial r} \neq 0$$

for some $r \in R(U)$. For every $V \in \mathrm{Vec}$, take

$$Z(V) := \{q \in X(U \oplus V) \mid h(P(\pi_U)(q)) \neq 0\}$$

and note that $Z(V)$ is a dense open subset of $X(U \oplus V)$ since X is irreducible. So

$$\dim Z(V) = \dim X(U \oplus V)$$

and therefore it suffices to prove that $\dim Z(V)$ is a polynomial in $\dim V$ for $\dim V \gg 0$. Write $P' := P \circ \mathrm{Sh}_U = Q' \oplus R$. Then the map $h \circ P(\pi_U)\colon P'(V) \to K$ factors through the projection map $\pi_{Q'(V)}\colon P'(V) \to Q'(V)$. We get a map

$$Z(V) \to \{q' \in Q'(V) \mid h(P(\pi_U)(q, 0)) \neq 0\} =: Z'(V)$$

and this map is known to be a closed embedding. So we can view Z as a closed subset of Z', which in turn we can view as a closed subset of $K \oplus Q' < P$. Therefore, we conclude that $\dim X(U \oplus V) = \dim Z(V)$ is a polynomial in $\dim V$ for $\dim V \gg 0$. □

Now suppose that X is the closure of the image of a polynomial transformation $\alpha\colon Q \to P$. Let $g, h \in \mathbb{Q}[x]$ be the polynomials such that $\dim X(V) = g(\dim V)$ and $\dim Q(V) = h(\dim V)$ for all $V \in \mathrm{Vec}$ with $\dim V \gg 0$. Then $g(n) \leq h(n)$ for all integers $n \gg 0$ and hence $\deg(g) \leq \deg(h) = \deg(Q)$. The following proposition tell us what happens when the difference $h - g$ still has the same degree as Q.

Proposition 4.7.2. *If* $\deg(h - g) = \deg(Q)$*, then the polynomial transformation* $\alpha\colon Q \to P$ *factors through* Q/R *for some nonzero subfunctor* $R \subseteq Q$*.*

Proof. Let $U, V \in \mathrm{Vec}$ be vector spaces and view $Q(U), Q(V)$ as subspaces of $Q(U \oplus V)$ via the natural maps. Take $q \in Q(U)$ and $r \in Q(V)$ and let ε be a variable. Then we can write

$$\alpha_{U\oplus V}(q + \varepsilon r) \equiv \alpha_{U\oplus V}(q) + \varepsilon\beta_{(U,V)}(q, r) \bmod \varepsilon^2$$

for some polynomial transformation

$$\beta\colon (Q \circ T_1) \oplus (Q \circ T_2) \to P \circ (T_1 \oplus T_2)$$

between bivariate polynomial functors. Recall here that $T_1(U, V) = U$ and $T_2(U, V) = V$ for all $U, V \in \mathrm{Vec}$. Note that

$$V \mapsto \ker(\beta_{(U,V)}(q, -))$$

is a (linear) subfunctor of Q for all $U \in \mathrm{Vec}$ and $q \in Q(U)$. Let $U_1, U_2 \in \mathrm{Vec}$ be vector spaces and take $q_1 \in Q(U_1)$ and $q_1 \in Q(U_2)$. Then we claim that

$$\ker(\beta_{(U_1\oplus U_2,V)}(q_1 + q_2, -)) \subseteq \ker(\beta_{(U_1,V)}(q_1, -)) \cap \ker(\beta_{(U_2,V)}(q_2, -))$$

where we view q_1 and q_2 as elements of $Q(U_1 \oplus U_2)$ via the natural maps. Indeed, for $r \in Q(V)$ such that

$$\beta_{(U_1\oplus U_2,V)}(q_1 + q_2, r) = 0$$

we see that $\beta_{(U_1,V)}(q_1, r) = 0$ from the fact that the diagram

$$\begin{tikzcd}
Q(U_1 \oplus U_2) \oplus Q(V) \arrow[r, "\beta_{(U_1\oplus U_2,V)}"] \arrow[d, "Q(\pi_{U_1})\oplus Q(\mathrm{id}_V)"'] & P((U_1 \oplus U_2) \oplus V) \arrow[d, "P(\pi_{U_1}\oplus \mathrm{id}_V)"] \\
Q(U_1) \oplus Q(V) \arrow[r, "\beta_{(U_1,V)}"'] & P(U_1 \oplus V)
\end{tikzcd}$$

commutes. We similarly see that $\beta_{(U_2,V)}(q_2, r) = 0$. So the containment holds. And, from this follows that there are $U_0 \in \mathrm{Vec}$ and $q_0 \in Q(U_0)$ such that

$$\ker(\beta_{(U_0,V)}(q_0, -)) = \bigcap_{\substack{U\in\mathrm{Vec}\\ q\in Q(U)}} \ker(\beta_{(U,V)}(q, -))$$

for all $V \in \mathrm{Vec}$. Now consider the (linear) subfunctor $R \subseteq Q$ defined by

$$R(V) := \ker(\beta_{(U_0,V)}(q_0, -))$$

for all $V \in \mathrm{Vec}$. We claim that $\alpha\colon Q \to P$ factors through Q/R. To see this, we have to prove that α_V factors through $Q(V)/R(V)$ for every $V \in \mathrm{Vec}$. Note that $\beta_{(V,V)}$ restricts to the zero map on $Q(V) \oplus R(V)$. This means that

$$\alpha_V(q + \varepsilon r) - \alpha_V(q) \equiv \varepsilon\beta_{(V,V)}(q, r) \equiv 0 \bmod \varepsilon^2$$

for all $q \in Q(V)$ and $r \in R(V)$. So the partial derivative of α_V at any point $q \in Q(V)$ in any direction $r \in R(V)$ is zero. So α_V factors through $Q(V)/R(V)$ for every $V \in \mathrm{Vec}$ and hence α factors through Q/R. If the subfunctor $R \subseteq Q$ is nonzero, then we are done. This leaves the case where $R = 0$. So we assume that $\beta_{(U_0,V)}(q_0, -)$ is injective for every $V \in \mathrm{Vec}$. This means that the dimension of the tangent space

$$T_{\alpha_{U_0\oplus V}(q_0)} X(U_0 \oplus V)$$

is at least $\dim Q(V)$. So $\dim Q(V) \leq \dim X(U_0 \oplus V) \leq \dim Q(U_0 \oplus V)$. This shows that $\deg(h - g) < \deg(h) = \deg(Q)$ must hold. □

By repeated use of the proposition, we see that when X is the closure of the image of a polynomial transformation from a minimal Q, then $\deg(h - g) < \deg(Q)$ must hold.

4.7.2 A counterexample $(S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to S^4$

Consider the polynomial transformation $\alpha\colon (S^2)^{\oplus 3} \to S^4$ defined by the maps

$$\begin{array}{rcl} \alpha_V\colon S^2(V)^{\oplus 3} & \to & S^4(V) \\ (f,g,h) & \mapsto & fg - h^2 \end{array}$$

for $V \in \mathrm{Vec}$ and the closed subset $X = \overline{\mathrm{im}(\alpha)}$ of S^4. One would hope that $X = \mathrm{im}(\alpha)$. We will first show that this is not the case. Let $\beta\colon (S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to S^4$ be the polynomial transformation defined by the maps

$$\begin{array}{rcl} \beta_V\colon V^{\oplus 2} \oplus S^2(V)^{\oplus 3} & \to & S^4(V) \\ (x,y,f,g,h) & \mapsto & x^2 f + y^2 g + xyh \end{array}$$

for $V \in \mathrm{Vec}$.

Lemma 4.7.3. *We have* $\mathrm{im}(\beta) \subseteq X$.

Proof. Take $V \in \mathrm{Vec}$, $t \in K^*$, $x, y \in V$ and $f, g, h \in S^2(V)$. Then we have

$$\begin{array}{rcl} t^{-1}\alpha_V\left(y^2 + tf, x^2 + tg, xy - \frac{1}{2}th\right) & = & t^{-1}\left((y^2+tf)(x^2+tg) - (xy - \frac{1}{2}th)^2\right) \\ & = & x^2 f + yg^2 + xyh + t(\cdots) \in \mathrm{im}(\alpha_V). \end{array}$$

So since the field K is infinite, we find that $\mathrm{im}(\beta_V) \subseteq \overline{\mathrm{im}(\alpha_V)} = X$. □

Lemma 4.7.4. *There is no polynomial transformation* $\gamma\colon (S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to (S^2)^{\oplus 3}$ *such that* $\beta = \alpha \circ \gamma$. *In particular, we have* $\mathrm{im}(\beta) \not\subseteq \mathrm{im}(\alpha)$

Proof. We know that if $\mathrm{im}(\beta) \subseteq \mathrm{im}(\alpha)$, then $\beta = \alpha \circ \gamma$ for some polynomial transformation

$$\gamma\colon (S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to (S^2)^{\oplus 3}$$

by Proposition 4.5.17. Such a polynomial transformation has to be defined by polynomial maps of the form

$$\gamma_V(x,y,f,g,h) = \begin{pmatrix} c_{11}x^2 + c_{12}xy + c_{13}y^2 + c_{14}f + c15g + c_{16}h \\ c_{21}x^2 + c_{22}xy + c_{23}y^2 + c_{24}f + c25g + c_{26}h \\ c_{31}x^2 + c_{32}xy + c_{33}y^2 + c_{34}f + c35g + c_{36}h \end{pmatrix}$$

for $V \in \mathrm{Vec}$ for some constants $c_{ij} \in K$. This turns the equation $\beta = \alpha \circ \gamma$ into a system of polynomial equations in the c_{ij}. Now, one can check that this system has no solutions using a Gröbner basis calculation. □

Proposition 4.7.5. *We have* $\mathrm{im}(\alpha) \neq X$.

Proof. This follows from the previous two lemmas. □

This example leads to several open questions.

Question 4.7.6. *Is* $\mathrm{im}(\beta)$ *closed?*

Question 4.7.7. *Is* $\mathrm{im}(\alpha) \cup \mathrm{im}(\beta)$ *closed?*

Let $\alpha\colon Q \to P$ be a polynomial transformation and take $X = \overline{\mathrm{im}(\alpha)}$. In our example, we found a polynomial transformation $\beta = \beta_0$ such that $\mathrm{im}(\beta) \subseteq X$ by taking a limit of polynomial transformations $\beta_t = \alpha \circ \gamma_t$ where the $\gamma_t\colon (S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to S^4$ were defined by the maps

$$\begin{array}{rcl} \gamma_{t,V}\colon V^{\oplus 2} \oplus S^2(V)^{\oplus 3} & \to & S^4 \\ (x, y, f, g, h) & \mapsto & (t^{-1}y^2 + tf, t^{-1}x^2 + tg, t^{-1}xy - th/2) \end{array}$$

for $V \in \mathrm{Vec}$ and $t \in K^*$. One can ask whether this is the only way to define such β.

Question 4.7.8. *Can we always write X as the union of images* $\mathrm{im}(\beta)$ *of polynomial transformations $\beta_0\colon R \to P$ that are limits of families of polynomial transformations $\beta_t\colon R \to P$ factoring though α?*

Now again consider the polynomial transformation $\beta\colon (S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3} \to S^4$ defined by the maps

$$\begin{array}{rcl} \beta_V\colon V^{\oplus 2} \oplus S^2(V)^{\oplus 3} & \to & S^4(V) \\ (x, y, f, g, h) & \mapsto & x^2 f + y^2 g + xyh \end{array}$$

for $V \in \mathrm{Vec}$ as an element of the space $\mathrm{Mor}((S^1)^{\oplus 2} \oplus (S^2)^{\oplus 3}, S^4)$. Given polynomial functors Q, P, one might hope that the set of polynomial transformations $Q \to P$ that factor through some quotient of Q is a closed subset of the space of all polynomial transformations $Q \to P$. The polynomial transformation β shows that this is not always the case. Indeed, the polynomial transformation β is a limit of polynomial transformations factoring through $(S^2)^{\oplus 3}$, but β itself does not factor through either $S^1 \oplus (S^2)^{\oplus 3}$ or $(S^1)^{\oplus 2} \oplus (S^2)^{\oplus 2}$. Here we see that β does not factor through $S^1 \oplus (S^2)^{\oplus 3}$ since otherwise the coefficients x^2, y^2, xy of f, g, h in the maps defining β would be linearly dependent. And, we see that β does not factor through $(S^1)^{\oplus 2} \oplus (S^2)^{\oplus 2}$ since $\dim \mathrm{im}(\beta_V) > 2 \dim V + 2 \dim S^2(V)$ for $V \in \mathrm{Vec}$ with $\dim V \gg 0$.

4.7.3 Unbounded slice rank of strength ≤ 1 polynomials

Consider the polynomial transformation $\alpha\colon (S^2)^{\oplus 2} \to S^4$ defined by the maps

$$\begin{array}{rcl} \alpha_V\colon S^2(V)^{\oplus 2} & \to & S^4(V) \\ (g, h) & \mapsto & g \cdot h \end{array}$$

for $V \in \mathrm{Vec}$. Let $k \in \mathbb{N}$ be an integer and also consider the polynomial transformation $\beta\colon (S^1)^{\oplus k} \oplus (S^3)^{\oplus k} \to S^4$ defined by the maps

$$\begin{array}{rcl} \beta_V\colon V^{\oplus k} \oplus S^3(V)^{\oplus k} & \to & S^4(V) \\ (\ell_1, \dots, \ell_k, f_1, \dots, f_k) & \mapsto & \ell_1 \cdot f_1 + \dots + \ell_k \cdot f_k \end{array}$$

for $V \in \mathrm{Vec}$. Since the only polynomial transformation $(S^2)^{\oplus 2} \to (S^1)^{\oplus k} \oplus (S^3)^{\oplus k}$ is zero, we see that α does not factor through β. It follows that $\mathrm{im}(\alpha) \not\subseteq \mathrm{im}(\beta)$ by Proposition 4.5.17. Hence the slice rank of the strength ≤ 1 polynomials in $\mathrm{im}(\alpha)$ is not bounded by k. Since this holds for every $k \in \mathbb{N}$, we see that the slice rank of polynomials of the form $g \cdot h$ with $\deg(g) = \deg(h) = 2$ is unbounded. This shows that the gap between the strength and slice rank of a polynomial can be arbitrarily big.

Bibliography

[1] S. Abeasis, A. Del Fra, *Young diagrams and ideals of Pfaffians*, Adv. Math. 35 (1980), no. 2, pp. 158–178.

[2] T. Ananyan, M. Hochster, *Small subalgebras of polynomial rings and Stillman's Conjecture*, J. Am. Math. Soc. 33 (2020), no. 1, pp. 291–309.

[3] A.A. Baranov, A.G. Zhilinskii, *Diagonal direct limits of simple Lie algebras*, Comm. Algebra 27 (1999), no. 6, pp. 2749–2766.

[4] A. Bik, *Noetherianity up to conjugation of locally diagonal inverse limits*, Linear Algebra Appl. 582 (2019), pp. 237-290.

[5] A. Bik, A. Czapliński, M. Wageringel, *Semi-algebraic properties of Minkowski sums of a twisted cubic segment*, Collect. Math. (2020).

[6] A. Bik, J. Draisma, *A note on ED degrees of group-stable subvarieties in polar representations*, Isr. J. Math. 228 (2018), no. 1, pp. 353–377.

[7] A. Bik, J. Draisma, R.H. Eggermont, *Polynomials and tensors of bounded strength*, Commun. Contemp. Math. 21 (2019), no. 7, 1850062 (24 pages).

[8] A. Bik, J. Draisma, R.H. Eggermont, A. Snowden, *The geometry of polynomial representations*, preprint.

[9] A. Bik, J. Draisma, A. Oneto, E. Ventura, *The monic rank*, Math. Comput. (2020).

[10] A. Bik, A. Oneto, *On the strength of general polynomials*, preprint.

[11] M.V. Catalisano, A.V. Geramita, A. Gimigliano, B. Harbourne, J. Migliore, U. Nagel, Y.S. Shin, *Secant varieties of the varieties of reducible hypersurfaces in* $\mathbb{P}^n$, J. Algebra 528 (2019), pp. 381–438.

[12] H.L. Bodlaender, B. van Antwerpen-de Fluiter, *Reduction Algorithms for Graphs of Small Treewidth*, Inf. Comput. 167 (2001), no. 2, pp. 86–119.

[13] J.V. Chipalkatti, *Decomposable ternary cubics*, Exp. Math. 11 (2002), no. 1, pp. 69–80.

[14] D.E. Cohen, *On the laws of a metabelian variety*, J. Algebra 5 (1967), pp. 267–273.

[15] H. Derksen, R.H. Eggermont, A. Snowden, *Topological noetherianity for cubic polynomials*, Algebra & Number Theory 11 (2017), no. 9, pp. 2197–2212.

[16] J. Draisma, *Noetherianity up to symmetry*, Combinatorial algebraic geometry, pp. 33–61, Springer, Cham, 2014.

[17] J. Draisma, *Topological Noetherianity of polynomial functors*, J. Am. Math. Soc. 32 (2019), no. 3, pp. 691–707.

[18] J. Draisma, R.H. Eggermont, *Plücker varieties and higher secants of Sato's Grassmannian*, J. Reine Angew. Math. 737 (2018), pp. 189–215.

[19] R.H. Eggermont, *Finiteness properties of congruence classes of infinite matrices*, Linear Algebra Appl. 484 (2015), pp. 290–303.

[20] R.H. Eggermont, A. Snowden, *Topological noetherianity for algebraic representations of infinite rank classical groups*, preprint.

[21] D. Erman, S.V. Sam, A. Snowden, *Big polynomial rings and Stillman's conjecture*, Invent. math. 218 (2019), no. 2, pp 413–439.

[22] E. Friedlander, A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. 127 (1997), no. 2, pp. 209–270.

[23] C.J. Hillar, S. Sullivant, *Finite Gröbner bases in infinite dimensional polynomial rings and applications*, Adv. Math. 229 (2012), no. 1, pp. 1–25.

[24] D. Kazhdan, T. Ziegler, *On ranks of polynomials*, Algebr. Represent. Theory 21 (2018), no. 5, pp. 1017–1021.

[25] D. Kazhdan, T. Ziegler, *Extending weakly polynomial functions from high rank varieties*, preprint.

[26] J.M. Landsberg, *Tensors: geometry and applications*, Graduate Studies in Mathematics 128, AMS, Providence, RI, 2012.

[27] J.M. Landsberg, M. Michałek, *Towards finding hay in a haystack: explicit tensors of border rank greater than* $2.02m$ *in* $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, preprint.

[28] S. Lang, *Hilbert's nullstellensatz in infinite-dimensional space*, Proc. Am. Math. Soc. 3 (1952), pp. 407–410.

[29] J.M. Starr, *A fact about linear spaces on hypersurfaces*, Proc. Lond. Math. Soc. 93 (2006), no. 2, pp. 301–303.

[30] A. Stasinski, *Similarity and commutators of matrices over principal ideal rings*, Trans. Am. Math. Soc. 368 (2016), no. 4, pp. 2333–2354.

[31] B. Sturmfels, S. Sullivant, *Combinatorial secant varieties*, Pure Appl. Math. Q. 2 (2006), no. 3, pp. 867–891.

[32] T. Tao, W. Sawin, *Notes on the "slice rank" of tensors*, `https://terrytao.wordpress.com/2016/08/24/notes-on-the-slice-rank-of-tensors/`, 2016.

[33] O. Taussky, *The role of symmetric matrices in the study of general matrices*, Linear Algebra Appl. 5 (1972), no. 2, pp. 147–154.

[34] A. Touzé, *Foncteurs strictement polynomiaux et applications*, Habilitation Thesis, 2014.

Erklärung

gemäss Art. 28 Abs. 2 RSL 05

Name:	Bik
Vornamen:	Michel Arthur
Matrikelnummer:	16-128-514
Studiengang:	Mathematik
Art des Abschlusses:	Dissertation
Titel der Arbeit:	Strength and Noetherianity for infinite Tensors
Leiter der Arbeit:	Prof. dr. ir. Jan Draisma

Ich erkläre hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen benutzt habe. Alle Stellen, die wörtlich oder sinngemäss aus Quellen entnommen wurden, habe ich als solche gekennzeichnet. Mir ist bekannt, dass andernfalls der Senat gemäss Artikel 36 Absatz 1 Buchstabe r des Gesetzes vom 5. September 1996 über die Universität zum Entzug des auf Grund dieser Arbeit verliehenen Titels berechtigt ist. Ich gewähre hiermit Einsicht in diese Arbeit.

Bern, 29. Juni 2020 Michel Arthur Bik

Lebenslauf

2016 - 2020	Doktorand der Mathematik an der Universität Bern
2014 - 2016	Masterstudium Mathematik an der Universität Leiden
2011 - 2014	Bachelorstudium Angewandte Mathematik an der Technischen Universität Delft
2005 - 2011	Mittelschule Krimpenerwaard College
1997 - 2005	Grundschule Prinses Ireneschool
1993	Geboren in Rotterdam