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Statistical Analysis of Antenna Array Systems with Perturbations in Phase, Gain and Element Positions

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Abstract—In this paper, we statistically analyze the effect of hardware impairments on power pattern of antenna array systems. We consider a linear array and formulate the stochastic beam pattern as a function of variations in phase, gain and element positions. By deriving a closed-form expression for the variance of the power pattern, we express how the performance of antenna array can be degraded in each angle, allowing for investigation of the role of each parameter in the final power pattern variations. The proposed closed-form expression serves as a tractable tool for analyzing the effect of perturbations in different settings.

Index Terms—Antenna array, phase errors, hardware impairments, array factor, power pattern variation.

I. INTRODUCTION

The performance of antenna array systems is unavoidably affected by different impairments including perturbations in phase, gain and array positions which are introduced due to non-ideal manufacturing process, aging and environmental conditions [1]. These perturbations make the beam pattern deviate from the nominal one which can considerably degrade the system performance and typically, several steps of calibration are required for compensating their effects.

To quantify how much degradation can be tolerated due to the effect of such perturbations, and then to design proper calibration algorithms, one first needs to know the characteristics of the power pattern variations. Since the perturbations are random variables, it is desirable to characterize the power pattern variations in terms of the first and second order statistics. Several works have been done in the literature about analytical study of the impact of random perturbations on the performance of antenna array systems [2]–[7]. In [2], the mean of power pattern is calculated in terms of the mean of random perturbations in phase (steering vector). The work in [3] models the sidelobe level of an antenna array using Rician distribution and presents an approximation for the variance of beam pattern as a function of the variance of perturbations in phase and gain (complex gain vector). In [4], [5], the authors model the combined effect of different perturbations using a single multidimensional random variable and calculate the

variance of beam pattern with the aid of a-priori information on the statistics of perturbations. In [6], a closed-form function for the mean of power pattern is derived in terms of variances of phase, gain, and element position perturbations. In [7], the beam pattern in presence of gain and phase perturbations is assumed to be a circular-symmetric complex Gaussian distributed function and the mean and variance of the beam pattern are calculated based on that approximate assumption.

There are also a few other works in the literature which analyze the perturbations in antenna arrays based on numerical methods and Monte-Carlo simulations [8]–[12]. In [8], the authors consider random gain and phase perturbations, and use Monte-Carlo simulations to numerically find the variations of the beam pattern. Similarly, in [9], Monte-Carlo simulations are used to obtain the mean of power pattern, the variance of beam pattern, and the bias of beam pattern for random perturbations in complex gain, element positions and orientations. In [10], [11], authors employ the interval arithmetic (IA) to compute power pattern bounds for antenna array perturbations in complex gain but with a reduced computational effort compared to Monte-Carlo methods. In [12], the authors propose a method to calculate the variations of power pattern by approximating the antenna power values using the first order Taylor expansion with respect to the uncertain amplitude in order to further reduce the computational complexity of the IA methods.

Although numerical calculations of the variations of power pattern by using Monte-Carlo simulations are effective for finding the bounds of variations [8]–[12], they are not tractable for analyzing the effects of all possible error combinations with all their different settings. It is more desirable to use analytical methods which are tractable for any combination of perturbations, allowing for investigation of the role of each parameter in the final power pattern variations. The previous works on analytical analysis [2]–[7], have not presented any closed-form expression for the variations of the power pattern. In this paper, we will follow the previous works done in [6] for calculation of mean of power pattern, and calculate the variance of power pattern for perturbations in phase, gain and element positions. We present a closed-form expression for the variance of power pattern in terms of variances of those three

perturbation parameters, and express analytical bounds for the power pattern.

The paper is organized as follows. In Section II, we make a brief review on antenna array systems and present the formulations for statistical definition of perturbations. In Section III, we explain the analytical derivations for the variance of power pattern with all its details. In Section IV, we validate the analytical study with simulations, and finally, the paper is concluded in Section V.

II. SYSTEM MODEL

A. Antenna Array System Review

Consider a linear array of N elements on z-axis with positions $\mathbf{p}_0, \mathbf{p}_1, ..., \mathbf{p}_{N-1}$ where $\mathbf{p}_n = [0 \quad 0 \quad p_{nz}]^T$. The array manifold vector is defined as [6]

$$\mathbf{v}(\mathbf{k}) = \begin{bmatrix} e^{-j\mathbf{k}\mathbf{p}_0} & e^{-j\mathbf{k}\mathbf{p}_1} & \ldots & e^{-j\mathbf{k}\mathbf{p}_{N-1}} \end{bmatrix}^T, \quad (1)$$

where $\mathbf{k}$ is the wavenumber and is used to express the direction of any array with any coordinates which is defined as

$$\mathbf{k} = \frac{2\pi}{\lambda} \begin{bmatrix} sin(\theta)cos(\psi) & sin(\theta)sin(\psi) & cos(\theta) \end{bmatrix}, \quad (2)$$

where λ is the wavelength, θ is the angle towards the z-axis and ψ is the angle from the x-axis. Now, if we consider a weight filter for each element, the complex gain vector is defined as

$$\mathbf{w} = \begin{bmatrix} w_0 & w_1 & \ldots & w_{N-1} \end{bmatrix}^T, \quad (3)$$

where,

$$w_i = g_i e^{j\phi_i}, \quad (4)$$

is the complex gain of each element in the phase shifter network (PSN) and arises after vector modulator (VM) which includes a phase shifter (PS) and a variable gain amplifier (VGA). In case of uniform linear array, all element gains are equal to $1/N$. The complex gain vector is the multiplication of real gains g_i with steering vector elements $e^{j\phi_i}$. These two components have different effects on the beam pattern. The steering vector is used for changing the direction of the main beam, and subsequently will change the direction of sidelobes. The vector of real gains on the other hand, is usually used for tapering and reducing the side-lobe levels, which also has some effects on the main lobe and can change the shape of the beam pattern. Beam pattern is defined as [6]

$$B(\theta, \psi) = B(\mathbf{k}) = \mathbf{w}^H \mathbf{v}(\mathbf{k}) = \sum_{i=0}^{N-1} g_i e^{j(\phi_i)} e^{-j\mathbf{k}\mathbf{p}_i}. \quad (5)$$

B. Statistical Modeling of Perturbations

If we consider the effect of perturbations in gain and phase of each element in the complex gain vector (2), then for each element we can write

$$w_i = g_i(1 + \Delta g_i)e^{j(\phi_i + \Delta\phi_i)}, \quad (6)$$

where Δg_i and $\Delta\phi_i$ are random variables with variances σ_g^2 and σ_ϕ^2.

We can define the perturbations in the position of array elements as

$$\mathbf{p}_i = \mathbf{p}_i^c + [0 \quad 0 \quad \Delta p_i]^T, \quad (7)$$

where Δp_i is a random parameter with the variance of σ_λ^2 in $z - axis$, and $\mathbf{p}_i^c$ is the vector of nominal element position in 3D space. Therefore, the stochastic beam pattern function can be written as

$$B(\mathbf{k}) = \sum_{i=0}^{N-1} g_i(1 + \Delta g_i)e^{(j(\phi_i + \Delta\phi_i) - j\mathbf{k}\mathbf{p}_i)}. \quad (8)$$

Now, if we consider Δp_i, Δg_i and $\Delta\phi_i$ for $i = 0, 1, \ldots, N-1$ as statistically independent, zero-mean, and Gaussian random variables [6], then the beam pattern in (8) is actually a random function of σ_g^2, σ_ϕ^2 and σ_λ^2 as the variances of perturbations in gain, phase, and element positions. The expectation of magnitude square of beam pattern (power pattern) can be obtained as [6]

$$\mathbb{E}\big[|B(\mathbf{k})|^2\big]$$
$$= |B^c(\mathbf{k})|^2 e^{-(\sigma_\phi^2 + \sigma_\lambda^2)} + \big((1 + \sigma_g^2) - e^{-(\sigma_\phi^2 + \sigma_\lambda^2)}\big) \sum_{i=0}^{N-1} g_i^2, \quad (9)$$

where $|B^c(\mathbf{k})|$ is the nominal beam pattern without any perturbation.

III. STATISTICAL ANALYSIS OF POWER PATTERN VARIATIONS

Eq. (9) gives us the average power, but we need to know the variance of the power pattern function to characterize its probable variations in the presence of perturbations. To mathematically formulate the variations, we need to calculate the variance of magnitude square of the beam pattern in (8). Namely, we need to find

$$\Lambda = \mathbb{E}[|B(\mathbf{k})|^4] - (\mathbb{E}[|B(\mathbf{k})|^2])^2. \quad (10)$$

For the second term, we can directly use (9), and for the first term, according to the expression of expected value of a random variable to the power of four, we have

$$\mathbb{E}[|B(\mathbf{k})|^4] = \mathbb{E}[B(\mathbf{k})^H B(\mathbf{k}) B(\mathbf{k})^H B(\mathbf{k})]$$
$$= \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{q=0}^{N-1} \mathbb{E}\bigg[g_i(1 + \Delta g_i)g_l(1 + \Delta g_l)g_m(1 + \Delta g_m)g_q$$
$$(1 + \Delta g_q)e^{j(\phi_i + \Delta\phi_i - \phi_l - \Delta\phi_l + \phi_m + \Delta\phi_m - \phi_q - \Delta\phi_q)}$$
$$e^{-j\mathbf{k}(\mathbf{p}_i - \mathbf{p}_l + \mathbf{p}_m - \mathbf{p}_q)}\bigg]. \quad (11)$$

Now, if we consider all random variables as statistically independent, zero-mean, and Gaussian, then by using method of moments, for the first part, we can write

$$\mathbb{E}\big[(1+\Delta g_i)(1+\Delta g_l)(1+\Delta g_m)(1+\Delta g_q)\big]$$

$$= \mathbb{E}\big[1 + \Delta g_m \Delta g_q + \Delta g_i \Delta g_m + \Delta g_i \Delta g_q + \Delta g_l \Delta g_m$$

$$+\Delta g_l \Delta g_q + \Delta g_i \Delta g_l + \Delta g_i \Delta g_m \Delta g_q + \Delta g_l \Delta g_m \Delta g_q$$

$$+\Delta g_i \Delta g_l \Delta g_m + \Delta g_i \Delta g_l \Delta g_q + \Delta g_i \Delta g_l \Delta g_m \Delta g_q\big], \tag{12}$$

where by any coordination of indices, the expectation of product of three random variables are zero and for the last term, by using the expression for the expected value of multiplication of four jointly Gaussian random variables [13], we can write

$$\mathbb{E}[\Delta g_i \Delta g_l \Delta g_m \Delta g_q]$$

$$= \mathbb{E}[\Delta g_i \Delta g_l]\mathbb{E}[\Delta g_m \Delta g_q] + \mathbb{E}[\Delta g_i \Delta g_m]\mathbb{E}[\Delta g_l \Delta g_q]$$

$$+\mathbb{E}[\Delta g_i \Delta g_q]\mathbb{E}[\Delta g_l \Delta g_m] = (\delta_{il}\delta_{mq} + \delta_{im}\delta_{lq} + \delta_{iq}\delta_{lm})\sigma_g^4, \tag{13}$$

which is derived using characteristic function of zero-mean jointly normal random variables, and $\delta_{il} = 1$ when $i = l$, and otherwise is zero. Now, we can rewrite equation (13) as

$$\mathbb{E}[(1+\Delta g_i)(1+\Delta g_l)(1+\Delta g_m)(1+\Delta g_q)]$$

$$= 1 + (\delta_{mq} + \delta_{im} + \delta_{iq} + \delta_{lm} + \delta_{lq} + \delta_{il})\sigma_g^2 \tag{14}$$

$$+(\delta_{il}\delta_{mq} + \delta_{im}\delta_{lq} + \delta_{iq}\delta_{lm})\sigma_g^4.$$

By using (23) in the Appendix, the second term in (12) can be written as

$$\mathbb{E}[e^{j(\Delta\phi_i - \Delta\phi_l + \Delta\phi_m - \Delta\phi_q)}] = e^{\mathbf{a}^T\mathbf{m} + 0.5\mathbf{a}^T\boldsymbol{\Sigma}\mathbf{a}}, \tag{15}$$
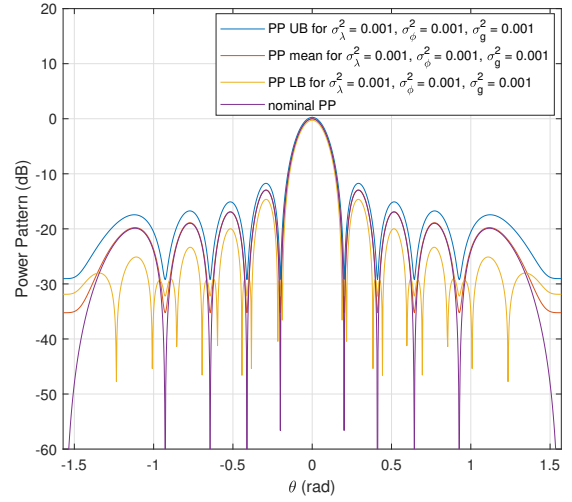
where $\mathbf{m}$ is the zero vector, since we have considered zero-mean random variables, and the covariance matrix $\boldsymbol{\Sigma}$ and the vector $\mathbf{a}$ are defined as

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{ii}^2 & \sigma_{il}^2 & \sigma_{im}^2 & \sigma_{iq}^2 \\ \sigma_{li}^2 & \sigma_{ll}^2 & \sigma_{lm}^2 & \sigma_{lq}^2 \\ \sigma_{mi}^2 & \sigma_{ml}^2 & \sigma_{mm}^2 & \sigma_{mq}^2 \\ \sigma_{qi}^2 & \sigma_{ql}^2 & \sigma_{qm}^2 & \sigma_{qq}^2 \end{bmatrix}, \tag{16}$$
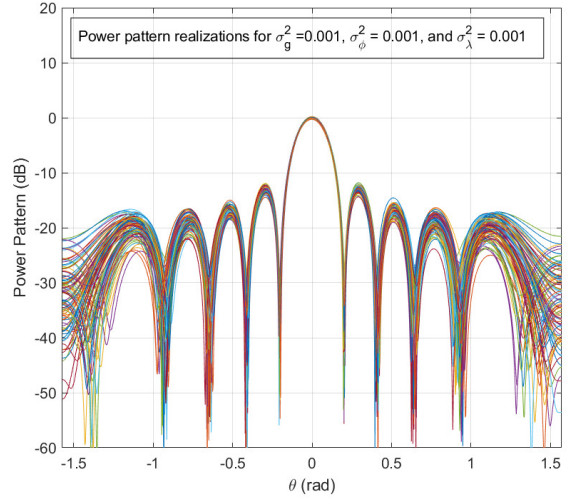
$$\mathbf{a} = \begin{bmatrix} j & -j & j & -j \end{bmatrix}^T, \tag{17}$$

respectively. We can rewrite (15) as

$$\mathbb{E}[e^{j(\Delta\phi_i - \Delta\phi_l + \Delta\phi_m - \Delta\phi_q)}]$$

$$= e^{0.5[-(\sigma_{ii}^2 + \sigma_{ll}^2 + \sigma_{mm}^2 + \sigma_{qq}^2) + 2(\sigma_{il}^2 - \sigma_{im}^2 + \sigma_{iq}^2 + \sigma_{lm}^2 - \sigma_{lq}^2 + \sigma_{mq}^2)]}$$

$$= e^{\sigma_\phi^2(-2 + (\delta_{mq} - \delta_{im} + \delta_{iq} + \delta_{lm} - \delta_{lq} + \delta_{il}))}. \tag{18}$$

(a)

(b)

Fig. 1: Power pattern analysis for a) analytical derivations based on (9) and (20), and for b) Monte-Carlo realizations.

Finally, for the last multiplicative term in (12), by using (23) in the Appendix, we can write

$$\mathbb{E}[e^{-j\mathbf{k}[0 \quad 0 \quad \Delta p_i - \Delta p_l + \Delta p_m - \Delta p_q]]^T}$$

$$= e^{\sigma_\lambda^2(-2 + (\delta_{mq} - \delta_{im} + \delta_{iq} + \delta_{lm} - \delta_{lq} + \delta_{il}))}. \tag{19}$$

Thus, according to equation (12) and by using the results in equations (14), (18), and (19), we can express the variance of power pattern as given in (20) on top of the next page. This closed-form expression can give us very useful insights through the variations of the power pattern in presence of perturbations which is explained in the next section.

IV. NUMERICAL RESULTS

For numerical analysis, we have considered an array of $N = 10$ antenna elements symmetrically positioned on the z-axis with steering angle of $\theta = 0$ in the broadside. The curves

$$\Lambda = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} \sum_{m=0}^{N-1} \sum_{q=0}^{N-1} \left[\left(g_i g_l g_m g_q e^{j(\phi_i - \phi_l + \phi_m - \phi_q)} e^{-j\mathbf{k}(\mathbf{p}_i^c - \mathbf{p}_l^c + \mathbf{p}_m^c - \mathbf{p}_q^c)} \right) \left(1 + (\delta_{mq} + \delta_{im} + \delta_{iq} + \delta_{lm} + \delta_{lq} + \delta_{il}) \sigma_g^2 \right. \right.$$

$$\left. \left. + (\delta_{il}\delta_{mq} + \delta_{im}\delta_{lq} + \delta_{iq}\delta_{lm}) \sigma_g^4 \right) \left(e^{\sigma_\phi^2(-2 + (\delta_{mq} - \delta_{im} + \delta_{iq} + \delta_{lm} - \delta_{lq} + \delta_{il}))} \right) \left(e^{\sigma_\lambda^2(-2 + (\delta_{mq} - \delta_{im} + \delta_{iq} + \delta_{lm} - \delta_{lq} + \delta_{il}))} \right) \right]$$

$$- \left(|B^c(\mathbf{k})|^2 e^{-(\sigma_\phi^2 + \sigma_\lambda^2)} + [(1 + \sigma_g^2) - e^{-(\sigma_\phi^2 + \sigma_\lambda^2)}] \sum_{i=0}^{N-1} g_i^2 \right)^2 \quad (20)$$
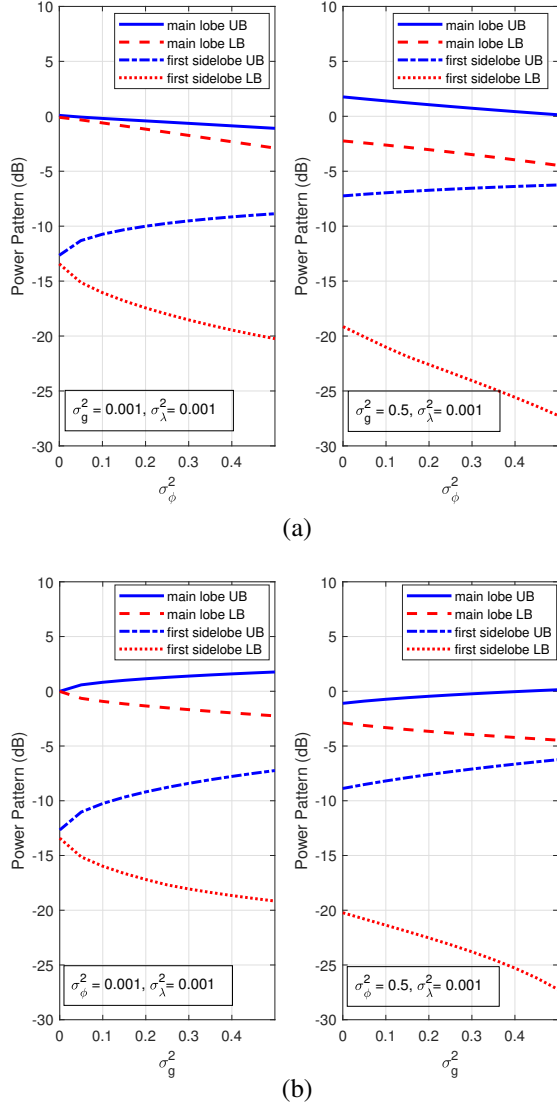

(a)

(b)

Fig. 2: Power pattern for the main lobe level and first sidelobe level with $\sigma_\lambda^2 = 0.001$ in a) span of σ_ϕ^2, and b) span of σ_g^2.

for nominal power pattern (nominal PP), the mean of power pattern (PP mean) based on (9), three standard deviations higher than the mean of the power pattern (PP UB), and three standard deviations lower than the mean of the power pattern (PP LB) based on (20) are depicted in Fig. 1-a. In Fig. 2-b, the power pattern is depicted for 100 realizations based on (8).

The standard deviation based on (20) is calculated as 0.020 for the main-lobe level at $\theta = 0$, which is closely matched to 0.0188 as the standard deviation calculated numerically from 100 realizations for the same angle ($\theta = 0$). These results reveal the accuracy of (20).

In Fig. 2, the upper bounds and lower bounds of power pattern are depicted for the main-lobe and the first side-lobe levels in different settings of perturbations. According to the figures, we can investigate in each combination of perturbations, which one is the most affecting one that deteriorates the performance. The proposed formulation also facilitates error tolerance studies. For example, according to Fig. 2-b, we can say that for the tolerance level of up to one dB in the main beam (1 dB loss), the variance of perturbations in gain should be less than 0.1 ($\sigma_g^2 < 0.1$) with $\sigma_\lambda^2 = 0.001$ and $\sigma_\phi^2 = 0.001$.

V. CONCLUSION

We have formulated a closed-form expression for variance of power pattern in antenna array systems. This closed-form function can predict the behavior of beam pattern for any combination of perturbations in gain, phase and antenna positions, and is a reliable metric for antenna array performance analysis, distinguishing the dominant perturbations, and it can be used for impairment calibrations.

APPENDIX

Considering a Normal random variable as $Z \sim \mathcal{N}(\mu, \sigma^2)$, then the nth moment of Z will be [13]

$$M_Z(n) = e^{(\mu n + \sigma^2 n^2 / 2)}, \quad (21)$$

and then we have

$$\mathbb{E}[e^Z] = M_Z(1) = e^{(\mu + \sigma^2 / 2)}. \quad (22)$$

In general, with $\mathbf{a} = [a_1 \quad \dots \quad a_K]^T$ and $\mathbf{z} = [z_1 \quad \dots \quad z_K]^T$ and by using general moment function, we can write the expected value of product of normal random variables as

$$\mathbb{E}\left[\prod_{i=1}^{K} e^{a_i z_i} \right] = \mathbb{E}[e^{\mathbf{a}^T \mathbf{z}}] = e^{\mathbf{a}^T \mathbf{m} + 0.5 \mathbf{a}^T \Sigma \mathbf{a}}, \quad (23)$$

where $\mathbf{m} = [\mu_1 \quad \dots \quad \mu_K]^T$ is the vector of mean values, and Σ is the covariance matrix.

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