Thesis for the Degree of Doctor of Philosophy

# Random Walk Boundaries: Their Entropies and Connections with Hecke Pairs

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CHALMERS | UNIVERSITY OF GOTHENBURG

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## Abstract

We present three papers in non-singular dynamics concerning boundaries of random walks on locally compact, second countable groups. One common theme is entropy. Paper II and III are concerned with boundary entropy spectra, while Paper I studies topological properties of entropy. In Paper II we moreover establish a technique to relate random walks on locally profinite groups to random walks on dense discrete subgroups, by the concept of Hecke pairs, which is also used in Paper III.

In Paper I we introduce different perspectives and extensions of Furstenberg's entropy and show semi-continuity and continuity results in these contexts. In particular we apply these to upper and lower limits of non-nested sequences of $\sigma$-algebras in the sense of Kudo.

Paper II relates certain random walks on locally profinite groups to random walks on dense discrete subgroups, using a Hecke subgroup, such that the Poisson boundary of the first becomes a boundary of the second one. If the Poisson boundaries of these two walks happen to coincide, then the Hecke subgroup in charge has to be amenable. For some random walks on lamplighter and solvable Baumslag-Solitar groups we obtain that their Poisson boundary is prime and the quasi-regular representation is reducible. Moreover, we find a group such that for any given summable sequence of positive numbers there is a random walk whose boundary entropy spectrum equals the subsum set of this sequence. In particular we obtain a boundary entropy spectrum which is a Cantor set and one which is an interval.

In Paper III we study the boundary entropy spectra of finitely supported, generating random walks on a certain affine group, realizing them as finite subsum sets. We show that the averaged information function of a stationary probability measure does not change when passing to a non-singular, absolutely continuous $\sigma$-finite measure and deduce an entropy formula.

**Key words and phrases:** Non-singular dynamical systems, random walks on groups, Poisson boundary, Furstenberg entropy, Hecke pairs, Schlichting completion, non-monotone sequences of $\sigma$-algebras.

# List of appended papers

The following three papers are appended to the thesis:

I Michael Björklund, Yair Hartman, Hanna Oppelmayer,
*Kudo-Continuity of Entropy Functionals.*
preprint.

II Michael Björklund, Yair Hartman, Hanna Oppelmayer,
*Random Walks on Dense Subgroups of Locally Compact Groups.*
preprint.

III Hanna Oppelmayer,
*Boundary Entropy Spectra as Finite Subsums.*
submitted to *Stochastics and Dynamics* (in revision).

Contributions

I All authors wrote parts of the paper, the main structure is due to Michael. I helped developing the proofs, discussed alternative directions with Yair.

II I took part in the development of the project, worked on proofs and wrote several pre-versions of the paper.

III I worked out the proofs, structured the paper and did all the writing.

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Hanna Oppelmayer
Göteborg, September 22, 2020

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# Part I

# Introduction

# Chapter 1

# Introduction

Random walks on topological groups are important objects in Probability Theory and Dynamics as well as in Group Theory. Understanding the behaviour of a generating random walk can give rise to purely group theoretical properties and vice versa. In the early sixties Furstenberg introduced in [14] the concept of *boundaries* for random walks on locally compact, second countable groups and showed that there always exists a (unique) maximal one, which he called *Poisson boundary*, due to its useful connection to harmonic functions.

In this thesis we present three papers with applications on qualitative and quantitative questions of boundaries, in particular we

- find cases where there are no non-trivial boundaries (primeness),
- prove reducibility of the quasi-regular representation of some Poisson boundaries,
- design various boundary entropy spectra,
- relate (Poisson) boundaries of some random walks on locally profinite groups with (Poisson) boundaries of random walks on dense discrete subgroups,
- show continuity statements for some entropy functions.

In Paper I we study notions of entropy from various perspectives and their behaviour with limits. In particular we make use of Kudo's notion of upper and lower limits for $\sigma$-algebras (introduced 1974 in [23]) and prove lower and upper semi-continuity for an extended version of Furstenberg's entropy for boundaries. This was motivated by the still widely open question about the shape of the boundary entropy spectrum, which we address in Paper II and Paper III for specific groups. A construction of a random walk attaining any desired entropy value is included in Paper II. Even more, we realize the collection of all boundary entropy values as the subsum set of any given summable positive sequence. Besides, Paper II links certain random walks on discrete groups to random walks on some locally profinite groups, which allows us to deduce primeness and reducibility statements for some lamplighter and Baumslag-Solitar groups.

The thesis is organised as follows. First we introduce the main objects focusing on concepts that are relevant for the appended papers, boundaries, primeness and the quasi-regular representation of the Poisson boundary. In Chapter 3 we relate a random walk of a discrete group to one of a totally disconnected, locally compact group by fixing a so-called Hecke subgroup and require that the random walk in charge “absorbs” this subgroup (see Definition 3.1). The Poisson boundaries of such random walks are nicely related which we use in several proofs in Paper II and Paper III. The succeeding chapter is devoted to various notions of entropy, preparing the reader for Paper I. Finally, we explain the main ideas of Paper II in a concrete example, namely the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$, in Chapter 5.

# Chapter 2

# Preliminaries

In this chapter we introduce the main objects of study for the present thesis. Basic knowledge of Measure Theory is assumed and we may refer the reader to [5] for an extensive introduction to the theory in concern.

## 2.1 Random walks on groups and their boundaries

A *topological group* is a group $G$ equipped a Hausdorff topology such that the group operations, multiplication of two elements and taking inverses, are continuous, i.e. $(g_1,g_2) \mapsto g_1 g_2^{-1}$ as a map from $G \times G$ to $G$ is jointly continuous w.r.t. the product topology on $G \times G$. The $\sigma$-algebra $\mathcal{B}(G)$ generated by the open subsets of a topological group is called *Borel-$\sigma$-algebra*. This shall always be the $\sigma$-algebra we refer to when talking about measurable subsets of a group. We will only study topological groups which are locally compact and second countable.

A *random walk* of a topological group $G$ is a probability measure $\mu$ on the Borel-$\sigma$-algebra of $G$. From a probabilistic point of view a random walk can be interpreted as the distribution of a sequence of identically distributed independent $G$-valued random variables $\{V_n\}_{n\in\mathbb{N}}$. If $G$ is second countable, one can take $(G^{\mathbb{N}},\mu^{\otimes\mathbb{N}})$ as underlying probability space for the random variables and define

$$V_n : G^{\mathbb{N}} \longrightarrow G$$

to be the projection to the $n$-th coordinate. The random product

$$U_n := V_1 \cdot \ldots \cdot V_n$$

models the "path" (i.e. multiplication of group elements) of the random process after $n$ steps following the law $\mu^{*n}$. Note that the sequence $\{U_n\}$

forms a Markov process. Heuristically, a *boundary* of a random walk $\mu$ is a space $X$ such that the random products $U_n$ "converge" in $G \cup X$ for $\mu^{\otimes \mathbb{N}}$-almost every point in $G^{\mathbb{N}}$ in a way that the limit preserves the group action, which we will make more precise subsequently. Also we refer to Furstenberg's paper [15] for a detailed description of this approach.

A measurable space $(X,\mathcal{A})$ is called *$G$-space* if $G$ acts on $X$ measurably, i.e. the map $(g,x) \mapsto gx$ is $\mathcal{A} \otimes \mathcal{B}(G)$-$\mathcal{A}$-measurable. Whenever $X$ is a topological space, the $\sigma$-algebra $\mathcal{A}$ is set to be the Borel-$\sigma$-algebra $\mathcal{B}(X)$ on $X$ and will mostly be omitted in the notations. A measure $\nu$ on $\mathcal{A}$ is called *quasi-invariant* if the image measure $g\nu$ and $\nu$ share the same null-sets for every $g \in G$. In this case we say that the space is *non-singular*. The measure is called *$K$-invariant* for a subgroup $K$ of $G$ if $k\nu = \nu$ for all $k \in K$.

A random walk boundary will be a certain topological $G$-space $X$ equipped with a probability measure, the so-called *hitting measure*. If $V_1 \cdot \ldots \cdot V_n$ converges to a $X$-valued random variable $W =: W_1$ then $V_k \cdot \ldots \cdot V_{k+n}$ converges too to some $X$-valued random variable $W_k$ for every $k \in \mathbb{N}$. It is natural to assume that the limit "preserves" the path of the random walk in the sense that

$$W_i = V_i \cdot W_{i+1}$$

for every $i \in \mathbb{N}$, and further that all $W_i$ have the same distribution. This implies in particular that $\nu = \mu * \nu$ when $\nu$ denotes the probability measure on $\mathcal{B}(X)$ which is the distribution of each $W_i$. Hence the boundaries we consider are probability spaces which lie in the following category.

**Definition 2.1.** A probability measure $\nu$ on a $G$-space $(X,\mathcal{A})$ is called *$\mu$-stationary* if

$$\nu = \mu * \nu$$

i.e. $\nu(A) = \int_G \nu(g^{-1}A)\, d\mu(g)$ for all $A \in \mathcal{A}$.

If $G$ is a locally compact, second countable group acting measurably on a standard space $X$, then there is a compact space $X'$ on which $G$ acts continuously such that $(X,\mathcal{B}(X))$ is $G$-equivariant isomorphic to $(X',\mathcal{B}(X'))$, i.e. there is a bijective Borel-measurable map $\phi : X \longrightarrow X'$ whose inverse is Borel-measurable too and such that $\phi(gx) = g\phi(x)$ for all $g \in G$, $x \in X$. (Proven for general Polish groups by Bekker-Kechris in [4].) The $G$-space $(X',\mathcal{B}(X'))$ is called a *compact $G$-model* of the $G$-space $(X,\mathcal{B}(X))$. Similar if $X$ is equipped with a probability measure, then $(X',\nu')$ is a compact model of $(X,\nu)$ if $X'$ is a compact model of $X$ with $G$-equivariant isomorphism $\phi$ on co-null-sets such that $\nu' = \nu \circ \phi^{-1}$.

We will from now on assume that $G$ is locally compact and second countable and when we consider actions on a compact space we shall always mean continuous actions. One can show (see e.g [15, Corollary 3.1] or [13, Lemma 1.33]) that for a compact, second countable Hausdorff space $X$ equipped with a $\mu$-stationary probability measure $\nu$ the weak-*-limit of the image measures

$$\lim_{n\to\infty} U_n(\omega)\nu =: \nu_\omega$$

exists for $\mu^{\otimes\mathbb{N}}$-a.e. $\omega \in G^{\mathbb{N}}$. Moreover, $\int_{G^{\mathbb{N}}} \nu_\omega \, d\mu^{\otimes\mathbb{N}}(\omega) = \nu$.

**Definition 2.2.** A standard probability $G$-space $(X,\nu)$ is called a *$\mu$-boundary* if $\nu$ is $\mu$-stationary and there exists a compact $G$-model $(X',\nu')$ of $(X,\nu)$ such that

$$\lim_{n\to\infty} U_n(\omega)\nu' = \delta_{W(\omega)}$$

with $W(\omega) \in X'$ for $\mu^{\otimes\mathbb{N}}$-a.e. $\omega \in G^{\mathbb{N}}$.

Note that the one-point space $(\{x\},\delta_x)$ with $gx = x$, $\forall g \in G$ is always a $\mu$-boundary. Such a $G$-space is called *trivial*.

The map $W : G^{\mathbb{N}} \to X'$ from the above definition is Borel-measurable hence a $\nu$-distributed random variable, so we get $\lim U_n(\omega) = W(\omega)$ for $\mu^{\otimes\mathbb{N}}$-a.e. $\omega \in G^{\mathbb{N}}$ in the path-model of the random walk, when putting the weakest topology on $G \cup X$ such that the identity maps $G \longrightarrow G \cup X$ and $X \longrightarrow G \cup X$ are continuous and the maps $g \mapsto g\nu$ and $x \mapsto \delta_x$ from $G \cup X$ to the space of measures on $X$ are continuous. (See [15], Definition 3.3.)

## 2.2 The maximal boundary: The Furstenberg-Poisson boundary

A standard probability $G$-space $(Y,\mathcal{Y},\eta)$ is called a *$G$-factor* of a non-singular $G$-space $(X,\mathcal{B},\nu)$, denoted by

$$\begin{array}{c} G \curvearrowright (X,\mathcal{B},\nu) \\ \big\downarrow \\ G \curvearrowright (Y,\mathcal{Y},\eta), \end{array}$$

if there exists a measurable $G$-equivariant map $\phi : X_0 \longrightarrow Y_0$ such that $\nu \circ \phi^{-1} = \eta$ where $X_0$, $Y_0$ are $G$-invariant measurable sets with $X_0 = X$ mod $\nu$ and $Y_0 = Y$ mod $\eta$. It is not hard to see that every non-singular $G$-factor of a non-singular $\mu$-boundary is a $\mu$-boundary again. Let us put some regularity constraints on the measure $\mu$ such that every $\mu$-stationary probability measure is non-singular, namely we shall from now on assume that $\mu$ is *generating*, i.e. the semi-group generated by the support of $\mu$ equals the whole group $G$.

**Definition 2.1.** The *Furstenberg-Poisson boundary* of $(G,\mu)$, denoted by $\mathrm{Poi}(G,\mu)$ is a $\mu$-boundary of $G$ such that every other $\mu$-boundary is a $G$-factor of the latter.

The existence and uniqueness (up to $G$-isomorphy) of such a "maximal" boundary for locally compact, second countable measured groups is due to Fursetenberg, see [15, Theorem 3.1] for a proof. Moreover, the Poisson boundary has the useful property that it characterizes the bounded $\mu$-harmonic functions on the group $G$ as $L^\infty$-function on the $G$-space Poi($G$,$\mu$).

**Definition 2.2.** A measurable function $h : G \longrightarrow \mathbb{R}$ is called *$\mu$-harmonic* if

$$h(g) = \int_G h(gg')\, d\mu(g')$$

for every $g \in G$. The set of all bounded $\mu$-harmonic functions on $G$ is denoted by $\mathcal{H}^\infty(G,\mu)$.

The set $\mathcal{H}^\infty(G,\mu)$ is a Banach space w.r.t. pointwise addition and the sup-norm. It even becomes a Banach algebra when quipped with the multiplication

$$h_1 \boxtimes h_2(g) := \lim_{n\to\infty} \int_G h_1(gg')h_2(gg')\, d\mu^{*n}(g') \tag{2.1}$$

for $h_1,h_2 \in \mathcal{H}^\infty(G,\mu)$. Let us moreover assume that $\mu$ is *spread-out*, which means that there exists a convolution power $k \in \mathbb{N}$ such that $\mu^{*k}$ is absolutely continuous w.r.t. a left-Haar measure $m_G$ (which exists due to locally compactness of $G$). Then every bounded $\mu$-harmonic function is (right-uniformly) continuous (see e.g. [1, Lemma 1.2]).

Given a $\mu$-stationary probability $G$-space $(X,\nu)$ the so-called *Poisson transform*

$$P_\nu : L^\infty(X,\nu) \longrightarrow \mathcal{H}^\infty(G,\mu)$$

defined by

$$P_\nu(f)(g) := \int_X f(gx)\, d\nu(x)$$

is a unital, positive, linear function, which is $G$-equivariant w.r.t. the right action on the function spaces, given by $gf(x) = f(gx)$ for $g \in G$, $f \in L^\infty(X,\nu)$, $x \in X$ and analogous of $\mathcal{H}^\infty(G,\mu)$. If $(X,\nu)$ is a compact $\mu$-boundary then $P$ is an isometry and multiplicative. Moreover, if $(X,\nu)$ is the Poisson boundary of $(G,\mu)$ then $P$ becomes an algebra isomorphism, i.e.

$$L^\infty(X,\nu) \cong \mathcal{H}^\infty(G,\mu).$$

In particular, the Poisson boundary is trivial if and only if the only bounded $\mu$-harmonic functions are the constant ones. This is for example the case whenever $G$ is a finitely generated virtually nilpotent group (e.g. abelian) for any generating probability measure $\mu$. The converse direction is proven by Frisch-Hartman-Tamuz-Ferdowsi in [12].

Let us close this section by mentioning that the connection to harmonic functions seems natural, when looking for limits of the random products $V_1 \cdot \ldots \cdot V_n$, since for every $h \in \mathcal{H}^\infty(G,\mu)$ the limit $\lim_{n\to\infty} h(gV_1(\omega)\cdot\ldots\cdot V_n(\omega))$ exists for every $g \in G$ and $\mu^{\mathbb{N}}$-a.e. $\omega \in G^{\mathbb{N}}$. We refer to [15] or [13] for details and proofs of all the above claims.

## 2.3 Prime boundaries

Let $\mu$ be a generating, spread-out probability measure on a locally compact, second countable group $G$. It is a natural question to ask whether $G$ has *non-trivial* $\mu$-boundaries, i.e. $\mu$-boundaries which are not $G$-isomorphic to a one-point $G$-space or to the Poisson boundary. If there are none, we call $(G,\mu)$ *prime*. Moreover, we shall call a $\mu$-boundary $(X,\nu)$ *prime* if it has up to $G$-isomorphy only the one-point $G$-space and the space $(X,\nu)$ itself as $G$-factors.

If $N$ is a closed normal subgroup of $G$ and $\alpha_N : G \longrightarrow G/N$ denotes the canonical map, then the Poisson boundary of $(G/N,\mu \circ \alpha_N^{-1})$ is a $G$-factor of the Poisson boundary of $(G,\mu)$, i.e. a $\mu$-boundary of $G$. If $N$ is amenable then $\mathrm{Poi}(G/N,\mu \circ \alpha_N^{-1})$ is $G$-isomorphic to $\mathrm{Poi}(G,\mu)$. A certain converse statement is true as well, shown by Kaimanovich in [18]. In Paper II, Corollary 1.9, we will give a to some extend related statement for a so-called *almost normal* or *Hecke* subgroup (defined in Chapter 3) in a certain setting. Of interest might as well be Corollary 1.12 in Paper II on this topic.

If $G$ is a discrete group with Kazhdan's Property (T) such that $\mathrm{Poi}(G,\mu)$ is non-trivial and has finite entropy (see Chapter 4 for a definition) (e.g. if $\mathrm{supp}(\mu)$ is finite), then there exists at least one $\mu$-boundary (which might be $\mathrm{Poi}(G,\mu)$ itself) which is prime. This is due to Nevo, [26, Theorem 4.3]. In Paper II, Corollary 1.10, we will show that there is a random walk on a free group of finite rank which acts essentially free on a prime boundary. Moreover, we will construct random walks on the lamplighter group and on solvable Baumslag-Solitar groups such that they become prime measured groups, see Corollary 1.16 in Paper II.

## 2.4 The quasi-regular representation

Let $G$ be a locally compact, second countable group acting non-singularly on a standard probability space $(X,\nu)$. Fix $p \in [1,\infty)$. We consider the action of $G$ on $L^p(X,\nu)$ defined by $gf(x) := f(g^{-1}x)$ for $f \in L^p(X,\nu)$, $g \in G$, $x \in X$. The *quasi-regular representation* on $L^p(X,\nu)$ (also called *Koopman representation* for $p = 2$)

$$\pi : G \longrightarrow \mathcal{U}(L^p(X,\nu))$$

is given by

$$\pi(g)(f)(x) := \left(\frac{dg\nu}{d\nu}(x)\right)^{\frac{1}{p}} f(g^{-1}x)$$

for $f \in L^p(X,\nu)$, $x \in X$, $g \in G$, where $\mathcal{U}(L^p(X,\nu))$ denotes the group of norm-preserving continuous invertible linear operators on the Banach space $(L^p(X,\nu), \|\cdot\|_p)$, in particular $\|\pi(g)f\|_p = \|f\|_p$ for every $g \in G$, $f \in L^p(X,\nu)$.

The representation is called *irreducible* if it is non-trivial and cannot be decomposed into $G$-invariant closed subspaces, i.e. there is no closed subspace $V \subseteq L^p(X,\nu)$ such that $\pi(g)V \subseteq V$ for all $g \in G$. Otherwise the representation is called *reducible*. Thus the quasi-regular representation on $L^p(X,\nu)$ is reducible iff there exists a function $f \in L^p(X,\nu)$ such that $\overline{\pi(G)\operatorname{span}(f)}$ is a proper subspace of $L^p(X,\nu)$.

Of special interest is the case when $(X,\nu)$ is the Poisson boundary of $(G,\mu)$. Bader-Muchnik proved in [2] that for a certain class of measured groups $(G,\mu)$, the quasi-regular representation on $L^2(\operatorname{Poi}(G,\mu))$ is irreducible. It was there conjectured that this might hold for all locally compact groups with admissible probability measure, which we disprove by a concrete example in Paper II, Corollary 1.16.

# Chapter 3

# From discrete to locally profinite groups

Let $\Gamma$ be a countable infinite discrete group. We will equip $\Gamma$ with a random walk $\tau$ in such a way that we can relate its boundaries to boundaries of a totally disconnected, locally compact, second countable (*t.d.l.c.s.c.*) group $H$, also called *locally profinite* group, in which $\Gamma$ can be densely embedded. This can be done in two ways: either by fixing an open, compact subgroup in the t.d.l.c.s.c. group, or by considering a so-called Hecke subgroup of the discrete group and obtain the t.d.l.c.s.c. group as the Schlichting completion w.r.t. this subgroup.

## 3.1 Hecke absorbing random walks

Let $H$ be a t.d.l.c.s.c. group such that there exists a homomorphism

$$\rho : \Gamma \longrightarrow H$$

with dense image in $H$. Further let $L$ be an open, compact subgroup of $H$ (which always exists in t.d.l.c. groups, by van Dantzig's theorem). We set $\Lambda := \rho^{-1}(L)$. Then $(\Gamma,\Lambda)$ is a so-called *Hecke pair*, which means that the $\Lambda$-orbits on the coset space $\Gamma/\Lambda$ are finite. (Alternatively, we may start with a Hecke pair $(\Gamma,\Lambda)$ and set $(H,L)$ to be its *Schlichting completion*, see e.g. [22] for a construction of this completion.) We will call $\Lambda$ a *Hecke subgroup* of $\Gamma$. This in particular enables us to "uniformize" a given random walk on $\Gamma$ along these orbits, which is done in Theorem 1.4 (III) (ii) in Paper II for instance. The resulting measure $\tau$ has the important property that it is $\Lambda$-invariant on elements of the coset space, i.e.

$$\tau(\lambda\gamma\Lambda) = \tau(\gamma\Lambda), \; \forall \lambda \in \Lambda \tag{3.1}$$

for every $\gamma\Lambda \in \Gamma/\Lambda$. Probability measures which fulfill the above equation will be a central object for our studies.

**Definition 3.1.** Let $(\Gamma,\Lambda)$ be a Hecke pair. A probability measure $\tau$ on $\Gamma$ is called $\Lambda$-*absorbing* if it fulfills 3.1.

Every $\Lambda$-absorbing probability measure $\tau$ on $\Gamma$ gives rise to a bi-$L$-invariant probability measure $\theta_\tau$ on $H$ as follows. First note that there is a bijection

$$\psi : \Gamma/\Lambda \longrightarrow H/L$$

given by $\gamma\Lambda \mapsto \rho(\gamma)L$ (confer [22, Proposition 3.9]). Let $\overline{\tau}$ denote the probability measure on $\Gamma/\Lambda$ given by $\overline{\tau}(\{\gamma\Lambda\}) := \tau(\gamma\Lambda) = \sum_{\lambda\in\Lambda}\tau(\gamma\lambda)$, i.e. $\overline{\tau} := \alpha\tau$ for $\alpha : \Gamma \longrightarrow \Gamma/\Lambda$ being the canonical map $\gamma \mapsto \gamma\Lambda$. We set

$$\theta_\tau(f) := \int_{H/L}\int_L f(hl)\, dm_L(l)\, d\psi\overline{\tau}(hL) \tag{3.2}$$

for every $f \in C_c(H)$, where $m_L$ denotes the Haar probability measure on $L$. Due to left-$L$-invariance of $m_L$, the above quantity is well-defined and due to right-$L$-invariance of $m_L$ the measure $\theta_\tau$ is right-$L$-invariant. Since $\tau$ is $\Lambda$-absorbing, and $\Lambda$ is dense in $L$ one sees that $\theta_\tau$ is left-$L$-invariant. Moreover, this construction is optimal in the sense that every bi-$L$-invariant probability measure $\theta$ on $H$ can be obtained by 3.2 for some $\Lambda$-absorbing probability measure $\tau$, i.e. $\theta = \theta_\tau$. Moreover, if $\tau$ is generating, then $\theta_\tau$ is admissible (see Paper II, Theorem 1.6). We shall assume from now on that $\tau$ is generating.

A crucial feature of the above construction is that for every $L$-invariant probability measure $\nu$ on an $H$-space $X$ we obtain

$$\theta_\tau * \nu = \tau * \nu$$

where we define the action of $\Gamma$ on $X$ by the map $\rho$. In particular if $\nu$ is $\theta_\tau$-stationary – which implies $L$-invariance – then $\nu$ has to be $\tau$-stationary. We show in Corollary 1.9 in Paper II even more, namely that every $\theta_\tau$-boundary is a $\tau$-boundary. In particular, Poi$(H,\theta_\tau)$ (viewed as a $\Gamma$-space) is a $\Gamma$-factor of Poi$(\Gamma,\tau)$,

$$\begin{array}{c} \Gamma \curvearrowright \mathrm{Poi}(\Gamma,\tau) \\ \big\downarrow \\ \Gamma \curvearrowright \mathrm{Poi}(H,\theta_\tau). \end{array}$$

# Chapter 4

# Entropy

Let $G$ be a l.c.s.c. group equipped with an admissible probability measure $\mu$. Let $(X,\mathcal{B},\nu)$ be a non-singular standard probability $G$-space, for example a $\mu$-boundary of $G$. We shall consider different notions of entropy of such spaces and related quantities.

## 4.1 Furstenberg's entropy

We are interested in non-$G$-invariant probability spaces, since $\mu$-boundaries which are $G$-invariant are $G$-isomorphic to the trivial one-point space. One way to quantify such spaces is thus by "how far away" they are from being invariant. To this end we look at the Radon-Nikodym-derivatives $\frac{dg^{-1}\nu}{d\nu}$ for every $g \in G$. They are constant 1 for all $g \in G$ if and only if $\nu$ is $G$-invariant. Furstenberg defined in [14] the *entropy* of $(X,\nu)$ as

$$h_\mu(X,\nu) := \int_G \int_X -\log\left(\frac{dg^{-1}\nu}{d\nu}(x)\right) d\nu(x)\, d\mu(g)$$

whenever the integrals are defined, i.e. the functions in charge are in $L^1(X,\nu)$ and $L^1(G,\mu)$, respectively. By Jensen's inequality this quantity is always non-negative. Clearly it attains zero if the measure $\nu$ is $G$-invariant.

Let us consider for now the case that $G$ is discrete and $(X,\nu)$ is a $\mu$-boundary with finite entropy. Kaimanovich-Vershik proved in [20, Theorem 1.1], a converse of the above statement, namely zero entropy implies that $(X,\nu)$ is a trivial one-point space. Moreover, they showed that the maximal entropy value among all boundaries is precisely attained at the Poisson boundary. In this case it equals the *random walk entropy*

$$h_{RW}(G,\mu) := \inf_{n\in\mathbb{N}} -\frac{1}{n} \sum_{\gamma\in\Gamma} \mu^{*n}(\gamma) \log(\mu^{*n}(\gamma)).$$

So we have

$$0 = h_\mu(\{*\},\delta_*) \leq h_\mu(X,\nu) \leq h_\mu(\mathrm{Poi}(G,\mu)) = h_{RW}(G,\mu)$$

for any $\mu$-boundary $(X,\nu)$. It is a natural question to ask which real values can be attained as the entropy of some $\mu$-boundary. In particular the following subset of $\mathbb{R}_{\geq 0}$, called the *boundary entropy spectrum*

$$\mathrm{BndEnt}(G,\mu) := \{h_\mu(X,\nu) \,:\, (X,\nu) \text{ a } \mu\text{-boundary}\}$$

will be studied in this thesis. The collection of the entropy values of all $\mu$-stationary, ergodic $G$-spaces $\mathrm{Ent}(G,\mu)$ – which clearly contains $\mathrm{BndEnt}(G,\mu)$ – has been intensively studied, i.e. by [6], [17], [7], [27], [26] and many more. One reason why it is an interesting object to understand is that a *full entropy realization*, i.e. $\mathrm{Ent}(G,\mu) = [0,h_{RW}(G,\mu)]$ implies a purely group theoretical property, namely that the group cannot have Property (T).

## 4.2 A different point of view: From the space of sub-$\sigma$-algebras

Let $(X,\mathcal{B},\nu)$ denote the Poisson boundary of $(G,\mu)$ and let us assume that $\mathcal{B}$ is complete, i.e. contains all subsets of $\nu$-null-sets and let $L^1(X,\mathcal{B},\nu)$ be separable. Every $\mu$-boundary $(Y,\mathcal{Y},\eta)$ is a $G$-factor of $(X,\mathcal{B},\nu)$ and as such it can be represented as a $G$-invariant sub-$\sigma$-algebra $\mathcal{A}$ of $\mathcal{B}$, namely $\mathcal{A} = \phi^{-1}(\mathcal{Y})$ (mod $\nu$), where $\phi$ denotes the factor map. Restricting our considerations to sub-$\sigma$-algebras which contain all $\nu$-null sets gives a unique assignment of a $G$-invariant sub-$\sigma$-algebra for every $\mu$-boundary. Let us denote

$$\Sigma_\mathcal{B} := \{\sigma(\mathcal{A} \cup \mathcal{N}) \,:\, \mathcal{A} \text{ a sub-}\sigma\text{-algebra of } \mathcal{B}\}$$

where $\mathcal{N} := \{N \in \mathcal{B} \,:\, \nu(N) = 0\}$. The reverse direction of obtaining a $G$-factor when given a $G$-invariant sub-$\sigma$-algebra of $\mathcal{B}$ is given by Mackey's point realization [24]. Thus we can identify $\mu$-boundaries (up to isomorphic spaces) with $G$-invariant elements of $\Sigma_\mathcal{B}$, i.e.

$$\{(Y,\mathcal{Y},\nu)\ \mu\text{-boundaries}\}/\mathrm{iso} \longleftrightarrow \{\mathcal{A} \in \Sigma_\mathcal{B}\ G\text{-invariant}\}.$$

Further we will observe that the Furstenberg entropy of a $\mu$-boundary $(Y,\eta)$ can be expressed in terms of the corresponding sub-$\sigma$-algebra $\mathcal{A} \in \Sigma_\mathcal{B}$. For every $g \in G$ the Radon-Nikodym derivative $\frac{dg^{-1}\eta}{d\eta}$ can be formulated as the conditional expectation of $\frac{dg^{-1}\nu}{d\nu}$ w.r.t $\mathcal{A}$ by

$$\frac{dg^{-1}\eta}{d\eta} \circ \phi = \mathbb{E}_\nu\Big[\frac{dg^{-1}\nu}{d\nu}\Big|\mathcal{A}\Big] \quad \eta\text{-a.e.}$$

where $\phi$ denotes the factor map (see e.g. [27, Lemma 1.6 (2)]). Therefore we can rewrite the inner integral of Furstenberg's entropy – which is referred to as *Kullback-Leibler divergence* in some contexts – as follows. For every $g \in G$

$$\int_Y -\log\Big(\frac{dg^{-1}\eta}{d\eta}(y)\Big)\,d\eta(y) = \int_X -\log\Big(\mathbb{E}_\nu\Big[\frac{dg^{-1}\nu}{d\nu}\Big|\mathcal{A}\Big](x)\Big)\,d\nu(x)$$

since $\eta = \nu \circ \phi^{-1}$. In particular

$$h_\mu(Y,\eta) = \int_G\int_X -\log\Big(\mathbb{E}_\nu\Big[\frac{dg^{-1}\nu}{d\nu}\Big|\mathcal{A}\Big](x)\Big)\,d\nu(x)\,d\mu(g).$$

Having this in mind we re-define Furstenberg's entropy as a function on $\Sigma_{\mathcal{B}}$. For $G$-invariant sub-$\sigma$-algebras this will just be the usual Furstenberg entropy of the corresponding $G$-factors.

**Definition 4.1.** For $\mathcal{A} \in \Sigma_{\mathcal{B}}$ we set

$$h_\mu(\mathcal{A}) := \int_G\int_X -\log\Big(\mathbb{E}_\nu\Big[\frac{dg^{-1}\nu}{d\nu}\Big|\mathcal{A}\Big](x)\Big)\,d\nu(x)\,d\mu(g)$$

when the integrals are defined.

The advantage of this point of view is that the entropy becomes a continuous function when we equip the space of sub-$\sigma$-algebras of $\mathcal{B}$ with the *strong topology* defined by

$$\mathcal{A}_n \to \mathcal{A} \ \text{ iff } \ \|\mathbb{E}_\nu[f|\mathcal{A}_n] - \mathbb{E}_\nu[f|\mathcal{A}]\|_1 \to 0,\ \forall f \in L^1(X,\mathcal{B},\nu)$$

for $\mathcal{A}_n,\mathcal{A} \in \Sigma_{\mathcal{B}}$. In particular this gives $\|\mathbb{E}_\nu[f|\mathcal{A}_n]\|_1 \to \|\mathbb{E}_\nu[f|\mathcal{A}]\|_1$ for all $f \in L^1(X,\mathcal{B},\nu)$. In Paper I we prove an even stronger statement than continuity, namely that the above notion of entropy respects upper and lower limits of sequences in $\Sigma_{\mathcal{B}}$ in the sense of Kudo [23], which are defined as follows.

**Definition 4.2.** A sub-$\sigma$-algebra $\mathcal{A} \in \Sigma_{\mathcal{B}}$ is an *upper limit* of a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $\Sigma_{\mathcal{B}}$ if

$$\limsup_{n\to\infty} \|\mathbb{E}_\nu[f|\mathcal{A}_n]\|_1 \le \|\mathbb{E}_\nu[f|\mathcal{A}]\|_1,\ \forall f \in L^1(X,\mathcal{B},\nu).$$

It is called *lower limit* if

$$\liminf_{n\to\infty} \|\mathbb{E}_\nu[f|\mathcal{A}_n]\|_1 \ge \|\mathbb{E}_\nu[f|\mathcal{A}]\|_1,\ \forall f \in L^1(X,\mathcal{B},\nu).$$

There is a partial order on the set of all sub-$\sigma$-algebras by inclusion, so we can talk about a *minimal* upper limit and a *maximal* lower limit. Kudo showed in [23] there there always exists a unique minimal upper limit and a unique maximal lower limit of a sequence in $\Sigma_{\mathcal{B}}$, these elements in $\Sigma_{\mathcal{B}}$ we will denoted by

$$\limsup_{n\to\infty} \mathcal{A}_n := \min\{\mathcal{A} \in \Sigma_{\mathcal{B}} \,:\, \mathcal{A} \text{ an upper limit of } (\mathcal{A}_n)_{n\in\mathbb{N}}\}$$

and

$$\liminf_{n\to\infty} \mathcal{A}_n := \max\{\mathcal{A} \in \Sigma_{\mathcal{B}} \,:\, \mathcal{A} \text{ a lower limit of } (\mathcal{A}_n)_{n\in\mathbb{N}}\}.$$

The sequence converges in the strong topology if and only if $\limsup_{n\to\infty} \mathcal{A}_n = \liminf_{n\to\infty} \mathcal{A}_n$. In this case one can replace the $L^1$-norm in the definition by any $L^p$-norm for $p \in [1,\infty]$, as shown by Piccinini in [28, Theorem 2.3.1]. Note that if $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a monotone sequence in $\Sigma_{\mathcal{B}}$, i.e. $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ $\forall n \in \mathbb{N}$ or $\mathcal{A}_n \supseteq \mathcal{A}_{n+1}$ $\forall n \in \mathbb{N}$, the above notion of upper and lower limits coincides with the set-theoretical one, while for a general sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ one only has

$$\sigma\Big(\bigcup_{k=1}^{\infty}\bigcap_{k=n}^{\infty} \mathcal{A}_n\Big) \subseteq \liminf_{n\to\infty} \mathcal{A}_n \subseteq \limsup_{n\to\infty} \mathcal{A}_n \subseteq \bigcap_{k=1}^{\infty} \sigma\Big(\bigcup_{k=n}^{\infty} \mathcal{A}_n\Big)$$

as observed by Kudo in [23, Remark 3.1].

## 4.3 Entropy functionals

The notion of entropy of the previous subsection can be generalized to the *set of probability densities* of $(X,\nu)$, which is nothing else than the unit ball of non-negative functions of $L^1(X,\nu)$, which we will denote by $\mathcal{P} := \{f \in L^1(X,\nu) \,:\, f \geq 0, \int_X f\, d\nu = 1\}$. Note that this set in particular contains all functions $\mathbb{E}_\nu[\frac{dg^{-1}\nu}{d\nu}|\mathcal{A}]$ for every $g \in G$, assuming again $\frac{dg^{-1}\nu}{d\nu} \in L^1(X,\nu)$ throughout.

**Definition 4.1.** The *entropy functional* of $f \in \mathcal{P}$ is defined by

$$\operatorname{Ent}(f) := \int_X -\log(f(x))\, d\nu(x).$$

The connection to Definition 4.1 is

$$h_\mu(\mathcal{A}) = \int_G \operatorname{Ent}(\mathbb{E}_\nu[\frac{dg^{-1}\nu}{d\nu}|\mathcal{A}])\, d\mu(g).$$

As a notion of upper and lower limits in this context we choose the following.

**Definition 4.2.** A function $f \in \mathcal{P}$ is called *upper limit* of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{P}$ if

$$\limsup_{n\to\infty} \int_t^\infty \nu(f_n \geq s)\, ds \leq \int_t^\infty \nu(f \geq s)\, ds \quad \text{for all } t \geq 0$$

where the integrals are taken w.r.t. the Lebsegue measure on $[0,\infty)$. It is called *lower limit* if

$$\liminf_{n\to\infty} \int_t^\infty \nu(f_n \geq s)\, d \geq \int_t^\infty \nu(f \geq s)\, ds \quad \text{for all } t \geq 0.$$

An element $\mathcal{A} \in \Sigma_{\mathcal{B}}$ is an upper limit of a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $\Sigma_{\mathcal{B}}$ in the sense of Definition 4.2 if and only if $\mathbb{E}_\nu[f|\mathcal{A}]$ is an upper limit in the above definition of $(\mathbb{E}_\nu[f|\mathcal{A}_n])_{n\in\mathbb{N}}$ for every $f \in L^\infty(X,\nu)$. Analogously, one can show the correspondence for lower limits. For a proof we refer to Paper I, Corollary 5.5.

# Chapter 5

# A concrete example: The lamplighter group

*Walking back and forth on an infinite street, lighting and dousing lamps, this is what the lamplighter does.*

Let

$$\Gamma = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \rtimes \mathbb{Z}.$$

One might interpret $(a_n) \in \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ as a sequence of lamps over $\mathbb{Z}$, where 1 symbolizes that a lamp is on and 0 that it is off. The right most coordinate in the semi-direct product of $\Gamma$ represents the position where the lamplighter stands at. Two elements $((a_n), m)$ and $((b_n), k)$ in $\Gamma$ are multiplied according to the semi-direct product as

$$((a_n), m)((b_n), k) = ((a_n + b_{n-m}), m + k)$$

i.e. the sequence $(b_n)$ is shifted to the right by $m$ and then added to the sequence $(a_n)$. In particular $((a_n), m)^{-1} = ((a_{n+m}), -m)$ and $e = ((0), 0)$, where $(0)$ denotes the sequence of only 0s. Let us denote by $(1_i)$ the sequence of zeros at all places, beside on the position $i$ where it is 1. A set of generators of the group is for example

$$S := \{((0), 1), ((0), -1), ((1_0), 0)\}$$

where $a := ((0),1)$ moves the lamplighter one position to the right $b := ((0), -1)$ moves him/her to the left and $c := ((1_0), 0)$ does not move the lamplighter but changes the configuration of the lamp at the origin.

We consider a random walk $\tau$ on $\Gamma$ which is supported on the set $S$. If $\tau$ is symmetric, i.e. $\tau(\gamma) = \tau(\gamma^{-1})$ then all bounded $\tau$-harmonic functions are constant and the Poisson boundary of $(\Gamma, \tau)$ is trivial. In the symmetric case the drift of $\tau$ is 0. The *drift* of $\tau$ on $\Gamma$ is defined as

$$\phi_\tau := \sum_{n\in\mathbb{Z}} n\,\tau(\mathrm{pr}_{\mathbb{Z}}^{-1}(\{n\})).$$

If the drift is positive, then the lamplighter will more likely go to the right. Therefore, for every position there will be a certain time where he will not return to the left hand side of this position, hence cannot change lamps there anymore. This means that the configuration of lamps will stabilize while the lamplighter will drift off to $+\infty$. This is the heuristic argument (see [19] for a proof) that the Poisson boundary of $(G,\tau)$ with $\phi_\tau < 0$ is the space of configurations, with finitely many ones at the left hand side together with the hitting measure $\nu$:

$$\mathrm{Poi}(\Gamma,\tau) = \Big(\bigoplus_{-\infty}^{-1} \mathbb{Z}_2 \oplus \prod_{0}^{\infty} \mathbb{Z}_2,\nu\Big).$$

Moreover, the measure $\nu$ is the unique $\tau$-stationary probability measure on that space. The action of $\Gamma$ on that space is defined as the group action projected to the space i.e. $((a_n),m)(b_n) := (a_n + b_{n-m})$ for $((a_n),m) \in \Gamma$ acting on the configuration sequence $(b_n)$ of the space.

## 5.1 Hecke absorbing measures

We consider the following subgroup in $\Gamma$

$$\Lambda := \bigoplus_{0}^{\infty} \mathbb{Z}_2 \rtimes \{0\}.$$

Then $(\Gamma,\Lambda)$ is a Hecke pair. Indeed, the coset space $\Gamma/\Lambda$ consists of elements of the form

$$((a_n),m)\Lambda = \Big((a_n)_{n=-\infty}^{m-1} + \bigoplus_{m}^{\infty} \mathbb{Z}_2\Big) \rtimes \{m\}$$

for $(((a_n),m) \in \Gamma$, hence all $\Lambda$-orbits on $\Gamma/\Lambda$ are finite, since elements in $\Lambda$ only change lamps over the positive integers and there are only finitely many which can be determined. For example

$$|\,\mathrm{Orb}_\Lambda(a\Lambda)| = 2,\ |\,\mathrm{Orb}_\Lambda(b\Lambda)| = 1,\ \ |\,\mathrm{Orb}_\Lambda(c\Lambda)| = 1$$

with $\mathrm{Orb}_\Lambda(a\Lambda) = \{a\Lambda, ((1_0),0)a\Lambda\}$ and $c\Lambda = \Lambda$.

We will produce a simple example of a $\Lambda$-absorbing probability measure $\widetilde{\tau}$ on $\Gamma$, which is finitely supported and generating, by following the construction in the proof of Theorem 1.4 in Paper II. To this end let us consider a generating, finitely supported random walk $\tau$ on $\Gamma$, e.g.

$$\tau = d_1\delta_a + d_2\delta_b + d_3\delta_c$$

with $d_1 + d_2 + d_3 = 1$ and $d_i \neq 0$. First, we project $\tau$ to the coset space $\Gamma/\Lambda$ and get a probability measure $\overline{\tau}$ on $\Gamma/\Lambda$ given by

$$\overline{\tau} = d_1\delta_{a\Lambda} + d_2\delta_{b\Lambda} + d_3\delta_{\Lambda}.$$

Averaging this measure along the $\Lambda$-orbits results in

$$\hat{\tau} = \frac{d_1}{2}\delta_{a\Lambda} + \frac{d_1}{2}\delta_{(1_0),0)a\Lambda} + d_2\delta_{b\Lambda} + d_3\delta_{\Lambda}.$$

Now we shall choose a finitely supported probability measure $\widetilde{\tau}$ on $\Gamma$ such that $\sum_{\lambda\in\Lambda}\widetilde{\tau}(\gamma\lambda) = \hat{\tau}(\gamma\Lambda)$. Hence one example of a $\Lambda$-absorbing probability measure would be

$$\widetilde{\tau} = \frac{d_1}{2}\delta_a + \frac{d_1}{2}\delta_{((1_0),0)a} + d_2\delta_b + d_3\delta_c. \tag{5.1}$$

### 5.1.1 A corresponding random walk on a t.d.l.c. group

Consider the groups

$$H = E \rtimes \mathbb{Z} \text{ and } L = \prod_0^\infty \mathbb{Z}_2 \rtimes \{0\}$$

where $E := \bigoplus_{-\infty}^{-1}\mathbb{Z}_2 \oplus \prod_0^\infty \mathbb{Z}_2$ denotes the collection of all lamp-configurations over $\mathbb{Z}$ with at most finitely many lamps turend on over the negative integers and possibly infinitely many lamps turned on over the positive integers. The subgroup $L$ is open and compact in the group topology generated by the identity neighborhoods $\{\prod_n^\infty \mathbb{Z}_2 \rtimes \{0\} : n \in \mathbb{N}\}$. Moreover, $\Gamma$ is a dense subgroup of $H$ and $\Lambda$ a dense subgroup of $L$ in this topology.

Recall from Chapter 3 that there is a bijection

$$\psi : \Gamma/\Lambda \longrightarrow H/L$$

given by $\gamma\Lambda \mapsto \gamma L$, which we use to construct a bi-$L$-invariant probability measure $\theta_{\widetilde{\tau}}$ on $H$, when given a $\Lambda$-absorbing probability measure $\widetilde{\tau}$ on $\Gamma$ such that $\theta_{\widetilde{\tau}} = \widetilde{\tau} \circ \psi^{-1}$ on $H/L$. As in Theorem 1.6 in Paper II we define

$$\theta_{\widetilde{\tau}}(A) := \sum_{gL\in H/L} m_L(L \cap g^{-1}A)\widetilde{\tau}(\psi^{-1}(gL)) \tag{5.2}$$

which is well-defined since the Haar measure $m_L$ on $L$ is left-$L$-invariant. In particular

$$\theta_{\widetilde{\tau}}(\gamma L) = \widetilde{\tau}(\gamma\Lambda)$$

for every $\gamma \in \Gamma$.

With $\widetilde{\tau}$ as in 5.1 we thus obtain

$$\theta_{\widetilde{\tau}}(A) = m_L(L \cap ((0), -1)A)\frac{d_1}{2} + m_L(L \cap ((1_{-1}), -1)A)\frac{d_1}{2}$$
$$+ m_L(L \cap ((0),1)A)d_2 + m_L(L \cap A)d_3.$$

Hence

$$\theta_{\widetilde{\tau}}\big((1, a_{l+1}, \dots, a_{l+k}] \rtimes \{n\}\big) = \begin{cases} 2^{-k-1}d_1 & \text{if } l = 0, n = 1, \\ 2^{-(k+l)-1}d_1 & \text{if } l \geq 1,\ n = 1, \\ 2^{-(k+l+2)}d_2 & \text{if } l \geq -1,\ n = -1, \\ 2^{-(k+l+1)}d_3 & \text{if } l \geq 0,\ n = 0, \\ 0 & \text{else} \end{cases}$$

where $(1,a_{l+1}, \dots, a_{l+k}]$ denotes the cylinder set of all sequences in $E$ which are 0 until position $l$, where they attain the value 1, the following $k$ entries are fixed, namely $a_{l+1}, \dots, a_{l+k}$, and the remaining ones are arbitrary free.

### 5.1.2 Harmonic functions for absorbing measures

The advantage of a bi-$L$-invariant probability measure $\theta_{\widetilde{\tau}}$ as in 5.2 is that we can identify $\theta_{\widetilde{\tau}}$-harmonic functions on $H$ as $\widetilde{\tau}$-harmonic functions on $\Gamma$, which are right-$\Lambda$-invariant . Thus

$$\mathcal{H}^\infty(H,\theta_{\widetilde{\tau}}) \hookrightarrow \mathcal{H}^\infty(\Gamma, \widetilde{\tau})$$

by a homomorphism that respects the Banach algebra structure with the multiplication defined by 2.1. A heuristic argument can be given as follows. Let $h \in \mathcal{H}^\infty(H,\theta_{\widetilde{\tau}})$, then we see that $h$ is right-$L$-invariant, since $\theta_{\widetilde{\tau}}$ is right-$L$-invariant. Therefore we can think of $h$ as a function $\overline{h}$ on $H/L$, which we can transfer to $\Gamma/\Lambda$ via the map $\psi$. Thus we get a right-$\Lambda$-invariant function $\widetilde{h}$ on $\Gamma$ by setting $\widetilde{h}(\gamma) := \overline{h}(\gamma L)$. This function is $\widetilde{\tau}$-harmonic, since for all $\gamma \in \Gamma$,

$$\sum_{\gamma \in \Gamma} \widetilde{h}(\gamma\gamma')\widetilde{\tau}(\gamma') = \sum_{t \in T}\sum_{\lambda \in \Lambda} \widetilde{h}(\gamma t \lambda)\widetilde{\tau}(t\lambda) = \sum_{t \in T} \widetilde{h}(\gamma t)\widetilde{\tau}(t\Lambda)$$
$$= \sum_{t \in T} \widetilde{h}(\gamma t)\theta_{\widetilde{\tau}}(tL) = \sum_{t \in T} \int_L h(\gamma t l)\, dt^{-1}\theta_{\widetilde{\tau}}(l) = h(\gamma) = \widetilde{h}(\gamma),$$

where $T \subseteq \Gamma$ denotes a set of representatives of $\Gamma/\Lambda$.

This allows to deduce that there is a $\Gamma$-factor map

$$\begin{array}{c} \Gamma \curvearrowright \mathrm{Poi}(\Gamma,\widetilde{\tau}) \\ \big\downarrow \\ \Gamma \curvearrowright \mathrm{Poi}(H,\theta_{\widetilde{\tau}}), \end{array}$$

using that $\mathcal{H}^\infty(H,\theta_{\widetilde{\tau}}) \cong L^\infty(\mathrm{Poi}(H,\theta_{\widetilde{\tau}}))$ and $\mathcal{H}^\infty(\Gamma,\widetilde{\tau}) \cong L^\infty(\mathrm{Poi}(\Gamma,\widetilde{\tau}))$ by algebra structure preserving, Γ-equivariant isomorphisms. In particular, every $\theta_{\widetilde{\tau}}$-boundary is a $\widetilde{\tau}$-boundary when viewed as a Γ-space. In Paper II, Theorem 1.6, we provide rigorous proofs of the above statements using a different approach.

## 5.2 Prime Poisson boundary

Let $\widetilde{\tau}$ be a Λ-absorbing, generating probability measure on Γ with positive drift and let $\theta_{\widetilde{\tau}}$ be as in 5.2. We claim that

$$\mathrm{Poi}(\Gamma,\widetilde{\tau}) = (E,\nu) = \mathrm{Poi}(H,\theta_{\widetilde{\tau}}) \tag{5.3}$$

as Γ-spaces with $E = \bigoplus_{-\infty}^{-1} \mathbb{Z}_2 \oplus \prod_0^\infty \mathbb{Z}_2$ and $\nu$ being the unique $\widetilde{\tau}$-stationary probability measure on $E$. The first equality we already know from the beginning of this chapter. To see the second one we use

**Lemma 5.1** (Theorem 1.6 (P3) in Paper II). *Every $\theta_{\widetilde{\tau}}$-stationary probability measure is $\widetilde{\tau}$-stationary.*

Hence if we find a $\theta_{\widetilde{\tau}}$-stationary probability measure $\nu'$ on $E$ then the above lemma states that $\nu'$ is $\widetilde{\tau}$-stationary too and thus by uniqueness we conclude $\nu' = \nu$. That there exists a $\theta_{\widetilde{\tau}}$-stationary probability measure on $E$ is for example proven in [11]. Thus 5.3 is established.

Note that we obtain an $H$-homogeneous space since

$$E \cong H/P$$

where $P := \{(0)\} \rtimes \mathbb{Z}$ is a closed subgroup of $H$, thus $H$ acts transitively on $E$. Moreover, $P$ is a maximal closed subgroup of $H$, meaning that the only proper closed over-group of $P$ is $H$ itself. Using uniqueness of the quasi-invariant measure class one can show that all $H$-factors of $(E,\nu)$ are of the form $H/C$ for some closed subgroup $C$ of $H$ which contains $P$ (see e.g. [25, Ch. IV, Sec. 2, Proposition 2.4 (b)]). Thus, by maximality of $P$ there are no non-trivial $H$-factors of $(E,\nu)$. Since Γ-factors of $(E,\nu)$ are Γ-equivariant isomorphic to $H$-factors (confer e.g. Corollary 4.14 in Paper II). We deduce that $(E,\nu)$ has no non-trivial Γ-factors, i.e. $(\Gamma,\widetilde{\tau})$ is prime.

## 5.3 Reducible quasi-regular representation

Let $\tau$ be as above a Λ-absorbing probability measure with positive drift on the lamplighter group Γ. We will show that the quasi-regular representation on $L^2(\mathrm{Poi}(\Gamma,\tau))$ is reducible. It is sufficient to find non-zero functions $f_1,f_2 \in L^2(E,m_E)$ such that $\langle \pi(g)f_1,f_2\rangle = 0$ for all $g \in H$ (see proof of Theorem 1.13 (IV) in Paper II). Following Lemma 9.3 in Paper II we may take for example

$$f_1 := 1_{\mathcal{O}\cap[1_1]} - 1_{\mathcal{O}\cap[0_1]}$$

and

$$f_2((a_n)) := \begin{cases} 1_{\mathcal{O}} & \text{if } a_1 + a_0 = 1 \\ -1_{\mathcal{O}} & \text{if } a_1 + a_0 = 0 \end{cases},$$

where $\mathcal{O} := \prod_0^\infty \mathbb{Z}_2$ denotes the collection of lamp configurations only supported over non-negative integers, and $[j_i]$ denotes the set of all sequences with the value $j$ at the position $i$. Indeed for $g = ((b_n),k) \in H$ we see that

$$\langle \pi(g)f_1,f_2\rangle = \int_E \Big(\frac{dgm_E}{dm_E}(x)\Big)^{\frac{1}{2}} f_1(g^{-1}x)\overline{f_2(x)}\, dm_E(x)$$

$$= \Delta(g)^{\frac{1}{2}} m_E(\{(a_n) \in \mathcal{O} \,:\, a_{1+k}+b_{1+k} = a_1+a_0, \text{ and } b_{n+k}+a_{n+k} = 0, \forall n \leq -1\})$$

$$-\Delta(g)^{\frac{1}{2}} m_E(\{(a_n) \in \mathcal{O} \,:\, a_{1+k}+b_{1+k} \neq a_1+a_0 \text{ and } b_{n+k}+a_{n+k} = 0, \forall n \leq -1\})\big)$$

$$= 0$$

where $\Delta(g) := \frac{dgm_E}{dm_E} = 2^{-k}$ is constant on $E$.

# Chapter 6

# Summary of papers

## 6.1 Summary of Paper I

Paper I is a joint work with Michael Björklund and Yair Hartman. We provide upper and lower semi-continuity results of several notions of entropy and thus in particular continuity statements. For Furstenberg's entropy viewed on the space of sub-$\sigma$-algebras we obtain for instance compatibility w.r.t. upper and lower limits in the sense of Kudo in [23], in Theorem 6.1 below.

Let $G$ be a compactly generated, locally compact, second countable group and $\mu$ be an admissible probability measure on $G$. Further let $(X,\mathcal{B},\nu)$ be a non-singular standard $G$-space with a $\nu$-complete $\sigma$-algebra $\mathcal{B}$. Assume that $\forall g \in G : \exists c_g > 1$ such that $\frac{dg^{-1}\nu}{d\nu} \in [c_g^{-1}, c_g]$ and $g \mapsto c_g$ is bounded on the support of $\mu$. As in Section 4.2 we denote by $\Sigma_{\mathcal{B}}$ the space of all sub-$\sigma$-algebras of $\mathcal{B}$ which contain all $\nu$-null-sets and by $h_\mu$ the entropy on $\Sigma_{\mathcal{B}}$ given by Definition 4.1. Under these assumptions we obtain

**Theorem 6.1** (by Theorem 1.8 in Paper I)**.** *For any sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ in $\Sigma_{\mathcal{B}}$ we have*

$$h_\mu(\liminf_{n\to\infty} \mathcal{A}_n) \leq \liminf_{n\to\infty} h_\mu(\mathcal{A}_n) \leq \limsup_{n\to\infty} h_\mu(\mathcal{A}_n) \leq h_\mu(\limsup_{n\to\infty} \mathcal{A}_n).$$

*In particular if $(\mathcal{A}_n)_{n\in\mathbb{N}}$ converges in the strong topology to a $\sigma$-algebra in $\Sigma_{\mathcal{B}}$, denoted by $\lim_{n\to\infty} \mathcal{A}_n$, then*

$$\lim_{n\to\infty} h_\mu(\mathcal{A}_n) = h_\mu(\lim_{n\to\infty} \mathcal{A}_n).$$

Let us now consider the case that $(X,\mathcal{B},\nu)$ is the Poisson boundary of $(G,\mu)$ (alternatively, some $\mu$-boundary). If $(\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence of $G$-invariant sub-$\sigma$-algebras in $\Sigma_{\mathcal{B}}$ – i.e. corresponds to $\mu$-boundaries

$(Y_n,\eta_n)$ – then $\limsup_{n\to\infty} \mathcal{A}_n$ and $\limsup_{n\to\infty} \mathcal{A}_n$ are $G$-invariant, too (proven in Lemma 3.3. of Paper I), hence correspond to $\mu$-boundaries, which we will denote by $(Y^+,\eta^+)$ and $(Y^-,\eta^-)$. This gives

**Corollary 6.2** (Theorem 1.16 in Paper I)**.** *With the notation as above we get*

$$h_\mu(Y^-,\eta^-) \leq \liminf_{n\to\infty} h_\mu(Y_n,\eta_n) \leq \limsup_{n\to\infty} h_\mu(Y_n,\eta_n) \leq h_\mu(Y^+,\eta^+).$$

*In particular if* $(Y^+,\eta^+)$ *is* $G$*-isomorphic to* $(Y^-,\eta^-)$ *then*

$$\lim_{n\to\infty} h_\mu(Y_n,\eta_n) = h_\mu(Y^+,\eta^+) = h_\mu(Y^-,\eta^-).$$

**Remark 6.3.** If $\Sigma_{\mathcal{B}}$ with the strong topology were compact (which it is not, see e.g. [23, Example 3.1]), then we would obtain that $h_\mu(\Sigma_{\mathcal{B}})$ is compact and since the set of $G$-invariant $\sigma$-algebras of $\Sigma_{\mathcal{B}}$ is a closed (hence compact) set we would gain that $\mathrm{BndEnt}(G,\mu)$ is compact (in particular closed), which is an open conjecture.

The reverse direction of Theorem 6.1 holds at least in the "extreme" cases:

**Theorem 6.4** (Theorem 1.15 in Paper I)**.** *If* $h_\mu(\mathcal{A}_n) \to 0$ *for* $n \to \infty$, *then* $\limsup_{n\to\infty} \mathcal{A}_n = \liminf_{n\to\infty} \mathcal{A}_n = \sigma(\mathcal{N})$*. Whereas if* $h_\mu(\mathcal{A}_n) \to h_\mu(\limsup_{n\to\infty} \mathcal{A}_n)$ *then* $\limsup_{n\to\infty} \mathcal{A}_n = \liminf_{n\to\infty} \mathcal{A}_n$*.*

Translating the above theorem into the language of boundaries we see that if $h_\mu(Y_n,\eta_n) \to 0$ then $(Y^+,\eta^+) = (Y^-,\eta^-) = (\{*\},\delta_*)$. While if $h_\mu(Y_n,\eta_n) \to h_\mu(Y^+,\eta^+)$ then $(Y^+,\mathcal{D}^+,\eta^+) = (Y^-,\eta^-)$. (This is Theorem 1.18 in Paper I.)

All this comes from a much more general statement for entropy functionals of upper and lower limits of probability densities in the sense of Subsection 4.3:

**Theorem 6.5** (Theorem 1.2 and Corollary 1.4 in Paper I)**.** *Let* $f_n \in L^1(X,\nu)$ *be non-negative functions such that* $\int_X f_n \, d\nu = 1$ *and let* $f^{up}$ *and* $f^{low}$ *be upper and lower limits in the sense of Definition 4.2, respectively. Then*

$$\liminf_{n\to\infty} \mathrm{Ent}(f_n) \leq \mathrm{Ent}(f^{low}) \leq \mathrm{Ent}(f^{up}) \leq \limsup_{n\to\infty} \mathrm{Ent}(f_n).$$

*In particular if* $f^{up} = f^{low}$ *then*

$$\lim_{n\to\infty} \mathrm{Ent}(f_n) = \mathrm{Ent}(f).$$

## 6.2 Summary of Paper II

In this joint work with Michael Björklund and Yair Hartman we develop a technique to establish a relation between boundaries of a totally disconnected, locally compact, second countable group $H$ and a densely embedded countable discrete group $\Gamma$ by using Hecke subgroups to link the measures in concern. More precisely let

$$\rho : \Gamma \longrightarrow H$$

be a homomorphism with dense image in $H$. Fix a compact open subgroup $L$ of $H$ and set $\Lambda := \rho^{-1}(L)$ (alternatively, we could start with a Hecke subgroup $\Lambda$ of $\Gamma$ and set $(H, K)$ to be the corresponding Schlichting completion, confer Chapter 3). The crucial condition we impose on the measures on $\Gamma$ is the following.

**Definition 6.1.** A probability measure $\tau$ on $\Gamma$ is called $\Lambda$-*absorbing* if

$$\tau(\lambda\gamma\Lambda) = \tau(\gamma\Lambda)$$

for every $\lambda \in \Lambda$, $\gamma\Lambda \in \Gamma/\Lambda$.

The above condition is precisely what is needed to obtain that the projected random walk on the coset space $\Gamma/\Lambda$ (which is isomorphic to $H/L$) becomes a Markov chain, as shown in [21, Proposition 2.10].

In Theorem 1.6 in Paper II we construct admissible bi-$L$-invariant probability measures $\theta_\tau$ for every $\Lambda$-absorbing probability measure $\tau$ on $\Gamma$ such that every $\theta$-boundary is a $\tau$-boundary, where we let $\Gamma$ act on $H$-spaces via the map $\rho$. In particular, if $\tau$ is generating then $\theta_\tau$ is generating, too and the Poisson boundary of $(H,\theta_\tau)$ is a $\Gamma$-factor of the Poisson boundary of $(\Gamma,\tau)$, in symbols

$$\begin{array}{c} \Gamma \curvearrowright \mathrm{Poi}(\Gamma,\tau) \\ \big\downarrow \\ \Gamma \curvearrowright \mathrm{Poi}(H,\theta_\tau) \end{array}$$

shown in Corollary 1.9. Our construction of $\theta_\tau$ is natural in the sense that every bi-$L$-invariant probability measures on $H$ arises from some $\Lambda$-absorbing probability measure on $\Gamma$ in this way, as provided in Theorem 1.6. We show that the $\Lambda$-invariant $\tau$-boundaries are exactly the $\theta_\tau$-boundaries when viewed as $\Gamma$-spaces, assuming that $\tau$ is generating (Corollary 1.9 (II) and Theorem 1.6). Furthermore, if $\mathrm{Poi}(H,\theta_\tau)$ is $\Gamma$-isomorphic to $\mathrm{Poi}(\Gamma,\tau)$ for some generating probability measure $\tau$, then $\Lambda$ has to be amenable, proven in Corollary 1.9. A converse holds for cases where there cannot be more than one $\tau$-stationary probability measure on the space of $\mathrm{Poi}(\Gamma,\tau)$.

### 6.2.1 Primeness

For certain (classes of) examples some boundaries of $\Lambda$-balanced, generating random walks on $\Gamma$ are homogeneous spaces for the $H$-action, i.e. of the form $(H/P,\nu)$ for some closed subgroup $P$ of $H$. If $P$ is a maximal closed proper subgroup, then we obtain primness w.r.t. $H$-factors. Using that $\Gamma$-factors are ($\Gamma$-isomorphic) to $H$-factors (confer e.g. Proposition 4.12 in Paper II) we may deduce primeness results for the action of $\Gamma$ provided that $\nu$ is $H$-quasi-invariant (confer Corollary 4.15).

For instance if $H = SL_2(\mathbb{Q}_p)$ and $L = SL_2(\mathbb{Z}_p)$ for some prime number $p$ then from Guivarc'h-Ji-Taylor's [16, Theorem 15.5] we can deduce that

$$\mathrm{Poi}(H,\theta_\tau) = (H/P, \nu)$$

where $P$ denotes the subgroup of upper triangular matrices in $SL_2(\mathbb{Q}_p)$. Note that here $H/P = L$ and $\nu$ is the unique $L$-invariant probability measure on $H/P$, namely the Haar measure on $L$. There is a densely embedded free group of finite rank inside $H$ (confer e.g. [8]), which will play the role of $\Gamma$. Now choosing a finite supported, generating, $\Lambda$-absorbing probability measure $\tau$ on $\Gamma$ we obtain a prime $\tau$-boundary, namely $\mathrm{Poi}(H,\theta_\tau)$. Summing up we have sketched the proof of

**Theorem 6.2** (Corollary 1.10 in Paper II). *There exists $r \in \mathbb{N}_{\geq 2}$ and a finitely supported generating probability measure $\tau$ on $\mathbb{F}_r$ such that* $\mathrm{Poi}(\Gamma,\tau)$ *admits a non-trivial prime $\Gamma$-factor, which is essentially free.*

Let us now turn to examples where

$$\mathrm{Poi}(\Gamma,\tau) = \mathrm{Poi}(H,\theta_\tau) = (H/P, \nu)$$

for some maximal closed proper subgroup $P$ of $H$. Here we can even deduce primeness of the Poisson boundary of $(\Gamma,\tau)$. In order to apply Kaimanovich's results in [19] we study the following two cases: Let $\tau$ be a generating, $\Lambda$-absorbing, finitely supported random walk with positive drift on

1. the lamplighter group
$$\Gamma = \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z} \tag{6.1}$$
in combination with
$$\Lambda = \bigoplus_{0}^{\infty} \mathbb{Z}/p\mathbb{Z} \rtimes \{0\},\ H = \Big(\prod_{-\infty}^{0} \mathbb{Z}/p\mathbb{Z} \oplus \bigoplus_{1}^{\infty} \mathbb{Z}/p\mathbb{Z}\Big) \rtimes \mathbb{Z},$$
$$L = \Big(\prod_{-\infty}^{0} \mathbb{Z}/p\mathbb{Z} \oplus \bigoplus_{1}^{\infty} \mathbb{Z}/p\mathbb{Z}\Big) \rtimes \{0\}$$
or

2. on the solvable Baumslag-Solitar group

$$\Gamma = BS(1,p) = \langle a,b \,:\, bab^{-1} = a^p \rangle \cong \mathbb{Z}[\frac{1}{p}] \rtimes \mathbb{Z} \tag{6.2}$$

with

$$\Lambda = \mathbb{Z} \rtimes \{0\},\ H = \mathbb{Q}_p \rtimes \mathbb{Z},\ L = \mathbb{Z}_p \rtimes \{0\}$$

for some prime number $p$. For these two classes of examples it is known (due to Kaimanovich [19]) that $\mathrm{Poi}(\Gamma,\tau) = (K,\nu)$ for a uniquely $\tau$-stationary probability measure $\nu$ on the field $K$, where $K = \prod_{-\infty}^{0} \mathbb{Z}/p\mathbb{Z} \oplus \bigoplus_1^\infty \mathbb{Z}/p\mathbb{Z}$ in the first example and $K = \mathbb{Q}_p$ in the second example. This leads to

**Theorem 6.3** (Corollary 1.16 in Paper II). *Let* $\Gamma$ *and* $\Lambda$ *be as in the lamplighter 6.1 or the Baumslag-Solitar 6.2 example above. Then* $\mathrm{Poi}(\Gamma,\tau)$ *is prime (w.r.t.* $\Gamma$*) for any finitely supported, generating,* $\Lambda$*-absorbing probability measure* $\tau$ *on* $\Gamma$.

### 6.2.2 Reducibility

Again using homogeneity (w.r.t. the t.d.l.c.s.c. group $H$) of the Poisson boundaries of positive drifting random walks on the lamplighter or the solvable Baumslag-Solitar examples above (6.1 and 6.2), we can investigate the quasi-regular representation on $L^p(K,\nu)$ for $(K,\nu) = \mathrm{Poi}(H,\theta_\tau) = \mathrm{Poi}(\Gamma,\tau)$. Since reducibility of the quasi-regular representation only depends on the measure class on $K$ we can pass the the Haar measure $m_K$ on $K$, due to our construction. This helps us proving

**Theorem 6.4** (Corollary 1.16 in Paper II). *Let* $(\Gamma,\tau)$ *be as in Theorem 6.3 above. Then the quasi-regular representation on* $L^p(\mathrm{Poi}(\Gamma,\tau))$ *is reducible for any* $p \in [1,\infty]$.

### 6.2.3 Boundary entropy spectra

We use the primeness result for $BS(1,2)$ for probability measures with positive vertical drift and a certain flexibility in choosing the absorbing measures in order to construct $\tau$ such that it attains explicitly given values for the Fustenberg entropy for every $\tau$-boundary of $\Gamma = \bigoplus_{\mathbb{N}} BS(1,2)$. This leads to

**Theorem 6.5** (in Theorem 1.19 in Paper II). *Given a sequence of summable, positive real numbers* $\beta = (\beta_n)_{n\in\mathbb{N}}$ *and let* $\Gamma = \bigoplus_{\mathbb{N}} BS(1,2)$. *Then there exists a generating (not necessary finitely supported) probability* $\tau$ *on* $\Gamma$ *such that*

$$\mathrm{BndEnt}(\Gamma,\tau) = \mathrm{SubSum}(\beta).$$

*In particular there exist two probability measures* $\tau_1$ *and* $\tau_2$ *on* $\Gamma$ *such that*

$$\mathrm{BndEnt}(\Gamma,\tau_1) = [0,h_{RW}(\Gamma,\tau)] \text{ and } \mathrm{BndEnt}(\Gamma,\tau_2) = \text{a Cantor set.}$$

Instead of the restricted product of the Baumslag-Solitar groups we can in fact take any countable restricted product of discrete countable groups such that on each one exists a generating probability measure which makes it prime and has finite random walk entropy.

## 6.3 Summary of Paper III

We consider

$$\Gamma = \mathbb{Z}[p_1^{-1}, \dots p_l^{-1}] \rtimes \langle p_1^{n_1} \cdots p_l^{n_l} \ : \ n_i \in \mathbb{Z} \rangle,$$

for given prime numbers $p_1, \dots, p_l$. This group is is not covered by the cases of Theorem 1.19 in Paper II, but we can deduce a similar result by slightly different means.

**Theorem 6.1** (Theorem 1 in Paper III)**.** *Given a sequence of positive real numbers $\beta := (\beta_1, \dots, \beta_l)$ we can find a finitely supported, generating probability measure $\tau$ on $\Gamma$ such that*

$$\mathrm{BndEnt}(\Gamma, \tau) = \mathrm{SubSum}(\beta).$$

Unlike in Theorem 1.19 in Paper II, we cannot utilize a direct sum structure of $\Gamma$ to construct a random walk with the desired boundary entropy spectrum here. Instead we observe that the boundaries are absolutely continuous w.r.t. some product measure, using the construction of Paper II to obtain homogeneous spaces (w.r.t. a certain t.d.l.c.s.c. group) by means of Hecke-subgroup-absorbing random walks and Brofferio's realization of the Poisson boundary of a negative drifting random walk $\tau$ of $\Gamma$ [10, Theorem 1] as

$$\Big( \prod_{i=1}^{l} \mathbb{Q}_{p_i}, \nu \Big)$$

for a unique $\tau$-stationary probability measure $\nu$ on that space. Here we say that the drift is negative if $\sum_{(r,s) \in \Gamma} \mathrm{pr}_2\, \tau(s) \log(|s|_{p_i}) < 0$ for all $i = 1, \dots, l$.

We observe that the average of the information function for the entropy does not change when passing to ($\sigma$-finite) measure which is absolutely continuous to the given one, meaning that

$$h_\mu(X, \nu) = \int_G \int_X I_\nu(g, x)\, d\nu(x)\, d\mu(g) = \int_G \int_X I_\xi(g, x)\, d\nu(x)\, d\mu(g),$$

when defined, for $\nu$ a $\mu$-stationary probability measure which is absolutely continuous w.r.t. a $\sigma$-finite, non-singular measure $\xi$ on $X$, given an admissible probability measure $\mu$ on a locally compact, second countable group

$G$ (Proposition 3.1. in Paper III). In particular, if $\xi$ is a product measure $\bigotimes_{i=1}^{l} \xi_i$ with constant Radon-Nikodym derivatives $\frac{dg^{-1}\xi_i}{d\xi_i} = \Delta_i(g)$ for every given $g \in G$, then we immediately obtain an entropy formula

$$h_\mu(X,\nu) = \sum_{i=1}^{l} \int_G -\log(\Delta_i(g))\, d\mu(g).$$

Constructing a certain Hecke-subgroup-absorbing probability measure on $\Gamma$ (similar to Paper II, Remark 1.15) helps us deducing Theorem 1 of paper III.

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