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PARTICLE DECAY IN THE EARLY UNIVERSE

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Physics was never my first choice of profession. But as often happens, life has a peculiar way of arranging things to ultimately be in the best way they can be. It feels like only yesterday that a young man, who barely passed the only compulsory course of high school physics, found himself watching a documentary about the universe dreaming of becoming a cosmologist. And what a dream it eventually turned out to be. All of this could not have been accomplished if not for a very special group of people to whom I owe my sincerest gratitude.

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Finally, I would like to acknowledge my profound gratitude to my father, my grandmother, my grandfathers and my aunt, who all taught and showed me by their example that with hard work and persistence anything in life is possible. Time was cut too short for us, but I may find peaceful consolation from knowing the pride that would be yours if only you would have seen the completion of this thesis.

At the time of writing, the world is coming to grips with a global coronavirus pandemic, reminding that the world around us is always in a state of constant motion; the only thing which is certain is change. As I look back at my years in the university, change is perhaps the only word adequate enough to describe the incredible journey I have taken. I cannot help but perceiving a chapter coming to an end, but, as I have so often witnessed in my life, the end of something always marks the beginning of something new.

Contents

Abstract

Particle decay processes have deep and profound implications in cosmology and especially on various early universe particle processes. Common decay rates, which may be experimentally verified e.g., in particle accelerators, are calculated using a theory which relies on Einstein's theory of special relativity thereby neglecting gravity which is contained in the general theory of relativity. The early universe is, however, a place where the curvature of the spacetime cannot be neglected anymore. Hence, the flat space quantum field theory becomes only an approximation and of limited applicability. A more precise picture which includes the role of gravity forces one to view things more generally from the perspective of quantum field theory in curved spacetime. As a result, particle decay rates, cross sections and lifetimes may be modified from the common decay rates obtained from flat spacetime theory.

The aim of this thesis is to investigate how the known flat space decay rates are modified in the presence of a gravitational field and what implications these have on early universe particle processes. Using quantum field theory in curved spactime and a conceptually clear method for calculating decay rates in curved spacetime, the decay of a massive scalar is studied in a realistic and cosmologically relevant scenarios in an expanding universe. The results have significance when studying early universe cosmological situations but also as the cosmological data and measurements become increasingly more accurate, there might arise a necessity in the future to include the effects of curved spacetime also in particle decay rates.

Tiivistelmä

Hiukkasten hajoamisprosesseilla on syvällisiä ja perustavanlaatuisia seurauksia kosmologiassa ja erityisesti varhaisen maailmankaikkeuden hiukkasprosesseissa. Tavanomaiset hajoamisnopeudet, joita voidaan esimerkiksi hiukkaskiihdyttimissä kokeellisesti todentaa, ovat laskettu käyttäen teoriaa joka nojautuu Einsteinin erityiseen suhteellisuusteoriaan jättäen näin huomioimatta painovoiman, joka sisältyy yleiseen suhteellisuusteoriaan. Varhainen maailmankaikkeus on kuitenkin paikka, jossa avaruusajan kaarevuutta ei voida enää jättää huomioimatta. Tällöin tavanomainen litteän (laakean) avaruuden kvanttikenttäteoria on vain approksimaatio ja sen käyttö rajallista. Tarkempi kuvaus ja painovoiman roolin huomioiminen pakottaakin tarkastelemaan asioita laajemmin kaarevan avaruuden kvanttikenttäteorian näkökulmasta. Tämän seurauksena hiukkasten hajoamisnopeudet, vaikutusalat ja eliniät saattavat kuitenkin muuttua tavanomaisista litteän avaruuden teoriasta saaduista tuloksista.

Tämän väitöskirjan tarkoitus on tutkia miten tunnetut litteän avaruuden hajoamisnopeudet muuttuvat painovoiman vaikutuksen alaisena ja mitä seurauksia tällä on varhaisen maailmankaikkeuden hiukkasprosesseihin. Käyttämällä kaarevan avaruuden kvanttikenttäteoriaa ja käsitteellisesti selkeää tapaa laskea hajoamisnopeuksia kaarevassa avaruudessa, massiivisen skalaarihiukkasen hajoamista on tarkasteltu realistisissa ja kosmologisesti merkityksellisissä skenaarioissa avaruuden laajetessa. Tuloksilla on merkitystä tutkittaessa varhaisen maailmankaikkeuden kosmologisia tapahtumia mutta myös kosmologisen datan ja mittausten tullessa yhä tarkemmiksi, voi tulevaisuudessa syntyä tarve ottaa huomioon myös kaarevan avaruuden seuraukset hiukkasten hajoamisnopeuksiin.

List of papers

This thesis consists of a review of the subject and the following original research articles:

I **Decay of a massive particle in a stiff-matter-dominated universe**,
J. Lankinen, I. Vilja, Phys. Rev. D **96**, 105026 (2017)

II **Decaying massive particles in the matter and radiation dominated eras**,
J. Lankinen, I. Vilja, Phys. Rev. D **97**, 065004 (2018)

III **Particle decay in expanding Friedmann-Robertson-Walker universes**,
J. Lankinen, I. Vilja, Phys. Rev. D **98**, 045010 (2018)

IV **Fermionic decay of a massive scalar in the early Universe**,
J. Lankinen, J. Malmi, I. Vilja, Eur. Phys. J. C **80**, 502 (2020)

V **Reheating via gravitational particle production in kination epoch**,
J. Lankinen, O. Kerppo, I. Vilja, Phys. Rev. D **101**, 063529 (2020)

Introduction

Particle decay processes in the early universe have deep and profound implications in cosmology, from baryogenesis [1, 2] to Big Bang nucleosynthesis [3] and reheating scenarios after inflation [4]. When considering these and other early universe processes in depth, the effect of curvature of the spacetime cannot be neglected anymore and the Minkowskian quantum field theory is ultimately only an approximation and of limited applicability. In this region, quantum field theory in curved spacetime must be used. As a result of spacetime curvature, the particle decay rates, cross sections and lifetimes are modified compared to the usual flat space results [5, 6]. New particle processes, forbidden in Minkowski space, are to be considered leading to new Feynman diagrams even at first-order [5–8].

Quantum field theory in curved spacetime in itself has proven to be a rich theory full of new physical phenomena. One of the most striking features arising from this theory is gravitational particle production; the creation of particles by the expansion of spacetime even without particle interactions. The foundations were laid by quantization of free fields around the second half of the last century [9–17] and a considerable amount of work has been devoted to this aspect; see e.g., [18–20] and references therein. Although free fields are an important facet of study, realistic fields tend to interact with each other which naturally led scientists to investigate interacting quantum fields on a curved spacetime as well [21, 22].

But while the investigations were focused mainly on the problem of renormalization, a closely related topic of mutually interacting fields has still been only scarcely investigated. A few extensive studies were made some time ago which established the fact that the gravitational particle creation severely interferes with the process of mutual interaction making the usual in-out approach non-applicable in curved spacetime [5–8, 23, 24]. Lately interest in the subject has resurfaced with the applications to QED processes in de Sitter spacetime [25–27]. The de Sitter spacetime is of considerable interest in scenarios involving e.g., cosmological inflation, but so far there is no experimental verification for this scenario. The postinflationary universe on the other hand includes universes dominated by radiation and ordinary matter with experimental data verifying the existence of these eras [28, 29], yet it has only scarcely been investigated from particle decay point of view [30, 31].

This thesis aims to increase our knowledge on how gravity affects particle decay particularly in the postinflationary universe. We begin by introducing the subject in question in the first three chapters with the rest of the thesis concerning the results obtained in Publications I-V. Chapter 1 concerns cosmology which sets up the background framework for this thesis. The introduction of the Robertson-Walker universe and the Einstein and Friedmann equations form the basis for describing the evolution of the universe. We discuss thermodynamics in the early universe with the Boltzmann equations and derive the equations which govern the time evolution of energy densities of a non-relativistic particle decaying into radiation. The chapter concludes by a brief review on the concept of cosmological inflation and reheating. These last two subjects give sufficient knowledge to understand the framework for the results on reheating in Chap. 5.

From the cosmological background we move to quantum theory and put quantized fields to propagate on the curved spacetime in Chap. 2. The focus will be on quantizing free fields on a curved spacetime revealing the profound phenomenon of gravitational particle creation. In a curved spacetime the vacuum becomes observer dependent, therefore the question of defining the vacuum state and particle concept in curved spacetime is also touched upon with the focus on a spatially flat Robertson-Walker universe. A discussion on conformal invariance concludes this chapter for its importance for later chapters.

With the free fields quantized, we introduce interactions into the mix in Chap. 3. The treatment of interacting fields in curved spacetime is very superficial in this chapter with the main focus being on conceptual issues arising when the background is curved. We will see in this chapter that the traditional in-out scheme from Minkowskian field theory cannot be straightforwardly transferred to curved spacetime because the gravitational particle creation interferes so severely with the mutual interaction. A method to calculate meaningful transition amplitudes in curved spacetime is then introduced via the added-up formalism introduced in [5] and a procedure to calculate decay rates in curved spacetime is given within this formalism.

The decay rate of a massive scalar particle to decay into massless particles in an expanding universe is discussed in Chap. 4 where the results of Publications I-IV are presented. We will find that the Minkowskian decay rates are modified and become time dependent. The case of a massive scalar to decay into massless scalars or fermions is considered within the added-up formalism and we present the results for the transition amplitudes and decay rates. We discuss some of the phenomena these introduce and find that for a conformally coupled massive particle, the decay into scalar channel is diminished and for fermionic channel enhanced by the spacetime expansion. Moreover, we find that in the very early universe the fermionic channel is the dominant channel of decay for these particles.

In Chap. 5 we present the results of Publication V. In it we considered a reheating scenario in a kination epoch through the Boltzmann equations, gravitational particle creation and the modified decay rates. It is seen that if the particle content is created by the expanding universe alone, this mechanism is able to reheat the universe to temperatures of about 10^4-10^{12} GeV. Finally, in Chap. 6 the key findings of this thesis are summarized and take a look at future regarding these findings.

Chapter 1

The Early Universe

The study of the early universe began taking shape somewhere in the 1980s when ideas from modern particle physics were adopted into cosmological context in a wide range of topics[1] to what is nowadays known as particle cosmology. The Standard Model of particle physics provided an understanding of physics up to the electroweak scale of about 250 GeV and using ideas from particle physics, models could be constructed to discuss physics even up to the Planck scale of about 10^{19} GeV. Applying these ideas and models, a wealth of interesting and compelling scenarios which might have taken place in the early universe began to emerge. These included events with intriguing names such as baryogenesis, leptogenesis and inflation to name just a few. Even the matter content may be envisaged to be something very strange and exotic like that of stiff matter [35–37] which can be realized in a variety of scenarios.

In this chapter, we will embark on a brief journey to these earliest of times. Topics will be picked up only from those needed for sufficient understanding of the results discussed in this thesis. We will consider the spatially flat Robertson-Walker cosmological model as it is well supported by observational evidence [28, 29, 38]. To lay the background for the results of Chap. 5, we consider the decay of a massive particle into massless particles with the help of the Boltzmann equations. In doing so, special care is taken with the fact in mind that in a curved spacetime the decay rates become time dependent. Finally, we will take a brief look at cosmological inflation which gives the background for studying reheating in Chap. 5 through the Boltzmann equations. Throughout the thesis we will use natural units with $\hbar = c = 1$ unless otherwise stated.

[1] See e.g., [32–34] and references therein.

1.1 Standard Cosmology

The cosmological principle, a premise at the heart of contemporary cosmological models, states that no observer occupies a preferred position in the universe. It looks the same no matter who you are or wherefrom you look at it. To put it more precisely, the universe is said to be spatially homogeneous and isotropic. But certainly the universe cannot share these properties on all scales. Indeed, on a small scale the universe looks anything but; it is filled with stars, galaxies and galaxy clusters ranging from a mere 1 light year in size of a star to the enormous sizes of about 10^7 light years for galaxy clusters. But increasing still the cosmic length scale uniformity does begin to emerge and on scales larger than those of the galaxy clusters the universe begins to look spatially homogenous and isotropic, a fact which is supported by many experiments, most manifestly in the uniformity of temperature in the CMB [28, 29] and on the large-scale structure of the universe [38]. Taking this large scale viewpoint, a natural starting place for a cosmological model is therefore a mathematical description of the spacetime based on this principle.

1.1.1 The Robertson-Walker Metric

Based on the cosmological principle, a suitable metric describing a universe with homogeneous and isotropic spatial sections is given by the Robertson-Walker metric [39, 40]

$$ds^2 = dt^2 - a(t)^2\Big[\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\Big], \tag{1.1.1}$$

where the scale factor $a(t)$ is a dimensionless function of the coordinate time t and (r, θ, ϕ) are the spherical coordinates. The sign convention $(+,-,-,-)$ is used. The spatial curvature of the universe is characterized by the parameter κ with $\kappa < 0$ corresponding to a negatively curved, $\kappa > 0$ to positively curved and $\kappa = 0$ to a spatially flat space. However, experimental data from various independent sources confirm the universe to be spatially flat to a high degree [28, 29] and these observations give credence for a study of a spatially flat Robertson-Walker metric in particular. In this case $\kappa = 0$ and the metric simplifies into the form

$$ds^2 = dt^2 - a(t)^2 d\mathbf{x}^2, \tag{1.1.2}$$

where the bold face denotes the spatial section given in cartesian coordinates (x, y, z). In this form, the metric is said to be given in comoving coordinates. Often it is convenient to define a new time variable η, the conformal time, as $dt = a(\eta)d\eta$ and express the metric (1.1.2) as

$$ds^2 = a(\eta)^2(d\eta^2 - d\mathbf{x}^2). \tag{1.1.3}$$

It then follows directly that the spatially flat Robertson-Walker metric is conformal to the Minkowski metric, i.e., $g_{\mu\nu}^{RW} = a(\eta)^2\eta_{\mu\nu}$ with a new time variable η. Spacetimes sharing this property are said to be conformally flat. This is an example of a conformal transformation in which the metric tensor $g_{\mu\nu}$ is transformed as $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}$, for some continuous, real-valued function $\Omega(x)$ of the spacetime coordinates [41].

1.1.2 Einstein's Equations

The mathematical description for a spatially homogeneous and isotropic metric forms only one part of the cosmological model and is geometrical without any reference to gravity itself. To complete the cosmological model, we still need to incorporate gravity into it. Gravity and the curvature of spacetime are intertwined by the theory of general relativity in which the curvature of the spacetime is related to the matter distribution in it. The content of general theory of relativity is embodied in a set of differential equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{1.1.4}$$

known as Einstein's equations, where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar and G denotes Newton's gravitational constant. The energy and momentum content of the universe is contained in the the stress-energy tensor $T_{\mu\nu}$. Contracting both sides of the Eq. (1.1.4) gives a useful form as

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}), \tag{1.1.5}$$

where T denotes the trace of the energy-momentum tensor. Being a set of second-order nonlinear partial differential equations for the metric $g_{\mu\nu}$, the Einstein's equations are extremely difficult to solve in any sort of generality. Therefore, the first step to finding solutions to the Einstein's equations is to describe the matter content of the universe in terms of $T_{\mu\nu}$.

A good approximation to the form of matter is found by applying the cosmological principle to the tensor $T_{\mu\nu}$ itself. In this case the underlying symmetries force the energy-momentum tensor of the universe to take the form of a perfect fluid which is completely characterized by the rest-frame energy density ρ and an isotropic pressure p [34]. From the general form of the stress-energy tensor for a perfect fluid

$$T_{\mu\nu} = (\rho(t) + p(t))u_\mu u_\nu - p(t)g_{\mu\nu}, \tag{1.1.6}$$

where the four velocity of the fluid is denoted by u^μ, one obtains in the rest frame $u^\mu = (1, 0, 0, 0)$ a convenient form,

$$T^\mu{}_\nu = \mathrm{diag}(\rho, -p, -p, -p), \tag{1.1.7}$$

by raising an index in Eq. (1.1.6). The time dependence of the functions ρ and p has been suppressed to clarify notation, a convention which will mostly be used from henceforth. With this perfect fluid approximation, the Robertson-Walker metric (1.1.2) can be inserted into the Einstein's equations (1.1.5) to obtain the relationship between the expansion of the universe and its matter content, the Friedmann equations. For the 00-component of (1.1.5) we have

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{1.1.8}$$

With the help of this equation, the ii-components may be written as

$$H^2 = \frac{8\pi G}{3}\rho, \tag{1.1.9}$$

where the Hubble parameter H is defined as $H \equiv \dot{a}/a$ and the dot refers to derivation with respect to the coordinate time t. Finally, from the conservation of the stress-energy tensor $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ denotes the covariant derivative, one obtains the continuity equation as

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{1.1.10}$$

With this basic formalism, we turn to the description of the evolution of the universe.

1.1.3 Matter Content and Evolution of the Universe

Once the evolution of the energy density ρ from Eq. (1.1.10) is known, the actual dynamical evolution of the universe is obtained by solving the Friedmann equation (1.1.9) for the scale factor $a(t)$. To solve the continuity equation (1.1.10), a relationship between the pressure and energy density can be introduced in the form of an equation of state,

$$p = \omega\rho, \tag{1.1.11}$$

where the equation of state parameter ω is assumed to be a constant. With this assumption, Eq. (1.1.11) describes most of the common cosmological fluids like radiation and matter. The conservation of energy (1.1.10) is now given by

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}(1+\omega)\rho = 0 \tag{1.1.12}$$

with a solution of the form

$$\rho(t) = \rho_0 a^{-3(1+\omega)}, \tag{1.1.13}$$

where the value of the constant ρ_0 is fixed by initial conditions. Plugging this solution into the Friedmann equation (1.1.9), one obtains in the case of a spatially flat universe

$$a(t) = bt^{\frac{2}{3(1+\omega)}}, \tag{1.1.14}$$

where b is a positive constant controlling the expansion rate of the universe again fixed by initial conditions.

The evolution of the universe is therefore seen to depend on the matter content through the equation of state parameter ω and it is of utmost importance to understand the nature of the matter driving the expansion. Direct evidence from the CMB provides support for an early universe dominated by radiation [28, 29] and after that the universe is known to be matter-dominated as the structures were formed. These universes have the equation of state parameters $\omega = 1/3$ and $\omega = 0$, respectively and are governed by the scale parameter $a(t) \propto t^{1/2}$ for radiation and $a(t) \propto t^{2/3}$ for matter. Of course, in the very earliest of times it is by no means unreasonable to envisage a universe dominated by a matter of very exotic nature. One type of exotic matter in particular has received much attention lately; a fluid obeying the stiff equation of state $\rho = p$. The intrigue in this particular equation of state lies in the possibility that an early stiff-matter-dominated era may well have existed, since the stiff matter energy density scales as $\rho \propto a^{-6}$ diluting much faster than radiation, for which $\rho \propto a^{-4}$. Indeed, the implications of an early stiff matter era near the singularity has been much discussed [35–37] and also a more general stiff matter cosmology has been studied [42]. An early stiff matter era may also have implications for baryogenesis [43] and Big Bang Nucleosynthesis [44] and it also appears on various inflationary and reheating scenarios [45–53].

1.2 Thermodynamics in the Early Universe

The early universe was extremely hot and dense and in this environment particle interactions Γ occurred very rapidly compared to the expansion rate H i.e., $\Gamma \gg H$. The interactions were so frequent that any fluctuation in their energy density would be smoothed out and thermal equilibrium would be reached. This makes the equilibrium description of the early universe is a very good approximation. But this is not always the case. Once $\Gamma \sim H$, the equilibrium description is no longer a good approximation because the interactions do not occur rapidly enough to bring the system into thermal equilibrium. The particles will then decouple from the thermal plasma, but they will do so at different times because the interaction rates Γ are different for different particle interactions and masses. The equilibrium thermodynamic description therefore fails at $\Gamma \sim H$ and an out-of-equilibrium description is needed. This description is provided by the Boltzmann equations which will be reviewed following [32].

1.2.1 The Boltzmann Equations

The evolution of a particle's phase space distribution function $f_i(p^\mu, x^\mu)$ is governed by the Boltzmann equation [32]

$$L[f_i] = C[f_i], \tag{1.2.1}$$

where C is the collision operator characterising interactions between particles and L is the Liouville operator. The index i represents a particle species. Due to homogeneity and isotropy of a Robertson-Walker universe, the phase space density function $f_i(E(\mathbf{p}), t)$ is a function of energy E and time t only and the Liouville operator is given by [32]

$$L[f_i] = E\frac{\partial f_i}{\partial t} - \frac{\dot{a}}{a}|\mathbf{p}^2|\frac{\partial f_i}{\partial E}. \tag{1.2.2}$$

Using the definition of number density n in terms of phase space density f,

$$n_i = \frac{g_i}{(2\pi)^3}\int f_i(E,t)d^3p, \tag{1.2.3}$$

the Boltzmann equation for a non-relativistic particle in a Robertson-Walker universe can be cast in integral form as

$$\dot{n}_i + 3Hn_i = \frac{g_i}{(2\pi)^3}\int C[f_i]\frac{d^3p}{E}, \tag{1.2.4}$$

where g_i denotes the internal degrees of freedom of the particle. Given in the form (1.2.4), the Boltzmann equations are a set of coupled integral-partial differential equations for all species present. In many cases simplifications can be made to reduce the problem to a single integro-differential equation for the particle species of interest. One of the most common uses of Boltzmann equations encountered is an out-of-equilibrium decay of a massive particle.

1.2.2 Out-of-Equilibrium Decay

The out-of-equilibrium decay process where a non-relativistic ϕ particle decays into φ particles is described by the Boltzmann equation [32]

$$\dot{n}_\phi + 3Hn_\phi = -\Gamma_\varphi n_\phi. \tag{1.2.5}$$

Here Γ_φ denotes the decay rate of the ϕ particles to decay into φ particles. The physical significance of the terms is manifest. The second term on the left side accounts for the dilution of the ϕ particles due to expansion and the right-hand side accounts for their decay into φ particles. For a non-relativistic ϕ particle with mass m, the energy density is given by $\rho_\phi = mn_\phi$, and the evolution of the energy density is

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\varphi \rho_\phi. \tag{1.2.6}$$

The solution of (1.2.6) is readily found as

$$\rho_\phi(t) = \Big(\frac{a(t_0)}{a(t)}\Big)^3 \rho_\phi(t_0) e^{-\int_{t_0}^t \Gamma_\varphi dt'}, \tag{1.2.7}$$

where t_0 indicates the initial time. It is worth noting that in the absence of interactions, Eq. (1.2.6) reduces to the energy conservation equation (1.1.10). If we assume that the energy released by the decay of the ϕ particles is rapidly converted into the energy of some relativistic particles φ, the energy density of the universe resides only in these two components. The Boltzmann equations for the relativistic species is given by

$$\dot{\rho}_\varphi + 4H\rho_\varphi = \Gamma_\varphi \rho_\phi, \tag{1.2.8}$$

where the right-hand side now accounts for the creation of φ particles via the decay. For relativistic particles, the factor of 4 in the second term on the left is due to redshift. A formal solution is again easily found,

$$\rho_\varphi(t) = \frac{1}{a(t)^4}\int_{t_0}^t \Gamma_\varphi \rho_\phi(t') a(t')^4 dt' + \Big(\frac{a(t_0)}{a(t)}\Big)^4 \rho_\varphi(t_0). \tag{1.2.9}$$

The solutions (1.2.7) and (1.2.9) describe the evolution of the energy densities of these two components in the universe. Eventually, if the non-relativistic particles decay, their energy density is transferred to radiation. If the universe is initially dominated by non-relativistic matter, it will later become the subdominant form due to decays and radiation will start to dominate at some point $\rho_\phi(t_{eq}) = \rho_\varphi(t_{eq})$, where t_{eq} denotes the equilibrium time. The universe then transforms from a matter-dominated one into a radiation-dominated universe.

In the preceding calculations it is usually assumed that the decay rate Γ_φ is time-independent simplifying considerably the integrations. This assumption is of course valid when using decay rates obtained in Minkowskian field theory, but in a curved spacetime when using quantum field theory in curved spacetime to calculate the decay rates, this does not need to be so. We have therefore left the integral over the decay rate explicit in Eq. (1.2.7) in anticipation of later chapters.

1.3 Inflationary Universe

Although the standard cosmological model has been very succesful in describing the evolution of the universe and direct evidence supports its validity all the way back to primordial nucleosynthesis, it is not without its shortcomings. A major issue concerns the initial conditions which have produced the universe as we see it today. Although the spatial geometry of the universe is observed to be very flat to a high degree, the initial conditions required for this to happen would have had to be set to an accuracy of dozens of orders of magnitude. That such finely tuned initial conditions would have occurred seems extremely unlikely. This is known as the flatness problem. Second, there is the horizon problem. According to CMB, the regions in the whole observable universe, even two regions opposite of one another, are to a high degree in thermal equilibrium. But because of the constancy of the speed of light, these two opposite regions have had no time to interact with each other in order to bring them to thermal equilibrium. The standard cosmological model has difficulty explaining how these causally disconnected regions have reached thermal equilibrium.

These problems may be overcome by introducing a concept of cosmological inflation [54–56]. The basic idea behind this proposition is that in the early stages of its evolution the universe underwent a rapid period of accelerated expansion. Inflation, as originally proposed in 1981 by Guth [54], provided answers to the shortcomings of standard cosmology but lead to a situation where the universe would inflate forever [57]. A year later, Linde [55] and independently Albrecht and Steinhardt [56] proposed an inflation model in which inflation occurred by a scalar field rolling slowly down a potential as inflation occurred.

While inflation is able to describe how the universe ended up being spatially flat and homogeneous, its true merit lies in its ability to describe how structure formation took place via small perturbations and at present the main use of inflation lies in investigating this structure formation [33, 58]. But, even though this thesis does not concern the inflation scenario in these applications, we give a short review of the ”new inflation” model.

1.3.1 Standard Inflation

In the most basic inflationary picture one has a single scalar field ϕ called the inflaton, whose potential energy can lead to the accelerated expansion of the universe. The energy density and pressure for a scalar field in flat Robertson-Walker universe are given by [32]

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{1.3.1}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{1.3.2}$$

The spatial gradient terms in the above equations have been suppressed, since they scale as $1/a^2$ and quickly become irrelevant as the universe expands rapidly. The Friedmann equation and the continuity equation now give

$$\frac{1}{3M_{Pl}^2}\Big(\frac{1}{2}\dot{\phi}^2 + V(\phi)\Big) = H^2, \tag{1.3.3}$$

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0, \tag{1.3.4}$$

where the reduced Planck mass is defined as $M_{Pl} \equiv (8\pi G)^{-1/2}$. Since inflation is a period of accelerated expansion $\ddot{a} > 0$, the first Friedmann equation (1.1.8) leads to the requirement $\rho + 3p < 0$. The implication for a scalar field ϕ with energy density and pressure given by (1.3.1) and (1.3.2) is that the potential energy of the inflaton dominates its kinetic energy. Furthermore, in this case $\rho_\phi = -p_\phi$, so $\omega = -1$. With the potential energy dominating, the standard way for analyzing inflation is the so-called slow-roll approximation [33],

$$\frac{\dot{\phi}^2}{2} \ll V(\phi), \quad |\ddot{\phi}| \ll 3H|\dot{\phi}| \tag{1.3.5}$$

so that Eqs. (1.3.3) and (1.3.4) are approximately given by

$$H^2 \approx \frac{1}{3M_{Pl}^2}V(\phi), \tag{1.3.6}$$

$$3H\dot{\phi} \approx -V'(\phi). \tag{1.3.7}$$

Defining the slow-roll parameters η and ε as

$$\varepsilon \equiv \frac{M_{Pl}^2}{2}\Big(\frac{V'(\phi)}{V(\phi)}\Big)^2, \quad \eta \equiv M_{Pl}^2\frac{V''(\phi)}{V(\phi)}, \tag{1.3.8}$$

the slow-roll conditions (1.3.5) are given by $\varepsilon \ll 1$ and $|\eta| \ll 1$ which is seen by a direct substitution of Eqs. (1.3.6) and (1.3.7) into the slow-roll conditions. Once these conditions are violated, inflation effectively ends.

1.3.2 Reheating the Universe

During inflation the size of the universe has increased tremendously and as a result all the particles which might have been present have been diluted away leaving behind a cold universe practically void of all matter. To recover the standard description of the evolution of the universe, a mechanism for creating radiation to drive its expansion is needed, i.e., one needs to reheat the universe.

Several methods to achieve this have been investigated in the literature. In the original paper of Guth, this was achieved through collision of bubbles [54], but the eternal inflation from which the model suffered prevented the universe from reheating [57]. In the new inflationary models, reheating was accomplished by a perturbative decay of oscillations near the minimum of the inflaton potential [55]. Reheating may also occur by parametric resonance in a process known as preheating in which the produced particles decay extremely rapidly and the oscillation phase ends almost instantly [59]. In this thesis we will consider a reheating mechanism via gravitational particle creation, but as a prelude and to introduce some concepts, the main aspects of reheating via oscillations are briefly reviewed.

Once the slow-roll conditions are violated, the motion of the scalar field is once again described by Eq. (1.3.4). The scalar field moves rapidly into the minimum of its potential and begins to oscillate. In this process particles are created which subsequently decay ultimately producing radiation. To take this decay into account, a dissipative term $\Gamma_\varphi \dot{\phi}$ is added into Eq. (1.3.4). By multiplying the ensuing equation by $\dot{\phi}$ and averaging over an oscillation cycle, we obtain [32]

$$\dot{\rho}_\phi + 3H\rho_\phi + \Gamma_\varphi \rho_\phi = 0, \tag{1.3.9}$$

which describes the evolution of the energy density of the scalar field. This is recognized as the Boltzmann equation (1.2.6) for the decay of a nonrelativistic particle. Indeed, the oscillations truly behave as non-relativistic matter [32] and the results of Sec. 1.2.2 apply.

Assuming that the decay products φ are relativistic, the evolution of the energy density is described by the Boltzmann equation (1.2.9) for relativistic particles. At the beginning of the oscillations the energy loss into the φ particles is initially negligible compared to the energy loss from dilution [4]. Hence, from the start of the oscillations up until $t \sim \Gamma_\varphi^{-1}$, the universe is dominated by ordinary matter. In matter-dominated era $a(t) \propto t^{2/3}$, so Eq. (1.2.9) gives

$$\rho_\varphi(t) = \frac{9M_{Pl}^2\Gamma_\varphi}{5t}\Big(1 - \Big(\frac{t_0}{t}\Big)^{5/3}\Big), \tag{1.3.10}$$

where t_0 denotes the end of inflation, i.e., the beginning of oscillations. It has also been assumed that the initial density $\rho_\varphi(t_0) = 0$ because initially there is no production of radiation. It is seen that the energy density ρ_φ increases from 0 to about $9M_{Pl}^2\Gamma_\varphi^2/5$.

Neither the perturbative decay nor the preheating mechanism produce a thermal spectrum of the decay products [4]. In many situations it is also important to know the temperature at which the universe takes on a thermal distribution. Therefore, the decay products need to interact with each other and thermalize to bring the system into thermal equilibrium. On that moment, the energy density for these relativistic particles is given by $\rho_\varphi = g_*\pi^2T^4/30$, where T denotes the temperature and g_* the effective degrees of freedom. The universe becomes radiation dominated at the time $t \sim \Gamma_\varphi^{-1}$ when the ϕ particles decay rapidly. The temperature, known as reheating temperature, at this time is

$$T(t = \Gamma_\varphi^{-1}) \approx 1.5g_*^{-1/4}\sqrt{M_{Pl}\Gamma_\varphi}, \tag{1.3.11}$$

and after the evolution of the universe is described by the standard Hot Big Bang model.

Chapter 2

Gravitation and Quantum Fields

Quantum field theory in curved spacetime is the study of propagating quantum fields on a curved spacetime, where the background metric is left unquantized [18, 19]. Since the Minkowskian metric familiar from flat space quantum field theory is replaced by a more general metric governed by Einstein's equations, gravity is explicitly taken into account. From the early studies of particle production [10–17] to results on quantum black holes [60–64], this theory has proven to be rich and full of new physical phenomena. Though all the aspects of the theory are interesting in their own regard, in this thesis the main ingredient picked up is the gravitational particle creation in a spatially flat Robertson-Walker universe, which was extensively studied by Parker in the 1960s. In his articles [10–12], among other things, it was shown that particles are always created in pairs with opposite momenta. Studies were not only constrained to isotropic universes, but also on anisotropic universes and indeed, it may even be that particle production leads to rapid isotropization of the metric in the very early universe [13, 14]. While the current creation of particles due to expansion of space is quite insignificant, it may have been an important phenomenon in the early times of the universe when cosmic expansion proceeded rapidly [65]. To describe this phenomenon will lead us into concepts like causality of spacetime and a special type of transformations called Bogoliubov transformations. These concepts arise when quantization on a classical curved spacetime is performed.

The starting point therefore lies on the quantization of a free field in a curved spacetime. Although nature has shown us that most important phenomena occur when particles are interacting with each other, understanding the quantization of free fields provides the basis for studying these interactions because once the free particle is understood, mutual interactions can be described by a perturbative scheme based on the free field. Even more crucially, the free field quantization in curved spacetime also brings out the most striking features of curved spacetime quantum field theory in a clear manner. Once a grasp of the fundamental notions is gained by the quantization procedure of the free field one is able to move to interactions with physically motivated procedures. Without loss of generality we will consider a real scalar field because any complex scalar field can be expressed in terms of two real scalar fields. This not only helps to clarify the procedure but also brings out the features characteristic to curved spacetime field theory more transparently. Quantizing the spinor field is also touched upon from the parts necessary for the later chapters.

2.1 Canonical Quantization in Curved Space

To quantize fields in curved spacetime one tries to transfer the quantization procedure of the Minkowskian quantum field theory to curved space in the most general fashion possible. While the procedure proceeds in quite similar fashion, a time-dependent metric tensor and curved spacetime introduces some subtleties peculiar to curved space alone. In the following quantization procedure we will focus mainly on understanding concepts rather than on mathematical rigor.

Speaking of these peculiarities, the common method in physics to describe the evolution of a system as an initial value problem, i.e., the evolution of the system is described uniquely once the initial state of the system is known at some initial time, needs to be stated more precicely. This is because in curved spacetime this is not necessarily always true since causality may be violated in certain spacetimes. To be able to describe the system and to find a unique solution for the field equations along with a consistent quantization prescription, we will restrict to a class of spacetimes which are globally hyperbolic i.e., those admitting a Cauchy surface[1]. Given initial conditions on a Cauchy surface determine the future (and the past) uniquely. While it is possible to try to quantize the theory without this restriction [67, 68], for the purpose of this thesis, we are only interested in

[1] For technical definitions see e.g., [66].

the globally hyperbolic spacetime like the Robertson-Walker universe.

To begin, the physics of flat spacetime must be first generalized into curved spacetime. This means that the action functional for the system must be constructed so that it is invariant under general coordinate transformations according to the principle of general covariance instead of ordinary (global) Lorentz invariance. The simplest way to achieve this is the minimal coupling prescription, where the Minkowskian metric $\eta_{\mu\nu}$ is replaced by a general metric $g_{\mu\nu}$, the ordinary derivatives ∂_μ by the covariant derivative ∇_μ and $d^n x$ by the invariant volume element $\sqrt{-g}\, d^n x$, where g denotes the determinant of the metric. With the assumption of global hyperbolicity and a generally covariant action, the quantization may proceed. We will consider a field ϕ_a, where a denotes the different fields, propagating in a classical spacetime with a line element $ds^2 = -g_{\mu\nu} dx^\mu dx^\nu$.

2.1.1 Canonical Formalism

The generally covariant classical action functional S for the system in an n-dimensional spacetime is given by

$$S[\phi_a(x)] = \int \mathcal{L}(\phi_a(x), \nabla\phi_a(x), g_{\mu\nu}(x)) d^n x, \tag{2.1.1}$$

where the Lagrangian density is denoted by $\mathcal{L}$ and ∇ denotes collectively the covariant derivatives of the field which reduce to ordinary partial derivatives for a scalar field. The factor of $\sqrt{-g}$ of the volume element has been absorbed into $\mathcal{L}$ and the spacetime coordinate x includes both spatial and temporal coordinates. The equations of motion are found by varying the action (2.1.1) with respect to the fields ϕ_a and requiring the variation to vanish. This yields the Euler-Lagrange equations of motion for the scalar field

$$\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Big) = 0, \tag{2.1.2}$$

while variation with respect to the metric yields the energy-momentum tensor $T^{\mu\nu}$ defined by

$$T^{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}(x)}, \tag{2.1.3}$$

where $\delta S/\delta g_{\mu\nu}$ referes to the functional derivative.

To proceed with the introduction of the canonical commutation relations and canonical momentum, one must proceed with more subtlety because of the dynamical nature of spacetime and causality already alluded above. It is here where the assumption on global hyperbolicity becomes necessary as it allows one to choose a time coordinate t such that every surface of constant time is a Cauchy surface [69, 70]. One would therefore expect the theory to have a well defined classical evolution from the initial conditions given at a Cauchy surface and indeed it can be shown to be true [71]. The canonical momentum π can therefore be defined as

$$\pi_a(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_a)}, \tag{2.1.4}$$

and with the help of π_a the Hamiltonian H is defined as

$$H = \int [\pi(x)\partial_0\phi(x) - \mathcal{L}(x)] d^{n-1}x. \tag{2.1.5}$$

In the canonical quantization, the field ϕ and its conjugate momentum π are promoted to operators acting on a Hilbert space and satisfying the commutation relations

$$\begin{aligned} &[\phi_a(t,\mathbf{x}), \phi_b(t,\mathbf{x}')] = 0, \\ &[\pi_a(t,\mathbf{x}), \pi_b(t,\mathbf{x}')] = 0, \\ &[\phi_a(t,\mathbf{x}), \pi_b(t,\mathbf{x}')] = i\delta_{ab}\delta(\mathbf{x}-\mathbf{x}'), \end{aligned} \tag{2.1.6}$$

defined on the same constant time hypersurface. These relations can be shown to be independent of the spacelike hypersurfaces chosen to represent the equal time and of coordinate systems within them, i.e., they are covariant [72]. With this basic formalism, we can now proceed with the quantization of a free scalar field.

2.1.2 The Free Scalar Field

In generalizing the Lagrangian density from the special relativistic theory into curved spacetime, one often seeks the simplest generalization which reduces to the ordinary free field Lagrangian in the absent of gravity. The simplest curved space generalization of a Lagrangian density for a neutral real scalar field ϕ is given by

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2), \tag{2.1.7}$$

where ξ is a dimensionless constant coupling the field to gravity via the Ricci scalar R. The Euler-Lagrange equations (2.1.2) yield the Klein-Gordon equation in curved spacetime,

$$\Box\phi(t, \mathbf{x}) + (m^2 + \xi R)\phi(t, \mathbf{x}) = 0, \tag{2.1.8}$$

where $\Box$ is the d'Alembert operator $\Box \equiv \nabla_\mu\nabla^\mu$. The coupling ξ appearing in (2.1.7) and (2.1.8) is essentially arbitrary although there have been attempts to restrict its value [73]. Two values most often used in the literature are the minimal coupling $\xi = 0$ and the conformal coupling $\xi = (n-2)/(4n-4)$ which in four spacetime dimensions gives the value $\xi = 1/6$. The former turns off the interaction with R and the latter makes the field equations (2.1.8) conformally invariant for a free massless field. We will return to this conformal invariance more in depth later in the chapter.

To quantize the theory, we must first find solutions of the Klein-Gordon equation (2.1.8). To do that, we enclose the field ϕ in a box with a coordinate length L and coordinate volume $V = L^3$ and impose periodic boundary conditions. This is to obtain a mathematically consistent theory and after physical quantities have been calculated, the volume V may be taken to infinity [19]. To begin, we introduce an inner product, the Klein-Gordon inner product, for two solutions as

$$\langle\phi|\psi\rangle = -i\int_\Sigma [\phi(x)\partial_\mu\psi^*(x) - (\partial_\mu\phi(x))\psi^*(x)]\sqrt{-g}d\Sigma^\mu, \tag{2.1.9}$$

where $d\Sigma^\mu = n^\mu d\Sigma$ with $d\Sigma$ being the volume element in a given spacelike hypersurface Σ and n^μ a future-directed unit vector orthogonal to this hypersurface. The scalar product (2.1.9) is independent of the choice of the hypersurface [71]. Suppose then that $\{u_\mathbf{k}, u^*_\mathbf{k}\}$ form a complete set of solutions of the Klein-Gordon equation (2.1.8) which are orthonormal with respect to the inner product (2.1.9), i.e., they satisfy the relations

$$\langle u_\mathbf{k}|u_{\mathbf{k}'}\rangle = \delta_{\mathbf{k}\mathbf{k}'}, \quad \langle u^*_\mathbf{k}|u^*_{\mathbf{k}'}\rangle = -\delta_{\mathbf{k}\mathbf{k}'}, \quad \langle u_\mathbf{k}|u^*_{\mathbf{k}'}\rangle = 0. \tag{2.1.10}$$

The mode $u_\mathbf{k}$ is said to be a positive mode in the sense that $\langle u_\mathbf{k}|u_\mathbf{k}\rangle > 0$

and similarly the mode $u^*_{\mathbf{k}}$ is said to be negative mode because $\langle u^*_{\mathbf{k}}|u^*_{\mathbf{k}}\rangle < 0$. The field ϕ can then be expanded in terms of these solutions as

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(t, \mathbf{x})]. \tag{2.1.11}$$

The quantization proceeds as described above by imposing the canonical commutation relations which are equivalent to having the commutation relations

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k}\mathbf{k}'} \tag{2.1.12}$$

for the annihilation operator $a_{\mathbf{k}}$ and creation operator $a^\dagger_{\mathbf{k}}$ with other commutators vanishing. The vacuum state $|0\rangle$ is defined in the usual way

$$a_{\mathbf{k}} |0\rangle = 0, \quad \forall\, \mathbf{k} \tag{2.1.13}$$

and the one-particle state $|1_{\mathbf{k}}\rangle$ is obtained by operating on the vacuum state by the creation operator,

$$a^\dagger_{\mathbf{k}} |0\rangle = |1_{\mathbf{k}}\rangle\,. \tag{2.1.14}$$

In quantum field theory one usually deals with many particle systems, so we also introduce the Fock multi-particle states defined by acting repeatedly with the creation operators on the vacuum state to produce a multi-particle state

$$a^\dagger_{\mathbf{k}_1} a^\dagger_{\mathbf{k}_2} \cdots a^\dagger_{\mathbf{k}_j} |0\rangle = |1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, ..., 1_{\mathbf{k}_j}\rangle\,, \tag{2.1.15}$$

with all $\mathbf{k}_j$ distinct. If some of the $\mathbf{k}_j$ are repeated n_j times, there will be a corresponding normalization factor of $1/\sqrt{n_j!}$ on the left-hand side.

The quantization in a globally hyperbolic spacetime has so far been carried out in a fashion similar to Minkowski spacetime generalizing the procedure straightforwardly to curved spacetime. There is, however, a major exception in curved spacetime responsible of all the novel phenomena associated with quantum field theory in curved spacetime, namely there is no unique choice analogous to the positive mode solutions in Minkowski space; the mode solutions $u_{\mathbf{k}}$ appearing in (2.1.11) are not unique [74, 75].

In Minkowskian field theory the positive mode solution $u_{\mathbf{k}} \propto e^{-i\omega t}$, with $\omega^2 = k^2 + m^2$, so that $\partial_t u_{\mathbf{k}} = -i\omega u_{\mathbf{k}}$. Moreover, the operator ∂_t is a global timelike Killing vector so a positive mode solution defined via this Poincaré invariant criterion stays positive throughout the whole spacetime [18]. This is to say, the vacuum is invariant under the Poincaré group of transformations, while in a curved spacetime the Poincaré group is no longer a symmetry group of the spacetime [72]. In a general spacetime there may not be any timelike Killing vectors to define positive modes, or they may be defined only in some specific regions of the spacetime. Hence, there exists an unlimited number of different mode solutions, each equally viable to be used in the expansion of the field operator ϕ. These expansions are, however, related with one another leading us naturally to Bogoliubov coefficients.

2.1.3 Bogoliubov Transformations

Consider therefore two distinct sets of mode solutions each with a corresponding set of creation and annihilation operators satisfying the commutation relations (2.1.12). Let these be $\{u_{\mathbf{k}}, u^*_{\mathbf{k}}\}$ with $\{a^\dagger_{\mathbf{k}}, a_{\mathbf{k}}\}$ and $\{v_{\mathbf{k}}, v^*_{\mathbf{k}}\}$ with $\{b^\dagger_{\mathbf{k}}, b_{\mathbf{k}}\}$. Let $|0\rangle$ denote the vacuum state of the $u_{\mathbf{k}}$ modes defined by

$$a_{\mathbf{k}}\,|0\rangle = 0, \quad \forall\, \mathbf{k}. \tag{2.1.16}$$

and let $|\tilde{0}\rangle$ denote the vacuum state of the $v_{\mathbf{k}}$ modes defined by

$$b_{\mathbf{k}}\,|\tilde{0}\rangle = 0, \quad \forall\, \mathbf{k}, \tag{2.1.17}$$

with the many-particle states are constructed in the usual way. The field operator ϕ may be expanded in both basis solutions

$$\phi(t, \mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}} u_{\mathbf{k}}(t, \mathbf{x}) + a^\dagger_{\mathbf{k}} u^*_{\mathbf{k}}(t, \mathbf{x})] = \sum_{\mathbf{k}'} [b_{\mathbf{k}'} v_{\mathbf{k}'}(t, \mathbf{x}) + b^\dagger_{\mathbf{k}'} v^*_{\mathbf{k}'}(t, \mathbf{x})]. \tag{2.1.18}$$

The annihilation operators may be written as $a_{\mathbf{k}} = \langle\phi|u_{\mathbf{k}}\rangle$, $b_{\mathbf{k}} = \langle\phi|v_{\mathbf{k}}\rangle$ (cf. (2.1.10)) and the relationship between the creation and annihilation operators in different regions are given by

$$a_{\mathbf{k}} = \sum_{\mathbf{k}'} (\alpha_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'} - \beta^*_{\mathbf{k}\mathbf{k}'} b^\dagger_{\mathbf{k}'}). \tag{2.1.19}$$

This is known as a Bogoliubov transformation and the complex coefficients

$\alpha_{\mathbf{kk}'}$ and $\beta_{\mathbf{kk}'}$ are known as Bogoliubov coefficients. In an isotropic spacetime their explicit form may be given in terms of the Wronskian of the positive mode solutions [20]. Using the commutation relations for the annihilation and creation operators, or alternatively the scalar product (2.1.10), one obtains the following relations for the Bogoliubov coefficients,

$$\sum_{\mathbf{s}}(\alpha_{\mathbf{ks}}\alpha^*_{\mathbf{k}'\mathbf{s}} - \beta^*_{\mathbf{ks}}\beta_{\mathbf{k}'\mathbf{s}}) = \delta_{\mathbf{kk}'} \tag{2.1.20}$$

$$\sum_{\mathbf{s}}(\alpha_{\mathbf{ks}}\alpha_{\mathbf{k}'\mathbf{s}} - \beta_{\mathbf{ks}}\beta_{\mathbf{k}'\mathbf{s}}) = 0 \tag{2.1.21}$$

Moreover, even though the modes $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ define different vacuum states, they are still both solutions of the Klein-Gordon equation (2.1.8) and are therefore related to each other. We may therefore express the mode $u_{\mathbf{k}}$ in terms of the modes $v_{\mathbf{k}}$ in a linear combination as

$$u_{\mathbf{k}} = \sum_{\mathbf{k}'}[\alpha_{\mathbf{kk}'}v_{\mathbf{k}'} + \beta_{\mathbf{kk}'}v^*_{\mathbf{k}'}]. \tag{2.1.22}$$

The number operator $N_{\mathbf{k}}$ may be constructed in the usual way by defining $N_{\mathbf{k}} = a^\dagger_{\mathbf{k}}a_{\mathbf{k}}$ and indeed it easily seen that $\langle 0|N_{\mathbf{k}}|0\rangle = 0$. But these $a_{\mathbf{k}}$ operators are related by Eq. (2.1.19) to the $b_{\mathbf{k}}$ operators. Therefore, we may also ask what the number operator gives if applied to the vacuum $|\tilde{0}\rangle$ of these modes. In this case, by the same relation, we have

$$\langle\tilde{0}|N_{\mathbf{k}}|\tilde{0}\rangle = \sum_{\mathbf{k}'}|\beta_{\mathbf{kk}'}|^2. \tag{2.1.23}$$

This result can be interpreted as the vacuum $|\tilde{0}\rangle$ of the $v_{\mathbf{k}'}$ modes containing $\sum_{\mathbf{k}'}|\beta_{\mathbf{kk}'}|^2$ particles in the $u_{\mathbf{k}}$ mode. The Bogoliubov coefficient $\beta_{\mathbf{kk}'}$ therefore becomes the essential quantity in evaluating particle creation in curved spacetime and effectively it gives the average number of particles created during the expansion of spacetime [10, 11].

2.1.4 Quantization of the Spinor Field

In generalizing the action for the scalar field above, we followed the minimal coupling principle and replaced the flat spacetime quantities with their curved spacetime counterparts. In truth, this procedure is valid only for objects which transform as tensors under Lorentz transformation and not for objects like spinors [34]. To incorporate spinors into general relativity, one must adopt a different prescription, the formalism of tetrads or vielbeins.

In the tetrad formalism[2] one takes advantage over the principle of equivalence and attaches to every point in the spacetime an inertial coordinate system ξ^a and introduces the tetrad fields e^a_μ as

$$d\xi^a = \frac{\partial \xi^a}{\partial x^\mu} dx^\mu = e^a_\mu dx^\mu, \tag{2.1.24}$$

where the tetrad field is defined by $\partial\xi^a/\partial x^\mu \equiv e^a_\mu$. We adopt the convention that the latin indices refer to the local inertial coordinates and the greek indices to the general curved coordinates. The inverse tetrad is defined analogously as

$$dx^\mu = \frac{\partial x^\mu}{\partial \xi^a} d\xi^a = e^\mu_a d\xi^a, \tag{2.1.25}$$

where $\partial x^\mu/\partial\xi^a \equiv e^\mu_a$ and the metric satisfies the relation $g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu$. The tetrad field may therefore be viewed as a transformation between the arbitrary coordinates x^μ and the inertial coordinates ξ^a. The tetrad field has the property of transforming a vector with a general coordinate index μ (or more generally a tensor) into a set of scalars via a contraction e.g., $A^a = e^a_\mu A^\mu$, where A^a is a set of four scalars. The spinor may then be brought into the framework of general relativity by focusing on these scalars instead.

The Lagrangian density for a massive spinor field ψ in curved spacetime can now be given by [18]

$$\mathcal{L}_\psi = \frac{i}{2}\sqrt{-g}(\overline{\psi}\gamma^\mu\nabla_\mu\psi - (\nabla_\mu\overline{\psi})\gamma^\mu\psi - m\overline{\psi}\psi), \tag{2.1.26}$$

where $\overline{\psi}$ denotes the Dirac conjugate spinor $\overline{\psi} = \psi^\dagger\gamma^0$. The curved space gamma matrices are defined via the tetrad as $\gamma^\mu = e^\mu_a\gamma^a$, where γ^a denotes the usual flat space gamma matrix. The curved space gamma matrices satisfy the usual anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{2.1.27}$$

and the covariant derivative is defined with the help of a spin-connection

[2] See e.g., [34, 76].

Γ_μ as

$$\nabla_\mu := \partial_\mu + \Gamma_\mu, \tag{2.1.28}$$

where (assuming no torsion),

$$\Gamma_\mu = \frac{1}{8}[\gamma^a, \gamma^b] e_a{}^\nu \partial_\mu e_{b\nu}. \tag{2.1.29}$$

By varying the action one obtains the Dirac equation for a ψ-particle in curved space

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0. \tag{2.1.30}$$

A corresponding equation will be obtained for the $\overline{\psi}$-particle aswell. The canonical quantization of the spinor field now follows closely the procedure given above. Instead of the commutation relations, the spinors satisfy the anticommutation relations

$$\{\psi_a(t, \mathbf{x}), \psi_b(t, \mathbf{x}')\} = 0, \tag{2.1.31}$$
$$\{\pi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} = 0, \tag{2.1.32}$$
$$\{\psi_a(t, \mathbf{x}), \pi_b(t, \mathbf{x}')\} = i\delta_{ab}\delta(\mathbf{x} - \mathbf{x}').$$

Introducing the inner product for the spinors over a hypersurface Σ chosen to be a constant time surface,

$$\langle \psi | \phi \rangle = \int_t d^{n-1}x \overline{\psi}(t, \mathbf{x}) \gamma_0 \phi(t, \mathbf{x}), \tag{2.1.33}$$

the spinor field ψ may be expanded with time-dependent modes as

$$\psi(t, \mathbf{x}) = \sum_{\pm s} \sum_{\mathbf{k}} [b^s_{\mathbf{k}} u^s_{\mathbf{k}}(t, \mathbf{x}) + d^{s\dagger}_{\mathbf{k}} v^s_{\mathbf{k}}(t, \mathbf{x}), \tag{2.1.34}$$

where a summation over the spinor indices s is made. Likewise, the conjugate field $\overline{\psi}$ may be expanded as

$$\overline{\psi}(t, \mathbf{x}) = \sum_{\pm s} \sum_{\mathbf{k}} [d^s_{\mathbf{k}} \bar{v}^s_{\mathbf{k}}(t, \mathbf{x}) + b^{s\dagger}_{\mathbf{k}} \bar{u}^s_{\mathbf{k}}(t, \mathbf{x})]. \tag{2.1.35}$$

Moreover, the spinors are normalized according to

$$u(\mathbf{k},s)^\dagger u(\mathbf{k},s') = v(\mathbf{k},s)^\dagger v(\mathbf{k},s') = 2|\mathbf{k}|\delta_{ss'}. \tag{2.1.36}$$

This completes the quantization procedure for scalars and spinors in curved spacetime sufficient for this thesis. A more detailed study may be found in [18, 19] along with the quantization of the vector field. The vector field quantization has been omitted because for the results of the Publications in this thesis, it is not needed.

2.2 Particle and the Vacuum

As one might have foreseen from the previous quantization procedure, the lack of a unique choice for the positive modes in curved spacetime, and the vacua they define, lead to an interpretational problem of a particle in curved spacetime. The question then becomes how to define the concept of a particle[3]. One approach would be to define the particle operationally by using particle detectors; a particle is something a particle detector detects. The theory behind particle detectors has been studied in numerous papers [77–81] and while it has its own merits, it is one which is not pursued here. In this thesis, we are more concerned about a method which would be more in the spirit of conventional in-out quantum field theory. The main tool would be to use Bogoliubov coefficients and trying to establish a particle concept.

But it is here, in this in-out formalism, where a possible solution lies when trying to establish a well defined particle concept. Even though a particle concept in the usual plane-wave sense cannot be established when the effect of gravity is considerable, one may focus on a particular class of spacetimes in which the effects of gravity disappear in these in- and out-regions. It is not necessary to assume that the geometry is flat in these regions, but if it is not some physically motivated criterion for defining positive frequency solutions must be given [22].

More generally, if the spacetime admits regions where the particle concept may be unambiguously established, as is the case for stationary spacetimes [82], one may construct the vacuum states for these regions and com-

[3]The particle concept defined in terms of field modes is a global concept. For fields taken as fundamental objects see e.g., [18, 19].

pare them via the Bogoliubov transformation to find the Bogoliubov coefficients. The problem therefore reduces to finding solutions of the Klein-Gordon equation (2.1.8) and on identifying the positive field modes which would reduce to the correct positive modes in the regions where the particle concept may be established. In the following, these notions will be put into a mathematical form which will serve as a basis of discussion in Chap. 4, where the field modes obtained in Publications I-IV are presented.

2.2.1 In and Out Vacuum States

Based on the above discussion, we assume that a stable particle concept may be unambiguously defined in the in- and out-regions. In the in-region, the field can be expanded in terms of the in-region modes $\{u_{\mathbf{k}}^{\text{in}}, u_{\mathbf{k}}^{\text{in}*}\}$ as

$$\phi(t,\mathbf{x}) = \sum_{\mathbf{k}} [a_{\mathbf{k}}^{\text{in}} u_{\mathbf{k}}^{\text{in}}(t,\mathbf{x}) + a_{\mathbf{k}}^{\text{in}^\dagger} u_{\mathbf{k}}^{\text{in}*}(t,\mathbf{x})], \tag{2.2.1}$$

where the set of creation and annihilation operators $\{a_{\mathbf{k}}^{\text{in}^\dagger}, a_{\mathbf{k}}^{\text{in}}\}$ obey the commutation relations $[a_{\mathbf{k}}^{\text{in}}, a_{\mathbf{k}'}^{\text{in}^\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}$ in the in-region. These modes then define a vacuum state in the in-region denoted as $|0,\text{in}\rangle$ in the usual way

$$a_{\mathbf{k}}^{\text{in}} |0,\text{in}\rangle = 0, \quad \forall\, \mathbf{k}. \tag{2.2.2}$$

In the out-region we consider another set of mode solutions $\{u_{\mathbf{k}}^{\text{out}}, u_{\mathbf{k}}^{\text{out}*}\}$ and the corresponding creation and annihilation operators $\{a_{\mathbf{k}}^{\text{out}^\dagger}, a_{\mathbf{k}}^{\text{out}}\}$ obeying the commutation relations $[a_{\mathbf{k}}^{\text{out}}, a_{\mathbf{k}'}^{\text{out}^\dagger}] = \delta_{\mathbf{k}\mathbf{k}'}$ now in the out-region. The field can be expanded similarly in terms of these modes in the out-region. The out-vacuum $|0,\text{out}\rangle$ is defined as

$$a_{\mathbf{k}}^{\text{out}} |0,\text{out}\rangle = 0, \quad \forall\, \mathbf{k}. \tag{2.2.3}$$

While the in-vacuum and the corresponding operators $\{u_{\mathbf{k}}^{\text{in}}, u_{\mathbf{k}}^{\text{in}*}\}$ are defined throughout the spacetime, they possess a known physical interpretation only in the in-region and not necessarily so in the out-region. A corresponding interpretation is also attached to the out-vacuum and the corresponding operators $\{u_{\mathbf{k}}^{\text{out}}, u_{\mathbf{k}}^{\text{out}*}\}$.

The creation and annihilation operators in the in- and out-regions obey the usual commutation relations (2.1.12) in their respective regions, but they do not do so with respect to each other in different regions. Hence, the commutation relations between the operators in the in- and out-regions are given by [5],

$$\begin{aligned}
[a_{\mathbf{k}}^{\text{out}}, a_{\mathbf{k}'}^{\text{in}}] &= \beta_{\mathbf{k}} \delta_{\mathbf{k},-\mathbf{k}'}, \quad [a_{\mathbf{k}}^{\text{out}\dagger}, a_{\mathbf{k}'}^{\text{in}}] = -\alpha_{\mathbf{k}} \delta_{\mathbf{k},\mathbf{k}'} \\
[a_{\mathbf{k}}^{\text{out}}, a_{\mathbf{k}'}^{\text{in}\dagger}] &= \alpha_{\mathbf{k}}^{*} \delta_{\mathbf{k},\mathbf{k}'}, \quad [a_{\mathbf{k}}^{\text{out}\dagger}, a_{\mathbf{k}'}^{\text{in}\dagger}] = -\beta_{\mathbf{k}}^{*} \delta_{\mathbf{k},-\mathbf{k}'}
\end{aligned} \tag{2.2.4}$$

The multi-particle states are constructed in the usual way in both regions and the annihilation operators are related by the Bogoliubov transformation (2.1.19). The focus is now on finding the positive field modes which corresponds to these vacua.

In passing, we do note that as the vacuum state is not unique, there are various ways to try to define it in curved spacetime and the vacuum given above is only one. In the literature, many different vacuum states are found and they each have their own purposes and places [61, 64, 83, 84]. One must therefore be careful when talking about a vacuum in curved spacetime. In this thesis, when talking about the vacuum, it will mean vacuum as defined in the in- and out-regions by the above prescription.

2.2.2 Field Modes in Flat Robertson-Walker Universe

Although some general statements may be made on particle creation [11, 12], most often the explicit field modes are required in order to obtain the form of the Bogoliubov coefficients $\beta_{\mathbf{k}\mathbf{k}'}$ and the corresponding particle spectrum. We will present the method for finding the field modes in a spatially flat Robertson-Walker metric (1.1.3) given in conformal time.

First, the field ϕ may be decomposed as usual,

$$\phi(\eta, \mathbf{x}) = \sum_{\mathbf{p}} [a_{\mathbf{p}} u_{\mathbf{p}}(\eta, \mathbf{x}) + a_{\mathbf{p}}^{\dagger} u_{\mathbf{p}}^{*}(\eta, \mathbf{x})], \tag{2.2.5}$$

and because of the homogeneity of the spatial sections in a Robertson-Walker spacetime, the mode solutions $u_{\mathbf{p}}$ are separable [18],

$$u_{\mathbf{p}}(\eta, \mathbf{x}) = \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2} a(\eta)} \chi_p(\eta), \tag{2.2.6}$$

where $p \equiv |\mathbf{p}|$. The Klein-Gordon equation (2.1.8) for the positive mode then reduces to the differential equation for the temporal part χ_p,

$$\chi_p''(\eta) + \Big(p^2 + a(\eta)^2 m^2 + \Big(\xi - \frac{1}{6}\Big) R a(\eta)^2\Big)\chi_p(\eta) = 0, \tag{2.2.7}$$

where χ_p satisfies the Wronskian condition $\dot{\chi}_p \chi_p^* - \chi_p \dot{\chi}_p^* = i$. Denoting $C(\eta) \equiv a(\eta)^2$, the Ricci scalar is given by [18],

$$R = \frac{1}{C}\Big(\frac{3\ddot{C}}{C} - \frac{3\dot{C}^2}{C^2}\Big). \tag{2.2.8}$$

To find the positive field modes, Eq. (2.2.7) must be solved. From the solution one needs to find the correctly normalized combination, which behaves like the field mode with which the in- and out-region particles have been defined, e.g., if the in- and out-regions are Minkowskian, the solution must reduce to the Minkowskian field modes in these regions. This construction defines the field modes $u_{\mathbf{p}}^{\text{in}}$ and $u_{\mathbf{p}}^{\text{out}}$. The caveat is, that even though Eq. (2.2.7) looks like a quite simple differential equation, there is no known general solution to this for a general scale factor $a(\eta)$ even for the conformal coupling. Therefore, a scale factor must be exactly chosen and even if a solution for the differential equation is then found, it may be that the positive mode solution which would reduce to known solution in the in- and out-regions cannot be recognized.

To illustrate this proceduce, and later chapters in mind, we choose a power-law expansion law $a(\eta) = b\eta^{n/2}$, with $b > 0$ giving the Ricci scalar

$$R = \frac{3n(n-2)}{2b^2\eta^{n+2}}. \tag{2.2.9}$$

It is now seen that as $\eta \to \pm\infty$, the scalar curvature vanishes, the spacetime is slowly varying[4] and a stable particle concept may be established in these asymptotic in- and out-regions. The positive mode solutions in these regions may be obtained by the prescription described above. But even for a general power-law expansion $a(\eta) = b\eta^{n/2}$ and conformal coupling, Eq. (2.2.7) does not possess a solution when the parameter n is left unspecified, i.e., the equation $\chi_p''(\eta) + (p^2 + m^2 b^2 \eta^n)\chi_p(\eta) = 0$ is not exactly solvable although for some special values of n solutions may be found. Most notably, solutions may be found and positive modes recognized e.g., for universes

[4] i.e., $\frac{d^l}{d\eta^l}\frac{\dot{C}(\eta)}{C(\eta)} \overset{\eta\to\pm\infty}{\longrightarrow} 0, \ \forall l \geq 0.$

which are dominated by stiff matter with $n = 1$ [85] and the de Sitter space with $n = -2$ [84].

From the particle creation standpoint, it is important to solve Eq. (2.2.7) for a general momentum p in order to find the explicit spectrum of the created particles. In some instances, however, knowledge of only the rest frame field mode may still be of importance in some calculations. Most notably this may be used to study the decay of a massive particle in its rest frame, for which knowing the rest frame field modes is sufficient. It turns out that for a scale factor $a(\eta) = b\eta^{n/2}$, Eq. (2.2.7) has an explicit solution for $p = 0$ for any n. We will return to this in Chap. 4.

Before moving on we comment on the construction of the in- and out-states. The use of $\eta \to \pm\infty$ is the usual way for determining the in- and out-regions, but is of course a little dissatisfying from a physical perspective because the spacetime needs to be completed by passing through the singularity while for a realistic model the time should advance from zero onwards. There is a way, however, to construct a stable particle concept in the vicinity of the singularity $\eta = 0$ as was noted in [85]. Consider again Eq. (2.2.7). As $\eta \to 0$, for our chosen scale factor, the solutions χ_p are given as plane waves in the neighborhood of $\eta = 0$. Hence, the field modes are exact and a stable particle concept may be established. It must be stressed though that this is not viable *at* the singularity, only in some neighborhood of it where the scale factor is $a(\eta_0)$ with η_0 in the neighborhood of $\eta = 0$. It is this construction of the in-region near the singularity which will be used in the results of this thesis in Chap. 4 and 5.

2.3 Conformal Transformations and Particle Creation

Conformal transformations are special kinds of transformations of the metric which were studied by Penrose in the 1960s [41]. Essentially, it is a local change of the metric scale along with an appropriate conformally transformed field. But what Penrose showed in [41] was that under a conformal transformation, the field equations for a free scalar field are invariant if the mass is zero and the coupling $\xi = 1/6$ in four spacetime dimensions. The connection between conformal transformations and particle creation appeared in the early works of Parker [10–12] where he noticed that there was no gravitational particle creation for particles of zero mass which are conformally coupled to gravity. This very fundamental notion is of great importance and will ultimately help us in providing a way to appropriately discuss about particle decay in curved spacetime.

On the other hand, if there is a mass term, a non-conformal coupling, or something else which breaks the conformal invariance of the field equations, the field modes get mixed and particle creation is indeed possible [19]. We will therefore take a closer look at this interconnectivity with conformal transformations and particle creation in curved spacetime. For simplicity this will be done for the scalar field only in four dimensions, but generalizations to fields of higher spin and to higher dimensions yield similar results, if not so straigthforwardly [41, 86].

2.3.1 Conformal Invariance

We start by considering a conformal transformation $\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}$ of the metric. For the equations of motion to be invariant, it is sufficient that the Lagrangian density is invariant under the transformation[5], i.e. $\tilde{\mathcal{L}} = \mathcal{L}$. Recall the free field Lagrangian density for the scalar field given by

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2). \tag{2.3.1}$$

We assume the field to transform as $\tilde{\phi} = \Omega^q\phi$, where q is a parameter, known as conformal weight, yet to be determined. Under the conformal transformation the transformed Lagrangian density reads as

$$\begin{aligned}\tilde{\mathcal{L}} =& \frac{\Omega^4}{2}\sqrt{-g}\{\Omega^{-2}g^{\mu\nu}(\Omega^{2q}\partial_\nu\phi\partial_\mu\phi + q^2\Omega^{2q-2}\phi^2\partial_\mu\Omega\partial_\nu\Omega + 2q\Omega^{2q-1}\phi\partial_\mu\Omega\partial_\nu\phi) \\ &- \xi\Omega^{2q-2}R\phi^2 - 6\xi\Omega^{2q-3}\phi^2 g^{\mu\nu}\partial_\nu\Omega\partial_\mu\Omega - m^2\Omega^{-2}\phi^2\}.\end{aligned} \tag{2.3.2}$$

In order for the first term to become invariant, $q = -1$ must be chosen. Moreover choosing $\xi = 1/6$ and performing a partial integration this equation can be written as

$$\tilde{\mathcal{L}} = \mathcal{L} + \frac{1}{2}\sqrt{-g}\partial_\nu(\phi^2 g^{\mu\nu}\partial_\mu(\ln\Omega)) - \frac{1}{2}\sqrt{-g}m^2\Omega^2\phi^2. \tag{2.3.3}$$

The second term on the right hand side is now a surface term upon integration and therefore does not have effect on the equations of motion

[5] Note that $\sqrt{-g}$ is included in the definition of $\mathcal{L}$.

and may be discarded. The key point to emphasize is that only with the value $\xi = 1/6$ can the surface term be formed. Otherwise, for arbitrary values of ξ, there would remain terms which could not be made invariant by any choice of q. It is now evident from Eq. (2.3.3) that if the mass m is non-zero, the Lagrangian density is not invariant under the conformal transformation; the presence of a mass term in the Lagrangian breaks the conformal symmetry.

The conditions under which the field equations are invariant in conformal transformations may therefore be established: the scalar field must transform as $\tilde{\phi} = \Omega^{-1}\phi$, the mass $m = 0$ and the gravitation coupling $\xi = 1/6$. The conformal weight in this case is $q = -1$. Equivalent calculations may be made for other fields aswell, and in the case of a spinor field ψ it can be shown that the weight is $q = -2/3$, i.e., $\tilde{\psi} = \Omega^{-2/3}\psi$ [86].

To obtain the field equations, we note that under the conformal transformation, the functional derivative of the action transforms as

$$\frac{\delta S}{\delta \phi} = \frac{\delta \tilde{S}}{\delta \phi} = \frac{1}{\Omega}\frac{\delta \tilde{S}}{\delta \tilde{\phi}} \tag{2.3.4}$$

so the equations of motion transform as

$$\Big(\Box + \frac{1}{6}R\Big)\phi(x) = \Omega^3\Big(\tilde{\Box} + \frac{1}{6}\tilde{R}\Big)\tilde{\phi}(x). \tag{2.3.5}$$

Therefore, if ϕ is a solution of the original field equation

$$\Big(\Box + \frac{1}{6}R\Big)\phi(x) = 0, \tag{2.3.6}$$

then $\tilde{\phi} = \Omega^{-1}\phi$ is a solution of the transformed equation

$$\Big(\tilde{\Box} + \frac{1}{6}\tilde{R}\Big)\tilde{\phi}(x) = 0, \tag{2.3.7}$$

or vice versa. While all this has been for a general metric, it turns out that this formalism has quite important consequences for the spacetime which is conformally flat, like the Robertson-Walker spacetime.

2.3.2 Conformally Flat Spacetimes

The notion of a conformally flat spacetime was already introduced in Chap. 1. What we will do next is to make a conformal transformation which takes us from the spatially flat Robertson-Walker spacetime into Minkowski space. To use the results of last section, we adopt the notation where the tilde refers to Minkowski space, i.e., we identify $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$ and choose $\Omega(x) = a(\eta)^{-1}$. We then have $\eta_{\mu\nu} = a(\eta)^{-2} g^{RW}_{\mu\nu}$, with RW referring to the Robertson-Walker metric. In this case $\tilde{R} = 0$ and the transformed equations of motion (2.3.7) read as $\tilde{\Box}\tilde{\phi} = 0$. This has a solution in terms of normal plane waves with η as the time variable. The solution in the Robertson-Walker space can then be obtained by the relation $\phi = a^{-1}\tilde{\phi}$.

All this means that the solution of the field equation for the conformally coupled massless particles in the flat Robertson-Walker spacetime is just the ordinary plane-wave solution times times the conformal factor,

$$\phi(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}\sqrt{2k}a(\eta)} \int d^3k (a_{\mathbf{k}} e^{-ik\eta + i\mathbf{k}\cdot\mathbf{x}} + a^{\dagger}_{\mathbf{k}} e^{ik\eta - i\mathbf{k}\cdot\mathbf{x}}). \qquad (2.3.8)$$

The positive mode solutions may readily be identified. In this case however, the positive mode solutions defining the in-vacuum (2.2.2) are the same as the ones defining the out-vacuum (2.2.3). In this case there is no mixing of the modes, $\beta_{\mathbf{kk'}} = 0$, and there is no particle creation. As a consequence, there exists a global preferred vacuum in the spatially flat Robertson-Walker universe simply from the existence of one in the Minkowski space [87]. A vacuum state constructed in this way appears sometimes in the literature under the name of conformal vacuum [18].

We are therefore led to the important conclusion that *in a conformally flat spacetime conformally coupled massless particles are not created by the expansion of spacetime.* This notion appeared first in the papers of Parker [10, 11] and is readily generalized to hold for massless spinors aswell [12], but is not valid e.g., in certain anisotropic spacetimes where massless conformally coupled particles may be created [14] or in the presence of a conformal symmetry breaking interaction terms [88]. Hence, in order for particles to be produced, the conformal symmetry must be broken. This conclusion will take on an even more important meaning and place when we talk about particle decay in curved spacetime.

Chapter 3

Mutually Interacting Fields in Curved Spacetime

The goal of any physical theory is ultimately to match theory with experiment. While the free field quantization has revealed novel phenomena in the form of gravitationally created particles, this is a phenomenon involving no matter interactions in itself. Therefore, in order to get a closer description of the real world, interactions must be explicitly taken into account.

The development of the theory of interacting quantum fields in curved spacetime was a natural extension to the free field quantization that took place in the later decades of the 1900s. The focus was mainly on renormalization issues and self-interacting quantum fields [21, 22]. A closely related topic of mutual interaction, where interaction takes place between different species of particles, has been only scarcely investigated due to the more difficult nature of the problem. Few extensive studies were made in the early days, focusing not only on the physical interpretation but also on the effect of the mutual interaction on particle creation processes [5, 7, 8, 23, 24, 89, 90]. Interest in the mutual interactions waned for a while, but lately the subject has gained renewed attention and found itself on studies of quantum electrodynamic processes in de Sitter spacetime [25–27].

In contrast to the Minkowskian situation, the dynamical nature of the spacetime brings new aspects to interacting field theory. New processes forbidden in flat spacetime become possible due to lack of energy conservation [91] and the field may even decay into its own quanta because of this [92, 93]. Moreover, the interaction itself may induce particle production by breaking the conformal invariance of an otherwise conformally invariant free field theory [90] and even CPT invariance may be broken as the global Lorentz invariance of the theory is lost [94, 95]. It is perhaps no surprise that the physically relevant quantities, like cross sections, decay rates and lifetimes, may greatly differ from those obtained in the usual flat spacetime quantum field theory. Indeed, studies on these quantities have hinted on the restricted applicability of the Minkowskian results in a curved spacetime setting [5, 30, 31, 92, 93].

In this chapter we will take a dive into the vast subject of interacting field theory. While generalizing the flat spacetime formalism, many technical aspects and details will be largely overlooked[1] in favor of a clear conceptual understanding on interpretational issues presented by the curved background. These issues are found to be linked to the fundamental phenomena of particle creation and a method to overcome these will be developed. The main goal of this chapter is to arrive at a physically and conceptually clear definition of a particle decay in curved spacetime.

3.1 Interacting Field Theory in Curved Spacetime

The introduction of field interactions to the theory has the effect of adding nonlinear terms to the ensuing field equations. In flat spacetime solving these equations exactly has proved to be an unsurmountable task and the situation is hardly any simpler in curved spacetime.

Since the field equations cannot be exactly solved when interactions are present, the standard method in both flat or curved spacetime, see e.g., [18, 97], is to assume that the interaction is adiabatically switched off in some distant in and out regions. The generic field ϕ will then reduce to free fields ϕ_{in} and ϕ_{out} respectively in these regions. In curved spacetime it is in addition required that a stable particle concept may be established in these regions. The free fields ϕ_{in} and ϕ_{out} can be quantized in the manner described in Chap. 2 and the solutions may then be used to construct the in and out particle states as described in Sec. 2.2.1.

With interactions one is interested in calculating the scattering amplitudes, or the S-matrix elements. The main quantities of interest to be

[1] For more detail in technical aspects, see e.g., [96].

calculated are therefore the transition amplitudes such as

$$\langle \text{out}, 1_{\mathbf{p}_1}, 1_{\mathbf{p}_2}, \ldots, 1_{\mathbf{p}_j} | S | 1_{\mathbf{k}_l}, \ldots, 1_{\mathbf{k}_2}, 1_{\mathbf{k}_1}, \text{in} \rangle \,, \tag{3.1.1}$$

describing the evolution of a particular in state into a particular out state. For example, the transition amplitude for a decay of a massive ϕ particle into two massless ψ particles would given by $\langle \text{out}, 1^{\psi}_{\mathbf{k}_1}, 1^{\psi}_{\mathbf{k}_2} | S | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle$. The probability for this process to occur is then given as the absolute value squared of this amplitude.

The problem therefore is to calculate the transition amplitudes. Similar to Minkowski space theory, reduction formulas for the transition amplitude (3.1.1) can be obtained in curved spacetime with special care taken for the fact that the in and out vacuum states are in general inequivalent [21]. The reduction method does have its disadvantages though as it is not possible to prescribe the vacuum state which is used in the calculation [18]. Fortunately, the interaction picture approach is easily adapted from Minkowskian field theory to curved spacetime setting to allow the calculation of the transition amplitudes [18, 22, 90]. It has the advantage of facilitating practical calculations when interactions are present and is the one used in this thesis.

The curved spacetime nevertheless provides its own set of subtleties even with this approach. Although these concepts generalize mathematically quite nicely to curved spacetime, interpretational problems are encountered in actually using them because, contrast to Minkowskian situation, the in and out vacuum states are generally inequivalent. An interpretational issue is therefore attached to a transition amplitude such as $\langle \text{out}, 1^{\psi}_{\mathbf{k}_1}, 1^{\psi}_{\mathbf{k}_2} | S | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle$ as will be later shown. But first we establish the form for the S-matrix suitable for calculations and show how to perturbatively calculate the transition amplitudes.

3.1.1 The Interaction Picture in Curved Spacetime

Suppose that the Hamiltonian density of the system $\mathcal{H}$ can be decomposed into a free part $\mathcal{H}_0$ and interaction part $\mathcal{H}_I$ as $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$. Let $|\Psi\rangle$ be a state vector, which in the interaction picture satisfies the Schrödinger-like equation [18],

$$\mathcal{H}_I(x) \, |\Psi[\Sigma]\rangle = i \frac{\delta \, |\Psi[\Sigma]\rangle}{\delta \Sigma(x)}, \tag{3.1.2}$$

where $\Sigma(x)$ is taken to be a spacelike Cauchy surface through x. The solution of (3.1.2) is found in terms of a unitary operator U defined by

$$|\Psi[\Sigma]\rangle = U[\Sigma, \Sigma_0]\, |\Psi[\Sigma_0]\rangle \,, \tag{3.1.3}$$

where Σ_0 denotes the initial Cauchy surface taken to be at some fixed time. Inserting this expression into (3.1.2), one obtains the equation

$$\mathcal{H}_I(x) U[\Sigma, \Sigma_0] = i \frac{\delta U[\Sigma, \Sigma_0]}{\delta \Sigma(x)}, \tag{3.1.4}$$

with an initial condition $U[\Sigma_0, \Sigma_0] = 1$ as easily seen from Eq. (3.1.3). With this initial condition, equation (3.1.4) can be written in integral form as

$$U[\Sigma, \Sigma_0] = 1 - i \int_{\Sigma_0}^{\Sigma} \mathcal{H}_I(x') U[\Sigma', \Sigma_0] d^n x', \tag{3.1.5}$$

where the integration is understood to be taken between two Cauchy surfaces. The solution may be found by iteration, e.g., the term $U[\Sigma', \Sigma_0]$ in the integrand is given as

$$U[\Sigma', \Sigma_0] = 1 - i \int_{\Sigma_0}^{\Sigma'} \mathcal{H}_I(x'') U[\Sigma'', \Sigma_0] d^n x'', \tag{3.1.6}$$

where $\Sigma \geq \Sigma' \geq \Sigma''$. Substitution into (3.1.5) gives

$$\begin{aligned} U[\Sigma, \Sigma_0] =& 1 - i \int_{\Sigma_0}^{\Sigma} \mathcal{H}_I(x') d^n x' \\ &+ (-i)^2 \int_{\Sigma_0}^{\Sigma} \int_{\Sigma_0}^{\Sigma'} \mathcal{H}_I(x') \mathcal{H}_I(x'') U[\Sigma'', \Sigma_0] d^n x'' d^n x'. \end{aligned} \tag{3.1.7}$$

The term $U[\Sigma'', \Sigma_0]$ is then again solved from (3.1.5), inserted into (3.1.7) and the process continues. Because the integrations are time ordered $\Sigma \geq \Sigma' \geq \Sigma'' \geq \cdots$, we may introduce the usual time-ordering operator $\hat{T}$ in the Hamiltonian products without changing anything. As in flat spacetime, the integration intervals may be symmetrized [18] so that

$$U[\Sigma, \Sigma_0] = 1 + \sum_{m=1}^{\infty} \frac{(-i)^m}{m!} \int_{\Sigma_0}^{\Sigma} \hat{T}(\mathcal{H}_I(x_1) \cdots \mathcal{H}_I(x_m)) d^n x_1 \cdots d^n x_m$$
$$\equiv \hat{T} \exp[-i \int_{\Sigma_0}^{\Sigma} \mathcal{H}_I(x') d^n x']. \tag{3.1.8}$$

We may therefore formally define the S-matrix describing the transition from an initial state into a final state via the unitary operator U,

$$S \equiv U[\Sigma_{\text{out}}, \Sigma_{\text{in}}], \tag{3.1.9}$$

for some suitably chosen in- and out-regions of the spacetime. Using the relations (3.1.3) and (3.1.9), the state vector in the out region may be expressed via the S-matrix as

$$|\Psi[\Sigma_{\text{out}}]\rangle = S\, |\Psi[\Sigma_{\text{in}}]\rangle\, . \tag{3.1.10}$$

3.1.2 Perturbative Calculations

With the S-matrix defined, we can return to the calculation of the transition amplitudes. At this point we assume that we are dealing with non-derivative couplings so that $\mathcal{H}_I = -\mathcal{L}_I$ and that the spacetime is globally hyperbolic. The initial state is chosen as a multi-particle state, $|\Psi[\Sigma_{\text{in}}]\rangle = |1_{\mathbf{k}_1}, 1_{\mathbf{k}_2}, \ldots, 1_{\mathbf{k}_j}, \text{in}\rangle$. The transition amplitude (3.1.1) can now be calculated within the formalism presented above. The S-matrix is defined via Eqs. (3.1.9) and (3.1.8) as

$$S = \lim_{\beta \to 0^+} \hat{T} \exp[i \int_{\Sigma_{\text{in}}}^{\Sigma_{\text{out}}} e^{-\beta |x^0|} \mathcal{L}_I(x') d^n x']. \tag{3.1.11}$$

In the above a switch off $e^{-\beta|x^0|}$ for the interaction has been introduced to ensure convergence[2] and x^0 denotes the time coordinate. The S-matrix can be expanded as

$$S = 1 + i \lim_{\beta \to 0^+} \int \hat{T} e^{-\beta |x^0|} \mathcal{L}_I d^n x + \cdots\, , \tag{3.1.12}$$

[2] If the time interval is finite, this switch-off is not needed.

as in Eq. (3.1.8). In this thesis we are only concerned with 1st order processes, also known as tree-level processes, so the higher order terms in Eq. (3.1.12) are of no relevant use to us. The first term, which is unity, simply describes the possibility that no interaction takes place. We refer to this as the zeroth order term of the expansion. The first order transition amplitude is then given by

$$\mathcal{A} \equiv \langle \text{out}, 1_{\mathbf{p}_1}, 1_{\mathbf{p}_2}, \dots, 1_{\mathbf{p}_j} | A | 1_{\mathbf{k}_l}, \dots, 1_{\mathbf{k}_2}, 1_{\mathbf{k}_1}, \text{in} \rangle \,, \tag{3.1.13}$$

where

$$A = i \lim_{\beta \to 0^+} \int \hat{T} e^{-\beta |x^0|} \mathcal{L}_I d^n x. \tag{3.1.14}$$

In principle the calculation of the transition amplitude proceeds as in flat spacetime, where one substitutes the free field solutions into $\mathcal{L}_I$ with the exception of replacing the plane-wave solutions with their curved space counterparts [18]. However, the curved spacetime presents some subtleties on its own complicating this rather straightforward generalization.

3.1.3 In-Out Probability Amplitudes

It is tempting to generalize this in-out scheme into curved spacetime as presented previously. One would take an initial state containing finite amount of particles and with the S-matrix scheme compare it with an out-state containing finite amount of particles. As simple as it sounds, in curved spacetime this approach leads to severe consequences which can ultimately be tracked to the phenomena of gravitational particle creation.

To see how particle creation affects the in-out formalism, consider, as was already shown by Parker in 1969, the vacuum amplitude squared $| \langle \text{out}, 0 | 0, \text{in} \rangle |^2$, which can be expressed as [11],

$$| \langle \text{out}, 0 | 0, \text{in} \rangle |^2 = \exp\Big(- \frac{V}{(2\pi)^3} \int \log |\alpha_{\mathbf{p}}|^2 d^3 p \Big). \tag{3.1.15}$$

Here $\alpha_{\mathbf{p}}$ is the Bogoliubov coefficient. With V being the normalization volume, this expression diverges as $V \to \infty$. This is a direct consequence of particle creation: since the expansion creates particles in all modes, there is a vanishing probability for the vacuum state to stay as a vacuum state in the infinite limit.

But the vacuum amplitude appears also in the S-matrix amplitude for any general particle state. Consider e.g., a state containing a finite amount of c massive ϕ particles and r massless ψ particles with definite momenta which can be scalars, fermions or vectors. Construct for these an in-state $|1^{\phi}_{\mathbf{p}} \cdots 1^{\psi}_{\mathbf{k}} \cdots\rangle \equiv |c^{\phi} r^{\psi}\rangle$ as described in Chap. 2 and likewise for the out-state. Hence the transition amplitude is $\langle \text{out}, d^{\phi} s^{\psi}|S|c^{\phi} r^{\psi}, \text{in}\rangle$. Inserting a complete set of in-states in between, the amplitude can be expressed as

$$\langle \text{out}, d^{\phi} s^{\psi}|S^{(z)}|c^{\phi} r^{\psi}, \text{in}\rangle = \sum_{g,t} \langle \text{out}, \mathrm{d}^{\phi} \mathrm{s}^{\psi}|g^{\phi} t^{\psi}, \text{in}\rangle \, \langle \text{in}, \mathrm{g}^{\phi} \mathrm{t}^{\psi}|S^{(z)}|c^{\phi} r^{\psi}, \text{in}\rangle \,, \tag{3.1.16}$$

where z denotes the order of the S-matrix operator, i.e., $z = 0$ corresponds to no interaction, $z = 1$ to first order etc. Consider then the first factor on the right-hand side of Eq. (3.1.16). As shown by Audretsch and Spangehl [5], this term may be expressed as

$$\langle \text{out}, \mathrm{d}^{\phi} \mathrm{s}^{\psi}|g^{\phi} t^{\psi}, \text{in}\rangle = \langle \text{out}, 0|0, \text{in}\rangle \, f(\alpha, \beta), \tag{3.1.17}$$

using commutation relations for the in and out annihilation and creation operators (2.2.4) where f is a finite function of the Bogoliubov coefficients and independent of the normalization volume V. The problem now becomes evident: if one were to square the transition amplitude of Eq. (3.1.16), the result would be a vanishing probability as $V \to \infty$ for all orders of the mutual interaction generalizing the earlier result of Parker in [11]. As a consequence of (3.1.15), dividing the transition amplitude (3.1.17) by the vacuum amplitude is a quantity with no direct physical meaning, although this method has also been used in the literature [30, 31].

In the same manner as for the vacuum amplitude, the probability of finding a final state $|d^{\phi} s^{\psi}, \text{out}\rangle$ with finite amount of particles and other modes empty, has a vanishing probability. The reason for this stems from the particle creation which fills all the modes already in the zeroth order. We are therefore led to the conclusion that in curved spacetime the normal Minkowskian type in-out transition amplitudes lose their physical meaning. The fact that particle creation so severely interferes with the mutual interaction process in curved spacetime forces us to consider different approaches to the description of particle decay. Fortunately, such approaches have been developed in curved spacetime context while still retaining the conceptual basis of the Minkowskian in-out transition amplitude scheme.

3.2 The Added-Up Probability

As seen in the last section, particle production in curved spacetime severely interferes with mutual interaction and the decay process itself. As the concept of particle loses its meaning in curved spacetime, it sounds reasonable that it also renders the decay process itself an ambiguous one. Since the decay rate does not have the same conceptual meaning as in Minkowski spacetime, the normal in-out method is not directly transferable to curved spacetime and instead one must use an alternative method of calculation. Indeed, as noted already in [98, 99], the possibility of particle creation from the vacuum points out that the standard Feynman rules could be modified.

The problem relates back to the non-uniqueness of the vacuum states $|0, \mathrm{in}\rangle$ and $|0, \mathrm{out}\rangle$ which are in general different from each other. So while one can calculate a transition amplitude like $\langle \mathrm{out}, 0|P|0, \mathrm{in}\rangle$, for some operator P, it is not the same as the physically relevant expectation values $\langle \mathrm{in}, 0|P|0, \mathrm{in}\rangle$ taken with respect to the same vacuum state. This type of ambiquity in the formalism requires alternative formulations of the problem. The in-in, closed-time-path, or Schwinger-Keldysh formalism seems particulary suitable for curved spacetime [100, 101] but leads quickly to extremely difficult expressions and calculations. Figuratively speaking, in the in-in formalism one propagates the field forward in time and then "backwards in time", forming a closed time loop. As a consequence, this formalism also increases the number of Feynman diagram to be included in the calculations [100]. Moreover, as noted by Kay, the in-in formalism automatically takes care of the infinite particle creation divergences [99]. A deeper dive into this subject is beyond the scope of this thesis.

In the spirit of the in-in formalism, Audretsch and Spangehl proposed some time ago a method for calculating transition amplitudes in curved spacetime which they called the added-up probability [5]. This procedure was physically motivated on the basis of the issues expressed in the last section and is more tied to a traditional detector problem of flat spacetime. It is also the method used in the calculations in this thesis and we will therefore give a sufficient introduction into it following [5]. We will work within the interaction picture presented in the last section and assume that the spacetime is conformally flat, for reason soon to be given.

3.2.1 What Can You Detect?

The starting point in establishing a physically motivated decay rate formula is to consider the whole problem as a detector problem. Like in flat spacetime theory, the final state particles are something that a particle detector would detect. In a curved spacetime, however, a single particle counter can only register the combined effect of the background and mutual interaction [23]. If one has a detector detecting massive particles, there is no way for the detector to discern whether these massive particles are a result of the interaction, or whether they were created gravitationally by the expansion. It is impossible to seperate out these two contributions and here lies the root of the whole problem. Moreover, the gravitationally created particles decay themselves complicating the detection problem. We must therefore pose the question of decay a little differently.

To obtain information regarding the mutual interaction only, one should fix the in-state and concentrate on the out-states containing a specific indicator configuration which the detector could detect and which would be minimally disturbed by particle creation from the vacuum. An indicator configuration is therefore a configuration which cannot be produced from the background. On a conformally flat spacetime, one such configuration is provided by massless conformally coupled particles. As seen in Sec. 2.3, these are not produced gravitationally by the expansion of the spacetime. Therefore, a detection of a massless particle[3] in the out-state means that it has solely been created or influenced by the mutual interaction only.

A physically reasonable question therefore is what would be the probability of detecting a massless particle state *regardless* of what has happened to the massive particle states, since these are always created by the expansion. Technically this is achieved by summing over all the massive out states. Consider therefore again a state $|1^{\phi}_{\mathbf{p}} \cdots 1^{\psi}_{\mathbf{k}} \cdots\rangle \equiv |c^{\phi} r^{\psi}\rangle$. The added-up transition probability is defined as [5],

$$w^{\text{add}}(s^{\psi}|c^{\phi}r^{\psi}) = \sum_{\text{all } d} |\langle \text{out}, d^{\phi} s^{\psi}|S|c^{\phi}r^{\psi}, \text{in}\rangle|^2, \qquad (3.2.1)$$

where the sum is over all massive out-states. Because there is no creation of massless particles, $|s^{\psi}, \text{out}\rangle = |s^{\psi}, \text{in}\rangle$. Moreover, both the in and out

[3] From hereon, the term conformally will be dropped when talking about massless particles, but it should be understood that when talking about massless particles for the rest of this thesis, they always refer to conformally coupled ones.

massive states are complete,

$$\sum_{\text{all } c} |c^{\phi}, \text{in}\rangle \langle \text{in}, c^{\phi}| = \mathbb{1} = \sum_{\text{all } d} |d^{\phi}, \text{out}\rangle \langle \text{out}, d^{\phi}| \,. \tag{3.2.2}$$

Inserting this completeness relation twice into the right-hand side of Eq. (3.2.1) and using $|s^{\psi}, \text{out}\rangle = |s^{\psi}, \text{in}\rangle$, the added-up probability can be reduced to in-in amplitudes [5]

$$w^{\text{add}}(s^{\psi}|c^{\phi}r^{\psi}) = \sum_{\text{all } d} |\langle \text{in}, d^{\phi}s^{\psi}|S|c^{\phi}r^{\psi}, \text{in}\rangle|^2. \tag{3.2.3}$$

Because of (3.2.2), the added-up probability may also be given in terms of out-out states. It is now seen that the added-up probability can be expressed in terms of in-in amplitudes for which an in-in Feynman diagram technique applies [5].

3.2.2 Particle Decay in the $\phi\psi^2$ Theory

Without performing any explicit calculations, an example is perhaps in order to illuminate the added-up method. With the Chap. 4 in mind, we consider a massive particle decaying into two massless particles. The spacetime is chosen to be flat Robertson-Walker universe and the interaction term is $\mathcal{L}_I = -\sqrt{-g}\lambda\phi\psi^2$, where λ is the coupling constant. At this point, it is not necessary to restrict into any specific species of particles. As usual, we assume that a stable concept of a particle may be established in the in- and out-regions. The S-matrix may be expanded as

$$S = 1 - i\lambda A + \mathcal{O}(\lambda^2), \tag{3.2.4}$$

where

$$A \equiv \lim_{\beta \to 0^+} \int \hat{T}\phi\psi^2 e^{-\beta|\eta|}\sqrt{-g}d^4x. \tag{3.2.5}$$

To make the theory more manageable, we consider only first order, tree-level processes. As shown in [5], the added-up transition probability (3.2.3) reduces for this type of interaction to

$$w^{\text{add}} = \lambda^2 \Big\{ |\langle \text{in}, 1^{\psi}_{\mathbf{k}_1} 1^{\psi}_{\mathbf{k}_2} | A | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle|^2 + \sum_{\mathbf{q}} |\langle \text{in}, 1^{\phi}_{\mathbf{p}} 1^{\phi}_{\mathbf{q}} 1^{\psi}_{\mathbf{k}_1} 1^{\psi}_{\mathbf{k}_2} | A | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle|^2 \Big\}. \tag{3.2.6}$$

The corresponding Feynman diagrams, now with the in-in amplitudes, are given in Fig. 3.1. The second diagram appears because there is no energy conservation in curved spacetime making this graph possible. In a loose sense, one may think that the energy stored in the gravitational field is ”taken” to make the creation of all three particles possible as the spacetime expands. In addition the particle may also pass through undisturbed, which is depicted by the dashed line in diagram b.

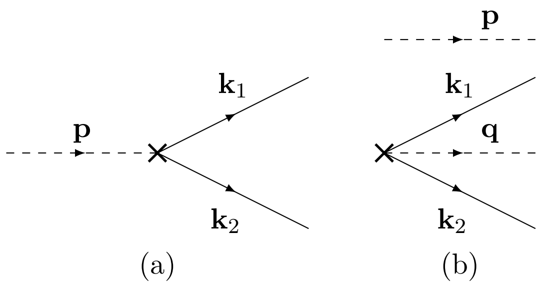


Fig. 3.1: Diagrams of the added-up transition probability. The dashed line corresponds to the massive particle and the solid lines to massless particles.

The second amplitude in (3.2.6) can further be simplified by using the property $\langle \text{in}, 1^{\phi}_{\mathbf{p}} 1^{\phi}_{\mathbf{q}} 1^{\psi}_{\mathbf{k}_1} 1^{\psi}_{\mathbf{k}_2} | A | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle = \langle \text{in}, 1^{\phi}_{\mathbf{q}} 1^{\psi}_{\mathbf{k}_1} 1^{\psi}_{\mathbf{k}_2} | A | 0, \text{in} \rangle$ [5]. This second diagram complicates the interpretation of the decay of a single massive particle with momenta $\mathbf{p}$. Therefore, the three-momentum conservation law $\mathbf{p} = \mathbf{k}_1 + \mathbf{k}_2$ is imposed so that the decay probability would correspond most closely to a decay of a massive particle with momenta $\mathbf{p}$. In a universe with spatial translational symmetry, the three-momentum conservation law is always valid [32]. The added-up *decay probability* is defined with this restriction [5] and

$$w^{\text{add}} = \lambda^2 \Big\{ |\langle \text{in}, 1^{\psi}_{\mathbf{k}} 1^{\psi}_{\mathbf{p}-\mathbf{k}} | A | 1^{\phi}_{\mathbf{p}}, \text{in} \rangle|^2 + |\langle \text{in}, 1^{\phi}_{\mathbf{p}} 1^{\psi}_{\mathbf{k}} 1^{\psi}_{\mathbf{p}-\mathbf{k}} | A | 0, \text{in} \rangle|^2 \Big\}, \tag{3.2.7}$$

and the corresponding Feynman diagrams in Fig. 3.2 below.

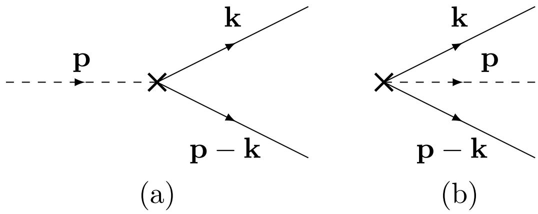


Fig. 3.2: Diagrams contributing to the added-up decay probability.

The total decay probability is then obtained by summing over the **k** momenta as

$$w^{\mathrm{tot}} = \sum_{\mathbf{k}} w^{\mathrm{add}}. \tag{3.2.8}$$

We have therefore arrived at something which closest resembles the decay for a massive particle with momentum **p** in curved spacetime. The added-up method offers a generalization of flat spacetime decay to curved spacetime which contains a minimal admixture of particle creation processes which are indistinquishable from the actual decay process in the spirit of the in-out formalism.

Chapter 4

Decaying Massive Particle in the Early Universe

As already mentioned in the introduction, particle decay processes in the early universe play an important role in cosmology. The calculations performed have commonly used the Minkowskian decay rates which brings up the question about how good these Minkowskian approximations really are when the spacetime is curved. In this chapter we try to answer this question by presenting the results of Publications I-IV where the decay of a massive particle using quantum field theory in curved spacetime were studied. The study was conducted using the physically motivated added-up probability introduced in the last chapter. The decay into both scalar and fermionic channels was considered in spatially flat Robertson-Walker universes and it was found that the decay rates are considerably modified for early times, but asymptotically only by an additive term. We will consider these decay channels first separately and then compare the differences.

4.1 Decay into Scalar Channel

The decay of a massive scalar particle into two massless scalars was the subject of research in Publications I-III. The case of a conformally coupled massive particle was studied in Publication I for a stiff-matter-dominated universe and in Publication II for universes dominated by matter and radiation. In Publication III the decay was generalized to contain not only a non-conformally coupled massive scalar but also a more general power-law expansion for the universe. In the following the results for the transition amplitudes and decay rates will be given with this more general formalism.

First, the necessary background information must be established. Consider a massive real scalar field ϕ with mass m and a gravitational coupling ξ and a conformally coupled massless real scalar field φ. The Lagrangian for the theory is given by

$$\mathcal{L} = \frac{\sqrt{-g}}{2}\{\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 - \xi R\phi^2 + \partial_\mu\varphi\partial^\mu\varphi - \frac{R}{6}\varphi^2\} + \mathcal{L}_I. \quad (4.1.1)$$

For the interaction term, we choose

$$\mathcal{L}_I = -\sqrt{-g}\lambda\phi\varphi^2, \quad (4.1.2)$$

where $\lambda \neq 0$ is the interaction coupling constant. We will consider a spatially flat four-dimensional Robertson-Walker spacetime with the line element $ds^2 = a(\eta)^2(d\eta^2 - d\mathbf{x}^2)$ for which the conformal time coordinate $\eta \in (0, \infty)$. From the point of view of quantization and particle definition, this is the same type of universe as considered in Sec. 2.2.2. The definition of the in- and out-regions and construction of the field modes follows the same procedure with the in-region estalished in the neighborhood of $\eta = 0$. Recall that the massive scalar field modes were given by

$$u_{\mathbf{p}}(\eta, \mathbf{x}) = \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}a(\eta)}\chi_p(\eta), \quad (4.1.3)$$

with χ_p satisfying Eq. (2.2.7) and the positive modes for the massless field as

$$v_{\mathbf{k}}(\eta, \mathbf{x}) = \frac{1}{(2\pi)^{3/2}a(\eta)}\frac{e^{i\mathbf{k}\cdot\mathbf{x}-ik\eta}}{\sqrt{2k}} \quad (4.1.4)$$

due to conformal invariance. As described in Sec. 3.1.2, the S-matrix is expanded to first order $S = 1 - i\lambda A + \mathcal{O}(\lambda^2)$ where

$$A = \lim_{\beta\to 0^+}\int \hat{T}e^{-\beta\eta}\phi\varphi^2\sqrt{-g}d^4x. \quad (4.1.5)$$

Throughout we work with the established added-up formalism.

4.1.1 Total Transition Probability

In Publication II, a general form for the total transition probability for the massive scalar to decay into two massless scalars was derived. Since this derivation introduces a couple of important features, we present its derivation in the briefest of forms.

Referring back to Fig. 3.2, which gives the Feynman diagrams for the decay process in curved spacetime, we define for the amplitude of diagram a $\mathcal{A}^a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}) \equiv -i\lambda \langle \text{out}, 1^{\varphi}_{\mathbf{k}_1} 1^{\varphi}_{\mathbf{k}_2} | \text{A} | 1^{\phi}_{\mathbf{p}}, \text{out} \rangle$ and $\mathcal{A}^b(\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}) \equiv -i\lambda \langle \text{out}, 1^{\phi}_{\mathbf{q}} 1^{\varphi}_{\mathbf{k}_1} 1^{\varphi}_{\mathbf{k}_2} | \text{A} | 0, \text{out} \rangle$ for diagram b. With the solutions (4.1.3) and (4.1.4) this gives

$$\mathcal{A}^a(\mathbf{k}_1, \mathbf{k}_2, \mathbf{p}) = \frac{-i\lambda\delta(\mathbf{p} - \mathbf{k}_1 - \mathbf{k}_2)}{(2\pi)^{3/2} 2\sqrt{k_2 k_2}} \lim_{\beta \to 0^+} \int_0^\infty e^{-\beta\eta} a(\eta) e^{i(k_1+k_2)\eta} \chi_p(\eta) d\eta. \tag{4.1.6}$$

For diagram b, the same amplitude is obtained with the following changes, $\delta(\mathbf{p} - \mathbf{k}_1 - \mathbf{k}_2) \to \delta(\mathbf{q} + \mathbf{k}_1 + \mathbf{k}_2)$ and $\chi_p(\eta) \to \chi_p(\eta)^*$. The delta-function expresses the three-momentum conservation law. Note that the amplitudes are defined with respect to the out-modes. As was noted in Sec. 3.2.1, the added-up transition amplitude can also be given in terms of the out-out amplitudes. The choice to use out-mode solutions stems from the fact that the positive solutions for the field modes are easily recognized in the asymptotic future for our model. Imposing the three-momentum conservation law, performing the $\mathbf{k}_2$ integration and passing to continuum limit, we have for the total probability

$$w^{\text{tot}} = \int_{\mathbb{R}^3} d^3\mathbf{k} \left(|\mathcal{A}^a(k, |\mathbf{p} - \mathbf{k}|)|^2 + |\mathcal{A}^b(k, |\mathbf{p} - \mathbf{k}|)|^2 \right). \tag{4.1.7}$$

This integral as it is, is extremely difficult to solve for a general momentum parameter $\mathbf{p}$ and for a general χ_p the k integration cannot be performed. For this reason, we go into the rest frame of the massive particle $\mathbf{p} = 0$, which allows the use of powerful methods of distributions. Using the symmetry property $|\mathcal{A}^a(k, |\mathbf{p} - \mathbf{k}|)|^2 = |\mathcal{A}^b(-k, -|\mathbf{p} - \mathbf{k}|)|^2$ and spherical coordinates,

$$w^{\text{tot}} = \frac{\lambda^2}{8\pi^2} \int_{-\infty}^{\infty} dk \left| \int_0^T a(\eta) e^{2ik\eta} \chi_{p=0}(\eta) d\eta \right|^2. \tag{4.1.8}$$

Even with the switch-off, the time integral may diverge, as is the case for the stiff-matter-dominated universe, so we have introduced a cutoff at $\eta = T$ and taken the β-limit inside the integral. In this form, the k integral can be treated as a distribution so we finally arrive at the general form for the total transition probability

$$w^{\text{tot}} = \frac{\lambda^2}{8\pi} \int_0^T a(\eta)^2 \big|\chi_{p=0}(\eta)\big|^2 d\eta, \tag{4.1.9}$$

which can be used to calculate the total added-up probability for a known scale factor and known rest frame field modes.

4.1.2 Scalar Modes in Rest Frame

Even though the total transition probability (4.1.9) is restricted to only massive modes in their rest frames, it also presents us with some new possibilities. Since we are only considering decay in the rest frame, the p^2-term in the differential equation (2.2.7) for χ_p is zero which helps us in finding the exact mode solutions. The differential equation (2.2.7) then becomes exactly solvable for a general power-law expansion. We then have the advantage of also finding field modes and studying decay for universes for which the exact solutions for general p cannot be found, like the matter-dominated universe.

In Publication III, the differential equation (2.2.7) with $p^2 = 0$ was solved exactly for a general power-law scale factor

$$a(\eta) = b\eta^{n/2}, \tag{4.1.10}$$

where b is a positive constant, n essentially unrestricted and we work in conformal time. The Ricci scalar is given by Eq. (2.2.9) and the interesting cases of universes dominated by stiff matter, radiation and ordinary matter are given by the values $n = 1$, $n = 2$ and $n = 4$, respectively. The solutions for the massive scalar field modes in the rest frame were found in terms of Hankel functions $H_\alpha^{(1,2)}$, with the normalization depending on whether the index α was real or purely imaginary. For real index, the normalized positive solution was found in a manner as described in Sec. 2.2.2 and was found to be

$$\chi_{p=0}(\eta) = \sqrt{\frac{\pi\eta}{2(2+n)}} e^{-\frac{i\pi}{4}(1-2\alpha)} H_\alpha^{(2)}\Big(\frac{2bm\eta^{(2+n)/2}}{2+n}\Big), \tag{4.1.11}$$

where α is defined as

$$\alpha \equiv \frac{\sqrt{1-n(n-2)(6\xi-1)}}{2+n}. \tag{4.1.12}$$

For purely imaginary index α, we write $\alpha = i\tilde{\alpha}$, where

$$\tilde{\alpha} = \frac{\sqrt{n(n-2)(6\xi-1)-1}}{2+n} \tag{4.1.13}$$

is real. Their normalized positive mode solutions were found to be given as

$$\chi_{p=0}(\eta) = \sqrt{\frac{\pi\eta}{2(2+n)}}\, e^{-\frac{i\pi}{4}+\frac{\pi\tilde{\alpha}}{2}} H^{(2)}_{i\tilde{\alpha}}\Big(\frac{2bm\eta^{(2+n)/2}}{2+n}\Big). \tag{4.1.14}$$

The Hankel function of the second kind behaves asymptotically like $H^{(2)}_{\nu}(z) \sim \sqrt{2/(\pi z)}\, e^{-i(z-\nu\pi/2-\pi/4)}$, so the only difference between these two solutions is a different exponential factor. These mode solutions are not defined when $n = -2$, which would correspond to the de Sitter spacetime. The differential equation for $\chi_{p=0}$ may be solved for this special value, but the positive modes cannot still be recognized. Hence, we have to exclude de Sitter spacetime from our study. Given these positive mode solutions for the massive scalar field, we may now insert these with the scale factor $a(\eta) = b\eta^{n/2}$ into the general formula (4.1.9) to obtain the transition probability.

4.1.3 Transition Probabilities

The total transition probability is obtained by inserting the positive rest frame field modes into Eq. (4.1.9). In both cases the solution was found as

$$w = \frac{\lambda^2}{32m^2}\int_0^{mt} uH^{(1)}_{\alpha}(u)H^{(2)}_{\alpha}(u)du, \tag{4.1.15}$$

where u is a dimensionless variable and t denotes the standard coordinate time given by the relation $dt = a(\eta)d\eta$.

Although the lower limit of the integral is taken from 0 onwards, we note that it may also be taken as mt_0 where t_0 is the initial time near the singularity. For the scalar channel decay, the integral may be exactly solved with the lower limit of zero and this limit does present us with some interesting features when the decay rate is calculated. Before moving on, we note that the integrand of Eq. (4.1.15) can be interpreted as the differential decay rate

$$\Gamma_{\text{diff}} = \frac{\lambda^2}{32} t H_\alpha^{(1)}(mt) H_\alpha^{(2)}(mt), \tag{4.1.16}$$

when $t > 0$. This form will come in handy as we consider the reheating scenario via Boltzmann equations in the next chapter.

Returning now to the integral (4.1.15), we found that it can be evaluated exactly in terms of Bessel functions and the solutions are given by[1]

$$\begin{aligned} w_{\text{Re}} = \frac{\lambda^2}{64m^2} \{ &(mt)^2 [J_\alpha(mt)^2 - J_{\alpha-1}(mt) J_{\alpha+1}(mt) + Y_\alpha(mt)^2 \\ &- Y_{\alpha-1}(mt) Y_{\alpha+1}(mt)] - \frac{4\alpha \cot(\pi\alpha)}{\pi} \}, \end{aligned} \tag{4.1.17}$$

for the real index for which the integral (4.1.15) converges when $|\alpha| < 1$. When $\alpha = 0$, the constant term is given by its limiting value $2/\pi^2$. For the imaginary index the exact solution was found as

$$\begin{aligned} w_{\text{Im}} = &\frac{\lambda^2}{64m^2} \{ (mt)^2 [J_\alpha(mt)^2 - J_{\alpha-1}(mt) J_{\alpha+1}(mt) + Y_\alpha(mt)^2 \\ &- Y_{\alpha-1}(mt) Y_{\alpha+1}(mt)] - \frac{4\tilde{\alpha} \coth(\pi\tilde{\alpha})}{\pi} \}, \end{aligned} \tag{4.1.18}$$

where α is given by (4.1.12) with the same condition $|\alpha| < 1$. The constant terms arise from the lower limit of the integral which was set to zero. Since the integral (4.1.15) converges only when $|\alpha| < 1$, this condition restricts the pair (n, ξ) by the inequalities

$$\frac{(n+3)(n+1)}{n(2-n)} < 6\xi - 1 \leq \frac{1}{n(n-2)}, \; n \notin (0,2) \tag{4.1.19}$$

$$\frac{1}{n(n-2)} \leq 6\xi - 1 < \frac{(n+3)(n+1)}{n(2-n)}, \; n \in (0,2). \tag{4.1.20}$$

These restrictions divide the $n\xi$ plane into regions where a finite transition probability can be calculated (Fig. 4.1).

[1] The following results differ by a multiplication by 2 in the additive term from those of Publication III due to an error found in the calculations.

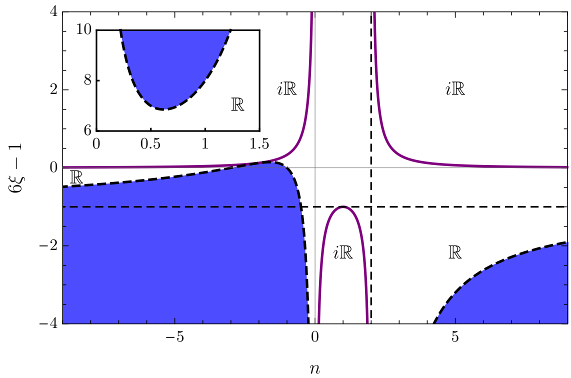


Fig. 4.1: Allowed regions for the total transition probability. The shaded area corresponds to values of the pair (n, ξ) for which the transition amplitude is not finite. Regions indicated by $\mathbb{R}$ are for real index solution, while regions with $i\mathbb{R}$ are for imaginary index solutions. The dashed lines correspond to asymptotes $n = 2$ and $6\xi - 1 = -1$.

There are two regions where all values of ξ are allowed, namely $n = 0$ the Minkowski space and $n = 2$ the radiation-dominated universe. This is hardly surprising, since the Ricci scalar (2.2.9) vanishes at these values and along with it the coupling to gravity. Moreover, there does exist one region where all values of n are allowed. This band is located between the maximum and minimum points of the non-allowed region, where

$$\xi \in \Big(\frac{3-\sqrt{5}}{4}, \frac{3+\sqrt{5}}{4}\Big). \tag{4.1.21}$$

It is somewhat puzzling why the mean decay rate is not finite for all the values, excluding the line $n = -2$. Currently there exists no measurable restrictions on the parameter ξ which would allow us to exclude e.g., the lower part of Fig. 4.1 altogether. Although we have calculated the transition amplitudes for all n, in this thesis we are mainly interested in the region where n is positive, since it contains the cosmologically interesting cases of stiff-matter-, matter- and radiation-dominated universes. In this region there are no restrictions for a positive gravitational coupling ξ. We do note, however, that the differential decay rate (4.1.16) may be calculated without any restrictions.

4.1.4 Asymptotic Decay Rates

Besides the exact transition probabilities, we can also look at their asymptotic forms in the long-time limit. The exact probabilities can be expanded in asymptotic series and we found that the leading terms are given by

$$w \sim \frac{\lambda^2}{16\pi m}\Big(t - \frac{|\alpha|\cot(\pi|\alpha|)}{m}\Big), \tag{4.1.22}$$

where the result has been combined to include both the imaginary and real solutions.

In Minkowskian space field theory, the decay rate is obtained by dividing the total probability by the time t. This procedure is now a little more problematic due to the appearance of the constant term in Eq. (4.1.22) so it is not clear how to define the decay rate and with it the lifetime of the particle. To this end, we notice that there are two different time scales involved: that of the duration of the mutual interaction and that of the gravitational influence. The first one is characterized by the infinite time while the second is finite because the gravitational fields exerts its influence only in the region between the in- and out-regions. This second time scale may be defined as $t_{\rm grav} := t_f - t_i$, where t_i indicates the time when the gravitational field begins its influence and t_f its end [5]. To deal with the constant term we adopt the same procedure as was introduced in [5] where this constant term is divided by $t_{\rm grav}$. With this definition, the mean decay rate was found to be

$$\Gamma_\phi \sim \frac{\lambda^2}{16\pi m}\Big(1 - \frac{|\alpha|\cot(\pi|\alpha|)}{mt_{\rm grav}}\Big). \tag{4.1.23}$$

It should be stressed that while there seems to be no way to define $t_{\rm grav}$ exactly [5, 6], this procedure will at least give some quantitative idea of the gravitational influence on particle decay.

It is now seen that the influence of the gravitational field modifies the Minkowskian decay rates, at least asymptotically, by an additive time-dependent term. In the next section we will see that the first term truly corresponds to the Minkowskian decay rate. The question then arises on the significance of this correction term. In a real setting the time $t_{\rm grav}$ is usually much longer than the inverse of mass, which implies that the correction term is quite small. Equation (4.1.23) is, however, only the first order asymptotic expansion and as the time gets smaller, next to leading order contributions must be taken into account aswell. Therefore it cannot be said outright that relative correction term in Eq. (4.1.23) can be neglected altogether, since in particular when $m \sim t$, the full equation (4.1.15) for the transition probability must be used. While the sign in front of this relative correction term in (4.1.23) is negative, the term itself can be either negative or positive depending on the values of α. We now take a closer look at how the value of α influences the decay rate and show how the Minkowskian result is reproduced.

4.1.5 Minkowskian Decay Rate and Gravitational Correction Term

It is natural to assume that the general result in curved spacetime reproduces the Minkowskian results when the appropriate limit is taken. For our model, this limit is given when the spacetime becomes static, i.e., $a(t) = 1$. This is achieved by taking $n = 0$ and the parameter b may be chosen as $b = 1$ although it does not explicitly appear in the transition probabilities. In this case the parameter $\alpha = 1/2$, the constant term in Eq. (4.1.17) is zero and the rest of the expression reduces to

$$w_{\text{Mink}} = \frac{\lambda^2 t}{16\pi m} \tag{4.1.24}$$

which, when divided by t, gives the correct Minkowskian limit. It was also found that using the added-up method with the Minkowskian plane-waves inserted, the above result is also produced validating the procedure.

However, if the value $\alpha = 1/2$ gives us the Minkowskian decay rate when $n = 0$, this immediately implies that there are values of (n, ξ) which give the Minkowskian decay rate when $\alpha = 1/2$. By the properties of Bessel function, the same rate is found when $\alpha = -1/2$ aswell. Hence, solving the parameter α in Eq. (4.1.12) for $\alpha = \pm 1/2$ gives two solutions. The one is the $n = 0$ case and the other is

$$\xi(n) = \frac{n-4}{8(n-2)}. \tag{4.1.25}$$

Along this curve then, the decay rate is exactly and always Minkowskian. There is one special point on this curve corresponding to $n = 4$, for which $\xi = 0$. This means that the decay rate in a matter-dominated universe for minimally coupled scalars is exactly that of Minkowskian space.

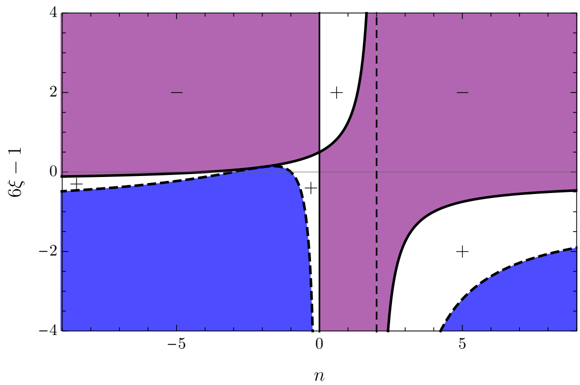


Fig. 4.2: Plot of the Minkowskian rate curves (solid). The gravitational correction term is negative in the shaded purple region indicated by the −-sign. The +-sign indicates the areas where the contribution to the decay rate is enchancing and the shaded blue areas correspond to the values of (n, ξ) where the total decay rate is not defined.

The change of the gravitational correction term from negative to positive happens when crossing the boundary curve (4.1.25). This curve has vertical asymptote at $n = 2$ and horizontal asymptotes at $6\xi - 1 = \pm 1/4$. For the cosmologically interesting cases of stiff-matter-, radiation- and matter-dominated universes, we see that the effect of the gravitational field is always to diminish the decay as long as the gravitational coupling is positive, except for the minimally coupled field in matter-dominated universe for which the decay rate is Minkowskian. In the regions where the integrated transition amplitude cannot be calculated, it seems plausible that the effect of the spacetime would still be enhancing as the differential rate can be calculated in these regions (Fig. 4.2). It can be inferred, with some caution, that the effect of the gravitational coupling to the decay is mostly to enhance it, when the coupling is negative, and to decrease it when it is positive. While the preceding analysis was done for a single real scalar field ϕ, a generalization to multicomponent scalar field is straigthforward.

4.2 Decay into Fermionic Channel

In Publication IV, the decay of a massive scalar into a massless fermion-antifermion pair was considered. The formalism follows closely that of the scalar channel decay; solve the field modes and use the added-up probability to obtain the transition amplitude. The scalar particle will be taken in its rest frame so results from previous section can be applied straightforwardly and the decay product particles are massless spinors as reguired by the added-up probability. In this case we may also further elaborate the model and consider the implications these assumptions might have. From the point of symmetry, one may consider a complex scalar field with non-zero charge, a chiral spinor field ψ_L with opposite charge and a zero-charge field ψ_R. In a $U(1)$-symmetric theory a mass term in the fermion Lagrangian density violates this symmetry, so for a globally $U(1)$-symmetric theory the fermions should be massless [97]. Also, considering the scalar field ϕ as an $SU(2)$ doublet, by defining the (global) $U(1)$-charges of the theory in a suitable way, the fermionic fields may be thought as fermions of the standard model [97].

We consider then a Lagrangian for the theory consisting of three parts: the Lagrangian density $\mathcal{L}_\phi$ for the complex scalar field, $\mathcal{L}_\psi$ for the fermion field and the interaction term $\mathcal{L}_I$. The Lagrangian density for a complex scalar field is given by

$$\mathcal{L}_\phi = \sqrt{-g}(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \xi R\phi^*\phi). \tag{4.2.1}$$

As the complex scalar field can always be decomposed into two real scalar fields as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, we will only consider real scalar fields to make the theory more manageable and it also allows us to use results from the previous section. The only effect this has is that the total transition amplitude should be multiplied by a factor of two in the complex case. The massless spinor Lagrangian is given by

$$\mathcal{L}_\psi = \frac{i}{2}\sqrt{-g}(\overline{\psi}\gamma^\mu\nabla_\mu\psi - (\nabla_\mu\overline{\psi})\gamma^\mu\psi), \tag{4.2.2}$$

where γ^μ denotes the gamma matrices in curved spacetime. The quantization of the spinor field in curved spacetime goes as described in Sec. 2.1.4.

For the interaction we choose a Yukawa type interaction

$$\mathcal{L}_{\mathcal{I}} = -\sqrt{-g}h\phi\psi\overline{\psi}, \tag{4.2.3}$$

where the dimensionless coupling constant is denoted by h and chosen to be real. The perturbative expansion of the S-matrix to first order is given as $S = 1 - ihA + \mathcal{O}(h^2)$, where

$$A = \lim_{\beta\to 0^+} \int \hat{T}e^{-\beta\eta}\phi\psi\overline{\psi}\sqrt{-g}d^4x. \tag{4.2.4}$$

The problem now is to find the correctly normalized positive mode solutions for the massless spinor in spatially flat Robertson-Walker universe.

4.2.1 Spinor Modes in Rest Frame

Solving the spinor modes in curved spacetime is considerably harder than the scalar modes owing to the more complex nature of the spinor [102–104]. In order to obtain exact solutions, we must fix the scale factor outright even for a massless field. The solutions are most easily found when using standard coordinate time t at the outset. We therefore choose as the scale factor a power-law expansion $a(t) = b't^{n'}$ with $n' \in [0, 1)$. This covers the cosmologically interesting cases like stiff-matter-, radiation- and matter-dominated universes for $n' = 1/3, n' = 1/2$ and $n' = 2/3$, respectively. It should be noted that the parameters b', n' are not the same as those used in previous section for the scalar channel decay. These may be related to the conformal time scale factor (4.1.10) by

$$n' = \frac{n}{2+n}, \quad b' = b\Big(\frac{2+n}{2b}\Big)^{\frac{n}{2+n}}. \tag{4.2.5}$$

The results of the previous section for the scalar field may be straigthforwardly transformed into standard coordinate time by using these relations and $dt = a(\eta)d\eta$.

The spinor modes were solved using the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{4.2.6}$$

where σ_0 is the identity matrix and σ_i are the Pauli matrices. This representation decouples the Dirac equation into sets of two second-order differential equations allowing for exact solutions to be found. By solving the Dirac equation (2.1.30) in curved spacetime for the massless spinor, the following set of mode solutions were found

$$u^s_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{[2\pi a(t)]^{3/2}\sqrt{2k}} u(\mathbf{k},s) e^{i\mathbf{k}\cdot\mathbf{x} - \frac{ik}{b'(1-n')} t^{1-n'}} \tag{4.2.7}$$

$$v^s_{\mathbf{k}}(t,\mathbf{x}) = \frac{1}{[2\pi a(t)]^{3/2}\sqrt{2k}} v(\mathbf{k},s) e^{-i\mathbf{k}\cdot\mathbf{x} + \frac{ik}{b'(1-n')} t^{1-n'}}. \tag{4.2.8}$$

The positive and negative energy spinors were normalized according to Eq. (2.1.36) and were found to be

$$u(\mathbf{k},+) = \begin{pmatrix} \sqrt{k-k_3} \\ \frac{-k_+}{\sqrt{k-k_3}} \\ 0 \\ 0 \end{pmatrix}, \quad v(\mathbf{k},+) = \begin{pmatrix} \sqrt{k+k_3} \\ \frac{-k_+}{\sqrt{k-k_3}} \\ 0 \\ 0 \end{pmatrix},$$

$$u(\mathbf{k},-) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k+k_3} \\ \frac{k_+}{\sqrt{k+k_3}} \end{pmatrix}, \quad v(\mathbf{k},-) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k-k_3} \\ \frac{k_+}{\sqrt{k+k_3}} \end{pmatrix}, \tag{4.2.9}$$

where we have defined $k \equiv |\mathbf{k}|$ and $k_\pm \equiv k_1 \pm ik_2$ with $\mathbf{k} = (k_1,k_2,k_3)^T$. The spinor field ψ and its conjugate field $\overline{\psi}$ may now be expanded as in Eqs. (2.1.34) and (2.1.35). For the massive scalar field mode we may use the results of the previous section with the parameters converted to the primed variables.

4.2.2 Transition Probability and Decay Rate

The total transition amplitude for the massive scalar to decay into a massless fermion-antifermion pair was calculated using the added-up method and was found to be

$$w^{\text{tot}} = \frac{h^2}{32}\int_{mt_0}^{mt} \left| \frac{d}{ds}\left(s^{\frac{1-n'}{2}} H^{(2)}_\alpha(s)\right)\right|^2 s^{n'}\,ds, \tag{4.2.10}$$

where s is a dimensionless variable. The Hankel function arises from the scalar field mode (4.1.11) and the index α may given in terms of the primed parameters as

$$\alpha = \frac{\sqrt{(1-n')^2 - 4n'(2n'-1)(6\xi-1)}}{2}. \tag{4.2.11}$$

Contrast to the scalar channel, the total transition probability diverges in the lower limit $t_0 \to 0$ unless the spacetime is Minkowskian, radiation-dominated or the coupling ξ is conformal. But since the theory of gravitation we are dealing with is classical, the spacetime singularity poses its own problems. For a physical explanation of this divergence a more complete theory of quantum gravity might be needed to explain it, but as long as t_0 is in the neighborhood of the singularity, Eq. (4.2.10) has a well defined lower limit.

The differential decay rate may again be interpreted as the integrand of Eq. (4.2.10), which gives

$$\Gamma_{\psi}^{\text{diff}} = \frac{h^2 t^{n'}}{32} \left| \frac{d}{dt}\left(t^{\frac{1-n'}{2}} H_{\alpha}^{(2)}(mt) \right) \right|^2 \tag{4.2.12}$$

and we found that with the same Minkowskian limit $n' = 0, b' = 1$, the transition amplitude reduces to the Minkowskian decay rate

$$\Gamma_{\text{Mink}} = \frac{h^2 m}{16\pi} \tag{4.2.13}$$

when divided by the time $\Delta t \equiv t - t_0$. Having transition amplitudes defined for both scalar and fermion channel decays, we may now compare them with each other.

4.3 Modification of Minkowskian Results

It is already clear from the results of the last section, that Minkowskian decay rates are modified in curved spacetime. In the case of scalar channel decay, the decay rate obtains an additive term arising from the lower limit of the integral taken to be the limit when $t_0 \to 0$. For the fermionic channel we found a same kind of additive term when the massive particle is conformally coupled so the integration may be performed with the lower limit $t_0 \to 0$. In this section we will take a closer look at how the decay rates are modified in curved spacetime and compare the differences in decay into either scalar or fermionic channels. We will begin by considering the case where the massive particle is conformally coupled because this allows for the comparison of these two channels of decay all the way from the initial singularity where the additive term originates. It is found that there is an ordering in these additive terms depending on the universe type in question. Finally, we take a look at what happens with the differential decay rates as the time approaches singularity.

4.3.1 Conformally Coupled Massive Particles

We start by considering the special case where the decaying particle is conformally coupled to the background gravitational field in the integration limit $t_0 \to 0$. For the fermionic channel we found that the total transition amplitude is then given in terms of Bessel functions as

$$\begin{aligned} w^{\text{tot}} =& \frac{h^2}{64}\{(mt)^2[J_{-\frac{1+n'}{2}}(mt)^2 - J_{-\frac{3+n'}{2}}(mt)J_{\frac{1-n'}{2}}(mt) + Y_{-\frac{1+n'}{2}}(mt)^2 \\ & - Y_{-\frac{3+n'}{2}}(mt)Y_{\frac{1-n'}{2}}(mt)] + \frac{2(1+n')\tan(n'\pi/2)}{\pi}\}, \end{aligned} \tag{4.3.1}$$

where the last constant term again arises from the lower limit of $t_0 \to 0$. For further insight, we also take a look at the asymptotic, long-time behavior which we found to be given by

$$w^{\text{tot}} \sim \frac{h^2 m}{16\pi}\Big(t + \frac{(1+n')}{2m}\tan(\frac{n'\pi}{2})\Big), \tag{4.3.2}$$

valid when $n' \in [0,1)$, and the asymptotic decay rate is given by the same prescription as for the scalar channel by dividing by the time of the gravitational influence t_{grav},

$$\Gamma_\psi \sim \frac{h^2 m}{16\pi}\Big(1 + \frac{(1+n')\tan(n'\pi/2)}{2mt_{grav}}\Big). \tag{4.3.3}$$

The asymptotic decay rate for the scalar channel Eq. (4.1.23) for the conformal coupling and in terms of the primed parameters is now given by

$$\Gamma_\phi \sim \frac{\lambda^2}{16m\pi}\Big(1 - \frac{(1-n')\cot[(1-n')\pi/2]}{2mt_{grav}}\Big), \tag{4.3.4}$$

also valid when $n' \in [0,1)$.

A major difference can now be seen when comparing the two expressions Eqs. (4.3.3) and (4.3.4) corresponding to the sign of the additive term. For a decay into scalars the sign is negative and for fermionic channel it is positive. This indicates that for a conformally coupled massive scalar, the effect of gravitation is to enhance the decay into the fermionic channel while diminishing the decay into the scalar channel. Although these inferences are made from the asymptotic formulas, in Publication IV it was found numerically that they hold also for the exact mean decay rates. Moreover, both functions in the correction terms are increasing functions on the interval $n' \in [0,1)$, so the correction term increases as n' increases. Therefore the decay into scalar channel is diminished and decay into fermionic channel enhanced as n' increases. This was already noted in Publications I and II for the stiff-matter-, radiation- and matter-dominated universes, where it was noted that the correction term is largest for matter-dominated universe and smallest for stiff matter out of these three. A speculative idea was also put forth that the faster the universe is expanding the smaller is the decay rate into the scalar particles.

This idea can be quantified more easily by taking a look at the Hubble parameter $H = \dot{a}/a = n'/t$. This shows us that as n' increases the universe expands relatively faster. Hence, the faster the universe is expanding the faster is the decay rate into the fermionic channel. The reason for this behavior may be speculated from a statistical point of view. As the universe expands faster and faster, more states are becoming available for the fermions to occupy. On the other hand, for bosons the Bose enhancement factor is reduced by the expansion as more states become available in total thereby diminishing the decay rate.

4.3.2 A Dominant Channel of Decay

As was already noted, the integral (4.2.10) for the fermionic transition probability diverges in the limit $t_0 \to 0$ while the corresponding integral

(4.1.15) for the scalar channel does not. This singular behavior may be investigated by taking a look at the behavior of the integrand, interpreting it as the differential decay rate, as $t \to 0$. What we found was that the differential decay rate of the fermionic channel (4.2.12) behaves like $t^{-2\alpha-1}$ in the vicinity of the spacetime singularity, while the scalar differential decay rate (4.1.16) behaves like $t^{-2\alpha+1}$. Taken together, these observations imply that the fermionic decay channel dominates the scalar one in the early universe, because the inequality $2\alpha + 1 > 2\alpha - 1$ is always true.

This inequality is true regardless of the value of the parameters n and ξ which define the matter content of the universe and the coupling of the massive particle to the gravitational field. It must be stressed though that this inference holds only near the singularity of the spacetime and it is not evident from the analysis just what this exact time when this is true is. This is because decay rates are differently proportional to masses and to couplings which affect the decay rate. Also this holds only when the decay products are massless and conformally coupled, so a more general notion on this property cannot be made.

Chapter 5

Reheating in the Kination Epoch

In the previous chapter we took a theoretical look into particle decay in curved spacetime within the framework of quantum field theory in curved spacetime. Now it is time to take these results and apply them to a cosmological scenario. The inflationary universe, and especially reheating, provides a good application for these results. As inflation leaves behind a cold universe void of matter, it needs to be reheated as described in Chap. 1. At the end of inflation the spacetime metric changes, usually from de Sitter spacetime into a spatially flat Robertson-Walker metric. But the theory of quantum fields in curved spacetime predicts that particles are created in this transition due to the change of the metric itself. This insight provided a basis for investigation on whether this mechanism might be capable of reheating the universe.

In the 1980s, Ford investigated a situation where massless nonconformally coupled particles were created as the metric changed from de Sitter into matter- or radiation-dominated era and indeed found it to be capable of reheating the universe [105]. This idea of reheating by gravitational particle production was expanded by Spokoiny to an inflationary scenario where the kinetic energy of the inflaton field dominates at the end of inflation. In his studies, Spokoiny introduced some potentials which, contrary to to the conventional models, do not have minimums but rather just rapidly fall for large values of the field [106]. In this case the potential energy of the inflaton is converted to its kinetic energy and the universe ends up being driven by the kinetic term. Again, it was found that this mechanism is capable of reheating the universe. While Spokoiny called this situation deflationary, as opposed to inflationary, it was Joyce who supposedly coined the term kination to describe this phase of kinetic energy dominated expansion [43]. Recently, the idea of a kination phase has extensively been used in quintessential models [45–53].

The kination phase has a close connection with a universe dominated by stiff matter. After inflation, the universe emerges in a kination phase where the kinetic energy dominates the potential energy, so from Eqs. (1.3.1) and (1.3.2) we have $p_\phi = \rho_\phi$. As this equation corresponds to the stiff equation of state, it can be inferred that the kination phase corresponds to a universe where the energy density is dominated by stiff matter.

Though much investigated, particle creation from a sudden or smooth change of the metric into the kination era is not the only way to produce particles into the universe because there is also gravitational particle creation by the expansion of spacetime itself. In this chapter, we present the results of Publication V concerning reheating in a kination epoch in which, contrast to previous studies, the starting point will be gravitational particle creation in a stiff-matter-dominated era. The study will be conducted through non-equilibrium decay using the Boltzmann equations. The aim is for a more precise calculation, so the particle decay will also be described in the context of curved spacetime field theory and the decay rates for stiff-matter, radiation- and matter-dominated universes from previous chapter will be used.

5.1 Reheating via Gravitational Particle Production

The aim of Publication V was to study reheating in the kination era where particles were produced by the expansion of spacetime in the stiff-matter-dominated era and to obtain the reheating temperature. More precisely, we considered a massive scalar particle ϕ interacting with massless scalar particles φ as described by the Lagrangian densities (4.1.1) and (4.1.2). The massive particle then decays into massless particles which ultimately thermalize and reheat the universe.

Described in this way, there exists two different scenarios which might happen. As the massive non-relativistic particle decays into massless particles, described as radiation, it may happen that either one of these energy densities dominate when the equilibrium with the background stiff matter energy density $\rho_{\rm stiff}$ is reached. If at this equilibrium point the massive particle energy density ρ_ϕ is larger, the universe becomes matter-dominated for a while and if the massless particle energy density ρ_φ dominates the universe transforms directly into radiation-dominated universe. We had to consider the possibility of both of these scenarios to occur. In both cases, the effective reheating temperature T_{rh} was obtained as a maximum of the function $\rho_\varphi^{\rm max}$ in the radiation-dominated era through the relation $\frac{\pi^2}{30} g_* T_{rh}^4 = \rho_\varphi^{\rm max}$. In both cases the evolution of the energy densities was described by the Boltzmann equations in a manner presented in Sec. 1.2.2 with the curved space differential decay rates for the matter-, radiation- and stiff-matter-dominated eras given by Eq. (4.1.16). The spacetime was taken as a four-dimensional spatially flat Robertson-Walker universe with the familiar scale factor $a(\eta) = b\eta^{n/2}$ where n is assumed to be non-negative.

Regarding the Boltzmann equations, the evolution of the energy densities in the radiation- and matter-dominated eras is described by the ordinary Boltzmann equations (1.2.6) as given in Chap. 1, but in the stiff-matter era there is an additional term contributing to the energy density coming from the gravitational particle production. Hence, the Boltzmann equations (1.2.6) in the stiff-matter era are given by

$$\dot{\rho}_\phi + 3H\rho_\phi = -\Gamma_\varphi \rho_\phi + w_\phi, \tag{5.1.1}$$

where w_ϕ is the contribution from the gravitationally created particles to the energy density and Γ_φ is given by Eq. (4.1.16) with $n = 1$ corresponding to the stiff-matter era. In the radiation- and matter-dominated eras, the decay rate in the ordinary Boltzmann equations is given by Eq. (4.1.16) with $n = 2$ and $n = 4$, respectively. These decay rates were derived in the rest frame of the decaying particle but, as was shown in [85], particle production in a stiff-matter-dominated universe is peaked at low-momentum modes and is greater the more massive the particle is. Therefore the use of zero-momentum decay rate serves as a valid approximation to be used in the Boltzmann equations.

The differential equation (5.1.1) has the formal solution

$$\rho_\phi(t) = \frac{1}{a(t)^3} e^{-\int_{t_0}^{t} \Gamma_\varphi(t')dt'} \int_{t_0}^{t} a(t')^3 w_\phi(t') e^{\int_{t_0}^{t'} \Gamma_\varphi(t'')dt''} dt', \tag{5.1.2}$$

where t_0 denotes the initial time taken to be the time when inflation ends. We have also assumed that the initial energy density $\rho_\phi(t_0)$ is zero, a reasonable assumption because inflation would inflate out any existing matter. Since we are using curved space decay rates, the decay rate Γ_φ has explicit time dependence. The contribution from the gravitationally created particles in the stiff-matter era can be obtained from their differential creation rate $d|\beta_k|^2/d\eta$ as

$$w_\phi(\eta) = \frac{1}{2\pi}\int_0^\infty k^2\omega_k(\eta)\frac{d|\beta_k|^2}{d\eta}dk, \tag{5.1.3}$$

where $\omega_k(\eta) = \sqrt{k^2 + m^2b^2\eta}$. Using the differential creation rate per mode k obtained in [85],

$$\begin{aligned}\frac{d|\beta_k|^2}{d\eta} =& \frac{\pi(mb)^2\eta}{2k}\Big[\mathrm{Ai}\Big(\frac{-k^2-(mb)^2\eta}{(mb)^{4/3}}\Big)\mathrm{Ai}'\Big(\frac{-k^2-(mb)^2\eta}{(mb)^{4/3}}\Big) \\ &+ \mathrm{Bi}\Big(\frac{-k^2-(mb)^2\eta}{(mb)^{4/3}}\Big)\mathrm{Bi}'\Big(\frac{-k^2-(mb)^2\eta}{(mb)^{4/3}}\Big)\Big]\end{aligned} \tag{5.1.4}$$

we found that the contribution to the energy density of the massive scalars was given in coordinate time t by

$$w_\phi(t) = \frac{3(mb)^{13/3}}{32b}t\big[\mathrm{Ai}(-(3mt/2)^{2/3})^2 + \mathrm{Bi}(-(3mt/2)^{2/3})^2\big], \tag{5.1.5}$$

under the approximation $\omega_k(\eta) \approx mb\sqrt{\eta}$, which was proven to be sound in [85]. After the stiff-matter-dominated era, the evolution of the massive particles is described by the usual Boltzmann equations (1.2.6), i.e., Eq. (5.1.1) without the w_ϕ term, because there is no gravitational particle production anymore. The energy density for the massless decay products is given by Eq. (1.2.8) with the formal solution

$$\rho_\varphi(t) = \frac{1}{a(t)^4}\int_{t_0}^t \Gamma_\varphi\rho_\phi(t')a(t')^4dt', \tag{5.1.6}$$

where the initial energy density is set to zero because there are no decay products present. Finally, the energy density of the background in the stiff-matter era is given by [42],

$$\rho_{\text{stiff}}(t) = \frac{1}{24\pi G_N t^2}, \tag{5.1.7}$$

where G_N is the gravitational constant.

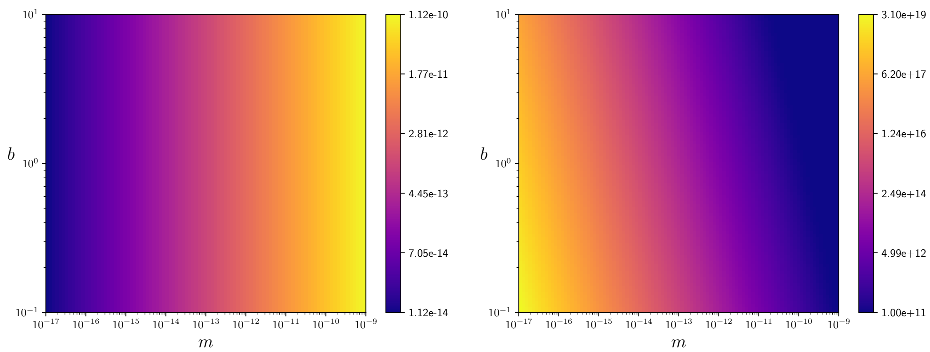
5.2 Numerical Results

To obtain the reheating temperature, we used PYTHON programming to numerically evaluate the integrals of the energy densities of the previous section with a 200×200 grid of the parameters m and b. The analysis was done in Planck units, where $G_N = 1$ in addition to $\hbar = c = 1$, for numerical reasons. The Planck units may be converted to GeV after the calculations. The initial time t_0 was fixed to $t_0 = 10^{11}$ corresponding to $t_0 \sim 10^{-32}$ sec, a value commonly found in the literature [32, 33], with no observable changes in the numerical values if changed a few orders of magnitude around this value. The gravitational coupling ξ used was between the minimal and conformal couplings and the parameter b was calculated in the range of $b \in [10^{-1}, 10^{1}]$. These presented no observable changes to the reheating temperature although they do affect on the time of transition between different phases of the universe. The coupling constant λ was assumed to be small for the perturbation theory to work and it was fixed to run with the mass through the relation $\lambda = \gamma m$, with five values for γ, namely $\gamma = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$.

For the two different scenarios which could happen, either the universe becomes immediately radiation-dominated or is temporarily matter-dominated, we found that with the full range of parameters used the universe always ends up being matter-dominated for a while. This implies that the decay into massless particles is not fast enough to increase their energy density sufficiently above the energy density of the massive particles as the equilibrium with the background stiff matter is reached. The possibility of a straight transition into radiation-dominated universe cannot be ruled out completely, because there might exist a range of parameters with which this could be achieved, e.g., increasing the coupling λ sufficiently, although in this case the perturbative expansion might not work.

5.2.1 Reheating Temperature

For the actual reheating temperature, we found it to lie in the interval of about 10^{-15}-10^{-7} in Planck units depending on the parameters used. In natural units, this corresponds to about the order of 10^{4}-10^{12} GeV. In Fig. 5.1a the reheating temperature T_{rh} is presented for selected parameter values. From this figure it is also evident that the reheating temperature does not depend on the expansion parameter b. The reason behind this can be traced back to the fact that the parameter b appears explicitly only in the differential creation rate (5.1.4) thereby directly affecting the massive particle energy density in the stiff-matter era alone. Combined with the knowledge that in a stiff- matter-dominated universe particle creation is peaked to high-mass modes [85] and the decay rate to massless particles is faster the more massive the particle is, it is reasonable to expect that the energy densities ρ_ϕ and ρ_φ follow each other closely in the stiff-matter era. In the matter- and radiation-dominated phases the evolution is independent of the parameter b, which would explain the independence of this parameter in the reheating temperature.



(a) (b)

Fig. 5.1: a) Reheating temperature T_{rh} as a function of mass m and expansion parameter b given in Planck units with the ratio $\lambda/m = 10^{-2}$ and $\xi = 1/6$ b) Time of transition to matter dominated era as a function of mass m and the expansion parameter b given in Planck units. The ratio λ/m is 10^{-4} and the coupling ξ is conformal.

The expansion rate b does, however, affect on the time the universe transitions into the matter-dominated era but has no effect on the transition time from matter- to radiation-dominated era. A clearer picture of this is presented in Fig. 5.1b, where the time of transition to the matter-dominated era is given as a function of the mass of the decaying particle and the expansion rate b. As can be seen, increasing the mass m or the parameter b generally shortens the time when matter domination is reached. Since in the stiff-matter-dominated era, the particle creation is most effective for large m and b [85], this is a reasonable result. Hence, for a very massive particle, the particle creation from the vacuum is so explosive that the universe transforms almost immediately into the matter-dominated phase.

The interaction coupling λ, or rather its ratio with the mass m, does have an effect on the reheating temperature as seen from Fig. 5.2. With a fixed mass m, an increase in this ratio corresponds to increasing λ by the same amount. An order of magnitude increase corresponds to roughly an order of magniture increase in the reheating temperature. This sort of relation is quite natural because the λ affects directly the decay rate and its increase increases the decay into massless particles which in turn increases their energy density ρ_φ. Hence, if the energy density is larger to begin with, it is a natural consequence for the reheating temperature to be higher also.

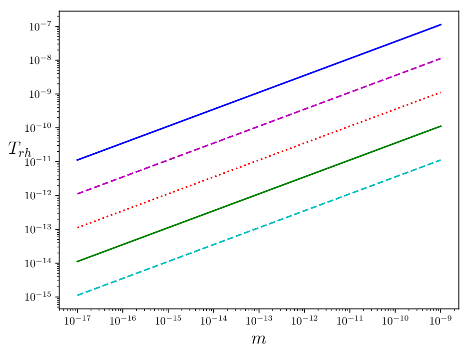


Fig. 5.2: Reheating temperature in Planck units as a function of mass m with five values of the ratio λ/m; 10^{-1} (blue), 10^{-2} (magenta), 10^{-3} (red), 10^{-4} (green) and 10^{-5} (cyan) with $b = 10$ and $\xi = 1/6$.

We also ran the simulation with the Minkowskian decay rates and found that both produce results within the same order of magnitude with only small numerical differences. Although the Minkowskian decay rates are modified in curved spacetime, it is likely that at least in the scalar decay case, the timescales in question are so small that the effect of curved space modification has not produced any noticeable effects yet.

In conclusion, we showed that the gravitational particle creation mechanism is able to reheat the universe to temperatures of about 10^4-10^{12} GeV and that this temperature is independent on the expansion rate of the universe and the coupling of the decaying particle into gravity. The advantage of our numerical approach is that we obtain an exact numerical value for the reheating temperature which does not explicitly depend on the time when inflation ends, which is the case in models where the particles are produced by the change of the metric [49–51]. At this point a question naturally comes to mind how the inclusion of the fermionic decay channel with the curved space decay rates of Chap. 4 affects the above reheating scenario. This question is currently investigated in ongoing research involving both the scalar and fermionic channels of decay.

Chapter 6

Concluding Remarks

In this thesis we have studied particle decay in the presence of a gravitational field using quantum field theory in curved spacetime. The results we have presented indicate that the Minkowskian decay rates are modified by the expanding universe; significantly for early times near the spacetime singularity and by an additive term in the long-time limit. As the topic of this thesis is a mix of both modern cosmology as well as quantum field theory in curved spacetimes, a sufficient background on these topics was presented in Chap. 1-3. By no means were they meant to be comprehensive but rather the purpose was to allow the reader to follow the thesis. In Chap. 1 we reviewed the key topics of modern cosmology from those parts which were needed. The focus was given to a spatially flat Robertson-Walker spacetime and investigation into an exotic stiff matter phase was given. Non-equilibrium decay was considered with the help of Boltzmann equations for a massive particle to decay into two massless particles and finally a short review on the concepts of cosmological inflation and reheating was given.

In Chap. 2 we moved to quantum fields propagating on curved spacetimes. The quantization of a free scalar and spinor fields was given in the most general way as possible and the phenomena of gravitational particle creation was discussed. In this chapter the persisting feature of quantum field theory in curved spacetime of not having a unique solution for the positive field modes was discussed. In the case of a spatially flat Robertson-Walker universe, the construction of a meaningful in- and out-regions of spacetime in which the particle concept could be defined was given. As the notion of conformal invariance is of utmost importance in regard to the concepts presented in this thesis, a section to it was also devoted.

Leaving the free fields behind, in Chap. 3, the case of mutually interacting fields in curved spacetime was considered. Although the theory of interacting fields in curved spacetime was considered, weight was given to the interpretational problems arising because the gravitational particle creation severely interferes with the mutual interaction. To overcome these issues a concept of added-up transition probabilities introduced by Audretsch and Spangehl was reviewed with the goal of reaching a formula for the decay rate in curved spacetime which can physically be motivated. The issues encountered in this chapter suggest that considerable care must be exercised when studying mutually interacting fields in curved spacetime.

In the following chapters the key findings of Publications I-V were given. In Chap. 4 the results were presented for the transition amplitudes and decay rates for a massive scalar particle to decay into two massless scalars or a massless fermion-antifermion pair in the rest frame of the decaying particle. It was found that decay rates are considerably modified at early times as compared to the Minkowskian decay rates but only by an additive term in the long-time limit. Moreover, for conformally coupled particles, the decay into fermionic channel was found to be enhanced and the decay into scalar channel diminished by the expanding spacetime with the fermionic channel dominating the scalar channel at early times. These results were used in Chap. 5 when a reheating scenario was considered within the kination era after inflation. The particles were created by gravitational particle creation, which subsequently decay and reheat the universe to temperatures of about 10^4-10^{12} GeV. Within the parameter range used, the scalar channel decay results in the universe always being dominated by ordinary matter for a short while before the radiation-dominated universe begins.

To conclude, the results presented in this thesis reinforce the notion that Minkowskian field theory is ultimately only an approximation in a regime where the effects of gravitation cannot be neglected anymore. We have seen how the changing spacetime metric modifies the Minkowskian decay rates and this modification depends on the value of the gravitational coupling and the channel of decay for a massive scalar. The results open up a possibility to do more precise cosmological calculations by replacing the Minkowskian decay rates commonly used by the curved space counterparts. And we must not forget that as cosmological data and measurements become increasingly more accurate, there might be a necessity in the future to include the effects of curved space also in particle decay rates.

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