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The $\bar{\partial}$-Equation, Duality, and Holomorphic Forms on a Reduced Complex Space

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Abstract
We solve the $\bar{\partial}$-equation for (p, q)-forms locally on any reduced pure-dimensional complex space and we prove an explicit version of Serre duality by introducing suitable concrete fine sheaves of certain (p, q)-currents. In particular this gives a condition for the $\bar{\partial}$-equation to be globally solvable. Our results also give information about holomorphic p-forms on singular spaces.

Keywords Complex space · $\bar{\partial}$ equation · Holomorphic forms · Serre duality

Mathematics Subject Classification 32A26 · 32A27 · 32C15 · 32C30 · 32C37

1 Introduction

Let X be a reduced complex space of pure dimension n. A smooth form on X is locally the pullback to X_{reg} of a smooth form in some ambient complex manifold. It is well known that this is an intrinsic notion and we denote the corresponding sheaf by $\mathscr{E}_X$. It is proved in [8] that if φ is a smooth $\bar{\partial}$-closed $(0, q)$-form, $q > 0$, on X and X is Stein, then there is a smooth $(0, q-1)$-form ψ on X_{reg} such that $\bar{\partial}\psi = \varphi$; if $q = 0$ then φ is strongly holomorphic. In general ψ cannot be smooth on X, see, e.g., [8, Example 1.1]. However, the local solution operators constructed in [7,8] provide solutions ψ with certain mild singularities at X_{sing}. In particular it is shown that ψ is a current on X and that $\bar{\partial}\psi = \varphi$ in the current sense also across X_{sing}.

In case X is smooth, local existence results for the $\bar{\partial}$-equation for $(0, q)$-forms easily carry over to (p, q)-forms. The reason is that the holomorphic p-forms in this

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case are the sections of a vector bundle, i.e., a locally free sheaf. However, in the presence of singularities the situation is much more involved since, in this case, there are several natural notions of holomorphic p-forms and usually the corresponding sheaves are not locally free. We will be particularly interested in two notions, $\widehat{\Omega}_X^p$ and ω_X^p. The sheaf $\widehat{\Omega}_X^p$ is the sheaf of Kähler–Grothendieck p-forms modulo torsion. Alternatively one can define $\widehat{\Omega}_X^p$ in the same way as $\mathscr{E}_X$ above replacing "smooth form" by "holomorphic p-form". Notice that $\widehat{\Omega}_X^0 = \mathscr{O}_X$. The sheaf ω_X^p was introduced by Barlet [13]; for $p = n$ it is the Grothendieck dualizing sheaf.

Our main result is that locally on X the $\bar{\partial}$-equation for (p, q)-forms is always solvable, if interpreted in the sense of currents even across X_{sing}. For other results about the $\bar{\partial}$-equation in the singular setting see, e.g., [14,21,29,31–33,37,39]. Recall that the (p, q)-currents on X are the dual of the compactly supported sections of $\mathscr{E}_X^{n-p,n-q}$; given an embedding $X \hookrightarrow M$, currents on X can also be identified with certain currents in ambient space, see Sect. 2.

Theorem 1.1 *Let X be a pure n-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ be a relatively compact open subset, and set $X' := X \cap D'$. There are integral operators $\mathscr{K} : \mathscr{E}^{p,q}(X) \to \mathscr{E}^{p,q-1}(X'_{reg})$ and $\mathscr{P} : \mathscr{E}^{p,0}(X) \to \widehat{\Omega}^p(X')$ such that $\mathscr{K}\varphi$ has a current extension to X' and, as currents on X',*

$$\varphi = \mathscr{K}(\bar{\partial}\varphi) + \mathscr{P}\varphi, \quad \varphi \in \mathscr{E}^{p,0}(X),$$
$$\varphi = \bar{\partial}\mathscr{K}\varphi + \mathscr{K}(\bar{\partial}\varphi), \quad \varphi \in \mathscr{E}^{p,q}(X), \ q \geq 1.$$

The construction of $\mathscr{P}$ shows that $\mathscr{P}\varphi$ in fact has a holomorphic extension to D'. The integral operators $\mathscr{K}$ and $\mathscr{P}$ are given by kernels $k(\zeta, z)$ and $p(\zeta, z)$ which are currents on $X \times X'$ that are respectively integrable and smooth on $X_{reg} \times X'_{reg}$ and that have principal value-type singularities at the singular locus of $X \times X'$. Since a current locally has finite order we get the following result.

Corollary 1.2 *Let φ be a smooth $\bar{\partial}$-closed (p, q)-form on X_{reg} such that there is a C^ℓ-smooth form in D whose pullback to X_{reg} equals φ. There is an $M_{D'} \geq 0$, independent of φ, such that the following holds:*

(i) *If $q = 0$ and $\ell \geq M_{D'}$ then there is a $\tilde{\varphi} \in \widehat{\Omega}^p(X')$ such that $\varphi\restriction_{X'_{reg}} = \tilde{\varphi}\restriction_{X'_{reg}}$.*

(ii) *If $q \geq 1$ and $\ell \geq M_{D'}$ then there is a smooth $(p, q-1)$-form u on X'_{reg} such that $\bar{\partial}u = \varphi$ on X'_{reg}.*

Part (i) for $p = 0$ and $M_{D'} = \infty$ is a classical result by Malgrange [30, Théorème 4] answering a question by Grauert; for $M_{D'} < \infty$ it is due to Spallek [42]. Part (ii) for $p = 0$ was proved by Henkin and Polyakov [25] in case X is a reduced complete intersection and in general in [7]. We remark that Corollary 1.2 is explicit in the sense that $\mathscr{P}\varphi$ (resp. $\mathscr{K}\varphi$) provides an explicit holomorphic extension of φ to D' (resp. explicit solution to $\bar{\partial}u = \varphi$ on X'_{reg}).

As already mentioned, $\bar{\partial}\psi = \varphi$ is in general not smoothly solvable in neighborhoods of singular points even if φ is smooth (and $\bar{\partial}$-closed), i.e., the complex $(\mathscr{E}_X^{p,\bullet}, \bar{\partial})$ is in general not exact. Therefore $\mathscr{K}\varphi$ cannot be smooth in general. However, the singularities of $\mathscr{K}\varphi$ are not worse than that one can apply another $\mathscr{K}$-operator. In fact, one can apply $\mathscr{K}$-operators repeatedly. In a similar way as in [8], starting with smooth forms and iteratively applying $\mathscr{K}$-operators and multiplying by smooth forms, we construct fine sheaves $\mathscr{A}_X^{p,q}$ of certain currents, which are closed under $\mathscr{K}$-operators and $\bar\partial$, see Sect. 6.1 below for details. We have the following generalization of [8, Theorem 1.2].

Theorem 1.3 *Let X be a reduced complex space of pure dimension n. For each $p = 0, \dots, n$ there are fine sheaves $\mathscr{A}_X^{p,q}$, $q = 0, \dots, n$, of (p, q)-currents on X with the standard extension property such that*

(i) $\mathscr{E}_X^{p,q} \subset \mathscr{A}_X^{p,q}$ *and* $\oplus_q \mathscr{A}_X^{p,q}$ *is a module over* $\oplus_q \mathscr{E}_X^{0,q}$,
(ii) $\mathscr{A}_{X_{reg}}^{p,q} = \mathscr{E}_{X_{reg}}^{p,q}$,
(iii) *the following sheaf complex is exact*

$$0 \to \widehat{\Omega}_X^p \hookrightarrow \mathscr{A}_X^{p,0} \xrightarrow{\bar\partial} \mathscr{A}_X^{p,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \mathscr{A}_X^{p,n} \to 0. \tag{1.1}$$

That a current has the standard extension property (SEP) means roughly speaking that it is determined by its restriction to any dense Zariski open subset, see Sect. 2 for the precise definition.

Since the $\mathscr{A}_X$-sheaves are fine, the de Rham theorem gives the following generalization to the singular setting of the classical Dolbeault isomorphism.

Corollary 1.4 *Let X be a reduced complex space of pure dimension, let $F \to X$ be a holomorphic vector bundle, and let $\mathscr{F}$ be the associated locally free $\mathscr{O}_X$-module. Then*

$$H^q(X, \mathscr{F} \otimes \widehat{\Omega}_X^p) \simeq H^q\big(\mathscr{A}^{p,\bullet}(X, F), \bar\partial\big).$$

Notice that since $(\mathscr{A}_X^{p,\bullet}, \bar\partial)$ is a resolution of $\widehat{\Omega}_X^p$, whose sections in particular are smooth, and since $\mathscr{E}_X^{p,\bullet} \subset \mathscr{A}_X^{p,\bullet}$, it follows from a well-known construction that each cohomology class in $H^q\big(\mathscr{A}^{p,\bullet}(X), \bar\partial\big)$ has a smooth representative (cf., e.g., [38, Sect. 7]).

The operators $\mathscr{K}$ and $\mathscr{P}$ in Theorem 1.1 extend to operators $\mathscr{A}^{p,q}(X) \to \mathscr{A}^{p,q-1}(X')$ and $\mathscr{A}^{p,0}(X) \to \widehat{\Omega}^p(X')$, respectively, and the integral formulas continue to hold; it is this generalized version of Theorem 1.1 that we will prove below.

The operators $\mathscr{K}$ and $\mathscr{P}$ can be applied to, for instance, semi-meromorphic currents. However, the integral formulas of Theorem 1.1 then cannot hold in general. Indeed, if this were the case then, in particular, any $\bar\partial$-closed meromorphic p-form on X would be in $\widehat{\Omega}^p(X')$. This is to say that $\omega_X^p = \widehat{\Omega}_X^p$, which is not true in general. On the other hand, the obstruction to the integral formulas to hold is explicit and gives a residue criterion, formulated in Theorem 5.5 below, for a meromorphic p-form to be a section of $\widehat{\Omega}_X^p$. This generalizes results by Tsikh [43], Andersson [5], and Henkin-Passare [24]. The residue criterion leads to a geometric criterion, Proposition 5.6, which in turn gives the following geometric characterization of complex spaces with the property that any holomorphic p-form on the regular part extends to a section of $\widehat{\Omega}_X^p$. Recall that to a coherent analytic sheaf $\mathscr{G}$ on X there are associated *singularity subvarieties* $S_0(\mathscr{G}) \subset S_1(\mathscr{G}) \subset \cdots \subset X$, see, e.g., [41, §1] or Sect. 2.3 below.

Proposition 1.5 *Let X be a reduced complex space of pure dimension n. Then the following conditions are equivalent:*

(i) $codim_X X_{sing} \geq 2$ *and* $codim_X S_{n-k}(\widehat{\Omega}^p_X) \geq k+2$ *for* $k \geq 1$.
(ii) *For any open* $U \subset X$ *the restriction map* $\widehat{\Omega}^p(U) \to \widehat{\Omega}^p(U_{reg})$ *is bijective.*

This result is a variation on [41, Theorem 1.14], see also [40], that is explicit in the sense mentioned above. Notice that, for $p = 0$, Proposition 1.5 is a normality criterion. It is in fact possible to verify directly that condition (i) with $p = 0$ is equivalent to Serre's conditions $R1$ and $S2$. From Proposition 1.5 we get the following result, see the end of Sect. 6.1.

Corollary 1.6 *Assume that X is a reduced complete intersection. Then X is smooth if and only if condition* (i)*, or equivalently* (ii)*, of Proposition* 1.5 *with $p = n$ holds.*

In view of Corollary 1.4, $H^q(X, \widehat{\Omega}^p_X)$ encodes the global obstructions to solving the $\bar{\partial}$-equation. To get some control of these obstructions we will describe the dual of $H^q(X, \widehat{\Omega}^p_X)$ as Dolbeault cohomology of fine sheaves $\mathscr{B}_X^{n-p,n-q}$ of certain $(n-p, n-q)$-currents on X. This description of the dual of $H^q(X, \widehat{\Omega}^p_X)$ provides a concrete analytic realization, Theorem 1.9 below, of Serre duality in the singular setting analogous to the classical one in the non-singular case. The operators $\mathscr{K}$ and $\mathscr{P}$ correspond to integrating the kernels $k(\zeta, z)$ and $p(\zeta, z)$, respectively, with respect to ζ; integrating with respect to z instead gives operators $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ with different properties. Applying $\check{\mathscr{K}}$-operators repeatedly gives the $\mathscr{B}_X$-sheaves, which in a sense are dual to the $\mathscr{A}_X$-sheaves. The case $p = 0$ of the following result is proved in [38].

Theorem 1.7 *Let X be a reduced complex space of pure dimension n. For each $p = 0, \ldots, n$ there are fine sheaves $\mathscr{B}_X^{n-p,q'}$, $q' = 0, \ldots, n$, of $(n-p, q')$-currents on X with the SEP such that*

(i) $\mathscr{E}_X^{n-p,q'} \subset \mathscr{B}_X^{n-p,q'}$ *and* $\oplus_{q'} \mathscr{B}_X^{n-p,q'}$ *is a module over* $\oplus_{q'} \mathscr{E}_X^{0,q'}$,
(ii) $\mathscr{B}_{X_{reg}}^{n-p,q'} = \mathscr{E}_{X_{reg}}^{n-p,q'}$,
(iii) $0 \to \mathscr{B}_X^{n-p,0} \xrightarrow{\bar{\partial}} \mathscr{B}_X^{n-p,1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{B}_X^{n-p,n} \to 0$ *is a sheaf complex with coherent cohomology sheaves* $\omega_X^{n-p,q'} := \mathscr{H}^{q'}(\mathscr{B}_X^{n-p,\bullet}, \bar{\partial})$ *and* $\omega_X^{n-p} = \omega_X^{n-p,0}$. *If $\widehat{\Omega}^p_X$ is Cohen–Macaulay then* $(\mathscr{B}_X^{n-p,\bullet}, \bar{\partial})$ *is a resolution of* ω_X^{n-p}.

The proof of Theorem 1.7 will show that if $i\colon X \hookrightarrow D \subset \mathbb{C}^N$, then $i_*\omega_X^{n-p,q'} \simeq \mathscr{E}xt_{\mathscr{O}}^{\kappa+q'}(\widehat{\Omega}^p_X, \Omega^N)$, where $\kappa = N - n$, $\mathscr{O} = \mathscr{O}_{\mathbb{C}^N}$, and $\Omega^N = \Omega^N_{\mathbb{C}^N}$ is the sheaf of holomorphic p-forms in $\mathbb{C}^N$; we will use this notation throughout.

Theorem 1.8 *Let X be a pure n-dimensional analytic subset of a pseudoconvex domain $D \subset \mathbb{C}^N$, let $D' \Subset D$ and set $X' := X \cap D'$. There are integral operators $\check{\mathscr{K}}\colon \mathscr{B}^{n-p,q'}(X) \to \mathscr{B}^{n-p,q'-1}(X')$ and $\check{\mathscr{P}}\colon \mathscr{B}^{n-p,q'}(X) \to \mathscr{B}^{n-p,q'}(X')$ such that*

$$\psi = \bar{\partial}\check{\mathscr{K}}\psi + \check{\mathscr{K}}(\bar{\partial}\psi) + \check{\mathscr{P}}\psi$$

on X'. If $\widehat{\Omega}_X^p$ is Cohen–Macaulay and $\psi \in \mathscr{B}^{n-p,q'}(X)$ then $\check{\mathscr{P}}\psi \in \omega^{n-p}(X')$ if $q'=0$ and $\check{\mathscr{P}}\psi = 0$ if $q' \geq 1$.

Notice that if $\psi \in \omega^{n-p}(X)$ then, on X', $\psi = \check{\mathscr{P}}\psi$ is a representation formula for sections of ω_X^{n-p}.

If $\varphi \in \mathscr{A}^{p,q}(X)$ and $\psi \in \mathscr{B}^{n-p,n-q}(X)$ then the product $\varphi\wedge\psi$ exists, Theorem 7.1. On X_{reg} it is just the exterior product of smooth forms, and this form turns out to have a unique extension to X as a current with the SEP. Moreover, $\bar\partial(\varphi\wedge\psi) = \bar\partial\varphi\wedge\psi + (-1)^{p+q}\varphi\wedge\bar\partial\psi$. Hence, there is a pairing, the trace map, $(\varphi,\psi)\mapsto \int_X \varphi\wedge\psi$ and it descends to a trace map on cohomology.

Theorem 1.9 *Let X be a compact reduced complex space of pure dimension n and let $F \to X$ be a holomorphic vector bundle. Then the following pairing is non-degenerate:*

$$H^q\big(\mathscr{A}^{p,\bullet}(X,F),\bar\partial\big) \times H^{n-q}\big(\mathscr{B}^{n-p,\bullet}(X,F^*),\bar\partial\big) \to \mathbb{C}, \tag{1.2}$$

$$([\varphi]_{\bar\partial},[\psi]_{\bar\partial}) \mapsto \int_X \varphi\wedge\psi. \tag{1.3}$$

The case $p=0$ is proved in [38]. Notice that it follows from Theorem 1.9 together with Corollary 1.4 and Theorem 1.7 that if $\widehat{\Omega}_X^p$ is Cohen–Macaulay, then there is a non-degenerate pairing $H^q(X,\widehat{\Omega}_X^p)\times H^{n-q}(X,\omega_X^{n-p}) \to \mathbb{C}$. For $p=0$ this is the well-known duality on Cohen–Macaulay spaces. For $p>0$ it follows that Barlet's sheaf ω_X^{n-p} is dualizing with respect to $\widehat{\Omega}_X^p$ in the same way as the Grothendieck sheaf ω_X^n is dualizing with respect to $\mathscr{O}_X$. If $\widehat{\Omega}_X^p$ is not Cohen–Macaulay, then ω_X^{n-p} does not suffice to describe the dual of $H^q(X,\widehat{\Omega}_X^p)$; higher $\mathscr{E}xt$'s come into play. This is also the case in the classical duality by Ramis and Ruget [36]: Given a coherent sheaf $\mathscr{F}$ on X they describe the dual of $H^q(X,\mathscr{F})$ as $\mathrm{Ext}^{-q}(X;\mathscr{F},\mathbf{K}_X^\bullet)$, where $\mathbf{K}_X^\bullet$ is the dualizing complex in the sense of [36].

We notice the following consequence of Theorem 1.9: If φ is a smooth $\bar\partial$-closed (p,q)-form on X, then there is a smooth solution to the $\bar\partial$-equation on X_{reg} if $\int_X\varphi\wedge\psi=0$ for all $\bar\partial$-closed smooth $(n-p,n-q)$-forms ψ on X. Indeed, φ defines an element in $H^q(\mathscr{B}^{p,\bullet}(X),\bar\partial)$ and each element in $H^{n-q}(\mathscr{A}^{n-p,\bullet}(X),\bar\partial)$ has a smooth representative.

With a slight modification of the statement, the Serre duality, Theorem 1.9, continues to hold on paracompact spaces provided certain separability conditions are fulfilled. In fact, instead of proving Theorem 1.9, we will prove the following slightly more general result:

If X is a reduced paracompact complex space of pure dimension n and we replace $\mathscr{B}^{n-p,\bullet}(X,F^*)$ in Theorem 1.9 by the corresponding space of sections with compact support, then the conclusion of Theorem 1.9 holds provided that $H^q(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$ and $H^{q+1}(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$ are Hausdorff.

We remark that the Hausdorff assumption is automatically fulfilled if X is compact or holomorphically convex; this follows from the Cartan–Serre theorem and Prill's result, [35], respectively. Moreover, by the Andreotti–Grauert theorem, $H^q(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$ and $H^{q+1}(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$ are Hausdorff for $q\geq k$ if X is k-convex.

2 Preliminaries

Let X be a pure n-dimensional reduced complex space. Following [26, Sect. 4.2], the vector space of (p,q)-currents on X is the dual of the vector space of $(n-p,n-q)$-test forms $\mathscr{D}^{n-p,n-q}(X)$, i.e., the compactly supported sections of $\mathscr{E}_X^{n-p,n-q}$. More concretely, if $i\colon X\hookrightarrow D\subset\mathbb{C}^N$ is an embedding and μ is a (p,q)-current on X, then $\nu:=i_*\mu$ is a $(p+\kappa,q+\kappa)$-current in D (recall that $\kappa:=N-n$) and $\nu.\xi=0$ for any test form ξ in D whose pullback to X_{reg} vanishes. Conversely, if ν is such a current in D then there is a current μ on X such that $\nu=i_*\mu$.

Let χ be any smooth regularization of the characteristic function of $[1,\infty)\subset\mathbb{R}$; throughout the paper, χ will denote such a function. A current μ on X is said to have the *standard extension property* (SEP) with respect to a subvariety $Z\subset X$ if $\chi(|h|^2/\epsilon)\mu\restriction_U\to\mu\restriction_U$ as $\epsilon\to 0$ for any open $U\subset X$, where h is any holomorphic tuple on U not vanishing identically on any irreducible component of $Z\cap U$. If $Z=X$ we simply say that μ has the SEP (on X).

2.1 Meromorphic Forms

Let here X be a pure-dimensional analytic subset of some domain $D\subset\mathbb{C}^N$ and let W be an analytic subset containing X_{sing} but not any irreducible component of X. It is proved in [24] that the following conditions on a holomorphic p-form φ on $X\backslash W$ are equivalent. (1) φ is locally the pullback to $X\backslash W$ of a meromorphic p-form in a neighborhood of X. (2) For any desingularization $\pi\colon\tilde{X}\to X$ such that $\pi^{-1}X_{reg}\simeq X_{reg}$, $\pi^*\varphi$ has a meromorphic extension to $\tilde{X}$. (3) There is a current T in D such that $i_*\varphi=T\restriction_{D\backslash W}$, where $i\colon X\hookrightarrow D$ is the inclusion. (4) For any $h\in\mathscr{O}(X)$ that vanishes on W, but not identically on any component of X, the current

$$\mathscr{D}^{n-p,n}(X)\ni\xi\mapsto\lim_{\epsilon\to 0}\int_X\chi(|h|^2/\epsilon)\varphi\wedge\xi \tag{2.1}$$

exists and is independent of h.

The sheaf of germs of p-forms satisfying these conditions is called the sheaf of germs of *meromorphic* p*-forms* on X; we will denote it by $\mathscr{M}_X^p$. One can check that if $x\in X$ is an irreducible point then $\mathscr{M}_{X,x}^0$ is (isomorphic to) the field of fractions of $\mathscr{O}_{X,x}$. We usually make no distinction between a meromorphic form φ and the associated principal value current (2.1).

2.2 Pseudomeromorphic Currents

Pseudomeromorphic currents were introduced in [10]; the definition we need and will use is from [8]. In one complex variable z it is elementary to see that the principal value current $1/z^m$ exists and can be defined, e.g., as the limit as $\epsilon\to 0$ of $\chi(|h(z)|^2/\epsilon)/z^m$, where h is a holomorphic function (or tuple) vanishing at $z=0$, or as the value at $\lambda=0$ of the analytic continuation of the current-valued function $\lambda\mapsto|h(z)|^{2\lambda}/z^m$. It follows that the *residue current* $\bar\partial(1/z^m)$ can be computed as the limit of $\bar\partial\chi(|h(z)|^2/\epsilon)/z^m$

or as the value at $\lambda = 0$ of $\lambda \mapsto \bar\partial|h(z)|^{2\lambda}/z^m$. Since tensor products of currents are well-defined we can form the current

$$\tau = \bar\partial \frac{1}{z_1^{m_1}} \wedge \cdots \wedge \bar\partial \frac{1}{z_r^{m_r}} \wedge \frac{\gamma(z)}{z_{r+1}^{m_{r+1}} \cdots z_n^{m_n}} \tag{2.2}$$

in $\mathbb{C}^n$, where $m_1, \ldots, m_r$ are positive integers, $m_{r+1}, \ldots, m_n$ are nonnegative integers, and γ is a smooth compactly supported form. Notice that τ is anti-commuting in the residue factors $\bar\partial(1/z_j^{m_j})$ and commuting in the principal value factors $1/z_k^{m_k}$. We say that a current of the form (2.2) is an *elementary pseudomeromorphic current*. Let X be a pure-dimensional reduced complex space and let $x \in X$. We say that a germ of a current μ at x is *pseudomeromorphic* if it is a finite sum of pushforwards $\pi_*\tau = \pi_*^1 \cdots \pi_*^\ell \tau$, where $\mathcal{U}$ is a neighborhood of x,

$$\mathcal{U}^\ell \xrightarrow{\pi^\ell} \cdots \xrightarrow{\pi^2} \mathcal{U}^1 \xrightarrow{\pi^1} \mathcal{U}^0 = \mathcal{U},$$

each π^j is either a modification, a simple projection $\mathcal{U}^j = \mathcal{U}^{j-1} \times Z \to \mathcal{U}^{j-1}$, or an open inclusion, and τ is an elementary pseudomeromorphic current on $\mathcal{U}^\ell \subset \mathbb{C}^N$. The union of all germs of pseudomeromorphic currents on X forms an open subset of the sheaf of germs of currents on X and thus defines a subsheaf $\mathcal{PM}_X$. Notice that since $\bar\partial$ maps an elementary pseudomeromorphic current to a sum of such currents it follows that $\bar\partial$ maps $\mathcal{PM}_X$ to itself.

The following result is fundamental and will be used repeatedly in this paper.

Dimension principle. *Let X be a reduced pure-dimensional complex space, let $\mu \in \mathcal{PM}(X)$, and assume that μ has support contained in a subvariety $V \subset X$. If μ has bidegree $(*, q)$ and* $\operatorname{codim}_X V > q$, *then $\mu = 0$.*

This result is from [10], see also [8, Proposition 2.3]. In connection to the dimension principle we also mention that if $\mu \in \mathcal{PM}(X)$, $\operatorname{supp}\mu \subset V$, and h is a holomorphic function vanishing on V, then $\bar h\mu = 0$ and $d\bar h \wedge \mu = 0$. An arbitrary current μ with $\operatorname{supp}\mu \subset V$ is of the form $\mu = i_*\tau$, where i is the inclusion of V, for some current τ on V if and only if $h\mu = dh \wedge \mu = \bar h\mu = d\bar h \wedge \mu = 0$ for all holomorphic h vanishing on V. Thus, if $\mu \in \mathcal{PM}(X)$, there is such a τ if and only if $h\mu = dh \wedge \mu = 0$ for all holomorphic functions h vanishing on V.

Another fundamental property of pseudomeromorphic currents is that they can be "restricted" to analytic (or constructible) subsets: Let $\mu \in \mathcal{PM}(X)$, let $V \subset X$ be an analytic subset, and set $V^c := X \backslash V$. Then the restriction of μ to the open subset V^c has a natural pseudomeromorphic extension $\mathbf{1}_{V^c}\mu$ to X. It follows that $\mathbf{1}_V\mu := \mu - \mathbf{1}_{V^c}\mu$ is a pseudomeromorphic current with support contained in V. In [10] $\mathbf{1}_{V^c}\mu$ is defined as the value at 0 of the analytic continuation of the current-valued function $\lambda \mapsto |h|^{2\lambda}\mu$, where h is any holomorphic tuple with zero set V; $\mathbf{1}_{V^c}\mu$ can also be defined as $\lim_{\epsilon\to 0} \chi(|h|^2 v/\epsilon)\mu$, where v is any smooth strictly positive function, see [11, Lemma 3.1], cf. also [28, Lemma 6].[1] Taking restrictions is commutative, in

[1] ϵ-Approximations and λ-approximations can be used interchangeably; λ-approximations are often computationally easier to work with while we believe that ϵ-approximations are conceptually easier. For the rest of this paper we will work with ϵ-approximations.

fact, if V and W are any constructible subsets then $\mathbf{1}_V\mathbf{1}_W\mu = \mathbf{1}_{V\cap W}\mu$. Let us also notice that $\mu \in \mathcal{PM}(X)$ has the SEP (on X) precisely means that $\mathbf{1}_V\mu = 0$ for all germs of analytic subsets $V \subset X$ of positive codimension. We will denote by $\mathcal{W}_X$ the subsheaf of $\mathcal{PM}_X$ of currents with the SEP on X. From [11, Sect. 3] it follows that if $\pi: X' \to X$ is either a modification, a simple projection, or an open inclusion, and $\mu \in \mathcal{W}(X')$ then $\pi_*\mu \in \mathcal{W}(X)$.

Lemma 2.1 *Let X be a reduced complex space and let $Y \subset X$ be an analytic nowhere dense subset. If $\mu \in \mathcal{PM}(X) \cap \mathcal{W}(X\backslash Y)$ then $\mathbf{1}_{X\backslash Y}\mu \in \mathcal{W}(X)$.*

Proof Let $V \subset X$ be a germ of an analytic nowhere dense subset. Since $\mu \in \mathcal{W}(X\backslash Y)$ we see that supp $\mathbf{1}_V\mu \subset Y \cap V$ and so $\mathbf{1}_V\mathbf{1}_{X\backslash Y}\mu = \mathbf{1}_{X\backslash Y}\mathbf{1}_V\mu = 0$. □

For future reference we give the following simple lemma, part (i) of which is almost tautological.

Lemma 2.2 *Let X be a germ of a reduced complex space and let $\mu \in \mathcal{W}(X)$.*

(i) *We have that $\bar\partial\mu \in \mathcal{W}(X)$ if and only if $\lim_{\epsilon\to 0}\bar\partial\chi(|h|^2/\epsilon)\wedge\mu = 0$ for all generically non-vanishing holomorphic tuples h on X.*
(ii) *Let $Y \subset X$ be an analytic nowhere dense subset, let h be a holomorphic tuple such that $Y = \{h = 0\}$, and assume that $\bar\partial\mu \in \mathcal{W}(X\backslash Y)$. Then $\bar\partial\mu \in \mathcal{W}(X)$ if and only if $\lim_{\epsilon\to 0}\bar\partial\chi(|h|^2/\epsilon)\wedge\mu = 0$.*

Proof Since $\mu \in \mathcal{W}(X)$ we have that $\mu = \lim_{\epsilon\to 0}\chi(|h|^2/\epsilon)\mu$ for any generically non-vanishing h. It follows that

$$\bar\partial\mu = \lim_{\epsilon\to 0}\bar\partial(\chi(|h|^2/\epsilon)\mu) = \lim_{\epsilon\to 0}\bar\partial\chi(|h|^2/\epsilon)\wedge\mu + \lim_{\epsilon\to 0}\chi(|h|^2/\epsilon)\bar\partial\mu. \quad (2.3)$$

Now, $\bar\partial\mu \in \mathcal{W}(X)$ if and only if the last term on the right-hand side equals $\bar\partial\mu$ for all generically non-vanishing h and part (i) of the lemma follows. The "only if" part of (ii) also follows directly from (2.3). On the other hand, if $\lim_{\epsilon\to 0}\bar\partial\chi(|h|^2/\epsilon)\wedge\mu = 0$ then, by (2.3), $\bar\partial\mu = \mathbf{1}_{X\backslash Y}\bar\partial\mu$ and so the "if" part of (ii) follows from Lemma 2.1. □

Recall that a current on X is said to be semi-meromorphic if it a principal value current of the form α/f, where α is a smooth form and f is a holomorphic function or section of a complex line bundle such that f does not vanish identically on any component of X. Following [8], see also [11], we say that a current a on X is *almost semi-meromorphic* if there is a modification $\pi: X' \to X$ and a semi-meromorphic current α/f on X' such that $a = \pi_*(\alpha/f)$; if f takes values in $L \to X'$ we need also α to take values in $L \to X'$ if we want a to be scalar valued. If a is almost semi-meromorphic on X then the smallest Zariski-closed set outside of which a is smooth has positive codimension and is denoted $ZSS(a)$, the *Zariski-singular support* of a, see [11].

For proofs of the statements in this paragraph we refer to [11, Sect. 3], see also [8, Sect. 2]. Let a be an almost semi-meromorphic current on X and let $\mu \in \mathcal{PM}(X)$. Then there is a unique pseudomeromorphic current T on X coinciding with $a \wedge \mu$ outside of $ZSS(a)$ and such that $\mathbf{1}_{ZSS(a)}T = 0$. If h is a holomorphic tuple, or section of a Hermitian vector bundle, such that $\{h = 0\} = ZSS(a)$, then $T = \lim_{\epsilon\to 0} \chi(|h|^2/\epsilon) a \wedge \mu$; henceforth we will write $a \wedge \mu$ in place of T. One defines $\bar{\partial} a \wedge \mu$ so that Leibniz' rule holds, i.e., $\bar{\partial} a \wedge \mu := \bar{\partial}(a \wedge \mu) - (-1)^{\deg a} a \wedge \bar{\partial}\mu$. If $\mu \in \mathcal{W}(X)$ then $a \wedge \mu \in \mathcal{W}(X)$; in this case $a \wedge \mu = \lim_{\epsilon\to 0} \chi(|h|^2/\epsilon) a \wedge \mu$ if h is any generically non-vanishing holomorphic section of a Hermitian vector bundle such that $\{h = 0\} \supset ZSS(a)$. If μ is almost semi-meromorphic then $a \wedge \mu$ is almost semi-meromorphic and, in fact, $a \wedge \mu = (-1)^{\deg a \deg \mu} \mu \wedge a$.

Let X be an analytic subset of pure codimension κ of some complex N-dimensional manifold D. The subsheaves of $\mathcal{PM}_D$ of germs of $\bar{\partial}$-closed (k, κ)-currents, $k = 0, \ldots, N$, with support on X are the sheaves of Coleff–Herrera currents with support on X and are denoted $\mathscr{CH}_X^k$. Coleff–Herrera currents were originally introduced by Björk as the $\bar{\partial}$-closed currents μ on D of bidegree (N, κ) such that $\bar{h}\mu = 0$ for any holomorphic function h vanishing on X and with the SEP with respect to X, see, e.g., [15]. It is proved in [4] that the definitions are equivalent. The model example is the Coleff–Herrera product: Assume that $f_1, \ldots, f_\kappa \in \mathscr{O}(D)$ defines a regular sequence. Then the iteratively defined product $\bar{\partial}(1/f_1) \wedge \cdots \wedge \bar{\partial}(1/f_\kappa)$ is the Coleff–Herrera product originally introduced by Coleff and Herrera in [17] in a slightly different way; cf. also [16].

Let us also notice that if X and Z are reduced pure-dimensional complex spaces and $\mu \in \mathcal{PM}(X)$, then $\mu \otimes 1 \in \mathcal{PM}(X \times Z)$, see, e.g., [8, Sect. 2]. We will usually omit "$\otimes 1$" and simply write, e.g., $\mu(\zeta)$ to denote which coordinates μ depends on.

2.3 Residue Currents Associated with Generically Exact Complexes

Let E_j, $j = 0, \ldots, M$, be trivial vector bundles over an open subset of $\mathbb{C}^N$, let $f_j : E_j \to E_{j-1}$ be holomorphic mappings, and assume that

$$0 \to E_M \xrightarrow{f_M} \cdots \xrightarrow{f_2} E_1 \xrightarrow{f_1} E_0 \xrightarrow{f_0} 0 \tag{2.4}$$

is a complex that is pointwise exact outside of an analytic subset V of positive codimension. The bundle $E := \oplus_j E_j$ gets a natural superstructure by setting $E^+ := \oplus_j E_{2j}$ and $E^- := \oplus_j E_{2j+1}$. Following [9] we define currents U and R with values in $\text{End}(E)$ associated with (2.6) and a choice of Hermitian metrics on the E_k.[2] Notice that $\text{End}(E)$ gets an induced superstructure and so spaces of forms and currents with values in E or $\text{End}(E)$ get superstructures as well. Let $f := \oplus_j f_j$ and set $\nabla := f - \bar{\partial}$, which then becomes an odd mapping on spaces of forms or currents with values in E such that $\nabla^2 = 0$; notice that ∇ induces an odd mapping ∇_{End} on $\text{End}(E)$-valued forms or currents such that $\nabla_{\text{End}}^2 = 0$. Outside of V, let $\sigma_k : E_{k-1} \to E_k$ be the pointwise minimal inverse of f_k, i.e., for each $z \notin V$,

$$\sigma_k(z) f_k(z) = \Pi_{(\text{Ker} f_k(z))^\perp}, \quad f_k(z)\sigma_k(z) = \Pi_{\text{Im} f_k(z)},$$

where Π denotes orthogonal projection. Let $\sigma := \sigma_1 + \sigma_2 + \cdots$; it is an odd element in $\text{End}(E)$ and $f\sigma + \sigma f = \text{Id}_E$. Let $u := \sigma + \sigma\bar{\partial}\sigma + \sigma(\bar{\partial}\sigma)^2 + \cdots$. Outside of V we have

[2] That a current takes values in a vector bundle F means that it acts on test-forms with values in F^*.

$fu + uf = \mathrm{Id}_E + \bar\partial u$, i.e., $\nabla_{\mathrm{End}} u = \mathrm{Id}_E$, see [9]. Notice that $u = \sum_{0<\ell} \sum_{0\le k<\ell} u_\ell^k$, where $u_\ell^k := \sigma_\ell \bar\partial \sigma_{\ell-1} \cdots \bar\partial \sigma_{k+1}$, is a smooth $\mathrm{Hom}(E_k, E_\ell)$-valued $(0, \ell-k-1)$-form outside of V. We extend u as a current across V by setting $U := \lim_{\epsilon\to 0} \chi(|F|^2/\epsilon)u$, where F is a (non-trivial) holomorphic tuple vanishing on V, cf., [9, Sect. 2] and [2, Theorem 5.1]. As with u we will write $U = \sum_{0<\ell} \sum_{0\le k<\ell} U_\ell^k$, where now U_ℓ^k is a $\mathrm{Hom}(E_k, E_\ell)$-valued $(0, \ell-k-1)$-current.

Remark 2.3 The procedure of taking pointwise minimal inverses produce almost semi-meromorphic currents, see, e.g., [11, Sect. 4]. Thus the σ_j have almost semi-meromorphic extensions across V and, letting σ_j denote the extension as well, we have $U_\ell^k := \sigma_\ell \bar\partial \sigma_{\ell-1} \cdots \bar\partial \sigma_{k+1}$, where the products are in the sense of Sect. 2.2 above. In particular, each U_ℓ^k is an almost semi-meromorphic current in (some domain in) $\mathbb{C}^N$.

The current R is defined by the equation $\nabla_{\mathrm{End}} U = \mathrm{Id}_E - R$, and hence R is supported on V. Since $\nabla_{\mathrm{End}}^2 = 0$, we have $\nabla_{\mathrm{End}} R = 0$. Notice that R is an almost semi-meromorphic current plus $\bar\partial$ of such a current. One can check that

$$R = \lim_{\epsilon\to 0} \big(1 - \chi(|F|^2/\epsilon)\big)\mathrm{Id}_E + \bar\partial\chi(|F|^2/\epsilon) \wedge u. \tag{2.5}$$

We write $R = \sum_{0<\ell} \sum_{0\le k<\ell} R_\ell^k$, where R_ℓ^k is a $\mathrm{Hom}(E_k, E_\ell)$-valued $(0, \ell-k)$-current.

Now consider the sheaf complex

$$0 \to \mathscr{O}(E_M) \xrightarrow{f_M} \cdots \xrightarrow{f_2} \mathscr{O}(E_1) \xrightarrow{f_1} \mathscr{O}(E_0) \tag{2.6}$$

associated with (2.4). Assume that (2.6) is exact so that it provides a free resolution of the sheaf $\mathscr{F} = \mathscr{O}(E_0)/\mathscr{I\!m}\, f_1$. Recall that any coherent sheaf is of this form and has a free resolution locally. By definition, $\mathscr{F}$ has (co)dimension r if the associated primes of each stalk $\mathscr{F}_x$ all have (co)dimension $\le r$ ($\ge r$); $\mathscr{F}$ has pure (co)dimension if all associated primes are equidimensional. Let Z_k be the set where f_k does not have optimal rank; it is well known that the Z_k are analytic and independent of the choice of free resolution, thus invariants of $\mathscr{F}$. Let $\kappa = \operatorname{codim} \mathscr{F}$. By, e.g., [19, Corollary 20.12],

$$\cdots \subset Z_k \subset Z_{k-1} \subset \cdots \subset Z_{\kappa+1} \subsetneq Z_\kappa = \cdots = Z_1.$$

Moreover, by [19, Corollary 20.14], $\operatorname{codim} Z_k \ge k+1$ for $k \ge \kappa+1$ if and only if $\mathscr{F}$ has pure codimension κ. We recall also that $\mathscr{F}$ is Cohen–Macaulay if and only if $Z_k = \emptyset$ for $k \ge \kappa+1$, i.e., if and only if there is a resolution (2.6) of $\mathscr{F}$ with $M = \kappa$.

By definition, see [41, §1], the singularity subvarieties $S_\ell(\mathscr{G})$ of $\mathscr{G} := \mathscr{F}\restriction_{Z_1}$ are the set of points $x \in Z_1$ such that $\mathrm{depth}_{\mathscr{O}_{Z_1,x}}(\mathscr{G}_x) \le \ell$. It is straightforward to check that Z_k is the set of points $x \in \mathbb{C}^N$ such that the projective dimension of $\mathscr{F}_x$ is $\ge k$ and so, from the Auslander–Buchsbaum formula, it follows that $S_{N-\ell}(\mathscr{G}) = Z_\ell$.

It is proved in [9] that if (2.6) is exact and R is the associated current, then $R = \sum_{\ell \geq \kappa} R_\ell^0$. Moreover, a section φ of $\mathscr{O}(E_0)$ is in $\mathscr{I}m\, f_1$ if and only if (the E-valued) current $R\varphi$ vanishes. Thus, if (2.6) is exact, $Rf = Rf_1 = 0$ and hence

$$0 = \nabla_{\mathrm{End}} R = fR - Rf - \bar{\partial} R = fR - \bar{\partial} R. \tag{2.7}$$

In what follows we will only be concerned with currents associated to exact complexes (2.6). We will therefore write $R_\ell := R_\ell^0$.

Example 2.4 The model example is the Koszul complex: Let $f_1, \dots, f_\kappa \in \mathscr{O}(D)$ (D a domain in $\mathbb{C}^N$) be a regular sequence and let (2.6) with $M = \kappa$ be the associated Koszul complex, which then is a resolution of $\mathscr{O}/\langle f_1, \dots, f_\kappa \rangle$. With the trivial metric on the bundles E_j the resulting R is $R_{BM} \wedge e_\kappa \wedge e_0^*$, where R_{BM} is the residue current of Bochner–Martinelli type introduced in [34] and e_0 and e_κ are suitable frames for the line bundles E_0 and E_κ respectively. It is shown in [34], see also [4], that R_{BM} equals the Coleff–Herrera product in the present situation. By [9, Theorem 4.1], R is in fact independent of the choice of Hermitian metric and so the above procedure always produce the Coleff–Herrera product (times $e_\kappa \wedge e_0^*$) in the case of regular sequences.

3 The Sheaves $\widehat{\Omega}_X^p$ and Associated Residue Currents

Let $X = \{h_1 = \cdots = h_r = 0\}$ be a pure n-dimensional analytic subset of an open set $D \subset \mathbb{C}^N$ and set $\kappa := N - n$; assume that $0 \in X$. Let $\tilde{\mathscr{J}}_X^p \subset \Omega_D^p$ be the subsheaf generated over $\mathscr{O}_D$ by $h_i \mathrm{d}z^I$ and $\mathrm{d}h_j \wedge \mathrm{d}z^J$, $1 \leq i, j \leq r$, $|I| = p$, $|J| = p - 1$. By definition, $\Omega_X^p := \Omega_D^p / \tilde{\mathscr{J}}_X^p$ is the sheaf of germs of Kähler–Grothendieck differential p-forms on X. It is clear that it is a coherent analytic sheaf of codimension κ in D and that it coincides with the standard sheaf of holomorphic p-forms on X_{reg}. In general, Ω_X^p has non-trivial torsion; recall that a torsion element of Ω_X^p is represented by a form φ in ambient space such that $\varphi \in \tilde{\mathscr{J}}_X^p$ generically on each irreducible component of X, i.e., the pullback of φ to X_{reg} vanishes.

Example 3.1 Let $X = \{z_1^2 = z_2^3\} \subset \mathbb{C}^2$ and $\varphi = 2z_2 \mathrm{d}z_1 - 3z_1 \mathrm{d}z_2$. Then φ defines a torsion element in Ω_X^1. In fact, it is straightforward to check that φ is not in $\tilde{\mathscr{J}}_X^1$, and using the parametrization $t \mapsto (t^3, t^2)$ of X it is immediate that φ vanishes on X_{reg}.

We set $\widehat{\Omega}_X^p := \Omega_X^p / torsion$ and we call the sections of $\widehat{\Omega}_X^p$ *strongly holomorphic p-forms*. Let $\mathscr{J}_X^p \subset \Omega_D^p$ be the subsheaf of forms whose pullback to X_{reg} vanishes. Since $\tilde{\mathscr{J}}_X^p \subset \mathscr{J}_X^p$ there is a natural surjective map $\Omega_X^p \to \Omega_D^p / \mathscr{J}_X^p$ with kernel $\mathscr{J}_X^p / \tilde{\mathscr{J}}_X^p$, which consists of the torsion elements of Ω_X^p. Hence, $\widehat{\Omega}_X^p = \Omega_D^p / \mathscr{J}_X^p$. Notice that the sections of $\widehat{\Omega}_X^p$ define $\bar{\partial}$-closed currents on X with the SEP. Notice also that $\widehat{\Omega}_X^p$ has pure codimension κ. In fact, for any $\varphi \in \widehat{\Omega}_X^p$, $\mathrm{ann}(\varphi)$ is the ideal corresponding to those irreducible components of X where φ is generically non-vanishing. The associated primes of $\widehat{\Omega}_X^p$ are thus the ideals of the irreducible components of X. We remark that strongly holomorphic forms have been studied by several authors, e.g., in [20] and [24].

For simplicity we will for the rest of this section assume that D is a neighborhood of the closure of the unit ball $\mathbb{B}$ of $\mathbb{C}^N$ and we denote the inclusion $X \hookrightarrow \mathbb{B}$ by i. Moreover, we let (2.6) be a resolution of $\widehat{\Omega}_X^p = \Omega_{\mathbb{B}}^p/\mathcal{J}_X^p$ with $E_0 = T^*_{p,0}\mathbb{B}$ so that $\mathscr{O}(E_0) = \Omega^p_{\mathbb{B}}$; recall also the associated sets Z_k, cf. Sect. 2.3. Since $\widehat{\Omega}_X^p$ has pure codimension κ we have codim $Z_k \geq k+1$, for $k = \kappa+1, \kappa+2, \ldots$, and in particular $Z_N = \emptyset$. Hence, we can, and will, assume that $M \leq N-1$ in (2.6). The resolution (2.6) induces a complex (2.4) that is pointwise exact outside of X. A choice of Hermitian metrics on the E_j gives us associated $\mathrm{Hom}(E_0, E)$-valued currents U and R so that, in particular, a holomorphic p-form φ is a section of $\mathcal{J}_X^p$ if and only if the E-valued current $R\varphi$ vanishes.

Notice that $R_k \wedge \mathrm{d}z$, where $\mathrm{d}z = \mathrm{d}z_1, \ldots, \mathrm{d}z_N$, is a $\mathrm{Hom}(E_0, E_k)$-valued $(0, k)$-current, i.e., a distribution-valued section of $E_k \otimes E_0^* \otimes T^*_{N,k}\mathbb{B}$. Recalling that $E_0 = T^*_{p,0}\mathbb{B}$, interior multiplication induces a natural isomorphism $E_0^* \otimes T^*_{N,k}\mathbb{B} \to T^*_{N-p,k}\mathbb{B}$. Hence we can view $R_k \wedge \mathrm{d}z$ as a distribution-valued section of $E_k \otimes T^*_{N-p,k}\mathbb{B}$, i.e., as an E_k-valued $(N-p, k)$-current. Unless explicitly said, we will use the second point of view (even though the notation might suggest otherwise).

To explain the two view-points of $R_k \wedge \mathrm{d}z$ in some more detail, let φ be an $E_k^* \otimes E_0$-valued test form of bidegree $(0, N-k)$. Then, since $E_0 = T^*_{p,0}\mathbb{B}$, φ can as well be seen as an E_k^*-valued test form $\tilde{\varphi}$ of bidegree $(p, N-k)$. Consider the diagram

$$\begin{array}{ccc} E_k \otimes E_0^* \otimes T^*_{N,k}\mathbb{B} & \xrightarrow{\varphi} & T^*_{N,N}\mathbb{B} \\ \downarrow & & || \\ E_k \otimes T^*_{N-p,k}\mathbb{B} & \xrightarrow{\wedge\tilde{\varphi}} & T^*_{N,N}\mathbb{B} \end{array}$$

where φ also denotes the natural map induced by φ, and the map $\wedge\tilde{\varphi}$ is defined by taking wedge product with $\tilde{\varphi}$. One checks that the diagram commutes. With the original view-point, $R_k \wedge \mathrm{d}z$ acts on φ; with the second view-point, $R_k \wedge \mathrm{d}z$ acts on $\tilde{\varphi}$.

For future reference we also note that with the first point of view $R \wedge \mathrm{d}z$ can be naturally multiplied by smooth E_0-valued $(0, *)$-forms yielding E-valued currents; with the second point of view $R \wedge \mathrm{d}z$ can be naturally multiplied by scalar-valued $(p, *)$-forms yielding the same E-valued currents.

Example 3.2 Assume that $X = \{w_1 = \cdots = w_\kappa = 0\}$, where $(z_1, \ldots, z_n; w_1, \ldots, w_\kappa)$ are local coordinates in an open subset $\mathcal{U}$ of $\mathbb{C}^N$. A basis for the $(p, 0)$-forms in $\mathcal{U}$ is given by the union of $\{\mathrm{d}z_I \wedge \mathrm{d}w_J\}$, where I and J range over increasing multiindices such that $|I| + |J| = p$. Let E_0' and E_0'' be the subbundles of $T^*_{p,0}\mathcal{U}$ generated by $\mathrm{d}z_I$, $|I| = p$, and $\mathrm{d}z_J \wedge \mathrm{d}w_K$, $|J| < p$, respectively. It is clear that $\mathcal{J}_X^p$ is generated by $w_i \mathrm{d}z_J$, $i = 1, \ldots, \kappa$, $|J| = p$ and $\mathrm{d}z_I \wedge \mathrm{d}w_J$, $|J| \geq 1$. To get a resolution of $\widehat{\Omega}_X^p$ we let, for each increasing multiindex $J \subset \{1, \ldots, n\}$ with $|J| = p$, $(E_\bullet^J, f_\bullet^J)$ be the Koszul complex corresponding to $w_1, \ldots, w_\kappa$, where E_0^J is (identified with) the line bundle generated by $\mathrm{d}z_J$; notice that $\oplus_{|J|=p} E_0^J = E_0'$. It is well known that $(\mathscr{O}(E_\bullet^J), f_\bullet^J)$ is a resolution of the quotient $\mathscr{O}\mathrm{d}z_J/\langle w_1, \ldots, w_\kappa\rangle\mathscr{O}\mathrm{d}z_J$. Let $(E_\bullet', f_\bullet')$ be the direct sum of the complexes $(E_\bullet^J, f_\bullet^J)$ over all increasing multiindices J with $|J| = p$. Then

$$0 \to \mathscr{O}(E'_\kappa) \xrightarrow{f'_\kappa} \cdots \xrightarrow{f'_3} \mathscr{O}(E'_2) \xrightarrow{f'_2} \mathscr{O}(E'_1) \oplus \mathscr{O}(E''_0) \xrightarrow{f'_1 \oplus \mathrm{Id}} \mathscr{O}(E'_0) \oplus \mathscr{O}(E''_0) \tag{3.1}$$

is a resolution of $\widehat{\Omega}^p_X$ since (3.1) is exact (as a direct sum of exact complexes) and the cokernel of the map $f'_1 \oplus \mathrm{Id}$ equals $\widehat{\Omega}^p_X$.

Since $w_1, \ldots, w_\kappa$ is a regular sequence it follows that, for any choice of Hermitian metrics on the E^J_i, the current R^J associated with $\left(E^J_\bullet, f^J_\bullet\right)$ equals

$$R^J = \varepsilon^J \otimes (\mathrm{d}z_J)^* \otimes \bar{\partial}\frac{1}{w_1} \wedge \cdots \wedge \bar{\partial}\frac{1}{w_\kappa},$$

where ε^J is a frame for E^J_κ, $(\mathrm{d}z_J)^*$ is the dual of $\mathrm{d}z_I$, and $\bar{\partial}(1/w_1) \wedge \cdots \wedge \bar{\partial}(1/w_\kappa)$ is the Coleff–Herrera product, cf. Example 2.4. Choosing a metric that respects the direct sum structure we get that $R = \sum'_{|J|=p} R^J$ is the current associated with (3.1).

The two view-points mentioned before this example are illustrated by

$$\begin{aligned} R \wedge \mathrm{d}z \wedge \mathrm{d}w &= \sum_{|J|=p}{}' \varepsilon^J \otimes (\mathrm{d}z_J)^* \otimes \bar{\partial}\frac{1}{w_1} \wedge \cdots \wedge \bar{\partial}\frac{1}{w_\kappa} \wedge \mathrm{d}z \wedge \mathrm{d}w \\ &= \sum_{|J|=p}{}' \varepsilon^J \otimes \bar{\partial}\frac{1}{w_1} \wedge \cdots \wedge \bar{\partial}\frac{1}{w_\kappa} \wedge \mathrm{d}z_{J^c} \wedge \mathrm{d}w, \end{aligned} \tag{3.2}$$

where $J^c = \{1, \ldots, n\} \backslash J$.

4 Barlet's Sheaf $\omega^\bullet_X$ and Structure Forms on X

The sheaf $\omega^\bullet_X$ was introduced by Barlet in [13] as the kernel of a natural map $j_* j^* \Omega^\bullet_X \to \mathscr{H}^1_{X_{sing}}\big(\mathscr{E}xt^\kappa_{\mathscr{O}}(\mathscr{O}_X, \Omega^{\kappa+\bullet})\big)$, where $j\colon X_{reg} \hookrightarrow X$ is the inclusion. It is proved, [13, Proposition 4], that the sections of ω^p_X can be identified with the holomorphic p-forms on X_{reg} that have an extension to X as a $\bar{\partial}$-closed current with the SEP. Moreover, it is shown that ω^p_X is coherent. Hence, $\omega^p_X/\widehat{\Omega}^p_X$ is a coherent sheaf supported on X_{sing}. It follows that locally, for a suitable generically non-vanishing holomorphic function h, one has $h\omega^p_X \subset \widehat{\Omega}^p_X$. Therefore ω^p_X can be identified with the sheaf of germs of meromorphic p-forms on X that are $\bar{\partial}$-closed considered as principal value currents; we will use this as the definition of ω^p_X. This analytic point of view was emphasized and explored by Henkin and Passare [24], and therefore we sometimes call sections of ω^p_X Barlet–Henkin–Passare holomorphic p-forms.

From Barlet's definition, since $j_* j^* \Omega^\bullet_X$ is torsion free (and from the one we use as well), it is clear that $\omega^\bullet_X$ is torsion free. Moreover, from [13, p. 195] it follows that if $\mathrm{codim}_X X_{sing} \geq 2$, then any holomorphic form on X_{reg} extends (necessarily uniquely) to a section of $\omega^\bullet_X$ over X. Thus, by [23, Proposition 1.6], if X is normal then $\omega^\bullet_X$ is reflexive. On a normal space the reflexive hull of any reasonable sheaf of holomorphic forms therefore coincides with $\omega^\bullet_X$.

Let $i: X \hookrightarrow D$ be a pure n-dimensional analytic subset of a neighborhood D of $\overline{\mathbb{B}} \subset \mathbb{C}^N$, $\kappa = N - n$. As in the previous section, let (2.6) be a resolution of $\widehat{\Omega}_X^p = \Omega_{\mathbb{B}}^p / \mathcal{J}_X^p$ with $\mathscr{O}(E_0) = \Omega_{\mathbb{B}}^p$, recall the associated sets Z_k, and let $R = R_\kappa + \cdots$ be the associated current (for some choice of Hermitian metrics). Recall that $R \wedge \mathrm{d}z$ is (considered as) an $(N - p, *)$-current with values in E_*, cf. the paragraph before Example 3.2. The following proposition is the analogue of [8, Proposition 3.3] and the proof is essentially the same, we therefore omit it.

Proposition 4.1 *There is a unique almost semi-meromorphic current $\omega = \omega_0 + \omega_1 + \cdots + \omega_{n-1}$ on X, where ω_k is an $E_{\kappa+k}$-valued $(n - p, k)$-current, such that*

$$R \wedge \mathrm{d}z = i_* \omega.$$

Moreover, cf. (2.7),

$$f_{\restriction X} \omega = \bar{\partial} \omega.$$

The current ω has the following additional properties:

(i) *If $\widehat{\Omega}_X^p$ is Cohen–Macaulay, then ω_0 is an E_κ-valued section of ω_X^{n-p} over X. In general, there is a vector $\tilde{\omega}_0 = (\tilde{\omega}_{01}, \ldots, \tilde{\omega}_{0\nu})$ of sections of ω_X^{n-p} over X and a vector $\alpha_0 = (\alpha_{01}, \ldots, \alpha_{0\nu})$ of almost semi-meromorphic E_κ-valued* $(0, 0)$-*current in $\mathbb{B}$, smooth outside of $Z_{\kappa+1}$, such that $\omega_0 = \alpha_{0\restriction X} \cdot {}^t\tilde{\omega}_0$ as currents on X.*

(ii) *For $k \geq 1$ there are almost semi-meromorphic* $(0, 1)$-*currents α_k in $\mathbb{B}$ with values in $Hom(E_{\kappa+k-1}, E_{\kappa+k})$ that are smooth outside of $Z_{\kappa+k}$ and such that $\omega_k = \alpha_{k\restriction X} \omega_{k-1}$ as currents.*

The form ω will be called an $(n - p)$-*structure form.*

Since $R \wedge \mathrm{d}z = i_* \omega$, where ω is almost semi-meromorphic on X, it follows that R has the SEP with respect to X.

Example 4.2 (Example 3.2 continued) We use the notation of Example 3.2 and we set $\mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and $\mathrm{d}w = \mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_\kappa$. From (3.2) and the Poincaré–Lelong formula we get

$$\begin{aligned} R \wedge \mathrm{d}w \wedge \mathrm{d}z &= \sideset{}{'}\sum_{|J|=p} \varepsilon^J \otimes \bar{\partial} \frac{\mathrm{d}w_1}{w_1} \wedge \cdots \wedge \bar{\partial} \frac{\mathrm{d}w_\kappa}{w_\kappa} \wedge \mathrm{d}z_{J^c} \\ &= \pm (2\pi i)^\kappa \sideset{}{'}\sum_{|J|=p} \varepsilon^J \otimes [X] \wedge \mathrm{d}z_{J^c}. \end{aligned}$$

The $(n - p)$-structure form thus is $\pm (2\pi i)^\kappa \sum'_{|J|=p} \varepsilon^J \otimes \mathrm{d}z_{J^c}$ in this case.

Using our $(n - p)$-structure form ω we now give various descriptions of $\omega_X^\bullet$. Dualizing our resolution (2.6) of $\widehat{\Omega}_X^p$ and then tensoring by $\Omega_{\mathbb{B}}^N$ we get the sheaf complex $\big(\mathscr{O}(E_\bullet^*)\otimes\Omega_{\mathbb{B}}^N,\ f_\bullet^*\otimes\mathrm{Id}\big)$ with associated cohomology sheaves $\mathscr{H}^\ell\big(\mathscr{O}(E_\bullet^*)\otimes\Omega_{\mathbb{B}}^N\big)$; it is well known that $\mathscr{H}^\ell\big(\mathscr{O}(E_\bullet^*)\otimes\Omega_{\mathbb{B}}^N\big)\simeq\mathscr{E}xt^\ell_{\mathscr{O}_{\mathbb{B}}}\big(\widehat{\Omega}_X^p,\Omega_{\mathbb{B}}^N\big)$. Let $\xi\in\mathscr{O}(E_\kappa^*)$ be such that $f_{\kappa+1}^*\xi=0$. Then, in view of (2.7),

$$\bar\partial(\xi\cdot i_*\omega_0)=\xi\cdot\bar\partial R_\kappa\wedge dz=\xi\cdot f_{\kappa+1}R_{\kappa+1}\wedge dz=f_{\kappa+1}^*\xi\cdot R_{\kappa+1}\wedge dz=0.$$

It follows that the current $i^*\xi\cdot\omega_0$ is $\bar\partial$-closed on X. Hence, $i^*\xi\cdot\omega_0$ is a section of ω_X^{n-p}. If $\xi=f_\kappa^*\xi'$ one checks in a similar way that $i^*\xi\cdot\omega_0=0$ and we see that we have a mapping,

$$\mathscr{H}^\kappa\big(\mathscr{O}(E_\bullet^*)\otimes\Omega_{\mathbb{B}}^N\big)\to\omega_X^{n-p},\quad[\xi]\otimes dz\mapsto i^*\xi\cdot\omega_0. \tag{4.1}$$

Proposition 4.3 *The mapping (4.1) is an isomorphism and it induces a natural isomorphism* $\omega_X^{n-p}\simeq\mathscr{E}xt^\kappa_{\mathscr{O}_{\mathbb{B}}}\big(\widehat{\Omega}_X^p,\Omega_{\mathbb{B}}^N\big)$.

Proof Let φ be a section of ω_X^{n-p}. Then $i_*\varphi$ is a $\bar\partial$-closed $(N-p,\kappa)$-current in $\mathbb{B}$ and it induces a map $\Omega_{\mathbb{B}}^p\to\mathscr{CH}_X^N$ by

$$\psi\mapsto i_*\varphi\wedge\psi, \tag{4.2}$$

whose kernel clearly contains $\mathcal{J}_X^p$. Hence, (4.2) induces a map $\Omega_{\mathbb{B}}^p/\mathcal{J}_X^p\to\mathscr{CH}_X^N$. Thus, we get a map $\omega_X^{n-p}\to\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\widehat{\Omega}_X^p,\mathscr{CH}_X^N)$, which one easily checks is injective. In view of (4.1) we get a commutative diagram

$$\begin{array}{ccc}\mathscr{H}^\kappa\big(\mathscr{O}(E_\bullet^*)\otimes\Omega_{\mathbb{B}}^N\big) & \longrightarrow & \omega_X^{n-p}\\ & \searrow & \downarrow\\ & & \mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}\big(\widehat{\Omega}_X^p,\mathscr{CH}_X^N\big),\end{array} \tag{4.3}$$

where the diagonal map is the composition, i.e., the map given by $[\xi]\otimes dz\mapsto\xi\cdot R_\kappa\wedge dz$, where we here temporarily view $R_\kappa\wedge dz$ as a $\mathrm{Hom}(E_0,E_\kappa)$-valued (N,κ)-current; cf. the paragraphs preceding Example 3.2. By [6, Theorem 1.5] this map is an isomorphism and since the vertical map is injective it follows that both the horizontal map and the vertical map are isomorphisms. From *ibid*. we also know that the diagonal map is independent of the choices of Hermitian resolution of $\widehat{\Omega}_X^p$ and of dz. □

Notice that the isomorphism $\omega_X^{n-p}\simeq\mathscr{E}xt^\kappa_{\mathscr{O}_{\mathbb{B}}}\big(\widehat{\Omega}_X^p,\Omega_{\mathbb{B}}^N\big)$ of Proposition 4.3 is explicitly realized by our $(n-p)$-structure form ω. An elegant algebraic proof of the isomorphism was recently found by Barlet. He has communicated his proof to us and generously let us include it here.

Alternative proof of Proposition 4.3 Consider the natural map $\Omega_X^p\to\widehat{\Omega}_X^p$. Denote the kernel by $\mathscr{T}$ and notice that it has codimension $>\kappa$; it is the torsion submodule of Ω_X^p, cf. Sect. 3. It follows that $\mathscr{E}xt^k_{\mathscr{O}_{\mathbb{B}}}(\mathscr{T},\Omega_{\mathbb{B}}^N)=0$ for $k\le\kappa$. Applying the functor

$\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(-, \Omega^N_{\mathbb{B}})$ to the exact sequence $0 \to \mathscr{T} \to \Omega^p_X \to \widehat{\Omega}^p_X \to 0$ we get a long exact sequence of $\mathscr{E}xt$-sheaves. From this, and the vanishing of $\mathscr{E}xt^k(\mathscr{T}, \Omega^N_{\mathbb{B}})$ for $k \leq \kappa$, it follows that $\mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}\big(\widehat{\Omega}^p_X, \Omega^N_{\mathbb{B}}\big) \simeq \mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}\big(\Omega^p_X, \Omega^N_{\mathbb{B}}\big)$.

Let $\mathscr{G} := (d\mathcal{J}^0_X \wedge \Omega^{p-1}_{\mathbb{B}}) \cap (\mathcal{J}^0_X \Omega^p_{\mathbb{B}})$, let $\mathscr{F} := d\mathcal{J}^0_X \wedge \Omega^{p-1}_{\mathbb{B}}/\mathscr{G}$, and notice that $\mathscr{F}$ and $\mathscr{G}$ are $\mathscr{O}_X$-modules; $\mathcal{J}^0_X \subset \mathscr{O}_{\mathbb{B}}$ is the ideal defining X, cf. Sect. 3. We have a natural short exact sequence of $\mathscr{O}_X$-modules in $\mathbb{B}$

$$0 \to \mathscr{F} \longrightarrow \mathscr{O}_X \otimes \Omega^p_{\mathbb{B}} \longrightarrow \Omega^p_X \to 0.$$

Applying $\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(-, \Omega^N_{\mathbb{B}})$ we again obtain a long exact sequence of $\mathscr{E}xt$-sheaves. Since $\operatorname{codim} X = \kappa$ these sheaves vanish until level κ and in particular one gets the exact sequence

$$0 \to \mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\Omega^p_X, \Omega^N_{\mathbb{B}}) \longrightarrow \mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X \otimes \Omega^p_{\mathbb{B}}, \Omega^N_{\mathbb{B}}) \xrightarrow{\ b\ } \mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{F}, \Omega^N_{\mathbb{B}}).$$

Since $\Omega^p_{\mathbb{B}}$ is a free $\mathscr{O}_{\mathbb{B}}$-module and since $\mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X, \Omega^N_{\mathbb{B}}) \simeq i_*\omega^n_X$ by [13, Lemma 4], one has

$$\begin{aligned} &\mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X \otimes \Omega^p_{\mathbb{B}}, \Omega^N_{\mathbb{B}}) \\ &\quad \simeq \mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\Omega^p_{\mathbb{B}}, \mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X, \Omega^N_{\mathbb{B}})) \\ &\quad \simeq \mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\Omega^p_{\mathbb{B}}, i_*\omega^n_X). \end{aligned}$$

Since $\omega^{n-p}_X \simeq \mathscr{H}om_{\mathscr{O}_X}(\Omega^p_X, \omega^n_X)$ by [13, Proposition 3], we will be done if we can show that the kernel of the map b above consists of those homomorphisms $\Omega^p_{\mathbb{B}} \to i_*\omega^n_X$ which in fact are homomorphisms $\Omega^p_X \to \omega^n_X$; since $\mathcal{J}^0_X i_*\omega^n_X = 0$, a homomorphism $\Omega^p_{\mathbb{B}} \to i_*\omega^n_X$ is a homomorphism $\Omega^p_X \to \omega^n_X$ if and only if it vanishes on $d\mathcal{J}^0_X \wedge \Omega^{p-1}_{\mathbb{B}}$. To understand the map b one can for instance use that $(\mathscr{C}^{N,\bullet}, \bar{\partial})$, where $\mathscr{C}^{N,\bullet}$ is the sheaf of germs of $(N, \bullet)$-currents in $\mathbb{B}$, is a resolution of $\Omega^N_{\mathbb{B}}$ by stalk-wise injective sheaves. In fact, then

$$\mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X \otimes \Omega^p_{\mathbb{B}}, \Omega^N_{\mathbb{B}}) \simeq \mathscr{H}^{\kappa}\Big(\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\Omega^p_{\mathbb{B}}, \mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X, \mathscr{C}^{N,\bullet})), \bar{\partial}\Big)$$

and, since $\mathscr{F} = \mathscr{O}_X \otimes \mathscr{F}$,

$$\mathscr{E}xt^{\kappa}_{\mathscr{O}_{\mathbb{B}}}(\mathscr{F}, \Omega^N_{\mathbb{B}}) \simeq \mathscr{H}^{\kappa}\Big(\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\mathscr{F}, \mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\mathscr{O}_X, \mathscr{C}^{N,\bullet})), \bar{\partial}\Big)$$

and the map b is induced by restricting homomorphisms defined on $\Omega^p_{\mathbb{B}}$ to the subsheaf $d\mathcal{J}^0_X \wedge \Omega^{p-1}_{\mathbb{B}}$. □

It follows from Proposition 4.3 that $\omega^{\bullet}_X$ is coherent, which, as mentioned above, also is proved in [13]. In addition to Proposition 4.3 we have the following descriptions of ω^{n-p}_X; the second one is [13, Lemma 4].

Proposition 4.4 *We have*

(i) $i_*\omega_X^{n-p} = \{\mu \in \mathscr{CH}_X^{N-p};\ \mathcal{J}_X^p \wedge \mu = 0\}$,
(ii) $i_*\omega_X^{n-p} = \{\mu \in \mathscr{CH}_X^{N-p};\ \mathcal{J}_X^0\mu = 0, \mathrm{d}\mathcal{J}_X^0 \wedge \mu = 0\}$,
(iii) *the map* $\omega_X^{n-p} \to \mathscr{H}om_{\mathscr{O}_X}(\widehat{\Omega}_X^p, \omega_X^n)$, $\mu \mapsto (\varphi \mapsto \mu \wedge \varphi)$*, is an isomorphism.*

Proof Part (i) follows since the vertical map in (4.3) is an isomorphism. As mentioned, part (ii) is [13, Lemma 4] (expressed in our terminology, cf. [6, Theorem 1.5]). To show part (iii), first notice that the map clearly is injective. To see surjectivity, let λ be a homomorphism $\widehat{\Omega}_X^p \to \omega_X^n$. Then $i_* \circ \lambda$ is a homomorphism $\widehat{\Omega}_X^p \to \mathscr{CH}_X^N$. Since the vertical map in (4.3) is an isomorphism there is a $\mu \in \omega_X^{n-p}$ such that $i_* \circ \lambda(\varphi) = i_*(\mu \wedge \varphi)$ and thus the map in (iii) is surjective. □

We remark that one may replace $\widehat{\Omega}_X^p$ in part (iii) by Ω_X^p, cf. the proof above; then we recover [13, Proposition 3]. We remark also that [13, Proposition 3] implies that ω_X^{n-p} coincides with the differential $n-p$-forms considered by Kersken in [27]; Proposition 4.3 is [27, Korollar 6.2 (2)].

We conclude this section with the following lemma.

Lemma 4.5 *If φ is a smooth $(n-p, q)$-form on X, then there is a smooth $(0, q)$-form ϕ on X with values in $E_\kappa^* \restriction_X$ such that $\varphi = \omega_0 \wedge \phi$.*

Proof Consider a smooth extension of φ to $\mathbb{B}$ and write the extension on the form $\sum_j \varphi_j' \wedge \varphi_j''$, where φ_j' is a holomorphic $n-p$-form in $\mathbb{B}$ and φ_j'' is a smooth $(0, q)$-form in $\mathbb{B}$. The $(N-p, \kappa)$-current $\varphi_j' \wedge [X]$ defines a section of $\mathscr{H}om_{\mathscr{O}_{\mathbb{B}}}(\widehat{\Omega}_X^p, \mathscr{CH}_X^N)$ by $\widehat{\Omega}_X^p \ni \psi \mapsto \psi \wedge \varphi_j' \wedge [X]$. From the proof of Proposition 4.3 it follows that there is a section ξ_j of $\mathscr{O}(E_\kappa^*)$ such that $i_*(i^*\xi_j \cdot \omega_0) = \varphi_j' \wedge [X]$. It follows that $i^*\varphi_j' = i^*\xi_j \cdot \omega_0$ and so $\varphi = \sum_j i^*\varphi_j' \wedge i^*\varphi_j'' = \sum_j i^*\xi_j \cdot \omega_0 \wedge i^*\varphi_j'' = \omega_0 \wedge i^* \sum_j \xi_j \varphi_j''$. □

5 Integral Operators on an Analytic Subset

Let $D \subset \mathbb{C}^N$ be a domain (not necessarily pseudoconvex at this point), let $k(\zeta, z)$ be an integrable $(N, N-1)$-form in $D \times D$, and let $p(\zeta, z)$ be a smooth (N, N)-form in $D \times D$. Assume that k and p satisfy the equation of currents

$$\bar\partial k(\zeta, z) = [\Delta^D] - p(\zeta, z) \tag{5.1}$$

in $D \times D$, where $[\Delta^D]$ is the current of integration along the diagonal. Applying (5.1) to test forms $\psi(z) \wedge \varphi(\zeta)$ it is straightforward to verify that for any compactly supported (p, q)-form φ in D one has the following Koppelman formula:

$$\varphi(z) = \bar\partial_z \int_{D_\zeta} k(\zeta, z) \wedge \varphi(\zeta) + \int_{D_\zeta} k(\zeta, z) \wedge \bar\partial\varphi(\zeta) + \int_{D_\zeta} p(\zeta, z) \wedge \varphi(\zeta).$$

In [1] Andersson introduced a very flexible method of producing solutions to (5.1). Let $\eta = (\eta_1, \ldots, \eta_N)$ be a holomorphic tuple in $D \times D$ that defines the diagonal and let

Λ_η be the exterior algebra spanned by $T^*_{0,1}(D\times D)$ and the $(1,0)$-forms $d\eta_1,\dots,d\eta_N$. On forms with values in Λ_η interior multiplication with $2\pi i\sum \eta_j\partial/\partial\eta_j$, denoted δ_η, is defined; set $\nabla_\eta=\delta_\eta-\bar\partial$.

Let s be a smooth $(1,0)$-form in Λ_η such that $|s|\lesssim|\eta|$ and $|\eta|^2\lesssim|\delta_\eta s|$ and let $B=\sum_{k=1}^N s\wedge(\bar\partial s)^{k-1}/(\delta_\eta s)^k$. It is proved in [1] that then $\nabla_\eta B=1-[\Delta^D]$. Identifying terms of top degree we see that $\bar\partial B_{N,N-1}=[\Delta^D]$ and we have found a solution to (5.1). For instance, if we take $s=\partial|\zeta-z|^2$ and $\eta=\zeta-z$, then the resulting B is sometimes called the full Bochner–Martinelli form and the term of top degree is the classical Bochner–Martinelli kernel.

A smooth section $g(\zeta,z)=g_{0,0}+\cdots+g_{N,N}$ of Λ_η, where the subscript means bidegree, defined for $z\in D'\subset D$ and $\zeta\in D$, such that $\nabla_\eta g=0$ and $g_{0,0}\restriction_{\Delta^D}=1$ is called a *weight* with respect to $z\in D'$. It follows that $\nabla_\eta(g\wedge B)=g-[\Delta^D]$ and, identifying terms of bidegree $(N,N-1)$, we get that

$$\bar\partial(g\wedge B)_{N,N-1}=[\Delta^D]-g_{N,N} \tag{5.2}$$

in $D_\zeta\times D'_z$ and hence another solution to (5.1). If D is pseudoconvex and K is a holomorphically convex compact subset, then one can find a weight g with respect to z in some neighborhood $D'\Subset D$ of K such that $z\mapsto g(\zeta,z)$ is holomorphic in D' and $\zeta\mapsto g(\zeta,z)$ has compact support in D independently of $z\in D'$; see, e.g., [3, Example 2] or [8, Example 5.1] in case $D=\mathbb{B}$. Weights with compact support in ζ will be used in the construction of the operators $\mathscr{K}$ and $\mathscr{P}$. In the construction of the "dual operators" $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$, see Sect. 5.2 below, the roles of z and ζ will be interchanged and we then use weights $g(\zeta,z)$ with respect to $\zeta\in D'$ such that $z\mapsto g(\zeta,z)$ has compact support in D independently of $\zeta\in D'$.

Let $V\to D$ be a vector bundle, let $\pi_\zeta\colon D_\zeta\times D_z\to D_\zeta$ and $\pi_z\colon D_\zeta\times D_z\to D_z$ be the natural projections and set $V_z\otimes V^*_\zeta:=\pi_z^*V\otimes\pi_\zeta^*V^*$. Then a weight may take values in $V_z\otimes V^*_\zeta\simeq \mathrm{Hom}(V_\zeta,V_z)$. Such a weight should satisfy the same properties but with the condition $g_{0,0}\restriction_{\Delta^D}=1$ replaced by $g_{0,0}\restriction_{\Delta^D}=\mathrm{Id}_V$, cf. [22] and [3]. If g is a weight with values in $V_z\otimes V^*_\zeta$ then (5.2) holds with $[\Delta^D]$ replaced by $\mathrm{Id}_V\otimes[\Delta^D]$.

The main difference in the construction of our operators $\mathscr{K}$, $\mathscr{P}$, $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ compared to the ones in [8] and [38] is that in the present setting we need to use weights with values in the vector bundle $T^*_{p,0}D_z\otimes T_{p,0}D_\zeta$. Weights with values in $T^*_{p,0}D_z\otimes T_{p,0}D_\zeta$ is necessary for us since we need weights for division-interpolation with respect to the submodule $\mathscr{J}^p_X\subset\Omega^p_D$. The construction of these weights is as follows, cf. [9].

Let $\tilde X$ be an analytic subset of pure codimension κ of a neighborhood of $\overline{D}$, where D now is assumed to be strictly pseudoconvex, and set $X=\tilde X\cap D$. Let (2.6) be a free resolution of $\widehat{\Omega}^p_X$ in D with $E_0=T^*_{p,0}D$ and let $U=U(\zeta)$ and $R=R(\zeta)$ be the associated currents (for some choice of Hermitian metrics on the E_k's). Let $E^z_k:=\pi^*_zE_k$ and $E^\zeta_k:=\pi^*_\zeta E_k$. One can find Hefer morphisms $H^\ell_k=H^\ell_k(\zeta,z)$, which depend holomorphically on $(\zeta,z)\in D\times D$ and are $\mathrm{Hom}(E^\zeta_k,E^z_\ell)$-valued $(k-\ell,0)$-forms such that

$$H^k_k\restriction_{\Delta_D}=\mathrm{Id}_{E_k}\quad\text{and}\quad \delta_\eta H^\ell_k=H^\ell_{k-1}f_k-f_{\ell+1}(z)H^{\ell+1}_k,\ \ k>\ell,$$

where $f_k = f_k(\zeta)$; see [3, Proposition 5.3]. Let $F = F(\zeta)$ be a holomorphic tuple such that $X = \{F = 0\}$ and set $\chi^\epsilon := \chi(|F|^2/\epsilon)$; we regularize U and R as in Sect. 2 so that $U^\epsilon := \chi^\epsilon u$ and

$$R^\epsilon := \mathrm{Id}_E - \nabla_{\mathrm{End}} U^\epsilon = (1 - \chi^\epsilon)\mathrm{Id}_E + \bar{\partial}\chi^\epsilon \wedge u.$$

We write U_k^ϵ and R_k^ϵ for the parts of U^ϵ and R^ϵ that take values in $\mathrm{Hom}(E_0, E_k)$ and we define

$$G^\epsilon := \sum_{k\geq 0} H_k^0 R_k^\epsilon + f_1(z) \sum_{k\geq 1} H_k^1 U_k^\epsilon, \tag{5.3}$$

which one can check is a weight with values in $\mathrm{Hom}(E_0^\zeta, E_0^z) = T^*_{p,0}D_z \otimes T_{p,0}D_\zeta$.

Remark 5.1 One can use the λ-regularizations $U^\lambda = |F|^{2\lambda}u$ and $R^\lambda = (1 - |F|^{2\lambda})\mathrm{Id}_E + \bar{\partial}|F|^{2\lambda} \wedge u$ of U and R, respectively, and define the weight $G^\lambda = HR^\lambda + f_1(z)HU^\lambda$. Our integral operators can then be obtained as the value at $\lambda = 0$ via analytic continuation, cf. Sect. 2.2 and in particular Footnote 1.

Letting g be any scalar-valued weight with respect to, say, $z \in D' \subset D$ it follows that $G^\epsilon \wedge g$ is a $\mathrm{Hom}(E_0^\zeta, E_0^z)$-valued weight and (5.2) holds with g replaced by $G^\epsilon \wedge g$ and $[\Delta^D]$ replaced by $\mathrm{Id}_{E_0} \otimes [\Delta^D]$. Let $\nabla^z = \oplus_j f_j(z) - \bar{\partial}$ and let ∇^z_{End} be the corresponding endomorphism-valued operator. Recall that $\nabla^z_{\mathrm{End}} R(z) = 0$ and notice that, since $f(z) \restriction_{E_0} = 0$,

$$\begin{aligned}
&\nabla^z_{\mathrm{End}}(G^\epsilon \wedge g \wedge B)_{N,N-1} \\
&\quad = -\bar{\partial}(G^\epsilon \wedge g \wedge B)_{N,N-1} \\
&\quad = -\mathrm{Id}_{E_0} \otimes [\Delta^D] + (G^\epsilon \wedge g)_{N,N}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&-\nabla^z_{\mathrm{End}}\big(R(z) \wedge \mathrm{d}z \wedge (G^\epsilon \wedge g \wedge B)_{N,N-1}\big) \\
&\quad = R(z) \wedge \mathrm{d}z \wedge [\Delta^D] - R(z) \wedge \mathrm{d}z \wedge (G^\epsilon \wedge g)_{N,N}.
\end{aligned} \tag{5.4}$$

Notice that $R(z) \wedge [\Delta^D]$ and $R(z) \wedge B$ are well-defined; they are simply tensor products of currents since z and $\zeta - z$ are independent variables on $D \times D$. In view of (5.3), since $R(z) f_1(z) = 0$, (5.4) becomes

$$\begin{aligned}
&-\nabla^z_{\mathrm{End}}\big(R(z) \wedge \mathrm{d}z \wedge (HR^\epsilon \wedge g \wedge B)_{N,N-1}\big) \\
&\quad = R(z) \wedge \mathrm{d}z \wedge [\Delta^D] - R(z) \wedge \mathrm{d}z \wedge (HR^\epsilon \wedge g)_{N,N},
\end{aligned} \tag{5.5}$$

where $HR^\epsilon := \sum_{k\geq 0} H_k^0 R_k^\epsilon$. Let $\iota \colon X \simeq \Delta^X \hookrightarrow X \times X$ be the diagonal embedding and let $\mathrm{i} \colon X \times X \hookrightarrow D \times D$ be the inclusion. By Proposition 4.1 we have

$$\mathrm{i}_* \iota_* \omega = R(z) \wedge \mathrm{d}z \wedge [\Delta^D], \tag{5.6}$$

where ω is the $(n-p)$-structure form corresponding to R.

Consider now the term $(HR^{\epsilon}\wedge g)_{N,N}$. Noticing that R^{ϵ} contains no $d\eta_j$ we see that

$$(HR^{\epsilon}\wedge g)_{N,N}=\tilde{p}(\zeta,z)\wedge R^{\epsilon}\wedge d\eta, \tag{5.7}$$

for some $\mathrm{Hom}(E^{\zeta},E_0^{z})$-valued form $\tilde{p}(\zeta,z)$ that is smooth for $(\zeta,z)\in D\times D'$; if g is chosen holomorphic in z (respectively ζ), then $\tilde{p}$ is holomorphic in z (respectively ζ). To further reveal the structure of $\tilde{p}$, let $\varepsilon_1,\ldots,\varepsilon_N$ be a frame for an auxiliary trivial vector bundle $F\to D\times D$, replace each occurrence of $d\eta_j$ in H and g by ε_j, and denote the result by $\hat{H}$ and $\hat{g}$. We get

$$\begin{aligned}\tilde{p}(\zeta,z)\wedge R^{\epsilon}\wedge\varepsilon=(\hat{H}R^{\epsilon}\wedge\hat{g})_{N,N}&=\sum_{k\geq 0}\hat{H}_k^0R_k^{\epsilon}\wedge\hat{g}_{N-k,N-k}\\&=\sum_{k\geq 0}\tilde{p}_k(\zeta,z)\wedge R_k^{\epsilon}\wedge\varepsilon,\end{aligned} \tag{5.8}$$

where $\tilde{p}_k(\zeta,z)=\pm\epsilon^*\lrcorner\hat{H}_k^0\wedge\hat{g}_{N-k,N-k}$ is a smooth $(0,N-k)$-form in $D\times D'$ with values in $\mathrm{Hom}(E_k^{\zeta},E_0^{z})$; it is holomorphic in z (or ζ) if g is chosen so. For degree reasons it follows that

$$R(z)\wedge dz\wedge(HR^{\epsilon}\wedge g)_{N,N}=R(z)\wedge dz\wedge\sum_{k\geq 0}\tilde{p}_k(\zeta,z)\wedge R_k^{\epsilon}\wedge d\zeta. \tag{5.9}$$

Since $R(z)\wedge R$ is well-defined (as a tensor product) we may set $\epsilon=0$ in (5.9) and since $R=R_{\kappa}+R_{\kappa+1}+\cdots$ we then sum only over $k\geq\kappa$. In view of Proposition 4.1 it follows that

$$\lim_{\epsilon\to 0}R(z)\wedge dz\wedge(HR^{\epsilon}\wedge g)_{N,N}=i_*\omega(z)\wedge p(\zeta,z), \tag{5.10}$$

where

$$p(\zeta,z):=\sum_{k\geq\kappa}i^*\tilde{p}_k(\zeta,z)\wedge\omega_{k-\kappa}(\zeta)=\sum_{k\geq\kappa}\pm i^*\big(\varepsilon^*\lrcorner\hat{H}_k^0\wedge\hat{g}_{N-k,N-k}\big)\wedge\omega_{k-\kappa}(\zeta).$$

We here, and in the following, view $\tilde{p}_k$ not as $(0,N-k)$-form with values in $\mathrm{Hom}(E_k^{\zeta},E_0^{z})$ but as a $(p,N-k)$-form with values in $(E_k^{\zeta})^*$; cf. the paragraphs before Example 3.2. Thus, $p(\zeta,z)$ is a scalar valued almost semi-meromorphic current on $X\times X'$ of bidegree (n,n) such that $z\mapsto p(\zeta,z)$ is, or rather, has a natural extension that is smooth in D (or holomorphic if $z\mapsto g(\zeta,z)$ is); notice that $p(\zeta,z)$ has degree p in dz_j and degree $n-p$ in $d\zeta_j$.

We proceed in an analogous way with the current $R(z)\wedge dz\wedge(HR^{\epsilon}\wedge g\wedge B)_{N,N-1}$ and we get, cf. (5.9), that

$$R(z)\wedge dz\wedge (HR^{\epsilon}\wedge g\wedge B)_{N,N-1}=R(z)\wedge dz\wedge \sum_{j\geq 0}\tilde{k}_j(\zeta,z)\wedge R^{\epsilon}_j\wedge d\zeta, \tag{5.11}$$

where $\tilde{k}_j(\zeta,z):=\pm\varepsilon^*\lrcorner\hat{H}^0_j\wedge(\hat{g}\wedge\hat{B})_{N-j,N-j-1}$ is a $(0,N-j-1)$-form with values in $\mathrm{Hom}(E^{\zeta}_j,E^{z}_0)$. From Sect. 2 we know that the limit as $\epsilon\to 0$ of (5.11) exists and yields a pseudomeromorphic current in $D\times D'$. Moreover, precisely as in [8, Lemma 5.2] one shows that

$$\lim_{\epsilon\to 0}R(z)\wedge dz\wedge (HR^{\epsilon}\wedge g\wedge B)_{N,N-1}=\lim_{\epsilon\to 0}R(z)\wedge dz\wedge (HR\wedge g\wedge B^{\epsilon})_{N,N-1},$$

where $B^{\epsilon}:=\chi(|\eta|^2/\epsilon)B$, holds in the sense of current on $(D\backslash X_{sing})\times(D'\backslash X_{sing})$. In view of (5.11) and Proposition 4.1 we thus get

$$\lim_{\epsilon\to 0}R(z)\wedge dz\wedge (HR^{\epsilon}\wedge g\wedge B)_{N,N-1}=\lim_{\epsilon\to 0}\chi(|\eta|^2/\epsilon)\mathrm{i}_*\omega(z)\wedge k(\zeta,z) \tag{5.12}$$

in $(D\backslash X_{sing})\times(D'\backslash X_{sing})$, where

$$\begin{aligned} k(\zeta,z) &:=\sum_{j\geq\kappa}\mathrm{i}^*\tilde{k}_j(\zeta,z)\wedge\omega_{j-\kappa}(\zeta)\\ &=\pm\sum_{j\geq\kappa}\mathrm{i}^*\big(\varepsilon^*\lrcorner\hat{H}^0_j\wedge(\hat{g}\wedge\hat{B})_{N-j,N-j-1}\big)\wedge\omega_{j-\kappa}(\zeta). \end{aligned} \tag{5.13}$$

As with $\tilde{p}_j(\zeta,z)$, we here and in the following view $\tilde{k}_j(\zeta,z)$ as a $(p,N-j-1)$-form with values in $(E^{\zeta}_j)^*$ so that $k(\zeta,z)$ becomes a scalar valued almost semi-meromorphic $(n,n-1)$-current on $X\times X'$; the degree in dz_j being p and the degree in $d\zeta_j$ being $n-p$. Recall that $B_{\ell,\ell-1}=s\wedge(\bar\partial s)^{\ell-1}/(\delta_\eta s)^{\ell}$ and that $|s|\lesssim|\eta|$ and $|\eta|^2\lesssim|\delta_\eta s|$. Since $\hat{B}_{\ell,\ell-1}$, $\ell=1,\cdots,n$, are the only components of $\hat{B}$ that enter in the expression for $k(\zeta,z)$ it follows that $k(\zeta,z)$ is integrable on $X_{reg}\times X'_{reg}$. Hence, the limit on the right-hand side of (5.12) is just the locally integrable form $k(\zeta,z)\wedge\omega(\zeta)$ on $X_{reg}\times X'_{reg}$. From (5.5), (5.6), (5.10), and (5.12) we thus see that

$$-\nabla\omega(z)\wedge k(\zeta,z)=\iota_*\omega-\omega(z)\wedge p(\zeta,z) \tag{5.14}$$

as currents on $X_{reg}\times X'_{reg}$, where ∇ here means the endomorphism-version of $f(z)\restriction_X -\bar\partial$. Since R is ∇_{End}-closed it follows that $\omega(z)$ is ∇-closed and so the left-hand side of (5.14) equals $\omega(z)\wedge\bar\partial k(\zeta,z)$. By Lemma 4.5 we have thus proved.

Proposition 5.2 *In $X_{reg}\times X'_{reg}$ we have that $\bar\partial k(\zeta,z)=[\Delta^X]-p(\zeta,z)$ as currents.*

The following technical lemma corresponds to [8, Lemma 6.4]; cf. also [38, Proposition 4.3 (ii)]. It is a statement on $X^{\nu+1}:=X\times\cdots\times X$ ($\nu+1$ factors); X_{z^ℓ} refers to the ℓ^{th} factor and z^ℓ are points on X_{z^ℓ}.

Lemma 5.3 *Let ω be any $(n-p)$-structure form and let $k^{(\ell)}(z^{\ell-1}, z^{\ell})$, $\ell = 1, \dots, \nu$, be given by (5.13) for possibly different choices of H's, g's, B's, and $(n-p)$-structure forms ω's. Then*

$$T := \omega(z^{\nu}) \wedge k^{(\nu)}(z^{\nu-1}, z^{\nu}) \wedge k^{(\nu-1)}(z^{\nu-2}, z^{\nu-1}) \wedge \cdots \wedge k^{(1)}(z^{0}, z^{1}) \quad (5.15)$$

is an almost semi-meromorphic current on $X^{\nu+1}$. If $h = h(z^{\ell})$ is a generically non-vanishing holomorphic tuple on $X_{z^{\ell}}$ then $\bar\partial\chi(|h|^2/\epsilon) \wedge T \to 0$ as $\epsilon \to 0$.

5.1 The Integral Operators $\mathscr{K}$ and $\mathscr{P}$ on $(p, *)$-Forms

In order to construct the integral operators $\mathscr{K}$ we choose the weight g in the definitions of $p(\zeta, z)$ and $k(\zeta, z)$ to be a weight with respect to $z \in D' \Subset D$ such that $\zeta \mapsto g(\zeta, z)$ has compact support in D. Let φ be a pseudomeromorphic (p, q)-current on X. In view of Sect. 2.2, $k(\zeta, z) \wedge \varphi(\zeta)$ and $p(\zeta, z) \wedge \varphi(\zeta)$ are well-defined pseudomeromorphic currents in $X_{\zeta} \times X'_{z}$, where $X' = X \cap D'$. Let $\pi^{z}\colon X_{\zeta} \times X_{z} \to X_{z}$ be the natural projection and set

$$\mathscr{K}\varphi(z) := \pi^{z}_{*}\, k(\zeta, z) \wedge \varphi(\zeta), \quad \mathscr{P}\varphi(z) := \pi^{z}_{*}\, p(\zeta, z) \wedge \varphi(\zeta). \qquad (5.16)$$

Since $\zeta \mapsto g(\zeta, z)$ has compact support in D it follows that $\mathscr{K}\varphi$ and $\mathscr{P}\varphi$ are well-defined pseudomeromorphic currents in X'. Notice that $\mathscr{P}\varphi$ has a natural smooth extension to D' since $z \mapsto p(\zeta, z)$ has; notice also that if φ has the SEP then $\mathscr{K}\varphi$ has the SEP in view of Sect. 2.2. Moreover, as in [8, Lemma 6.1] one shows that if $\varphi = 0$ in a neighborhood of a point $x \in X'$, or if φ is smooth in a neighborhood of x and $x \in X'_{reg}$, then $\mathscr{K}\varphi$ is smooth in a neighborhood of x.

If φ is a pseudomeromorphic (p, q)-current with compact support in X, then one can choose any weight g in the definitions of $k(\zeta, z)$ and $p(\zeta, z)$ and define $\mathscr{K}\varphi$ and $\mathscr{P}\varphi$ by (5.16); the outcome has the same general properties.

The following proposition is proved in the same way as [8, Proposition 6.3].

Proposition 5.4 *Let $\varphi \in \mathcal{W}^{p,q}(X)$, let ω be the $(n-p)$-structure form that enters in the definitions of $k(\zeta, z)$ and $p(\zeta, z)$, and assume that $\bar\partial(\omega \wedge \varphi)$ has the SEP. Let g be a weight with respect to $z \in D' \subset D$. If either g has compact support in D_{ζ} or φ has compact support in X then $\varphi = \bar\partial\mathscr{K}\varphi + \mathscr{K}(\bar\partial\varphi) + \mathscr{P}\varphi$ as currents on X'_{reg}.*

Notice that the condition that $\bar\partial(\omega \wedge \varphi)$ has the SEP implies that $\bar\partial\varphi$ has the SEP. In fact, from Sect. 2.2 we know that $\omega \wedge \varphi$ has the SEP and so, in view of Lemma 2.2, $\bar\partial(\omega \wedge \varphi)$ has the SEP if and only if $\bar\partial\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi \to 0$ for all generically non-vanishing h. In particular, $\bar\partial\chi(|h|^2/\epsilon) \wedge \omega_0 \wedge \varphi \to 0$ and so, by Lemma 4.5, $\bar\partial\chi(|h|^2/\epsilon) \wedge \varphi \to 0$. By Lemma 2.2 again we conclude that $\bar\partial\varphi$ has the SEP.

From Proposition 5.4 it is easy to prove the following residue criterion for a meromorphic p-form to be strongly holomorphic. Recall the operator $\nabla = \oplus_j f_j - \bar\partial$. attached to (2.6).

Theorem 5.5 *Let X be a pure n-dimensional analytic subset of some neighborhood of the closure of a strictly pseudoconvex domain $D \Subset \mathbb{C}^N$ and let ω be an $(n-p)$-structure* *form on $X \cap D$ corresponding to a resolution (2.6) of $\widehat{\Omega}^p_X$. Then a meromorphic p-form φ on $X \cap D$ is strongly holomorphic if and only if*

$$\nabla(\omega \wedge \varphi) = 0. \tag{5.17}$$

Moreover, if (5.17) holds, $D' \Subset D$, and $\mathscr{P}$ is an integral operator constructed using ω and a weight $g(\zeta, z)$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D, then $\mathscr{P}\varphi$ is a holomorphic extension of $\varphi_{\restriction_{X \cap D'}}$ to D'.

Proof Notice first that if φ is strongly holomorphic then (5.17) holds since $\nabla\omega = 0$.

For the converse, notice that $\omega \wedge \varphi$ has the SEP so that $\chi(|h|^2/\epsilon)\omega \wedge \varphi \to \omega \wedge \varphi$ for all generically non-vanishing h. Hence, if (5.17) holds, we get

$$0 = \nabla(\omega \wedge \varphi) = \lim_{\epsilon \to 0} \nabla\big(\chi(|h|^2/\epsilon)\omega \wedge \varphi\big) = -\lim_{\epsilon \to 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi$$

for all such h. From Lemma 2.2 it thus follows that $\bar{\partial}(\omega \wedge \varphi)$ has the SEP. From the paragraph after Proposition 5.4 it then follows that $\bar{\partial}\varphi$ has the SEP and since φ is holomorphic generically we see that $\bar{\partial}\varphi = 0$. By Proposition 5.4 we get that $\varphi = \mathscr{P}\varphi$ on $X_{reg} \cap D'$. However, both φ and $\mathscr{P}\varphi$ have the SEP so this holds on $X \cap D'$. □

Theorem 5.5 gives the following geometric criterion for a meromorphic p-form to be strongly holomorphic.

Proposition 5.6 *Let X be a pure n-dimensional reduced complex space and let φ be a meromorphic p-form on X with pole set $P_\varphi \subset X$. Suppose that (i) $\operatorname{codim}_X P_\varphi \geq 2$, and that (ii) $\operatorname{codim}_X S_{n-k}(\widehat{\Omega}^p_X) \cap P_\varphi \geq k + 2$ for $k \geq 1$. Then φ is strongly holomorphic.*

Proof Since $\widehat{\Omega}^p_X$ is torsion free a strongly holomorphic extension of φ, if such exist, is unique. Therefore the statement of the proposition is local and we may assume that X is an analytic subset of a neighborhood of $\overline{\mathbb{B}} \subset \mathbb{C}^N$. Let $\omega = \omega_0 + \cdots$ be an $(n-p)$-structure form on $X \cap \mathbb{B}$. By Theorem 5.5 we need to show that $\nabla(\omega \wedge \varphi) = 0$. Since ω and φ are almost semi-meromorphic we have $\pm\omega \wedge \varphi = \varphi \wedge \omega = \lim_{\epsilon \to 0} \chi(|h|^2/\epsilon)\varphi \wedge \omega$, where h is a generically non-vanishing holomorphic function such that $\{h = 0\} \supset P_\varphi$. Thus, since $\nabla\omega = 0$, we see that $\nabla(\omega \wedge \varphi) = \pm \lim_{\epsilon \to 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \varphi \wedge \omega$ and so we need to show that

$$\lim_{\epsilon \to 0} \bar{\partial}\chi(|h|^2/\epsilon) \wedge \varphi \wedge \omega_\ell = 0 \tag{5.18}$$

for $\ell = 0, 1, 2, \ldots$. For $\ell = 0$ the left-hand side of (5.18) is a pseudomeromorphic $(n, 1)$-current on X with support contained in P_φ; hence it vanishes by the dimension principle and assumption (i).

Recall from Sect. 2.3 the sets Z_k associated with a resolution (2.6) of $\widehat{\Omega}^p_X$ and that $S_{N-k}(\widehat{\Omega}^p_X) = Z_k$. Assumption (ii) is thus equivalent to $\operatorname{codim} Z_k \cap P_\varphi \geq k + 2$ for $k \geq N - n + 1$. Now, assume that (5.18) holds for $\ell = m$. Since, by Proposition 4.1 (ii), ω_{m+1} is a smooth form times ω_m outside of Z_{m+1} it follows that for $\ell = m + 1$ the left-hand side of (5.18) is a pseudomeromorphic $(n, m + 2)$-current with support contained in $Z_{m+1} \cap P_\varphi$. Thus, (5.18) holds for $\ell = m + 1$ by assumption (ii) and the dimension principle. □

5.2 The Integral Operators $\check{\mathscr{K}}$ and $\check{\mathscr{P}}$ on $(n-p,*)$-Forms

A general integral operator $\check{\mathscr{K}}$ is constructed by choosing the weight g in the definitions of $k(\zeta,z)$ and $p(\zeta,z)$ to be a weight with respect to $\zeta \in D' \Subset D$ such that $z \mapsto g(\zeta,z)$ has compact support in D. Let ψ be a pseudomeromorphic $(n-p,q)$-current on X. In the same way as above $k(\zeta,z) \wedge \varphi(z)$ and $p(\zeta,z) \wedge \varphi(z)$ are well-defined pseudomeromorphic currents in $X'_\zeta \times X_z$ and we set

$$\check{\mathscr{K}}\psi(\zeta) := \pi_*^\zeta\, k(\zeta,z) \wedge \varphi(z), \quad \check{\mathscr{P}}\psi(\zeta) := \pi_*^\zeta\, p(\zeta,z) \wedge \varphi(z),$$

which become pseudomeromorphic currents on X'. Notice that $\check{\mathscr{P}}\psi$ has the SEP if ψ has, and moreover, is of the form $\sum_{\ell\geq 0} A_\ell(\zeta) \wedge \omega_\ell(\zeta)$, where A_ℓ is a smooth form with values in $E^*_{\kappa+\ell}$; if g is chosen so that $\zeta \mapsto g(\zeta,z)$ is holomorphic then the A_ℓ are holomorphic. The current $\check{\mathscr{K}}\psi$ has the SEP if ψ has, and it has the form $\sum_{\ell\geq 0} C_\ell(\zeta) \wedge \omega_\ell(\zeta)$, where the C_ℓ take values in $E^*_{\kappa+\ell}$ and are: (i) smooth close to $x \in X'$ if $\psi = 0$ close to x, and (ii) smooth close to $x \in X'_{reg}$ if ψ is smooth close to x.

As for $\mathscr{K}$ and $\mathscr{P}$, if ψ happens to have compact support in X then any weight g may be used to define $\check{\mathscr{K}}\psi$ and $\check{\mathscr{P}}\psi$.

Proposition 5.7 *Let $\psi \in \mathcal{W}^{n-p,q}(X)$, assume that $\bar\partial\psi \in \mathcal{W}^{n-p,q+1}(X)$, and let g be a weight with respect to $\zeta \in D' \subset D$. If either g has compact support in D_z or ψ has compact support in X then $\psi = \bar\partial\check{\mathscr{K}}\psi + \check{\mathscr{K}}(\bar\partial\psi) + \check{\mathscr{P}}\psi$ as currents on X'_{reg}.*

This is proved in the same way as [38, Proposition 3.1].

6 The Sheaves $\mathscr{A}_X^{p,q}$ and $\mathscr{B}_X^{n-p,n-q}$

6.1 The Sheaves $\mathscr{A}_X^{p,\bullet}$

Let X be a reduced complex space of pure dimension n. Following [8, Definition 7.1] we say that a (p,q)-current φ on X on an open subset $U \subset X$ is a section of $\mathscr{A}_X^{p,q}$ over U if for every $x \in U$ the germ φ_x can be written as a finite sum of terms

$$\xi_\nu \wedge \mathscr{K}^{(\nu)}(\cdots \xi_2 \wedge \mathscr{K}^{(2)}(\xi_1 \wedge \mathscr{K}^{(1)}(\xi_0))\cdots), \tag{6.1}$$

where ξ_0 is a smooth $(p,*)$-form and the ξ_ℓ, $\ell \geq 1$, are smooth $(0,*)$-forms such that ξ_ℓ has support where $z \mapsto k^{(\ell)}(\zeta,z)$ is defined.

Proposition 6.1 *The sheaf $\mathscr{A}_X^{p,q}$ has the following properties:*

(i) $\mathscr{E}_X^{p,q} \subset \mathscr{A}_X^{p,q} \subset \mathcal{W}_X^{p,q}$ *and* $\oplus_q \mathscr{A}_X^{p,q}$ *is a module over* $\oplus_q \mathscr{E}_X^{0,q}$,
(ii) $\mathscr{A}_{X_{reg}}^{p,q} = \mathscr{E}_{X_{reg}}^{p,q}$,
(iii) *for any operator* $\mathscr{K}$ *on* $(p,*)$*-forms as in Sect. 5.1* $\mathscr{K}: \mathscr{A}_X^{p,q} \to \mathscr{A}_X^{p,q-1}$,

(iv) *if φ is a section of $\mathscr{A}_X^{p,q}$ and ω is any $(n-p)$-structure form, then $\bar{\partial}(\omega \wedge \varphi)$ has the SEP.*

Proof (i), (ii), and (iii) are immediate from the definition of $\mathscr{A}_X^{p,q}$ and the general properties of the $\mathscr{K}$-operators in Sect. 5.1. To prove (iv) we may assume that φ is of the form (6.1). Then $\omega \wedge \varphi$ is a push-forward of $T \wedge \xi$, where T is of the form (5.15) and ξ is a smooth form on $X^{\nu+1}$. Choosing $h = h(z^\nu)$ in Lemma 5.3 it follows that $\bar{\partial}\chi(|h|^2/\epsilon) \wedge \omega \wedge \varphi \to 0$ as $\epsilon \to 0$ and so, by Lemma 2.2, $\bar{\partial}(\omega \wedge \varphi)$ has the SEP. □

Proof of Theorem 1.1 Let $D'' \Subset D$ be a strictly pseudoconvex neighborhood of $\overline{D}'$ and carry out the construction of $k(\zeta, z)$ and $p(\zeta, z)$ in Sect. 5 in $D'' \times D''$ using a weight $g(\zeta, z)$ with respect to $z \in D'$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D''. Notice that then $\mathscr{P}\varphi$ is holomorphic and that g, and hence also $p(\zeta, z)$, has bidegree $(*, 0)$ in the z-variables so that $\mathscr{P}\varphi = 0$ if φ has bidegree (p, q) with $q \geq 1$. Let $\varphi \in \mathscr{A}^{p,q}(X)$. By Proposition 6.1 (iv), $\bar{\partial}(\omega \wedge \varphi)$ has the SEP and so Proposition 5.4 shows that

$$\varphi = \bar{\partial}\mathscr{K}\varphi + \mathscr{K}(\bar{\partial}\varphi) + \mathscr{P}\varphi \tag{6.2}$$

in the sense of currents on X'_{reg}. Now, $\mathscr{K}\varphi \in \mathscr{A}^{p,q-1}(X')$ by Proposition 6.1 (iii). Hence, by Proposition 6.1 (iv) and the comment after Proposition 5.4, $\bar{\partial}\mathscr{K}\varphi$ has the SEP. In the same way $\bar{\partial}\varphi$ has the SEP and so $\mathscr{K}(\bar{\partial}\varphi)$ has the SEP. All terms in (6.2) thus have the SEP and therefore (6.2) holds on X', concluding the proof. □

Proposition 6.2 *Let X be a reduced complex space of pure dimension n. Then $\bar{\partial}\colon \mathscr{A}_X^{p,q} \to \mathscr{A}_X^{p,q+1}$ and the sheaf complex (1.1) is exact.*

Proof Let φ be a $\bar{\partial}$-closed section of $\mathscr{A}_X^{p,q}$ over some small neighborhood U of a given point $x \in X$; we may assume that U is an analytic subset of some pseudoconvex domain in some $\mathbb{C}^N$. As in the proof of Theorem 1.1 above one shows that, for suitable operators $\mathscr{K}$ and $\mathscr{P}$, $\varphi = \bar{\partial}\mathscr{K}\varphi$ if $q \geq 1$ and $\varphi = \mathscr{P}\varphi$ is a section of $\widehat{\Omega}_X^p$ if $q = 0$.

It remains to see that $\bar{\partial}\colon \mathscr{A}_X^{p,q} \to \mathscr{A}_X^{p,q+1}$. Let φ be a $\bar{\partial}$-closed section of $\mathscr{A}_X^{p,q}$ over some small neighborhood U of a given point $x \in X$; we may assume that φ is of the form (6.1) and we will use induction over ν. If $\nu = 0$ then $\varphi = \xi_0$ is smooth and so $\bar{\partial}\varphi$ is in $\mathscr{E}_X^{p,q+1} \subset \mathscr{A}_X^{p,q+1}$. Assume that $\bar{\partial}\varphi'$ is in $\mathscr{A}_X^{p,*}$ for any φ' of the form (6.1). Since φ' is a section of $\mathscr{A}_X^{p,*}$ it follows from Proposition 5.4 that

$$\varphi' = \bar{\partial}\mathscr{K}^{(\nu+1)}\varphi' + \mathscr{K}^{(\nu+1)}(\bar{\partial}\varphi') + \mathscr{P}^{(\nu+1)}\varphi' \tag{6.3}$$

as currents on U'_{reg} for some sufficiently small neighborhood U' of x, cf. the proof of Theorem 1.1 above. As in that same proof, (6.3) extends to hold on U'. The left-hand side as well as the last term on the right-hand side of (6.3) are clearly in $\mathscr{A}_X^{p,*}$ and, since $\bar{\partial}\varphi'$ is in $\mathscr{A}_X^{p,*}$ by assumption and $\mathscr{K}$-operators preserve $\mathscr{A}_X^{p,*}$, also the second term on the right-hand side is in $\mathscr{A}_X^{p,*}$. Hence, $\bar{\partial}\mathscr{K}^{(\nu+1)}\varphi'$ is a section of $\mathscr{A}_X^{p,*}$ over U' showing that $\bar{\partial}\varphi$ is in $\mathscr{A}_X^{p,*}$ for φ of the form (6.1) with ν replaced by $\nu + 1$. □

Notice that Theorem 1.3 follows from Propositions 6.1 and 6.2.

Proof of Proposition 1.5 Assume that condition (i) of Proposition 1.5 holds. Then, in view of the last paragraph in Sect. 4, any holomorphic p-form on the regular part at least extends to a section of ω_X^p; in particular, such forms are meromorphic. It is thus clear from Proposition 5.6 that $\widehat{\Omega}^p(U) \to \widehat{\Omega}^p(U_{reg})$ is surjective for any open $U \subset X$; the injectivity is obvious. We remark that the implication (i) $\Rightarrow$ (ii) also follows from [40, Satz III].

Assume that condition (ii) of Proposition 1.5 holds. In view of [41, Theorem 1.14, $(d) \Rightarrow (b)$] it is sufficient to show that the restriction map $H^1(U, \widehat{\Omega}_X^p) \to H^1(U_{reg}, \widehat{\Omega}_X^p)$ is injective for any open $U \subset X$. By Corollary 1.4, $H^1(U, \widehat{\Omega}_X^p) \simeq H^1(\mathscr{A}^{p,\bullet}(U), \bar{\partial})$, so let $\varphi \in \mathscr{A}^{p,\bullet}(U)$ be $\bar{\partial}$-closed and assume that its image in $H^1(\mathscr{A}^{p,\bullet}(U_{reg}), \bar{\partial})$ vanishes, i.e., that there is $\psi \in \mathscr{A}^{p,0}(U_{reg})$ such that $\varphi = \bar{\partial}\psi$ on U_{reg}. Let $x \in U_{sing}$. By Theorem 1.3, there is a neighborhood $V \subset U$ of x and a $\psi' \in \mathscr{A}^{p,0}(V)$ such that $\varphi = \bar{\partial}\psi'$ in V. Then $\psi - \psi'$ is holomorphic on V_{reg} and so, by condition (ii), $\psi - \psi' \in \widehat{\Omega}^p(V)$. Hence, $\psi = \psi' + \psi - \psi'$ can be locally extended across U_{sing} to a section of $\mathscr{A}_X^{p,0}$. In view of the SEP, extensions are unique and so $\psi \in \mathscr{A}^{p,0}(U)$ and consequently $\bar{\partial}\psi \in \mathscr{A}^{p,1}(U)$. The equality $\varphi = \bar{\partial}\psi$ on U_{reg} therefore extends to hold on U by the SEP and so φ defines the zero element in $H^1(U, \widehat{\Omega}_X^p)$. □

Proof of Corollary 1.6 Assume that $X = \{f_1 = \cdots = f_\kappa = 0\} \subset D \subset \mathbb{C}^N$ has codimension κ and that $df_1 \wedge \cdots \wedge df_\kappa \neq 0$ on X_{reg}. Let $\tilde{\omega}$ be a meromorphic n-form in D such that the polar set of $\tilde{\omega}$ intersects X properly and such that, outside of the polar set of $\tilde{\omega}$, $df_1 \wedge \cdots \wedge df_\kappa \wedge \tilde{\omega} = dz$ for some local coordinates z in D. Let ω be the pullback of $\tilde{\omega}$ to X. Then ω is a holomorphic n-form on X_{reg} that is uniquely determined by dz and X; in fact, ω is the Poincaré-Leray residue of the meromorphic form $dz/(f_1 \cdots f_\kappa)$. If ω has a strongly holomorphic extension to X, then, since $df_1 \wedge \cdots \wedge df_\kappa \wedge \omega = dz$, it follows that $df_1 \wedge \cdots \wedge df_\kappa \neq 0$ on X. □

Some *a priori* assumption on X is necessary for Corollary 1.6. In fact, if $X = \{z_1 = z_4 = 0\} \cup \{z_2 = z_3 = 0\} \subset \mathbb{C}^4$ then one can check that any holomorphic 2-form on X_{reg} extends across X_{sing} to a section of $\widehat{\Omega}_X^2$.

6.2 The Sheaves $\mathscr{B}_X^{n-p,\bullet}$

To define $\mathscr{B}_X^{n-p,\bullet}$ we follow [38, Definition 4.1] and we say that an $(n-p, q')$-current ψ on an open subset $U \subset X$ is a section of $\mathscr{B}_X^{n-p,q'}$ over U if for every $x \in U$ the germ ψ_x can be written as a finite sum of terms

$$\xi_\nu \wedge \check{\mathscr{K}}^{(\nu)}(\cdots \xi_2 \wedge \check{\mathscr{K}}^{(2)}(\xi_1 \wedge \check{\mathscr{K}}^{(1)}(\omega \wedge \xi_0)) \cdots), \tag{6.4}$$

where ω is an $(n-p)$-structure form and the ξ_ℓ are smooth $(0, q')$-forms with support where $\zeta \mapsto k^{(\ell)}(\zeta, z)$ is defined. Recall that ω is an $(n-p, *)$-current with values in a bundle $\oplus_k E_k \restriction_X$ so we need ξ_0 to take values in $\oplus_k E_k^* \restriction_X$ to make $\omega \wedge \xi_0$ scalar-valued.

It is immediate from the definition and from the general properties of the $\check{\mathscr{K}}$-operators that $\mathscr{B}_X^{n-p,q'} \subset \mathcal{W}_X^{n-p,q'}$, that $\mathscr{B}_{X_{reg}}^{n-p,q'} = \mathscr{E}_{X_{reg}}^{n-p,q'}$, that the $\check{\mathscr{K}}$-operators and $\check{\mathscr{P}}$-operators preserve $\oplus_q \mathscr{B}_X^{n-p,q'}$, and that $\oplus_{q'} \mathscr{B}_X^{n-p,q'}$ is a module over $\oplus_{q'} \mathscr{E}_X^{0,q'}$. Let ψ be a smooth $(n-p,q')$-form and let ω be an $(n-p)$-structure form in a neighborhood of some point in X. Then, by Lemma 4.5, there is a smooth $(0,q')$-form ψ' (with values in the appropriate bundle) such that $\psi = \omega_0 \wedge \psi'$. Hence we see that $\mathscr{E}_X^{n-p,q'} \subset \mathscr{B}_X^{n-p,q'}$. Let us also notice that if ψ is in $\mathscr{B}_X^{n-p,q'}$ then $\bar\partial\psi$ has the SEP. In fact, we may assume that ψ is of the form (6.4) so that $\psi = \pi_* T \wedge \xi$, where T is given by (5.15), ξ is a smooth form, and π is the natural projection $X^{\nu+1} \to X_{z^0}$. Letting $h = h(z^0)$ be a generically non-vanishing holomorphic tuple on X_{z^0}, we have that $\bar\partial\chi(|h|^2/\epsilon) \wedge T \wedge \xi \to 0$ by Lemma 5.3. Hence, by Lemma 2.2, we see that $\bar\partial\psi$ has the SEP.

Proof of Theorem 1.8 Let $D'' \Subset D$ be a strictly pseudoconvex neighborhood of $\overline{D}'$ and carry out the construction of $k(\zeta,z)$ and $p(\zeta,z)$ in Sect. 5 in $D'' \times D''$ using a weight $g(\zeta,z)$ with respect to $\zeta \in D'$ such that $\zeta \mapsto g(\zeta,z)$ is holomorphic in D' and $z \mapsto g(\zeta,z)$ has compact support in D''. Let $\psi \in \mathscr{B}^{n-p,q'}(X)$. By Proposition 5.7 we have

$$\psi = \bar\partial\check{\mathscr{K}}\psi + \check{\mathscr{K}}(\bar\partial\psi) + \check{\mathscr{P}}\psi \tag{6.5}$$

as currents on X'_{reg}. From what we noticed just before the proof all terms have the SEP and so (6.5) holds on X'. Notice that $\check{\mathscr{P}}\psi = A_{q'}(\zeta) \wedge \omega_{q'}(\zeta)$, where $\mathscr{A}_{q'}$ is holomorphic. Since if $\widehat{\Omega}_X^p$ is Cohen–Macaulay we may choose $\omega = \omega_0$ to be $\bar\partial$-closed it follows that $\check{\mathscr{P}}\psi \in \omega^{n-p}(X')$ if $q' = 0$ and $\check{\mathscr{P}}\psi = 0$ if $q' \geq 1$. □

Proof of Theorem 1.7 We have already noted that (i) and (ii) hold.

To show that $\bar\partial\colon \mathscr{B}_X^{n-p,q'} \to \mathscr{B}_X^{n-p,q'+1}$ let ψ be a section of $\mathscr{B}_X^{n-p,q'}$ in a neighborhood of some $x \in X$; we may assume that ψ is of the form (6.4) and we use induction over ν. If $\nu = 0$ then $\psi = \omega \wedge \xi_0$ and it is enough to see that $\bar\partial\omega$ is a section of $\mathscr{B}_X^{n-p,*}$ (with values in $E\restriction_X$); but since $\bar\partial\omega = f\omega$ this is clear. The induction step is done in the same way as in the proof of Proposition 6.2.

To show that $\omega_X^{n-p,q'}$ is coherent and that $\omega_X^{n-p} = \omega_X^{n-p,0}$ assume that X can be identified with an analytic subset of a strictly pseudoconvex domain $D \subset \mathbb{C}^N$. Recall that (2.6) is a resolution of $\widehat{\Omega}_X^p$ in D. Taking $\mathscr{H}om$ into Ω^N we get a complex isomorphic to $(\mathscr{O}(E_\bullet^*) \otimes \Omega^N, \bar\partial)$ with associated cohomology sheaves isomorphic to $\mathscr{E}xt^\bullet(\widehat{\Omega}_X^p, \Omega^N)$, which are coherent; cf. Sect. 4. We define the map

$$\varrho_{q'}\colon \mathscr{O}(E_{\kappa+q'}^*) \otimes \Omega^N \to \mathscr{B}_X^{n-p,q'}, \quad \varrho_{q'}(\xi dz) = i^*\xi \cdot \omega_{q'}.$$

Since

$$\begin{aligned}\bar\partial\varrho_{q'}(\xi dz) &= i^*\xi \cdot \bar\partial\omega_{q'} = i^*\xi \cdot f_{\kappa+q'+1}\restriction_X \omega_{q'+1} = i^* f_{\kappa+q'+1}^* \restriction_X \xi \cdot \omega_{q'+1} \\ &= \varrho_{q'+1}(f_{\kappa+q'+1}^* \xi dz),\end{aligned}$$

the map $\varrho_\bullet$ is a map of complexes and so induces a map on cohomology. In view of Proposition 4.3 the proof will be complete if we show that $\varrho_\bullet$ is a quasi-isomorphism.

Since $i_*\omega_{q'} = R_{\kappa+q'} \wedge dz$ it follows from [6, Theorem 7.1] that the map on cohomology is injective. For the surjectivity, let $\psi \in \mathscr{B}^{n-p,q'}(X)$ be $\bar{\partial}$-closed and choose a weight $g(\zeta, z)$ in the kernels $k(\zeta)$ and $p(\zeta, z)$ with respect to ζ in some $D' \Subset D$ such that $\zeta \mapsto g(\zeta, z)$ is holomorphic in D' and $z \mapsto g(\zeta, z)$ has compact support in D. As in the proof of Theorem 1.8 we get that $\psi = \bar{\partial}\check{\mathscr{K}}\psi + \check{\mathscr{P}}\psi$ on $X'_{reg} := X_{reg} \cap D'$ and so the cohomology class of ψ is represented by $\check{\mathscr{P}}\psi$. From the definition of $p(\zeta, z)$ in Sect. 5 we see that

$$\check{\mathscr{P}}\psi(\zeta) = \pm\omega_{q'}(\zeta) \wedge \int_{X_z} \tilde{p}_{\kappa+q'}(\zeta, z) \wedge \psi(z)$$

and $\zeta \mapsto \tilde{p}_{\kappa+q'}(\zeta, z)$ is a section of $\mathscr{O}(E^*_{\kappa+q'})$ over D' by the choice of g. We finally show that

$$f^*_{\kappa+q'+1} \int_{X_z} \tilde{p}_{\kappa+q'}(\zeta, z) \wedge \psi(z) = 0. \tag{6.6}$$

First notice that it follows from (5.7) and (5.8) that, for each k, $\tilde{p}_k(\zeta, z) \wedge d\eta = H^0_k \wedge g_{N-k}$. Moreover,

$$\begin{aligned}
f^*_{k+1} H^0_k \wedge g_{N-k} &= H^0_k f_{k+1} \wedge g_{N-k} = \big(f_1(z) H^1_{k+1} + \delta_\eta H^0_{k+1}\big) \wedge g_{N-k} \\
&= f_1(z) H^1_{k+1} \wedge g_{N-k} \pm H^0_{k+1} \wedge \delta_\eta g_{N-k} \\
&= f_1(z) H^1_{k+1} \wedge g_{N-k} \pm H^0_{k+1} \wedge \bar{\partial} g_{N-k-1} \\
&= f_1(z) H^1_{k+1} \wedge g_{N-k} + \bar{\partial}(H^0_{k+1} \wedge g_{N-k-1}) \\
&=: (f_1(z) A_k + \bar{\partial} B_k) \wedge d\eta,
\end{aligned}$$

where A_k and B_k take values in $\mathrm{Hom}(E^\zeta_{k+1}, E^z_1)$ and $\mathrm{Hom}(E^\zeta_{k+1}, E^z_0)$ respectively; the second equality follows from the properties of the Hefer morphisms, the third by noting that $0 = \delta_\eta(H^0_{k+1} \wedge g_{N-k}) = \delta_\eta H^0_{k+1} \wedge g_{N-k} \pm H^0_{k+1} \wedge \delta_\eta g_{N-k}$, the fourth since g is a weight, the fifth since the Hefer morphisms are holomorphic, and the sixth by collecting all $d\eta_j$. Hence, we get that $f^*_{k+1}\tilde{p}_k(\zeta, z) = f_1(z) A_k + \bar{\partial} B_k$. Since $f_{1\restriction X} = 0$ and by Stokes' theorem, (6.6) follows. □

7 Serre Duality

7.1 The Trace Map

The following result is the generalization of [38, Theorem 5.1] from the case $p = 0$ to the general case $0 \leq p \leq n$. The proof of [38, Theorem 5.1] goes through in the general case essentially verbatim.

Theorem 7.1 *Let X be a reduced complex space of pure dimension n. There is a unique map*

$$\wedge\colon \mathscr{A}_X^{p,q} \times \mathscr{B}_X^{n-p,q'} \to \mathcal{W}_X^{n,q+q'}$$

extending the exterior product on X_{reg}. Moreover, if φ and ψ are sections of $\mathscr{A}_X^{p,q}$ and $\mathscr{B}_X^{n-p,q'}$, respectively, then $\bar{\partial}(\varphi \wedge \psi)$ has the SEP.

It follows that $\bar{\partial}(\varphi \wedge \psi) = \bar{\partial}\varphi \wedge \psi + (-1)^{p+q}\varphi \wedge \bar{\partial}\psi$ since both sides have the SEP and it clearly holds on X_{reg}.

Let $\varphi \in \mathscr{A}^{p,q}(X)$ and $\psi \in \mathscr{B}^{n-p,n-q}(X)$ and assume that at least one of φ and ψ has compact support. By Theorem 7.1, $\varphi \wedge \psi$ is a well-defined section of $\mathcal{W}_X^{n,n}$ with compact support and we may define the trace map $(\varphi, \psi) \mapsto \int_X \varphi \wedge \psi$; the integral is interpreted as the action of $\varphi \wedge \psi$ on the constant function 1 on X. We notice that if h is a generically non-vanishing holomorphic section of a Hermitian vector bundle such that $\{h = 0\} \supset X_{sing}$ then the trace map may be computed as $\lim_{\epsilon \to 0} \int_X \chi(|h|^2/\epsilon)\, \varphi \wedge \psi$. We get an induced trace map on the level of cohomology since if, say, $\varphi = \bar{\partial}\tilde{\varphi}$ for some $\tilde{\varphi} \in \mathscr{A}^{p,q-1}(X)$ with compact support if φ has, then $\varphi \wedge \psi = \bar{\partial}(\tilde{\varphi} \wedge \psi)$ by the Leibniz rule and so $\int_X \varphi \wedge \psi = 0$.

7.2 Local Duality

Let $\tilde{X}$ be an analytic subset of $\overline{D} \subset \mathbb{C}^N$, where D is pseudoconvex, and set $X := \tilde{X} \cap D$. Let F be a holomorphic vector bundle on X and let $\mathscr{F}$ be the associated locally free $\mathscr{O}_X$-module. Since X is Stein and $\mathscr{F} \otimes \widehat{\Omega}_X^p$ is coherent it follows from Corollary 1.4 that the complex

$$0 \to \mathscr{A}^{p,0}(X, F) \xrightarrow{\bar{\partial}} \mathscr{A}^{p,1}(X, F) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{A}^{p,n}(X, F) \to 0$$

is exact except at the level 0 where the cohomology is $\widehat{\Omega}^p(X, F)$. We endow $\widehat{\Omega}^p(X, F)$ with the standard canonical Fréchet space topology, see, e.g., [18, Chapter IX].

Theorem 7.2 *Let $\mathscr{F}^*$ be the sheaf of sections of F^* and let $\mathscr{B}_c^{n-p,q'}(X, F^*)$ be the space of sections of $\mathscr{F}^* \otimes \mathscr{B}_X^{n-p,q'}$ with compact support in X. The complex*

$$0 \to \mathscr{B}_c^{n-p,0}(X, F^*) \xrightarrow{\bar{\partial}} \mathscr{B}_c^{n-p,1}(X, F^*) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathscr{B}_c^{n-p,n}(X, F^*) \to 0 \tag{7.1}$$

is exact except at the level n and the pairing

$$\widehat{\Omega}^p(X, F) \times H^n\big(\mathscr{B}_c^{n-p,\bullet}(X, F^*), \bar{\partial}\big) \to \mathbb{C}, \quad (\varphi, [\psi]) \mapsto \int_X \varphi \cdot \psi \tag{7.2}$$

makes $H^n\big(\mathscr{B}_c^{n-p,\bullet}(X, F^*), \bar{\partial}\big)$ the topological dual of $\widehat{\Omega}^p(X, F)$.

Sketch of proof Since we are in the local situation we may assume that an element in $\mathscr{B}_c^{n-p,q'}(X, F^*)$ is just a tuple of elements in $\mathscr{B}_c^{n-p,q'}(X)$ and carry out the following argument component-wise. Let $\psi \in \mathscr{B}_c^{n-p,q'}(X)$ be $\bar{\partial}$-closed. Let $D' \Subset D'' \subset D$, where $\operatorname{supp}\psi \subset D'$ and D'' is strictly pseudoconvex, and construct $k(\zeta, z)$ and $p(\zeta, z)$ as in Sect. 5 with a weight $g(\zeta, z)$ with respect to $z \in D'$ such that $z \mapsto g(\zeta, z)$ is holomorphic in D' and $\zeta \mapsto g(\zeta, z)$ has compact support in D''. Then $p(\zeta, z) = \sum_k \tilde{p}_{\kappa+k}(\zeta, z) \wedge \omega_k(\zeta)$, where $\zeta \mapsto \tilde{p}_{\kappa+k}(\zeta, z)$ has compact support in D'' and $z \mapsto \tilde{p}_{\kappa+k}(\zeta, z)$ is a section of $\widehat{\Omega}_X^p$ over $X' := X \cap D'$.

As in the proof of Theorem 1.8 we get $\psi = \bar{\partial}\check{\mathscr{K}}\psi + \check{\mathscr{P}}\psi$ in X'. From the properties of $p(\zeta, z)$ we get that $\check{\mathscr{P}}\psi = 0$ if $q' < n$ so (7.1) is exact except at the level n. If $q' = n$ then the cohomology class of ψ is represented by $\check{\mathscr{P}}\psi$ and

$$\check{\mathscr{P}}\psi = \pm \sum_{k \geq 0} \omega_k(\zeta) \wedge \int_{X_z} \tilde{p}_{\kappa+k}(\zeta, z) \wedge \psi(z).$$

Hence, if $\int_X \varphi\psi = 0$ for all $\varphi \in \widehat{\Omega}^p(X)$ then $\check{\mathscr{P}}\psi = 0$ and the cohomology class of ψ thus is 0. It follows that $H^n(\mathscr{B}_c^{n-p,\bullet}(X), \bar{\partial})$, via (7.2), is a subset of the topological dual of $\widehat{\Omega}^p(X)$.

Let λ be a continuous linear functional on $\widehat{\Omega}^p(X)$. Then λ induces a continuous functional $\tilde{\lambda}$ on $\Omega^p(D)$ that has to be carried by some compact $K \Subset D$. By the Hahn–Banach theorem there is an $(N-p, N)$-current μ of order 0 in D with support in a neighborhood $U(K) \Subset D$ of K such that $\tilde{\lambda}(\tilde{f}) = \int \tilde{f} \wedge \mu$ for all $\tilde{f} \in \Omega^p(D)$. Now choose a weight $g(\zeta, z)$ with respect to $z \in U(K)$ that is holomorphic for $z \in U(K)$ and has compact support in D_ζ and let $p(\zeta, z) = \sum_k \tilde{p}_{\kappa+k}(\zeta, z) \wedge \omega_k(\zeta)$ be a corresponding integral kernel. We set

$$\check{\mathscr{P}}\mu := \sum_{k \geq 0} \omega_k(\zeta) \wedge \int_{D_z} \tilde{p}_{\kappa+k}(\zeta, z) \wedge \mu(z)$$

and observe that $\check{\mathscr{P}}\mu \in \mathscr{B}_c^{n-p,n}(X)$. Let $\varphi \in \widehat{\Omega}^p(X)$ and set $\tilde{\varphi} := \mathscr{P}\varphi$. Then $\tilde{\varphi} \in \Omega^p(U(K))$ by the choice of weight and moreover, $\tilde{\varphi}{\restriction_{U(K)\cap X}} = \varphi{\restriction_{U(K)\cap X}}$. We get

$$\lambda(\varphi) = \tilde{\lambda}(\tilde{\varphi}) = \int_{D_z} \tilde{\varphi} \wedge \mu = \int_{D_z} \mathscr{P}\varphi \wedge \mu = \int_{X_\zeta} \varphi \wedge \check{\mathscr{P}}\mu$$

and so λ is given by integration against $\check{\mathscr{P}}\mu \in \mathscr{B}_c^{n-p,n}(X)$. For more details of the last part of the proof see the proof of [38, Theorem 6.1]. □

7.3 Global Duality

Let us briefly recall how one can patch up the local duality to the global one of Theorem 1.9 using Čech cohomology; cf., e.g., [38, Sect. 6.2]. Let $\mathscr{U} := \{U_j\}$ be a locally finite open covering of X such that each U_j can be identified with an analytic subset of some pseudoconvex domain in some $\mathbb{C}^N$. In view of Theorem 1.3 and Corollary 1.4 this gives us a Leray covering for $\mathscr{F}\otimes\widehat{\Omega}_X^p$. Recall that spaces of sections of $\mathscr{F}\otimes\widehat{\Omega}_X^p$ have a standard Fréchet space structure. Let $C^k(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)$ be the group of formal sums

$$\sum_{i_0\cdots i_k}\varphi_{i_0\cdots i_k}U_{i_0}\wedge\cdots\wedge U_{i_k},\quad \varphi_{i_0\cdots i_k}\in\mathscr{F}\otimes\widehat{\Omega}^p(U_{i_0}\cap\cdots\cap U_{i_k}),$$

with the product topology; $U_{i_0}\wedge\cdots\wedge U_{i_k}$ is the formal exterior product of the symbols U_i with the suggestive formal computation rules, e.g., $U_1\wedge U_2=-U_2\wedge U_1$. Each element of $C^k(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)$ thus has a unique representation of the form $\sum_{i_0<\cdots<i_k}\varphi_{i_0\cdots i_k}U_{i_0}\wedge\cdots\wedge U_{i_k}$ that we will abbreviate as $\sum'_{|I|=k+1}\varphi_I U_I$. Let $\delta\colon C^k(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)\to C^{k+1}(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)$ be the coboundary operator

$$\delta\sum_{|I|=k+1}^{\prime}\varphi_I U_I:=\sum_{|I|=k+1}^{\prime}\varphi_I U_I\wedge\sum_j U_j=\sum_{|I|=k+1}^{\prime}\sum_j\varphi_I\restriction_{U_I\cap U_j}U_I\wedge U_j.$$

This operator is continuous, and we get the following complex of Fréchet spaces

$$0\to C^0(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)\xrightarrow{\delta}C^1(\mathscr{U},\mathscr{F}\otimes\widehat{\Omega}_X^p)\xrightarrow{\delta}\cdots. \tag{7.3}$$

The cohomology of this complex is the Čech cohomology of $\mathscr{F}\otimes\widehat{\Omega}_X^p$ with respect to the covering $\mathscr{U}$. Since $\mathscr{U}$ is a Leray covering, the q^{th} cohomology group is isomorphic to $H^q(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$. The standard topology on $H^q(X,\mathscr{F}\otimes\widehat{\Omega}_X^p)$ is defined so that the isomorphism is also a homeomorphism.

Let $\mathcal{B}_{n-p}^*$ be the precosheaf (see, e.g., [12, Sect. 3]) defined by assigning to each open $U\subset X$ the space $\mathcal{B}_{n-p}^*(U):=H^n\big(\mathscr{B}_c^{n-p,\bullet}(U,F^*),\bar\partial\big)$ and for $U'\subset U$ the inclusion map $i_U^{U'}:\mathcal{B}_{n-p}^*(U')\to\mathcal{B}_{n-p}^*(U)$ given by extension by 0. Let, for $k\geq 0$, $C_c^{-k}(\mathscr{U},\mathcal{B}_{n-p}^*)$ be the group of formal sums

$$\sum_{i_0\cdots i_k}[\psi_{i_0\cdots i_k}]_{\bar\partial}U_{i_0}^*\wedge\cdots\wedge U_{i_k}^*,\quad \psi_{i_0\cdots i_k}\in\mathscr{B}_c^{n-p,n}(U_{i_0}\cap\cdots\cap U_{i_k},F^*),$$

with the suggestive computation properties and only finitely many $[\psi_{i_0\cdots i_k}]_{\bar\partial}$ non-zero. Let $\delta^*\colon C_c^{-k}(\mathscr{U},\mathcal{B}_{n-p}^*)\to C_c^{-k+1}(\mathscr{U},\mathcal{B}_{n-p}^*)$ be the coboundary operator

$$\delta^*\sum_{|I|=k+1}^{\prime}[\psi_I]U_I^*:=\sum_j U_j\lrcorner\sum_{|I|=k+1}^{\prime}[\psi_I]U_I^*=\sum_{|I|=k+1}^{\prime}\sum_j i_{U_{I\setminus\{j\}}}^{U_I}[\psi_I]U_j\lrcorner U_I^*,$$

where $\lrcorner$ is formal interior multiplication. We get the complex

$$0\leftarrow C_c^0(\mathscr{U},\mathcal{B}_{n-p}^*)\xleftarrow{\delta^*}C_c^{-1}(\mathscr{U},\mathcal{B}_{n-p}^*)\xleftarrow{\delta^*}\cdots. \tag{7.4}$$

By Theorem 7.2, $C_c^{-k}(\mathscr{U}, \mathcal{B}^*_{n-p})$ is the topological dual of $C^k(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X)$ via the pairing $C^k(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X) \times C_c^{-k}(\mathscr{U}, \mathcal{B}^*_{n-p}) \to \mathbb{C}$ given by

$$(\varphi, [\psi]_{\bar\partial}) = \left(\sum_{|I|=k+1}' \varphi_I U_I, \sum_{|I|=k+1}' [\psi_I]_{\bar\partial} U_I^* \right) \mapsto \int_X \varphi \lrcorner \psi = \sum_{|I|=k+1}' \int_X \varphi_I \wedge \psi_I. \tag{7.5}$$

Moreover, if $\varphi \in C^{k-1}(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X)$ and $[\psi] \in C_c^{-k}(\mathscr{U}, \mathcal{B}^*_{n-p})$, then

$$\int_X \varphi \lrcorner \delta^* \psi = \int_X \varphi \lrcorner \left(\sum_j U_j \lrcorner \psi \right) = \int_X \left(\varphi \wedge \sum_j U_j \right) \lrcorner \psi = \int_X \delta\varphi \lrcorner \psi$$

and so (7.4) is the dual complex of (7.3). It follows, see, e.g., [36, Lemme 2], that

$$\mathrm{Ker}\big(\delta^* \colon C_c^{-k}(\mathscr{U}, \mathcal{B}^*_{n-p}) \to C_c^{-k+1}(\mathscr{U}, \mathcal{B}^*_{n-p})\big) / \overline{\delta^* C_c^{-k-1}(\mathscr{U}, \mathcal{B}^*_{n-p})} \tag{7.6}$$

is the topological dual of

$$\mathrm{Ker}\big(\delta \colon C^k(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X) \to C^{k+1}(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X)\big) / \overline{\delta C^{k-1}(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}^p_X)}. \tag{7.7}$$

If $H^k(X, \mathscr{F} \otimes \widehat{\Omega}^p_X)$ and $H^{k+1}(X, \mathscr{F} \otimes \widehat{\Omega}^p_X)$ are Hausdorff, then the closure signs in (7.6) and (7.7) are superfluous and so $H^{-k}(C_c^\bullet(\mathscr{U}, \mathcal{B}^*_{n-p}), \delta^*)$ is the topological dual of $H^k(X, \mathscr{F} \otimes \widehat{\Omega}^p_X)$, via the pairing induced by (7.5); cf., e.g., [38, Lemma 6.4].

Proof of Theorem 1.9 Consider the double complex

$$K^{-i,j} := C_c^{-i}(\mathscr{U}, \mathscr{B}_c^{n-p,j}).$$

Here $\mathscr{B}_c^{n-p,j}$ is the precosheaf $U \mapsto \mathscr{B}_c^{n-p,j}(U, F^*)$ with inclusion maps given by extending by 0, the map $K^{-i,j} \to K^{-i+1,j}$ is δ^*, and the map $K^{-i,j} \to K^{-i,j+1}$ is $\bar\partial$.

For each $i \geq 0$ the "row" $K^{-i,\bullet}$ is, by Theorem 7.2, exact except at the level n where the cohomology is $C_c^{-i}(\mathscr{U}, \mathcal{B}^*_{n-p})$. For each $j, 0 \leq j \leq n$, the "column" $K^{\bullet,j}$ is exact except at the level 0 where the cohomology is $\mathscr{B}_c^{n-p,j}(X, F^*)$; this follows from, e.g., [38, Lemma 6.3] since the $\mathscr{B}_X$-sheaves are fine. Hence, by standard homological algebra, e.g., a spectral sequence argument, it follows that

$$H^{n-q}\big(\mathscr{B}_c^{n-p,\bullet}(X, F^*), \bar\partial\big) \simeq H^{-q}\big(C_c^\bullet(\mathscr{U}, \mathcal{B}^*_{n-p}), \delta^*\big). \tag{7.8}$$

Explicitly, if $\psi \in \mathscr{B}_c^{n-p,n-q}(X, F^*)$ is $\bar\partial$-closed and $\{\chi_j\}_j$ is a partition of unity subordinate to $\mathscr{U}$, then its image in $H^{-q}\big(C_c^\bullet(\mathscr{U}, \mathcal{B}^*_{n-p}), \delta^*\big)$ is the class of

$$\sum_{|I|=q+1}' [\chi_{i_0}\bar{\partial}\chi_{i_1} \wedge \cdots \wedge \bar{\partial}\chi_{i_q} \wedge \psi]_{\bar{\partial}} U_I^*.$$

In view of (7.8) and the paragraph before this proof, there is a non-degenerate pairing

$$H^{n-q}\big(\mathscr{B}_c^{n-p,\bullet}(X, F^*), \bar{\partial}\big) \times H^q(X, \mathscr{F} \otimes \widehat{\Omega}_X^p) \to \mathbb{C}. \tag{7.9}$$

Therefore, by Corollary 1.4, there is a non-degenerate pairing (1.2).

The pairing (7.9) is induced by (7.5) via the isomorphisms (7.8) and $H^q(X, \mathscr{F} \otimes \widehat{\Omega}_X^p) \simeq H^q(C^\bullet(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}_X^p, \delta)$. One can use the explicit description of (7.8) and a similar explicit description of $H^q(\mathscr{A}^{p,\bullet}(X, F), \bar{\partial}) \simeq H^q(C^\bullet(\mathscr{U}, \mathscr{F} \otimes \widehat{\Omega}_X^p, \delta)$ to show that the pairing (1.2) is given by (1.3). This is done in the proof of [38, Theorem 1.3] in the case $p = 0$. It is straightforward to adapt that proof to the general situation $0 \leq p \leq n$ and we omit the details. □

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