

THESIS FOR THE DEGREE OF PHILOSOPHY

Multipoint Okounkov bodies,
strong topology of
 ω -plurisubharmonic
functions and Kähler-Einstein
metrics with prescribed
singularities

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Göteborg, 2020

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Final Defence: 29th September 2020, a.a. 2019/2020.

Tor Vergata: XXXII^o ciclo, tutor prof. F. Bracci, coordinatore prof. A. Braides.

Chalmers: assistant supervisor prof. B. Berndtsson, examiner prof. R. Berman.

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Göteborg, 2020

ISBN 978-91-7905-372-7

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Doktorsavhandlingar vid Chalmers tekniska högskola

Ny series nr 4839

ISSN 0346-718X

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Typeset with L^AT_EX.

Printed in Gothenburg, Sweden 2020

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Abstract

The most classical topic in Kähler Geometry is the study of Kähler-Einstein metrics as solution of complex Monge-Ampère equations. This thesis principally regards the investigation of a strong topology for ω -plurisubharmonic functions on a fixed compact Kähler manifold (X, ω) , its connection with complex Monge-Ampère equations with prescribed singularities and the consequent study of singular Kähler-Einstein metrics. However the first part of the thesis, Paper I, provides a generalization of Okounkov bodies starting from a big line bundle over a projective manifold and a bunch of distinct points. These bodies encode renowned global and local invariants as the volume and the multipoint Seshadri constant.

In Paper II the set of all ω -psh functions slightly more singular than a fixed singularity type are endowed with a complete metric topology whose distance represents the analog of the L^1 Finsler distance on the space of Kähler potentials. These spaces can be also glued together to form a bigger complete metric space when the singularity types are totally ordered. Then Paper III shows that the corresponding metric topology is actually a strong topology given as coarsest refinement of the usual topology for ω -psh functions such that the relative Monge-Ampère energy becomes continuous. Moreover the main result of Paper III proves that the extended Monge-Ampère operator produces homeomorphisms between these complete metric spaces and natural sets of singular volume forms endowed their strong topologies. Such homeomorphisms extend Yau's famous solution to the Calabi's conjecture and the strong topology becomes a significant tool to study the stability of solutions of complex Monge-Ampère equations with prescribed singularities. Indeed Paper IV introduces a new continuity method with movable singularities for classical families of complex Monge-Ampère equations typically attached to the search of log Kähler-Einstein metrics. The idea is to perturb the prescribed singularities together with the Lebesgue densities and asking for the strong continuity of the solutions. The results heavily depend on the sign of the so-called cosmological constant and the most difficult and interesting case is related to the search of Kähler-Einstein metrics on a Fano manifold. Thus Paper V contains a first analytic characterization of the existence of Kähler-Einstein metrics with prescribed singularities on a Fano manifold in terms of the relative Ding and Mabuchi functionals. Then extending the Tian's α -invariant into a function on the set of all singularity types, a first study of the relationships between the existence of singular Kähler-Einstein metrics and

genuine Kähler-Einstein metrics is provided, giving a further motivation to study these singular special metrics since the existence of a genuine Kähler-Einstein metric is equivalent to an algebrico-geometric stability notion called K -stability which in the last decade turned out to be very important in Algebraic Geometry.

Keywords: Kähler Geometry, Complex Monge-Ampère equations, Pluripotential theory, Kähler-Einstein metrics, Canonical metrics, Fano manifolds, Okounkov bodies, Seshadri constant, Kähler packing.

Preface

This thesis consists of the following papers:

- Paper I** Antonio Trusiani,
“*Multipoint Okounkov bodies*” ,
arxiv preprint, <https://arxiv.org/abs/1804.02306>;
- Paper II** Antonio Trusiani,
“ *L^1 metric geometry of potentials with prescribed singularities on compact Kähler manifolds*” ,
arxiv preprint, <https://arxiv.org/abs/1909.03897>;
- Paper III** Antonio Trusiani,
“*The strong topology of ω -plurisubharmonic functions*” ,
arxiv preprint, <https://arxiv.org/abs/2002.00665>;
- Paper IV** Antonio Trusiani,
“*Continuity method with movable singularities for classical complex Monge-Ampère equations*” ,
arxiv preprint, <https://arxiv.org/abs/2006.09120>;
- Paper V** Antonio Trusiani,
“*Kähler-Einstein metrics with prescribed singularities on Fano manifolds*” ,
arxiv preprint, <https://arxiv.org/abs/2006.09130>.

Acknowledgments

A PhD thesis does not reflect the real journey of a PhD student. Mine was not easy at all, although I can bet almost all PhD students usually claim the same. However you all I am going to thank in the sequel only know tips of icebergs of the feelings, problems, doubts, gratifications and fears I have experienced during the last four years. Anyway I know it would have been impossible for me to conclude this journey without your support and your presence. You all were helping me bringing my burdens during the the steepest climbing and getting closer during the darkest silent nights. So, thank you.

Grazie a Stefano Trapani. Il merito (o la colpa) del mio dottorato di ricerca è in gran parte il tuo. Mi hai supportato fin dall'ultimo anno della magistrale e sei stato prima di tutto un advisor con i controfiocchi, che mi ha contattato più di una volta durante i miei periodi di silenzio. Non è una cosa che gli advisors solitamente fanno. Inoltre sei sempre stato presente nel darmi consigli matematici e non. Un ottimo amico. Spero di poter lavorare insieme in futuro come tu ben sai.

“Tack” to David Witt Nyström. I am sorry to not have learnt Swedish to thank you in your first language but you know the reasons. Thank you to have welcomed me in Chalmers as your PhD student although we basically did not know each other and there were a lot of bureaucracy problems. Thank you to have been also a patient advisor full of suggestions and advice, and to have spent a lot of time trying to teach me how to write a good introduction of a paper and to set its structure.

Thanks to prof. S. Kolodziej for accepting to be the opponent of this thesis, as to the members of committee: prof. C. Arezzo, prof. L. Turowska, prof. A. Sola and prof. C. Spotti.

Grazie ai miei fratelli matematici di Roma, Simone Diverio ed Eleonora Di Nezza, per essere stati presenti per domande matematiche e per consigli sul mondo della ricerca sia prima dell'inizio del mio dottorato che durante.

Riguardo Tor Vergata, vorrei ringraziare i dottorandi, i postdocs e i professori incontrati nelle (poche) mie presenze in dipartimento. In particolare grazie ai componenti del gruppo whatsapp DenteTor per piacevoli pranzi insieme, ai miei compagni di ufficio Gianluca, Antonio e Davide per avermi sopportato, a Matteo e Josias per

interessanti discussioni matematiche ed a Daniele per le conversazioni random.

About Chalmers, I would like to thank the group of complex geometry and the others PhD students, postdocs and professors I met during these years: in particular thank to (my mathematical brother) Jakob, (my officemates) Jimmy and Mingchen, Mattias, Richard, Hossein, Bo, Elizabeth, Martin, Lucas, Zakarias, Mats, Håkan, Jinciao, Jonathan, Valentina, Kristian, Umberto and Simone. You made me feel relaxed at the department, and with some of you I spent jovial afterworks and other beautiful moments. A special thank to my examiner Robert Berman to have contributed to make this joint PhD possible.

Grazie a voi, Amici di sempre. Quelli del paesino: Stefano, Alessio, Marco e Andrea. Quelli del liceo: Fabio e Michele. E il corridore Daniele. Siete stati, siete e sarete sempre presenti nella mia vita. Voi mi conoscete meglio di quanto io potrò mai... e mi siete ancora amici! Grazie anche per essere stati degli autisti impeccabili nei miei frequenti viaggi in aeroporto ad orari assurdi.

Grazie ad Alighieri e al gruppo di Atletica per avermi aiutato a scaricare l'inevitabile stress tramite sudore e crampi.

Thank to all my new friends I have met in Göteborg, who made me feel happy a lot of times (maybe too many times). Specially thank to Iris & Suraj, catalanass Berta, cocoach Antonio, Milo Normale, orsetto/toro di Verona Massimo, gatto Eric, volpe Matteo, popeye Matteo, scroccoboy Anton, piopao Paolo, Sonali, Gaurav, Luigi e Adriana.

Grazie a te, Amore mio. A modo tuo mi hai stretto la mano in questo percorso, facendomi distrarre dal lavoro vivendo indimenticabili momenti insieme.

Infine vorrei ringraziare la mia Famiglia. Prima di tutto i miei nonni. Siete sempre stati entusiasti della strada che stavo intraprendendo. Nonna Bianca, grazie per il tuo immenso calore. Spero di riuscire a trascorrere più tempo con te nell'imminente futuro. Nonno Geppetto Renato, Nonna Danda Iolanda, è molto doloroso realizzare che non potrete assistere alla conclusione di questo dottorato ma sono certo che ne sareste infinitamente orgogliosi. Vi porto nel cuore.

Grazie Alice per essere stata una sorella semplicemente ideale. In questi importanti quattro anni per entrambi ci siamo fiancheggiati a vicenda, sempre di comune accordo. Ovviamente grazie anche a Gino e alla piccola Penelope!

Grazie Mamma, e grazie Papà. Avete creduto nelle mie capacità molto prima di me e mi avete continuamente spronato in questo viaggio anche se in questi anni la distanza da casa ha avuto il suo peso. Vi voglio un mondo di bene e ve ne vorrò sempre.

Antonio Trusiani
Göteborg, September 2020

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Part I

INTRODUCTION

Chapter 1

Introduction

In these my first years in the mathematical world, I pictured Mathematics as an infinite dimensional puzzle where the pieces are given by theorems, conjectures and theories. This thesis regards Kähler Geometry, an area in Pure Mathematics which includes some of the most connected pieces.

Part of this work has a very classical flavor.

There are three names of scientists in the title, and one of them is maybe the most famous scientist in the history: A. Einstein. Through his general relativity theory in 1915, he revolutionized our vision of the universe, linking space and time in a precise geometrical way. In the vacuum the Einstein field equation simply asks that the Ricci curvature is a multiple of the metric, condition that afterward was called Einstein condition. The notion of metric and of its Ricci curvature come from Riemannian Geometry. A Riemannian manifold is indeed the data of a geometrical object which locally looks like the Euclidean space (i.e. a manifold), and of a metric which is, roughly speaking, a way to measure distances, angles, volumes and in the general to transfer the usual differential analysis of the Euclidean space to the manifold. Then the Ricci curvature of a metric intuitively measures how much computing volumes on the Riemannian manifold differs from the analog on the Euclidean space.

The amount of the proportionality between the Ricci curvature and the metric in the Einstein field equation depends on the so-called cosmological constant, which plays a determinate role to estimate the age, to describe the motion and to predict the future of our universe (i.e. it is the core of Cosmology). For the purposes of the thesis it is also important to underline that the cosmological constant influences the shape of the universe in the sense of its global geometry.

Theoretical Physics was reformed since Einstein's work as can be easily imagined, but it may appear surprising to find out that also Complex Geometry were strongly

inspired. The latter is the study of complex manifolds, which can be thought as manifolds provided with some rigidity.

Indeed in 1933 E. Kähler introduced the concept of special metrics (now called Kähler metrics) for a complex manifold to study the Einstein condition. His work gave the birth to Kähler Geometry which is a huge very active area in Pure Mathematics. A Kähler-Einstein metric is a Kähler metric which satisfies the Einstein condition.

In the compact case there is a topological obstruction (i.e. given directly from the geometry of the manifold) to the existence of Kähler-Einstein metric, which splits the problem to three very different cases based on the sign of the cosmological constant. When the latter is negative the existence and uniqueness of a Kähler-Einstein metric was completely solved by T. Aubin and S.T. Yau in 1976. S.T. Yau also showed the existence and the uniqueness when the cosmological constant is null (Ricci-flat) as special case of his solution to the famous Calabi's conjecture posed by E. Calabi in 1954 during the International Congress of Mathematicians in Amsterdam. The complex compact manifolds admitting a Kähler-Einstein metric with null cosmological constant are now called Calabi-Yau's manifold and in the last decades they come back to be relevant in Theoretical Physics for String Theory.

By what said above a Kähler metric induces a way to measure volumes on the complex manifold. In fact to any Kähler metric on a compact complex manifold is associated a volume form, and this correspondence is given through the so-called Monge-Ampère operator. S.T. Yau proved that the Monge-Ampère operator produces a bijection, 1-1 correspondence, between the set of all Kähler metrics which are similar to a fixed Kähler metric (namely, for those who knows, in the same cohomology class) and the set of all volume forms with the right total mass.

The third paper in this thesis extends Yau's bijection considering also some singular metrics in the sense of J.P. Demailly (1990) and some singular volume forms. Actually these two sets can also be endowed with natural geometrical structures (topologies) and the bijection in the third paper respects these structures. Let me stress to say that these topologies are related to another classical problem: the continuity of the Monge-Ampère operator.

The remaining case of positive cosmological constant (Fano manifolds) is more challenging since there are obstructions to the existence of Kähler-Einstein metrics as first proved by Y. Matsushima in 1957. Moreover the uniqueness holds modulo the identity connected component of the automorphism group as S. Bando and T. Mabuchi showed in 1987. However in 2015 X. Chen, S. Donaldson and S. Sun completed the characterization of the existence of Kähler-Einstein metrics in terms of an algebrico-geometric stability notion called K -(poly)stability. As predicted by S.T. Yau in 1993, their result links Riemannian and Algebraic Geometry, underlying the strength and the beauty of Kähler Geometry.

Recall that Algebraic Geometry studies projective manifolds, i.e. manifolds given as common zero sets of some polynomial equations. Thus, roughly speaking, a man-

ifold can be equipped with its projective structure given by its algebraic definition and with a Riemannian structure induced by the choice of a metric. This basically corresponds to studying the manifold using Algebra or using Analysis and Kähler Geometry can be thought as the meeting point of Algebraic Geometry and Riemannian Geometry as explained better in the sequel.

The fifth article consists of an analytic characterization of the existence of singular Kähler-Einstein metric with positive cosmological constants. A first comparison changing the singularities type is also provided. This gives a further motivation to study these singular Kähler-Einstein metrics. Indeed in the last decade K -stability turned out to be a very important notion in Algebraic Geometry since it is strongly related to other classical problems like the classification of projective varieties, but it is still hard to detect if a projective manifold is K -stable.

The work of S.T. Yau mentioned above and a different proof due to V. Datar and G. Székelyhidi of the characterization of Kähler-Einstein metrics on Fano manifolds in terms of K -stability are based on continuity methods. Namely they constructed a continuous family of complex Monge-Ampère equations varying a parameter $t \in [0, 1]$ such that at time $t = 0$ the equation is easier while the solutions at $t = 1$ give the Kähler-Einstein metrics. In this way they basically reduced the problem to the convergence of solutions, i.e. if a given family of solutions u_t converges for $t \rightarrow t_0$ to a solution u_{t_0} . **The fourth article** suggests new continuity methods where one also requires some prescribed singular behavior of the solutions, i.e. some further constraints. One advantage of this method is that one can choose to move the singularities without modifying the complex Monge-Ampère equation and a natural application is the stability of Kähler-Einstein metrics with different prescribed singularities.

Any PhD student deals with different difficulties during his/her research. **The second article** is the conclusion of mine most challenging period. Its final version, as often happens, does not keep track of all fails attempts and of all studies. It contains the metric structure on the space of singular metrics which was essential for all the sequel of my thesis, and it will be the starting point of some projects I have in mind for my immediate future.

The other name on the title of this title is A. Okounkov. He won the field medal recently in 2006 for his works and in particular because he found a way to construct a simplified image in an Euclidean space of an abstract geometric object. More precisely he associated to a n -dimensional projective manifold X endowed with an ample line bundle L (i.e. a manifold embedded in \mathbb{P}^N for some N) a convex bounded set $\Delta(L) \subset \mathbb{R}^n$ with interior not-empty. This object is now called Okounkov body, and it provides a way to study important algebrico-geometrical invariants of (X, L) , like the volume of L (a global measure of the positivity of L), through convex geometry. His construction comes back to the well-known correspondence between

toric polarized manifolds and their polytopes in toric geometry. $\Delta(L)$ depends on the choice of a point $x \in X$ and it is possible to study the local positivity of L at x recovering the Seshadri constant of L at x directly from the shape of $\Delta(L)$.

The first article of this thesis regards the construction of $N \geq 1$ Okounkov bodies $\Delta_1(L), \dots, \Delta_N(L)$ associated to the choice of an ample line bundle $L \rightarrow X$ and of N different points $x_1, \dots, x_N \in X$. These multipoint Okounkov bodies contain global and local positivity properties of $(L \rightarrow X; x_1, \dots, x_N)$, for instance in terms of the volume of L and of the multipoint Seshadri constant of L at x_1, \dots, x_N . It is worth to recall that the latter is related to several renewed conjectures in Algebraic Geometry and in particular to the Nagata's Conjecture which was introduced by M. Nagata in 1959 after he found a counterexample to the 14th Hilbert's problem.

1.1 Kähler geometry

Among all complex manifolds, Kähler manifolds have rich geometry and represent the transcendental variants of the more classical Projective manifolds.

All the thesis regards *compact* Kähler manifolds. Compactness is a key property that forces the global geometry of a complex manifold to have some further rigidity, gaining many global beautiful properties which can be investigated using analytic and algebraic tools.

The following exposition is necessarily sketchy. Good references are [GH], [Har] for classical results about Kähler and Algebraic Geometry, while [GZ17] for the pluripotential description of complex Monge-Ampère equations.

1.1.1 Projective manifolds

Algebraic Geometry is the study of algebraic varieties. The most classical of them are the projective varieties over \mathbb{C} .

Let \mathbb{P}^n denote the n -dimensional projective space defined as $\mathbb{C}^{n+1} \setminus \{0\} / \sim$ where the equivalence relation \sim is given as $(z_0, \dots, z_n) \sim (w_0, \dots, w_n)$ if there exists $\lambda \in \mathbb{C}^*$ such that $(z_0, \dots, z_n) = \lambda(w_0, \dots, w_n)$. Namely, any point in \mathbb{P}^n is represented by a complex line in \mathbb{C}^{n+1} , and in homogeneous coordinates this is expressed as $[Z_0 : \dots : Z_n]$. \mathbb{P}^n is an example of a n -dimensional complex manifold which is compact. Indeed it is the compactification of \mathbb{C}^n adding all the points at infinity, i.e. a copy of \mathbb{P}^{n-1} .

The zero set of any homogeneous polynomial in z_0, \dots, z_n descends to the quotient and defines a locus in \mathbb{P}^n . A projective algebraic set in \mathbb{P}^n is then the common zero set of a family of homogeneous polynomials in $n+1$ variables, and a *projective variety* is a projective algebraic set which is not the union of two distinct proper projective algebraic sets (i.e. it is irreducible). All projective varieties presented in this thesis are smooth, i.e. they are *projective manifolds*.

There is a natural topology for projective manifolds, called Zariski topology, which

arises by their algebraic definition and which is advantageous to explore all the algebrico-geometrical properties. However we are more interested in the topology induced by holomorphic coordinates, i.e. in the structure as complex manifolds. Note that this analytic structure is intrinsic, i.e. it does not depend on the particular embedding into the projective space. It is also significant to emphasize that there is a nice correspondence between the algebrico-geometrical and the analytic-geometrical point of view as the renowned Serre's GAGA Theorem explains.

1.1.2 Divisors

Many of the geometrical features of a projective manifold X can be described investigating the geometry of all its subvarieties of codimension 1, i.e. of all its *divisors*. An homogeneous polynomial f on \mathbb{C}^{n+1} cuts out a projective variety in \mathbb{P}^n of dimension $n - 1$, i.e. $D := \{f = 0\} \subset \mathbb{P}^n$ is a projective subvariety of dimension $n - 1$. For instance $f = z_0$ defines, in homogeneous coordinates, the locus $D = \{[0 : Z_1 : \dots : Z_n]\} \simeq \mathbb{P}^{n-1}$. Observe that the zero set of f^2 coincides with D , but the homogeneous polynomials are different. In the first case the polynomial is irreducible while in the second case the multiplicity of $f = 0$ is double. It is then convenient to keep track of the multiplicity saying that in the second case f^2 defines $2D$.

More generally a prime divisor of a projective manifold X of dimension n is a projective subvariety $D \subset X$ of dimension $n - 1$. In particular by the notation of the previous subsection any prime divisor is irreducible. Note that if $\dim X = 1$, then any point is a prime divisor, if $\dim X = 2$ then the prime divisors consists of all the irreducible projective curves. A divisor D is then given by a finite formal \mathbb{Z} -linear combination of prime divisors.

The additive group $\text{Div}(X)$ of all divisors is naturally endowed with a equivalence relation \sim_{lin} called the linear equivalence. Recall first that to any rational function $f := g_1/g_2 : X \dashrightarrow \mathbb{C}$, i.e. the ratio of two homogeneous polynomial g_1, g_2 of the same degree where $g_2|_X \not\equiv 0$, is associated a divisor $(f) := \text{div } g_1 - \text{div } g_2$ where similarly as before $\text{div } g_i$ is the divisor attached to g_i considering its zero locus counted with multiplicity. Then $D_1 \sim_{lin} D_2$ if there exists a not trivial rational function $f : X \dashrightarrow \mathbb{C}$ such that $D_1 = D_2 + (f)$.

The definition of divisors and their linear equivalence transfer to compact complex manifolds replacing projective subvariety with irreducible analytic subvariety and rational functions with global meromorphic functions. The latter are locally given by the ratio of holomorphic functions. It is also useful to recall that any divisor D can be described by the data $\{(U_j, f_j)\}_{j \in J}$ where U_j form an open cover of X induced by holomorphic coordinates and f_j are meromorphic functions on U_j such that $g_{j,k} := f_j - f_k$ is holomorphic with no zeros on $U_j \cap U_k$ for any $j \neq k$. Indeed (f_j) define local divisors which glue together into the divisor D . For those who know, this description has a precise interpretation in the language of sheaves. In fact denoting with \mathcal{O}_X (respectively with \mathfrak{M}_X) the sheaf on X of holomorphic (resp.

meromorphic) functions and with \mathcal{O}_X^* (resp. \mathfrak{M}_X^*) the subsheaf of the multiplicative elements, the space of global sections $H^0(X, \mathfrak{M}_X^*/\mathcal{O}_X^*)$ represents $\text{Div}(X)$ as follows from the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_X^* \rightarrow \mathfrak{M}_X^* \rightarrow \mathfrak{M}_X^*/\mathcal{O}_X^* \rightarrow 0. \quad (1.1)$$

A generic compact complex manifold may have very few divisors. For instance there are some complex tori with no divisors. However this is not the case when the manifold is projective since the intersection with hyperplanes on \mathbb{P}^N produce a lot of divisors on $X \subset \mathbb{P}^n$. Indeed we will see in the sequel that projective manifolds are characterized by the property of having many divisors.

1.1.3 Line bundles and sections

A *line bundle* L over a compact complex manifold X is a complex manifold of dimension $\dim X + 1$ with an holomorphic surjective map $p : L \rightarrow X$ such that $L_x \simeq \mathbb{C}$ for any $x \in X$ where $L_x := p^{-1}(x)$ is the fiber over x , and such that locally L looks like the product of the base X times \mathbb{C} . In other words, there exists a open cover $\{U_j\}_{j \in J}$ of X such that $L|_{p^{-1}(U_j)} \simeq U_j \times \mathbb{C}$ for any $j \in J$. Obviously by compactness of X the open cover can be assumed to be finite.

As immediate consequence of the maximum principle, all holomorphic functions on X are necessarily constant, but there may be many (global) holomorphic sections of a line bundle, i.e. $s : X \rightarrow L$ holomorphic map such that $p \circ s = \text{Id}_X$. The set of all holomorphic sections for a line bundle L is denoted with $H^0(X, L)$ and it is a finite-dimensional vector space over \mathbb{C} . Obviously any line bundle has a lot of local sections, but the existence of global section is a delicate matter which it is connected to the positivity of the line bundle as it will be more clear in the sequel.

When L twists, i.e. it is not given as $L = X \times \mathbb{C}$, any holomorphic section $s : X \rightarrow L$ has not-empty zero locus, i.e. s is associated to an effective divisor D . In this case L is isomorphic to the twisted line bundle $\mathcal{O}_X(D)$, whose holomorphic sections are given by all meromorphic functions which have poles at most D , i.e.

$$H^0(X, \mathcal{O}_X(D)) = \{f \text{ meromorphic on } X : (f) + D \geq 0\}$$

where $(f) + D \geq 0$ means that $(f) + D$ is effective. In particular if $D' \sim_{lin} D$ then the associated line bundles (i.e the twisted line bundles) $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ have the same space of global sections. Indeed it is possible to prove that two twisted line bundles are isomorphic if and only if the divisors are linearly equivalent, i.e. there is a well-defined injective map

$$\text{Div}(X)/\sim_{lin} \longrightarrow \text{Pic}(X) \quad (1.2)$$

where $\text{Pic}(X)$ is set of all line bundles over X modulo isomorphisms. The latter is a group, the Picard group, since line bundles are endowed with a multiplicative operation given by the tensor product whose inverse is obtained by considering the

dual. Indeed (1.2) is a monomorphism, i.e. it respects the group structures, and if X is projective then (1.2) is an isomorphism.

Recall also that the local trivializations of a line bundle are given by the choice of nowhere zero holomorphic local sections s_j . In fact a line bundle L can be expressed as an element in $H^1(X, \mathcal{O}_X^*)$ from the data $\{(U_j, s_j)\}_{j \in J}$, and more generally $\text{Pic}(X)$ is actually isomorphic to $H^1(X, \mathcal{O}_X^*)$.

As example, the projective space \mathbb{P}^n has a natural line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ whose fiber over $[Z_0 : \cdots : Z_n]$ is given by the dual of the complex line passing through (z_0, \dots, z_n) . This line bundle coincides with $\mathcal{O}_{\mathbb{P}^n}(H)$ where H is the hyperplane divisor cut by an homogeneous polynomial of degree 1 in $n+1$ variables.

1.1.4 Kähler metrics

An hermitian metric h on a compact complex manifold X is a smooth family of inner products (positive-definite Hermitian forms) on the holomorphic tangent spaces $T_x X$ for $x \in X$ locally generated by $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$. Locally

$$h = \sum_{j,k=1}^n h_{j,k} dz_j \otimes d\bar{z}_k$$

where $h_{j,k}$ are smooth functions and $n = \dim X$. The real part of a hermitian metric induces a Riemannian metric g on the underlying real manifold, while minus its imaginary part is a real 2-form ω called the fundamental form. In coordinates

$$\omega = \frac{i}{2} \sum_{j,k=1}^n h_{j,k} dz_j \wedge d\bar{z}_k. \quad (1.3)$$

The quantities h, g, ω preserve the complex structure J , i.e. $h(Ju, Jv) = h(u, v)$ and similarly for g, ω . We recall indeed that the underlying $2n$ -real dimensional manifold $X^{\mathbb{R}}$ is naturally endowed with a family of endomorphisms $J_x : T_x X^{\mathbb{R}} \rightarrow T_x X^{\mathbb{R}}$ on the real tangent space $TX^{\mathbb{R}}$, given in real coordinates as $J_x(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$, $J_x(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}$. Clearly $J_x^2 = -\text{Id}$ and by \mathbb{C} -linear extension $\{J_x\}_{x \in X^{\mathbb{R}}}$ splits $T_{\mathbb{C}} X^{\mathbb{R}} = TX^{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ into the holomorphic tangent bundle $T^{1,0}X$ (simply denoted TX as above) and the antiholomorphic tangent bundle $T^{0,1}X$ where the *almost complex structure* $J = \{J_x\}_{x \in X}$ corresponds respectively to the i -action and to minus the i -action. Observe that for ω preserving the complex structure is the same as saying that it is a $(1,1)$ -form, according to the natural decomposition of differential forms induced by J as in (1.3). Moreover ω is strictly positive, since for a $(1,1)$ -form its strict positivity is equivalent to the positive definiteness of the matrix $(h_{j,k})_{j,k=1}^n$ in the associated local expression (1.3).

The metric h is then said to be a Kähler metric if its fundamental form ω is closed. Equivalently, if ω is locally $\partial\bar{\partial}$ -exact, i.e. $\omega = \frac{i}{2} \partial\bar{\partial}u$ for a smooth function u called Kähler potential. This was the definition E. Kähler gave. Note that if h is Kähler

then ω is in particular a symplectic form.

Furthermore any strictly positive real closed $(1, 1)$ -form ω is said to be a *Kähler form*. Indeed any Kähler form is the fundamental form of a Kähler metric $h = g - i\omega$ setting $g(u, v) := \omega(u, Jv)$.

There are several other equivalent ways to introduce the Kähler condition. However a very brief upshot is that a Kähler manifold, namely a manifold admitting a Kähler metric, is a manifold which is endowed with three compatible structures: a Riemannian structure, a symplectic structure and a complex structure.

Moreover compact Kähler manifolds can be also thought as the transcendental analog of projective manifolds, and many algebraic properties of projective manifolds extend analytically to Kähler manifolds. In fact any projective manifold is a compact Kähler manifold. To see this observe that \mathbb{P}^N is naturally endowed with a Kähler metric called the Fubini-Study metric, whose Kähler form is expressed as

$$\omega_{FS} := dd^c \log ||z||^2 = \frac{i}{\pi} \partial \bar{\partial} \log ||z||^2 \quad (1.4)$$

where $||z||^2 = \sum_{j=0}^N |z_j|^2$ and $d^c := \frac{i}{4\pi}(\bar{\partial} - \partial)$. Then any projective manifold X inherits a Kähler metric given as restriction of the Fubini-Study metric for an embedding $X \subset \mathbb{P}^N$. However observe that different embeddings into projective spaces produce different metrics on X .

Finally it is useful to underline that a positive multiple of a Kähler form is still a Kähler form, as any convex linear combinations Kähler forms, i.e. the set of all the cohomology classes in $H^2(X, \mathbb{R})$ admitting a Kähler form as representative is a cone: the Kähler cone \mathcal{K} . A complex compact manifold is then Kähler if and only if $\mathcal{K} \neq \emptyset$.

1.1.5 Positivity of line bundles

A hermitian metric h on a line bundle $L \rightarrow X$ is the choice of a smooth family of inner products on the fibers $L_x \simeq \mathbb{C}$ varying $x \in X$. Locally h is determined specifying the length of the nowhere zero local sections s_j chosen for L , i.e. locally $||s_j||_h^2 = e^{-\phi_j}$ for $\phi_j : U_j \rightarrow \mathbb{R}$ smooth functions usually called the weights of h .

An important global objects attached to a hermitian metric h is its curvature defined locally as $dd^c \phi_j$. Indeed these local $(1, 1)$ -forms glue together to produce a closed smooth $(1, 1)$ -form on X , which depends on h but does not depend on the choice of the local sections s_j . If now one fixes a hermitian metric h_0 for L with curvature ω_{h_0} , then by definition it is easy to check that any other hermitian metric h for L is given as $h_0 e^{-\phi}$ for a global smooth function ϕ on X . In particular its curvature is given as $\omega_h = \omega_{h_0} + dd^c \phi$, which shows that the cohomology class of the curvature of hermitian metrics for a fixed line bundle does not depend on the choice of the metric. In fact the cohomology class of the curvature of any hermitian metric on L coincides with the first Chern class of L , $c_1(L)$, which is given through the cobordism

operator $\delta : H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z})$ obtained from the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^* \longrightarrow 0.$$

The whole image of the cobordism operator, i.e. the set of all first Chern classes, has a natural group structure and it is called Neron-Severi group, $NS(X) \subset H^2(X, \mathbb{Z})$. A line bundle L is said to be positive if $c_1(L) \in \mathcal{K}$, i.e. if there exists a hermitian metric on L whose curvature is a Kähler form. By a well-known Theorem of Kodaira (1954) being positive is related to the fact that multiples $L^{\otimes k}$ have enough global sections to produce an embedding $\Phi : X \hookrightarrow \mathbb{P}^N$ such that $L = \Phi^* \mathcal{O}_{\mathbb{P}^N}(1)$. At level of curvatures, $\omega_h = \Phi^* \omega_{FS|X}$, in fact ω_{FS} is a smooth representative of $c_1(\mathcal{O}_{\mathbb{P}^N}(1))$. In other words Kodaira's Theorem says that a complex compact manifold X is projective if and only if it admits an *ample* line bundle L , i.e. $c_1(L) \in NS(X) \cap \mathcal{K} \neq \emptyset$. The notion of ampleness can be also given in terms of divisors. In fact for a projective manifold X the Neron-Severi group can be expressed as $Div(X)/\equiv_{num}$, where \equiv_{num} is the numerical equivalence which grounds on Intersection Theory (see [Full]). For instance if $D_1 \equiv_{num} D_2$ then $D_1 \cdot C = D_2 \cdot C$ for any irreducible curve C on X , where the quantity $D_i \cdot C$ is equal to the number of points counted with multiplicity if D_i and C meet transversely. Obviously the linear equivalence implies the numerical equivalence. Then on a projective manifold by the Seshadri's criterion a divisor D is ample if and only if there exists a positive constant ϵ such that $D \cdot C \geq \epsilon \max_{x \in C} \text{mult}_x C$ for any irreducible curve C .

The definition of ampleness extends to \mathbb{Q}, \mathbb{R} -line bundles (resp. \mathbb{Q}, \mathbb{R} -divisors) taking finite \mathbb{K} -linear combinations for $\mathbb{K} = \mathbb{Q}, \mathbb{R}$. With obvious notations $NS(X)_{\mathbb{K}} = NS(X) \otimes_{\mathbb{K}} \mathbb{K}$, which is a finite-dimensional \mathbb{K} -vector space. Then the ample cone is naturally given as $\mathcal{A} := NS(X)_{\mathbb{R}} \cap \mathcal{K}$, and its closure is the algebraic part of the nef cone $\mathcal{N} = \overline{\mathcal{K}}$, i.e. $\mathcal{N}_{NS} := \mathcal{N} \cap NS(X)_{\mathbb{R}}$. It immediately follows from said above that on a projective manifold a \mathbb{R} -divisor D is nef if and only if $D \cdot C \geq 0$ for any irreducible curve C .

1.1.6 Quasi-plurisubharmonic functions

An important notion of convexity in several complex variables is given by *plurisubharmonicity*. Letting $\Omega \subset \mathbb{C}^n$ a domain, a plurisubharmonic function $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function, $u \not\equiv -\infty$ such that $u|_{L \cap \Omega}$ is subharmonic for any complex line $L \subset \mathbb{C}^n$.

The notion is local, and examples of plurisubharmonic functions are given by the pluriharmonic functions, which locally represent the real part of holomorphic functions and which in particular are analytic. Plurisubharmonic functions instead may be singular, but they have good integrability properties since $PSH(\Omega) \subset L^p_{loc}$ for any $p \in [1, +\infty)$ with gradients is L^q for any $1 \leq q < 2$.

When $u \in C^2(\Omega)$ the plurisubharmonicity condition is equivalent to the positivity

of the Levi form of u , namely

$$\mathcal{L}(u) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \otimes d\bar{z}_k \geq 0.$$

We recall that this positivity is equivalent to the semi-definite positiveness of the matrix $(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k})_{j,k=1,\dots,n}$. More generally any u locally integrable upper semicontinuous function is plurisubharmonic if and only if $\mathcal{L}(u) \geq 0$ in the weak sense of distributions (see also next subsection).

Another important feature of plurisubharmonicity is that it is preserved under holomorphic maps with image not contained in the $\{-\infty\}$ -locus, i.e. given $u \in PSH(X, \Omega)$ and $f : \Omega' \rightarrow \Omega$ holomorphic, $u \circ f \in PSH(X, \Omega')$ or $u \equiv -\infty$. In particular it make sense to talk of plurisubharmonic functions on complex manifolds, but as a consequence of the maximum principle all the global plurisubharmonic functions on a compact complex manifold are constant. However when X is also Kähler there are many *quasi-plurisubharmonic* (q-psh) functions, i.e. $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ such that locally u is given as the sum of a plurisubharmonic function and a smooth function. The natural topology on these functions is the L^1 -topology which in this thesis will be called weak topology. Note that for L^1 -topology we mean the $L^1(X, dV)$ -topology where dV is any fixed volume form on X . Finally recall that on the set of quasi-plurisubharmonic functions the weak topology is equivalent to the L^p -topology for any $p > 1$.

1.1.7 Currents

A current can be thought as a differential form with distributional coefficients. More precisely, on a compact complex manifold, a current of bidegree (p, q) (also said (p, q) -current) is a continuous linear functional on the space of smooth differential forms of bidegree $(n - p, n - q)$. The pairing is indicated with $\langle T, u \rangle$ or simply by $\int_X T \wedge u$. Indeed locally

$$T = \sum_{|I|=p, |J|=q} T_{I,J} dz_I \wedge d\bar{z}_J \quad (1.5)$$

where $T_{I,J}$ are distributions (against smooth functions) and where with obvious multi-index notation $dz_I = dz_{i_1} \wedge \dots \wedge dz_{i_p}$ and similarly for $d\bar{z}_J$.

The exterior derivative d naturally extends to currents, i.e. if T is a (p, q) -current then $\langle dT, u \rangle := (-1)^{p+q+1} \langle T, du \rangle$ for any u smooth differential form of the right bidegree. The currents ∂T and $\bar{\partial} T$ are defined similarly and $d = \partial + \bar{\partial}$. A current T is then said to be closed if $dT = 0$.

This thesis will principally regards $(1, 1)$ -currents which are *positive*. A $(1, 1)$ -current T is said to be positive if locally $T = dd^c u$ for u local plurisubharmonic function, where we recall that $dd^c = \frac{i}{2\pi} \partial \bar{\partial}$. This is equivalent to ask that $(T_{j,k})_{j,k=1,\dots,n}$

is semi-positive definite in the associated local description as in (1.5). Similarly a (n, n) -currents T is positive if $\int_X fT \geq 0$ for any $f \in C^\infty(X)$, $f \geq 0$. For a current of bidegree (p, p) the definition of positivity is slightly more complicated. However we underline that any positive current has order 0, i.e. it acts on continuous differential forms.

Combining the $\partial\bar{\partial}$ -lemma and the de-Rham's Theorem one also gets

$$H^{1,1}(X, \mathbb{R}) = \frac{\{T \text{ closed } (1, 1)\text{-current}\}}{\{T \text{ } dd^c\text{-exact } (1, 1)\text{-current}\}}.$$

As a consequence given a cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ which admits a closed and positive $(1, 1)$ -current T , i.e. α *pseudoeffective*, and given a smooth $(1, 1)$ -form θ representative of α the set

$$PSH(X, \theta) := \{u \text{ q-psh, : } \theta + dd^c u \geq 0\}$$

is not-empty and it is called the set of all θ -psh functions. At level of topologies $PSH(X, \theta)$ is homeomorphic to $\mathcal{T}_{\{\theta\}}X \times \mathbb{R}$ where the set $\mathcal{T}_{\{\theta\}}X$ of all closed and positive $(1, 1)$ -currents with cohomology class $\{\theta\}$ is naturally endowed with its weak topology. When $\theta = \omega$ is a Kähler form, the set of ω -psh functions will be one of the principal character of the sequel.

Observe also that since $\mathcal{T}_{\{\theta\}}X$ is weakly compact, any set $\{u \in PSH(X, \theta) : |\sup_X u| \leq C\}$ for $C \in \mathbb{R}$ is weakly compact. Moreover if $u_k \rightarrow u$ weakly as elements in $PSH(X, \theta)$ then $\sup_X u_k \rightarrow \sup_X u$ (it is often called Hartogs' Lemma). Important examples of closed $(1, 1)$ -currents are given by currents of integration along divisors. Letting D be a divisor, the current of integration $[D]$ is given as

$$\langle [D], u \rangle := \int_D u$$

for any smooth $(n-1, n-1)$ -form u . Note that if s a holomorphic section of $L := \mathcal{O}_X(D)$ cutting the divisor D (i.e. $(s) = D$ as mentioned in subsection 1.1.3), and h is a hermitian metric on L with curvature θ , then

$$[D] = \theta + dd^c \log |s|_h^2.$$

This follows from the Poincaré-Lelong equation: $dd^c \log |f|^2 = (f)$ for f holomorphic function.

1.1.8 Non-pluripolar product and Monge-Ampère operator

The wedge product among currents is not always well-defined, but the authors in [BEGZ10] found a way to define the wedge product of closed and positive $(1, 1)$ -currents through the so-called *non-pluripolar product*. The term non-pluripolar

means that such wedge product does not take mass on pluripolar sets, i.e. on Borel sets locally contained in $\{u = -\infty\}$ for u local psh function.

The construction in [BEGZ10] relies on the work of Bedford, Taylor ([BT87]), where the coarsest refinement of the usual topology such that all plurisubharmonic functions become continuous (plurifine topology) was introduced, and where the authors defined the wedge product $dd^c u_1 \wedge \cdots \wedge dd^c u_k$ for u_1, \dots, u_k locally bounded psh functions on a complex manifold. One main property of the Bedford-Taylor construction is that it is local with respect to the plurifine topology, i.e.

$$\mathbf{1}_U dd^c u_1 \wedge \cdots \wedge dd^c u_p = \mathbf{1}_U dd^c v_1 \wedge \cdots \wedge dd^c v_p$$

if $u_j = v_j$ on U plurifine open set. Imposing this property for unbounded psh functions u_1, \dots, u_p in [BEGZ10], the non-pluripolar product $\langle dd^c u_1 \wedge \cdots \wedge dd^c u_p \rangle$ is completely determined by

$$\mathbf{1}_{\bigcap_{j=1}^p \{u_j > -C\}} \langle \bigwedge_{j=1}^p dd^c u_j \rangle = \mathbf{1}_{\bigcap_{j=1}^p \{u_j > -C\}} \bigwedge_{j=1}^p dd^c \max(u_j, -C)$$

for any $C \in \mathbb{R}$ since $\bigcap_{j=1}^p \{u_j = -\infty\}$ is pluripolar. We recall that the maximum of a finite set of psh functions is psh.

The main problem for this construction is that the non-pluripolar product may not have locally finite mass as an example of Kiselman shows ([Kis84]), but on a compact Kähler manifold X this cannot happen. More precisely given T_1, \dots, T_p closed and positive $(1, 1)$ -currents, the non-pluripolar product $\langle T_1 \dots T_p \rangle$ locally defined as $\langle dd^c u_1 \wedge \cdots \wedge dd^c u_p \rangle$ for local psh potentials u_1, \dots, u_p has finite mass over X , it does not take mass over any pluripolar set, and it is a closed and positive (p, p) -current. A principal role in this thesis is played by the Monge-Ampère operator. Assuming ω Kähler, the Monge-Ampère operator is defined as $MA_\omega(u) := \langle (\omega + dd^c u)^n \rangle$ for any $u \in PSH(X, \omega)$. When $u \in C^2$ one has locally

$$MA_\omega(u) = \frac{n!}{(2\pi)^n} \det \left(\frac{\partial^2 (\varphi + u)}{\partial z_j \partial \bar{z}_k} \right) i^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

where $\omega = dd^c \varphi$. In particular this brief calculation explains the nomenclature since $MA_\omega(\cdot)$ represents the complex analog of the real Monge-Ampère operator first studied by G. Monge in the 1780 and A. Ampère in the 1820.

It is also important to underline that $\int_X MA_\omega(u) = \int_X \omega^n$ for any $u \in PSH(X, \omega) \cap C^2$ as an immediate consequence of Stokes' Theorem, i.e. the Monge-Ampère mass of smooth ω -psh functions is a cohomological quantity called volume (see also next subsection). More generally one has $\int_X MA_\omega(u) \leq \int_X \omega^n$ since roughly speaking the non-pluripolar product does not consider the mass contained in pluripolar sets. Bedford and Taylor also proved that the Monge-Ampère operator is continuous with respect to monotonic sequences, and this property keep holding for the non-

pluripolar product if all elements in the sequence and the limit have full Monge-Ampère mass, i.e. belong to

$$\mathcal{E}(X, \omega) := \left\{ u \in PSH(X, \omega) : \int_X MA_\omega(u) = \int_X \omega^n \right\}.$$

In the sequel we will omit the bracket notation for the non-pluripolar product to be coherent with the notations used in the papers.

1.1.9 Singular metrics and volume

In this thesis particular importance is given to the notion of singular metrics of a line bundle L introduced by J.P. Demailly in [Dem90]. Letting h_∞ be a hermitian metric on L with curvature θ , a singular metric on L is given as $h = h_\infty e^{-\varphi}$ for $\varphi \in L^1$. When φ belongs to $PSH(X, \theta)$ the singular metric h is said positive and its curvature $\theta + dd^c \varphi$ is a closed positive $(1, 1)$ -current. Indeed there is a one-one correspondence between positive singular metrics for L and $PSH(X, \theta)$, and all closed and positive $(1, 1)$ -currents representatives of $c_1(L)$ are given as curvatures of positive singular metrics for L .

Thus the algebraic part of the pseudoeffective cone $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of all cohomology classes which admits a closed and positive $(1, 1)$ -currents is given by $\mathcal{E} \cap NS(X)_\mathbb{R}$ and coincides with the closure of the cone of all (numerical equivalence class of) effective \mathbb{R} -divisors. In fact any effective \mathbb{R} -divisor D induces a singular metric with curvature $[D]$ on the associated \mathbb{R} -line bundle L .

The interior of the pseudoeffective cone is the big cone \mathcal{B} whose cohomology classes are characterized to admit a Kähler current as representative, i.e. a closed and positive $(1, 1)$ -current T such that $T \geq \epsilon \omega$ for $\epsilon > 0$ small enough and ω fixed Kähler form on X . Then the cone $\mathcal{B} \cap NS(X)_\mathbb{R}$ coincides with the cone generated by all numerical equivalence classes of big divisors/line bundles, and an analog of Kodaira's embedding theorem holds: X admits a big line bundle if and only if X is birational to a projective manifold (i.e. X is Moishezon). Indeed for any $L \rightarrow X$ big line bundle the space of global sections $H^0(X, kL)$ (using the additive notations for the tensor product) has maximal growth, i.e. its dimension as vector space grows as k^n where $n = \dim X$, and the quantity

$$\text{Vol}_X(L) = \limsup_{k \rightarrow \infty} \frac{\dim_{\mathbb{C}} H^0(X, kL)}{k^n/n!} \in \mathbb{R}_{>0}$$

is called the *volume* of $L \rightarrow X$.

The normalization is chosen so that $\text{Vol}_{\mathbb{P}^n}(\mathcal{O}_X(1)) = 1$ since the space of all global sections of $\mathcal{O}_{\mathbb{P}^n}(k) := k\mathcal{O}_X(1)$ is isomorphic to the space of all homogeneous polynomial of degree k in $n+1$ variables. Note that in this case the volume coincides with the top self-intersection $(\mathcal{O}_{\mathbb{P}^n}(1)^n)$ analytically described as $\int_X \theta^n$ when θ is a smooth $(1, 1)$ -form representative of $c_1(\mathcal{O}_{\mathbb{P}^n}(1))$. More generally if L is an ample line bundle then $\text{Vol}_X(L) = (L^n)$ by asymptotic Riemann-Roch Theorem.

To give a pluripotential description of $\text{Vol}_X(L)$ and to extend the notion of volume to big/pseudoeffective cohomology class, we first observe that, for θ smooth $(1,1)$ -form, $PSH(X, \theta)$ has a natural partial order \preceq given as $u \preceq v$ if $u \leq v + C$ for a constant $C \in \mathbb{R}$, i.e. the partial order is given comparing the singularities. The function

$$V_\theta := \sup\{u \in PSH(X, \theta) : u \leq 0\}$$

is θ -psh function and the associated closed and positive $(1,1)$ -current $T_{min} := \theta + dd^c V_\theta$ is said to have minimal singularities for clear reasons. If $\{\theta\} = c_1(L) \in \mathcal{E} \cap \text{NS}(X)$ for a pseudoeffective line bundle L then

$$\text{Vol}_X(L) = \int_X T_{min}^n \quad (1.6)$$

where the top wedge product on the right side of (1.6) is in the sense of the non-pluripolar product. Thus the volume of a pseudoeffective class α not necessarily integral can be naturally defined as $\text{Vol}_X(\alpha) := \int_X T_{min}^n$ and one gets that $\text{Vol}_X(\alpha) > 0$ if and only if α is big.

1.1.10 Canonical divisor and volume forms

The canonical divisor K_X possesses many clues about the geometry of X . Local trivializations of the associated line bundle $\det(T^*X)$, always denoted by K_X , are holomorphic $(n,0)$ -forms. In particular a positive metric h on $\pm K_X$ naturally determines a volume form μ_h given locally as

$$\mu_h = i^{n^2} e^{\pm \phi_j} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

where $\phi_j = -\log \|s_j\|_h^2$ are the local weights defining h , i.e. s_j are nowhere zero holomorphic local sections of $\pm K_X$. For instance when K_X is trivial there is a nowhere zero global section s and this leads to a volume form $dV = i^{n^2} s \wedge \bar{s}$. The first Chern class of the anticanonical bundle $-K_X$ is denoted with $c_1(X)$ and it is said the first Chern class of X .

1.1.11 Kähler-Einstein metrics

A Kähler-Einstein metric h is a Kähler metric on X with associated Riemannian metric g and fundamental form ω (see subsection 1.1.4) such that

$$\text{Ric}(g) = \lambda \omega \quad (1.7)$$

where $\text{Ric}(g)$ is the Ricci curvature of g and $\lambda \in \mathbb{R}$.

The Ricci curvature is given by taking the trace of the Riemannian curvature which contains the information about how the manifold is curved. The Ricci curvature

measures how much the volume form dV_g on the manifolds differs from the standard Euclidean volume form dV_{Eucl} . Indeed

$$dV_g = \left(1 - \frac{1}{6}R_{jk}x^jx^k + O(\|x\|^3)\right)dV_{Eucl},$$

where $\text{Ric}(g) = \sum_{j,k=1}^{2n} R_{j,k}dx_j \wedge dx_k$ in real coordinates. We also underline that classical theorems in Riemannian geometry connect the global geometry of the manifold to lower bounds of the Ricci curvature.

In Kähler geometry the Ricci form $\text{Ric}(g)$ is a closed $(1,1)$ -form with cohomology class $c_1(X)$. In fact

$$\text{Ric}(g) = -dd^c \log \det(g)$$

which can be seen as the curvature of an hermitian metric on $-K_X$. This important remark has key consequences. First, the first Chern class must have a sign to give a sense to (1.7), i.e. the search of Kähler-Einstein metrics can only be performed in the three following cases:

- i) K_X trivial;
- ii) K_X ample;
- iii) $-K_X$ ample.

The first case corresponds to $\lambda = 0$ and the problem reduces to find Ricci-flat metrics. In the remaining cases the cohomology class of the Kähler form ω must be proportional to $c_1(X)$, so up to rescaling the Kähler form we may suppose $\lambda = \pm 1$, i.e. $\pm\omega \in c_1(X)$. In particular ω coincides with the curvature of a positive hermitian metric on $\mp K_X$. So for ω_0 fixed Kähler form in $\mp K_X$ the problem to find a Kähler-Einstein metric in (ii), (iii) is equivalent to find an element in

$$\mathcal{H}_{\omega_0} := \{u \in PSH(X, \omega_0) \cap C^\infty(X) : \omega_0 + dd^c u > 0\},$$

i.e. a Kähler form $\omega = \omega_0 + dd^c u$ in $\pm c_1(X)$, such that

$$\text{Ric}(\omega) = \pm\omega$$

where with obvious notation $\text{Ric}(\omega) := \text{Ric}(g_\omega)$ for g_ω Riemannian metric associated to ω .

As said in the prelude, in the case (i) the problem was completely solved by Yau ([Yau78]) as particular consequence of the resolution of Calabi's Conjecture. Namely he proved that for any $\alpha \in \mathcal{K}$ and for any ρ closed $(1,1)$ -form with cohomology class $c_1(X)$ there exists a unique Kähler form ω in α such that $\text{Ric}(\omega) = \rho$. In the last five decades, manifolds with K_X trivial have been denominated Calabi-Yau manifolds. If K_X is ample then X is a manifold of general type, and there exists a unique Kähler-Einstein metric ([Yau78], [Aub78]), i.e. there exists a unique ω Kähler form in the cohomology class of K_X such that $\text{Ric}(\omega) = -\omega$.

Instead, in the Fano case the uniqueness is modulo the identity component of the automorphism group ([BM87]) while there are obstructions on the existence of Kähler-Einstein metrics. For instance Matsushima in [Mat57] showed that a necessary condition to the existence of Kähler-Einstein metrics is the reductiveness of the automorphism group. Recently Chen, Donaldson, Sun ([CDS15]) proved that the existence of Kähler-Einstein metrics for a Fano manifold is equivalent to an algebrico-geometrical notion called K -(poly)stability (see next subsection).

As consequence of the classical Uniformization Theorem, any Riemann Surface (i.e. a complex compact manifold of dimension 1) has a Kähler-Einstein metric. For Kähler surfaces the obstruction found by Matsushima is also sufficient for the existence of Kähler-Einstein metric as proved by Tian ([Tian90]). For instance \mathbb{P}^2 admits a Kähler-Einstein metric, the Fubini-Study metric, while the blow-up at one or at two distinct points of \mathbb{P}^2 does not.

In higher dimension the situation is much more complicated and already in dimension 3 there are Fano manifolds whose K -stability properties are unknown. In fact detecting K -stability is very hard (see again next subsection), which is one of the main motivation of the last two papers of this thesis and of future works.

1.1.12 Yau-Tian-Donaldson Conjecture

Given a polarization (X, L) , i.e. an ample line bundle L over a projective variety X (which might have some singularities), the Yau-Tian-Donaldson Conjecture states that (X, L) is K -stable if and only if $L \rightarrow X$ admits a hermitian metric h whose curvature ω determines a *constant scalar curvature Kähler metric* on X . The scalar curvature of a metric is obtained as trace of the Ricci curvature. In particular any Kähler-Einstein metric has constant scalar curvature, and when $c_1(X)$ has a sign and $c_1(L)$ is proportional to $c_1(X)$ then it is not difficult to prove that any constant scalar curvature Kähler metric is Kähler-Einstein.

It took several years to give the actual notion of K -stability which was completed by Tian in [Tian97] and Donaldson in [Don02]. It was inspired by Geometric Invariant Theory, and involves the positivity of all the weights, called Donaldson-Futaki invariants, associated to test configurations (see below). The right definition is quite technical and it is beyond the purpose of this Introduction, so we just give a sketchy presentation.

Given an ample line bundle $L \rightarrow X$ where X is just a projective variety, a test configuration $(\mathfrak{X}, \mathfrak{L})$ for the pair (X, L) is an equivariant \mathbb{C}^* -degeneration of the pair. More precisely it is the data of a family $p : \mathfrak{X} \rightarrow \mathbb{C}$ such that $p^{-1}(t)$ is isomorphic to X for any $t \neq 0$ through the natural \mathbb{C}^* -action, and of a \mathbb{C}^* -equivariant line bundle $\mathfrak{L} \rightarrow \mathfrak{X}$ such that $(X_t, \mathfrak{L}_{X_t}) \simeq (X, L)$ for any $t \neq 0$ through the \mathbb{C}^* -action where we set $X_t := p^{-1}(t)$. Then the central fiber X_0 , which may be singular even if X is smooth, is endowed with a \mathbb{C}^* -action and the Donaldson-Futaki invariant of $(\mathfrak{X}, \mathfrak{L})$, usually denoted $DF(\mathfrak{X}, \mathfrak{L})$, basically represents the Hilbert-Mumford weight of the \mathbb{C}^* -action. Then (X, L) is said K -semistable if $DF(\mathfrak{X}, \mathfrak{L}) \geq 0$ for any test

configuration $(\mathfrak{X}, \mathfrak{L})$, while (X, L) is said to be K -stable if it is K -semistable and the Donaldson-Futaki invariant vanishes only when the test configuration is almost trivial (i.e. the test configuration is close to be the trivial product).

It is worth to underline that any test configuration can be realized as an actual \mathbb{C}^* -degeneration of (X, L) into a fixed projective space \mathbb{P}^N , but the dimension N can be arbitrarily big which suggests the difficulties in detecting which pairs (X, L) are K -stable.

Although the definition of K -stability came from the problem to find special metrics for line bundles $L \rightarrow X$, in the last decade it gains a great importance in Algebraic Geometry because of its connection with the Minimal Model Program and with the construction of moduli spaces.

1.1.13 Complex Monge-Ampère equations

Monge-Ampère equations arise in different areas of Mathematics and in particular in Optimal Transport Theory. This thesis principally concerns the study of complex Monge-Ampère equations of the type

$$\begin{cases} MA_\omega(u) = \mu \\ u \in PSH(X, \omega) \end{cases} \quad (1.8)$$

for μ positive Borel measure, where ω is a fixed Kähler form on a compact manifold X and $MA_\omega(u) = (\omega + dd^c u)^n$ is the Monge-Ampère operator (see subsection 1.1.8). When the measure μ is smooth with total mass equal to $\int_X \omega^n$ it makes sense to look for a smooth solution, but in general weak solutions are requested to solve (1.8).

These equations, allowing also some twisting term on the right hand side, are strongly related to the search of Kähler-Einstein metrics. In fact, assuming X to be Calabi-Yau and ω Kähler, there exists $f \in C^\infty$ smooth function such that $\text{Ric}(\omega) = dd^c f$ while by definition

$$\text{Ric}(\omega + dd^c u) = \text{Ric}(\omega) - dd^c \log \left(\frac{(\omega + dd^c u)^n}{\omega^n} \right).$$

Thus since on a compact manifold any pluriharmonic function is constant, the search of a Kähler Ricci-flat $(1, 1)$ -form $\omega + dd^c u$ is equivalent to solve the complex Monge-Ampère equation

$$\begin{cases} MA_\omega(u) = e^{f+a} \omega^n \\ u \in \mathcal{H}_\omega \end{cases} \quad (1.9)$$

where a is a numerical constant given imposing the right total mass, i.e. $a = \log \int_X \omega^n - \log \int_X f \omega^n$. Yau showed the existence of an unique Kähler-Einstein metric solving (1.9). Indeed replacing e^f with any arbitrary smooth positive function, one obtains the pluripotential description of the Calabi's conjecture.

In the case $\mp K_X$ ample and ω with cohomology class $\pm c_1(X)$, let $f \in C^\infty$ such that

$\text{Ric}(\omega) = \pm(\omega + dd^c f)$, i.e. f Ricci potential of ω . Then finding a Kähler-Einstein metric is equivalent to solve the complex Monge-Ampère equation

$$\begin{cases} MA_\omega(u) = e^{\pm(f-u)}\omega^n \\ u \in \mathcal{H}_\omega. \end{cases} \quad (1.10)$$

Analytically the sign on the right hand side in (1.10) yields the possibility to use the maximum principle to get a C^0 -estimate. It is in fact worth to underline that the solution provided by Yau and by Aubin to the Kähler-Einstein problem in the ample canonical line bundle used a continuity method (see subsection 1.4.2) with a priori estimates, while in the Fano case the same method does not apply since the C^0 -estimate does not always hold.

1.2 Paper I

Paper I regards the data of a big line bundle L over a projective manifold X and of a choice of N distinct points on X .

1.2.1 Multipoint Seshadri constant

J.P. Demailly in [Dem90] introduced a way to measure the positivity of a nef line bundle L at a point x of a projective manifold X , the *Seshadri constant* of L at x :

$$\epsilon_S(L; x) := \inf \frac{L \cdot C}{\text{mult}_x C}$$

where the infimum is over all irreducible curves passing through x . Equivalently $\epsilon_S(L; x) = \sup\{t > 0 : p^*L - tE \text{ is nef}\}$, where $p : \text{Bl}_x X \rightarrow X$ is the blow-up at x while E is the exceptional divisor. Note in particular that the Seshadri constant is a cohomological invariant and that necessarily $\epsilon_S(L; x) \leq \sqrt[n]{L^n}$ for any ample line bundle. Moreover it is clear that $\epsilon_S(L; x) \geq 0$ and that the inequality is strict if L is ample (see also subsection 1.1.5). In fact the Seshadri criterion can be phrased as $\inf_{x \in X} \epsilon_S(L; x) > 0$ if and only if L is ample.

The Seshadri constant can be also described as the biggest asymptotic order at x which can be completely prescribed by the ring $R(X, L) := \bigoplus_{k \in \mathbb{N}} H^0(X, kL)$, i.e. in terms of jets. Namely, for any $k \in \mathbb{N}$ let $s_k(x) \in \mathbb{N}$ be the biggest natural number such that all jets of order less or equal to $s_k(x)$ can be prescribed by global sections in $H^0(X, kL)$. Then $\epsilon_S(L; x) = \lim_{k \rightarrow \infty} s_k(x)/k$.

This last interpretation easily generalizes to big line bundles. In fact Nakamaye in [Nak03] defined the *moving* Seshadri constants for big line bundles, which can be compute in terms of jets as in the nef case.

Considering more (distinct) points x_1, \dots, x_N , the analog of the Seshadri constant

is the *multipoint Seshadri constant*

$$\epsilon_S(L; x_1, \dots, x_N) := \inf \frac{L \cdot C}{\sum_{j=1}^N \text{mult}_{x_j} C}$$

where the infimum is over all irreducible curves passing at least one point among x_1, \dots, x_N . Similarly to the one point case, it describes the positivity of the nef line bundle L at the points chosen and $\epsilon_S(L; x_1, \dots, x_N) = \sup\{t \geq 0 : p^*L - \mathbb{E} \text{ is nef}\}$ where $p : \text{Bl}_{\{x_1, \dots, x_N\}} X \rightarrow X$ is the blow-up at $\{x_1, \dots, x_N\}$ while $\mathbb{E} := \sum_{j=1}^N E_j$ is the sum of the exceptional divisors. This yields $\epsilon_S(L; x_1, \dots, x_N) \leq \sqrt[n]{(L^n)/N}$. As before the multipoint Seshadri constant also coincides with the biggest asymptotic order at x_1, \dots, x_N which can be prescribed by $R(X, L)$, and this jets interpretation gives an equivalent version of the *moving* multipoint Seshadri constant for big line bundles.

Try to compute and/or to estimate the multipoint Seshadri constants is of considerable importance in Algebraic Geometry since there are several renowned conjectures and theories attached to this invariant. A classical example is given by the Nagata's Conjecture, stated by M. Nagata in 1958 ([Nag58]). It is equivalent to prove that the multipoint Seshadri constant of $\mathcal{O}_{\mathbb{P}^2}(1)$ at $N \geq 9$ points in very general position is maximal, i.e. $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$. Recall that $\epsilon_S(L; \cdot)$ is lower-semicontinuous for any nef line bundle L and its supremum is reached outside a countable union of proper subvarieties, i.e. when the points are in very general position. It is then reasonable to set $\epsilon_S(L; N)$ where the N points are in very general position.

1.2.2 Toric manifolds

A toric manifold of dimension n is a complex manifold which has an action $(\mathbb{C}^*) \curvearrowright X$ with a dense open orbit where $(\mathbb{C}^*)^n$ represents the n -dimensional complex torus. When $X \subset \mathbb{P}^N$ is also projective, it is given as compactification of the $(\mathbb{C}^*)^n$ -action which lifts to the line bundle $L := \mathcal{O}_{\mathbb{P}^N}(1)|_X$, and many of the geometrical properties of (X, L) are encoded in a Delzant polytope $P_L \subset \mathbb{R}^n$, i.e. in a convex hull of a finite number of points in \mathbb{Z}^n such that any vertex has exactly n edges starting from it. Indeed there is a 1-1 correspondence between Delzant polytopes and polarized toric manifolds (X, L) , namely X toric manifolds and L torus-invariant line bundles. More precisely given a Delzant polytope P , for any $k \in \mathbb{N}$ define the map $f_{kP} : (\mathbb{C}^*)^n \rightarrow \mathbb{P}^{N_k-1}$ as $f_{kP}(z) := [z^{\alpha_1} : \dots : z^{\alpha_{N_k}}]$ where $\alpha_1, \dots, \alpha_{N_k}$ is an enumeration of all points in $kP \cap \mathbb{Z}^n$ and where $z^{\alpha_j} := \prod_{k=1}^n z_k^{\alpha_{j,k}}$. Then for $k \gg 1$ big enough, f_{kP} produces an embedding and a polarized toric manifold (X_P, L_P) by compactification where clearly $L_P = \mathcal{O}_{\mathbb{P}^{N_k-1}}(1)|_{X_P}$. Observe also that

$$H^0(X_P, kL_P) \simeq \bigoplus_{\alpha \in kP \cap \mathbb{Z}^n} \langle z^\alpha \rangle,$$

namely there exists a basis $\{s_\alpha\}_{\alpha \in kP \cap \mathbb{Z}^n}$ such that $s_\alpha/s_\beta = z^{\alpha-\beta}$ on the torus \mathbb{C}^n . In particular

$$n! \text{Vol}_{\mathbb{R}^n}(P) = \text{Vol}_{X_P}(L_P). \quad (1.11)$$

as a consequence of a result of Khovanskii on semigroups ([Kho93]) and of the definition of the volume of a line bundle (subsection 1.1.9). See [Ful2] and [Cox] to know more about toric varieties.

1.2.3 Okounkov bodies

Passing from a polarized toric manifold (X_P, L_P) to its Delzant polytope P transfers many abstract geometric questions to convex geometric problems. A. Okounkov in [Oko96], [Oko03] found a natural way to mimic this nice correspondence to general polarized projective manifold (X, L) . Namely to any (X, L) is associated a convex body $\Delta(L) \subset \mathbb{R}^n$, now called *Okounkov body*, where n is the dimension of X , which is basically a simplified image of (X, L) .

The construction starts fixing a point $x \in X$ and an admissible flag centered at x or, equivalently, holomorphic coordinated on a trivializing open set U centered at x . Letting $t : U \rightarrow L$ be a nowhere zero local section of L , then any section $s \in H^0(X, kL)$ locally writes as $s|_U = ft^k$ for $f \in \mathcal{O}_X(U)$. The Okounkov body $\Delta(L)$ is then defined as

$$\Delta(L) := \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu(s)}{k} : s \in H^0(X, kL) \setminus \{0\} \right\}}$$

where $\nu(s) := \min_{lex} \{\alpha \in \mathbb{N}^n : a_\alpha \neq 0 \text{ where } f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha\}$, i.e. ν is a valuation which associates to any section its leading term exponent at x with respect to the lexicographical order. Note that $\Delta(L)$ does not depend on the local section t chosen, but it depends on the choice of x and of the holomorphic coordinates. However

$$n! \text{Vol}_{\mathbb{R}^n}(\Delta(L)) = \text{Vol}_X(L), \quad (1.12)$$

which extends (1.11). Indeed the Okounkov body essentially comes back to the polytope when (X, L) is a toric polarized manifold if the point chosen is a fixed point with respect to the torus action.

The Okounkov bodies' construction works in the more general setting of big line bundles as pointed out in [LM09], [KKh12]. Moreover $\Delta(L)$ is a cohomological invariant and (1.12) together with the variation of Okounkov bodies on the big cone gives the log-concavity of the volume as consequence of the Brunn-Minkowski inequality which was the main reason of A. Okounkov to introduce these invariants. The volume of a line bundle is clearly a global invariant, but the local aspect of the construction leads to the natural question if $\Delta(L)$ encodes also local properties of $(L \rightarrow X, x)$. Firstable observe that changing order by an unitary trasformation, the volume of the Okounkov body remains constant although the shape of $\Delta(L)$ may mutate. Küronya-Lozovanu proved in [KL15] and in [KL17] that, considering the

degree-lexicographic order, the Okounkov body is a finer invariant than the Seshadri constant of L at x since

$$\epsilon_S(L; x) = \sup\{t \geq 0 : t\Sigma_n \subset \Delta(L)\}$$

where Σ_n is the unit n -simplex.

Finally recall that Witt Nyström in [WN15] showed how a torus-invariant domain $D(L) \subset \mathbb{C}^n$, constructed from $\Delta(L)$, equipped with the standard Euclidean metric, approximates (X, L) in the sense that for any relatively compact open set $U \subset D(L)$ there exists an holomorphic embedding $f : U \rightarrow X$ centered at x such that the pushforward of the standard Euclidean metric extends to a metric on L and such that the volume of $D(L)$ is equal to the volume of L (a similar result holds in the big case). His result should be compared to the well-known characterization of the Seshadri constant $\epsilon_S(L; x)$ as the supremum of all radii r such that there exists an holomorphic embedding $f : B_r(0) \rightarrow X$ centered at x of the Euclidean complex ball of radius r into X with the properties that the standard Euclidean metric extends to a hermitian metric on L .

1.2.4 Main results

Given N different points x_1, \dots, x_N on a projective manifold X and a big line bundle L , it is natural to wonder if it is possible to construct N Okounkov bodies $\Delta_1(L), \dots, \Delta_N(L)$ which encode global invariant as the volume and local invariant as the multipoint Seshadri constant $\epsilon_S(L; x_1, \dots, x_N)$. This is part of the content of Paper I.

More precisely given $(L \rightarrow X; x_1, \dots, x_N)$ as above, the *multipoint Okounkov bodies* are defined as

$$\Delta_j(L) := \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu^{x_j}(s)}{k} : s \in V_{k,j} \right\}} \subset \mathbb{R}^n$$

where $V_{k,j} := \{s \in H^0(X, kL) : \nu^{x_j}(s) < \nu^{x_i}(s) \text{ for any } i \neq j\}$, and $\nu^{x_1}, \dots, \nu^{x_N}$ are valuations defined as in the one-point case considering the leading term exponents at x_1, \dots, x_N and $>$ is the lexicographic order (the valuations may also be more general).

Note that there may be sections which are not associated to any multipoint Okounkov body since $\Delta_j(L)$ is obtained considering all sections whose leading term exponent at x_j is *strictly* smaller than the leading term exponent at the other points. This is the main technical problem in proving the following result.

Theorem A (Theorem A of Paper I). *Let L be a big line bundle. Then*

$$n! \sum_{j=1}^N \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \text{Vol}_X(L).$$

Differently from the one-point case, it may happen that some of the multipoint Okounkov bodies has interior not-empty or it is empty. However they are numerical invariants and they well-behave under variations of the big cohomology class. Moreover these multipoint Okounkov bodies connect the geometry of (X, L) to the notion of *Kähler packing*.

Theorem B (Theorem C of Paper I) . *Let L be a big line bundle and let x_1, \dots, x_N be N distinct points. Then there exist N torus-invariant domains $D_1(L), \dots, D_N(L) \subset \mathbb{C}^n$ such that $\{(D_j(L), \omega_{st})\}_{j=1, \dots, N}$ packs perfectly into (X, L) . Namely for any family of relatively compact open sets $U_j \subseteq D_j(L)$ there exists an holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ and a Kähler form ω lying in $c_1(L)$ such that $f_*\omega_{st} = \omega|_{f(U_j)}$ for any $j = 1, \dots, N$. Moreover $\sum_{j=1}^N \text{Vol}_{\mathbb{C}^n}(D_j) = \text{Vol}_X(L)$.*

A similar result holds for big line bundles.

As a consequence, the multipoint Seshadri constant $\epsilon_S(L; x_1, \dots, x_N)$ can be described as the supremum of all radii r such that there exists an holomorphic embedding $f : \bigsqcup_{j=1}^N B_r(0) \rightarrow X$ centered at x_1, \dots, x_N with the properties that the standard Euclidean metric extends to a hermitian metric on L . In fact the domains $D_j(L)$ in Theorem B are defined as $D_j(L) := \mu^{-1}(\Delta_j(L)^{ess})$ where $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$, $\mu(z_1, \dots, z_n) := (|z_1|^2, \dots, |z_n|^2)$ and where $\Delta_j(L)^{ess}$ is the *essential* part of $\Delta_j(L)$ ([WN15]) which coincides with the interior of $\Delta_j(L)$ as subset of $\mathbb{R}_{\geq 0}^n$ with its induced topology when L is ample. While it is possible to read the multipoint Seshadri constants directly from the shape of the multipoint Okounkov bodies $\Delta_j(L)$ when the latter are constructed considering the degree-lexicographic order as the next result recalls.

Theorem C (Theorem B of Paper I) . *Let L be an ample line bundle and let x_1, \dots, x_N be N distinct points. Let also $\Delta_1(L), \dots, \Delta_N(L)$ be the multipoint Okounkov bodies constructed considering the degree-lexicographic order. Then*

$$\epsilon_S(L; x_1, \dots, x_N) = \sup\{t \geq 0 : t\Sigma_n \subset \Delta_j(L)^{ess} \text{ for any } j = 1, \dots, N\}$$

where Σ_n is the unit n -simplex.

As said previously, the multipoint Seshadri constant is connected to several conjectures like the Nagata's conjecture. For surfaces a more precise description of the shape of the multipoint Okounkov bodies is provided. Finally in the toric case, in many different situations, the multipoint Okounkov bodies can be directly recovered subdividing the polytope.

1.3 Paper II - III

In these two papers (X, ω) is a compact Kähler manifold.

1.3.1 Variational approach

A useful way to deal with complex Monge-Ampère equations is through variational approaches. More precisely to study equations of the type

$$\begin{cases} MA_\omega(u) = \mu \\ u \in \mathcal{E}(X, \omega), \end{cases} \quad (1.13)$$

for μ such that $\mu(X) = \int_X \omega^n$, recalling that $\mathcal{E}(X, \omega)$ is the set of all ω -psh functions with full Monge-Ampère mass, one defines a functional F_μ whose the critical points of F_μ are solutions of (1.13). Obviously F_μ depends on μ , but it would be natural to split it as sum of two functionals whose differentials coincide respectively with the right and with the left hand in (1.13). Indeed the Monge-Ampère measure of a smooth ω -psh function can be described as differential of a well-known functional E called Monge-Ampère energy, which was first introduced in [Aub84], [Mab86]. It is defined as

$$E(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X u(\omega + dd^c u)^j \wedge \omega^{n-j}.$$

Thanks to Bedford-Taylor theory, it is possible to extend it to locally bounded ω -psh functions, i.e. to elements in $PSH(X, \omega)$ with minimal singularities. Thus since E is non decreasing, it is natural to set

$$E(u) := \inf\{E(v) : v \in PSH(X, \omega) \text{ with minimal singularities, } v \geq u\}.$$

One problem with this variational approach is that the set

$$\mathcal{E}^1(X, \omega) := \{u \in \mathcal{E}(X, \omega) : E(u) > -\infty\}$$

is properly contained in $\mathcal{E}(X, \omega)$. Therefore we are actually restricting the set of solution in (1.13) from $\mathcal{E}(X, \omega)$ to $\mathcal{E}^1(X, \omega)$. Then if $u \in \mathcal{E}^1(X, \omega)$ one gets

$$\frac{d}{dt} E(P_\omega(u + tf))|_{t=0} = \int_X f MA_\omega(u) \quad (1.14)$$

for any $f \in C^0(X)$, where

$$P_\omega(u + tf) := (\sup\{v \in PSH(X, \omega) : v \leq u + tf\})^*$$

is a Perron-Bremermann envelope. Here the star is for the upper semicontinuous regularization.

Then, defining the action $L_\mu(u) := \int_X u d\mu$ and $V := \int_X \omega^n$, the critical points, when exist, of the translation invariant functional $F_\mu = E - VL_\mu$ solve (1.13) as showed in [BBGZ13]. More precisely, setting $\mathcal{E}_{norm}^1(X, \omega) := \{u \in \mathcal{E}^1(X, \omega) : \sup_X u = 0\}$,

$$\begin{cases} MA_\omega(u) = \mu \\ u \in \mathcal{E}_{norm}^1(X, \omega) \end{cases} \quad (1.15)$$

admits a unique solution if and only if $\mu \in \mathcal{M}^1(X, \omega) := \{V\mu : \mu \text{ is a probability measure with } E^*(\mu) < +\infty\}$ where

$$E^*(\mu) := \sup_{u \in \mathcal{E}^1(X, \omega)} F_\mu(u) < +\infty$$

is basically the Legendre transform of the Monge-Ampère energy. There are some key points in their proof. First the continuity of L_μ with respect to the weak topology, which yields the upper-semicontinuity of F_μ since the Monge-Ampère energy was known to be upper-semicontinuity. Second the fact that the boundedness of F_μ from above is equivalent to the coercivity of F_μ with respect to the J -functional $J(u) := -E(u) + \int_X u \omega^n$ ([Aub84]), namely the existence of $A > 0, B \geq 0$ such that

$$F_\mu(u) \leq -AJ(u) + B.$$

Note that to prove this last property the authors used the convexity of L_μ , i.e. the concavity of F_μ (E is concave).

The sets $\mathcal{E}^1(X, \omega)$ and $\mathcal{M}^1(X, \omega)$ are then endowed with two natural strong topologies given as the coarsest refinements of the weak topologies such that the energies E, E^* become continuous. In fact as showed in [BBEGZ19] the Monge-Ampère operator produces a homeomorphism

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega), strong) \rightarrow (\mathcal{M}^1(X, \omega), strong). \quad (1.16)$$

Recall that the Monge-Ampère operator is not continuous with respect to the weak topology even on $\mathcal{E}^1(X, \omega)$.

Finally it is remarkable to observe that a variational approach to solve complex Monge-Ampère equations was already settled in [Käh33].

1.3.2 L^1 -metric geometry

The space of Kähler metrics \mathcal{H}_ω possesses an infinite dimensional Riemannian structure as showed in the pioneering works of Semmes ([Sem92]) and Donaldson ([Don99]). It is given by

$$(f, g)_\varphi := \left(\int_X f g (\omega + dd^c \varphi)^n \right)^{1/2}$$

for any $\varphi \in \mathcal{H}_\omega$ and any $f, g \in T_\varphi \mathcal{H}_\omega \simeq C^\infty(X)$. It is important to underline that the geodesics are given as solutions of homogeneous complex Monge-Ampère equations (see [Chen00a]). More precisely given $\varphi_1, \varphi_2 \in \mathcal{H}_\omega$ the *weak* geodesic joining φ_1, φ_2 is the function

$$\Phi(z, t) := \left(\sup \left\{ U \in PSH(X \times S, \pi_X^* \omega) : \limsup_{t \rightarrow 0^+} U(\cdot, t) \leq \varphi_1 \text{ and } \limsup_{t \rightarrow 1^-} U(\cdot, t) \leq \varphi_2 \right\} \right)^* \quad (1.17)$$

where $S := \{t \in \mathbb{C} : 0 < \operatorname{Re} t < 1\}$ and where $\pi_X : X \times S \rightarrow X$ is the usual projection. These geodesics solve homogeneous Monge-Ampère equations on $X \times S$, but, although they always exist, they may be just $C^{1,1}$. Moreover \mathcal{H}_ω is not complete. Therefore in [Dar17] Darvas described the completion of \mathcal{H}_ω endowed with the Riemannian structure as the second energy class $\mathcal{E}^2(X, \omega)$, which in particular becomes a geodesic metric space.

Anyway in this thesis the main interest is on the metric structure on \mathcal{H}_ω given by the Finsler metric

$$|f|_{1,\varphi} := \int_X |f|(\omega + dd^c \varphi)^n$$

for any $\varphi \in \mathcal{H}_\omega$ and $f \in T_\varphi \mathcal{H}_\omega$. This structure was introduced in [Dar15] where the author showed that its metric completion coincides with $\mathcal{E}^1(X, \omega)$ and that the associated distance can be described as

$$d(u_1, u_2) = E(u_1) + E(u_2) - 2E(P_\omega(u_1, u_2)). \quad (1.18)$$

for any $u_1, u_2 \in \mathcal{E}^1(X, \omega)$. Here $P_\omega(u_1, u_2) := (\sup\{v \in PSH(X, \omega) : v \leq \min(u_1, u_2)\})^*$ is the largest ω -psh which is smaller than u_1, u_2 (recall that generally the minimum between two ω -psh is not ω -psh). The closed formula (1.18) is very useful to study $(\mathcal{E}^1(X, \omega), d)$ through pluripotential theory exploring the properties of the Monge-Ampère energy. Moreover the weak geodesics (1.17) are metric geodesics in $(\mathcal{E}^1(X, \omega), d)$, i.e. for any two potentials $u_1, u_2 \in \mathcal{E}^1(X, \omega)$ there is a unique weak geodesic joining u_1, u_2 . However $(\mathcal{E}^1(X, \omega), d)$ is not a CAT(0)-space since by (1.18) it immediately follows that

$$d(u_1, u_2) = d(u_1, P_\omega(u_1, u_2)) + d(P_\omega(u_1, u_2), u_2).$$

A great advantage to work with $(\mathcal{E}^1(X, \omega), d)$ is that its metric topology (usually called L^1 -metric topology for obvious reasons) coincides with the strong topology of [BBEGZ19] described in the previous subsection. In particular the coercivity measured through the J -functional can be replaced by the d -coercivity after an suitable normalization. In fact there is a constant $C \in \mathbb{R}$ such that

$$d(u, 0) - C \leq J(u) \leq d(u, 0)$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega)$.

1.3.3 Convergence of metric spaces

The main reference for this subsection is [BBI].

Given two subset A, B of a metric space (X, d) there is a well-known natural distance between A, B given as

$$d_H(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}$$

and it is called the Hausdorff distance. Note that $d(A, \overline{A}) = 0$, indeed all closed sets of X endowed with the Hausdorff distance produces a metric space.

This distance suggests a way to measure how much two metric spaces differs from being isometric, i.e. the *Gromov-Hausdorff distance* between metric spaces. The idea is to embed $(X, d_X), (Y, d_Y)$ isometrically into a third metric space (Z, d_Z) and compute the Hausdorff distance between the images of X and of Y . Obviously this distance depends on the metric space (Z, d_Z) and on the embeddings chosen, so one defines the Gromov-Hausdorff as infimum among all possible choices, i.e.

$$d_{GH}(X, Y) := \inf\{d_H^{d_Z}(X, Y) : (X, d_X), (Y, d_Y) \subset (Z, d_Z)\}$$

where $d_H^{d_Z}$ denotes the Hausdorff distance on (Z, d_Z) . It is then an easy exercise to prove that one can restrict to consider $Z := X \sqcup Y$ endowed with a distance d_Z such that $d_Z|_X = d_X, d_Z|_Y = d_Y$. Moreover d_{GH} descends to a distance on the set of all isometry classes of compact metric spaces, and a sequence of compact metric spaces $\{(X_k, d_{X_k})\}_{k \in \mathbb{N}}$ is said to converge in the Gromov-Hausdorff sense to a metric space (X, d_X) if $d_{GH}(X_k, X) \rightarrow 0$ as $k \rightarrow \infty$. Note that although this convergence is useful for compact metric spaces, it becomes too strong for non-compact metric spaces. For instance considering $X_k = B_k(0) \subset \mathbb{R}^n$ with the Euclidean distance, it easily follows that $d_{GH}(X_k, X_{k+1}) = 1$ for any $k \in \mathbb{N}$ while intuitively (X_k, d_{Eucl}) converges to (\mathbb{R}, d_{Eucl}) . Therefore for non-compact metric spaces it is more convenient to consider the *pointed Gromov-Hausdorff convergence*. For sequences of pointed length metric spaces this notion of convergence requires the convergence of balls centered at the points for any fixed radius. Namely a sequence of pointed length metric spaces $\{(X_k, p_k, d_{X_k})\}_{k \in \mathbb{N}}$ converges in the Gromov-Hausdorff sense to a pointed length metric space (X, p, d_X) if for any $r > 0$, $d_{GH}(B_r(p_k), B_r(p)) \rightarrow 0$ as $k \rightarrow \infty$. Alternatively, when the metric spaces considered are locally compact, one can define a pointed Gromov-Hausdorff distance for pointed compact metric spaces as

$$d_{GH}((X, p, d_X), (Y, q, d_Y)) := \inf\{d_H^{d_Z}(X, Y) + d_Z(p, q) : (X, d_X), (Y, d_Y) \subset (Z, d_Z)\},$$

and then a sequence (X_k, p_k, d_{X_k}) converges in the pointed Gromov-Hausdorff sense to (X, p, d_X) if and only if $(\overline{B_r(p_k)}, p_k)$ converges in the pointed Gromov-Hausdorff sense to $(\overline{B_r(p)}, p)$ for any $r > 0$ fixed.

Observe that the pointed Gromov-Hausdorff convergence is a refinement of the Gromov-Hausdorff convergence in the sense that if $(X_k, d_{X_k}), (X, d_X)$ are compact sets then $(X_k, p_k, d_{X_k}) \xrightarrow{p-GH} (X, p, d_X)$ implies $(X_k, d_{X_k}) \xrightarrow{GH} (X, d_X)$ while vice versa if $(X_k, d_{X_k}) \xrightarrow{GH} (X, d_X)$ and $p \in X$ then there exists a sequence $p_k \in X_k$ such that $(X_k, p_k, d_{X_k}) \xrightarrow{p-GH} (X, p, d_X)$.

Finally recall that the morphisms in the category of metric spaces are given by short maps, i.e. 1-Lipschitz maps.

1.3.4 Relative setting

Some constraints may be requested on the behavior of canonical metrics. Often this reduces to solve complex Monge-Ampère equations *with prescribed singularities*, i.e.

$$\begin{cases} MA_\omega(u) = \mu \\ u \in \mathcal{E}(X, \omega, \psi) \end{cases} \quad (1.19)$$

where μ is a positive non-pluripolar measure, $\psi \in PSH(X, \omega)$ represents the prescribed singularities and similarly to the absolute setting

$$\mathcal{E}(X, \omega, \psi) := \left\{ u \in PSH(X, \omega) : u \preceq \psi, \int_X MA_\omega(u) = \int_X MA_\omega(\psi) =: V_\psi \right\}$$

is the set of all ω -psh functions more singular than ψ with relative full Monge-Ampère mass. These spaces were introduced in [DDNL18] where the authors extended to the ψ -relative setting many known results of the absolute setting (see in particular [BEGZ10]). A key point in their theory is the fact that the Monge-Ampère mass respect the partial order \preceq given by the singularities, i.e.

$$u \preceq v \implies \int_X MA_\omega(u) \leq \int_X MA_\omega(v)$$

as fully showed in [WN17]. In [DDNL18] a deep investigation of (1.19) were presented and the authors found out that a necessary assumption to make (1.19) always solvable, under the hypothesis $\mu(X) = V_\psi$, is that ψ must be a *model type envelope* (as called in Paper II), i.e.

$$\psi = \left(\lim_{C \rightarrow \infty} P_\omega(\psi + C, 0) \right)^*. \quad (1.20)$$

The right hand in (1.20) is briefly denoted as $P_\omega[\psi]$. More generally for a couple of ω -psh functions u, v , the function

$$P_\omega[u](v) := \left(\lim_{C \rightarrow \infty} P_\omega(u + C, v) \right)^*$$

is the largest ω -psh function which is smaller than v and more singular than u , and $P_\omega[u] := P_\omega[u](0)$.

The authors in [DDNL18] also defined the ψ -relative Monge-Ampère energy on the set $PSH(X, \omega, \psi) := \{u \in PSH(X, \omega) : u \preceq \psi\}$ as

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X (u - \psi)(\omega + dd^c u)^j \wedge (\omega + dd^c \psi)^{n-j}$$

if $u - \psi$ is globally bounded, i.e. if u has ψ -relative minimal singularities, and, using the monotonicity property, as

$$E_\psi(u) := \inf \{E_\psi(v) : v \in PSH(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities, } v \geq u\}$$

otherwise. It is then naturally defined the set

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\},$$

and $E_\psi : \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R}$ keeps having the Monge-Ampère measure as differential, i.e. the ψ -relative analog of (1.14) holds.

1.3.5 Main results

In Papers II, III letting ψ be a fixed model type envelope with non-zero total mass V_ψ the set $\mathcal{E}^1(X, \omega, \psi)$ is endowed with a complete metric structure and the homeomorphism (1.16) is extended to the relative setting. More precisely, defining the ψ -relative energy E_ψ^* on the set of all probability measures as

$$E_\psi^*(\mu) := \sup_{u \in \mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi}(u)$$

where $F_{\mu, \psi} := E_\psi - V_\psi L_\mu$ and the action L_μ is extended to $PSH(X, \omega)$ basically as $L_\mu(u) := \int_X (u - P_\omega[u]) d\mu$, the set

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure with } E_\psi^*(\mu) < +\infty\}$$

has a natural strong topology given as the coarsest refinement of the weak topology such that E_ψ^* becomes continuous.

Theorem D (Theorem A of Paper II, Theorem A of Paper III) . *Let $\psi \in PSH(X, \omega)$ be a model type envelope with $V_\psi > 0$ and let*

$$d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v)).$$

Then $(\mathcal{E}^1(X, \omega, \psi), d)$ is a complete metric space and the Monge-Ampère operator produces an homeomorphism

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega, \psi), d) \rightarrow (\mathcal{M}^1(X, \omega, \psi), strong) \quad (1.21)$$

where $\mathcal{E}_{norm}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$.

As proved in Paper III the metric topology on $(\mathcal{E}^1(X, \omega, \psi), d)$ is a strong topology in the sense of subsection 1.3.1. Indeed it coincides with the coarsest refinement of the weak topology such that the energy E_ψ becomes continuous and the set $P_\omega[\psi](\mathcal{H}_\omega)$ is strongly dense. The main difficulties in Theorem D is on showing the homeomorphism (1.21). The bijectivity is an adaptation of the variational approach to the relative setting since a critical point of $F_{\mu, \psi}$ solves (1.19), while to prove the bicontinuity there are some deeper differences with respect to the absolute setting. Technically a key point in the absolute setting is that any potential $v \in \mathcal{E}^1(X, \omega)$ can be approximated with a decreasing sequence of ω -psh continuous functions v_j inside the class $\mathcal{E}^1(X, \omega)$, and this leads to the continuity of the action $\mathcal{M}^1(X, \omega) \ni MA_\omega(u) \rightarrow \int_X v MA_\omega(u)$ when restricted to

$\mathcal{M}_C^1(X, \omega) := \{V\mu \in \mathcal{M}^1(X, \omega) : E^*(\mu) \leq C\}$ for any $C \in \mathbb{R}$ fixed. Indeed such map turns out to be the uniform limit of the maps $\mathcal{M}_C^1(X, \omega) \ni MA_\omega(u) \rightarrow \int_X v_j MA_\omega(u)$ which are continuous by duality. This property does not a priori hold anymore in the relative setting, obviously considering $\int_X (v - \psi) MA_\omega(u)$. In fact although $P_\omega[\psi](\cdot) : \mathcal{E}^1(X, \omega) \rightarrow \mathcal{E}^1(X, \omega, \psi)$ is a projection with many nice properties it is still not clear if $P_\omega[\psi](u) - \psi$ is continuous for any $u \in PSH(X, \omega) \cap C^0(X)$.

It is then natural to wonder if it is possible to glue together different spaces $(\mathcal{E}^1(X, \omega, \psi), d)$ into an unique metric space, whose topology is a strong topology connected to the Monge-Ampère operator and hence to the stability of solutions of complex Monge-Ampère equations. Indicating with \mathcal{M} the set of all model type envelopes, and with \mathcal{M}^+ its subset of all elements with non-zero total Monge-Ampère mass, one has $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \emptyset$ for any $\psi_1, \psi_2 \in \mathcal{M}^+$ but they may have same total mass $V_{\psi_1} = V_{\psi_2}$. Thus, according to the homeomorphism (1.21) it is natural to consider totally ordered sets $\mathcal{A} \subset \mathcal{M}^+$.

Theorem E (Theorem B of Paper II) . *Let $\mathcal{A} \subset \mathcal{M}^+$ be totally ordered. Then*

$$X_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

is endowed with a complete distance $d_{\mathcal{A}}$ which restricts to d on $\mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$.

Observe that $\mathcal{M} \subset \{u \in PSH(X, \omega) : \sup u = 0\}$ is weakly closed, so $\overline{\mathcal{A}} \subset \mathcal{M}$, but the minimum element ψ_{\min} of $\overline{\mathcal{A}}$ may have zero mass. In this case the set $\mathcal{E}^1(X, \omega, \psi_{\min}) = PSH(X, \omega, \psi_{\min})$ is identified with a singleton $P_{\psi_{\min}}$ since $E_{\psi_{\min}} \equiv 0$ by definition. The construction of the distance $d_{\mathcal{A}}$ of Theorem E then relies on the properties of the projection $P_\omega\cdot$. Indeed if $\psi_1 \preceq \psi_2 \preceq \psi_3$ then

1. $P_\omega[\psi_1](P_\omega[\psi_2](u)) = P_\omega[\psi_1](u) \in \mathcal{E}^1(X, \omega, \psi_1)$ for any $u \in \mathcal{E}^1(X, \omega, \psi_3)$;
2. $\|P_\omega[\psi_1](u) - P_\omega[\psi_1](v)\|_{L^\infty} \leq \|u - v\|_{L^\infty}$ for any $u, v \in \mathcal{E}^1(X, \omega, \psi_2)$ such that $u - v$ is globally bounded;
3. $d(P_\omega[\psi_1](u), P_\omega[\psi_1](v)) \leq d(u, v)$ for any $u, v \in \mathcal{E}^1(X, \omega, \psi_2)$.

It seems then natural to define a distance $d_{\mathcal{A}}$ whose value at two potentials $u \in \mathcal{E}^1(X, \omega, \psi_1), v \in \mathcal{E}^1(X, \omega, \psi_2)$ is bigger than $d(u, P_\omega[\psi_1](v))$ and of $d(w, v)$ if $P_\omega[\psi_1](w) = u$. So, using the fact that $V_{\psi_1} < V_{\psi_2}$ if $\psi_1, \psi_2 \in \mathcal{M}^+, \psi_1 \preceq \psi_2$, a natural definition of $d_{\mathcal{A}}(u, v)$ would be

$$d(u, P_\omega[\psi_1](v)) + \sup_{\{w \in \mathcal{E}^1(X, \omega, \psi_2) : P_\omega[\psi_1](w) = u\}} \{d(w, v) - d(u, P_\omega[\psi_1](v))\} + V_{\psi_2} - V_{\psi_1}, \quad (1.22)$$

but there are some problem in this definition. First, not any element in $\mathcal{E}^1(X, \omega, \psi_1)$ is necessarily given as projection of elements in $\mathcal{E}^1(X, \omega, \psi_2)$, thus one first need

to define the distance $d_{\mathcal{A}}$ on a smaller dense subset and then to recover $X_{\mathcal{A}}$ as completion. Second, the supremum in (1.22) is not a priori finite and one may try to pick an element w at minimum distance with respect to ψ_2 but even in this case such supremum is very unstable since the space $(\mathcal{E}^1(X, \omega, \psi_2), d)$ is not locally compact. Therefore the idea is to adapt (1.22) using strongly compact sets given by the *entropy*. More precisely for any $C \in \mathbb{R}$ the set

$$\mathcal{K}_C := \left\{ \varphi \in \mathcal{E}^1(X, \omega) : \max \left(\left| \sup_X \varphi \right|, H_{\omega^n/V}(MA_{\omega}(\varphi)/V) \right) \leq C \right\}$$

is strongly compact in $\mathcal{E}^1(X, \omega)$ as proved in [BBEGZ19] where for any couple of probability measure μ, ν the entropy of ν with respect to μ is given as

$$H_{\mu}(\nu) := \int_X f \log f d\mu$$

if ν is absolutely continuous with respect to μ with density f such that $f \log f \in L^1(\mu)$, and as $H_{\mu}(\nu) := +\infty$ otherwise. By the Lipschitz property 3 of the distance d stated above, for any $\psi \in \mathcal{M}$ the set $\mathcal{P}_C(X, \omega, \psi) := P_{\omega}[\psi](\mathcal{K}_C)$ is compact in $(\mathcal{E}^1(X, \omega, \psi), d)$ and

$$\mathcal{P}(X, \omega, \psi) := \bigcup_{C \in \mathbb{R}} \mathcal{P}_C(X, \omega, \psi)$$

includes $P_{\omega}[\psi](\mathcal{H}_{\omega})$. Thus for $u \in \mathcal{P}(X, \omega, \psi_1), v \in \mathcal{P}(X, \omega, \psi_2), \psi_1 \preceq \psi_2$, one defines

$$\tilde{d}_{\mathcal{A}}(u, v) := d(u, P_{\omega}[\psi_1](v)) + \sup \left\{ d(a, b) - d(P_{\omega}[\psi_1](a), P_{\omega}[\psi_1](b)) \right\} + V_{\psi_2} - V_{\psi_1}$$

where the supremum is over $a, b \in \mathcal{P}_{\max(C_1, C_2)}(X, \omega, \psi_2)$ where C_1, C_2 are respectively the minimum positive values such that $u \in \mathcal{P}_{C_1}(X, \omega, \psi_1), v \in \mathcal{P}_{C_2}(X, \omega, \psi_2)$. Anyway $\tilde{d}_{\mathcal{A}}$ barely satisfies the triangle inequality, so with the usual trick $d_{\mathcal{A}}$ will be given on $\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi) \times \bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi)$ as infimum of the sum of the value $\tilde{d}_{\mathcal{A}}$ over all chains, i.e.

$$d_{\mathcal{A}}(u, v) := \inf_{\{u=w_0, \dots, w_m=v\}} \sum_{j=0}^{m-1} \tilde{d}_{\mathcal{A}}(w_j, w_{j+1}).$$

Then to conclude the proof of Theorem E it remains to prove that $d_{\mathcal{A}}$ is a distance which restrict to d over $\mathcal{P}(X, \omega, \psi)$ for any $\psi \in \mathcal{A}$ and then to check that its completion coincides with $X_{\mathcal{A}}$.

As consequence of Theorem E, given a decreasing sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ converging to $\psi \in \mathcal{M}^+$, the sequence of metric spaces $(\mathcal{E}^1(X, \omega, \psi_k), d)$ approximates $(\mathcal{E}^1(X, \omega, \psi), d)$. Indeed Theorem C in Paper II shows that the convergence holds in a *compact pointed Gromov-Hausdorff* sense. This new type of convergence mimics the characterization of the pointed Gromov-Hausdorff convergence described in

subsection 1.3.3 replacing the sequence of balls with the compact sets given as projection of element with bounded entropy. In fact since $(\mathcal{E}^1(X, \omega, \psi), d)$ is not locally compact the pointed Gromov-Hausdorff still seems a too strong convergence. Furthermore the projection maps $P_{i,j} : P_\omega[\psi_j](\cdot) : (\mathcal{E}^1(X, \omega, \psi_i), d) \rightarrow (\mathcal{E}^1(X, \omega, \psi_j), d)$ for $i \leq j$ produce a direct system in the category of metric space as immediate consequence of the contraction property of d , and the space $(\mathcal{E}^1(X, \omega, \psi), d)$ basically coincides with the direct limit of such direct system (Theorem D in Paper II).

In Paper III it is then shown that the metric topology is a strong topology since it coincides with the coarsest refinement of the weak topology such that $E(\cdot)$ becomes continuous. Namely $\{u_k\}_{k \in \mathbb{N}} \subset X_{\mathcal{A}}$ converges to $u \in X_{\mathcal{A}}$ in $(X_{\mathcal{A}}, d_{\mathcal{A}})$ if and only if $u_k \rightarrow u$ weakly and $E_{P_\omega[u_k]}(u_k) \rightarrow E_{P_\omega[u]}(u)$. In this case we say that $u_k \rightarrow u$ strongly, and we refer to the paper for the natural definition of the weak convergence to the point $P_{\psi_{\min}}$ in the case $V_{\psi_{\min}} = 0$. Note in particular that the strong convergence does not depend on the set \mathcal{A} chosen. In fact endowing the set

$$Y_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi)$$

with its natural strong topology given as the coarsest refinement of the weak topology such that E_{ψ}^* becomes continuous, the main theorem of Paper III is the following.

Theorem F (Theorem B, Paper III) . *Let $\mathcal{A} \subset \mathcal{M}^+$ be a totally ordered set. Then the Monge-Ampère operator*

$$MA_{\omega} : (X_{\mathcal{A}, \text{norm}}, d_{\mathcal{A}}) \rightarrow (Y_{\mathcal{A}}, \text{strong})$$

is an homeomorphism where with obvious notations $X_{\mathcal{A}, \text{norm}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}_{\text{norm}}^1(X, \omega, \psi)$.

The proof of this Theorem is obviously a bit more involved with respect to that of (1.21), but the idea is basically the same. Indeed the bijectivity is clear, while for the continuity the proof uses uniform estimates on the ψ -relative functional I_{ψ}, J_{ψ} (and in particular the analog of the so-called convergence in energy) and the upper-semicontinuity of $E(\cdot)$ with respect to the weak convergence. Note that this last property is quite the core of the proof, and the upper-semicontinuity of $E_{\psi}(\cdot)$ for any ψ seems to not be enough to conclude.

Finally it is worth to underline that the strong convergence implies the convergence in capacity (and in ψ -relative capacity for any $\psi \in \mathcal{M}^+$). In fact if $u_k \rightarrow u$ strongly for $V_{P_\omega[u]} > 0$ then there exists a subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ such that $v_h := (\sup\{u_{k_j} : j \geq h\})^*$, $w_h := P_\omega(u_{k_h}, u_{k_{h+1}}, \dots)$ converges to u monotonically.

1.4 Paper IV - V

As in the previous section (X, ω) is assumed to be a Kähler compact manifold though in Paper V ω will also be the curvature of a hermitian metric on the anticanonical bundle, i.e. $\{\omega\} = c_1(X)$ (in particular X will be Fano).

1.4.1 Ding & Mabuchi functionals

As stated in subsection 1.1.13, the study of Kähler-Einstein metrics in the curved cases reduces to solve

$$\begin{cases} MA_\omega(u) = e^{-\lambda u + C} \mu \\ u \in \mathcal{H}_\omega \end{cases} \quad (1.23)$$

for $\lambda = -1, 1$ according to $c_1(X) = \lambda\omega$, i.e. if X is canonically polarized or if X is anticanonically polarized, where μ is a volume form depending on λ . (1.23) can be split in two problems. First finding a weak solution, i.e. a solution of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u + C} \mu \\ u \in \mathcal{E}^1(X, \omega), \end{cases} \quad (1.24)$$

and then exploring its regularity.

The second problem classically consists in obtaining a C^0 -regularity and a Laplacian estimate $C^{-1}\omega \leq \omega + dd^c u \leq C\omega$ for $C \in \mathbb{R}$. Indeed thanks to the Evans-Krylov theory u would then be $C^{2,\alpha}$ which is enough regularity to apply Schauder's theorem and a bootstrap argument to get the smoothness. For the C^0 -regularity see the proof of Kolodziej ([Kol98]) where any solution of $MA_\omega(u) = f\omega^n$ is continuous if $f \in L^p$ for $p > 1$. Observe that in (1.24) the density in the right hand side belongs to L^p for $p > 1$ as a consequence of the resolution of the strong openness conjecture (see [GZ15]). Instead, a proof of the Laplacian estimate can be found for instance in Theorem 10.1 in [BBEGZ19]. As conclusion any weak Kähler-Einstein metric (i.e. a solution of (1.24)) is a Kähler-Einstein metric.

Thus through a variational approach similar to (1.3.1) one defines the functional $L_{\mu,\lambda} : \mathcal{E}^1(X, \omega) \rightarrow \mathbb{R}$

$$L_{\mu,\lambda}(u) := \frac{-1}{\lambda} \log \int_X e^{-\lambda u} \mu$$

so that the differential of the translation invariant functional $D_{\mu,\lambda} := VL_{\mu,\lambda} - E$ coincides with the equation (1.24). This functional is called *Ding functional* ([Ding88]) and its minimizers solve (1.24). Moreover any weak Kähler-Einstein metric minimizes $D_{\mu,\lambda}$. In fact in the canonically polarized case this follows from the convexity of $D_{\mu,-1}$ similarly as in subsection 1.3.1 for F_μ thanks to Hölder's inequality. Moreover $D_{\mu,-1}$ is also lower-semicontinuous and J -coercive in the usual sense (or equivalently d -coercive over $\mathcal{E}_{norm}^1(X, \omega)$). In the Fano case instead the fact that weak Kähler-Einstein metrics minimize $D := D_{\mu,1}$ is a consequence of a deep result of Berndtsson on the positivity of direct image bundles ([Bern15]). Indeed an application of his results yields the weak geodesic convexity of D in $(\mathcal{E}^1(X, \omega), d)$, i.e. the convexity along weak geodesics given as solutions of homogeneous complex Monge-Ampère equations (i.e. geodesics as in (1.17)), and the uniqueness of Kähler-Einstein metrics modulo the action of $\text{Aut}(X)^0$ (retrieving a result proved by Bando and Mabuchi, [BM87]). Here $\text{Aut}(X)^0$ is the connected component of the identity of the automorphism group. Note that key points in the proof are also given by

the linearity of the Monge-Ampère energy along weak geodesics and, as in subsection 1.3.1 and in the canonically polarized case, by the lower-semicontinuity of D with respect to the weak topology which follows from the continuity of $L_{\mu,1}$ and the upper-semicontinuity of E .

Weak Kähler-Einstein metrics can also be expressed as critical points of the Mabuchi functional M , first introduced in [Mab86]. Indeed M can be defined in a general setting of (X, ω) compact Kähler manifold and its critical points are constant scalar curvature Kähler metrics. Recall that this functional is weak geodesically convex as proved in [BDL15], and in the Fano case the Mabuchi functional can be described as

$$M(u) := \left(H_{\omega^n/V} - E^* \right) (MA_\omega(u)/V)$$

for any $u \in \mathcal{E}^1(X, \omega)$ thanks to the Chen-Tian formula ([Chen00b], [Tian]). Moreover the following statements are equivalent:

- i) $u \in \mathcal{E}^1(X, \omega)$ solves (1.24);
- ii) $D(u) = \inf_{\mathcal{E}^1(X, \omega)} D$;
- iii) $M(u) = \inf_{\mathcal{E}^1(X, \omega)} M$,

as summarized in [BBEGZ19]. Furthermore in the case $\text{Aut}(X)^0$ the existence of weak Kähler-Einstein metrics are equivalent to the J -coercivity of D, M or equivalently to the d -coercivity over $\mathcal{E}_{norm}^1(X, \omega)$, i.e. the following conditions are equivalent:

- i) there exists an (unique) solution u to (1.24) with $\sup_X u = 0$;
- ii) there exist $A > 0, B \geq 0$ such that $D(u) \geq Ad(u, 0) - B$ over $\mathcal{E}_{norm}^1(X, \omega)$;
- iii) there exist $A > 0, B \geq 0$ such that $M(u) \geq Ad(u, 0) - B$ over $\mathcal{E}_{norm}^1(X, \omega)$.

This result is part of what proved in [DR15], although there already were a lot of progresses in this direction (see for instance [Tian97], [DT92]).

Observe that the J -coercivity of the Mabuchi functionals is related to K -stability. Indeed to any test configurations is associated a geodesic ray in $(\mathcal{E}^1(X, \omega), d)$, i.e. an *algebraic* geodesic ray, and the slope at infinity of the Mabuchi functional along algebraic geodesic rays is strongly connected to the Donaldson-Futaki invariants of test configurations ([BJ19]). Moreover it is remarkable to say that a big difference between the Kähler-Einstein case with respect to the cscK case relies on the existence of the Ding functional, which thanks to the uniform Ding stability (i.e. the uniform positivity of the slope at infinity of the Ding functional along algebraic geodesic rays) connects the uniform Mabuchi stability (i.e. the *uniform* K -stability) to the existence of Kähler-Einstein metrics as proved in [BBJ18]. Namely a pluripotential proof of a slightly different version of the Yau-Tian-Donaldson conjecture in the Fano case, independent on the proof given in [CDS15].

1.4.2 Continuity method

One classical technique to solve complex Monge-Ampère equations as (1.23) is through the continuity method. Namely one defines a family of complex Monge-Ampère equations

$$\begin{cases} MA_\omega(u) = g_t(u)\omega^n \\ u \in \mathcal{E}^1(X, \omega) \end{cases} \quad (1.25)$$

where $\{g_t(\cdot)\}_{t \in [0,1]}$ are the densities and where $MA_\omega(u) = g_1(u)\omega^n$ is the requested Monge-Ampère equation to study. The basic idea is to show that the subset $S \subset I$ of all $t \in I$ such that (1.25) admits a solution is not-empty, closed and open so that $S = I$. Therefore the equation $MA_\omega(u) = g_0(u)\omega^n$ is usually the one which easily admits a solution, while the openness result often follows from the implicit function Theorem if the family $\{g_t\}_{t \in [0,1]}$ has enough regularity. It is important to underline that sometimes more regularity on the solutions is requested, which also leads to a stronger continuity of the family of solutions $\{u_t\}_{t \in S}$.

There are several natural continuity methods for different complex Monge-Ampère equations. Anyway, since their wide geometrical applications, the class of complex Monge-Ampère equations

$$\begin{cases} MA_\omega(u) = e^{-a_t u} f_t \omega^n \\ u \in \mathcal{E}^1(X, \omega) \end{cases} \quad (1.26)$$

is general enough. Here we assume $\{f_t\}_{t \in [0,1]} \subset L^1 \setminus \{0\}$ to be a continuous family of non-negative L^1 -functions, while $\{a_t\}_{t \in [0,1]}$ is a continuous family of real numbers. Observe for instance that in the case X Fano, $\{\omega\} = c_1(X)$, taking $a_t = t$ and $f_t = g e^{-\frac{(1-t)}{m} \log \|s\|_{h^m}}$ where $s \in H^0(X, -mK_X)$ is a holomorphic section cutting a smooth divisor D and g is a suitable smooth positive function, the solutions of (1.26) correspond to the search of (weak) log Kähler-Einstein metrics. More precisely $\omega_{u_t} := \omega + dd^c u_t$ for u_t solution of (1.26) satisfies

$$\text{Ric}(\omega_{u_t}) = t\omega_{u_t} + (1-t)[D], \quad (1.27)$$

see also subsection 1.1.13. Indeed recall that it is possible to extend the Ricci form to currents as in [BBJ18], i.e. $\text{Ric}(\omega_{u_t}) := \text{Ric}(MA_\omega(u_t))$ where we set $\text{Ric}(\mu) := dd^c f$ for any μ positive measure such that locally $\mu = e^{-f} i^{n^2} \Omega \wedge \bar{\Omega}$ for Ω nowhere zero local holomorphic section of K_X .

The path (1.27) was considered in the proof of the Yau-Tian-Donaldson conjecture for Fano manifold in [CDS15], although they did not use uniquely the continuity method. While if $[D]$ is replaced by a smooth Kähler form, (1.27) becomes the continuity path used by Datar and Székelyhidi ([DS18]) to give a proof of the Yau-Tian-Donaldson conjecture directly using the continuity method. The main point in their proof, and the unique obstacle to prove that $S = I$, relies on the so-called C^0 -partial estimate, which basically produces an uniform upper bound on the solutions u_t .

1.4.3 Analytic singularities

In Paper IV and Paper V particular importance will be given to model type envelopes $\psi \in \mathcal{M}$ with *analytic singularities type*, i.e. $\psi = P_\omega[u]$ where $u \in PSH(X, \omega)$ is locally given as

$$u = g + c \log (|f_1|^2 + \cdots + |f_k|^2)$$

for g smooth, $c \in \mathbb{R}_{>0}$ and $\{f_j\}_{j=1}^k$ local holomorphic functions, which are local generators of a coherent ideal sheaf \mathcal{I} . In this case given a resolution of the ideal \mathcal{I} , i.e. a map $p : Y \rightarrow X$ given by a sequence of blow-ups of smooth centers such that $p^{-1}\mathcal{I} = \mathcal{O}_Y(-D)$ for an effective divisor D over Y , we have

$$p^*(\omega + dd^c u) = \eta + c[D] \quad (1.28)$$

for a semipositive smooth $(1, 1)$ -form η on Y . The analytic singularities of u are then formally encoded in (\mathcal{I}, c) . Recall also that when $\{\eta\}$ is a big class, i.e. $\int_Y \eta^n > 0$, it is also possible to define the space $(\mathcal{E}^1(Y, \eta), d)$ similarly to the Kähler case. We will say that $\psi \in \mathcal{M}$ has algebraic singularities type if it has analytic singularities and $c \in \mathbb{Q}_{>0}$.

1.4.4 Tian's α -invariant

As said in subsection 1.1.11, in dimension 2 the unique obstacle to the existence of Kähler-Einstein metrics for Fano manifolds is the reductiveness of the automorphism group, i.e. the obstruction found by Matsushima, as proved by Tian in [Tian90]. His proof was based on a global invariant of (X, ω) he introduced in [Tian87], the so-called α -invariant:

$$\alpha_\omega(0) := \sup \left\{ \alpha > 0 : \sup_{u \in PSH(X, \omega), \sup_X u = 0} \int_X e^{-\alpha u} \omega^n < +\infty \right\}.$$

A version of the α -invariant to the prescribed singularities setting will be the key object in Paper V.

The main interests for the α -invariant is when $\{\omega\} = c_1(X)$. In fact as proved by Tian,

$$\alpha_\omega(0) > \frac{n}{n+1} \implies \text{there exists a Kähler-Einstein metric.}$$

Observe also that, as showed by Demailly, this invariant can be expressed algebraically through the *log canonical threshold*. Namely, assuming for instance $\{\omega\} = c_1(X)$,

$$\alpha_\omega(0) = \alpha(X, 0) := \inf_{F \sim_{lin, \mathbb{Q}} -K_X, F \geq 0} lct(X, 0, F)$$

where $lct(X, 0, F) := \sup\{\alpha > 0 : (X, \alpha F) \text{ is } klt\}$ is the log canonical threshold and where the \mathbb{Q} -linear equivalence means that there exists $r \in \mathbb{N}$ such that $rF \sim_{lin} -rK_X$. Recall that being *klt* (i.e. *Kawamata log terminal*) for a pair (X, F) is a notion coming from Birational Geometry which analytically means that

$e^{-\sum_{j=1}^m a_j \log |s_j|_{h_j}^2} \in L^1$ where $F = \sum_{j=1}^m a_j F_j$ for F_j prime divisors cut by s_j and where h_j are hermitian metrics on $\mathcal{O}_X(F_j)$.

1.4.5 Main Results of Paper IV

In the Paper IV a continuity method *with movable singularities* is provided for (X, ω) compact Kähler manifold. Namely, for $t \in [0, 1]$ we set

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f_t \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi_t) \end{cases} \quad (1.29)$$

where $\lambda \in \mathbb{R}$, $f_t \in L^1 \setminus \{0\}$ continuous family of non-negative functions, and $\{\psi_t\}_{t \in [0, 1]} \subset \mathcal{M}^+$ totally ordered set of model type envelopes.

The idea is to generalize many classical continuity methods using a variational approach and the strong topology introduced in Paper II and Paper III. Obviously the sign of λ determines three very different cases.

If $\lambda < 0$ then the existence of a unique solution of (1.29) is proved in [DDNL18], so the study of the continuity method with movable singularities reduces to a stability problem. In this case there are no obstruction to the strong convergence of solutions (see Theorem G below).

In the case $\lambda = 0$ the existence of a unique solution of (1.29) is related to the belongness to $\mathcal{M}^1(X, \omega, \psi_t)$ as recalled in Theorem D. Paper IV then provides a characterization of the closure of the continuity method with movable singularities.

Theorem G (Theorems A and B of Paper IV) . *Given the complex Monge-Ampère equations*

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k), \end{cases} \quad (1.30)$$

for $k \in \mathbb{N}$ and $\lambda \leq 0$, assume that

- i) $f_k, f \in L^1 \setminus \{0\}$ non-negative such that $f_k \rightarrow f$ in L^1 ;
- ii) $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered such that $\psi_k \rightarrow \psi \in \mathcal{M}^+$ weakly;
- iii) $f_k \omega^n \in \mathcal{M}^1(X, \omega, \psi_k)$ for any $k \in \mathbb{N}$ if $\lambda = 0$;
- iv) $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ be the unique solutions of (1.30), normalized in the case $\lambda = 0$ so that $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$, i.e. $\sup_X u_k = 0$.

If $\lambda < 0$ then $u_k \rightarrow u$ strongly where $u \in \mathcal{E}^1(X, \omega, \psi)$ is the unique solution of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (1.31)$$

While if $\lambda = 0$ then, letting u be a weak accumulation point of $\{u_k\}_{k \in \mathbb{N}}$, $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, $u_k \rightarrow u$ strongly and u solves (1.31) if and only if $E_{\psi_k}(u_k) \geq -C$ uniformly and

$$\limsup_{k \rightarrow \infty} \int_X (\psi_k - u_k) f_k \omega^n \leq \int_X (\psi - u) f \omega^n. \quad (1.32)$$

While the uniform boundedness of the energies in Theorem G for the case $\lambda = 0$ is obviously necessary, we believe that the condition (1.32) might not be, but in this generality the situation is quite tricky. However there are some interesting cases where (1.32) is shown to be unnecessary as, for instance, when $f_k \in L^p$ for $p > 1$ with $\|f_k\|_{L^p} \leq C$ uniformly.

The case $\lambda > 0$, finally, is much more complicated. However when $\{\psi_t\}_{t \in [0,1]}$ is increasing, i.e. the singularities decrease, Paper IV contains an openness result assuming $f_t \equiv f \in L^p$ for $p > 1$ and a closure result depending on a boundedness from above of the solutions which should be compared with the classical C^0 -partial estimate as said in subsection 1.4.2.

To state the results, it is necessary to introduce the functional $F_{f,\psi,\lambda} := E_\psi - V_\psi L_{f,\lambda}$ where $L_{f,\lambda}(u) := \frac{-1}{\lambda} \log \int_X e^{-\lambda u} f \omega^n$ for $u \in \mathcal{E}^1(X, \omega, \psi)$, which generalizes the Ding functional to the relative setting and to different densities. Indeed it is a translation invariant functional whose critical points solve the Monge-Ampère equation $MA_\omega(u) = e^{-\lambda u} f \omega^n$. Observe that, already at this point, there is an obvious necessary condition to add on ψ to solve $MA_\omega(u) = e^{-\lambda u} f \omega^n$ in $\mathcal{E}^1(X, \omega, \psi)$. Namely the singularities type of ψ must not be too nasty in relation with the singularities of f . For instance when $f \equiv 1$, the condition becomes $e^{-\lambda \psi} \in L^1$, i.e. $c(\psi) > \lambda$ where $c(\cdot)$ is the *complex singularity exponent* defined as

$$c(u) := \sup \left\{ c > 0 : \int_X e^{-cu} \omega^n < \infty \right\}$$

(see [DK01]). Another problem of the variational approach is that a priori there may be solutions of the Monge-Ampère equations which are not global maximizers of $F_{f,\psi,\lambda}$. Since the principal focus is on Kähler-Einstein metrics with prescribed singularities on Fano manifolds (see also Paper V), Paper IV does not include a further study about when solutions are maximizers of $F_{f,\psi,\lambda}$ and it often assumes the d -coercivity of $F_{f,\psi,\lambda}$ over $\mathcal{E}_{norm}^1(X, \omega, \psi)$.

Theorem H (Theorem C of Paper IV). *Let $\psi \in \mathcal{M}^+$, $\lambda > 0$ and $f \in L^p$ for $p \in (1, +\infty]$. Assume also that $c(\psi) > \frac{\lambda p}{p-1}$ where $\frac{\lambda p}{p-1} = \lambda$ if $p = +\infty$. If the functional $F_{f,\psi,\lambda}$ is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$, then there exists $A > 1$ such that the complex Monge-Ampère equation*

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi') \end{cases}$$

admits a solution for any $\psi' \succcurlyeq \psi$ such that $V_{\psi'} < AV_\psi$.

The bound for the complex singularity exponent is sharp in the case $p = +\infty$ as said above, while in general it allows to prove the continuity of $L_{f,\lambda}$ over $\mathcal{E}^1(X, \omega, \psi)$, and hence the upper-semicontinuity of $F_{f,\psi,\lambda}$ which implies the existence of maximizers given its d -coercivity. Let us stress that the coefficient $A > 1$ of Theorem H only depends on the slope at infinity of $F_{f,\psi,\lambda}$.

The closedness result is instead the following.

Theorem I (Theorem D of Paper IV) . *Let $\lambda > 0$, $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered set such that $\psi_k \preceq \psi_{k+1}$ for any $k \in \mathbb{N}$, and $f_k, f \geq 0$ such that $f_k \rightarrow f$ in L^p for $p \in (1, \infty]$. Assume also the following conditions:*

i) $c(\psi) > \frac{\lambda p}{p-1}$;

ii) the complex Monge-Ampère equations

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k) \end{cases}$$

admit solutions u_k given as maximizers of $F_{f_k, \psi_k, \lambda}$;

iii) $\sup_X u_k \leq C$ uniformly.

Then there exists a subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ which converges strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$ solution of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases}$$

It is important to remark that the (basically unique) obstacle given by the uniform bound in (iii) is necessary. Indeed if $\{\omega\} = c_1(X)$ and $\psi_t = P_\omega[(1-t)\varphi_D]$ for $\varphi_D \in PSH(X, \omega)$ such that $\omega + dd^c \varphi_D = \frac{1}{r}[D]$ for a smooth effective divisor $D \sim_{lin} -rK_X$, then solving $MA_\omega(u_t) = e^{-u} f \omega^n$ for f smooth in the class $\mathcal{E}^1(X, \omega, \psi_t)$ is equivalent to solve

$$\begin{cases} MA_\omega(w_t) = t^{-n} e^{-tw_t - (1-t)\varphi_D} f \omega^n \\ w_t \in \mathcal{E}^1(X, \omega). \end{cases} \quad (1.33)$$

The correspondence is given by $w_t := \frac{1}{t}v_t$ where $u_t = v_t + (1-t)\varphi_D$. For suitable $f > 0$ the path (1.33) coincides with (1.27), which is well-known to admit solutions for t small enough while $\sup_{t \in S} \sup_X w_t = +\infty$ when $(X, -K_X)$ is not K -stable. Here $S \subset (0, 1]$ is the set of parameters such that (1.33) admits a solution.

Basically what happened in this case is that we removed the fixed part given by the divisor $\frac{1-t}{r}D$ to any element $u \preceq \psi_t$ to get an equivalent Monge-Ampère equation in a different cohomology class (which in this specific case it is just a multiple of $\{\omega\}$). This is a more general fact about model type envelopes with analytic singularities type. Indeed in Paper IV it is shown that studying Monge-Ampère equations over (X, ω) in the class $\mathcal{E}^1(X, \omega, \psi)$ is the same as studying equivalent Monge-Ampère equations over (Y, η) in the class $\mathcal{E}^1(Y, \eta)$ where (Y, η) are given by the resolution of the ideal defining the analytic singularities (see subsection 1.4.3). This yields to a natural applications of the study of complex Monge-Ampère equations with prescribed singularities.

First, recall that given a divisor D such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ for $\lambda \in \mathbb{Q}$, it makes sense to look at (weak) D -log Kähler-Einstein metrics, i.e. to find $u \in \mathcal{E}^1(X, \omega)$ such that

$$\text{Ric}(\omega + dd^c u) - [D] = \lambda(\omega + dd^c u). \quad (1.34)$$

In pluripotential sense this corresponds to solve a Monge-Ampère equation of the type

$$\begin{cases} MA_\omega(u) = e^{-\lambda u + C} f_D \omega^n \\ u \in \mathcal{E}^1(X, \omega). \end{cases}$$

for $C \in \mathbb{R}$ where $f_D > 0$ encodes the singularities of the divisor D . It is then convenient to say that $\omega + dd^c u$ is a $(D, [\psi])$ -log Kähler-Einstein metric if $u \in \mathcal{E}^1(X, \omega, \psi)$ and $MA_\omega(u) = e^{-\lambda u + C} f_D \omega^n$ for $C \in \mathbb{R}$. Indeed $\omega + dd^c u$ is actually the curvature of singular hermitian metrics which differ each other by translation constants and which are D -log Kähler-Einstein metrics in the sense of (1.34).

Then as said above, in the case $\psi \in \mathcal{M}^+$ with analytic singularities type, i.e. briefly $\psi \in \mathcal{M}_{an}^+$, a $(D, [\psi])$ -log Kähler-Einstein metric in the class $\{\omega\}$ over X is basically the same as a D' -log semiKähler-Einstein metric in the class $\{\eta\}$ over Y . Here we use the word semiKähler to denote a big and semipositive smooth form. Namely there exists a map

$$\Phi : \mathcal{M}_{an}^+ \rightarrow \{(Y, \eta) : \eta \text{ semiKähler with } \omega \geq p_* \eta \text{ where } p : Y \rightarrow X \text{ is a resolution}\} / \sim$$

where $(Y, \eta) \sim (Y', \eta')$ if there exists a third element $(Z, \tilde{\eta})$ dominating $(Y, \eta), (Y', \eta')$ in the usual sense. Denoting with $\mathcal{K}_{(X, \omega)}$ the image of Φ , the study of the existence of $(D, [\psi])$ -log Kähler-Einstein metrics varying $\psi \in \mathcal{M}^+$ then includes the study of all log semiKähler-Einstein metrics in pairs (Y, η) . Note in particular that (the class of) all small perturbations of the class $\{\omega\}$ in the direction of exceptional divisors are contained in $\mathcal{K}_{(X, \omega)}$. Moreover $\mathcal{K}_{(X, \omega)}$ inherits a natural partial order, a notion of strong convergence, and Theorems G, H, I can be naturally translating in this particular setting (see Theorem E in Paper IV).

1.4.6 Main Results of Paper V

The last Paper of this thesis concerns the study of Kähler-Einstein metrics with prescribed singularities on a Fano manifold (X, ω) , i.e. solutions of

$$\begin{cases} MA_\omega(u) = e^{-u} \mu \\ u \in \mathcal{E}^1(X, \omega, \psi) \end{cases} \quad (1.35)$$

where μ is a suitable volume form (namely $\mu = e^f \omega^n$ for f Ricci potential so that $Ric(\mu) = \omega$). A necessary condition on $\psi \in \mathcal{M}^+$ is that $e^{-\psi} \in L^1$, i.e. $c(\psi) > 1$. For obvious reasons it is said that (X, ψ) is klt when this happens and \mathcal{M}_{klt}^+ represents the set of all model type envelopes with non-zero total Monge-Ampère mass such that (X, ψ) is klt. In other words \mathcal{M}_{klt}^+ is the admissible set of model type envelopes for the search of Kähler-Einstein metrics with prescribed singularities.

After having defined a ψ -relative version of the Ding and of the Mabuchi functional, i.e. D_ψ, M_ψ , one goal of Paper V is to generalize the characterizations of the existence of the Kähler-Einstein metrics to the relative setting. Unfortunately two

more assumptions on ψ are added before proving these results. First ψ is given as decreasing limit of model type envelopes with algebraic singularities types. \mathcal{M}_D briefly denotes this set. Note that $\mathcal{M}_{D,klt}^+$ seems to be the biggest subset of \mathcal{M}^+ where it may make sense wondering if a relative analog of the Yau-Tian-Donaldson conjecture holds. However the assumption $\psi \in \mathcal{M}_D$ is necessary to easily deduce the linearity of E_ψ along weak geodesic segments in $(\mathcal{E}^1(X, \omega, \psi), d)$ thanks to Demailly's Theorem of Regularization and previous known results in the big absolute setting.

The second assumption is instead more involved. Indeed ψ is supposed to have *small unbounded locus*, i.e. locally bounded outside a complete closed pluripolar set. This hypothesis is necessary to apply a particular case of Berndtsson's convexity result.

Theorem J (Theorems C and D of Paper V) . *Let $\psi \in \mathcal{M}_{D,klt}^+$ with small unbounded locus and let $u \in \mathcal{E}^1(X, \omega, \psi)$. Then the following statements are equivalent:*

- i) $\omega + dd^c u$ is a Kähler-Einstein metrics with prescribed singularities $[\psi]$;
- ii) $D_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi$;
- iii) $M_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} M_\psi$.

Moreover the uniqueness of Kähler-Einstein metrics with prescribed singularities $[\psi]$ is modulo the action of $\text{Aut}(X, [\psi])^\circ := \text{Aut}(X, [\psi]) \cap \text{Aut}(X)^\circ$, where $\text{Aut}(X, [\psi])$ is the subgroup of all automorphisms F such that $F^*\psi - \psi$ is globally bounded.

Furthermore when $\text{Aut}(X, [\psi])^\circ = \{\text{Id}\}$ then the following conditions are equivalent:

- i) there exists an unique Kähler-Einstein metric with prescribed singularities $[\psi]$;
- ii) D_ψ is d -coercive over $\mathcal{E}_{\text{norm}}^1(X, \omega, \psi)$;
- iii) M_ψ is d -coercive over $\mathcal{E}_{\text{norm}}^1(X, \omega, \psi)$.

This Theorem together with the continuity method with movable singularities developed in Paper IV gives the strong continuity of Kähler-Einstein metrics with prescribed singularities $[\psi_t]$ where $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}_{klt,D}^+$ is an increasing segment (i.e. the singularities of ψ_t decrease). Indeed this is the content of Theorem B in Paper V. The advantage is that one can choose $\psi \in \mathcal{M}_{klt}^+$ and consider for instance the natural path $\psi_t := (1-t)\psi$. It is then clear the importance to understand which prescribed singularities $\psi \in \mathcal{M}_{klt}^+$ admits a Kähler-Einstein metrics with prescribed singularities $[\psi]$. Therefore in Paper V the *Kähler-Einstein locus*

$$\mathcal{M}_{KE} := \{\psi \in \mathcal{M}_{klt}^+ : \text{there exists a Kähler-Einstein metrics with prescribed singularities } [\psi]\},$$

is introduced and a first study of its structure using the relative version of the α -invariant is provided. More precisely setting $\mathcal{M} \ni \psi \rightarrow \alpha_\omega(\psi) \in (0, +\infty)$,

$$\alpha_\omega(\psi) := \sup \left\{ \alpha > 0 : \sup_{\{u \preceq \psi, \sup_X u = 0\}} \int_X e^{-\alpha u} d\mu < +\infty \right\}$$

as the natural generalization of the α -invariant, the following result holds.

Theorem K (Theorem A of Paper V) . *Let (X, ω) be a Fano manifold. Then*

$$\left\{ \psi \in \mathcal{M}_{klt}^+ : \alpha_\omega(\psi) > \frac{n}{n+1} \right\} \subset \mathcal{M}_{KE}.$$

Moreover (i) \Rightarrow (ii) \Rightarrow (iii) in the following conditions:

i) there exists $\psi \in \mathcal{M}$, $t \in (0, 1]$ such that

$$\alpha_\omega(\psi_t) > \frac{n}{(n+1)t}$$

for $\psi_t := P_\omega[t\psi]$;

ii) $\alpha_\omega(0) > \frac{n}{n+1}$;

iii) $\mathcal{M}_{KE} = \mathcal{M}_{klt}^+$.

Furthermore if $\psi \in \mathcal{M}_{klt}^+$ satisfies $\text{lct}(X, 0, \psi) := \sup\{p > 1 : (X, p\psi) \text{ is } klt\} \geq \frac{n^2+1}{n^2-n}$ then

$$\alpha_\omega(\psi) > \frac{n^2+1}{n+1} \implies 0 \in \mathcal{M}_{KE}. \quad (1.36)$$

The main upshot of this Theorem is that \mathcal{M}_{KE} seems to be a very rigid locus, and therefore the study of Kähler-Einstein metrics with prescribed singularities may help to understand if there exists a genuine Kähler-Einstein metric (and hence to detect if $(X, -K_X)$ is K -stable). Note for instance that the implication (i) \Rightarrow (ii) and (1.36) (look at the paper for its sharpest version) give a direct result on the existence of genuine Kähler-Einstein metric estimating the α -invariant at singular elements $\psi \in \mathcal{M}_{klt}^+$. It is important to stress that, roughly speaking, the more ψ is singular the easier is the computation of $\alpha_\omega(\psi)$ as it follows from the definition. For instance in Paper V some rough estimates of $\alpha_\omega(\psi)$ for ψ with 0-dimensional equisingularities are presented, i.e. when ψ has analytic singularities at N points of the same weight. These estimates are given in terms of multipoint Seshadri constants and of pseudoeffective thresholds. Note that this kind of computations may be improved with the idea to produce new K -stable Fano manifolds.

On the other hand the implication (ii) \Rightarrow (iii) implies that all Fano manifolds with $\alpha_\omega(0) > \frac{n}{n+1}$ have many canonical metrics, i.e. for any admissible prescribed singularities there exists a Kähler-Einstein metrics with such prescribed singularities. In particular for these manifolds there are many log Kähler-Einstein metrics for weak Fano pairs (Y, η) given by resolution of integrally closed coherent analytic sheaves. However (ii) cannot be replaced with $\alpha_\omega(0) \geq \frac{n}{n+1}$ as a counterexample with $X = \mathbb{P}^2$ shows. This leads to the following conjecture.

Conjecture A (Conjecture A). *Let (X, ω) be a Fano manifold with $\text{Aut}(X)^\circ = \{Id\}$. Then*

$$0 \in \mathcal{M}_{KE} \iff \mathcal{M}_{KE} = \mathcal{M}_{klt}^+.$$

Conjecture A would yield that showing the non-existence of Kähler-Einstein metrics with prescribed singularities implies in many cases the non-existence of genuine Kähler-Einstein metrics.

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Part II

PAPERS

PAPER I

Multipoint Okounkov bodies

Antonio Trusiani

arxiv preprint

Chapter 2

Multipoint Okounkov bodies

Abstract

Starting from the data of a big line bundle L on a projective manifold X with a choice of $N \geq 1$ different points on X we give a new construction of N Okounkov bodies which encodes important geometric features of $(L \rightarrow X; p_1, \dots, p_N)$ such as the volume of L , the (moving) multipoint Seshadri constant of L at p_1, \dots, p_N , and the possibility to construct Kähler packings centered at p_1, \dots, p_N . Toric manifolds and surfaces are examined in detail.

Keywords: Okounkov body, Seshadri constant, packings problem, projective manifold, ample line bundle.

2010 Mathematics subject classification: 14C20 (primary); 32Q15, 57R17 (secondary).

2.1 Introduction

Okounkov in [Oko96] and [Oko03] found a way to associate a convex body $\Delta(L) \subset \mathbb{R}^n$ to a polarized manifold (X, L) where $n = \dim_{\mathbb{C}} X$. Namely,

$$\Delta(L) := \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu^p(s)}{k} : s \in H^0(X, kL) \setminus \{0\} \right\}}$$

where $\nu^p(s)$ is the *leading term exponent* at p with respect to a total additive order on \mathbb{Z}^n and holomorphic coordinates centered at $p \in X$ (see subsection 2.2.4). This convex body is now called *Okounkov body*.

Okounkov's construction was inspired by toric geometry, indeed in the toric case, if L_P is a torus-invariant ample line bundle, $\Delta(L_P)$ is essentially equal to the polytope

P .

The same construction works even if L is a big line bundle, i.e. a line bundle such that $\text{Vol}_X(L) := \limsup_{k \rightarrow \infty} \frac{n!}{k^n} \dim_{\mathbb{C}} H^0(X, kL) > 0$, as proved in [LM09], [KKh12] (see also [Bou14]) and the Okounkov body captures the volume of L since

$$\text{Vol}_X(L) = n! \text{Vol}_{\mathbb{R}^n}(\Delta(L)).$$

Moreover if $>$ is the lexicographical order then the $(n-1)$ -volume of any not trivial *slice* of the Okounkov body is related to the *restricted volume* of $L - tY$ along Y where Y is a smooth irreducible divisor such that $Y|_{U_P} = \{z_1 = 0\}$.

Another invariant which can be encoded by the Okounkov body is the *(moving) Seshadri constant* $\epsilon_S(\|L\|; p)$ (see [Dem90] in the ample case, or [Nak03] for the extension to the big case). Indeed, as K ronya-Lozovanu showed in [KL15a], [KL17], if the Okounkov body is defined using the *deglex order*¹, then

$$\epsilon_S(\|L\|; p) = \max \{0, \sup \{t \geq 0 : t\Sigma_n \subset \Delta(L)\}\}$$

where Σ_n is the unit n -simplex.

As showed by Witt Nystr m in [WN15], we can restrict to consider the *essential* Okounkov body $\Delta(L)^{\text{ess}}$ to get the same characterization of the moving Seshadri constant.

Recall that $\Delta(L)^{\text{ess}} := \bigcup_{k \geq 1} \Delta^k(L)^{\text{ess}}$, where $\Delta^k(L) = \text{Conv}(\{\frac{\nu(s)}{k} : s \in H^0(X, kL) \setminus \{0\}\})$ and the *essential* part of $\Delta^k(L)$ consists of its interior as subset of $\mathbb{R}_{\geq 0}^n$ with its natural induced topology.

Seshadri constants are also defined for a collection of different points. For a nef line bundle L , the *multipoint Seshadri constant* of L at p_1, \dots, p_N is given as

$$\epsilon_S(L; p_1, \dots, p_N) := \inf_C \frac{L \cdot C}{\sum_{j=1}^N \text{mult}_{p_j} C}.$$

In this paper we introduce a multipoint version of the Okounkov body. More precisely, for a fixed big line bundle L on a projective manifold X of dimension n and $p_1, \dots, p_N \in X$ different points, we construct N Okounkov bodies $\Delta_j(L) \subset \mathbb{R}^n$ for $j = 1, \dots, N$.

Definition 2.1.1. *Let L be a big line bundle and let $>$ be a fixed total additive order on \mathbb{Z}^n .*

$$\Delta_j(L) := \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu^{p_j}(s)}{k} : s \in V_{k,j} \right\}} \subset \mathbb{R}^n$$

*is called **multipoint Okounkov body** of L at p_j , where $V_{k,j} := \{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_j}(s) < \nu^{p_i}(s) \text{ for any } i \neq j\}$ for any $k \geq 0$.*

¹ $\alpha <_{\text{deglex}} \beta$ iff $|\alpha| := \sum_{j=1}^n \alpha_j < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha <_{\text{lex}} \beta$, where $<_{\text{lex}}$ is the lexicographical order

We observe that the multipoint Okounkov body of L at p_j is obtained by considering all sections whose leading term in p_j is strictly smaller than those at the other points. They are convex compact sets in \mathbb{R}^n but, unlike the one-point case, for $N \geq 2$ it can happen that some $\Delta_j(L)$ is empty (Remark 2.3.8). The definition does not depend on the order of the points.

Our first theorem concerns the relationship between the multipoint Okounkov bodies and the volume of the line bundle:

Theorem A. ² *Let L be a big line bundle. Then*

$$n! \sum_{j=1}^N \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \text{Vol}_X(L).$$

Furthermore, similar to section § 4 in [LM09], we show that $\Delta_j(\cdot)$ is a numerical invariant and that there exists of a open subset of the big cone containing $B_+(p_j)^C = \{\alpha \in N^1(X)_{\mathbb{R}} : p_j \notin \mathbb{B}_+(\alpha)\}$ over which $\Delta_j(\cdot)$ can be extended continuously (see section § 2.3.2). Recall that the points, and more in general the valuations ν^{p_j} , are fixed.

Moreover when $>$ is the lexicographical order and Y_1, \dots, Y_N are smooth irreducible divisors such that $Y_j|_{U_{p_j}} = \{z_{j,1} = 0\}$, the fibers of $\Delta_j(L)$ are related to the restricted volumes of $L - t \sum_{i=1}^N Y_i$ along Y_j (see section § 2.3.3).

The multipoint Okounkov bodies can be finer invariants than the *moving* multipoint Seshadri constant (a natural generalization of the multipoint Seshadri constant to big line bundles, see section § 2.5) as our next Theorem shows.

Theorem B. *Let L be a big line bundle and let $>$ be the deglex order. Then*

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \max \{0, \xi(L; p_1, \dots, p_N)\}$$

where $\xi(L; p_1, \dots, p_N) := \sup\{t \geq 0 : t\Sigma_n \subset \Delta_j(L)^{\text{ess}} \text{ for any } j = 1, \dots, N\}$

Next we recall another interpretation of the one point Seshadri constant: $\epsilon_S(L; p)$ is equal to the supremum of r such that there exists an holomorphic embedding $f : (B_r(0), \omega_{st}) \rightarrow (X, L)$ with the property that $f_*\omega_{st}$ extends to a Kähler form ω with cohomology class $c_1(L)$ (see Theorem 5.1.22 and Proposition 5.3.17. in [Laz04]). This result is a consequence of a deep analysis in symplectic geometry by McDuff-Polterovich ([MP94]), where they dealt with the *symplectic packings problem* (in the same spirit, Biran in [Bir97] proved the symplectic analogues of the Nagata's conjecture).

Successively Kaveh in [Kav16] showed how the one-point Okounkov body can be used to construct a symplectic packing. On the same line Witt Nyström in [WN15]

²The theorem holds in the more general setting of a family of *faithful valuations* $\nu^{p_j} : \mathcal{O}_{X, p_j} \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ respect to a fixed total additive order $>$ on \mathbb{Z}^n .

introduced the torus-invariant domain $D(L) := \mu^{-1}(\Delta(L)^{\text{ess}})$ (called *Okounkov domain*) for $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$, $\mu(z_1, \dots, z_n) := (|z_1|^2, \dots, |z_n|^2)$, and showed how it approximates the manifold.

To get a similar characterization of the multipoint Seshadri constant, we give the following definition of *Kähler packing*.

Definition 2.1.2. *We say that a finite family of n -dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs into (X, L) for L ample line bundle on a n -dimensional projective manifold X if for any family of relatively compact open set $U_j \Subset M_j$ there is a holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ and a Kähler form ω lying in $c_1(L)$ such that $f_*\eta_j = \omega|_{f(U_j)}$. If, in addition,*

$$\sum_{j=1}^N \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs perfectly into (X, L) .

Following [WN15] we define the *multipoint Okounkov domains* as the torus-invariant domains of \mathbb{C}^n given by $D_j(L) := \mu^{-1}(\Delta_j(L)^{\text{ess}})$.

Theorem C. ³ *Let L be an ample line bundle. Then $\{(D_j(L), \omega_{st})\}_{j=1, \dots, N}$ packs perfectly into (X, L) .*

Note that for big line bundles a similar theorem holds, given a slightly different definition of *packings* (see section 2.4.2).

As a consequence (Corollary 2.5.17), if $>$ is the deglex order then

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \max \{0, \sup \{r > 0 : B_r(0) \subset D_j(L) \forall j = 1, \dots, N\}\}.$$

This result was known in dimension 2 by the work of Eckl ([Eckl17]).

Moving to particular cases, for toric manifolds we prove that, chosen torus-fixed points and the deglex order, the multipoint Okounkov bodies can be obtained *subdividing* the polytope (Theorem 2.6.4). If we consider all torus-invariant points the subdivision is *barycentric* (Corollary 2.6.6). As a consequence we get that the multipoint Seshadri constant of N torus-fixed points is in $\frac{1}{2}N$ (Corollary 2.6.7).

Finally in the surface case, we extend the result in [KLM12] showing, for the lexicographical order, the polyhedrality of $\Delta_j(L)$ (Theorem 2.6.9). Moreover for $\mathcal{O}_{\mathbb{P}^2}(1)$ over \mathbb{P}^2 we completely characterize $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1))$ in function of $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N)$ obtaining an explicit formula for the restricted volume of $\mu^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ for $t \in \mathbb{Q}$ where $\mu : \tilde{X} \rightarrow X$ is the blow-up at N very general points and $\mathbb{E} := \sum_{j=1}^N E_j$ is the sum of the exceptional divisors (Theorem 2.6.14). As a consequence we independently get a result present in [DKMS15]: the ray $\mu^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ meets at most two *Zariski chambers*.

³the theorem holds even if ν^{p_j} is a family of faithful *quasi-monomial* valuations respect to the same linearly independent vectors $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{N}^n$.

2.1.1 Organization

Section 2.2 contains some preliminary facts on singular metrics, base loci of divisors and Okounkov bodies.

In section 2.3 we develop the theory of multipoint Okounkov bodies: the goal is to generalize some results in [LM09] for $N \geq 1$. We prove here Theorem A.

Section 2.4 is dedicated to show Theorem C.

In section 2.5 we introduce the notion of *moving* multipoint Seshadri constants. Moreover we prove Theorem B, connecting the moving multipoint Seshadri constant in a more analytical language in the spirit of [Dem90], and deduce the connection between the moving multipoint Seshadri constant and Kähler packings.

The last section 2.6 deals with the two aforementioned particular cases: toric manifolds and surfaces.

2.1.2 Related works

In addition to the already mentioned papers of Witt Nyström ([WN15]), Eckl ([Eckl17]), and Kürona-Lozovanu ([KL15a], [KL17]), during the final revision of this paper the work of Shin [Sh17] appeared as a preprint. Starting from the same data of a big divisor over a projective manifold of dimension n and the choice of r different points, he gave a construction of an *extended Okounkov Body* $\Delta_{Y^1, \dots, Y^r}(D) \subset \mathbb{R}^{rn}$ from a valuation associated to a family of admissible or infinitesimal flags Y^1, \dots, Y^r . In the ample case thanks to the Serre's vanishing Theorem, the multipoint Okounkov bodies can be recovered from the extended Okounkov body as projections after suitable subdivisions. Precisely, with the notation given in [Sh17], we get

$$F(\Delta_j(D)) = \pi_j \left(\Delta_{Y^1, \dots, Y^r}(D) \cap H_{1,j} \cap \dots \cap H_{j-1,j} \cap H_{j+1,j} \cap \dots \cap H_{r,j} \right)$$

where $\pi_j : \mathbb{R}^{rn} \rightarrow \mathbb{R}^n$, $\pi_j(\vec{x}_1, \dots, \vec{x}_r) := \vec{x}_j$, $H_{i,j} := \{(\vec{x}_1, \dots, \vec{x}_r) \in \mathbb{R}^{rn} : x_{i,1} \geq x_{j,1}\}$ and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F(y_1, \dots, y_n) := (|y|, y_1, \dots, y_{n-1})$. Note that $x_{i,1}$ means the first component of the vector \vec{x}_i while $|y| = y_1 + \dots + y_n$. The same equality holds if $L := \mathcal{O}_X(D)$ is big and $c_1(L) \in \text{Supp}(\Gamma_j(X))^\circ$ (see section 2.3.2).

2.1.3 Acknowledgements

I want to thank David Witt Nyström and Stefano Trapani for proposing the project to me and for their suggestions and comments. It is also a pleasure to thank Bo Berndtsson for reviewing this article, Valentino Tosatti for his interesting comments and Christian Schultes for pointing out a mistake in the previous version.

2.2 Preliminaries

2.2.1 Singular metrics and (currents of) curvature

Let L be an holomorphic line bundle over a projective manifold X . A smooth (hermitian) metric φ is the collection of an open cover $U_{j \in J}$ of X and of smooth functions $\varphi_j \in C^\infty(U_j)$ such that on each not-empty intersection $U_i \cap U_j$ we have $\varphi_i = \varphi_j + \ln |g_{i,j}|^2$ where $g_{i,j}$ are the transition function defining the line bundle L . Note that if s_j are nowhere zero local sections with respect to which the transition function are calculated then $|s_j| = e^{-\varphi_j}$. The curvature of a smooth metric φ is given on each open U_j by $dd^c \varphi_j$ where $d^c = \frac{1}{4\pi}(\partial - \bar{\partial})$ so that $dd^c = \frac{i}{2\pi} \partial \bar{\partial}$. We observe that it is a global $(1,1)$ -form on X , so for convenience we use the notation $dd^c \varphi$. The metric is called *positive* if the $(1,1)$ -form $dd^c \varphi$ is a Kähler form, i.e. if the functions φ_j are strictly plurisubharmonic. By the well-known Kodaira Embedding Theorem, a line bundle admits a positive metric iff it is ample.

Demaily in [Dem90] introduced a weaker notion of metric: a (hermitian) singular metric φ is given by a collection of data as before but with the weaker condition that $\varphi_j \in L^1_{loc}(U_j)$. If the functions φ_j are also plurisubharmonic, then we say that φ is a singular positive metric. Note that $dd^c \varphi$ exists in the weak sense, indeed it is a closed positive $(1,1)$ -current (we will call it the current of curvature of the metric φ). We say that $dd^c \varphi$ is a Kähler current if it dominates some Kähler form ω . By Proposition 4.2. in [Dem90] a line bundle is big iff it admits a singular positive metric whose current of curvature is a Kähler current.

In this paper we will often work with \mathbb{R} -line bundles, i.e. with formal linear combinations of line bundles. Moreover since we will work exclusively with projective manifolds, we will often consider an \mathbb{R} -line bundle as a class of \mathbb{R} -divisors modulo linear equivalence and its first Chern class as a class of \mathbb{R} -divisors modulo numerical equivalence.

2.2.2 Base loci

We recall here the construction of the *base loci* (see [ELMNP06]).

Given a \mathbb{Q} -divisor D , let $\mathbb{B}(D) := \bigcap_{k \geq 1} \text{Bs}(kD)$ be the *stable base locus* of D where $\text{Bs}(kD)$ is the base locus of the linear system $|kD|$. The base loci $\mathbb{B}_+(D) := \bigcap_A \mathbb{B}(D - A)$ and $\mathbb{B}_-(D) := \bigcup_A \mathbb{B}(D + A)$, where A varies among all ample \mathbb{Q} -divisors, are called respectively *augmented* and *restricted base locus* of D . They are invariant under rescaling and $\mathbb{B}_-(D) \subset \mathbb{B}(D) \subset \mathbb{B}_+(D)$. Moreover as described in the work of Nakamaye, [Nak03], the restricted and the augmented base loci are numerical invariants and can be considered as defined in the Neron-Severi space (for a real class it is enough to consider only ample \mathbb{R} -divisors A such that $D \pm A$ is a \mathbb{Q} -divisor). The stable base loci do not, see Example 1.1. in [ELMNP06], although by Proposition 1.2.6. in [ELMNP06] the subset where the augmented and restricted base loci are equal is open and dense in the Neron-Severi space $N^1(X)_{\mathbb{R}}$.

Thanks to the numerical invariance of the restricted and augmented base loci, we will often talk of restricted and/or augmented base loci of a \mathbb{R} -line bundle L . Moreover the restricted base locus can be thought as a measure of the *nefness* since D is nef iff $\mathbb{B}_-(D) = \emptyset$, while the augmented base locus can be thought as a measure of the *ampleness* since D is ample iff $\mathbb{B}_+(D) = \emptyset$. Moreover $\mathbb{B}_-(D) = X$ iff D is not pseudoeffective while $\mathbb{B}_+(D) = X$ iff D is not big.

2.2.3 Additive Semigroups and their Okounkov bodies

We briefly recall some notions about the theory of the Okounkov bodies constructed from additive semigroups (the main references are [KKh12] and [Bou14], see also [Kho93]).

Let $S \subset \mathbb{Z}^{n+1}$ be an additive subsemigroup not necessarily finitely generated. We denote by $C(S)$ the closed cone in \mathbb{R}^{n+1} generated by S , i.e. the closure of the set of all linear combinations $\sum_i \lambda_i s_i$ with $\lambda_i \in \mathbb{R}_{\geq 0}$ and $s_i \in S$. In this paper we will work exclusively with semigroups S such that the pair $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is *admissible*, i.e. $S \subset \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, or *strongly admissible*, i.e. it is admissible and $C(S)$ intersects the hyperplane $\mathbb{R}^n \times \{0\}$ only in the origin (see section §1.2 in [KKh12]). We recall that a closed convex cone C with *apex* the origin is called *strictly convex* iff the biggest linear subspace contained in C is the origin, so if $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is strongly admissible then $C(S)$ is strictly convex.

Definition 2.2.1. *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be an admissible pair. Then*

$$\Delta(S) := \pi(C(S) \cap \{\mathbb{R}^n \times \{1\}\})$$

*is called **Okounkov convex set of** $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$, where $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the projection to the first n coordinates. If $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ is strongly admissible, $\Delta(S)$ is also called **Okounkov body of** $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$.*

Remark 2.2.2. The convexity of $\Delta(S)$ is immediate, while it is not hard to check that it is compact iff the pair is strongly admissible. Moreover S generates a subgroup of \mathbb{Z}^{n+1} of maximal rank iff $\Delta(S)$ has interior not-empty.

Defining $S^k := \{\alpha : (k\alpha, k) \in S\} \subset \mathbb{R}^n$ for $k \in \mathbb{N}$, we get

Proposition 2.2.3 ([WN15]). *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be an admissible pair. Then*

$$\Delta(S) = \overline{\bigcup_{k \geq 1} S^k}.$$

Moreover if $K \subset \Delta(S)^\circ \subset \mathbb{R}^n$ compact subset then $K \subset \text{Conv}(S^k)$ for $k \geq 1$ divisible enough, where Conv denotes the closed convex hull. In particular

$$\Delta(S)^\circ = \bigcup_{k \geq 1} \text{Conv}(S^k)^\circ = \bigcup_{k \geq 1} \text{Conv}(S^{k!})^\circ$$

with $\text{Conv}(S^{k!})$ non-decreasing in k .

Proof. It is clear that $\Delta(S) \supset \overline{\bigcup_{k \geq 1} S^k}$. The reverse implication follows from Theorem 1.4. in [KKh12] if S is finitely generated, while in general we can approximate $\Delta(S)$ by Okounkov bodies of finitely generated subsemigroups of S . The second statement is the content of Lemma 2.3 in [WN14] when S is finitely generated, while the general case follows observing that $\text{Conv}(S^{k!})$ is non-decreasing in k by definition. \square

When a strong admissible pair $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ satisfies the further hypothesis $\Delta(S) \subset \mathbb{R}_{\geq 0}^n$ then we denote with

$$\Delta(S)^{\text{ess}} := \bigcup_{k \geq 1} \text{Conv}(S^k)^{\text{ess}}$$

the *essential* Okounkov body where $\text{Conv}(S^k)^{\text{ess}}$ represents the interior of $\text{Conv}(S^k)$ as subset of $\mathbb{R}_{\geq 0}^n$ with its induced topology ([WN15]). Note that if S is finitely generated then $\Delta(S)^{\text{ess}}$ coincides with the interior of $\Delta(S)$ as subset of $\mathbb{R}_{\geq 0}^n$, but in general they may be different since points in the hyperplanes $\{x_i = 0\}$ may be contained in $\Delta(S)^{\text{ess}}$ but not in $\Delta(S)^{\circ}$.

Proposition 2.2.4. *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be a strongly admissible pair such that $\Delta(S) \subset \mathbb{R}_{\geq 0}^n$, and let $K \subset \Delta(S)^{\text{ess}}$ be a compact set. Then there exists $k \gg 1$ divisible enough such that $K \subset \text{Conv}(S^k)^{\text{ess}}$. In particular*

$$\Delta(S)^{\text{ess}} = \bigcup_{k \geq 1} \text{Conv}(S^{k!})^{\text{ess}}$$

with $\text{Conv}(S^{k!})^{\text{ess}}$ non-decreasing in k , and $\Delta(S)^{\text{ess}}$ is an open convex set of $\mathbb{R}_{\geq 0}^n$.

Proof. We may assume that $\Delta(S)^{\text{ess}} \neq \emptyset$ otherwise it is trivial. Therefore the subgroup of \mathbb{Z}^{n+1} generated by S has maximal rank. Then as in Proposition 2.2.3 it is enough to prove the Proposition assuming S finitely generated. Thus we conclude exactly as in Lemma 2.3 in [WN14] using Theorem 1.4. in [KKh12]. \square

We also recall the following key Theorem:

Theorem 2.2.5 ([Bou14], Théorème 1.12.; [KKh12], Theorem 1.14.) . *Let $(S, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ be a strongly admissible pair, let $G(S) \subset \mathbb{Z}^{n+1}$ be the group generated by S and let ind_1 and ind_2 be respectively the index of the subgroups $\pi_1(G(S))$ and $\pi_2(G(S))$ in \mathbb{Z}^n and in \mathbb{Z} where π_1 and π_2 are respectively the projection to the first n -coordinates and to the last coordinate. Then*

$$\frac{\text{Vol}_{\mathbb{R}^n}(\Delta(S))}{\text{ind}_1 \text{ind}_2^n} = \lim_{m \rightarrow \infty, m \in \mathbb{N}(S)} \frac{\#S^m}{m^n}$$

where $\mathbb{N}(S) := \{m \in \mathbb{N} : S^m \neq \emptyset\}$ and the volume is respect to the Lebesgue measure.

Finally we need to introduce the *valuations* :

Definition 2.2.6. *Let V be an algebra over \mathbb{C} . A **valuation** from V to \mathbb{Z}^n equipped with a total additive order $>$ is a map $\nu : V \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ such that*

- i) $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ for any $f, g \in V \setminus \{0\}$ such that $f+g \neq 0$;*
- ii) $\nu(\lambda f) = \nu(f)$ for any $f \in V \setminus \{0\}$ and any $\mathbb{C} \ni \lambda \neq 0$;*
- iii) $\nu(fg) = \nu(f) + \nu(g)$ for any $f, g \in V \setminus \{0\}$.*

Often ν is defined on the whole V adding $+\infty$ to the group \mathbb{Z}^n and imposing $\nu(0) := +\infty$.

For any $\alpha \in \mathbb{Z}^n$ the α -leaf of the valuation is defined as the quotient of vector spaces

$$\hat{V}_\alpha := \frac{\{f \in V \setminus \{0\} : \nu(f) \geq \alpha\} \cup \{0\}}{\{f \in V \setminus \{0\} : \nu(f) > \alpha\} \cup \{0\}}.$$

A valuation is said to have *one-dimensional leaves* if the dimension of any leaf is at most 1.

Proposition 2.2.7 ([KKh12], Proposition 2.6.) . *Let V be an algebra over \mathbb{C} , and let $\nu : V \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ be a valuation with one-dimensional leaves. Then for any no trivial subspaces $W \subset V$,*

$$\#\nu(W \setminus \{0\}) = \dim_{\mathbb{C}} W.$$

We will say that a valuation $\nu : V \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ is *faithful* if the field of fractions K of V has transcendental degree n and the extension $\nu : K \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ defined as $\nu(f/g) := \nu(f) - \nu(g)$ (see Lemme 2.3 in [Bou14]) has the whole \mathbb{Z}^n as image. Note that any faithful valuation has one-dimensional leaves (see Remark 2.26. in [Bou14]).

2.2.4 The Okounkov body associated to a line bundle

In this section we recall the construction and some known results of the Okounkov body associated to a line bundle L around a point $p \in X$ (see [LM09],[KKh12] and [Bou14]).

Consider the abelian group \mathbb{Z}^n equipped with a total additive order $>$, let $\nu : \mathbb{C}(X) \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ be a faithful valuation with *center* $p \in X$ (see the previous subsection). We recall that $p \in X$ is the (unique) center of ν if $\mathcal{O}_{X,p} \subset \{f \in \mathbb{C}(X) : \nu(f) \geq 0\}$ and $\mathfrak{m}_{X,p} \subset \{f \in \mathbb{C}(X) : \nu(f) > 0\}$, and that the semigroup $\nu(\mathcal{O}_{X,p} \setminus \{0\})$ is well-ordered by the induced order (see §2 in [Bou14]).

Assume that $L|_U$ is trivialized by a non-zero local section t . Then any section $s \in H^0(X, kL)$ can be written locally as $s = ft^k$ with $f \in \mathcal{O}_X(U)$. Thus we define $\nu(s) := \nu(f)$, where we identify $\mathbb{C}(X)$ with the meromorphic function field and $\mathcal{O}_{X,p}$ with the stalk of \mathcal{O}_X at p . We observe that $\nu(s)$ does not depend on the trivialization

t chosen since any other trivialization t' of $L|_V$ differs from t on $U \cap V$ by an unit $u \in \mathcal{O}_X(U \cap V)$. We define an additive semigroup associated to the valuation by

$$\Gamma := \{(\nu(s), k) : s \in H^0(X, kL) \setminus \{0\}, k \geq 0\} \subset \mathbb{Z}^n \times \mathbb{Z}.$$

We call the **Okounkov body**, $\Delta(L)$, the Okounkov convex set of $(\Gamma, \mathbb{R}^n \times \mathbb{R}_{\geq 0})$ (see Definition 2.2.1), i.e.

$$\Delta(L) := \pi(C(\Gamma) \cap \{\mathbb{R}^n \times \{1\}\})$$

where $\pi : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is the projection to the first n coordinated. By Proposition 2.2.3 we have

$$\begin{aligned} \Delta(L) &= \overline{\bigcup_{k \geq 1} \left\{ \frac{\nu(s)}{k} : s \in H^0(X, kL) \setminus \{0\} \right\}} \\ &= \text{Conv} \left(\left\{ \frac{\nu(s)}{k} : s \in H^0(X, kL) \setminus \{0\}, k \geq 1 \right\} \right), \end{aligned}$$

and we note that it is a convex set of \mathbb{R}^n but it has interior not-empty iff Γ generates a subgroup of \mathbb{Z}^{n+1} of maximal rank (Remark 2.2.2). Furthermore for a prime divisor $D \in \text{Div}(X)$ we will denote $\nu(D) = \nu(f)$ for f any local equation for D near p , and the map $\nu : \text{Div}(X) \rightarrow \mathbb{Z}^n$ extends to a \mathbb{R} -linear map from $\text{Div}(X)_{\mathbb{R}}$.

Theorem 2.2.8 ([LM09],[KKh12]) . *The following statements hold:*

- i) $\Delta(L)$ is a compact convex set lying in \mathbb{R}^n ;
- ii) $n! \text{Vol}_{\mathbb{R}^n}(\Delta(L)) = \text{Vol}_X(L)$, and in particular L is big iff $\Delta(L)^\circ \neq \emptyset$, i.e $\Delta(L)$ is a convex body;
- iii) if L is big then $\Delta(L) = \overline{\{D \in \text{Div}_{\geq 0}(X)_{\mathbb{R}} : D \equiv_{\text{num}} L\}}$ and, in particular, the Okounkov body depends only on the numerical class of the big line bundle.

Quasi-monomial valuation Equip \mathbb{Z}^n of a total additive order $>$, fix $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{Z}^n$ linearly independent and fix local holomorphic coordinates $\{z_1, \dots, z_n\}$ around a fixed point p . Then we can define the *quasi-monomial* valuation $\nu : \mathcal{O}_{X,p} \setminus \{0\} \rightarrow \mathbb{Z}^n$ by

$$\nu(f) := \min \left\{ \sum_{i=1}^n \alpha_i \vec{\lambda}_i : a_\alpha \neq 0 \text{ where locally around } p, f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \right\}$$

where the minimum is taken respect to the $>$ order fixed on \mathbb{Z}^n . Note that it is faithful iff $\det(\vec{\lambda}_1, \dots, \vec{\lambda}_n) = \pm 1$.

For instance if we equip \mathbb{Z}^n of the lexicographical order and we take $\vec{\lambda}_j = \vec{e}_j$ (j -th vector of the canonical base of \mathbb{R}^n) we get

$$\nu(f) := \min_{lex} \{ \alpha : a_\alpha \neq 0 \text{ where locally around } p, f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \}.$$

This is the valuation associated to an admissible flag $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{p\}$, in the sense of [LM09]⁴, such that locally $Y_i := \{z_1 = \cdots = z_i = 0\}$ (see also [WN15]).

A change of coordinates with the same local flag produces the same valuation, i.e. the valuation depends uniquely on the local flag.

Note: *In the paper a valuation associated to an admissible flag Y will be the valuation constructed by the local procedure starting from local holomorphic coordinates as just described.*

On the other hand if we equip \mathbb{Z}^n of the deglex order and we take $\vec{\lambda}_i = \vec{e}_i$, we get the valuation $\nu : \mathcal{O}_{X,p} \setminus \{0\} \rightarrow \mathbb{Z}^n$,

$$\nu(f) := \min_{\text{deglex}} \{\alpha : a_\alpha \neq 0 \text{ where locally around } p, f = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha\}.$$

This is the valuation associated to an *infinitesimal* flag Y in p : given a flag of subspaces $T_p X =: V_0 \supset V_1 \supset \cdots \supset V_n = \{0\}$ such that $\dim_{\mathbb{C}} V_i = n - i$, consider on $\tilde{X} := \text{Bl}_p X$ the flag

$$\tilde{X} =: Y_0 \supset \mathbb{P}(T_p X) = \mathbb{P}(V_0) =: Y_1 \supset \cdots \supset \mathbb{P}(V_{n-1}) =: Y_n =: \{\tilde{p}\}.$$

Note that Y is an admissible flag around \tilde{p} on the blow-up \tilde{X} . Indeed we recover the valuation on \tilde{X} associated to this admissible flag considering $F \circ \nu$ where the function $F : (\mathbb{Z}^n, >_{\text{deglex}}) \rightarrow (\mathbb{Z}^n, >_{\text{lex}})$ is the order-preserving isomorphism $F(\alpha) := (|\alpha|, \alpha_1, \dots, \alpha_{n-1})$, i.e. considering the quasi-monomial valuation given by the lexicographical order and $\vec{\lambda}_i := \vec{e}_1 + \vec{e}_i$.

Note: *In the paper a valuation associated to an infinitesimal flag Y will be the valuation ν constructed by the local procedure starting from local holomorphic coordinates as just described, and in particular the total additive order on \mathbb{Z}^n will be the deglex order in this case.*

2.2.5 A moment map associated to an $(S^1)^n$ -action on a particular manifold

In this brief subsection we recall some fact regarding a moment map for an $(S^1)^n$ -action on a symplectic manifold (X, ω) cosntructed from a convex hull of a finite set $\mathcal{A} \subset \mathbb{N}^n$ (see section §3 in [WN15]).

Let $\mathcal{A} \subset \mathbb{N}^n$ be a finite set, let $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$ be the map $\mu(z_1, \dots, z_n) := (|z_1|^2, \dots, |z_n|^2)$.

Then if $\text{Conv}(\mathcal{A})^{\text{ess}} \neq \emptyset$, we define

$$\mathcal{D}_{\mathcal{A}} := \mu^{-1}(\text{Conv}(\mathcal{A})^{\text{ess}}) = \mu^{-1}(\text{Conv}(\mathcal{A}))^\circ$$

⁴ Y_i smooth irreducible subvariety of X of codimension i such that Y_i is a Cartier divisor in Y_{i-1} for any $i = 1, \dots, n$.

where with $\text{Conv}(\mathcal{A})^{\text{ess}}$ we have indicated the interior of $\text{Conv}(\mathcal{A})$ respect to the induced topology on $\mathbb{R}_{\geq 0}^n$. Next we define $X_{\mathcal{A}}$ as the manifold we get removing from \mathbb{C}^n all submanifolds given by $\{z_{i_1} = \dots = z_{i_r} = 0\}$ which do not intersect $\mathcal{D}_{\mathcal{A}}$. We equip the manifold with the form $\omega_{\mathcal{A}} := dd^c \phi_{\mathcal{A}}$ where

$$\phi_{\mathcal{A}}(z) := \ln \left(\sum_{\alpha \in \mathcal{A}} |\mathbf{z}^{\alpha}|^2 \right).$$

Here $\mathbf{z} = \{z_1, \dots, z_n\}$ and $\mathbf{z}^{\alpha} = z_1^{\alpha_1} \dots z_n^{\alpha_n}$. Clearly, by construction, $\omega_{\mathcal{A}}$ is an $(S^1)^n$ -invariant Kähler form on $X_{\mathcal{A}}$, so in particular $(X_{\mathcal{A}}, \omega_{\mathcal{A}})$ can be thought as a symplectic manifold. Moreover defining $f(w_1, \dots, w_n) := (e^{w_1/2}, \dots, e^{w_n/2})$, the function $u_{\mathcal{A}}(w) := \phi_{\mathcal{A}} \circ f(w)$ is plurisubharmonic and independent of the imaginary part y_i , and $f^* \omega_{\mathcal{A}} = dd^c u_{\mathcal{A}}$. Thus an easy calculation shows that

$$dd^c u_{\mathcal{A}} = \frac{1}{4\pi} \sum_{j,k=1}^n \frac{\partial^2 u_{\mathcal{A}}}{\partial x_k \partial x_j} dy_k \wedge dx_j$$

which implies

$$d \frac{\partial}{\partial x_k} u_{\mathcal{A}} = dd^c u_{\mathcal{A}} \left((4\pi) \frac{\partial}{\partial y_k}, \cdot \right).$$

Therefore, setting $H_k := \frac{\partial u_{\mathcal{A}}}{\partial x_k} \circ f^{-1}$, since $(f^{-1})_* (2\pi \frac{\partial}{\partial \theta_k}) = 4\pi \frac{\partial}{\partial y_k}$, we get

$$dH_k = \omega_{\mathcal{A}} \left(2\pi \frac{\partial}{\partial \theta_k}, \cdot \right).$$

Hence $\mu_{\mathcal{A}} = (H_1, \dots, H_n) = \nabla u_{\mathcal{A}} \circ f^{-1}$ is a moment map for the $(S^1)^n$ -action on the symplectic manifold $(X_{\mathcal{A}}, \omega_{\mathcal{A}})$. Furthermore it is not hard to check that $\mu_{\mathcal{A}}((\mathbb{C}^*)^n) = \text{Conv}(\mathcal{A})^{\circ}$, that $\mu_{\mathcal{A}}(X_{\mathcal{A}}) = \text{Conv}(\mathcal{A})^{\text{ess}}$ and that for any $U \subset X_{\mathcal{A}}$, setting $f^{-1}(U) = V \times (i\mathbb{R}^n)$,

$$\begin{aligned} \int_U \omega_{\mathcal{A}}^n &= \int_{V \times (i[0, 4\pi]^n)} (dd^c u_{\mathcal{A}})^n = n! \int_V \det(\text{Hess}(u_{\mathcal{A}})) = \\ &= n! \int_{\nabla u_{\mathcal{A}}(V)} dx = n! \text{Vol}(\mu_{\mathcal{A}}(U)). \end{aligned}$$

Finally we quote here an useful result:

Lemma 2.2.9 ([WN15], Lemma 3.1.). *Let U be a relatively compact open subset of $\mathcal{D}_{\mathcal{A}}$. Then there exists a smooth function $g : X_{\mathcal{A}} \rightarrow \mathbb{R}$ with compact support such that $\omega = \omega_{\mathcal{A}} + dd^c g$ is Kähler and $\omega = \omega_{st}$ over U .*

2.3 Multipoint Okounkov bodies

We fix an additive total order $>$ on \mathbb{Z}^n and a family of faithful valuations $\nu^{p_j} : \mathbb{C}(X) \setminus \{0\} \rightarrow (\mathbb{Z}^n, >)$ centered at p_j , where recall that p_1, \dots, p_N are different

points chosen on the n -dimensional projective manifold X and L is a line bundle on X .

Definition 2.3.1. We define $V_{\cdot,j} \subset R(X, L)$ as

$$V_{k,j} = \{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_j}(s) < \nu^{p_i}(s) \text{ for any } i \neq j\}.$$

Remark 2.3.2. They are disjoint graded subsemigroups with respect to the multiplicative action since $\nu^{p_j}(s_1 \otimes s_2) = \nu^{p_j}(s_1) + \nu^{p_j}(s_2)$, but they are not necessarily closed under addition and $\bigcup_{j=1}^N V_{k,j}$ is typically strictly contained in $H^0(X, kL) \setminus \{0\}$ for some $k \geq 1$. Note that $V_{k,j}$ contains sections whose leading term at p_j with respect to ν^{p_j} is strictly smaller than the leading term at p_i with respect to ν^{p_i} for any $i \neq j$.

Clearly the properties of the valuations ν^{p_j} assure that

- i) $\nu^{p_j}(s) = +\infty$ iff $s = 0$ (by extension $\nu^{p_j}(0) := +\infty$);
- ii) for any $s \in V_{\cdot,j}$ and for any $0 \neq a \in \mathbb{C}$, $\nu^{p_j}(as) = \nu^{p_j}(s)$.

Thus we can define

$$\Gamma_j := \{(\nu^{p_j}(s), k) : s \in V_{k,j}, k \geq 0\} \subset \mathbb{Z}^n \times \mathbb{Z}.$$

Lemma 2.3.3. Γ_j is an additive subsemigroup of \mathbb{Z}^{n+1} and $(\Gamma_j, \mathbb{R}^n \times \mathbb{R})$ is a strongly admissible pair.

Proof. The first part is an immediate consequence of the definition, while the last part follows from the inclusion $\Gamma_j \subset \Gamma_{p_j} := \{(\nu^{p_j}(s), k) : s \in H^0(X, kL) \setminus \{0\}, k \geq 0\}$ (see subsection 2.2.4). \square

Definition 2.3.4. We call $\Delta_j(L) := \Delta(\Gamma_j)$ the **multipoint Okounkov body** of L at p_j , i.e. $\Delta_j(L) = \bigcup_{k \geq 1} \frac{\nu^{p_j}(V_{k,j})}{k}$ by Proposition 2.2.3.

The multipoint Okounkov bodies depend on the choice of the faithful valuations $\nu^{p_1}, \dots, \nu^{p_N}$, but we will omit the dependence to simplify the notations.

Remark 2.3.5. If we fix local holomorphic coordinates $\{z_{j,1}, \dots, z_{j,n}\}$ around p_j , we can consider any family of faithful quasi-monomial valuations with center p_1, \dots, p_N (see paragraph §2.2.4), where any ν^{p_j} is given by the same choice of a total additive order on \mathbb{Z}^n and the choice of a family of \mathbb{Z} -linearly independent vectors $\vec{\lambda}_{1,j}, \dots, \vec{\lambda}_{n,j} \in \mathbb{Z}^n$ (they may be different). For instance we can choose those associated to the family of admissible flags $Y_{j,i} := \{z_{j,1} = \dots = z_{j,i} = 0\}$ (with \mathbb{Z}^n equipped of the lexicographical order) or those associated to the family of infinitesimal flags Y . (with in this case \mathbb{Z}^n equipped of the deglex order).

Lemma 2.3.6. The following statements hold:

- i) $\Delta_j(L)$ is a compact convex set contained in \mathbb{R}^n ;

- ii) if $p_j \notin \mathbb{B}_+(L)$ then $\Gamma_j(L)$ generates \mathbb{Z}^{n+1} as a group. In particular $\Delta_j(L)^\circ \neq \emptyset$;
- iii) if $\Gamma_j(L)$ is not empty then it generates \mathbb{Z}^{n+1} as a group. In particular $\Delta_j(L)^\circ \neq \emptyset$ iff $\Delta_j(L) \neq \emptyset$.

Proof. The first point follows by construction (see Definition 2.2.1 and Remark 2.2.2).

For the second point, proceeding similarly to Lemma 2.2 in [LM09], let D be a big divisor such that $L = \mathcal{O}_X(D)$ and let A, B be two fixed ample divisors such that $D = A - B$. Since D is big there exists $\mathbb{N} \ni k \gg 1$ such that $kD - B$ is linearly equivalent to an effective divisor F .

Moreover, since by hypothesis $p_j \notin \mathbb{B}_+(L)$, by taking $k \gg 1$ big enough, we may assume that $p_j \notin \text{Supp}(F)$ (see Corollary 1.6. in [ELMNP06]), thus F is described by a global section f that is an unity in \mathcal{O}_{X, p_j} . Then, possibly adding a very ample divisor to A and B we may suppose that there exists sections $s_0, \dots, s_n \in V_{1,j}(B)$ such that $\nu^{p_j}(s_0) = \vec{0}$ and $\nu^{p_j}(s_l) = \vec{\lambda}_l$ for any $l = 1, \dots, n$ where $\vec{\lambda}_1, \dots, \vec{\lambda}_n$ are linearly independent vectors in \mathbb{Z}^n which generate all \mathbb{Z}^n as a group (remember that the valuations ν^{p_j} are faithful). Thus, since $s_i \otimes f \in V_{1,j}(kL)$ for any $i = 0, \dots, n$ and $\nu^{p_j}(f) = \vec{0}$, we get

$$(\vec{0}, k), (\vec{\lambda}_1, k), \dots, (\vec{\lambda}_n, k) \in \Gamma_j(L).$$

And, since $(k+1)D - F$ is linearly equivalent to A we may also assume that $(\vec{0}, k+1) \in \Gamma_j(L)$, which concludes the proof of (ii).

Finally to prove (iii), let $s \in V_{k,j}(L)$ such that $(\nu^{p_j}(s), k) \in \Gamma_j(L)$ and set $\vec{w} := \nu^{p_j}(s)$. Then by Lemma 2.2 in [LM09] there exists $m \in \mathbb{N}$ big enough and a vector $\vec{v} \in \mathbb{Z}^n$ such that

$$(\vec{v}, m), (\vec{v} + \vec{\lambda}_1, m), \dots, (\vec{v} + \vec{\lambda}_n, k), (\vec{v}, m+1) \in \Gamma(L) \quad (2.1)$$

where with $\Gamma(L)$ we denote the semigroup associated to ν^{p_j} for the one-point Okounkov body (see subsection 2.2.4) and where $\vec{\lambda}_1, \dots, \vec{\lambda}_n$ are linearly independent vectors in \mathbb{Z}^n as in (ii). The points in (2.1) correspond to sections $t_0, \dots, t_n \in H^0(X, mL) \setminus \{0\}$, $t_{n+1} \in H^0(X, (m+1)L)$. Next by definition of $V_{j,j}(L)$ there exists $N \gg 1$ big enough such that $s^N \otimes t_j \in V_{Nk+m,j}(L)$ for any $j = 0, \dots, n$ and $s^N \otimes t_{n+1} \in V_{Nk+m+1}(L)$. Therefore

$$(N\vec{w} + \vec{v}, m), (N\vec{w} + \vec{v} + \vec{\lambda}_1, m), \dots, (N\vec{w} + \vec{v} + \vec{\lambda}_n, k), (N\vec{w} + \vec{v}, m+1) \in \Gamma_j(L),$$

which concludes the proof. \square

Remark 2.3.7. Let X be a curve, L a line bundle of degree $\deg L = c$, and p_1, \dots, p_N are different points on X . Then by the proof of Lemma 2.3.6, $\Delta_j(L)$ are intervals in \mathbb{R} containing the origin. Moreover if the points are very general and the faithful valuations ν^{p_j} are associated to admissible or to infinitesimal flags, then $\Delta_j(L) = [0, c/N]$ for any $j = 1, \dots, N$ as a consequence of Theorem A.

Remark 2.3.8. In higher dimension, however, the situation is more complicated. Indeed it may happen that $\Delta_j(L) = \emptyset$ for some j as the following simple example shows.

Consider on $X = \text{Bl}_q \mathbb{P}^2$ two points $p_1 \notin \text{Supp}(E)$ and $p_2 \in \text{Supp}(E)$ (E exceptional divisor), and consider the big line bundle $L := H + aE$ for $a > 1$. Clearly, if we consider the family of admissible flags given by any fixed holomorphic coordinates centered at p_1 and holomorphic coordinates $\{z_{1,2}, z_{2,2}\}$ centered at p_2 where locally $E = \{z_{1,2} = 0\}$, then $\Delta_2(L) = \emptyset$. Indeed by the theory of one-point Okounkov bodies for surfaces (see section 6.2 in [LM09]) $\Delta_1(L) \subset \Delta^{p_1}(L) = \Sigma$ (where Σ is the standard 2-simplex and $\Delta^{p_1}(L)$ the one-point Okounkov body) while $\Delta_2(L) \subset \Delta^{p_2}(L) = (a, 0) + \Sigma^{-1}$ ($\Sigma^{-1} = \text{Conv}(\vec{0}, \vec{e}_1, \vec{e}_1 + \vec{e}_2)$ inverted simplex), and the conclusion follows by construction. Actually, from Theorem A we get $\Delta_1(L) = \Sigma$. We refer to subsection 2.6.2 for a detailed analysis on the multipoint Okounkov bodies on surfaces, and to subsection 2.6.1 for the toric case.

2.3.1 Proof of Theorem A

The goal of this section is to prove Theorem A.

Theorem A. *Let L be a big line bundle. Then*

$$n! \sum_{j=1}^N \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \text{Vol}_X(L)$$

We first introduce $W_{\cdot,j} \subset R(X, L)$ defined as

$$W_{k,j} := \{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_j}(s) \leq \nu^{p_i}(s) \text{ if } 1 \leq i \leq j \text{ and } \nu^{p_j}(s) < \nu^{p_i}(s) \text{ if } j < i \leq N\}$$

and we set $\Gamma_{W,j} := \{(\nu^{p_j}(s), k) : s \in W_{k,j}, k \geq 0\}$. It is clear $W_{\cdot,j}$ are graded subsemigroups of $R(X, L)$ and that Lemma 2.3.3 holds for $\Gamma_{W,j}$. Moreover they are closely related to $V_{\cdot,j}$ and $\bigsqcup_{j=1}^N W_{k,j} = H^0(X, kL) \setminus \{0\}$ for any $k \geq 0$, but they depend on the order chosen on the points.

Lemma 2.3.9. *For every $k \geq 1$ we have that*

$$\sum_{j=1}^N \#\Gamma_{W,j}^k = h^0(X, kL),$$

where we recall that $\Gamma_{W,j}^k := \{\alpha \in \mathbb{R}^n : (k\alpha, k) \in \Gamma_{W,j}\}$.

Proof. We define a new valuation $\nu : \mathbb{C}(X) \setminus \{0\} \rightarrow \mathbb{Z}^n \times \cdots \times \mathbb{Z}^n \simeq \mathbb{Z}^{Nn}$ given by $\nu(f) := (\nu^{p_1}(f), \dots, \nu^{p_N}(f))$, where we put on \mathbb{Z}^{Nn} the lexicographical order on the product of N total ordered abelian groups \mathbb{Z}^n , i.e.

$(\lambda_1, \dots, \lambda_N) < (\mu_1, \dots, \mu_N)$ if there exists $j \in \{1, \dots, N\}$ s.t. $\lambda_i = \mu_i \forall i < j$ and $\lambda_j < \mu_j$.

Fix $k \in \mathbb{N}$. For every $j = 1, \dots, N$, let $\Gamma_{W,j}^k = \{\alpha_{j,1}, \dots, \alpha_{j,r_j}\}$ and $s_{j,1}, \dots, s_{j,r_j} \in W_{k,j}$ be a set of sections such that $\nu^{p_j}(s_{j,l}) = \alpha_{j,l}$ for any $l = 1, \dots, r_j$.

We claim that $\{s_{1,1}, \dots, s_{N,r_N}\}$ is a base of $H^0(X, kL)$.

Let $\sum_{i=1}^r \mu_i s_i = 0$ be a linear relation in which $\mu_i \neq 0$, $s_i \in \{s_{1,1}, \dots, s_{N,r_N}\}$ for all $i = 1, \dots, r$ and $s_i \neq s_j$ if $i \neq j$. By construction we know that $\nu(s_1), \dots, \nu(s_r)$ are different points in \mathbb{Z}^{Nn} . Thus without loss of generality we can assume that $\nu(s_1) < \dots < \nu(s_r)$, but the relation

$$s_1 = -\frac{1}{\mu_1} \sum_{i=2}^r \mu_i s_i$$

implies that $\nu(s_1) \geq \min\{\nu(s_j) : j = 2, \dots, r\}$ which is the contradiction. Hence $\{s_{1,1}, \dots, s_{N,r_N}\}$ is a system of linearly independent vectors, thus to conclude the proof it is enough to show that it generates all $H^0(X, kL)$.

Let $t_0 \in H^0(X, kL) \setminus \{0\}$ be a section and set $\lambda_0 := (\lambda_{0,1}, \dots, \lambda_{0,N}) := \nu(t_0)$. By definition of $W_{\cdot,j}$ there exists a unique $j_0 \in 1, \dots, N$ such that $t_0 \in W_{k,j_0}$, which means that $\lambda_{0,i} \geq \lambda_{0,j_0}$ if $1 \leq i \leq j_0$, and that $\lambda_{0,i} > \lambda_{0,j_0}$ if $j_0 < i \leq N$. Therefore by construction there exists $l \in \{1, \dots, r_{j_0}\}$ such that $\lambda_{0,j_0} = \nu^{p_{j_0}}(s_{j_0,l})$, so we set $s_0 := s_{j_0,l}$. But

$$\dim \left(\frac{\{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_{j_0}}(s) \geq \lambda_{0,j_0}\} \cup \{0\}}{\{s \in H^0(X, kL) \setminus \{0\} : \nu^{p_{j_0}}(s) > \lambda_{0,j_0}\} \cup \{0\}} \right) \leq 1,$$

since $\nu^{p_{j_0}}$ has one-dimensional leaves, so there exists a coefficient $a_0 \in \mathbb{C}$ such that $\nu^{p_{j_0}}(t_0 - a_0 s_0) > \lambda_{0,j_0}$. Thus if $t_0 = a_0 s_0$ we can conclude the proof, otherwise we set $t_1 := t_0 - a_0 s_0$ and we want to iterate the process setting $\lambda_1 := (\lambda_{1,1}, \dots, \lambda_{1,N}) := \nu(t_1)$ and observing that $\min_j \lambda_{1,j} \geq \min_j \lambda_{0,j} = \lambda_{0,j_0}$ and that the inequality is strict if $t_1 \in W_{k,j_0}$.

Thus we get $t_0, t_1, \dots, t_l \in H^0(X, kL) \setminus \{0\}$ such that $t_l := t_{l-1} - a_{l-1} s_{l-1} \in W_{k,j_l}$ for an unique $j_l \in \{1, \dots, N\}$ where $s_{l-1} \in \{s_{j_{l-1},1}, \dots, s_{j_{l-1},r_{j_{l-1}}}\}$ satisfies $\nu^{p_{j_{l-1}}}(t_{l-1}) = \nu^{p_{j_{l-1}}}(s_{l-1})$, and $\min_j \lambda_{l,j} \geq \min_j \lambda_{l-1,j}$ for $\nu(t_l) =: \lambda_l$. Therefore we get a sequence of valutive points λ_l such that $\min_j \lambda_{l,j} \geq \min_j \lambda_{l-1,j} \geq \dots \geq \min_j \lambda_{0,j}$ where by construction there is at least one strict inequality if $l > N$. Hence we deduce that the iterative process have a conclusion since that the set of all valutive points of ν is finite as easy consequence of the finiteness of the cardinality of $\Gamma_{W,j}^k$ for each $j = 1, \dots, N$. \square

Proposition 2.3.10. *Let L be a big line bundle. Then $\Delta_j(mL) = m\Delta_j(L)$ and $\Delta_j^W(mL) = m\Delta_j^W(L)$ for any $m \in \mathbb{N}$ and for any $j = 1, \dots, N$ where $\Delta_j^W(L)$ is the Okounkov body associated to the additive semigroup $\Gamma_{W,j}(L)$.*

Proof. The proof proceeds similarly as the proof of Proposition 4.1.ii in [LM09], exploiting again the property of the total order on \mathbb{Z}^n .

We may assume $\Delta_j(L) \neq \emptyset$, otherwise it would be trivial, and we can choose $r, t \in \mathbb{N}$

such that $V_{r,j}, V_{tm-r,j} \neq \emptyset$, i.e. there exist sections $e \in V_{r,j}$ and $f \in V_{tm-r,j}$. Thus we get the inclusions

$$k\Gamma_j(mL)^k + \nu^{p_j}(e) + \nu^{p_j}(f) \subset (km+r)\Gamma_j(L)^{km+r} + \nu^{p_j}(f) \subset (k+t)\Gamma_j(mL)^{k+t}.$$

Letting $k \rightarrow \infty$, we find $\Delta_j(mL) \subset m\Delta_j(L) \subset \Delta_j(mL)$.

The same proof works for $\Delta_j^W(L)$. \square

Proposition 2.3.10 extends naturally the definition of the multipoint Okounkov bodies to \mathbb{Q} -line bundles.

We are now ready to prove Theorem A.

Proof of Theorem A. By Lemma 2.3.9 and Theorem 2.2.5 we get

$$\begin{aligned} n! \sum_{j=1}^N \frac{\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L))}{\text{ind}_{1,j}(L)\text{ind}_{2,j}(L)^n} &= \lim_{k \in \mathbb{N}(L), k \rightarrow \infty} \frac{n! \sum_{j=1}^N \#\Gamma_{W,j}^k}{k^n} = \\ &= \lim_{k \in \mathbb{N}(L), k \rightarrow \infty} \frac{h^0(X, kL)}{k^n/n!} = \text{Vol}_X(L). \end{aligned} \quad (2.2)$$

where we keep the same notations of Theorem 2.2.5 for the indexes $\text{ind}_{1,j}(L), \text{ind}_{2,j}(L)$ adding the j subscript to keep track of the points and the dependence on the line bundle because we want to perturb it.

Then we claim that

$$\Delta_j^W(L)^\circ = \Delta_j(L)^\circ, \quad (2.3)$$

for any $j = 1, \dots, N$. Note that since $\Gamma_{V,j} \subset \Gamma_{W,j}$ we only need to prove that $\Delta_j^W(L)^\circ \subset \Delta_j(L)^\circ$.

Let A be a fixed ample line bundle A such that there exist $s_1, \dots, s_N \in H^0(X, A)$ with $s_i \in V_{1,i}(A)$ and $\nu^{p_i}(s_i) = 0$. Thus we get $\Delta_j^W(mL - A) \subset \Delta_j(mL)$ for each $m \in \mathbb{N}$ and for any $j = 1, \dots, N$ since $s \otimes s_j^k \in V_{k,j}(mL)$ for any $s \in W_{k,j}(mL - A)$. Hence

$$\Delta_j^W(L - \frac{1}{m}A) \subset \Delta_j(L) \quad (2.4)$$

by Proposition 2.3.10.

Moreover since $m \rightarrow \text{ind}_{1,j}(L - \frac{1}{m}A)$ and $m \rightarrow \text{ind}_{2,j}(L - \frac{1}{m}A)$ are decreasing functions, (2.2) implies

$$\begin{aligned} \limsup_{m \rightarrow \infty} n! \sum_{j=1}^N \frac{\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L - \frac{1}{m}A))}{\text{ind}_{1,j}(L - \frac{1}{m}A)\text{ind}_{2,j}(L - \frac{1}{m}A)^n} &\geq \\ &\geq \limsup_{m \rightarrow \infty} n! \sum_{j=1}^N \frac{\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L - \frac{1}{m}A))}{\text{ind}_{1,j}(L - \frac{1}{m}A)\text{ind}_{2,j}(L - \frac{1}{m}A)^n} = \limsup_{m \rightarrow \infty} \text{Vol}_X(L - \frac{1}{m}A) = \\ &= \text{Vol}_X(L) = n! \sum_{j=1}^N \frac{\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L))}{\text{ind}_{1,j}(L)\text{ind}_{2,j}(L)^n} \end{aligned} \quad (2.5)$$

where we used the continuity of the volume function on line bundles. Thus since $\Delta_j^W(L - \frac{1}{m}A) \subset \Delta_j^W(L - \frac{1}{l}A)$ if $l > m$ for any $j = 1, \dots, N$, from (2.5) we deduce that $m \rightarrow \text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L - \frac{1}{m}A))$ is a continuous increasing function converging to $\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L))$ for any $j = 1, \dots, N$. Hence (2.3) follows from 2.4.

Finally combining (2.3) and Lemma 2.3.6 .(iii) we find out that $\text{ind}_{1,j}(L) = \text{ind}_{2,j}(L) = 1$ if $\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L)) \neq 0$. Thus using again (2.2) we obtain

$$n! \sum_{j=1}^N \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = n! \sum_{j=1}^N \frac{\text{Vol}_{\mathbb{R}^n}(\Delta_j^W(L))}{\text{ind}_{1,j}(L)\text{ind}_{2,j}(L)^n} = \text{Vol}_X(L),$$

which concludes the proof. \square

2.3.2 Variation of multipoint Okounkov bodies

Similarly to the section §4 in [LM09], we prove that for fixed faithful valuations ν^{p_j} centered at N different points the construction of the multipoint Okounkov Bodies is cohomological, i.e. $\Delta_j(L)$ depends only from the first Chern class $c_1(L) \in N^1(X)$ of the big line bundle L , where we have indicated with $N^1(X)$ the Neron-Severi group. Recall that $\rho(X) := \dim N^1(X)_{\mathbb{R}} < \infty$ where $N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 2.3.11. *Let L be a big line bundle. Then $\Delta_j(L)$ depends uniquely on the numerical class of the big line bundle L .*

Proof. Assume $\Delta_j(L)^\circ \neq \emptyset$, which by Lemma 2.3.6 is equivalent to $\Delta_j(L) \neq \emptyset$, and let L' such that $L' = L + P$ for P numerically trivial. Fix also an ample line bundle A . Then for any $m \in \mathbb{N}$ there exists $k_m \in \mathbb{N}$ and $s_m \in H^0(X, k_m m(P + \frac{1}{m}A))$ such that $s_m(p_i) \neq 0$ for any $i = 1, \dots, N$ since $P + \frac{1}{m}A$ is a ample \mathbb{Q} -line bundle. Hence we get $\Delta_j(L) \subset \Delta_j(L' + \frac{1}{m}A)$ by homogeneity (Proposition 2.3.10) because $s \otimes s_m^k \in V_{k,j}(k_m mL' + k_m A)$ for any section $s \in V_{k,j}(k_m mL)$. Therefore similarly to the proof of Theorem A, letting $m \rightarrow \infty$, we obtain $\Delta_j(L) \subset \Delta_j(L')$. Replacing L by $L + P$ and P by $-P$, Lemma 2.3.6 concludes the proof. \square

Setting $r := \rho(X)$ for simplicity, fix L_1, \dots, L_r line bundle such that $\{c_1(L_1), \dots, c_1(L_r)\}$ is a \mathbb{Z} -basis of $N^1(X)$; this lead to natural identifications $N^1(X) \simeq \mathbb{Z}^r$, $N^1(X)_{\mathbb{R}} \simeq \mathbb{R}^r$. Moreover by Lemma 4.6. in [LM09] we may choose L_1, \dots, L_r such that the pseudoeffective cone is contained in the positive orthant of \mathbb{R}^r .

Definition 2.3.12. *Letting*

$$\Gamma_j(X) := \Gamma_j(X; L_1, \dots, L_r) := \{(\nu^{p_j}(s), \vec{m}) : s \in V_{\vec{m},j}(L_1, \dots, L_r) \setminus \{0\}, \vec{m} \in \mathbb{N}^r\}$$

be the global multipoint semigroup of X at p_j with $p_1, \dots, \hat{p}_j, \dots, p_N$ fixed (it is an additive subsemigroup of \mathbb{Z}^{n+r}) where $V_{\vec{m},j}(L_1, \dots, L_r) := \{s \in H^0(X, \vec{m} \cdot (L_1, \dots, L_r)) \setminus \{0\} : \nu^{p_j}(s) < \nu^{p_i}(s) \text{ for any } i \neq j\}$, we define

$$\Delta_j(X) := C(\Gamma_j(X))$$

as the closed convex cone in \mathbb{R}^{n+r} generated by $\Gamma_j(X)$, and call it the **global multipoint Okounkov body** at p_j .

Lemma 2.3.13. *The semigroup $\Gamma_j(X)$ generates a subgroup of \mathbb{Z}^{n+r} of maximal rank.*

Proof. Since the cone $\text{Amp}(X)$ is open non-empty set in $N^1(X)_{\mathbb{R}}$ (we have indicated with $\text{Amp}(X)$ the *ample* cone, see [Laz04]), we can fix F_1, \dots, F_r ample line bundles that generate $N^1(X)$ as free \mathbb{Z} -module. Moreover, by the assumptions done for L_1, \dots, L_r we know that for every $i = 1, \dots, r$ there exists \vec{a}_i such that $F_i = \vec{a}_i \cdot (L_1, \dots, L_r)$. Thus, for any $i = 1, \dots, r$, the graded semigroup $\Gamma_j(F_i)$ sits in $\Gamma_j(X)$ in a natural way and it generates a subgroup of $\mathbb{Z}^n \times \mathbb{Z} \cdot \vec{a}_i$ of maximal rank by point *ii*) in Lemma 2.3.6 since $\mathbb{B}_+(F_i) = \emptyset$. We conclude observing that $\vec{a}_1, \dots, \vec{a}_r$ span \mathbb{Z}^r . \square

Next we need a further fact about additive semigroups and their cones. Let $\Gamma \subset \mathbb{Z}^n \times \mathbb{N}^r$ be an additive semigroup, and let $C(\Gamma) \subset \mathbb{R}^n \times \mathbb{R}^r$ be the closed convex cone generated by Γ . We call the *support* of Γ respect to the last r coordinates, $\text{Supp}(\Gamma)$, the closed convex cone $C(\pi(\Gamma)) \subset \mathbb{R}^r$ where $\pi : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^r$ is the usual projection. Then, given $\vec{a} \in \mathbb{N}^r$, we set $\Gamma_{\mathbb{N}\vec{a}} := \Gamma \cap (\mathbb{Z}^n \times \mathbb{N}\vec{a})$ and denote by $C(\Gamma_{\mathbb{N}\vec{a}}) \subset \mathbb{R}^n \times \mathbb{R}\vec{a}$ the closed convex cone generated by $\Gamma_{\mathbb{N}\vec{a}}$ when we consider it as an additive semigroup of $\mathbb{Z}^n \times \mathbb{Z}\vec{a} \simeq \mathbb{Z}^{n+1}$.

Proposition 2.3.14 ([LM09], Proposition 4.9.). *Assume that Γ generates a subgroup of finite index in $\mathbb{Z}^n \times \mathbb{Z}^r$, and let $\vec{a} \in \mathbb{N}^r$ be a vector lying in the interior of $\text{Supp}(\Gamma)$. Then*

$$C(\Gamma_{\mathbb{N}\vec{a}}) = C(\Gamma) \cap (\mathbb{R}^n \times \mathbb{R}\vec{a})$$

Now we are ready to prove the main theorem of this section:

Theorem 2.3.15. *The global multipoint Okounkov body $\Delta_j(X)$ is characterized by the property that in the following diagram*

$$\begin{array}{ccc} \Delta_j(X) & \subset & \mathbb{R}^n \times \mathbb{R}^r \simeq \mathbb{R}^n \times N^1(X)_{\mathbb{R}} \\ & \searrow & \swarrow \text{pr}_2 \\ & \mathbb{R}^r \simeq N^1(X)_{\mathbb{R}} & \end{array}$$

the fiber of $\Delta_j(X)$ over any cohomology class $c_1(L)$ of a big \mathbb{Q} -line bundle L such that $c_1(L) \in \text{Supp}(\Gamma_j(X))^{\circ}$ is the multipoint Okounkov body associated to L at p_j , i.e. $\Delta_j(X) \cap \text{pr}_2^{-1}(c_1(L)) = \Delta_j(L)$. Moreover $\text{Supp}(\Gamma_j(X))^{\circ} \cap N^1(X)_{\mathbb{Q}} = \{c_1(L) : \Delta_j(L) \neq \emptyset, L \text{ } \mathbb{Q}\text{-line bundle}\}$.

Remark 2.3.16. It seems a bit unclear what $\text{Supp}(\Gamma_j(X))^\circ$ is. By second point in Lemma 2.3.6, it contains the open convex set $B_+(p_j)^C$ where $B_+(p_j) := \{\alpha \in N^1(X)_\mathbb{R} : p \in B_+(\alpha)\}$ is closed respect to the metric topology on $N^1(X)_\mathbb{R}$ by Proposition 1.2. in [KL15a] and its complement is convex as easy consequence of Proposition 1.5. in [ELMNP06]. But in general $\text{Supp}(\Gamma_j(X))^\circ$ may be bigger: for instance if $N = 1$ $\text{Supp}(\Gamma_j(X))^\circ$ coincides with the big cone, and we can easily construct an example with $p_1, p_2 \in \mathbb{B}_-(L)$ and $\Delta_j(L)^\circ \neq \emptyset$ for $j = 1, 2$. For instance consider $X = \text{Bl}_q \mathbb{P}^2$, $L := H + E$ where E is the exceptional divisor and $p_1, p_2 \in \text{Supp}(E)$ different points. Then given two valuations associated to admissible flags $Y_{\cdot,j}$ for $j = 1, 2$ centered at p_1, p_2 such that $Y_{1,j} = E$ for any $j = 1, 2$, it is easy to check that $\Delta_j(L)^\circ \neq \emptyset$ for $j = 1, 2$ since by Lemma 2.3.6 this is equivalent to $\Delta_j(L) \neq \emptyset$.

Proof. For any vector $\vec{a} \in \mathbb{N}^r$ such that $L := \vec{a} \cdot (L_1, \dots, L_r)$ is a big line bundle in $\text{Supp}(\Gamma_j(X))^\circ$, we get $\Gamma_j(X)_{\mathbb{N}\vec{a}} = \Gamma_j(L)$, and so the base of the cone $C(\Gamma_j(X)_{\mathbb{N}\vec{a}}) = C(\Gamma_j(L)) \subset \mathbb{R}^n \times \mathbb{R}\vec{a}$ is the multipoint Okounkov body $\Delta_j(L)$, i.e.

$$\Delta_j(L) = \pi \left(C(\Gamma_j(X)_{\mathbb{N}\vec{a}}) \cap (\mathbb{R}^n \times \{1\}) \right).$$

Then Proposition 2.3.14 implies that the right side of the last equality coincides with the fiber $\Delta_j(X)$ over $c_1(L)$. Both sides rescale linearly, so the equality extends to \mathbb{Q} -line bundles.

Next by Lemma 2.3.6 it follows that $c_1(L) \in \text{Supp}(\Gamma_j(X))$ for any \mathbb{Q} -line bundle L such that $\Delta_j(L) \neq \emptyset$. On the other hand, by the first part of the proof we get

$$\text{Supp}(\Gamma_j(X))^\circ \cap N^1(X)_\mathbb{Q} \subset \{c_1(L) : \Delta_j(L) \neq \emptyset, L \text{ } \mathbb{Q}\text{-line bundle}\}. \quad (2.6)$$

Thus it remains to prove that the right hand in (2.6) is open in $N^1(X)_\mathbb{Q}$, which is equivalent to show that $\Delta_j(L - \frac{1}{k}A) \neq \emptyset$ for $k \gg 1$ big enough if A is a fixed very ample line bundle since the ample cone is open and not empty in $N^1(X)_\mathbb{R}$ and $N^1(X)_\mathbb{R}$ is a finite dimensional vector space. Considering the multiplication by a section $s \in H^0(X, A)$ such that $s(p_i) \neq 0$ for any $i = 1, \dots, N$, we obtain $\Delta_i(L - \frac{1}{k}A) \subset \Delta_i(L)$ for any $i = 1, \dots, N$. Therefore by Theorem A and Lemma 2.3.6 we necessarily have $\Delta_j(L - \frac{1}{k}A)^\circ \neq \emptyset$ for $k \gg 1$ big enough since $\text{Vol}_X(L - \frac{1}{k}A) \nearrow \text{Vol}_X(L)$ and $\Delta_j(L)^\circ \neq \emptyset$. This concludes the proof. \square

As a consequence of Theorem 2.3.15, we can define multipoint Okounkov bodies for \mathbb{R} -line bundles. Indeed it is natural to set $\Delta_j(L)$ as the limit (in the Hausdorff sense) of $\Delta_j(L_k)$ if $c_1(L) \in \text{Supp}(\Gamma_j(X))^\circ = \overline{\{c_1(L) : \Delta_j(L) \neq \emptyset, L \text{ } \mathbb{Q}\text{-line bundle}\}^\circ}$ where $\{L_k\}_{k \in \mathbb{N}}$ is any sequence of \mathbb{Q} -line bundles such that $c_1(L_k) \rightarrow c_1(L)$, and $\Delta_j(L) = \emptyset$ otherwise. This extension is well-defined and coherent with Lemma 2.3.6, since we obtain $\Delta_j(L)^\circ \neq \emptyset$ iff $\Delta_j(L) \neq \emptyset$.

Corollary 2.3.17. *The function $\text{Vol}_{\mathbb{R}^n} : \text{Supp}(\Gamma_j(X))^\circ \rightarrow \mathbb{R}_{>0}$, $c_1(L) \rightarrow \text{Vol}_{\mathbb{R}^n}(\Delta_j(L))$ is well-defined, continuous, homogeneous of degree n and log-concave, i.e.*

$$\text{Vol}_{\mathbb{R}^n}(\Delta_j(L + L'))^{1/n} \geq \text{Vol}_{\mathbb{R}^n}(\Delta_j(L))^{1/n} + \text{Vol}_{\mathbb{R}^n}(\Delta_j(L'))^{1/n}$$

Proof. The fact that it is well-defined and its homogeneity follow directly from Propositions 2.3.10 and 2.3.11, while the other statements are standard in convex geometry, using the Brunn-Minkowski Theorem and Theorem 2.3.15. \square

Finally we note that the Theorem 2.3.15 allows us to describe the multipoint Okounkov bodies similarly to Proposition 4.1. in [Bou14]:

Corollary 2.3.18. *If $L = \mathcal{O}_X(D)$ is a big line bundle such that $c_1(L) \in \text{Supp}(\Gamma_j(X))^\circ$, then*

$$\Delta_j(L) = \overline{\nu^{p_j} \{D' \in \text{Div}_{\geq 0}(X)_{\mathbb{R}} : D' \equiv_{\text{num}} D \text{ and } \nu^{p_j}(D') < \nu^{p_i}(D') \forall i \neq j\}}$$

where we have indicated with \equiv_{num} the numerical equivalence. In particular every rational point in $\Delta_j(L)^\circ$ is valutive and if it contains a small n -simplex with valutive vertices then any rational point in the n -simplex is valutive.

Proof. The first part follows directly from Theorem 2.3.15 since $D' \equiv_{\text{num}} D$ iff $c_1(L) = c_1(\mathcal{O}_X(D'))$ by definition (considering the \mathbb{R} -line bundle $\mathcal{O}_X(D')$). The statement about $\Delta_j(L)^\circ$ is a consequence of Proposition 3 in [Kho93].§3 and of Lemma 2.3.6.(iii) while the valutive property for the n -simplex is a consequence of the multiplicative rule of ν^{p_j} . \square

2.3.3 Geometry of multipoint Okounkov bodies

To investigate the geometry of the multipoint Okounkov bodies we need to introduce the following important invariant:

Definition 2.3.19. *Let L be a line bundle, $V \subset X$ a subvariety of dimension d and $H^0(X|V, kL) := \text{Im}\left(H^0(X, kL) \rightarrow H^0(V, kL|_V)\right)$. Then the quantity*

$$\text{Vol}_{X|V}(L) := \limsup_{k \rightarrow \infty} \frac{\dim H^0(X|V, kL)}{k^d/d!}$$

is called the **restricted volume of L along V** .

We refer to [ELMNP09] and reference therein for the theory about this new object. In the repeatedly quoted paper [LM09], given a valuation $\nu^p(s) = (\nu^p(s)_1, \dots, \nu^p(s)_n)$ associated to an admissible flag $Y = (Y_1, \dots, Y_n)$ such that $Y_1 = D$ and a line bundle L such that $D \not\subset \mathbb{B}_+(L)$, the authors also defined the one-point Okounkov body of the graded linear system $H^0(X|D, kL) \subset H^0(D, kL|_D)$ by

$$\Delta_{X|D}(L) := \Delta(\Gamma_{X|D})$$

with $\Gamma_{X|D} := \{(\nu^p(s)_2, \dots, \nu^p(s)_n, k) \in \mathbb{N}^{n-1} \times \mathbb{N} : s \in H^0(X|D, kL) \setminus \{0\}, k \geq 1\}$ and they proved the following

Theorem 2.3.20 ([LM09], Theorem 4.24, Corollary 4.25). *Let $D \notin \mathbb{B}_+(L)$ be a prime divisor with L big \mathbb{R} -line bundle and let Y_\bullet be an admissible flag such that $Y_1 =: D$. Let $C_{\max} := \sup\{\lambda \geq 0 : L - \lambda D \text{ is big}\}$. Then for any $0 \leq t < C_{\max}$*

$$\begin{aligned}\Delta(L)_{x_1 \geq t} &= \Delta(L - tD) + t\vec{e}_1 \\ \Delta(L)_{x_1 = t} &= \Delta_{X|D}(L - tD)\end{aligned}$$

Moreover

- i) $\text{Vol}_{\mathbb{R}^{n-1}}(\Delta(L)_{x_1=t}) = \frac{1}{(n-1)!} \text{Vol}_{X|D}(L - tD)$;
- ii) $\text{Vol}_X(L) - \text{Vol}_X(L - tD) = n \int_0^t \text{Vol}_{X|D}(L - \lambda D) d\lambda$;

In this section we suppose to have fixed a family of valuations ν^{p_j} associated to a family of admissible flags $Y_\bullet = (Y_{\bullet,1}, \dots, Y_{\bullet,N})$ on a projective manifold X , centered respectively in p_1, \dots, p_N (see paragraph 2.2.4 and Remark 2.3.5). Given a big line bundle L , and prime divisors D_1, \dots, D_N where $D_j = Y_{1,j}$ for any $j = 1, \dots, N$, we set

$$\mu(L; \mathbb{D}) := \sup\{t \geq 0 : L - t\mathbb{D} \text{ is big}\}$$

where $\mathbb{D} := \sum_{i=1}^N D_i$, and

$$\mu(L; D_j) := \sup\{t \geq 0 : \Delta_j(L - t\mathbb{D})^\circ \neq \emptyset\}.$$

Theorem 2.3.21. *Let L a big \mathbb{R} -line bundle, ν^{p_j} a family of valuations associated to a family of admissible flags Y_\bullet centered at p_1, \dots, p_N . Then, letting (x_1, \dots, x_n) be fixed coordinates on \mathbb{R}^n , for any $j \in \{1, \dots, N\}$ such that $\Delta_j(L)^\circ \neq \emptyset$ the followings hold:*

- i) $\Delta_j(L)_{x_1 \geq t} = \Delta_j(L - t\mathbb{D}) + t\vec{e}_1$ for any $0 \leq t < \mu(L; D_j)$, for any $j = 1, \dots, N$;
- ii) $\Delta_j(L)_{x_1 = t} = \Delta_{X|D_j}(L - t\mathbb{D})$ for any $0 \leq t < \mu(L; \mathbb{D})$, $t \neq \mu(L; D_j)$ and for any $j = 1, \dots, N$;
- iii) $\text{Vol}_{\mathbb{R}^{n-1}}(\Delta_j(L)_{x_1=t}) = \frac{1}{(n-1)!} \text{Vol}_{X|D_j}(L - t\mathbb{D})$ for any $0 \leq t < \mu(L; \mathbb{D})$, for any $j = 1, \dots, N$, and in particular $\mu(L; D_j) = \sup\{t \geq 0 : D_j \notin \mathbb{B}_+(L - t\mathbb{D})\}$.

Moreover

$$\text{iv) } \text{Vol}_X(L) - \text{Vol}_X(L - t\mathbb{D}) = n \int_0^t \sum_{i=1}^N \text{Vol}_{X|D_i}(L - \lambda \mathbb{D}) d\lambda \text{ for any } 0 \leq t < \mu(L; \mathbb{D}).$$

Proof. The first point follows as in Proposition 4.1. in [LM09], noting that if L is a big line bundle and $0 \leq t < \mu(L; D_j)$ integer then $\{s \in V_{k,j}(L) : \nu^{p_j}(s)_1 \geq kt\} \simeq V_{k,j}(L - t\mathbb{D})$ for any $k \geq 1$. Therefore $\Gamma_j(L)_{x_1 \geq t} = \varphi_t(\Gamma_j(L - t\mathbb{D}))$ where $\varphi_t : \mathbb{N}^n \times \mathbb{N} \rightarrow \mathbb{N}^n \times \mathbb{N}$ is given by $\varphi_t(\vec{x}, k) := (\vec{x} + tk\vec{e}_1, k)$. Passing to the cones we get $C(\Gamma_j(L)_{x_1 \geq t}) = \varphi_{t,\mathbb{R}}(C(\Gamma_j(L - t\mathbb{D})))$ where $\varphi_{t,\mathbb{R}}$ is the linear map between vector spaces associated to φ_t . Hence, taking the base of the cones, the equality

$\Delta_j(L)_{x_1 \geq t} = \Delta_j(L - t\mathbb{D}) + t\vec{e}_1$ follows. Finally, since both sides in (i) rescale linearly by Proposition 2.3.10, the equality holds for any L \mathbb{Q} -line bundle and $t \in \mathbb{Q}$. Both sides in (i) are clearly continuous in t if $0 \leq t < \mu(L; D_j)$ so it remains to extend it to \mathbb{R} -line bundles. We fix a decreasing sequence of \mathbb{Q} -line bundles $\{L_k\}_{k \in \mathbb{N}}$ such that $L_k \searrow L$, where for decreasing we mean $L_k - L_{k+1}$ is an pseudoeffective line bundle and where the convergence is in the Neron-Severi space $N^1(X)_{\mathbb{R}}$. Then, as a consequence of Theorem 2.3.15 $0 \leq t < \mu(L_k, D_j)$ for any $k \in \mathbb{N}$ big enough where t is fixed as in (i), and $\{\Delta_j(L_k)\}_{k \in \mathbb{N}}$ continuously approximates $\Delta_j(L)$ in the Hausdorff sense. Hence we obtain (i) letting $k \rightarrow \infty$.

Let us show point (ii), assuming first L \mathbb{Q} -line bundle and $0 \leq t < \mu(L; D_j)$ rational. We consider the additive semigroups

$$\begin{aligned} \Gamma_{j,t}(L) &= \{(\nu^{pj}(s), k) \in \mathbb{N}^n \times \mathbb{N} : s \in V_{k,j}(L) \text{ and } \nu^{pj}(s)_1 = kt\} \\ \Gamma_{X|D_j}(L - t\mathbb{D}) &:= \{(\nu^{pj}(s)_2, \dots, \nu^{pj}(s)_n, k) \in \mathbb{N}^{n-1} \times \mathbb{N} : \\ &\quad s \in H^0(X|D_j, k(L - t\mathbb{D})) \setminus \{0\}, k \geq 1\} \end{aligned}$$

and, setting $\psi_t : \mathbb{N}^{n-1} \times \mathbb{N} \rightarrow \mathbb{N}^n \times \mathbb{N}$ as $\psi_t(\vec{x}, k) := (kt, \vec{x}, k)$, we easily get $\Gamma_{j,t}(L) \subset \psi_t(\Gamma_{X|D_j}(L - t\mathbb{D}))$. Thus passing to the cones we have

$$C(\Gamma_j(L))_{x_1=t} = C(\Gamma_{j,t}(L)) \subset \psi_{t,\mathbb{R}}\left(C(\Gamma_{X|D_j}(L - t\mathbb{D}))\right)$$

where the equality follows from Proposition A.1 in [LM09]. Hence $\Delta_j(L)_{x_1=t} \subset \Delta_{X|D_j}(L - t\mathbb{D})$ for any $0 \leq t < \mu(L; D_j)$ rational. Moreover it is trivial that the same inclusion holds for any $\mu(L; D_j) < t < \mu(L; \mathbb{D})$.

Next let $0 \leq t < \mu(L; \mathbb{D})$ fixed and let A be a fixed ample line bundle such that there exists $s_j \in V_{1,j}(A)$ with $\nu^{pj}(s_j) = \vec{0}$ and $\nu^{pi}(s_j)_1 > 0$ for any $i \neq j$. Thus since to any section $s \in H^0(X|D_j, k(L - t\mathbb{D})) \setminus \{0\}$ we can associate a section $\tilde{s} \in H^0(X, kL)$ with $\nu^{pj}(\tilde{s}) = (kt, \nu^{pj}(s)_2, \dots, \nu^{pj}(s)_n)$ and $\nu^{pi}(\tilde{s})_1 \geq kt$ for any $i \neq j$, we get that $\tilde{s}^m \otimes s_j^k \in V_{k,j}(mL + A)$ for any $m \in \mathbb{N}$. By homogeneity this implies

$$\frac{\nu^{pj}(\tilde{s}^m \otimes s_j^k)}{mk} = \frac{\nu^{pj}(\tilde{s})}{k} = \left(t, \frac{\nu^{pj}(s)}{k}\right) =: x \in \Delta_j\left(L + \frac{1}{m}A\right)_{x_1=t}$$

for any $m \in \mathbb{N}$. Hence since $\Delta_j(L)^\circ \neq \emptyset$ we get $0 \leq t \leq \mu(L; D_j)$ and $x \in \Delta_j(L)_{x_1=t}$ by the continuity of $m \rightarrow \Delta_j(L + \frac{1}{m}A)$ (Theorem 2.3.15).

Summarizing we have showed that both sides of (ii) are empty if $\mu(L; D_j) < t < \mu(L; \mathbb{D})$ and that they coincide for any rational $0 \leq t < \mu(L; D_j)$. Moreover since by Theorem 2.3.20

$$\Delta_{X|D_j}(L - t\mathbb{D}) = \Delta\left(L - t \sum_{i=1, i \neq j}^N D_i\right)_{x_1=t}$$

with respect to the valuation ν^{pj} , we can proceed similarly as in (i) to extend the equality in (ii) first to t real and then to \mathbb{R} -line bundles using the continuity derived

from Theorem 2.3.15 and Theorem 4.5 in [LM09].

The point *iii*) is an immediate consequence of *ii*) using Theorem 2.3.20.i) and Theorem A and C in [ELMNP09], while last the point follows by integration using our Theorem A. \square

We observe that Theorem 2.3.21 may be helpful when one fixes a big line bundle L and a family of valuations associated to a family of infinitesimal flags centered at $p_1, \dots, p_N \notin \mathbb{B}_+(L)$. Indeed, similarly as stated in the paragraph § 2.2.4, componing with $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F(x) = (|x|, x_1, \dots, x_{n-1})$, the Theorem holds and in particular, for any $j = 1, \dots, N$, we get

- i) $F(\Delta_j(L))_{x_{j,1} \geq t} = \Delta_j(f^*L - t\mathbb{E}) + t\vec{e}_1$ for any $0 \leq t < \mu(f^*L; E_j)$;
- ii) $F(\Delta_j(L))_{x_{j,1}=t} = \Delta_{\tilde{X}|E_j}(f^*L - t\mathbb{E})$ for any $0 \leq t < \mu(f^*L; \mathbb{E})$;
- iii) $\text{Vol}_{\mathbb{R}^{n-1}}(F(\Delta_j(L))_{x_{j,1}=t}) = \frac{1}{(n-1)!} \text{Vol}_{\tilde{X}|E_j}(f^*L - t\mathbb{E})$ for any $0 \leq t < \mu(f^*L; \mathbb{E})$;

where we have set $f : \tilde{X} \rightarrow X$ for the blow-up at $Z = \{p_1, \dots, p_N\}$ and we have denoted with E_j the exceptional divisors. Note that $\mathbb{E} = \sum_{j=1}^N E_j$ and that the multipoint Okounkov body on the right side in *i*) is calculated from the family of valuations $\{\tilde{\nu}^{\tilde{p}_j}\}_{j=1}^N$ (it is associated to the family of admissible flags on \tilde{X} given by the family of infinitesimal flags on X).

This yields a new tool to study the *multipoint Seshadri constant* as stated in the Introduction (see Theorem B). And as application in the surfaces case we refer to subsection 2.6.2.

2.4 Kähler Packings

Recalling the notation of the subsection § 2.2.3, the *essential* multipoint Okounkov body is defined as

$$\Delta_j(L)^{\text{ess}} := \bigcup_{k \geq 1} \Delta_j^k(L)^{\text{ess}} = \bigcup_{k \geq 1} \Delta_j^{k!}(L)^{\text{ess}}$$

where $\Delta_j^k(L)^{\text{ess}} := \text{Conv}(\Gamma_j^k)^{\text{ess}} = \frac{1}{k} \text{Conv}(\nu^{p_j}(V_{k,j}))^{\text{ess}}$ is the interior of $\Delta_j^k(L) := \text{Conv}(\Gamma_j^k)$ as subset of $\mathbb{R}_{\geq 0}^n$ with its induced topology.

Fix a family of local holomorphic coordinates $\{z_{j,1}, \dots, z_{j,n}\}$ for $j = 1, \dots, N$ respectively centered at p_1, \dots, p_N and assume that the faithful valuations $\nu^{p_1}, \dots, \nu^{p_N}$ are quasi-monomial respect to the same additive total order $>$ on \mathbb{Z}^n and respect to the same vectors $\vec{\lambda}_1, \dots, \vec{\lambda}_n \in \mathbb{N}$ (see Remark 2.3.5). Thus similarly to the Definition 2.7. in [WN15], we give the following

Definition 2.4.1. *For every $j = 1, \dots, N$ we define $D_j(L) := \mu^{-1}(\Delta_j(L)^{\text{ess}})$ and call it the **multipoint Okounkov domains**, where $\mu(w_1, \dots, w_n) := (|w_1|^2, \dots, |w_n|^2)$.*

Note that, as stated in the subsection 2.2.5, we get $n! \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \text{Vol}_{\mathbb{C}^n}(D_j(L))$ for any $j = 1, \dots, N$.

We will construct *Kähler packings* (see Definition 2.4.2 and 2.4.6) of the multipoint Okounkov domains with the standard metric into (X, L) for L big line bundle. We will first address the ample case and then we will generalize to the big case in subsection § 2.4.2.

2.4.1 Ample case

Definition 2.4.2. *We say that a finite family of n -dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs into (X, L) for L ample if for every family of relatively compact open set $U_j \subseteq M_j$ there is a holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ and a Kähler form ω lying in $c_1(L)$ such that $f_*\eta_j = \omega|_{f(U_j)}$. If, in addition,*

$$\sum_{j=1}^N \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs perfectly into (X, L) .

Letting $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$ be the map $\mu(\mathbf{z}_j) := (|z_{j,1}|^2, \dots, |z_{j,n}|^2)$ where $\mathbf{z}_j = \{z_{j,1}, \dots, z_{j,n}\}$ are usual coordinates on \mathbb{C}^n and letting

$$\mathcal{D}_{k,j} := \mu^{-1}(k\Delta_j^k(L))^\circ = \mu^{-1}(k\Delta_j^k(L)^{\text{ess}}),$$

we define $M_{k,j}$ like the manifold we get by removing from \mathbb{C}^n all the submanifolds of the form $\{z_{j,i_1} = \dots = z_{j,i_m} = 0\}$ which do not intersect $\mathcal{D}_{k,j}$.

Thus

$$\phi_{k,j} := \ln \left(\sum_{\alpha_j \in \nu^{p_j}(V_{k,j})} |\mathbf{z}_j^{\alpha_j}|^2 \right)$$

is a strictly plurisubharmonic function on $M_{k,j}$ and we denote by $\omega_{k,j} := dd^c \phi_{k,j}$ the Kähler form associated (recall that $dd^c = \frac{i}{2\pi} \partial \bar{\partial}$, see subsection 2.2.1).

Lemma 2.4.3 ([And13], Lemma 5.2.) *For any finite set $\mathcal{A} \subset \mathbb{N}^m$ with a fixed additive total order $>$, there exists a $\gamma \in (\mathbb{N}_{>0})^m$ such that*

$$\alpha < \beta \quad \text{iff} \quad \alpha \cdot \gamma < \beta \cdot \gamma$$

for any $\alpha, \beta \in \mathcal{A}$.

Theorem 2.4.4. *If L is ample then for $k > 0$ big enough $\{(M_{k,j}, \omega_{k,j})\}_{j=1}^N$ packs into (X, kL) .*

Using the idea of the Theorem A in [WN15] we want to construct a Kähler metric on kL such that locally around the points p_1, \dots, p_N approximates the metrics $\phi_{k,j}$ after a suitable *zoom*. We observe that for any $\gamma \in \mathbb{N}^n$ and any section $s \in H^0(X, kL)$ with leading term $\alpha \in \mathbb{N}^n$ around a point $p \in X$ we have

$s(\tau^{\gamma_1} z_1, \dots, \tau^{\gamma_n} z_n) / \tau^{\gamma \cdot \alpha} \sim z_1^{\alpha_1} \dots z_n^{\alpha_n}$ for $\mathbb{R}_{>0} \ni \tau$ converging to zero. Therefore locally around p_j we have $\ln \left(\sum_{\alpha_j \in \nu^{p_j}(V_{k,j})} \left| \frac{s_{\alpha_j}(\tau^{\gamma} \mathbf{z}_j)}{\tau^{\gamma \cdot \alpha_j}} \right|^2 \right) \sim \phi_{k,j}$ where s_{α_j} are sections in $V_{k,j}$ with leading terms of their expansion at p_j equal to $\alpha_j \in \mathbb{N}^n$. Thus the idea is to consider the metric on kL given by $\ln(\sum_{i=1}^N \sum_{\alpha_i \in \nu^{p_i}(V_{k,i})} \left| \frac{s_{\alpha_i}}{\tau^{\gamma \cdot \alpha_i}} \right|^2)$ and define an opportune factor γ such that this metric approximates the local plurisubharmonic functions around the points p_1, \dots, p_N after the uniform zoom τ^{γ} for τ small enough. This will be possible thanks to Lemma 2.4.3 and the definition of $V_{k,j}$. Finally a standard regularization argument will conclude the proof.

Proof. We assume that the local holomorphic coordinates $\mathbf{z}_j = \{z_{j,1}, \dots, z_{j,n}\}$ centered at p_j contains the unit ball $B_1 \subset \mathbb{C}^n$ for every $j = 1, \dots, n$. Set $\mathcal{A}_j := \nu^{p_j}(V_{k,j})$ and $\mathcal{B}_i^j := \nu^{p_i}(V_{k,i})$ for $i \neq j$ to simplify the notation, let k be large enough so that $\Delta_j^k(L)^{\text{ess}} \neq \emptyset$ for any $j = 1, \dots, N$ (by Lemma 2.3.6 and Proposition 2.2.4) and let $\{U_j\}_{j=1}^N$ be a family of relatively compact open set (respectively) in $\{M_{k,j}\}_{j=1}^N$. Pick $\gamma \in \mathbb{N}^n$ as in Lemma 2.4.3 for $\mathcal{S} = \bigcup_{j=1}^N (\mathcal{A}_j \cup \bigcup_{i \neq j} \mathcal{B}_i^j)$ ordering with the total additive order $>$ induced by the family of quasi-monomial valuations, i.e. $\alpha > \beta$ iff $\alpha \cdot \gamma > \beta \cdot \gamma$. Next, for any $j = 1, \dots, N$, by construction we can choose a family of sections s_{α_j} in $V_{k,j}$, parametrized by \mathcal{A}_j , such that locally

$$s_{\alpha_j}(\mathbf{z}_j) = \mathbf{z}_j^{\alpha_j} + \sum_{\eta_j > \alpha_j} a_{j,\eta_j} \mathbf{z}_j^{\eta_j}$$

$$s_{\alpha_j}(\mathbf{z}_i) = a_{i,j} \mathbf{z}_i^{\beta_i^j} + \sum_{\eta_i > \beta_i^j} a_{i,\eta_i} \mathbf{z}_i^{\eta_i}$$

with $a_{i,j} \neq 0$ and $\alpha_j < \beta_i^j$ for any $i \neq j$.

Thus if we define, $\tau^{\gamma} \mathbf{z}_j := (\tau^{\gamma_1} z_{j,1}, \dots, \tau^{\gamma_n} z_{j,n})$ for $\tau \in \mathbb{R}_{\geq 0}$, then we get for any $\alpha_j \in \mathcal{A}_j$

$$s_{\alpha_j}(\tau^{\gamma} \mathbf{z}_j) = \tau^{\gamma \cdot \alpha_j} (\mathbf{z}_j^{\alpha_j} + O(|\tau|)) \quad \forall \tau^{\gamma} \mathbf{z}_j \in B_1 \quad (2.7)$$

$$s_{\alpha_j}(\tau^{\gamma} \mathbf{z}_i) = \tau^{\gamma \cdot \beta_i^j} (a_{i,j} \mathbf{z}_i^{\beta_i^j} + O(|\tau|)) \quad \forall \tau^{\gamma} \mathbf{z}_i \in B_1 \quad (2.8)$$

Let, for any $j = 1, \dots, N$, $g_j : M_{k,j} \rightarrow [0, 1]$ be a smooth function such that $g_j \equiv 0$ on U_j and $g_j \equiv 1$ on K_j^C for some smoothly bounded compact set K_j such that $U_j \Subset K_j \subset M_{k,j}$. Furthermore let U_j' be a relatively compact open set in $M_{k,j}$ such that $K_j \subset U_j'$.

Then pick $0 < \delta \ll 1$ such that $\phi_j := \phi_{k,j} - 4\delta g_j$ is still strictly plurisubharmonic for any $j = 1, \dots, N$.

Now we claim that for any j there is a real positive number $0 < \tau_j = \tau_j(\delta) \ll 1$ such

that for every $0 < \tau \leq \tau_j$ the following statements hold:

$$\begin{aligned} \tau^\gamma \mathbf{z}_j &\in B_1 \quad \forall \mathbf{z}_j \in U'_j, \\ \phi_j &> \ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}(\tau^\gamma \mathbf{z}_j)}{\tau^{\gamma \cdot \alpha_i}} \right|^2 \right) - \delta \quad \text{on } U_j, \\ \phi_j &< \ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}(\tau^\gamma \mathbf{z}_j)}{\tau^{\gamma \cdot \alpha_i}} \right|^2 \right) - 3\delta \quad \text{near } \partial K_j. \end{aligned}$$

Indeed it is sufficient that each request is true for $\tau \in (0, a)$ with a a positive real number. For the first request this is obvious, while the others follow from the equations (2.7) and (2.8) since $g_j \equiv 0$ on U_j and $g_j \equiv 1$ on K_j^C (recall that g_j is smooth and that $\gamma \cdot \alpha_i < \gamma \cdot \beta_i^j$ if $\alpha_i \in \mathcal{A}_i$ for any $j \neq i$).

So, since p_1, \dots, p_N are distinct points on X , we can choose $0 < \tau_k \ll 1$ such that the requests above hold for every $j = 1, \dots, N$ and $W_j \cap W_i$ for $j \neq i$ where $W_j := \varphi_j^{-1}(\tau_k^\gamma U'_j)$, where φ_j is the coordinate map giving the local holomorphic coordinates centered at p_j .

Next we define, for any $j = 1, \dots, N$,

$$\phi'_j := \max_{reg} \left(\phi_j, \ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}(\tau^\gamma \mathbf{z}_j)}{\tau^{\gamma \cdot \alpha_i}} \right|^2 \right) - 2\delta \right)$$

where $\max_{reg}(x, y)$ is a smooth convex function such that $\max_{reg}(x, y) = \max(x, y)$ whenever $|x - y| > \delta$. Therefore, by construction, we observe that ϕ'_j is smooth and strictly plurisubharmonic on $M_{k,j}$, identically equal to $\ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}(\tau^\gamma \mathbf{z}_j)}{\tau^{\gamma \cdot \alpha_i}} \right|^2 \right) - 2\delta$ near ∂K_j and identically equal to $\phi_{k,j}$ on U_j . So

$$\omega_j := dd^c \phi'_j$$

is equal to $\omega_{k,j}$ on U_j . Thus since for $k \gg 1$ big enough $\ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}}{\tau^{\gamma \cdot \alpha_i}} \right|^2 \right) - 2\delta$ extends as a positive hermitian metric of kL , with abuse of notation and unless restrict further τ , we get that $\{\omega_j\}_{j=1}^N$ extend to a Kähler form ω such that

$$\omega_{f(U_j)} = f_*(\omega_j|_{U_j}) = f_*\omega_{k,j}$$

where we are set $f : \bigsqcup_{j=1}^N U_j \rightarrow X$, $f|_{U_j} := \varphi_j^{-1} \circ \tau^\gamma$, the *uniform rescaling* for the embedding.

Since $\{U_j\}_{j=1}^N$ are arbitrary, this shows that $\{(M_{k,j}, \omega_{k,j})\}_{j=1}^N$ packs into (X, kL) . \square

Theorem C (Ample Case). *Let L be an ample line bundle. We have that $\{(D_j(L), \omega_{st})\}_{j=1}^N$ packs perfectly into (X, L) .*

Proof. If U_1, \dots, U_N are relatively compact open sets, respectively, in $D_j(L)$ then by Proposition 2.2.4 there exists $k > 0$ divisible enough such that U_j is compactly contained in $\mu^{-1}(\text{Conv}(\Delta_j^k(L)))^\circ$ for any $j = 1, \dots, N$, i.e. $\sqrt{k}U_j \Subset \mathcal{D}_{k,j} \Subset M_{k,j}$ for any $j = 1, \dots, N$.

By Lemma 2.2.9 there exist smooth functions $g_j : X_{k,j} \rightarrow \mathbb{R}$ with support on relatively compact open sets $U'_j \supset \sqrt{k}U_j$ such that $\tilde{\omega}_j := \omega_{k,j} + dd^c g_j$ is Kähler and $\tilde{\omega}_j = \omega_{st}$ holds on $\sqrt{k}U_j$.

Furthermore, fixing relatively compact open sets $V_j \subset M_{k,j}$ such that $U'_j \Subset V_j$ for any $j = 1, \dots, N$, by Theorem 2.4.4 we can find a holomorphic embedding $f' : \bigsqcup_{j=1}^N V_j \rightarrow X$ and a Kähler form ω' in $c_1(kL)$ such that $\omega'_{|f'(V)} = f'_* \omega_{k,j}$ for any $j = 1, \dots, N$.

Next, let χ_j be smooth cut-off functions on X such that $\chi_j \equiv 1$ on $f'(U'_j)$ and $\chi_j \equiv 0$ outside $\overline{f'(V_j)}$. Thus, since $f'(V_j) \cap f'(V_i) = \emptyset$ for every $j \neq i$ and since $g_j \circ f'^{-1}_{|f'(V_j)}$ has compact support in $f'(U'_j)$, the function $g = \sum_{j=1}^N \chi_j g_j \circ f'^{-1}$, extends to 0 outside $\bigcup_{j=1}^N \overline{f'(V_j)}$ and $g_{|f'(V_j)} = g_j \circ f'^{-1}_{|f'(V_j)}$.

Finally defining $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ by $f_{|U_j}(z_j) := f'_{|\sqrt{k}U_j}(\sqrt{k}z_j)$, we get

$$(\omega' + dd^c g)_{|f(U_j)} = f'_*(\omega_{k,j} + dd^c g_j)_{|\sqrt{k}U_j} = k f_* \omega_{st|U_j}$$

by construction. Hence $\omega := \frac{1}{k}(\omega' + dd^c g)$ is a Kähler form with class $c_1(L)$ that satisfies the requests since by Theorem A

$$\sum_{j=1}^N \int_{D_j(L)} \omega_{st}^n = n! \sum_{j=1}^N \text{Vol}_{\mathbb{R}^n}(\Delta_j(L)) = \text{Vol}_X(L) = \int_X \omega^n.$$

□

Remark 2.4.5. If the family of valuations fixed is associated to a family of admissible flags $Y_{j,i} = \{z_{j,1} = \dots = z_{j,i} = 0\}$ then each associated embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ can be chosen so that

$$f_{|f(U_j)}^{-1}(Y_{j,i}) = \{z_{j,1} = \dots = z_{j,i} = 0\}$$

In particular if $N = 1$ we recover the Theorem A in [WN15].

2.4.2 The big case

Definition 2.4.6. If L is big, we say that a finite family of n -dimensional Kähler manifolds $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs into (X, L) if for every family of relatively compact open set $U_j \Subset M_j$ there is a holomorphic embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ and

there exist a kähler current with analytical singularities T lying in $c_1(L)$ such that $f_*\eta_j = T|_{f(U_j)}$. If, in addition,

$$\sum_{j=1}^N \int_{M_j} \eta_j^n = \int_X c_1(L)^n$$

then we say that $\{(M_j, \eta_j)\}_{j=1, \dots, N}$ packs perfectly into (X, L) .

Reasoning as in the previous section we prove the following

Theorem C (Big Case). *Let L be a big line bundle. We have that $\{(D_j(L), \omega_{st})\}_{j=1}^N$ packs perfectly into (X, L) .*

Proof. By Lemma 2.3.6, $D_j(L) = \emptyset$ for any j such that $\Delta_j(L)^\circ = \emptyset$. So, unless removing some of the points we may assume that $\Delta_j(L)^\circ \neq \emptyset$ for any $j = 1, \dots, N$. Thus letting $k \gg 0$ big enough such that $\Delta_j^k(L)^{ess} \neq \emptyset$ for any j (Proposition 2.2.4) we can proceed similarly to the Theorem 2.4.4 with the unique difference that $\ln \left(\sum_{i=1}^N \sum_{\alpha_i \in \mathcal{A}_i} \left| \frac{s_{\alpha_i}}{\tau^{\gamma_i} \cdot \alpha_i} \right|^2 \right)$ extends to a positive singular hermitian metric, hence we get a (current of) curvature T that is a Kähler current with analytical singularities. Next, as in the ample case, we can show that $\{(D_j(L), \omega_{st})\}_{j=1}^N$ packs perfectly into (X, L) . \square

Remark 2.4.7. If the family of valuations fixed is associated to a family of admissible flags $Y_{j,i} = \{z_{j,1} = \dots = z_{j,i} = 0\}$ then each associated embedding $f : \bigsqcup_{j=1}^N U_j \rightarrow X$ can be chosen so that

$$f_{|f(U_j)}^{-1}(Y_{j,i}) = \{z_{j,1} = \dots = z_{j,i} = 0\}$$

In particular if $N = 1$ we recover the Theorem C in [WN15].

2.5 Local Positivity

2.5.1 Moving Multipoint Seshadri Constant

Definition 2.5.1. *Let L be a nef line bundle on X . The quantity*

$$\epsilon_S(L; p_1, \dots, p_N) := \inf \frac{L \cdot C}{\sum_{i=1}^N \text{mult}_{p_i} C}$$

where the infimum is on all irreducible curve $C \subset X$ passing through at least one of the points p_1, \dots, p_N is called the **multipoint Seshadri constant at p_1, \dots, p_N of L** .

This constant has played an important role in the last three decades and it is the natural extension of the Seshadri constant introduced by Demailly in [Dem90]. The following Lemma is well-known and its proof can be found for instance in [Laz04], [BDRH⁺09]:

Lemma 2.5.2. *Let L be a nef line bundle on X . Then*

$$\epsilon_S(L; p_1, \dots, p_N) = \sup\{t \geq 0 : \mu^*L - t \sum_{i=1}^N E_i \text{ is nef}\} = \inf\left(\frac{L^{\dim V} \cdot V}{\sum_{j=1}^N \text{mult}_{p_j} V}\right)^{\frac{1}{\dim V}}$$

where $\mu : \tilde{X} \rightarrow X$ is the blow-up at $Z = \{p_1, \dots, p_N\}$, E_i is the exceptional divisor above p_i and where the infimum on the right side is on all positive dimensional irreducible subvariety V containing at least one point among p_1, \dots, p_N .

The characterization of Lemma 2.5.2 allows to extend the definition to nef \mathbb{Q} -line bundles by homogeneity and to nef \mathbb{R} -line bundles by continuity.

Here we describe a possible generalization of the multipoint Seshadri constant for big line bundles:

Definition 2.5.3. *Let L be a big \mathbb{R} -line bundle, we define the **moving multipoint Seshadri constant at p_1, \dots, p_N of L** as*

$$\epsilon_S(\|L\|; p_1, \dots, p_N) := \sup_{f^*L = A + E} \epsilon_S(A; f^{-1}(p_1), \dots, f^{-1}(p_N))$$

if $p_1, \dots, p_N \notin \mathbb{B}_+(L)$ and $\epsilon_S(\|L\|; p_1, \dots, p_N) := 0$ otherwise, where the supremum is taken over all modifications $f : Y \rightarrow X$ with Y smooth such that f is an isomorphism around p_1, \dots, p_N and over all decomposition $f^*L = A + E$ where A is an ample \mathbb{Q} -divisor and E is effective with $f^{-1}(p_j) \notin \text{Supp}(E)$ for any $j = 1, \dots, N$.

For $N = 1$, we retrieve the definition given in [ELMNP09].

The following properties can be showed exactly as for the one-point case and they are left to the reader:

Proposition 2.5.4. *Let L, L' be big \mathbb{R} -line bundles. Then*

- i) $\epsilon_S(\|L\|; p_1, \dots, p_N) \leq \left(\frac{\text{Vol}_X(L)}{N}\right)^{1/n}$;
- ii) if $c_1(L) = c_1(L')$ then $\epsilon_S(\|L\|; p_1, \dots, p_N) = \epsilon_S(\|L'\|; p_1, \dots, p_N)$;
- iii) $\epsilon_S(\|\lambda L\|; p_1, \dots, p_N) = \lambda \epsilon_S(\|L\|; p_1, \dots, p_N)$ for any $\lambda \in \mathbb{R}_{>0}$;
- iv) if $p_1, \dots, p_N \notin \mathbb{B}_+(L) \cup \mathbb{B}_+(L')$ then $\epsilon_S(\|L + L'\|; p_1, \dots, p_N) \geq \epsilon_S(\|L\|; p_1, \dots, p_N) + \epsilon_S(\|L'\|; p_1, \dots, p_N)$.

We check that the moving multipoint Seshadri constant is an effective generalization of the multipoint Seshadri constant:

Proposition 2.5.5. *Let L be a big and nef \mathbb{Q} -line bundle. Then*

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \epsilon_S(L; p_1, \dots, p_N)$$

Proof. By homogeneity we can assume L line bundle and $p_1, \dots, p_N \notin \mathbb{B}_+(L)$ since if $p_j \in \mathbb{B}_+(L)$ for some j then by Proposition 1.1. and Corollary 5.6. in [ELMNP09] there exist an irreducible positive dimensional component $V \subset \mathbb{B}_+(L), p_j \in V$ such that $L^{\dim V} \cdot V = 0$ and Lemma 2.5.2 gives the equality.

Thus, fixed a modification $f : Y \rightarrow X$ as in the definition, we get

$$\frac{L \cdot C}{\sum_{i=1}^N \text{mult}_{p_i} C} = \frac{f^* L \cdot \tilde{C}}{\sum_{i=1}^N \text{mult}_{f^{-1}(p_i)} \tilde{C}} \geq \frac{A \cdot \tilde{C}}{\sum_{i=1}^N \text{mult}_{f^{-1}(p_i)} \tilde{C}}$$

since $f^{-1}(p_1), \dots, f^{-1}(p_N) \notin \text{Supp}(E)$ and $\epsilon_S(\|L\|; p_1, \dots, p_N) \leq \epsilon_S(L; p_1, \dots, p_N)$ follows.

For the reverse inequality, we can write $L = A + E$ with A ample \mathbb{Q} -line bundle and E effective such that $p_1, \dots, p_N \notin \text{Supp}(E)$, and we note that $L = A_m + \frac{1}{m}E$ for any $m \in \mathbb{N}$ for the ample \mathbb{Q} -line bundle $A_m := \frac{1}{m}A + (1 - \frac{1}{m})L$. Thus $\epsilon_S(\|L\|; p_1, \dots, p_N) \geq \epsilon_S(A_m; p_1, \dots, p_N)$ and letting $m \rightarrow \infty$ the inequality requested follows from the continuity of $\epsilon_S(\cdot; p_1, \dots, p_N)$ in the nef cone. \square

The following Proposition justifies the name given as generalization of the definition in [Nak03]:

Proposition 2.5.6. *If L is a big \mathbb{Q} -line bundle such that $p_1, \dots, p_N \notin \mathbb{B}(L)$ then*

$$\begin{aligned} \epsilon_S(\|L\|; p_1, \dots, p_N) &= \lim_{k \rightarrow \infty} \frac{\epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))}{k} = \\ &= \sup_{k \rightarrow \infty} \frac{\epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))}{k} \end{aligned}$$

where $M_k := \mu_k^*(kL) - E_k$ is the moving part of $|mL|$ given by a resolution of the base ideal $\mathfrak{b}_k := \mathfrak{b}(|kL|)$ (or set $M_k = 0$ if $H^0(X, kL) = \{0\}$).

Note that $\epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))$ does not depend on the resolution chosen and given k_1, k_2 divisible enough we may choose resolutions such that $M_{k_1+k_2} = M_{k_1} + M_{k_2} + E$ where E is an effective divisor with $p_1, \dots, p_N \notin \text{Supp}(E)$, so the existence of the limit in the definition follows from Proposition 2.5.4.iv).

Proof of Proposition 2.5.6. By homogeneity we can assume L big line bundle, $\mathbb{B}(L) = \text{Bs}(|L|)$ and that the rational map $\varphi : X \setminus \text{Bs}(|L|) \rightarrow \mathbb{P}^N$ associated to the linear system $|L|$ has image of dimension n .

Suppose first that there exist $j \in \{1, \dots, N\}$ and an integer $k_0 \geq 1$ such that $\mu_{k_0}^{-1}(p_j) \in \mathbb{B}_+(M_{k_0})$. Thus for any $\mathbb{N} \ni k \geq k_0$ we get $\mu_k^{-1}(p_j) \in \mathbb{B}_+(M_k)$. Then, since M_k is big and nef, there exists a subvariety V of dimension $d \geq 1$ such that

$M_k^d \cdot V = 0$ and $V \ni \mu_k^{-1}(p_j)$ (Corollary 5.6. in [ELMNP09]), thus by Lemma 2.5.2 $\epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N)) = 0$ and the equality follows.

Therefore we may assume $\mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N) \notin \mathbb{B}_+(M_k)$ for any $k \geq 1$ and we can write $M_k = A + E$ with A ample and E effective with $\mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N) \notin \text{Supp}(E)$. Clearly for any $m \in \mathbb{N}$, setting $A_m := \frac{1}{m}A + (1 - \frac{1}{m})M_k$, the equality $M_k = A_m + \frac{1}{m}E$ holds. Hence, since by definition $\epsilon_S(\|L\|; p_1, \dots, p_N) \geq \frac{1}{k} \epsilon_S(A_m; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))$ for any $m \in \mathbb{N}$, we finally obtain $\epsilon_S(\|L\|; p_1, \dots, p_N) \geq \frac{1}{k} \epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))$ letting $m \rightarrow \infty$.

For the reverse inequality, let $f: Y \rightarrow X$ be a modification as in the definition of the moving multipoint Seshadri constant, i.e. $f^*L = A + E$ with A ample \mathbb{Q} -divisor and E effective divisor with $p_1, \dots, p_N \notin \text{Supp}(E)$, and let $k \gg 1$ big enough such that kA is very ample. Thus, unless taking a log resolution of the base locus of $f^*(kL)$ that is an isomorphism around $f^{-1}(p_1), \dots, f^{-1}(p_N)$, we can suppose $f^*(kL) = M_k + E_k$ with $p_1, \dots, p_N \notin \text{Supp}(E_k)$ for E_k effective and M_k nef and big. Then, since kA is very ample, $M_k = kA + E'_k$ with E'_k effective and $E'_k \leq kE$. Hence we get $f^{-1}(p_1), \dots, f^{-1}(p_N) \notin \text{Supp}(E'_k)$ and $\frac{1}{k} \epsilon_S(M_k; f^{-1}(p_1), \dots, f^{-1}(p_N)) \geq \epsilon_S(A; f^{-1}(p_1), \dots, f^{-1}(p_N))$ by homogeneity, which concludes the proof. \square

Proposition 2.5.7. *Let L be a big \mathbb{Q} -line bundle. Then*

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \inf \left(\frac{\text{Vol}_{X|V}(L)}{\sum_{j=1}^N \text{mult}_{p_j} V} \right)^{1/\dim V}$$

where the infimum is over all positive dimensional irreducible subvarieties V containing at least one of the points p_1, \dots, p_N .

Proof. We may assume $p_1, \dots, p_N \notin \mathbb{B}_+(L)$ since otherwise the equality is a consequence of Corollary 5.9. in [ELMNP09]. Thus $V \not\subset \mathbb{B}_+(L)$ for any positive dimensional irreducible subvariety that pass through at least one of the points p_1, \dots, p_N , hence by Theorem 2.13. in [ELMNP09] it is sufficient to show that

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \inf \left(\frac{\|L^{\dim V} \cdot V\|}{\sum_{j=1}^N \text{mult}_{p_j} V} \right)^{1/\dim V}$$

where the infimum is over all positive dimensional irreducible subvarieties V that contain at least one of the points p_1, \dots, p_N . We recall that the *asymptotic intersection number* is defined as

$$\|L^{\dim V} \cdot V\| := \lim_{k \rightarrow \infty} \frac{M_k^{\dim V} \cdot \tilde{V}_k}{k^{\dim V}} = \sup_k \frac{M_k^{\dim V} \cdot \tilde{V}_k}{k^{\dim V}}$$

where M_k is the moving part of $\mu_k^*(kL)$ as in Proposition 2.5.6 and \tilde{V}_k is the proper transform of V through μ_k (the last equality follows from Remark 2.9. in

[ELMNP09]).

Lemma 2.5.2 and Proposition 2.5.6 (M_k is nef) imply

$$\begin{aligned} \epsilon_S(\|L\|; p_1, \dots, p_N) &= \sup_k \frac{\epsilon_S(M_k; \mu_k^{-1}(p_1), \dots, \mu_k^{-1}(p_N))}{k} = \\ &= \sup_k \inf_V \frac{1}{k} \left(\frac{(M_k^{\dim V} \cdot \tilde{V}_k)}{\sum_{j=1}^N \text{mult}_{p_j} V} \right)^{1/\dim V} \leq \inf_V \left(\frac{\|L^{\dim V} \cdot V\|}{\sum_{j=1}^N \text{mult}_{p_j} V} \right)^{1/\dim V}. \end{aligned}$$

Vice versa by the approximate Zariski decomposition showed in [Tak06] (Theorem 3.1.) for any $0 < \epsilon < 1$ there exists a modification $f: Y_\epsilon \rightarrow X$ that is an isomorphism around p_1, \dots, p_N , $f^*L = A_\epsilon + E_\epsilon$ where A_ϵ ample and E_ϵ effective with $f^{-1}(p_1), \dots, f^{-1}(p_N) \notin \text{Supp}(E_\epsilon)$, and

$$A_\epsilon^{\dim V} \cdot \tilde{V} \geq (1 - \epsilon)^{\dim V} \|L^{\dim V} \cdot V\|$$

for any $V \not\subset \mathbb{B}_+(L)$ positive dimensional irreducible subvariety (\tilde{V} proper transform of V through f). Therefore, passing to the infimum over all positive dimensional irreducible subvariety that pass through at least one of the points p_1, \dots, p_N we get

$$\begin{aligned} \epsilon_S(\|L\|; p_1, \dots, p_N) &\geq \epsilon_S(A_\epsilon; f^{-1}(p_1), \dots, f^{-1}(p_N)) \geq \\ &\geq (1 - \epsilon) \inf \left(\frac{\|L^{\dim V} \cdot V\|}{\sum_{j=1}^N \text{mult}_{p_j} V} \right)^{1/\dim V} \end{aligned}$$

which concludes the proof. \square

Theorem 2.5.8. *For any choice of different points $p_1, \dots, p_N \in X$, the function $N^1(X)_{\mathbb{R}} \ni L \rightarrow \epsilon_S(\|L\|; p_1, \dots, p_N) \in \mathbb{R}$ is continuous.*

Proof. The homogeneity and the concavity described in Proposition 2.5.4 implies the locally uniform continuity of $\epsilon_S(\|L\|; p_1, \dots, p_N)$ on the open convex subset $(\bigcup_{j=1}^N B_+(p_j))^C$ (see Remark 2.3.16). Therefore it is sufficient to check that $\lim_{L' \rightarrow L} \epsilon_S(\|L'\|; p_1, \dots, p_N) = 0$ if $c_1(L) \in \bigcup_{j=1}^N B_+(p_j)$. But this is a consequence of Proposition 2.5.7 using the continuity of the restricted volume described in Theorem 5.2. in [ELMNP09]. \square

To conclude the section we recall that for a line bundle L and for a integer $s \in \mathbb{Z}_{\geq 0}$, we say that L generates s -jets at p_1, \dots, p_N if the map

$$H^0(X, L) \rightarrow \bigoplus_{j=1}^N H^0(X, L \otimes \mathcal{O}_{X, p_j} / \mathfrak{m}_{p_j}^{s+1})$$

is surjective where we have set \mathfrak{m}_{p_j} for the maximal ideal in \mathcal{O}_{X, p_j} . And we report the following last characterization of the moving multipoint Seshadri constant:

Proposition 2.5.9 ([Ito13], Lemma 3.10.). *Let L be a big line bundle. Then*

$$\epsilon_S(||L||; p_1, \dots, p_N) = \sup_{k>0} \frac{s(kL; p_1, \dots, p_N)}{k} = \lim_{k \rightarrow \infty} \frac{s(kL; p_1, \dots, p_N)}{k}$$

where $s(kL; p_1, \dots, p_N)$ is 0 if kL does not generate s -jets at p_1, \dots, p_N for any $s \in \mathbb{Z}_{\geq 0}$, otherwise it is the biggest non-negative integer such that kL generates the $s(kL; p_1, \dots, p_N)$ -jets at p_1, \dots, p_N .

2.5.2 Proof of Theorem B

In the spirit of the aforementioned work of Demailly [Dem90], we want to describe the moving multipoint Seshadri constant $\epsilon_S(||L||; p_1, \dots, p_N)$ in a more analytical language.

Definition 2.5.10. *We say that a singular metric φ of a line bundle L has isolated logarithmic poles at p_1, \dots, p_N of coefficient γ if $\min\{\nu(\varphi, p_1), \dots, \nu(\varphi, p_N)\} = \gamma$ and φ is finite and continuous in a small punctured neighborhood $V_j \setminus \{p_j\}$ for every $j = 1, \dots, N$. We have indicated with $\nu(\varphi, p_j)$ the Lelong number of φ at p_j ,*

$$\nu(\varphi, p_j) := \liminf_{z \rightarrow x} \frac{\varphi_j(z)}{\ln|z - x|^2}$$

where φ_j is the local plurisubharmonic function defining φ around $p_j = x$.

We set $\gamma(L; p_1, \dots, p_N) := \sup\{\gamma \in \mathbb{R} : L \text{ has a positive singular metric with isolated logarithmic poles at } p_1, \dots, p_N \text{ of coefficient } \gamma\}$

Note that for $N = 1$ we recover the definition given in [Dem90].

Proposition 2.5.11. *Let L be a big \mathbb{Q} -line bundle. Then*

$$\gamma(L; p_1, \dots, p_N) = \epsilon_S(||L||; p_1, \dots, p_N)$$

Proof. By homogeneity we can assume L to be a line bundle, and we fix a family of local holomorphic coordinates $\{z_{j,1}, \dots, z_{j,n}\}$ in open coordinated sets U_1, \dots, U_N centered respectively at p_1, \dots, p_N .

Setting $z_j := (z_{j,1}, \dots, z_{j,n})$ and $s := s(kL; p_1, \dots, p_N)$ for $k \geq 1$ natural number, we can find holomorphic section f_α , parametrized by all $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^{Nn}$ such that $|\alpha_j| = s$ and $f_\alpha|_{U_j} = z_j^{\alpha_j}$ for any $j = 1, \dots, N$. In other words, we can find holomorphic sections of kL whose jets at p_1, \dots, p_N generates all possible combination of monomials of degree s around the points chosen. Thus the positive singular metric φ on L given by

$$\varphi := \frac{1}{k} \log \left(\sum_{\alpha} |f_\alpha|^2 \right)$$

has isolated logarithmic poles at p_1, \dots, p_N of coefficient s/k . Hence $\gamma(L; p_1, \dots, p_N) \geq s(kL; p_1, \dots, p_N)/k$, and letting $k \rightarrow \infty$ Proposition 2.5.9 implies $\gamma(L; p_1, \dots, p_N) \geq \epsilon_S(\|L\|; p_1, \dots, p_N)$.

Vice versa, assuming $\gamma(L; p_1, \dots, p_N) > 0$, let $\{\gamma_t\}_{t \in \mathbb{N}} \subset \mathbb{Q}$ be an increasing sequence of rational numbers converging to $\gamma(L; p_1, \dots, p_N)$ and let $\{k_t\}_{t \in \mathbb{N}}$ be an increasing sequence of natural numbers such that $\{k_t \gamma_t\}_{t \in \mathbb{N}}$ converges to $+\infty$. Moreover let A be an ample line bundle such that $A - K_X$ is ample, and let $\omega = dd^c \phi$ be a Kähler form in the class $c_1(A - K_X)$.

Thus for any positive singular metric φ_t of L with isolated logarithmic poles at p_1, \dots, p_N of coefficient $\geq \gamma_t$, $k_t \varphi_t + \phi$ is a positive singular metric of $k_t L + A - K_X$ with Kähler current $dd^c(k_t \varphi_t) + \omega$ as curvature and with isolated logarithmic poles at p_1, \dots, p_N of coefficient $\geq k_t \gamma_t$. Therefore, for $t \gg 1$ big enough, $k_t L_t + A$ generates all $(k_t \gamma_t - n)$ -jets at p_1, \dots, p_N by Corollary 3.3. in [Dem90], and thanks to Proposition 2.5.9 we obtain

$$\epsilon_S(\|L + \frac{1}{k_t} A\|; p_1, \dots, p_N) \geq \frac{k_t \gamma_t - n}{k_t} = \gamma_t - \frac{n}{k_t}.$$

Letting $t \rightarrow \infty$ we get $\epsilon_S(\|L\|; p_1, \dots, p_N) \geq \gamma(L; p_1, \dots, p_N)$ using the continuity of Theorem 2.5.8. \square

Remark 2.5.12. We observe that the same result cannot be true if we restrict to consider metric with logarithmic poles at p_1, \dots, p_N not necessarily isolated. Indeed Demailly in [Dem93] showed that for any nef and big \mathbb{Q} -line bundle L over a projective manifold, for any different points p_1, \dots, p_N , and for any τ_1, \dots, τ_N positive real numbers with $\sum_{j=1}^N \tau_j^n < (L^n)$ there exist a positive singular metric φ with logarithmic poles at any p_j of coefficient, respectively, τ_j . We thus conclude that the result in Proposition 2.5.11 holds considering metrics with logarithmic poles at p_1, \dots, p_N not necessarily isolated if and only if the multipoint Seshadri constant is maximal, i.e. $\epsilon_S(\|L\|, p_1, \dots, p_N) = (\text{Vol}_X(L)/N)^{1/n}$.

From now until the end of the section we fix a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered at p_1, \dots, p_N and the multipoint Okounkov bodies $\Delta_j(L)$ constructed from ν^{p_j} (see paragraph 2.2.4 and 2.3.5).

Definition 2.5.13. Let L be a big line bundle. We define

$$\xi(L; p_1, \dots, p_N) := \sup\{\xi \geq 0 \text{ s.t. } \xi \Sigma_n \subset \Delta_j(L)^{\text{ess}} \text{ for every } j = 1, \dots, N\}.$$

Remark 2.5.14. By definition, we note that $\xi(L; p_1, \dots, p_N) = \sup\{r > 0 : B_r(0) \subset D_j(L) \text{ for any } j = 1, \dots, N\}$.

If $N = 1$ then $\Delta_1(L) = \Delta(L)$, and it is well-known that the maximum δ such that $\delta \Sigma_n$ fits into the Okounkov body, coincides with $\epsilon_S(\|L\|; p)$ (Theorem C in [KL17]). The next theorem recover and generalize this result for any N :

Theorem B. *Let L be a big \mathbb{R} -line bundle, then*

$$\max\{\xi(L; p_1, \dots, p_N), 0\} = \epsilon_S(\|L\|; p_1, \dots, p_N)$$

Proof. By the continuity given by Theorem 2.3.15 and Theorem 2.5.8 and by the homogeneity of both sides we can assume L big line bundle. Moreover we may also assume $\Delta_j(L)^\circ \neq \emptyset$ for any $j = 1, \dots, N$ since otherwise it is a consequence of point *ii*) in Lemma 2.3.6.

Let $\{\lambda_m\}_{m \in \mathbb{N}} \subset \mathbb{Q}$ be an increasing sequence convergent to $\xi(L; p_1, \dots, p_N)$ (assuming that the latter is > 0). By Proposition 2.2.4, for any $m \in \mathbb{N}$ there exist $k_m \gg 1$ such that $\lambda_m \Sigma_n \subset \Delta_j^{k_m}(L)^{\text{ess}}$ for any $j = 1, \dots, N$. Therefore, chosen a set of section $\{s_{j,\alpha}\}_{j,\alpha} \subset H^0(X, k_m L)$ parametrized in a natural way by all valutive points in $\Delta_j^{k_m}(L)^{\text{ess}} \setminus \lambda_m \Sigma_n^{\text{ess}}$ for any $j = 1, \dots, N$ (i.e. $s_{j,\alpha} \in V_{k_m,j}$, $\nu^{p_j}(s_{j,\alpha}) = \alpha$ and $\alpha \notin \lambda_m \Sigma_n^{\text{ess}}$) the metric

$$\varphi_{k_m} := \frac{1}{k_m} \ln \left(\sum_{j=1}^N \sum_{\alpha} |s_{j,\alpha}|^2 \right)$$

is a positive singular metric on L such that $\nu(\varphi_{k_m}, p_j) \geq \lambda_m$ while φ_{k_m} is continuous and finite on a punctured neighborhood $V_j \setminus \{p_j\}$ for any $j = 1, \dots, N$ by Corollary 2.3.18. Hence letting $m \rightarrow \infty$, we get $\epsilon_S(\|L\|; p_1, \dots, p_N) = \gamma(L; p_1, \dots, p_N) \geq \xi(L; p_1, \dots, p_N)$, where the equality is the content of Proposition 2.5.11.

On the other hand, letting $\{\lambda_m\}_{m \in \mathbb{N}} \subset \mathbb{Q}$ be a increasing sequence converging to $\epsilon_S(\|L\|; p_1, \dots, p_N) > 0$, Proposition 2.5.9 implies that for any $m \in \mathbb{N}$ there exists $k_m \gg 0$ divisible enough such that $s(tk_m L; p_1, \dots, p_N) \geq tk_m \lambda_m$ for any $t \geq 1$. Thus, since the family of valuation is associated to a family of infinitesimal flags, we get

$$\frac{[tk_m \lambda_m]}{tk_m} \Sigma_n \subset \Delta_j^{k_m}(L)^{\text{ess}} \subset \Delta_j(L)^{\text{ess}} \quad \forall j = 1, \dots, N \text{ and } \forall t \geq 1.$$

Hence $\lambda_m \Sigma_n \subset \Delta_j(L)^{\text{ess}}$ for any $j = 1, \dots, N$, which concludes the proof. \square

Remark 2.5.15. In the case L is an ample line bundle, to prove the inequality $\epsilon_S(L; p_1, \dots, p_N) \geq \xi(L; p_1, \dots, p_N)$ we could have used Theorem C. In fact it implies that $\{(B_{\xi(L; p_1, \dots, p_N)}(0), \omega_{st})\}_{j=1}^N$ fits into (X, L) , and so by symplectic blow-up procedure for Kähler manifold (see section §5.3. in [MP94], or Lemma 5.3.17. in [Laz04]) we deduce $\xi(L; p_1, \dots, p_N) \leq \epsilon_S(L; p_1, \dots, p_N)$.

Remark 2.5.16. The proof of the Theorem shows that $\xi(L; p_1, \dots, p_N)$ is independent from the choice of the family of valuations given by a family of infinitesimal flags.

The following corollary extends Theorem 0.5 in [Eckl17] to all dimension (as Eckl claimed in his paper) and to big line bundles.

Corollary 2.5.17. *Let L be a big line bundle. Then*

$$\epsilon_S(\|L\|; p_1, \dots, p_N) = \max \left\{ 0, \sup \{ r > 0 : B_r(0) \subset D_j(L) \ \forall j = 1, \dots, N \} \right\}$$

For $N = 1$ it is the content of Theorem 1.3. in [WN15].

2.6 Some particular cases

2.6.1 Projective toric manifolds

In this section $X = X_\Delta$ is a smooth projective toric variety associated to a fan Δ in $N_{\mathbb{R}} \simeq \mathbb{R}^n$, so that the torus $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$ acts on X ($N \simeq \mathbb{Z}^n$ denote a lattice of rank n with dual $M := \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$, see [Ful93], [Cox11] for notation and basic fact about toric varieties).

It is well-known that there is a correspondence between toric manifolds X polarized by T_N -invariant ample divisors D and lattice polytopes $P \subset M_{\mathbb{R}}$ of dimension n . Indeed to any such divisor $D = \sum_{\rho \in \Delta(1)} a_\rho D_\rho$ (where we indicate with $\Delta(k)$ the cones of dimension k) the polytope P_D is given by $P_D := \bigcap_{\rho \in \Delta(1)} \{m \in M_{\mathbb{R}} : \langle m, v_\rho \rangle \geq -a_\rho\}$ where v_ρ indicates the generator of $\rho \cap N$. Vice versa any such polytope P can be described as $P := \bigcap_{F \text{ facet}} \{m \in M_{\mathbb{R}} : \langle m, n_F \rangle \geq -a_F\}$ where a *facet* is a 1-codimensional face of P and $n_F \in N$ is the unique primitive element that is normal to F and that point toward the interior of P . Thus the *normal fan associated to P* is $\Delta_P := \{\sigma_{\mathcal{F}} : \mathcal{F} \text{ face of } P\}$ where $\sigma_{\mathcal{F}}$ is the cone in $N_{\mathbb{R}}$ generated by all normal elements n_F as above for any facet containing the face \mathcal{F} . In particular vertices of P correspond to T_N -invariant points on the toric manifold X_P associated to Δ_P while facets of P correspond to T_N -invariant divisor on X_P . Finally the polarization is given by $D_P := \sum_{F \text{ facet}} a_F D_F$.

Thus, given an ample toric line bundle $L = \mathcal{O}_X(D)$ on a projective toric manifold X we can fix local holomorphic coordinates around a T_N -invariant point $p \in X$ (corresponding to a vertex $x_\sigma \in P$) such that $\{z_i = 0\} = D_i|_{U_\sigma}$ for D_i T_N -invariant divisor and we can assume $D|_{U_\sigma} = 0$.

Proposition 2.6.1 ([LM09], Proposition 6.1.(i)). *In the setting as above, the equality*

$$\phi_{\mathbb{R}^n}(P_D) = \Delta(L)$$

holds, where $\phi_{\mathbb{R}}$ is the linear map associated to $\phi : M \rightarrow \mathbb{Z}^n$, $\phi(m) := (\langle m, v_1 \rangle, \dots, \langle m, v_n \rangle)$, for $v_i \in \Delta_{P_D}(1)$ generators of the ray associated to D_i , and $\Delta(L)$ is the one-point Okounkov body associated to the admissible flag given by the local holomorphic coordinates chosen.

Moreover we recall that it is possible to describe the positivity of the toric line bundle at a T_N -invariant point x_σ corresponding to a vertex in P directly from the polytope:

Lemma 2.6.2. (Lemma 4.2.1, [BDRH⁺09]) *Let (X, L) be a toric polarized manifold, and let P be the associated polytope with vertices $x_{\sigma_1}, \dots, x_{\sigma_l}$. Then L generates k -jets at x_{σ_j} iff the length $|e_{j,i}|$ is bigger than k for any $i = 1, \dots, n$ where $e_{j,i}$ is the edge connecting x_{σ_j} to another vertex $x_{\sigma_{\tau(i)}}$.*

Remark 2.6.3. By assumption, we know that P is a Delzant polytope, i.e. there are exactly n edges originating from each vertex, and the first integer points on such edges form a lattice basis (for *integer* we mean a point belonging in M). Moreover if one fixes the first integer points on the edges starting from a vertex x_σ (i.e. a basis for $M \simeq \mathbb{Z}^n$), then the length of an edge starting from x_σ is defined as the usual length in \mathbb{R}^n . Observe that it is always an integer since the polytope is a *lattice* polytope.

Similarly to Proposition 2.6.1, chosen R T_N -invariants points corresponding to R vertices of the polytope P , we retrieve the multipoint Okounkov bodies of the corresponding R T_N -invariant points on X *directly* from the polytope:

Theorem 2.6.4. *Let (X, L) be a toric polarized manifold, and let P be the associated polytope with vertices $x_{\sigma_1}, \dots, x_{\sigma_l}$ corresponding, respectively, to the T_N -points p_1, \dots, p_l . Then for any choice of R different points ($R \leq l$) p_{i_1}, \dots, p_{i_R} among p_1, \dots, p_l , there exist a subdivision of P into R polytopes (a priori not lattice polytopes) P_1, \dots, P_R such that $\phi_{\mathbb{R}^n, j}(P_j) = \Delta_j(L)$ for a suitable choice of a family of valuations associated to infinitesimal (toric) flags centered at p_{i_1}, \dots, p_{i_R} , where $\phi_{\mathbb{R}^n, j}$ is the map given in the Proposition 2.6.1 for the point x_{σ_j} .*

Proof. Unless reordering, we can assume that the T_N -invariants points p_1, \dots, p_R correspond to the vertices $x_{\sigma_1}, \dots, x_{\sigma_R}$.

Next for any $j = 1, \dots, R$, after the identification $M \simeq \mathbb{Z}^n$ given by the choice of a lattice basis $m_{j,1}, \dots, m_{j,n}$ as explained in Remark 2.6.3, we retrieve the Okounkov Body $\Delta(L)$ at p_j associated to an infinitesimal flag given by the holomorphic coordinates $\{z_{1,j}, \dots, z_{n,j}\}$ as explained in Proposition 2.6.1 composing with the map $\phi_{\mathbb{R}^n, j}$. Thus, by construction, we know that any valutive point lying in the diagonal face of the n -simplex $\delta\Sigma_n$ for $\delta \in \mathbb{Q}$ correspond to a section $s \in H^0(X, kL)$ such that $\text{ord}_{p_j}(s) = k\delta$. Working directly on the polytope P , the diagonal face of the n -simplex $\delta\Sigma_n$ corresponds to the intersection of the polytope P with the hyperplane $H_{\delta, j}$ parallel to the hyperplane passing for $m_{1,j}, \dots, m_{n,j}$ and whose distance from the point x_{σ_j} is equal to δ (the *distance* is calculated from the identification $M \simeq \mathbb{Z}^n$).

Therefore defining

$$\begin{aligned} P_j &:= \overline{\bigcup_{(\delta_1, \dots, \delta_n) \in \mathbb{Q}_{\geq 0}^n, \delta_j < \delta_i \forall i \neq j} H_{\delta_1, 1} \cap \dots \cap H_{\delta_R, R} \cap P} = \\ &= \overline{\bigcup_{(\delta_1, \dots, \delta_n) \in \mathbb{Q}_{\geq 0}^n, \delta_j \leq \delta_i \forall i \neq j} H_{\delta_1, 1} \cap \dots \cap H_{\delta_R, R} \cap P} \end{aligned}$$

we get by Proposition 2.2.3 $\phi_{\mathbb{R}^n, j}(P_j) = \Delta_j(L)$ since any valutive point in $H_{\delta_1, 1} \cap \dots \cap H_{\delta_R, R} \cap P$ belongs to $\Delta_j(L)$ if $\delta_j < \delta_i$ for any $i \neq j$, while on the other hand any valutive point in $\Delta_j(L)$ belongs to $H_{\delta_1, 1} \cap \dots \cap H_{\delta_R, R} \cap P$ for certain rational numbers $\delta_1, \dots, \delta_R$ such that $\delta_j \leq \delta_i$. \square

Remark 2.6.5. As easy consequence, we get that for any polarized toric manifold (X, L) and for any choice of R T_N -invariants points p_1, \dots, p_R , the multipoint Okounkov bodies constructed from the infinitesimal flags as in Theorem 2.6.4 are polyhedral.

Corollary 2.6.6. *In the same setting of the Theorem 2.6.4, if $R = l$, then the subdivision is barycentric. Namely, for any fixed vertex x_{σ_j} , if F_1, \dots, F_n are the facets containing x_{σ_j} and b_1, \dots, b_n are their respective barycenters, then the polytope P_j is the convex body defined by the intersection of P with the n hyperplanes $H_{O, j}$ passing through the baricenter O of P and the barycenters $b_1, \dots, b_{j-1}, b_{j+1}, \dots, b_n$.*

Finally we retrieve and extend Corollary 2.3. in [Eck17] as consequence of Theorem 2.6.4 and Theorem B:

Corollary 2.6.7. *In the same setting of the Theorem 2.6.4, for any $j = 1, \dots, R$, let $\epsilon_{S, j} := \min_{i=1, \dots, n} \{\delta_{j, i}\}$ be the minimum among all the reparametrized length $|e_{j, i}|$ of the edges $e_{j, i}$ for $i = 1, \dots, n$, i.e. $\delta_{j, i} := |e_{j, i}|$ if $e_{j, i}$ connect x_{σ_j} to another point x_{σ_i} corresponding to a point $p \notin \{p_1, \dots, p_R\}$, while $\delta_{j, i} := \frac{1}{2}|e_{j, i}|$ if $e_{j, i}$ connect to a point x_{σ_i} corresponding to a point $p \in \{p_1, \dots, p_R\}$. Then*

$$\epsilon_S(L; p_1, \dots, p_R) = \min\{\epsilon_{S, j} : j = 1, \dots, R\}$$

In particular $\epsilon_S(L; p_1, \dots, p_R) \in \frac{1}{2}\mathbb{N}$.

2.6.2 Surfaces

When X has dimension 2, the following famous decomposition holds:

Theorem 2.6.8 (Zariski decomposition). *Let L be a pseudoeffective \mathbb{Q} -line bundle on a surface X . Then there exist \mathbb{Q} -line bundles P, N such that*

- i) $L = P + N$;
- ii) P is nef;
- iii) N is effective;
- iv) $H^0(X, kP) \simeq H^0(X, kL)$ for any $k \geq 1$;
- v) $P \cdot E = 0$ for any E irreducible curves contained in $\text{Supp}(N)$.

Moreover we recall that by the main theorem of [BKS04] there exists a locally finite decomposition of the big cone into rational polyhedral subcones (*Zariski chambers*)

such that in each interior of these subcones the negative part of the Zariski decomposition has constant support and the restricted and augmented base loci are equal (i.e. the divisors with cohomology classes in a interior of some Zariski chambers are *stable*, see [ELMNP06]).

Similarly to Theorem 6.4. in [LM09] and the first part of Theorem B in [KLM12] we describe the multipoint Okounkov bodies as follows:

Theorem 2.6.9. *Let L be a big line bundle over a surface X , let $p_1, \dots, p_N \in X$, and let ν^{p_j} a family of valuations associated to admissible flags centered at p_1, \dots, p_N with $Y_{1,i} = C_i|_{U_{p_i}}$ for irreducible curves C_i for $i = 1, \dots, N$. Then for any $j = 1, \dots, N$ such that $\Delta_j(L)^\circ \neq \emptyset$ there exist piecewise linear functions $\alpha_j, \beta_j : [t_{j,-}, t_{j,+}] \rightarrow \mathbb{R}_{\geq 0}$ for*

$$0 \leq t_{j,-} := \inf\{t \geq 0 : C_j \not\subset \mathbb{B}_+(L - t\mathbb{G})\} < \\ < t_{j,+} := \sup\{t \geq 0 : C_j \not\subset \mathbb{B}_+(L - t\mathbb{G})\} \leq \mu(L; \mathbb{G}) := \sup\{t \geq 0 : L - t\mathbb{G} \text{ is big}\}$$

where $\mathbb{G} = \sum_{j=1}^N C_j$, with α_j convex and β_j concave, $\alpha_j \leq \beta_j$, such that

$$\Delta_j(L) = \{(t, y) \in \mathbb{R}^2 : t_{j,-} \leq t \leq t_{j,+} \text{ and } \alpha_j(t) \leq y \leq \beta_j(t)\}$$

In particular $\Delta_j(L)$ is polyhedral for any $j = 1, \dots, N$.

Proof. By Lemma 2.3.6 and Theorem A we may assume $\Delta_j(L)^\circ \neq \emptyset$ for any $j = 1, \dots, N$ unless removing some of the points. Then by Theorem A and C in [ELMNP09] it follows that $0 \leq t_{j,-} < t_{j,+} \leq \mu(L; \mathbb{G})$ and that $[t_{j,-}, t_{j,+}] \times \mathbb{R}_{\geq 0}$ is the smallest vertical strip containing $\Delta_j(L)$. Thus by Theorem 2.3.21 and Lemma 6.3. in [LM09] we easily get $\Delta_j(L) = \{(t, y) \in \mathbb{R}^2 : t_{j,-} \leq t \leq t_{j,+} \text{ and } \alpha_j(t) \leq y \leq \beta_j(t)\}$ defining $\alpha_j(t) := \text{ord}_{p_j}(N_t|_{C_j})$ and $\beta_j(t) := \text{ord}_{p_j}(N_t|_{C_j}) + (P_t \cdot C_j)$ for $P_t + N_t$ Zariski decomposition of $L - t\mathbb{G}$ (N_t can be restricted to C_j since $\text{Supp}(N_t) = \mathbb{B}_-(L - t\mathbb{G})$). Next we proceed similarly to [KLM12] to show the polyhedrality of $\Delta_j(L)$, i.e. we set $L' := L - t_{j,+}\mathbb{G}$, $s = t_{j,+} - t$ and consider $L'_s := L' + s\mathbb{G} = L - t\mathbb{G}$ for $s \in [0, t_{j,+} - t_{j,-}]$. Thus the function $s \rightarrow N'_s$ is decreasing, i.e. $N'_{s'} - N'_s$ is effective for any $0 \leq s' < s \leq t_{j,+} - t_{j,-}$, where $L'_s = P'_s + N'_s$ is the Zariski decomposition of L'_s . Moreover, letting F_1, \dots, F_r be the irreducible (negative) curves composing N'_0 , we may assume (unless rearranging the F_i 's) that the support of $N'_{t_{j,+} - t_{j,-}}$ consists of F_{k+1}, \dots, F_r and that $0 =: s_0 < s_1 \leq \dots \leq s_k \leq t_{j,+} - t_{j,-} =: s_{k+1}$ where $s_i := \sup\{s \geq 0 : F_i \subset \mathbb{B}_-(L'_s) = \text{Supp}(N'_s)\}$ for any $i = 1, \dots, k$.

So, by the continuity of the Zariski decomposition in the big cone, it is enough to show that N'_s is linear in any not-empty open interval (s_i, s_{i+1}) for $i \in \{0, \dots, k\}$. But the Zariski algorithm implies that N'_s is determined by $N'_s \cdot F_l = (L' + s\mathbb{G}) \cdot F_l$ for any $l = i + 1, \dots, r$, and, since the intersection matrix of the curves F_{i+1}, \dots, F_r is non-degenerate, we know that there exist unique divisors A_i and B_i supported on $\cup_{l=i+1}^r F_l$ such that $A_i \cdot F_l = L' \cdot F_l$ and $B_i \cdot F_l = \mathbb{G} \cdot F_l$ for any $l = i + 1, \dots, r$. Hence $N'_s = A_i + sB_i$ for any $s \in (s_i, s_{i+1})$, which concludes the proof. \square

Remark 2.6.10. We observe that $\Delta_j(L) \cap [0, \mu(L; \mathbb{G}) - \epsilon] \times \mathbb{R}$ is *rational* polyhedral for any $0 < \epsilon < \mu(L; \mathbb{G})$ thanks to the proof and to the main theorem in [BKS04].

A particular case is when $p_1, \dots, p_N \notin \mathbb{B}_+(L)$ and ν^{p_j} is a family of valuations associated to infinitesimal flags centered respectively at p_1, \dots, p_N . Indeed in this case on the blow-up $\tilde{X} = \text{Bl}_{\{p_1, \dots, p_N\}} X$ we can consider the family of valuations $\tilde{\nu}^{\tilde{p}_j}$ associated to the admissible flags centered respectively at points $\tilde{p}_1, \dots, \tilde{p}_N \in \tilde{X}$ (see paragraph §2.2.4). Observe that $\tilde{Y}_{1,j} = E_j$ are the exceptional divisors over the points.

Lemma 2.6.11. *In the setting just mentioned, we have $t_{j,-} = 0$ and $t_{j,+} = \mu(f^*L; \mathbb{E})$ where $\mathbb{E} = \sum_{i=1}^N E_i$ and $f : \tilde{X} \rightarrow X$ is the blow-up map.*

Proof. Theorem B easily implies $t_{j,-} = 0$ for any $j = 1, \dots, N$ since $p_1, \dots, p_N \notin \mathbb{B}_+(L)$ and $F(\Delta_j(L)) = \Delta_j(f^*L)$ for $F(x_1, x_2) = (x_1 + x_2, x_1)$.

Next assume by contradiction there exists $j \in \{1, \dots, N\}$ such that $t_{j,+} < \mu(f^*L; \mathbb{E})$. Then by Theorem 2.3.21 and Theorem A and C in [ELMNP09] we obtain $\bar{t} := \sup\{t \geq 0 : E_j \not\subset \mathbb{B}_+(f^*L - t\mathbb{E})\} = \sup\{t \geq 0 : E_j \not\subset \mathbb{B}_-(f^*L - t\mathbb{E})\} < \mu(f^*L; \mathbb{E})$. Therefore setting $L_t := f^*L - t\mathbb{E}$ and letting $L_t = P_t + N_t$ be its Zariski decomposition, we get that $E_j \in \text{Supp}(N_t)$ iff $t > \bar{t}$ (see Proposition 1.2. in [KL15a]). But for any $\bar{t} < t < \mu(f^*L; \mathbb{E})$ we find out

$$0 = (L_t + t\mathbb{E}) \cdot E_j = L_t \cdot E_j + tE_j^2 < -t$$

where the first equality is justified by $P_t + N_t + t\mathbb{E} = f^*L$ while the inequality is a consequence of $L_t \cdot E_j < 0$ (since $E_j \in \text{Supp}(N_t)$) and of $E_i \cdot E_j = \delta_{i,j}$. Hence we obtain a contradiction. \square

About the Nagata's Conjecture: One of the version of the Nagata's conjecture says that for a choice of very general points $p_1, \dots, p_N \in \mathbb{P}^2$, for $N \geq 9$, the ample line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ has maximal multipoint Seshadri constant at p_1, \dots, p_N , i.e. $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$ where to simplify the notation we did not indicate the points since they are very general. We can read it in the following way:

Conjecture 2.6.12 ([Nag58], Nagata's Conjecture) . *For $N \geq 9$ very general points in \mathbb{P}^2 , let $\{\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1))\}_{j=1}^N$ be the multipoint Okounkov bodies calculated from a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered respectively at p_1, \dots, p_N . Then the following equivalent statements hold:*

- i) $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$;
- ii) $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1)) = \frac{1}{\sqrt{N}} \Sigma_2$, where Σ_2 is the standard 2-symplex;
- iii) $D_j(\mathcal{O}_{\mathbb{P}^2}(1)) = B_{\frac{1}{\sqrt{N}}}(0)$;

Remark 2.6.13. It is well know that the conjecture holds if $N \geq 9$ is a perfect square. And a similar conjecture (called Biran-Nagata-Szemberg's conjecture)

claims that for any ample line bundle L on a projective manifold of dimension n there exist $N_0 = N_0(X, L)$ big enough such that $\epsilon_S(L; N) = \sqrt[n]{\frac{L^n}{N}}$ for any $N \geq N_0$ very general points, i.e. it is maximal. This conjecture can be read through the multipoint Okounkov bodies as $\Delta_j(L) = \sqrt[n]{\frac{L^n}{N}} \Sigma_n$ for any $N \geq N_0$ very general points at X .

Theorem 2.6.14. *For $N \geq 9$ very general points in \mathbb{P}^2 , there exists a family of valuations ν^{p_j} associated to a family of infinitesimal flags centered respectively at p_1, \dots, p_N such that*

$$\begin{aligned} \Delta_j(\mathcal{O}_{\mathbb{P}^2}(1)) &= \left\{ (x, y) \in \mathbb{R}^2 : 0 \leq x \leq \epsilon \text{ and } 0 \leq y \leq \frac{1}{N\epsilon} \left(1 - \frac{x}{\epsilon}\right) \right\} = \\ &= \text{Conv}(\vec{0}, \epsilon \vec{e}_1, \frac{1}{N\epsilon} \vec{e}_2) \end{aligned}$$

where $\epsilon := \epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N)$. In particular $\mu(L, \mathbb{E}) = \frac{1}{N\epsilon}$ and

$$\text{Vol}_{X|E_j}(f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}) = \begin{cases} t & \text{if } 0 \leq t \leq \epsilon \\ \frac{\epsilon}{\frac{1}{N\epsilon} - \epsilon} \left(\frac{1}{N\epsilon} - t \right) & \text{if } \epsilon \leq t \leq \frac{1}{N\epsilon} \end{cases}$$

where $f : X = \text{Bl}_{\{p_1, \dots, p_N\}} \mathbb{P}^2 \rightarrow X$ is the blow-up at $Z = \{p_1, \dots, p_N\}$, E_1, \dots, E_N the exceptional divisors and $\mathbb{E} = \sum_{j=1}^N E_j$.

Proof. If $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = 1/\sqrt{N}$, i.e. maximal, then $\Delta_j(\mathcal{O}_{\mathbb{P}^2}(1)) = \frac{1}{\sqrt{N}} \Sigma_2$ as consequence of Theorem A and Theorem B. Thus we may assume $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) < 1/\sqrt{N}$, and we know that there exists $C = \gamma H - \sum_{j=1}^N m_j E_j$ sub-maximal curve, i.e. an irreducible curve such that $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) = \frac{\gamma}{M}$ where $M := \sum_{j=1}^N m_j$. Moreover, since the points are very general, for any cycle σ of lenght N there exists a curve $C_\sigma = \gamma H - \sum_{j=1}^N m_{\sigma(j)} E_j$, which implies $\mu(f^*\mathcal{O}_{\mathbb{P}^2}(1); \mathbb{E}) \geq \frac{M}{N\gamma} = \frac{1}{N\epsilon}$ since there exists a section $s \in H^0(\mathbb{P}^2, N\gamma)$ such that $\text{ord}_{p_j}(s) = M$ for any j . Recall that $\mu(f^*\mathcal{O}_{\mathbb{P}^2}(1); \mathbb{E}) = \sup\{t \geq 0 : f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E} \text{ is big}\}$. Next for any $j = 1, \dots, N$ we can easily fix holomorphic coordinates $(z_{1,j}, z_{2,j})$ such that $\nu^{p_j}(s) = (0, M)$ with respect to the deglex order. So considering an ample line bundle A such that there exist sections $s_1, \dots, s_N \in H^0(X, A)$ with $\nu^{p_j}(s_j) = (0, 0)$ and $\nu^{p_i}(s_j) > 0$ for any $i \neq j$ and for any $j = 1, \dots, N$, we get $s^l \otimes s_j^{N\gamma} \in V_{N\gamma, j}(lL + A)$, i.e. $(0, \frac{M}{N\gamma}) \in \Delta_j(L + \frac{1}{l}A)$ by homogeneity (Proposition 2.3.10) for any $l \in \mathbb{N}$ and any $j = 1, \dots, N$. Hence by Theorem 2.3.15 we get $(0, \frac{M}{N\gamma}) \in \Delta_j(L)$ for any $j = 1, \dots, N$. Finally since by Theorem B we know that $\epsilon_S(\mathcal{O}_{\mathbb{P}^2}(1); N) \Sigma_2 \subset \Delta_j(L)$ for any $j = 1, \dots, N$, Theorem A and the convexity imply that the multipoint Okounkov bodies have necessarily the shape requested. \square

Corollary 2.6.15. *The ray $f^*\mathcal{O}_{\mathbb{P}^2}(1) - t\mathbb{E}$ meet at most two Zariski chambers.*

This result was already showed in Proposition 2.5. of [DKMS15].

Remark 2.6.16. We recall that Biran in [Bir97] gave an homological criterion to check if a 4-dimensional symplectic manifold admits a full symplectic packings by N equal balls for large N , showing that $(\mathbb{P}^2, \omega_{FS})$ admits a full symplectic packings for $N \geq 9$. Moreover it is well-known that for any $N \leq 9$ the supremum r such that $\{(B_r(0), \omega_{st})\}_{j=1}^N$ packs into $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ coincides with the supremum r such that $(\mathbb{P}^2, \omega_{FS})$ admits a symplectic packings of N balls of radius r (called *Gromov width*), therefore by Theorem C and Corollary 2.5.17 the Nagata's conjecture is true iff the Gromov width of N balls on $(\mathbb{P}^2, \omega_{FS})$ coincides with the multipoint Seshadri constant of $\mathcal{O}_{\mathbb{P}^2}(1)$ at N very general points.

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PAPER II

L^1 metric geometry of potentials with prescribed singularities on compact Kähler manifolds

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arxiv preprint

Chapter 3

L^1 metric geometry of potentials with prescribed singularities on compact Kähler manifolds

Abstract

Given (X, ω) compact Kähler manifold and $\psi \in \mathcal{M}^+ \subset PSH(X, \omega)$ a *model type envelope* with non-zero mass, i.e. a fixed potential determining some singularities such that $\int_X (\omega + dd^c \psi)^n > 0$, we prove that the ψ -relative finite energy class $\mathcal{E}^1(X, \omega, \psi)$ becomes a complete metric space if endowed of a distance d which generalizes the well-known d_1 distance on the space of Kähler potentials.

Moreover, for $\mathcal{A} \subset \mathcal{M}^+$ total ordered, we equip the set $X_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$ of a natural distance $d_{\mathcal{A}}$ which coincides with the distance d on $\mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$. We show that $(X_{\mathcal{A}}, d_{\mathcal{A}})$ is a complete metric space.

As a consequence, assuming $\psi_k \searrow \psi$ and $\psi_k, \psi \in \mathcal{M}^+$, we also prove that $(\mathcal{E}^1(X, \omega, \psi_k), d)$ converges in a Gromov-Hausdorff sense to $(\mathcal{E}^1(X, \omega, \psi), d)$ and that there exists a direct system $\langle (\mathcal{E}^1(X, \omega, \psi_k), d), P_{k,j} \rangle$ in the category of metric spaces whose direct limit is dense into $(\mathcal{E}^1(X, \omega, \psi), d)$.

3.1 Introduction

In the last forty years it has become important to understand the space of Mabuchi \mathcal{H} , i.e. the space of Kähler potentials in a fixed Kähler cohomology class $\{\omega\} \in H^2(X, \mathbb{R}) \cap H^{1,1}(X)$ for (X, ω) compact Kähler manifold of dimension n :

$$\mathcal{H} := \{\varphi \in C^\infty : \omega + dd^c \varphi \text{ is a Kähler form}\},$$

where $d^c := \frac{i}{2\pi}(\partial - \bar{\partial})$, so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. By the pioneering papers [Mab86], [Sem92] and [Don99] \mathcal{H} can be endowed with a Riemannian structure given by the metric

$$(f, g)_\varphi := \left(\int_X fg(\omega + dd^c \varphi)^n \right)^{1/2}$$

where $\varphi \in \mathcal{H}$, $f, g \in T_\varphi \mathcal{H} \simeq C^\infty(X)$ and the metric geodesic segments are solutions of homogeneous complex Monge-Ampère equations (see also [Chen00]). Later Darvas introduced in [Dar15] the Finsler metric $|f|_{1, \varphi} := \int_X |f|(\omega + dd^c \varphi)^n$ on \mathcal{H} with associated distance d_1 , that we will simply denote by d . The metric completion of (\mathcal{H}, d) has a pluripotential description ([Dar15]) since it coincides with

$$\mathcal{E}^1(X, \omega) := \{u \in PSH(X, \omega) : E(u) > -\infty\}$$

where $E(\cdot)$ is the *Aubin-Mabuchi energy* defined as

$$E(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X u \omega^j \wedge (\omega + dd^c u)^{n-j}$$

if u is locally bounded and as $E(u) := \lim_{j \rightarrow \infty} E(\max(u, -j))$ otherwise (see [Mab86], [Aub84], [BB10] and [BEGZ10]). Here for the wedge product among $(1, 1)$ -currents we mean the *non-pluripolar product* (see [BEGZ10]). Moreover the d -distance can be expressed as

$$d(u, v) := E(u) + E(v) - 2E(P_\omega(u, v)),$$

where $P_\omega(u, v) := \sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\}$ is the *roof-top envelope* operator introduced in [RWN14]. The complete geodesic metric space $(\mathcal{E}^1(X, \omega), d)$ turned out to be very useful to formulate in analytic terms and in some cases to solve important conjectures regarding the search of special metrics (see [BBGZ13], [DR17], [BBEGZ19], [BDL16], [BBJ15], [DH17], [CC17], [CC18a], [CC18b]). Furthermore the metric topology is related to the continuity of the Monge-Ampère operator since it coincides with the so-called *strong topology* ([BBEGZ19]).

The space $\mathcal{E}^1(X, \omega)$ contains only potentials which are at most slightly singular (see [DDNL18a]). Thus Darvas, Di Nezza and Lu introduced in [DDNL18b] the analogous set $\mathcal{E}^1(X, \omega, \psi)$ with respect to a fixed ω -psh function ψ . More precisely,

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in PSH(X, \omega) : u \leq \psi + C \text{ for } C \in \mathbb{R} \text{ and } E_\psi(u) > -\infty\},$$

where

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X (u - \psi)(\omega + dd^c \psi)^j \wedge (\omega + dd^c u)^{n-j}$$

if $|u - \psi|$ is globally bounded and $E_\psi(u) := \lim_{j \rightarrow \infty} E_\psi(\max(u, \psi - j))$ otherwise. One of the reasons that leads them to investigate and develop the pluripotential theory of these sets was the search of solution with prescribed singularities $[\psi]$ for the complex Monge-Ampère equation $(\omega + dd^c u)^n = \mu$ (see also [DDNL18d]). They found out that there is a necessary condition to assume on ψ : $\psi - P_\omega[\psi](0)$ must be globally bounded where $P_\omega[\psi](0) := (\lim_{C \rightarrow \infty} P_\omega(\psi_j + C, 0))^*$, ([RWN14], the star is for the upper semicontinuous regularization). So, without loss of generality, one may assume that ψ is a *model type envelope*, i.e. $\psi = P_\omega[\psi](0)$ (see section 3.2). In this setting they were able to show the existence of Kähler-Einstein metric with prescribed singularities $[\psi]$ in the case of X manifold with ample canonical bundle and in the case X Calabi-Yau manifold.

Therefore one of the main motivations for this paper is to endow the set $\mathcal{E}^1(X, \omega, \psi)$ of a metric structure to address in a future work the problem of characterize analytically the existence of Kähler-Einstein metrics with prescribed singularities in the Fano case.

Thus, assuming ψ to be a model type envelope and defining

$$d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$$

on $\mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi)$, we prove the following theorem.

Theorem A. ¹ *Let $\psi \in PSH(X, \omega)$ be a model type envelope with non-zero mass $V_\psi = \int_X (\omega + dd^c \psi)^n > 0$. Then $(\mathcal{E}^1(X, \omega, \psi), d)$ is a complete metric space.*

The non-zero total mass $V_\psi > 0$ condition is a necessary hypothesis because otherwise $d \equiv 0$ (Remark 3.3.10).

The second main motivation of the paper is to set up a new way to compare and to study the solutions of a complex Monge-Ampère equation $(\omega + dd^c u)^n = \mu$ associated to different spaces $\mathcal{E}^1(X, \omega, \psi)$ (see [Tru20]). This leads to wonder, first of all, *how* a sequence of spaces $\mathcal{E}^1(X, \omega, \psi_k)$ *converges* to $\mathcal{E}^1(X, \omega, \psi)$ if $\psi_k \rightarrow \psi$. The most interesting case seems to be when $\{\psi_k\}_{k \in \mathbb{N}}$ is *totally ordered* with respect to the natural partial order \preceq on $PSH(X, \omega)$ given by $u \preceq v$ if $u \leq v + C$ for a constant $C \in \mathbb{R}$.

Thus in the second part of the paper, denoting with \mathcal{M} the set of all model type envelopes and with \mathcal{M}^+ its elements with non-zero mass, we assume to have a totally

¹The assumption on ω to be Kähler is unnecessary, i.e. this Theorem easily extends to the big case.

ordered subset $\mathcal{A} \subset \mathcal{M}^+$ and we define

$$X_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

where $\overline{\mathcal{A}} \subset \mathcal{M}$ is the closure of the set \mathcal{A} as subset of $PSH(X, \omega)$ with its L^1 -topology. Our next result regards the existence of a natural metric topology on $X_{\mathcal{A}}$ induced by a distance $d_{\mathcal{A}}$ which extends the distance d over $\mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$ (see section 3.4). Here for $\psi \in \mathcal{M} \setminus \mathcal{M}^+$ we identify the set $\mathcal{E}^1(X, \omega, \psi)$ with a singleton P_{ψ} .

Theorem B. *Let $\mathcal{A} \subset \mathcal{M}^+$ total ordered. Then $(X_{\mathcal{A}}, d_{\mathcal{A}})$ is a complete metric space and $d_{\mathcal{A}}$ restricts to d on $\mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$.*

The distance $d_{\mathcal{A}}$ is a natural generalization of the distances d , indeed in the companion paper [Tru20] we show how its metric topology defines a strong topology which is connected to the continuity of the Monge-Ampère operator.

As a consequence of Theorem B, considering a decreasing sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ converging to $\psi \in \mathcal{M}^+$, one immediately thinks that the metric spaces $(\mathcal{E}^1(X, \omega, \psi_k), d)$ essentially converges to $(\mathcal{E}^1(X, \omega, \psi), d)$. The problem here is that these metric spaces are not locally compact, therefore it is not clear what kind of convergence one should look at. In section 3.4 we introduce the *compact pointed Gromov-Hausdorff convergence* (*cp-GH*) which basically mimic the pointed Gromov-Hausdorff convergence (see [BH99] and [BB10]) replacing, for any space, the family of balls centered at the point with an increasing family with dense union of compact sets containing the point chosen (see Definition 3.4.19).

Theorem C. *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be a decreasing sequence converging to $\psi \in \mathcal{M}^+$. Then*

$$(\mathcal{E}^1(X, \omega, \psi_k), \psi_k, d) \xrightarrow{cp-GH} (\mathcal{E}^1(X, \omega, \psi), \psi, d).$$

Furthermore we show that the maps

$$P_{i,j} := P_{\omega}[\psi_j](\cdot) : (\mathcal{E}^1(X, \omega, \psi_i), d) \rightarrow (\mathcal{E}^1(X, \omega, \psi_j), d)$$

for $i \leq j$ are *short maps* (i.e. 1-Lipschitz). Hence $\langle (\mathcal{E}^1(X, \omega, \psi_i), d), P_{i,j} \rangle$ is a direct system in the category of metric spaces. We denote with $\mathfrak{m} - \lim_{\rightarrow}$ the direct limit in this category.

Theorem D. *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be a decreasing sequence converging to $\psi \in \mathcal{M}^+$. Then there is an isometric embedding*

$$\mathfrak{m} - \lim_{\rightarrow} \langle (\mathcal{E}^1(X, \omega, \psi_i), d), P_{i,j} \rangle \hookrightarrow (\mathcal{E}^1(X, \omega, \psi), d)$$

with dense image equal to $\bigcup_{k \in \mathbb{N}} P_{\omega}[\psi](\mathcal{E}^1(X, \omega, \psi_k))$.

3.1.1 Related Works

During the last period of the preparation of this paper, Xia in [X19b] independently showed Theorem A as particular case of his main Theorem.

3.1.2 Structure of the paper

After recalling some preliminaries in section 3.2, the third section is dedicated to prove Theorem A. In this section many of the proofs are just easily adapted to our setting from the absolute setting in the Kähler and in the big case (in particular [DDNL18c]).

Section 3.4 is the core of the paper, where we show Theorems B, C, and D.

3.1.3 Acknowledgments

The author is grateful to his two advisors S. Trapani and D. Witt Nyström for their comments and suggestions. He would also like to thank M. Xia for inspiring talks.

3.2 Preliminaries

Let (X, ω) be a compact Kähler manifold (ω fixed Kähler form on X). We denote with $PSH(X, \omega)$ the set of all ω -psh (ω -plurisubharmonic) functions on X , i.e. the set of all functions u given locally as sum of a plurisubharmonic function and a smooth function such that $\omega + dd^c u \geq 0$ as $(1, 1)$ -current. Here $d^c := \frac{i}{2\pi}(\partial - \bar{\partial})$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. We say that u is more singular than v if there exists a constant $C \in \mathbb{R}$ such that $u \leq v + C$. Being more/less singular is a partial order on $PSH(X, \omega)$. We use \preceq to denote such order, and we indicate with $[u]$ the class of equivalence with respect to this order. Moreover, according to the notations in [DDNL18b], $PSH(X, \omega, \psi)$ is the set of all $u \in PSH(X, \omega)$ such that $u \preceq \psi$, and $u \in PSH(X, \omega, \psi)$ is said to have *ψ -relative minimal singularities* if $u \in [\psi]$. To start the investigation of these functions we recall the construction of the envelopes introduced in [RWN14]: for any couple of ω -psh functions u, v , the function

$$P_\omega[u](v) := \left(\lim_{C \rightarrow +\infty} P_\omega(u + C, v) \right)^*$$

is ω -psh, where $P_\omega(u, v) := \sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\}$ is the *rooftop envelope* (the star is for the upper semicontinuous regularization). Roughly speaking if $u \preceq v$ then $P_\omega[u](v)$ is the largest ω -psh function that is bounded from above by v and that preserves the singularities type $[u]$. We say that an ω -psh function ψ is a *model type envelope* if $P_\omega[\psi] := P_\omega[\psi](0) = \psi$. There are plenty of these functions and $P_\omega[P_\omega[\psi]] = P_\omega[\psi]$. Hence $\psi \rightarrow P_\omega[\psi]$ may be thought as a projection from the set of ω -psh functions to the set of model type envelopes. We refer to Remark 1.6 in [DDNL18b] for some tangible examples of these functions. Denoting with \mathcal{M}

the set of model type envelopes, it is easy to see that if $\psi_1, \psi_2 \in PSH(X, \omega)$ satisfy $\psi_1 \preceq \psi_2$ then $\psi_1 \leq \psi_2$. Hence the partial orders \leq, \preceq coincide on \mathcal{M} .

Given T_1, \dots, T_p closed and positive $(1, 1)$ -currents, with $T_1 \wedge \dots \wedge T_p$ we will mean the *non-pluripolar* product (see [BEGZ10]). It is always well-defined on a compact Kähler manifold (Proposition 1.6 in [BEGZ10]) and it is *local in the plurifine topology*, i.e. in the coarsest topology with respect to which all psh functions on all open subsets of X become continuous (see also [BT87]). Moreover, setting $\omega_\varphi := \omega + dd^c \varphi$ if $\varphi \in PSH(X, \omega)$, the map

$$PSH(X, \omega) \ni \varphi \rightarrow V_\varphi := \int_X \omega_\varphi^n \in \mathbb{R}$$

respects the partial order defined before by the main theorem in [WN19], i.e. if $u \preceq v$ then $V_u \leq V_v$. Such monotonicity still holds considering the mixed product, i.e. $\int_X \omega_{u_1} \wedge \dots \wedge \omega_{u_n} \leq \int_X \omega_{v_1} \wedge \dots \wedge \omega_{v_n}$ if $u_j \preceq v_j$ for any $j = 1, \dots, n$ (Theorem 2.4 in [DDNL18b]). Generally we have the following principle:

Proposition 3.2.1. (Comparison Principle). *Let $u, v \in PSH(X, \omega)$ such that $u \preceq v$, and $w_1, \dots, w_{n-p} \in PSH(X, \omega)$ for $1 \leq p \leq n$ integer. Then*

$$\int_{\{v < u\}} \omega_u^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}} \leq \int_{\{v < u\}} \omega_v^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}}.$$

Proof. The proof proceeds as that of Corollary 1.4 in [WN19]. For any $\epsilon > 0$, set $v_\epsilon := \max(v, u - \epsilon)$. Thus

$$\begin{aligned} \int_X \omega_v^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}} &= \int_X \omega_{v_\epsilon}^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}} \geq \\ &\geq \int_{\{v > u - \epsilon\}} \omega_v^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}} + \int_{\{v < u - \epsilon\}} \omega_u^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}}, \end{aligned}$$

which implies

$$\int_{\{v < u - \epsilon\}} \omega_u^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}} \leq \int_{\{v < u\}} \omega_v^p \wedge \omega_{w_1} \wedge \dots \wedge \omega_{w_{n-p}}.$$

The result follows from letting $\epsilon \rightarrow 0$. □

We also recall some results of [DDNL18b] which will be very useful in the sequel:

Lemma 3.2.2 (Lemma 3.7, [DDNL18b]). *Let $u, v \in PSH(X, \omega)$. If $P_\omega(u, v) \neq -\infty$, then*

$$\omega_{P_\omega(u, v)}^n \leq \mathbb{1}_{\{P_\omega(u, v) = u\}} \omega_u^n + \mathbb{1}_{\{P_\omega(u, v) = v\}} \omega_v^n.$$

Theorem 3.2.3 (Theorem 3.8, [DDNL18b]). *Let $u, \psi \in PSH(X, \omega)$ such that u is less singular than ψ . Then*

$$MA_\omega(P_\omega[\psi](u)) \leq \mathbb{1}_{\{P_\omega[\psi](u)=u\}} MA_\omega(u).$$

In particular if ψ is a model type envelope then $MA_\omega(\psi) \leq \mathbb{1}_{\{\psi=0\}} MA_\omega(0)$.

Theorem 3.2.4 (Theorem 2.3, [DDNL18b]). *Let $\{u_j, u_j^k\}_{j=1, \dots, n} \in PSH(X, \omega)$ such that $u_j^k \rightarrow u_j$ in capacity as $k \rightarrow \infty$ for $j = 1, \dots, n$. Then for all bounded quasi-continuous function χ ,*

$$\liminf_{k \rightarrow \infty} \int_X \chi \omega_{u_1^k} \wedge \dots \wedge \omega_{u_n^k} \geq \int_X \chi \omega_{u_1} \wedge \dots \wedge \omega_{u_n}.$$

If additionally,

$$\int_X \omega_{u_1} \wedge \dots \wedge \omega_{u_n} \geq \limsup_{k \rightarrow \infty} \int_X \omega_{u_1^k} \wedge \dots \wedge \omega_{u_n^k}$$

then $\omega_{u_1^k} \wedge \dots \wedge \omega_{u_n^k} \rightarrow \omega_{u_1} \wedge \dots \wedge \omega_{u_n}$ in the weak sense of measures on X .

It is also useful to recall that if $PSH(X, \omega) \ni u_j \searrow u \in PSH(X, \omega)$ decreasing, then $u_j \rightarrow u$ in capacity, and that the convergence in capacity implies the L^1 -convergence (see [GZ17]).

3.2.1 Potentials with ψ -relative full mass.

If u, v belongs to the same class $[\psi]$ then $V_u = V_v$, but there are also examples of ω -psh functions u, v such that $u \prec v$ and $V_u = V_v$. Thus $u \in PSH(X, \omega, \psi)$ is said to have ψ -relative full mass if $V_u = V_\psi$, and the set of all ω -psh functions with ψ -relative full mass is denoted with $\mathcal{E}(X, \omega, \psi)$ (see [DDNL18b]).

Theorem 3.2.5. (Theorem 1.3, [DDNL18b]). *Suppose $\psi \in PSH(X, \omega)$ such that $V_\psi > 0$, and $u \in PSH(X, \omega, \psi)$. The followings are equivalent:*

- (i) $u \in \mathcal{E}(X, \omega, \psi)$;
- (ii) $P_\omega[u](\psi) = \psi$;
- (iii) $P_\omega[u] = P_\omega[\psi]$.

This result suggests that any function in the class $\mathcal{E}(X, \omega, \psi)$ is at most mildly more singular than ψ . Moreover this also implies that $\mathcal{E}(X, \omega, \psi_1) \cap \mathcal{E}(X, \omega, \psi_2) = \emptyset$ if ψ_1, ψ_2 are two different model type envelopes with non zero total masses $V_{\psi_1} > 0$, $V_{\psi_2} > 0$.

For any $u_1, \dots, u_p \in PSH(X, \omega)$, and for any $j_1, \dots, j_p \in \mathbb{N}$ such that $j_1 + \dots + j_p = n$ we introduce the notation

$$MA_\omega(u_1^{j_1}, \dots, u_p^{j_p}) := \omega_{u_1}^{j_1} \wedge \dots \wedge \omega_{u_p}^{j_p}$$

for the *(mixed) non-pluripolar complex Monge-Ampère measure* associated (it is a positive Borel measure) and we set $MA_\omega(u) := MA_\omega(u^n)$. Note that if $V_\psi > 0$ then the map

$$\mathcal{E}(X, \omega, \psi) \ni u \rightarrow MA_\omega(u)/V_\psi$$

has image contained in the set of non-pluripolar probability measures $\mathcal{M}(X)$. Moreover if ψ is also a model type envelope then this map is surjective and it descends to a bijection on the space of all closed and positive $(1, 1)$ -currents with ψ -relative full mass, i.e. on $\mathcal{E}(X, \omega, \psi)/\mathbb{R}$ (see Theorem A in [BEGZ10] when $\psi = 0$, Theorem 4.28 in [DDNL18b] when ψ has small unbounded locus, and Theorem 4.7 in [DDNL18d] for the general case). See the companion paper [Tru20] and references therein for a further analysis of the Monge-Ampère operator.

3.2.2 The ψ -relative finite energy class $\mathcal{E}^1(X, \omega, \psi)$.

From now until section 3.4 we will assume ψ model type envelope and $V_\psi > 0$, i.e. $\psi \in \mathcal{M}^+$ with the notations of the Introduction.

Definition 3.2.6. [DDNL18b] *The ψ -relative energy functional $E_\psi : PSH(X, \omega, \psi) \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as*

$$E_\psi(u) := \frac{1}{(n+1)} \sum_{j=0}^n \int_X (u - \psi) MA_\omega(u^j, \psi^{n-j})$$

if u has ψ -relative minimal singularities, and as

$$E_\psi(u) := \inf\{E_\psi(v) : v \in \mathcal{E}(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities, } v \geq u\}$$

otherwise.

When $\psi = 0$ this functional is, up to a multiplicative constant, the *Aubin-Mabuchi energy functional*, also called *Monge-Ampère energy* (see [Aub84], [Mab86]).

Remark 3.2.7. The authors in [DDNL18b] introduced this functional assuming ψ with small unbounded locus, but the integration by parts formula showed by Xia in [X19a] allows to work in the more general setting and all properties of E_ψ recalled below easily extend.

By Lemma 4.12 in [DDNL18b] $E_\psi(u) = \lim_{j \rightarrow \infty} E_\psi(u_j)$ for arbitrary $u \in PSH(X, \omega, \psi)$ where $u_j := \max(u, \psi - j)$ are the ψ -relative canonical approximants. Moreover, following the notations in [DDNL18b], we recall that

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}$$

and that $E_\psi(u) > -\infty$ is equivalent to $V_u = V_\psi$ and $\int_X (u - \psi) MA_\omega(u) > -\infty$ (compare also Proposition 2.11 in [BEGZ10]).

Proposition 3.2.8. [DDNL18b] *The ψ -relative energy functional is non-decreasing, concave along affine curves and continuous along decreasing sequences.*

Moreover we also have the following properties:

Proposition 3.2.9. *Suppose $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then:*

- i) $E_\psi(u) - E_\psi(v) = \frac{1}{(n+1)} \sum_{j=0}^n \int_X (u - v) MA_\omega(u^j, v^{n-j});$
- ii) if $u \leq v$ then $\int_X (u - v) MA_\omega(u) \leq E_\psi(u) - E_\psi(v) \leq \frac{1}{n+1} \int_X (u - v) MA_\omega(u);$
- iii) $\int_X (u - v) MA_\omega(u) \leq E_\psi(u) - E_\psi(v) \leq \int_X (u - v) MA_\omega(v).$

Proof. If u, v have ψ -relative minimal singularities then it is the content of Theorem 4.10 in [DDNL18b], while in the general case the proof is the same to that of Proposition 2.2 in [DDNL18c] replacing V_θ with ψ , using the Comparison Principle of Proposition 3.2.1 and the fact that for any $w \in \mathcal{E}^1(X, \omega, \psi)$

$$\begin{aligned} \lim_{j \rightarrow \infty} j \int_{\{w \leq \psi - j\}} MA_\omega(\max(w, \psi - j)) &= \lim_{j \rightarrow \infty} j \int_{\{w \leq \psi - j\}} MA_\omega(w) \leq \\ &\leq \lim_{j \rightarrow \infty} \int_{\{w \leq \psi - j\}} (\psi - w) MA_\omega(w) = 0 \end{aligned}$$

since $\int_X (w - \psi) MA_\omega(w) > -\infty$. \square

We conclude the subsection showing that the envelope operator $P_\omega(\cdot, \cdot)$ is an operator of the class $\mathcal{E}^1(X, \omega, \psi)$ (in the absolute setting, this problem was addressed by Darvas, see Corollary 3.5 in [Dar15]).

Proposition 3.2.10. *Assume $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then $P_\omega(u, v) \in \mathcal{E}^1(X, \omega, \psi)$. Moreover if $\{u_j, v_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^1(X, \omega, \psi)$ decreasing respectively to $u, v \in \mathcal{E}^1(X, \omega, \psi)$, then $E_\psi(P_\omega(u_j, v_j)) \searrow E_\psi(P_\omega(u, v))$.*

Proof. Up to rescaling we may assume $u, v \leq 0$. For any $j \in \mathbb{N}$ let $u_j := \max(u, \psi - j)$, $v_j := \max(v, \psi - j)$ be the ψ -relative canonical approximants of u, v . Then $w_j := P_\omega(u_j, v_j)$ is a decreasing sequence of potentials with ψ -relative minimal singularities. Moreover it is easy to check that $w_j \searrow P_\omega(u, v)$. Thus by Proposition 3.2.8 it is sufficient to find a uniform bound for $E_\psi(w_j)$, and by Proposition 3.2.9 this is equivalent to find $C > 0$ independent of j such that $\int_X (\psi - w_j) MA_\omega(w_j) \leq C$. But Lemma 3.2.2 implies

$$\begin{aligned} \int_X (\psi - w_j) MA_\omega(w_j) &\leq \int_{\{w_j = u_j\}} (\psi - u_j) MA_\omega(u_j) + \\ &+ \int_{\{w_j = v_j\}} (\psi - v_j) MA_\omega(v_j) \leq (n+1) |E_\psi(u_j) + E_\psi(v_j)| \leq (n+1) |E_\psi(u) + E_\psi(v)|. \end{aligned}$$

The second statement is now an easy consequence of the monotonicity of E_ψ since $P_\omega(u_j, v_j) \searrow P_\omega(u, v)$ for any couple of decreasing sequences $u_j \searrow u, v_j \searrow v$. \square

3.3 A metric geometry on $\mathcal{E}^1(X, \omega, \psi)$.

Recall that we are assuming $\psi \in \mathcal{M}^+$, i.e. ψ model type envelope with $V_\psi > 0$.

3.3.1 $\mathcal{E}^1(X, \omega, \psi)$ as metric space.

In this subsection we prove that $(\mathcal{E}^1(X, \omega, \psi), d)$ is a metric space where $d : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R}_{\geq 0}$ is defined as

$$d(u_1, u_2) := E_\psi(u_1) + E_\psi(u_2) - 2E_\psi(P_\omega(u_1, u_2)).$$

It follows from section 3.2 that d assumes finite non-negative values, and that d is continuous along decreasing sequences converging to elements in $\mathcal{E}^1(X, \omega, \psi)$.

Lemma 3.3.1. *Assume $u, v, w \in \mathcal{E}^1(X, \omega, \psi)$. Then the followings hold:*

- i) $d(u, v) = d(v, u)$;
- ii) if $u \leq v$ then $d(u, v) = E_\psi(v) - E_\psi(u)$;
- iii) if $u \leq v \leq w$ then $d(u, w) = d(u, v) + d(v, w)$;
- iv) $d(u, v) = d(u, P_\omega(u, v)) + d(v, P_\omega(u, v))$;
- v) $d(u, v) = 0$ iff $u = v$.

Proof. All points are straightforward except one implication in (v). Thus assume $d(u, v) = 0$. Then by (ii) and (iv) we get $E_\psi(u) = E_\psi(P_\omega(u, v))$, which implies $P_\omega(u, v) = u$ a.e. with respect to $MA_\omega(P_\omega(u, v))$ (Proposition 3.2.9). Hence by the domination principle (Proposition 3.11 in [DDNL18b]) we obtain $P_\omega(u, v) \geq u$, i.e. $P_\omega(u, v) = u$. The conclusion follows by symmetry. \square

To prove that $\mathcal{E}^1(X, \omega, \psi)$ is a metric space, it remains to prove the triangle inequality. We proceed as in section 3.1 in [DDNL18c], but for the courtesy of the reader we will report here many of their proofs adapted to our setting.

Proposition 3.3.2. *Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$ be potentials with ψ -relative minimal singularities. For $t \in [0, 1]$ set $\varphi_t := P_\omega((1-t)u + tv, v)$. Then for any $t \in [0, 1]$*

$$\frac{d}{dt} E_\psi(\varphi_t) = \int_X (v - \min(u, v)) MA_\omega(\varphi_t).$$

Proof. Let us prove the formula for the right derivative. The same argument easily works for the left derivative. Thus fix $t \in [0, 1]$, let $s > 0$ small and $f_t := \min((1-t)u + tv, v)$. Using Proposition 3.2.9 (iii) and Lemma 3.2.2 it is easy to check that

$$\int_X (f_{t+s} - f_t) MA_\omega(\varphi_{t+s}) \leq E_\psi(\varphi_{t+s}) - E_\psi(\varphi_t) \leq \int_X (f_{t+s} - f_t) MA_\omega(\varphi_t).$$

Moreover $\varphi_{t+s} \rightarrow \varphi_t$ uniformly as $s \rightarrow 0^+$ since $\|u - v\|_{L^\infty} \leq C$, thus $MA_\omega(\varphi_{t+s})$ converges weakly to $MA_\omega(\varphi_t)$. Therefore since $f_{t+s} - f_t = s(v - \min(u, v))$ and since and again $\|u - v\|_{L^\infty} \leq C$, Theorem 3.2.4 yields

$$\lim_{s \rightarrow 0^+} \frac{E_\psi(\varphi_{t+s}) - E_\psi(\varphi_t)}{s} = \int_X (v - \min(u, v)) MA_\omega(\varphi_t).$$

□

Proposition 3.3.3. *Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then $d(\max(u, v), u) \geq d(v, P_\omega(u, v))$.*

Proof. Setting $\varphi := \max(u, v)$ and $\phi := P_\omega(u, v)$, the inequality to prove is equivalent to $E_\psi(v) - E_\psi(\phi) \leq E_\psi(\varphi) - E_\psi(u)$. By Proposition 3.2.8 we may assume u, v having ψ -relative minimal singularities.

Next Proposition 3.3.2 implies

$$\begin{aligned} E_\psi(\varphi) - E_\psi(u) &= \int_0^1 \int_X (\varphi - u) MA_\omega((1-t)u + t\varphi) dt = \\ &= \int_0^1 \int_X (\varphi - u) MA_\omega(\max(w_t, u)) = \int_0^1 \int_{\{v > u\}} (v - u) MA_\omega(w_t) \end{aligned}$$

for $w_t := (1-t)u + tv$ for $t \in [0, 1]$, and where the last equality follows from the locality of the Monge-Ampère operator with respect to the plurifine topology.

On the other hand combining Proposition 3.3.2 with Lemma 3.2.2 and $\{w_t \leq v\} = \{u \leq v\}$ we get

$$\begin{aligned} E_\psi(v) - E_\psi(\phi) &= \int_0^1 \int_X (v - \min(u, v)) MA_\omega(P_\omega(w_t, v)) \leq \\ &\leq \int_0^1 \int_{\{v > u\}} (v - u) MA_\omega(w_t), \end{aligned}$$

which concludes the proof. □

Corollary 3.3.4. *Let $u, v, w \in \mathcal{E}^1(X, \omega, \psi)$. Then $d(u, v) \geq d(P_\omega(u, w), P_\omega(v, w))$.*

Proof. It follows from Lemma 3.3.1 .(iii) and Proposition 3.3.3 by an easy calculation (see Corollary 3.5 in [DDNL18c] for the details). □

We are now ready to prove the main theorem of this subsection:

Theorem 3.3.5. *$(\mathcal{E}^1(X, \omega, \psi), d)$ is a metric space.*

Proof. As said before, it remains only to prove the triangle inequality (see Lemma 3.3.1).

Let $u, v, w \in \mathcal{E}^1(X, \omega, \psi)$ and observe that the inequality $d(u, v) \leq d(u, w) + d(w, v)$ is equivalent to

$$E_\psi(P_\omega(u, w)) - E_\psi(P_\omega(u, v)) \leq E_\psi(w) - E_\psi(P_\omega(w, v)).$$

By Corollary 3.3.4 and using the monotonicity of the ψ -relative energy functional (Proposition 3.2.8) we get

$$\begin{aligned} E_\psi(w) - E_\psi(P_\omega(w, v)) &= d(w, P_\omega(w, v)) \geq d(P_\omega(w, u), P_\omega(w, v, u)) = \\ &= E_\psi(P_\omega(w, u)) - E_\psi(P_\omega(w, v, u)) \geq E_\psi(P_\omega(w, u)) - E_\psi(P_\omega(u, v)), \end{aligned}$$

which implies the Theorem. \square

3.3.2 Completeness of $(\mathcal{E}^1(X, \omega, \psi), d)$.

To show the completeness we first need to extend some results known in the absolute setting (i.e. if $\psi = 0$, see [BEGZ10], [DDNL18c]).

Proposition 3.3.6. *Assume $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then*

$$\begin{aligned} \frac{1}{3 \cdot 2^{n+2}(n+1)} \left(\int_X |u - v| (MA_\omega(u) + MA_\omega(v)) \right) &\leq d(u, v) \leq \\ &\leq \int_X |u - v| (MA_\omega(u) + MA_\omega(v)). \end{aligned}$$

Proof. The proof is the same as that of Theorem 3.7 in [DDNL18c] replacing their Theorem 2.1 and Lemma 3.1 by our Proposition 3.2.9 and Lemma 3.3.1. \square

Lemma 3.3.7. *There exist positive constants $A > 1, B > 0$ such that for any $u \in \mathcal{E}^1(X, \omega, \psi)$*

$$-d(\psi, u) \leq V_\psi \sup_X (u - \psi) = V_\psi \sup_X u \leq Ad(\psi, u) + B.$$

Proof. The equality follows from $u - \sup_X u \leq P_\omega[\psi] = \psi \leq 0$.

Next, if $\sup_X u \leq 0$ then the right inequality is trivial for any $A, B > 0$ while the left inequality is a consequence of $d(\psi, u) = -E_\psi(u) \geq -V_\psi \sup_X (u - \psi)$ (Proposition 3.2.9).

Therefore, let us assume $\sup_X u \geq 0$. By Proposition 2.7 in [GZ05] there exists a uniform bound $C > 0$ such that

$$\int_X |v - \sup_X v| MA_\omega(0) \leq C$$

for any $v \in PSH(X, \omega)$. Hence, since Theorem 4.2.2 gives $MA_\omega(\psi) \leq \mathbb{1}_{\{\psi=0\}} MA_\omega(0)$, we also have

$$\int_X |u - \sup_X u - \psi| MA_\omega(\psi) \leq \int_X |u - \sup_X u| MA_\omega(0) \leq C$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$. So, by Proposition 3.3.6,

$$\begin{aligned} d(u, \psi) &\geq D \int_X |u - \psi| MA_\omega(\psi) \geq \\ &\geq DV_\psi \sup_X u - D \int_X |u - \sup_X u - \psi| MA_\omega(\psi) \geq DV_\psi \sup_X u - DC. \end{aligned}$$

Take $A := 1/D$ and $B := C$ to conclude the proof. \square

Proposition 3.3.8. *Let $\{u_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^1(X, \omega, \psi)$ be an increasing sequence uniformly bounded by above, and let $u := (\lim_{j \rightarrow \infty} u_j)^* \in PSH(X, \omega)$. Then $u \in \mathcal{E}^1(X, \omega, \psi)$ and $E_\psi(u_j) \rightarrow E_\psi(u)$ as $j \rightarrow \infty$.*

Proof. Since $\sup_X u_j = \sup_X (u_j - \psi)$ is uniformly bounded, we immediately get that $u \leq \psi + C$ for a certain constant $C \in \mathbb{R}$, i.e. $u \in PSH(X, \omega, \psi)$. Furthermore since $u \geq u_j$ for any $j \in \mathbb{N}$ by construction, we also obtain $u \in \mathcal{E}^1(X, \omega, \psi)$ by [WN19] and the monotonicity of E_ψ . Thus since $E_\psi(u) = \lim_{k \rightarrow \infty} E_\psi(u^k)$, $E_\psi(u_j) = \lim_{k \rightarrow \infty} E_\psi(u_j^k)$ where $u^k := \max(u, \psi - k)$, $u_j^k := \max(u_j, \psi - k)$ are the ψ -relative canonical approximants, it is enough to check that $E_\psi(u_j^k) \searrow E_\psi(u_j)$ as $k \rightarrow \infty$ uniformly in j . Indeed this would imply

$$E_\psi(u) - E_\psi(u_j) \leq |E_\psi(u) - E_\psi(u^k)| + |E_\psi(u^k) - E_\psi(u_j^k)| + |E_\psi(u_j^k) - E_\psi(u_j)| \rightarrow 0$$

letting first $j \rightarrow \infty$ and then $k \rightarrow \infty$, since $|E_\psi(u^k) - E_\psi(u_j^k)| \rightarrow 0$ as $j \rightarrow \infty$ as a consequence of Lemma 4.1 in [DDNL18b] (see also Lemma 3.4.3 below).

Assume without loss of generality that $u \leq 0$. By Proposition 3.2.9 we have

$$0 \leq E_\psi(u_j^k) - E_\psi(u_j) \leq \int_X (u_j^k - u_j) MA_\omega(u_j) = \int_k^{+\infty} MA_\omega(u_j)(\{u_j \leq \psi - t\}) dt. \quad (3.1)$$

Next we set $v_{j,t} := \frac{u_j + \psi - t}{2}$ and we note that the following inclusions hold:

$$\{u_j \leq \psi - t\} \subset \{u_1 \leq v_{j,t}\} \subset \{u_1 \leq \psi - t/2\}.$$

Indeed the first inclusion follows from $u_1 \leq u_j$ while the last is a consequence $\sup_X u = \sup_X (u - \psi)$ (Lemma 3.3.7). Thus, by the comparison principle (Proposition 3.2.1) we have

$$\begin{aligned} MA_\omega(u_j)(\{u_j \leq \psi - t\}) &\leq MA_\omega(u_j)(\{u_1 \leq v_{j,t}\}) \leq 2^n MA_\omega(v_{j,t})(\{u_1 \leq v_{j,t}\}) \leq \\ &\leq 2^n MA_\omega(u_1)(\{u_1 \leq v_{j,t}\}) \leq 2^n MA_\omega(u_1)(\{u_1 \leq \psi - t/2\}). \end{aligned}$$

Therefore, continuing the estimates in (3.1),

$$\begin{aligned} 0 \leq E_\psi(u_j^k) - E_\psi(u_j) &\leq 2^{n+1} \int_{k/2}^{+\infty} MA_\omega(u_1)(\{u_1 \leq \psi - t\}) dt = \\ &= 2^{n+1} \int_X (u_1^{k/2} - u_1) MA_\omega(u_1), \end{aligned}$$

which concludes the proof since the right hand goes to 0 as $k \rightarrow +\infty$ (recall that $u_1 \in \mathcal{E}^1(X, \omega, \psi)$). \square

Theorem 3.3.9. *$(\mathcal{E}^1(X, \omega, \psi), d)$ is complete.*

Proof. Let $\{u_j\}_{j \in \mathbb{N}} \subset (\mathcal{E}^1(X, \omega, \psi), d)$ be a Cauchy sequence. Up to extract a subsequence we may assume that $d(u_j, u_{j+1}) \leq 2^{-j}$ for any $j \in \mathbb{N}$. Define $v_{j,k} := P_\omega(u_j, \dots, u_k)$ for $j, k \in \mathbb{N}, k \geq j$, i.e.

$$v_{j,k} = \sup\{v \in PSH(X, \omega) : v \leq \min(u_j, \dots, u_k)\}^*.$$

Clearly $v_{j,k} = P_\omega(u_j, v_{j+1,k}) \leq v_{j+1,k}$ if $k \geq j+1$ and $v_{j,k} \in \mathcal{E}^1(X, \omega, \psi)$ as consequence of Proposition 3.2.10 since $P_\omega(P_\omega(u, v), w) = P_\omega(u, v, w)$ for any $u, v, w \in PSH(X, \omega)$. Thus for any $k \geq j+1$

$$d(u_j, v_{j,k}) = d(u_j, P_\omega(u_j, v_{j+1,k})) \leq d(u_j, v_{j+1,k}) \leq d(u_j, u_{j+1}) + d(u_{j+1}, v_{j+1,k})$$

using Lemma 3.3.1. Iterating the argument we get

$$d(u_j, v_{j,k}) \leq \sum_{s=1}^{k-j} d(u_{j+s-1}, u_{j+s}) \leq \sum_{s=1}^{k-j} \frac{1}{2^{j+s-1}} \leq \sum_{s=j}^{\infty} \frac{1}{2^s} = \frac{1}{2^{j-1}}.$$

Moreover, since $v_{j,k}$ is decreasing in k , there exists a constant $C_j \in \mathbb{R}$ such that $v_{j,k} \leq \psi + C_j$ for any $k \geq j$. So

$$C_j - E_\psi(v_{j,k}) = d(\psi + C_j, v_{j,k}) \leq d(\psi + C_j, u_j) + d(u_j, v_{j,k}) \leq d(\psi + C_j, u_j) + 2^{-j+1},$$

which implies that $v_j := \lim_{k \rightarrow \infty} v_{j,k} \in \mathcal{E}^1(X, \omega, \psi)$ by Proposition 3.2.8, and $d(v_j, u_j) \leq 2^{-j+1}$ by continuity of the distance along decreasing sequences.

Next we observe that v_j is increasing in j and that

$$\begin{aligned} V_\psi \sup_X (v_j - \psi) &= V_\psi \sup_X v_j \leq Ad(\psi, v_j) + B \leq \\ &\leq A \left(d(\psi, u_1) + \sum_{s=1}^{j-1} d(u_s, u_{s+1}) + d(u_j, v_j) \right) + B \leq Ad(\psi, u_1) + 3A + B \end{aligned}$$

where the first inequality is the content of Lemma 3.3.7. Hence Proposition 3.3.8 leads to $u := (\lim_{j \rightarrow \infty} v_j)^* \in \mathcal{E}^1(X, \omega, \psi)$ and to

$$d(u_j, u) \leq d(u_j, v_j) + d(v_j, u) \leq 2^{-j+1} + d(v_j, u) \rightarrow 0$$

for $j \rightarrow \infty$. \square

Remark 3.3.10. In the case $\psi \in \mathcal{M} \setminus \mathcal{M}^+$, i.e. if ψ is a model type envelope with zero mass $V_\psi = 0$, then

$$PSH(X, \omega, \psi) = \mathcal{E}(X, \omega, \psi) = \mathcal{E}^1(X, \omega, \psi).$$

Indeed any function $u \in PSH(X, \omega, \psi)$ has zero mass $V_u = 0$ ([WN19]) and $E_\psi(u) = 0$ since for any ϕ with ψ -minimal singularities $|E_\psi(\phi)| \leq V_\psi \sup_X |\phi - \psi| = 0$. In particular $d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v)) = 0$ for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Moreover if $P_\omega(\psi_1, \psi_2) \not\equiv -\infty$ for ψ_1, ψ_2 model type envelopes with zero masses $V_{\psi_1} = V_{\psi_2} = 0$ then $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \mathcal{E}^1(X, \omega, P_\omega[\psi_1, \psi_2])$ is not empty.

3.4 The metric space $(X_{\mathcal{A}}, d_{\mathcal{A}})$ and consequences.

In this section we will prove the main Theorem B, i.e. assuming $\mathcal{A} \subset \mathcal{M}^+$ total ordered subset (recall the the partial order \preccurlyeq coincides with the order \leq on \mathcal{M}), we will endow the space $X_{\mathcal{A}} := \bigcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$ with a metric topology. Here $\overline{\mathcal{A}}$ denotes the closure of \mathcal{A} as subset of $PSH(X, \omega)$ with its L^1 -topology.

We will show that $\overline{\mathcal{A}} \subset \mathcal{M}$ and we will define a natural distance $d_{\mathcal{A}}$ on $X_{\mathcal{A}}$ which extends the distance d (Theorem A) on $\mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$ where if $\psi = \inf \mathcal{A} \in \mathcal{M} \setminus \mathcal{M}^+$ we identify the space $\mathcal{E}^1(X, \omega, \psi)$ with a point since in this case necessarily $d_{\mathcal{A}} = \tilde{d} \equiv 0$ (Remark 3.3.10).

We recall that the distance d on $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ is defined as

$$d(u, v) = E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v)).$$

Definition 3.4.1. Given $\psi \in \mathcal{M}^+$, the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ is defined as the metric topology given by the distance d .

In the case $\psi = 0$ the strong topology was introduced in [BBEGZ19] (Definition 2.1.), see also Proposition 5.9 and Theorem 5.5. in [Dar15].

The following Lemmas regarding the weak convergence of Monge-Ampère measures for functions belonging in different \mathcal{E}^1 -spaces will be essential in the sequel.

Lemma 3.4.2. Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ be a monotone sequence converging a.e. to $\psi \in PSH(X, \omega)$. Then $\psi \in \mathcal{M}$ and $MA_\omega(\psi_k) \rightarrow MA_\omega(\psi)$ weakly.

Proof. Assume first $\psi_k \searrow \psi$.

Since $\sup_X \psi_k = 0$ for any $k \in \mathbb{N}$ we immediately obtain $\psi \leq P_\omega[\psi] =: \tilde{\psi}$ which implies $\psi = P_\omega[\psi]$ since clearly $\tilde{\psi} \leq P_\omega[\psi_k]$ for any $k \in \mathbb{N}$. For the second statement, we first observe that

$$\int_X MA_\omega(\psi) \geq \limsup_{k \rightarrow \infty} \int_{\{\psi \geq -C\}} MA_\omega(\psi_k) \quad (3.2)$$

for any $C \in \mathbb{R}$ fixed since $MA_\omega(\psi_j) \rightarrow MA_\omega(\psi)$ weakly in the plurifine topology over $\{\psi > -\infty\}$ and $\{\psi \geq -C\}$ is a plurifine closed set (see [BT82] and [BT87]). On the other hand, by Theorem 4.2.2 $MA_\omega(\psi_k) \leq \mathbb{1}_{\{\psi_k=0\}} MA_\omega(0)$ for any $k \in \mathbb{N}$. Thus for any $C \geq 0$

$$\limsup_{k \rightarrow \infty} \int_{\{\psi < -C\}} MA_\omega(\psi_k) \leq \limsup_{k \rightarrow \infty} \int_{\{\psi < -C\} \cap \{\psi_k=0\}} MA_\omega(0) = 0, \quad (3.3)$$

where the last equality follows from $\bigcap_{k \in \mathbb{N}} \{\psi_k = 0\} = \{\psi = 0\}$ since $\psi_k \searrow \psi$ and $\psi, \psi_k \leq 0$. Hence combining (3.2) and (3.3) we obtain

$$\int_X MA_\omega(\psi) \geq \limsup_{k \rightarrow \infty} \int_X MA_\omega(\psi_k),$$

and Theorem 3.2.4 implies $MA_\omega(\psi_k) \rightarrow MA_\omega(\psi)$ weakly.

Assume now $\psi_k \nearrow \psi$ almost everywhere.

Again by Theorem 3.2.4 we immediately get $MA_\omega(\psi_k) \rightarrow MA_\omega(\psi)$ weakly since $\psi_k \rightarrow \psi$ in capacity. Moreover, similarly as before, $\psi \leq P_\omega[\psi]$. Thus to conclude the proof it remains to prove that $\psi \geq P_\omega[\psi]$.

We note that $MA_\omega(\psi) \leq \mathbb{1}_{\{\psi=0\}} MA_\omega(0)$ since $MA_\omega(\psi_k) \leq \mathbb{1}_{\{\psi_k=0\}} MA_\omega(0)$ for any $k \in \mathbb{N}$ (Theorem 4.2.2). Therefore

$$0 \leq \int_X (P_\omega[\psi] - \psi) MA_\omega(\psi) \leq \int_{\{\psi=0\}} (P_\omega[\psi] - \psi) MA_\omega(0) = 0$$

where the last equality follows from $\psi \leq P_\omega[\psi] \leq 0$. Hence by the domination principle (Proposition 3.11 in [DDNL18b]) we conclude that $P_\omega[\psi] \leq \psi$, i.e. $\psi \in \mathcal{M}$. \square

As a consequence of Lemma 3.4.2 we get that $\overline{\mathcal{A}} \subset \mathcal{M}$. Indeed since \mathcal{A} is totally ordered, any Cauchy sequence $\{\psi_k\}_{k \in \mathbb{N}}$ admits a subsequence monotonically converging a.e. to $(\lim_{k \rightarrow \infty} \psi_k)^*$. We also note that $\overline{\mathcal{A}}$ remains totally ordered.

Lemma 3.4.3. *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ total ordered such that $\psi_k \rightarrow \psi \in \mathcal{M}$ monotonically almost everywhere. Let also $u_1, u_2 \in \mathcal{E}^1(X, \omega, \psi)$, and let $\{u_{1,k}, u_{2,k}\}_{k \in \mathbb{N}}$ be two sequences converging in capacity respectively to u_1, u_2 such that $u_{1,k}, u_{2,k} \in \mathcal{E}^1(X, \omega, \psi_k)$. Then for any $j = 0, \dots, n$,*

$$MA_\omega(u_{1,k}^j, u_{2,k}^{n-j}) \rightarrow MA_\omega(u_1^j, u_2^{n-j})$$

weakly. Moreover if $u_{1,k} - u_{2,k}$ is uniformly bounded then, for any $j = 0, \dots, n$,

$$(u_{1,k} - u_{2,k}) MA_\omega(u_{1,k}^j, u_{2,k}^{n-j}) \rightarrow (u_1 - u_2) MA_\omega(u_1^j, u_2^{n-j}) \quad (3.4)$$

weakly. In particular if either $u_{1,k} \searrow u_1, u_{2,k} \searrow u_2$ a.e. or $u_{1,k} \nearrow u_1, u_{2,k} \nearrow u_2$ a.e. and $u_{1,k} - u_{2,k}$ uniformly bounded, then

$$d(u_{1,k}, u_{2,k}) \rightarrow d(u_1, u_2).$$

Proof. $MA_\omega(u_{1,k}^j, u_{2,k}^{n-j}) \rightarrow MA_\omega(u_1^j, u_2^{n-j})$ weakly as a consequence of Theorem 3.2.4 and Lemma 3.4.2. Thus the proof of the weak convergence in (3.4) is an adaptation of the proof of Lemma 4.1 in [DDNL18b] (and it is a particular case of Theorem 2.2. in [X19a]).

Next assume that $u_{1,k}, u_{2,k}$ converge monotonically almost everywhere to u_1, u_2 as in the statement. Thus to prove that $d(u_{1,k}, u_{2,k}) \rightarrow d(u_1, u_2)$ we only need to show that $P_\omega(u_{1,k}, u_{2,k}) \rightarrow P_\omega(u_1, u_2)$ almost everywhere since clearly $u_{i,k} - P_\omega(u_{1,k}, u_{2,k})$ is uniformly bounded. From $P_\omega(u_{1,k}, u_{1,k}) \leq u_{1,k}, u_{2,k}$ we immediately have $(\lim_{k \rightarrow \infty} P_\omega(u_{1,k}, u_{2,k}))^* \leq P_\omega(u_1, u_2)$. Therefore if the convergence is decreasing then $P_\omega(u_1, u_2) \leq P_\omega(u_{1,k}, u_{2,k})$ and we get the convergence of the d distances. If instead the convergence is increasing then, setting $\phi := (\lim_{k \rightarrow \infty} P_\omega(u_{1,k}, u_{2,k}))^*$, we observe that $\phi \in \mathcal{E}(X, \omega, \psi)$ and that $MA_\omega(P_\omega(u_{1,k}, u_{2,k})) \rightarrow MA_\omega(\phi)$ weakly as a consequence of Theorem 3.2.4 and Lemma 3.4.2. Moreover since by Lemma 3.2.2

$$\begin{aligned} MA_\omega(P_\omega(u_{1,k}, u_{2,k})) &\leq \mathbb{1}_{\{\phi \geq u_{1,k}\}} MA_\omega(u_{1,k}) + \mathbb{1}_{\{\phi \geq u_{2,k}\}} MA_\omega(u_{2,k}) \leq \\ &\leq \mathbb{1}_{\{\phi \geq u_{1,j}\}} MA_\omega(u_{1,k}) + \mathbb{1}_{\{\phi \geq u_{2,j}\}} MA_\omega(u_{2,k}) \end{aligned}$$

for any $j \leq k$, and $MA_\omega(u_{i,k}) \rightarrow MA_\omega(u_i)$ weakly for $i = 1, 2$, we get

$$MA_\omega(\phi) \leq \mathbb{1}_{\{\phi \geq u_{1,j}\}} MA_\omega(u_1) + \mathbb{1}_{\{\phi \geq u_{2,j}\}} MA_\omega(u_2).$$

Therefore letting $j \rightarrow \infty$ we obtain

$$\begin{aligned} 0 &\leq \int_X (P_\omega(u_1, u_2) - \phi) MA_\omega(\phi) \leq \int_{\{\phi \geq u_1\}} (P_\omega(u_1, u_2) - u_1) MA_\omega(u_1) + \\ &\quad + \int_{\{\phi \geq u_2\}} (P_\omega(u_1, u_2) - u_2) MA_\omega(u_2) \leq 0, \end{aligned}$$

which by the domination principle of Proposition 3.11. in [DDNL18b] implies $P_\omega(u_1, u_2) \leq \phi$. Hence $P_\omega(u_1, u_2) = \phi$ which as said above concludes the proof. \square

3.4.1 The contraction property of d .

Lemma 3.4.4. *Let $\psi, \psi_1, \psi_2 \in \mathcal{M}$ such that $\psi_2 \preceq \psi_1 \preceq \psi$. Then:*

- i) $P_\omega[\psi_2](P_\omega[\psi_1](u)) = P_\omega[\psi_2](u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$;
- ii) $P_\omega[\psi_1](\mathcal{E}^1(X, \omega, \psi)) \subset \mathcal{E}^1(X, \omega, \psi_1)$;
- iii) for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ such that $u - v$ is globally bounded, $\|P_\omega[\psi_1](u) - P_\omega[\psi_2](v)\|_{L^\infty} \leq \|u - v\|_{L^\infty}$ and in particular $P_\omega[\psi_1](u)$ has ψ_1 -relative minimal singularities for any $u \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities.

Proof. The inequality $P_\omega[\psi_1](u) \leq u$ immediately implies $P_\omega[\psi_2](P_\omega[\psi_1](u)) \leq P_\omega[\psi_2](u)$. Vice versa $P_\omega[\psi_2](u) \leq P_\omega[\psi_1](u)$ since $\psi_2 \preceq \psi_1$. Thus the first point

follows applying $P_\omega[\psi_2](\cdot)$ to both sides.

For the third statement, letting $C := \|u - v\|_{L^\infty}$, it is an easy consequence of the definition that

$$P_\omega[\psi_1](u) \geq P_\omega[\psi_1](v - C) = P_\omega[\psi_1](v) - C.$$

By symmetry, we also have $P_\omega[\psi_1](u) \leq P_\omega[\psi_1](v) + C$ which gives (iii). This immediately yields (ii) if $\psi_1 \in \mathcal{M} \setminus \mathcal{M}^+$. Therefore it remains to prove (ii) assuming $\psi_1 \in \mathcal{M}^+$. Letting $u_j := \max(u, \psi - j)$ be the ψ -relative canonical approximants of a generic $u \in \mathcal{E}^1(X, \omega, \psi)$, we get that $v := P_\omega[\psi_1](u)$ belongs to $\mathcal{E}^1(X, \omega, \psi_1)$ if and only if $\int_X (\psi_1 - v_j) MA_\omega(v_j)$ is uniformly bounded in j where $v_j := P_\omega[\psi_1](u_j)$ (see Proposition 3.2.8). But taking $D := \sup_X u > 0$ and using Theorem 4.2.2 we get

$$\begin{aligned} DV_{\psi_1} + \int_X (\psi_1 - v_j) MA_\omega(v_j) &\leq \int_{\{v_j = u_j\}} (\psi + D - u_j) MA_\omega(u_j) \leq \\ &\leq DV_\psi + \int_X (\psi - u_j) MA_\omega(u_j) < E \in \mathbb{R} \end{aligned}$$

for an uniform $E \in \mathbb{R}$ since $u \in \mathcal{E}^1(X, \omega, \psi)$ and $u_j \searrow u$. □

We are now ready to prove the following key property of the distance d .

Proposition 3.4.5. *Let $\psi, \psi' \in \mathcal{M}$ such that $\psi' \preceq \psi$. Then the map*

$$P_\omega[\psi'](\cdot) : (\mathcal{E}^1(X, \omega, \psi), d) \rightarrow (\mathcal{E}^1(X, \omega, \psi'), d)$$

is a Lipschitz map of Lipschitz constant equal to 1, i.e.

$$d(u, v) \geq d(P_\omega[\psi'](u), P_\omega[\psi'](v))$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$.

Proof. Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Set

$$\rho(u, v) := \int_X (u - v) MA_\omega(v).$$

if $u \geq v$ and $\rho(u, v) := \rho(v, u)$ if $v \geq u$. Proposition 3.2.9 implies $d(u, v) \leq \rho(u, v)$. Moreover assuming $\psi' \in \mathcal{M}^+$ such that $\psi' \preceq \psi$ as in the statement of the Proposition,

$$\begin{aligned} \rho(P_\omega[\psi'](u), P_\omega[\psi'](v)) &= \int_X (P_\omega[\psi'](u) - P_\omega[\psi'](v)) MA_\omega(P_\omega[\psi'](v)) \leq \\ &\leq \int_{\{P_\omega[\psi'](v) = v\}} (u - v) MA_\omega(v) \leq \rho(u, v) \end{aligned}$$

by Theorem 4.2.2. Therefore we define for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$

$$\tilde{d}(u, v) := \inf \left(\rho(u, w_1) + \sum_{j=1}^{m-1} \rho(w_j, w_{j+1}) + \rho(w_m, v) \right)$$

where the infimum is over all chain $\mathcal{C} = \{u = w_0, w_1, \dots, w_m, w_{m+1} = v\}$ for any $m \in \mathbb{N}$ such that any pair of consecutive elements in the chain w_j, w_{j+1} satisfies $w_j \geq w_{j+1}$ or $w_j \leq w_{j+1}$.

Clearly $\tilde{d}(u, v) \geq d(u, v)$ and it is straightforward to check that \tilde{d} is symmetric and that it satisfies the triangle inequality. Moreover by construction and Lemma 3.4.4 we also have $\tilde{d}(P_\omega[\psi'](u), P_\omega[\psi'](v)) \leq \tilde{d}(u, v)$ since ρ has the same property and $P_\omega[\psi'](u) \leq P_\omega[\psi'](v)$ if $u \leq v$. Thus to conclude the proof it remains to prove that $\tilde{d} \leq d$, which would imply $\tilde{d} = d$.

We first observe that it is enough to show that $\tilde{d}(u, v) \leq d(u, v)$ assuming $u \geq v$ since it would lead to

$$\begin{aligned} \tilde{d}(w_1, w_2) &\leq \tilde{d}(w_1, P_\omega(w_1, w_2)) + \tilde{d}(w_2, P_\omega(w_1, w_2)) \leq \\ &\leq d(w_1, P_\omega(w_1, w_2)) + d(w_2, P_\omega(w_1, w_2)) = d(w_1, w_2) \end{aligned}$$

by Lemma 3.3.1.(iv). Therefore let $u \geq v$, fix $N \in \mathbb{N}$ and set $w_{j,N} := \frac{j}{N}u + \frac{N-j}{N}v$ for $j = 0, \dots, N$. Then $\mathcal{C}_N := \{w_{0,N}, \dots, w_{N,N}\}$ is an admissible chain for the definition of $\tilde{d}(u, v)$. So

$$\begin{aligned} \tilde{d}(u, v) &\leq \sum_{j=0}^{N-1} \rho(w_{j,N}, w_{j+1,N}) = \sum_{j=0}^{N-1} \frac{1}{N} \int_X (u - v) M A_\omega(w_{j,N}) = \\ &= \sum_{s=0}^n \binom{n}{s} \left(\frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{j}{N} \right)^s \left(\frac{N-j}{N} \right)^{n-s} \right) \int_X (u - v) M A_\omega(u^s, v^{n-s}). \end{aligned}$$

Next by Lemma 3.4.6 below for any $s = 0, \dots, n$,

$$\frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{j}{N} \right)^s \left(\frac{N-j}{N} \right)^{n-s} \rightarrow \frac{1}{\binom{n}{s}(n+1)},$$

as $N \rightarrow \infty$. Hence we get

$$\tilde{d}(u, v) \leq \frac{1}{n+1} \sum_{s=0}^n \int_X (u - v) M A_\omega(u^s, v^{n-s}) = d(u, v),$$

which concludes the proof. \square

Lemma 3.4.6. *Let $n, N \in \mathbb{N}$ and let s be a non negative integer such that $s \leq n$. Then*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{j}{N} \right)^s \left(\frac{N-j}{N} \right)^{n-s} = \frac{1}{\binom{n}{s}(n+1)} \quad (3.5)$$

Proof. Consider the function $f : [0, 1] \rightarrow \mathbb{R}, x \rightarrow x^{n-s}(1-x)^s$, it is immediate to see that the sequence in (3.5) is the upper Darboux sum of f with respect to the partition $0 < \frac{1}{N} < \dots < \frac{j}{N} < \dots < 1$. Thus

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j=0}^{N-1} \left(\frac{j}{N}\right)^s \left(\frac{N-j}{N}\right)^{n-s} = \int_0^1 x^{n-s}(1-x)^s dx.$$

A brief calculation shows that $\int_0^1 x^{n-s}(1-x)^s dx = \frac{1}{\binom{n}{s}(n+1)}$. □

The contraction property showed above implies an uniform convergence on some compact sets.

Proposition 3.4.7. *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ be a sequence monotonically converging to $\psi \in \mathcal{M}$ almost everywhere. Then for any $\psi' \in \mathcal{M}$ such that $\psi' \succcurlyeq \psi_k$ for any $k \geq k_0$ big enough and for any compact set $\tilde{K} \subset \mathcal{E}^1(X, \omega, \psi')$ with respect to the strong topology on $\mathcal{E}^1(X, \omega, \psi')$, the sets $K := P_\omega[\psi](\tilde{K}) \subset (\mathcal{E}^1(X, \omega, \psi), d)$, $K_k := P_\omega[\psi_k](\tilde{K}) \subset (\mathcal{E}^1(X, \omega, \psi_k), d)$ are compact in their respective strong topologies for any $k \geq k_0$, and*

$$d(P_\omega[\psi_k](\varphi_1), P_\omega[\psi_k](\varphi_2)) \rightarrow d(P_\omega[\psi](\varphi_1), P_\omega[\psi](\varphi_2))$$

uniformly on $\tilde{K} \times \tilde{K}$, i.e. varying $(\varphi_1, \varphi_2) \in \tilde{K} \times \tilde{K}$.

Proof. It follows from Lemma 3.4.4 and Proposition 3.4.5 that $P_\omega[\psi_k](\tilde{K})$ is compact in $(\mathcal{E}^1(X, \omega, \psi_k), d)$ for any $k \in \mathbb{N}$, and similarly for K .

Next, we define $f_k : \tilde{K} \times \tilde{K} \rightarrow \mathbb{R}_{\geq 0}$ for $k \in \mathbb{N}$ and $f : \tilde{K} \times \tilde{K} \rightarrow \mathbb{R}_{\geq 0}$ respectively as

$$\begin{aligned} f_k(\varphi_1, \varphi_2) &:= d(P_\omega[\psi_k](\varphi_1), P_\omega[\psi_k](\varphi_2)) \\ f(\varphi_1, \varphi_2) &:= d(P_\omega[\psi](\varphi_1), P_\omega[\psi](\varphi_2)). \end{aligned}$$

We observe that f_k, f are Lipschitz continuous with respect to the strong topology on $\tilde{K} \times \tilde{K}$ (Proposition 3.4.5). Moreover by Lemma 3.4.3 $f_k \rightarrow f$ pointwise on a dense subset of $\tilde{K} \times \tilde{K}$ and $\{f_k\}_{k \in \mathbb{N}}$ is a monotone sequence. Hence Dini's Theorem implies that $f_k \rightarrow f$ uniformly on $\tilde{K} \times \tilde{K}$. □

3.4.2 The metric space $(\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi), d_A)$.

Definition 3.4.8. *Let $\psi \in \mathcal{M}$. We introduce the set*

$$\mathcal{P}_{\mathcal{H}}(X, \omega, \psi) := \{P_\omega[\psi](\varphi) : \varphi \in \mathcal{H}\}$$

where $\mathcal{H} := \{\varphi \in PSH(X, \omega) : \omega + dd^c \varphi \text{ Kähler form}\}$.

Observe that by Lemma 3.4.4 any $u \in \mathcal{P}_{\mathcal{H}}(X, \omega, \psi)$ has ψ -relative minimal singularities. This smaller set is dense in $(\mathcal{E}^1(X, \omega, \psi), d)$ as the next result shows:

Lemma 3.4.9. *Let $\psi \in \mathcal{M}$. Then $\mathcal{P}_{\mathcal{H}}(X, \omega, \psi)$ is dense in $\mathcal{E}^1(X, \omega, \psi)$ with respect to the strong topology.*

Proof. We can assume $\psi \in \mathcal{M}^+$, otherwise it is trivial. Let $u \in \mathcal{E}^1(X, \omega, \psi)$. We first observe that $v_j := P_{\omega}[\psi](\max(u, -j))$ belongs to $\mathcal{E}^1(X, \omega, \psi)$ and it has ψ -relative minimal singularities by Lemma 3.4.4. Moreover since

$$u = P_{\omega}[\psi](u) \leq v_j \leq \max(u, -j)$$

we also get that $v_j \searrow u$. Therefore $d(u, v_j) \rightarrow 0$ for $j \rightarrow \infty$ since d is continuous along decreasing sequences. Next, by density of \mathcal{H} into $\mathcal{E}^1(X, \omega) := \mathcal{E}^1(X, \omega, 0)$, for any $j \in \mathbb{N}$ there exists $\varphi_j \in \mathcal{H}$ such that $d(\varphi_j, \max(u, -j)) \leq 1/j$. Therefore, letting $w_j := P_{\omega}[\psi](\varphi_j) \in \mathcal{P}_{\mathcal{H}}(X, \omega, \psi)$, by Proposition 3.4.5 it follows that

$$d(w_j, u) \leq d(w_j, v_j) + d(v_j, u) \leq d(u, v_j) + \frac{1}{j} \rightarrow 0$$

as $j \rightarrow \infty$, which concludes the proof. \square

Remark 3.4.10. By the main Theorem in [Mc18], if the singularities of ψ are *analytic*, i.e. $\psi = P_{\omega}[u]$ for u with analytic singularities of type \mathfrak{a}^c for an analytic coherent ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$, $c \in \mathbb{R}_{>0}$, then any function $v \in \mathcal{P}_{\mathcal{H}}(X, \omega, \psi)$ is $C_{loc}^{1,1}(X \setminus V(\mathfrak{a}))$.

We need now to recall the definition of the *entropy*.

Definition 3.4.11. [Definition 2.9., [BBEGZ19]] *Let μ, ν two probability measures on X . The relative entropy $H_{\mu}(\nu) \in [0, +\infty]$ of ν with respect to μ is defined as follows. If ν is absolutely continuous with respect to μ and $f := \frac{d\nu}{d\mu}$ satisfies $f \log f \in L^1(\mu)$ then*

$$H_{\mu}(\nu) := \int_X f \log f d\mu = \int_X \log \left(\frac{d\nu}{d\mu} \right) d\nu.$$

Otherwise $H_{\mu}(\nu) := +\infty$.

The relative entropy provides compact sets in $\mathcal{E}^1(X, \omega)$ endowed with the strong topology (Definition 3.4.1).

Theorem 3.4.12. [Theorem 2.17., [BBEGZ19]] *Let C be a positive constant. Then the set*

$$\mathcal{K}_C := \left\{ \varphi \in \mathcal{E}^1(X, \omega) : \max \left(\left| \sup_X \varphi \right|, H_{MA_{\omega}(0)/V_0}(MA_{\omega}(\varphi)/V_0) \right) \leq C \right\}$$

is compact in $\mathcal{E}^1(X, \omega)$ with respect to the strong topology.

Definition 3.4.13. Given $\psi \in \mathcal{A}$, we define for any $C \geq 0$

$$\begin{aligned} \mathcal{P}_C(X, \omega, \psi) &:= P_\omega[\psi](K_C) = \\ &= \left\{ P_\omega[\psi](\varphi) \in \mathcal{E}^1(X, \omega, \psi) : \max(|\sup_X \varphi|, H_{MA_\omega(0)}(MA_\omega(\varphi))) \leq C \right\}, \end{aligned}$$

and

$$\mathcal{P}(X, \omega, \psi) := \bigcup_{C \in \mathbb{R}_{\geq 0}} \mathcal{P}_C(X, \omega, \psi).$$

As a consequence of Theorem 3.4.12 and Proposition 3.4.5 $\mathcal{P}_C(X, \omega, \psi)$ is compact in $(\mathcal{E}^1(X, \omega, \psi), d)$ and $\mathcal{P}_{C_1}(X, \omega, \psi) \subset \mathcal{P}_{C_2}(X, \omega, \psi)$ if $C_1 \leq C_2$. Moreover since $H_{MA_\omega(0)}(MA_\omega(\varphi)) < \infty$ for any $\varphi \in \mathcal{H}$, $\mathcal{P}_{\mathcal{H}}(X, \omega, \psi) = \bigcup_{C \in \mathbb{R}_{\geq 0}} \mathcal{P}_C[\psi](\mathcal{K}_C \cap \mathcal{H}) \subset \mathcal{P}(X, \omega, \psi)$. It is also clear that for any $u \in \mathcal{P}(X, \omega, \psi)$ there exists $C \in \mathbb{R}_{\geq 0}$ minimal such that $u \in \mathcal{P}_C(X, \omega, \psi)$. We set $\mathcal{P}(X, \omega) := \mathcal{P}(X, \omega, 0)$ and we call $\varphi \in \mathcal{P}(X, \omega)$ a *minimal entropy function* for $u \in \mathcal{P}(X, \omega, \psi)$ if $\varphi \in \mathcal{K}_C$ and $P_\omega[\psi](\varphi) = u$ for C minimal.

Definition 3.4.14. Let $u \in \mathcal{P}_{C_1}(X, \omega, \psi_1), v \in \mathcal{P}_{C_2}(X, \omega, \psi_2)$ for $\psi_1, \psi_2 \in \mathcal{A}$ such that $\psi_2 \preccurlyeq \psi_1$. Assume also that C_1 (respectively C_2) is minimal such that $u \in \mathcal{P}_{C_1}(X, \omega, \psi_1)$ (resp. $v \in \mathcal{P}_{C_2}(X, \omega, \psi_2)$). We define

$$\tilde{d}_{\mathcal{A}}(u, v) := d(v, P_\omega[\psi_2](u)) + \sup \left\{ d(a, b) - d(P_\omega[\psi_2](a), P_\omega[\psi_2](b)) \right\} + V_u - V_v$$

where the supremum is over all $a, b \in \mathcal{P}_{\max(C_1, C_2)}(X, \omega, \psi_1)$.

We observe that $\tilde{d}_{\mathcal{A}}$ takes finite values since the supremum in the definition is actually equal to

$$\max_{(\varphi_1, \varphi_2) \in \mathcal{K}_{\max(C_1, C_2)} \times \mathcal{K}_{\max(C_1, C_2)}} \left\{ d(P_\omega[\psi_1](\varphi_1), P_\omega[\psi_1](\varphi_2)) - d(P_\omega[\psi_2](\varphi_1), P_\omega[\psi_2](\varphi_2)) \right\}.$$

Proposition 3.4.15. Let $u \in \mathcal{P}(X, \omega, \psi_1), v \in \mathcal{P}(X, \omega, \psi_2)$ for $\psi_1, \psi_2 \in \mathcal{A}$ such that $\psi_2 \preccurlyeq \psi_1$. Then the followings hold:

- i) $\tilde{d}_{\mathcal{A}}(u, v) = \tilde{d}_{\mathcal{A}}(v, u)$;
- ii) $\tilde{d}_{\mathcal{A}}(u, v) \in \mathbb{R}_{\geq 0}$ and $\tilde{d}_{\mathcal{A}}(u, v) = 0$ if and only if $u = v$;
- iii) if $\psi_1 = \psi_2$ then $\tilde{d}_{\mathcal{A}}(u, v) = d(u, v)$;
- iv) $\tilde{d}_{\mathcal{A}}(u, v) \geq d(v, P_\omega[\psi_2](u))$ and $\tilde{d}_{\mathcal{A}}(u, v) \geq d(u, P_\omega[\psi_1](\varphi))$ where $\varphi \in \mathcal{P}(X, \omega)$ is a minimal entropy function for v .

Proof. The first point is trivial. By Proposition 3.4.5 and the main Theorem in [WN19] $\tilde{d}_{\mathcal{A}} \in \mathbb{R}_{\geq 0}$, and if $\psi_1 = \psi_2$ then $\tilde{d}_{\mathcal{A}}(u, v) = d(u, v)$. For (iv), instead, the first inequality is immediate, while the second inequality follows considering $a = u, b = P_\omega[\psi_1](\varphi)$ in the supremum.

Therefore it remains to prove that $\tilde{d}_{\mathcal{A}}(u, v) = 0$ implies $u = v$. But if $\tilde{d}_{\mathcal{A}}(u, v) = 0$ then in particular $V_{\psi_1} = V_{\psi_2}$, and Theorem 3.2.5 implies $\psi_1 = \psi_2$. Hence the third point and Theorem A conclude the proof. \square

The map $\tilde{d}_{\mathcal{A}}$ does not seem to be a distance on $\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi)$, since it hardly satisfies the triangle inequality. Indeed it is composed of three parts, and clearly two parts behaves well for the triangle inequality, but the part given by the supremum seem to be very unstable since the set of the supremum depends on the functions u, v taken. Therefore we want to modify $\tilde{d}_{\mathcal{A}}$ to get a distance $d_{\mathcal{A}}$ which still coincides with the d -distance on $\mathcal{P}(X, \omega, \psi)$ for any $\psi \in \mathcal{A}$. The next Lemma is the key point to proceed.

Lemma 3.4.16. *Let $u, v \in \mathcal{P}(X, \omega, \psi)$ for $\psi \in \mathcal{A}$. Then for any $m \in \mathbb{N}$ and any $w_1, \dots, w_m \in \bigsqcup_{\psi' \in \mathcal{A}} \mathcal{P}(X, \omega, \psi')$,*

$$d(u, v) \leq \tilde{d}_{\mathcal{A}}(u, w_1) + \sum_{j=1}^{m-1} \tilde{d}_{\mathcal{A}}(w_j, w_{j+1}) + \tilde{d}_{\mathcal{A}}(w_m, v).$$

The proof of this Lemma is quite laborious and it will be presented in the subsection 3.4.3.

Next we define $d_{\mathcal{A}} : \bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi) \times \bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi) \rightarrow \mathbb{R}_{\geq 0}$ as

$$d_{\mathcal{A}}(u, v) := \inf \left\{ \tilde{d}_{\mathcal{A}}(u, w_1) + \sum_{j=1}^{m-1} \tilde{d}_{\mathcal{A}}(w_j, w_{j+1}) + \tilde{d}_{\mathcal{A}}(w_m, v) \right\}$$

where the infimum is over all possible chains in $\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi)$.

We can now prove Theorem B:

Theorem B. *$\left(\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi), d_{\mathcal{A}} \right)$ is a metric space, and denoting with $X_{\mathcal{A}}$ its metric completion, we have*

$$X_{\mathcal{A}} = \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

where $\overline{\mathcal{A}} \subset \mathcal{M}$ is the closure of \mathcal{A} as subset of $PSH(X, \omega)$ with its L^1 -topology and where we identify $\mathcal{E}^1(X, \omega, \psi')$ with a singleton $P_{\psi'}$ if $\psi' := \inf \mathcal{A} \in \mathcal{M} \setminus \mathcal{M}^+$. In particular the complete metric space $(X_{\mathcal{A}}, d_{\mathcal{A}})$ restricts to $(\mathcal{E}^1(X, \omega, \psi), d)$ on $\mathcal{E}^1(X, \omega, \psi)$ for any $\psi \in \overline{\mathcal{A}}$.

Proof. Step 1: $\left(\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi), d_{\mathcal{A}} \right)$ is a metric space.

As a consequence of Lemma 3.4.16 we immediately get

$$d_{\mathcal{A}}|_{\mathcal{P}(X, \omega, \psi) \times \mathcal{P}(X, \omega, \psi)} = d$$

for any $\psi \in \mathcal{A}$. Therefore to prove that $d_{\mathcal{A}}$ is a distance on $\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi)$ it remains to prove that $d_{\mathcal{A}}(u, v) = 0$ implies $u = v$ since the triangle inequality easily follows from the construction (see also Proposition 3.4.15). But given $w_1, \dots, w_m \in \bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi)$, the uniform bound

$$\tilde{d}_{\mathcal{A}}(u, w_1) + \tilde{d}_{\mathcal{A}}(w_1, w_2) + \dots + \tilde{d}_{\mathcal{A}}(w_m, v) \geq |V_u - V_v|$$

holds. Therefore $d_{\mathcal{A}}(u, v) = 0$ leads to $V_u = V_v$, and since \mathcal{A} is totally ordered, by Theorem 3.2.5, we obtain that $u, v \in \mathcal{E}^1(X, \omega, \psi)$ for a common $\psi \in \mathcal{A}$. Hence $0 = d_{\mathcal{A}}(u, v) = d(u, v)$, which implies $u = v$ and concludes the first step.

Step 2: $\left(\bigsqcup_{\psi \in \overline{\mathcal{A}}} (\mathcal{E}^1(X, \omega, \psi), d) \right) \subset (X_{\mathcal{A}}, d_{\mathcal{A}})$.

For any $\psi \in (\overline{\mathcal{A}} \setminus \mathcal{A})$ there exists a monotone sequence $\{\psi_k\}_{k \in \mathbb{N}}$ such that $\psi = (\lim_{k \rightarrow \infty} \psi_k)^*$. Thus, letting $\mathcal{P}_C(X, \omega, \psi) \ni u = P_{\omega}[\psi](\varphi)$ for $\varphi \in \mathcal{K}_C$ minimal entropy function for u and letting $u_k := P_{\omega}[\psi_k](\varphi)$, we claim that $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence with respect the distance $d_{\mathcal{A}}$.

Indeed if $\{\psi_k\}_{k \in \mathbb{N}}$ is increasing, then for any j, k such that $j \geq k$ we have

$$\begin{aligned} d_{\mathcal{A}}(u_k, u_j) &\leq \tilde{d}_{\mathcal{A}}(u_k, u_j) \leq \\ &\leq \sup_{a, b \in \mathcal{P}_C(X, \omega, \psi_j)} \left\{ d(a, b) - d(P_{\omega}[\psi_k](a), P_{\omega}[\psi_k](b)) \right\} + V_{u_j} - V_{u_k} \leq \\ &\leq \sup_{a, b \in \mathcal{P}_C(X, \omega, \psi)} \left\{ d(a, b) - d(P_{\omega}[\psi_k](a), P_{\omega}[\psi_k](b)) \right\} + V_{\psi} - V_{\psi_k} \end{aligned}$$

by the definition of $d_{\mathcal{A}}$ and Proposition 3.4.5 since $\psi \succ \psi_k$ for any $k \in \mathbb{N}$. Therefore by Proposition 3.4.7 (see also Lemma 3.4.2) $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in $\left(\bigsqcup_{\psi \in \mathcal{A}} \mathcal{P}(X, \omega, \psi), d_{\mathcal{A}} \right)$.

If instead $\psi_k \searrow \psi$, we first denote with $C_1 \in \mathbb{R}_{\geq 0}$ the minimal constant such that $u_1 = P_{\omega}[\psi_1](\phi)$ for $\phi \in \mathcal{K}_{C_1}$. Thus for any j, k such that $j \geq k$ we have

$$\begin{aligned} d_{\mathcal{A}}(u_k, u_j) &\leq \tilde{d}_{\mathcal{A}}(u_k, u_j) \leq \\ &\leq \sup_{a, b \in \mathcal{P}_{C_1}(X, \omega, \psi_k)} \left\{ d(a, b) - d(P_{\omega}[\psi_j](a), P_{\omega}[\psi_j](b)) \right\} + V_{u_k} - V_{u_j} \leq \\ &\leq \sup_{a, b \in \mathcal{P}_{C_1}(X, \omega, \psi_k)} \left\{ d(a, b) - d(P_{\omega}[\psi](a), P_{\omega}[\psi](b)) \right\} + V_{\psi_k} - V_{\psi}, \end{aligned}$$

and as before Proposition 3.4.7 and Lemma 3.4.2 imply that $\{u_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence.

Hence we define the map

$$\tilde{\Phi} : \left(\bigsqcup_{\psi \in \overline{\mathcal{A}}} (\mathcal{P}(X, \omega, \psi), d) \right) \rightarrow (X_{\mathcal{A}}, d_{\mathcal{A}})$$

as $\tilde{\Phi}(u) := [u_k]$ where we recall that $(X_{\mathcal{A}}, d_{\mathcal{A}})$ is by definition the metric completion of $\bigsqcup_{\psi \in \mathcal{A}} (\mathcal{P}(X, \omega, \psi), d)$. We need to check that it is well-defined.

Let assume $u = P_{\omega}[\psi](\varphi) = P_{\omega}[\psi](\varphi')$ for $\varphi, \varphi' \in \mathcal{K}_C$ minimal entropy functions. Define also $u_k := P_{\omega}[\psi_k](\varphi)$, $u'_k := P_{\omega}[\psi_k](\varphi')$ where $\psi = (\lim_{k \rightarrow \infty} \psi_k)^*$ monotonically. Then by Proposition 3.4.7

$$\limsup_{k \rightarrow \infty} d_{\mathcal{A}}(u_k, u'_k) = \lim_{k \rightarrow \infty} d(u_k, u'_k) = d(u, u) = 0.$$

Next assume that $u = P_{\omega}[\psi](\varphi)$, $u_k := P_{\omega}[\psi_k](\varphi)$, $u'_k := P_{\omega}[\psi'_k](\varphi)$ for $\{\psi_k\}_{k \in \mathbb{N}}, \{\psi'_k\}_{k \in \mathbb{N}} \subset \mathcal{A}$ monotone sequence converging to ψ almost everywhere. We need to check that $d_{\mathcal{A}}(u_k, u'_k) \rightarrow 0$ as $k \rightarrow \infty$. Let $C_1, C'_1 \in \mathbb{R}$ be minimal constants such that $u_1 = P_{\omega}[\psi_1](\phi)$, $u_1 = P_{\omega}[\psi'_1](\phi')$ for $\phi \in \mathcal{K}_{C_1}$, $\phi' \in \mathcal{K}_{C'_1}$, and set $C_2 := \max(C, C_1, C'_1)$. Then for any $k \in \mathbb{N}$ such that $\psi'_k \preceq \psi_k$,

$$\begin{aligned} d_{\mathcal{A}}(u_k, u'_k) &\leq \tilde{d}_{\mathcal{A}}(u_k, u'_k) \leq \\ &\leq \sup_{(a,b) \in \mathcal{P}_{C_2}(X, \omega, \psi_k)} \left\{ d(a, b) - d(P_{\omega}[\psi'_k](a), P_{\omega}[\psi_k](b)) \right\} + V_{\psi_k} - V_{\psi'_k} \end{aligned}$$

and similarly if $\psi'_k \succ \psi_k$. Therefore, proceeding similarly as before, it is not difficult to check that $d_{\mathcal{A}}(u_k, u'_k) \rightarrow 0$ as $k \rightarrow \infty$ using again Proposition 3.4.7. Hence $\tilde{\Phi}$ is well-defined, $\mathcal{P}(X, \omega, \psi) \subset (X_{\mathcal{A}}, d_{\mathcal{A}})$ and if $u := P_{\omega}[\psi](\varphi_1)$, $v := P_{\omega}[\psi](\varphi_2)$ for minimal entropy functions φ_1, φ_2 then

$$d_{\mathcal{A}}(u, v) := \lim_{k \rightarrow \infty} d_{\mathcal{A}}(u_k, v_k) = \lim_{k \rightarrow \infty} d(u_k, v_k)$$

where $u_k := P_{\omega}[\psi_k](\varphi_1)$, $v_k := P_{\omega}[\psi_k](\varphi_2)$ for $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{A}$ monotonically converging a.e. to ψ . Therefore $d_{\mathcal{A}}(u, v) = d(u, v)$ by Proposition 3.4.7 and it easily follows from Lemma 3.4.9 that there is an unique continuous extension

$$\tilde{\Phi} : \left(\bigsqcup_{\psi \in \overline{\mathcal{A}}} (\mathcal{E}^1(X, \omega, \psi), d) \right) \rightarrow (X_{\mathcal{A}}, d_{\mathcal{A}})$$

which restricts to an isometric embedding on any metric space $(\mathcal{E}^1(X, \omega, \psi), d)$.

Step 3: set up the final strategy.

It remains to prove that $\bigsqcup_{\psi \in \overline{\mathcal{A}}} (\mathcal{E}^1(X, \omega, \psi), d_{\mathcal{A}})$ is complete. Thus let $\{u_j\}_{j \in \mathbb{N}} \subset \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{P}(X, \omega, \psi)$ be a Cauchy sequence. Up to extract a subsequence, we may assume $d_{\mathcal{A}}(u_j, u_{j+1}) \leq \frac{1}{2^j}$. For any $j \in \mathbb{N}$ let also $\varphi_j \in \mathcal{P}(X, \omega)$ a minimal entropy function for u_j and $\psi_j \in \overline{\mathcal{A}}$ such that $u_j \in \mathcal{P}(X, \omega, \psi_j)$. Since $\overline{\mathcal{A}}$ is totally ordered, up to consider a subsequence, we may assume that $\{\psi_j\}_{j \in \mathbb{N}}$ converges monotonically a.e. to $\psi \in \overline{\mathcal{A}}$.

Step 4: $\{\psi_j\}_{j \in \mathbb{N}}$ increasing.

Let for any $k \geq j$, $v_{j,k} := P_{\omega}(P_{\omega}[\psi_j](\varphi_j), \dots, P_{\omega}[\psi_j](\varphi_k))$ and let for any $k \geq j$,

$i \in \mathbb{N}$, $v_{j,k}^i := P_\omega\left(P_\omega[\psi_i](\varphi_j), \dots, P_\omega[\psi_i](\varphi_k)\right)$. Note that $v_{j,k}^j = v_{j,k}$ and that $v_{j,k} = P_\omega(u_j, v_{j+1,k}^j)$. Moreover we claim that $P_\omega[\psi_j](v_{j,k}^i) = v_{j,k}$ if $i \geq j$. Indeed $P_\omega[\psi_j](v_{j,k}^i) \leq v_{j,k}$ since $v_{j,k}^i \leq P_\omega[\psi_i](\varphi_j), \dots, P_\omega[\psi_i](\varphi_k)$ implies $P_\omega[\psi_j](v_{j,k}^i) \leq P_\omega[\psi_j](\varphi_j), \dots, P_\omega[\psi_j](\varphi_k)$ by Lemma 3.4.4. While the reverse inequality follows applying $P_\omega[\psi_j](\cdot)$ to the trivial inequality $v_{j,k} \leq v_{j,k}^i$. As a consequence we get that

$$d(u_j, v_{j,k}) = d(u_j, P_\omega(u_j, v_{j+1,k}^j)) \leq d(u_j, v_{j+1,k}^j) \leq d_A(u_j, v_{j+1,k})$$

where the last inequality follows from Proposition 3.4.15. (iv). Iterating, by the triangle inequality we have

$$d(u_j, v_{j,k}) \leq \sum_{l=0}^{k-j-1} d(u_{j+l}, u_{j+l+1}) \leq \sum_{l=0}^{k-j-1} \frac{1}{2^{j+l}} \leq \frac{1}{2^{j-1}}.$$

Clearly $v_{j,k}$ is decreasing in k , thus, letting $C_j \in \mathbb{N}$ such that $v_{j,k} \leq \psi_j + C_j$ for any $k \in \mathbb{N}$, we get

$$C_j - E_{\psi_j}(v_{j,k}) = E_{\psi_j}(\psi_j + C_j) - E_{\psi_j}(v_{j,k}) = d(\psi_j + C_j, v_{j,k}) \leq d(\psi_j + C_j, u_j) + \frac{1}{2^{j-1}},$$

which implies that $v_j := \lim_{k \rightarrow \infty} v_{j,k} \in \mathcal{E}^1(X, \omega, \psi_j)$ by Proposition 3.2.8. Moreover $d(u_j, v_j) \leq 2^{-j+1}$ by continuity along decreasing sequence. Observe also that $v_j \leq v_{j+1}$ by construction since $v_j \leq v_{j,k} \leq v_{j,k}^{j+1} \leq v_{j+1,k}$ for any $k \geq j+1$.

Then by Lemma 3.3.7 there exists two uniform constants $A > 1, B > 0$ such that

$$\sup_X (v_j - \psi_j) = \sup_X v_j \leq \frac{1}{V_{\psi_j}} (Ad(\psi_j, v_j) + B)$$

which implies that $u := (\lim_{j \rightarrow \infty} v_j)^* \in PSH(X, \omega, \psi)$. Therefore, assuming $\sup_X u = 0$ up to add a constant, by Theorem 3.2.4 we also have $MA_\omega(v_j) \rightarrow MA_\omega(u)$ weakly, which implies $V_u = V_\psi$ and, for any $m \in \mathbb{N}$ fixed,

$$\begin{aligned} \int_X (\psi - \max(u, \psi - m)) MA_\omega(\max(u, \psi - m)) &= \\ &= \lim_{j \rightarrow \infty} \int_X (\psi_j - \max(v_j, \psi_j - m)) MA_\omega(\max(v_j, \psi_j - m)) \leq \\ &\leq \lim_{j \rightarrow \infty} (n+1) d(\psi_j, \max(v_j, \psi_j - m)) \leq \lim_{j \rightarrow \infty} (n+1) d(\psi_j, v_j) \end{aligned}$$

using also that $\max(v_j, \psi_j - m) \nearrow \max(u, \psi - m)$ almost everywhere. Therefore $u \in \mathcal{E}^1(X, \omega, \psi)$ as a consequence of

$$d(\psi_j, v_j) \leq d_A(\psi_j, \psi_1) + d(\psi_1, u_1) + d_A(u_1, u_j) + d(u_j, v_j) \leq V_\psi - V_{\psi_1} + d(\psi_1, u_1) + 2.$$

Thus to finish this step it remains to check that $d_A(u, u_j) \rightarrow 0$ as $j \rightarrow \infty$, or equivalently that $d_A(u, v_j) \rightarrow 0$.

Set for any $k \geq j$, $v_k^j := P_\omega[\psi_j](v_k)$. Then by construction $\{v_k^j\}_{k \geq j}$ is an increasing sequence converging strongly in $\mathcal{E}^1(X, \omega, \psi_j)$ to $P_\omega[\psi_j](u)$. For $\epsilon > 0$ fixed, let also $\phi_\epsilon \in \mathcal{H}$ such that $d(u, P_\omega[\psi](\phi_\epsilon)) \leq \epsilon$. Next, for any j fixed let $s \in \mathbb{N}$ depending on j, ϵ such that $d(v_{j+s}^j, P_\omega[\psi_j](u)) \leq \epsilon$. Thus by the triangle inequality and Proposition 3.4.5 we have

$$\begin{aligned} d_{\mathcal{A}}(v_j, u) &\leq \sum_{l=0}^{s-1} d(v_{j+l}^j, v_{j+l+1}^j) + d(v_{j+s}^j, P_\omega[\psi_j](u)) + \\ &d(P_\omega[\psi_j](u), P_\omega[\psi_j](\phi_\epsilon)) + d_{\mathcal{A}}(P_\omega[\psi_j](\phi_\epsilon), P_\omega[\psi](\phi_\epsilon)) + d(P_\omega[\psi](\phi_\epsilon), u) \leq \\ &\leq \frac{1}{2^{j-2}} + 3\epsilon + d_{\mathcal{A}}(P_\omega[\psi_j](\phi_\epsilon), P_\omega[\psi](\phi_\epsilon)), \end{aligned}$$

which by Proposition 3.4.7 implies $\limsup_{j \rightarrow \infty} d_{\mathcal{A}}(v_j, u) \leq 3\epsilon$ since

$$\begin{aligned} d_{\mathcal{A}}(P_\omega[\psi_j](\phi_\epsilon), P_\omega[\psi](\phi_\epsilon)) &\leq \tilde{d}_{\mathcal{A}}(P_\omega[\psi_j](\phi_\epsilon), P_\omega[\psi](\phi_\epsilon)) \leq \\ &\leq \sup_{\varphi_1, \varphi_2 \in \mathcal{P}_{C_\epsilon}(X, \omega, \psi)} \{d(\varphi_1, \varphi_2) - d(P_\omega[\psi_j](\varphi_1), P_\omega[\psi_j](\varphi_2))\} + V_\psi - V_{\psi_j} \rightarrow 0 \end{aligned}$$

for a certain constant $C_\epsilon \in \mathbb{R}$.

Step 4: $\{\psi_j\}_{j \in \mathbb{N}}$ **decreasing.**

We define for any $j \in \mathbb{N}$, $w_j := P_\omega[\psi](u_j)$. Clearly $\{w_j\}_{j \in \mathbb{N}} \subset \mathcal{E}^1(X, \omega, \psi)$ is a Cauchy sequence since for any $j \geq k$

$$d(w_j, w_k) \leq d(u_j, P_\omega[\psi_j](u_k)) \leq d_{\mathcal{A}}(u_j, u_k) \leq \frac{1}{2^{k-1}}$$

by Proposition 3.4.5 and Proposition 3.4.15. Thus w_j converges strongly to a function u in $\mathcal{E}^1(X, \omega, \psi)$ and to conclude the proof it remains to prove that $d_{\mathcal{A}}(u_j, u) \rightarrow 0$ as $j \rightarrow \infty$. Therefore, letting for any $k \in \mathbb{N}$, $\phi_k \in \mathcal{H}$ such that $d(u_k, P_\omega[\psi_k](\phi_k)) \leq 1/k$, we get for any $k \leq j$

$$\begin{aligned} d_{\mathcal{A}}(u_j, u) &\leq d(u_j, P_\omega[\psi_j](u_k)) + d(P_\omega[\psi_j](u_k), P_\omega[\psi_j](\phi_k)) + \\ &+ d_{\mathcal{A}}(P_\omega[\psi_j](\phi_k), P_\omega[\psi](\phi_k)) + d(P_\omega[\psi](\phi_k), u) \leq d_{\mathcal{A}}(u_j, u_k) + \\ &+ d(u_k, P_\omega[\psi_k](\phi_k)) + d_{\mathcal{A}}(P_\omega[\psi_j](\phi_k), P_\omega[\psi](\phi_k)) + d(P_\omega[\psi](\phi_k), u) \leq \\ &\leq \frac{1}{2^{k-1}} + \frac{1}{k} + d_{\mathcal{A}}(P_\omega[\psi_j](\phi_k), P_\omega[\psi](\phi_k)) + d(P_\omega[\psi](\phi_k), u) \quad (3.6) \end{aligned}$$

combining Proposition 3.4.5 and Proposition 3.4.15. Therefore since clearly $P_\omega[\psi](\phi_k)$ converges strongly to u in $\mathcal{E}^1(X, \omega, \psi)$ and since, similarly to before, we have that $\limsup_{j \rightarrow \infty} d_{\mathcal{A}}(P_\omega[\psi_j](\phi_k), P_\omega[\psi](\phi_k)) = 0$, it follows from the inequality (3.6) that $\limsup_{j \rightarrow \infty} d_{\mathcal{A}}(u_j, u) = 0$ letting $j \rightarrow \infty$ and then $k \rightarrow \infty$. \square

3.4.3 Proof of Lemma 3.4.16.

The proof of Lemma 3.4.16 proceeds by induction on $m \in \mathbb{N}$ length of the chain.

Step 1 ($m = 1$): Assume $w \in \mathcal{P}(X, \omega, \psi')$ for $\psi' \in \mathcal{A}$. Then by Proposition 3.4.15.(iv) we get that

$$\tilde{d}_{\mathcal{A}}(u, w) + \tilde{d}_{\mathcal{A}}(w, v) \geq d(u, P_{\omega}[\psi](\varphi)) + d(P_{\omega}[\psi](\varphi), v) \geq d(u, v)$$

where $\varphi \in \mathcal{P}(X, \omega)$ is a minimal entropy function for w .

Step 2 ($m \rightarrow m+1$): reduce to an easier case 1. Assume now that the Lemma holds for any chain of length $n \leq m \in \mathbb{N}$, and let $w_1, \dots, w_{m+1} \in \bigsqcup_{\psi' \in \mathcal{A}} \mathcal{P}(X, \omega, \psi')$. To fix the notations assume $w_j \in \mathcal{P}(X, \omega, \psi_j)$ and that, for any $j = 0, \dots, n$, $\varphi_j \in \mathcal{K}_{C_j}$ is a choice of minimal entropy functions for w_j .

Next, using the definition of $\tilde{d}_{\mathcal{A}}$ and Proposition 3.4.15, if $\psi_{j+1} \preceq \psi_{j-1} \preceq \psi_j$ then

$$\tilde{d}_{\mathcal{A}}(w_{j-1}, w_j) + \tilde{d}_{\mathcal{A}}(w_j, w_{j+1}) \geq \tilde{d}_{\mathcal{A}}(w_{j-1}, P_{\omega}[\psi_{j-1}](\varphi_j)) + \tilde{d}_{\mathcal{A}}(P_{\omega}[\psi_{j-1}](\varphi_j), w_{j+1}).$$

Therefore we may assume there exists $j_0 \in \{1, \dots, m+1\}$ such that $\psi \succcurlyeq \psi_1 \succcurlyeq \dots \succcurlyeq \psi_{j_0}$ and $\psi_{j_0} \preceq \psi_{j_0+1} \preceq \dots \preceq \psi$.

Step 3 ($m \rightarrow m+1$): reduce to an easier case 2. We claim that if there exists $j \in \{1, \dots, m+1\}$ such that $C_j \geq \max(C_{j-1}, C_{j+1})$ (where we set $w_0 := u$, $w_{m+2} := v$) then

$$\tilde{d}_{\mathcal{A}}(w_{j-1}, w_j) + \tilde{d}_{\mathcal{A}}(w_j, w_{j+1}) \geq \tilde{d}_{\mathcal{A}}(w_{j-1}, w_{j+1}). \quad (3.7)$$

Indeed, if $j \neq j_0$, assuming by symmetry $j < j_0$, then by Lemma 3.4.4 and Proposition 3.4.5 the inequality (3.7) is an easy consequence of

$$\begin{aligned} d(P_{\omega}[\psi_j](w_{j-1}), w_j) + d(P_{\omega}[\psi_{j+1}](w_j), w_{j+1}) &\geq \\ &\geq d(P_{\omega}[\psi_{j+1}](w_{j-1}), P_{\omega}[\psi_{j+1}](w_j)) + d(P_{\omega}[\psi_{j+1}](w_j), w_{j+1}) \geq \\ &\geq d(P_{\omega}[\psi_{j+1}](w_{j-1}), w_{j+1}), \end{aligned}$$

and of

$$\begin{aligned} &\sup_{a, b \in \mathcal{P}_{C_j}(X, \omega, \psi_{j-1})} \left\{ d(a, b) - d(P_{\omega}[\psi_j](a), P_{\omega}[\psi_j](b)) \right\} + \\ &\quad + \sup_{a, b \in \mathcal{P}_{C_j}(X, \omega, \psi_j)} \left\{ d(a, b) - d(P_{\omega}[\psi_{j+1}](a), P_{\omega}[\psi_{j+1}](b)) \right\} \geq \\ &\geq \sup_{a, b \in \mathcal{P}_{\max(C_{j-1}, C_{j+1})}(X, \omega, \psi_{j-1})} \left\{ d(a, b) - d(P_{\omega}[\psi_{j+1}](a), P_{\omega}[\psi_{j+1}](b)) \right\}. \end{aligned}$$

In the case $j = j_0$, instead, assuming $\psi_{j-1} \preceq \psi_{j+1}$ the inequality (3.7) follows from

$$\begin{aligned} & d(P_\omega[\psi_j](w_{j-1}), w_j) + d(P_\omega[\psi_j](w_{j+1}), w_j) + \\ & \quad + \sup_{a, b \in \mathcal{P}_{C_j}(X, \omega, \psi_{j-1})} \left\{ d(a, b) - d(P_\omega[\psi_j](a), P_\omega[\psi_j](b)) \right\} \geq \\ & \geq d(P_\omega[\psi_j](w_{j-1}), P_\omega[\psi_j](w_{j+1})) + d(w_{j-1}, P_\omega[\psi_{j-1}](w_{j+1})) - \\ & \quad - d(P_\omega[\psi_j](w_{j-1}), P_\omega[\psi_j](w_{j+1})) = d(w_{j-1}, P_\omega[\psi_{j-1}](w_{j+1})). \end{aligned}$$

Indeed it implies

$$\begin{aligned} & \tilde{d}_A(w_{j-1}, w_j) + \tilde{d}_A(w_j, w_{j+1}) \geq d(w_{j-1}, P_\omega[\psi_{j-1}](w_{j+1})) + \\ & \quad + \sup_{a, b \in \mathcal{P}_{C_j}(X, \omega, \psi_{j+1})} \left\{ d(a, b) - d(P_\omega[\psi_j](a), P_\omega[\psi_j](b)) \right\} + V_{w_{j+1}} - V_{w_{j-1}} \geq \\ & \geq \tilde{d}_A(w_{j-1}, w_{j+1}). \end{aligned}$$

Therefore, using again the inductive hypothesis, we may assume there exists $i_0 \in \{0, \dots, m+2\}$ such that $C_0 > C_1 > \dots > C_{i_0-1} \geq C_{i_0}$ and $C_{i_0} \leq C_{i_0+1} < \dots < C_{m+1} < C_{m+2}$, where moreover $C_{i_0} < \max(C_{i_0-1}, C_{i_0+1})$ (in the extreme cases $i_0 = 0, m+2$ the last inequality obviously restricts respectively to $C_{i_0} = C_0 < C_1$ and to $C_{i_0} = C_{m+2} < C_{m+1}$).

Step 4 ($\mathbf{m} \rightarrow \mathbf{m} + 1$): case $|\mathbf{i}_0 - \mathbf{j}_0| > 1$. By symmetry we may assume $i_0 < j_0 - 1$. So $C_{j_0-2} \leq C_{j_0-1} < C_{j_0} < C_{j_0+1}$, which implies

$$\begin{aligned} & \tilde{d}_A(w_{j_0-1}, w_{j_0}) + \tilde{d}_A(w_{j_0}, w_{j_0+1}) \geq \\ & \geq \tilde{d}_A(w_{j_0-1}, P_\omega[\psi_{j_0}](w_{j_0-1})) + \tilde{d}_A(P_\omega[\psi_{j_0}](w_{j_0-1}), w_{j_0+1}) \end{aligned}$$

using the definition. Letting $\tilde{w} := P_\omega[\psi_{j_0}](w_{j_0-1})$ and \tilde{C} be the smallest non-negative real number such that $\tilde{w} \in \mathcal{P}_{\tilde{C}}(X, \omega, \psi_{j_0})$, we conclude this case by the argument exposed in the previous step since $\tilde{C} \leq C_{j_0-1}$ by construction and $C_{j_0-1} \geq C_{j_0-2}$.

Step 5 ($\mathbf{m} \rightarrow \mathbf{m} + 1$): case $|\mathbf{i}_0 - \mathbf{j}_0| = 1$ Let assume $i_0 = j_0 - 1$. Since $C_{j_0-1} \leq C_{j_0} < C_{j_0+1}$, as in Step 4, we can substitute w_{j_0} by $P_\omega[\psi_{j_0}](w_{j_0-1})$. Therefore, up to replace i_0 by $i_0 + 1$, we have $i_0 = j_0$ that is the last case addressed in the final step.

Step 6 ($\mathbf{m} \rightarrow \mathbf{m} + 1$): case $\mathbf{i}_0 = \mathbf{j}_0$ Since $C_0 > C_1 > \dots > C_{j_0-1} > C_{j_0}$, alternating several times Proposition 3.4.15 .(iv) and the triangle inequality for d on

$\mathcal{E}^1(X, \omega, \psi_i)$ for $i = 0, \dots, j_0 - 1$ we get

$$\begin{aligned}
 & \tilde{d}_{\mathcal{A}}(w_0, w_1) + \dots + \tilde{d}_{\mathcal{A}}(w_{j_0-1}, w_{j_0}) \geq \\
 & \geq \tilde{d}_{\mathcal{A}}(w_0, w_1) + \dots + \tilde{d}_{\mathcal{A}}(w_{j_0-2}, w_{j_0-1}) + d(w_{j_0-1}, P_{\omega}[\psi_{j_0-1}](\varphi_{j_0})) \geq \\
 & \geq \tilde{d}_{\mathcal{A}}(w_0, w_1) + \dots + \tilde{d}_{\mathcal{A}}(w_{j_0-3}, w_{j_0-2}) + d(P_{\omega}[\psi_{j_0-1}](\varphi_{j_0-2}), P_{\omega}[\psi_{j_0-1}](\varphi_{j_0})) + \\
 & \quad + \sup_{a, b \in \mathcal{P}_{C_{j_0-2}}(X, \omega, \psi_{j_0-2})} \left\{ d(a, b) - d(P_{\omega}[\psi_{j_0-1}](a), P_{\omega}[\psi_{j_0-1}](b)) \right\} \geq \\
 & \geq \tilde{d}_{\mathcal{A}}(w_0, w_1) + \dots + \tilde{d}_{\mathcal{A}}(w_{j_0-3}, w_{j_0-2}) + d(w_{j_0-2}, P_{\omega}[\psi_{j_0-2}](\varphi_{j_0})) \geq \\
 & \geq \dots \geq \tilde{d}_{\mathcal{A}}(w_0, w_1) + d(w_1, P_{\omega}[\psi_1](\varphi_{j_0})) \geq d(w_0, P_{\omega}[\psi_0](\varphi_{j_0})).
 \end{aligned}$$

Proceeding in the same way, by symmetry, we also get

$$\tilde{d}_{\mathcal{A}}(w_{j_0}, w_{j_0+1}) + \dots + \tilde{d}_{\mathcal{A}}(w_{m+1}, w_{m+2}) \geq d(P_{\omega}[\psi_0](\varphi_{j_0}), w_{m+2}).$$

Hence

$$\tilde{d}_{\mathcal{A}}(w_0, w_1) + \dots + \tilde{d}_{\mathcal{A}}(w_{m+1}, w_{m+2}) \geq d(w_0, w_{m+2}) = d(u, v),$$

which concludes the proof.

3.4.4 Gromov-Hausdorff types of convergences & direct limits: proof of Theorems C and D.

In this section we assume $\mathcal{A} = \{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ to be a total ordered subset such that $\psi_{k+1} \preceq \psi_k$ for any $k \in \mathbb{N}$. Moreover we suppose that $\psi_k \searrow \psi \in \mathcal{M}^+$.

Definition 3.4.17. *Let \mathcal{A} and $\psi \in \mathcal{M}^+$ as above. Then the elements of the family*

$$\mathcal{K}_{\mathcal{A}} := \bigcup_{k \in \mathbb{N}} \left\{ K \subset \mathcal{E}^1(X, \omega, \psi) \text{ compact such that } K \subset P_{\omega}[\psi](\tilde{K}) \right.$$

$$\left. \text{for } \tilde{K} \subset \mathcal{E}^1(X, \omega, \psi_k) \text{ compact} \right\}$$

are called \mathcal{A} -compact sets.

We recall that for a couple of compact metric spaces $(X, d_X), (Y, d_Y)$, the Gromov-Hausdorff distance between them is defined as

$$d_{GH}(X, Y) = \inf \{ d_H^d(X, Y) : d \text{ admissible distance on } X \sqcup Y \}$$

where a distance d on $X \sqcup Y$ is said to be admissible if $d|_{X \times X} = d_X$ and $d|_{Y \times Y} = d_Y$ and where d_H^d indicates the Hausdorff distance on the closed sets of $(X \sqcup Y, d)$.

A sequence of compact metric spaces (X_n, d_n) converges in the Gromov-Hausdorff sense to a compact metric space (X, d) if $d_{GH}(X_n, X) \rightarrow 0$. We will use the notation $(X_n, d_n) \xrightarrow{GH} (X, d)$ and we refer to [BBI01] and to [BH99] for this notion of convergence.

Proposition 3.4.18. *For any \mathcal{A} -compact set $K \subset \mathcal{K}_{\mathcal{A}}$ there exists a sequence of strongly compact sets $(K_k, d) \subset (\mathcal{E}^1(X, \omega, \psi_k), d)$ for $k \gg 1$ big enough such that*

$$(K_k, d) \xrightarrow{GH} (K, d).$$

Proof. Let $k_0 \in \mathbb{N}$ such that $K \subset P_{\omega}[\psi](\tilde{K})$ for a strongly compact set $\tilde{K} \subset (\mathcal{E}^1(X, \omega, \psi_{k_0}), d)$. Then we define

$$K_{k_0} := \tilde{K} \cap P_{\omega}[\psi]^{-1}(K),$$

noting that it is a compact set in $\mathcal{E}^1(X, \omega, \psi_{k_0})$. Therefore we define for any $k \geq k_0$

$$K_k := P_{\omega}[\psi_k](\tilde{K}) \cap P_{\omega}[\psi]^{-1}(K) = P_{\omega}[\psi_k](K_{k_0})$$

and a correspondence $\mathcal{R}_k \subset K_k \times K$ as $(u_k, u) \in \mathcal{R}_k$ if $u = P_{\omega}[\psi](u_k)$. Thus to prove that $d_{GH}(K_k, K) \rightarrow 0$ with respect to the d -distances it is enough to check that

$$\text{dis } \mathcal{R}_k := \sup \{ |d(u, v) - d(u_k, v_k)| : (u_k, u), (v_k, v) \in \mathcal{R}_k \} \rightarrow 0$$

as $k \rightarrow \infty$ (see Theorem 7.3.25. in [BBI01]). Hence Proposition 3.4.7 concludes the proof. \square

For non-compact metric spaces there is a weaker notion of convergence than the Gromov-Hausdorff convergence, that is the *pointed Gromov-Hausdorff* convergence. We recall that a sequence of pointed compact metric spaces (K_n, p_n, d_n) converges in the pointed Gromov-Hausdorff sense to (K, p, d) if $d_{GH}((K_n, p_n), (K, p)) \rightarrow 0$ as $n \rightarrow \infty$ where

$$d_{GH}((K_n, p_n), (K, p)) := \inf \{ d_H^d(K_n, K) + d(p_n, p) : d \text{ admissible metric on } X \sqcup Y \}.$$

Thus a sequence of non-compact pointed metric spaces (X_n, p_n, d_n) is said to converge in the pointed Gromov-Hausdorff sense to a non-compact pointed metric space (X, p, d) if for any $r > 0$

$$d_{GH}((\overline{B_r(p_n)}, p_n), (\overline{B_r(p)}, p)) \rightarrow 0$$

as $n \rightarrow \infty$ ². We will use the notation $(X_n, p_n, d_n) \xrightarrow{p-GH} (X, p, d)$.

If the pointed metric spaces are locally compact this convergence seems to be the most natural kind of convergence to look at. But if the pointed metric spaces are not locally compact, the pointed Gromov-Hausdorff convergence still seems a too strong kind of convergence. Thus we give the following general definition:

²This is actually not the right definition of point Gromov-Hausdorff convergence, but it is a characterization which holds when the sequence and the limit point are length spaces ([BBI01]).

Definition 3.4.19. A family of pointed metric spaces (X_n, p_n, d_n) converges in the compact pointed Gromov-Hausdorff convergence to a pointed metric space (X, p, d) if there exist a family of compact set $\{K_j\}_{j \in \mathbb{N}} \subset X$ and, for any $n \in \mathbb{N}$, a family of compact sets $\{K_{j,n}\}_{j \in \mathbb{N}} \subset X_n$ such that

- i) $p_n \in K_{j,n}$ for any $n \in \mathbb{N}$ and for any $j \in \mathbb{N}$;
- ii) $p \in K_j$ for any $j \in \mathbb{N}$;
- iii) for any $n \in \mathbb{N}$ fixed, $K_{j,n} \subset K_{j+1,n}$ for any $j \in \mathbb{N}$ and $\bigcup_{j \in \mathbb{N}} K_{j,n}$ is dense in X_n ;
- iv) $K_j \subset K_{j+1}$ for any $j \in \mathbb{N}$ and $\bigcup_{j \in \mathbb{N}} K_j$ is dense in X ;
- v) $d_{GH}((K_{j,n}, p_n), (K_j, p)) \rightarrow 0$.

We will use the notation $(X_n, p_n, d_n) \xrightarrow{cp-GH} (X, p, d)$.

We can now prove Theorem C:

Theorem C. Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ such that $\psi_k \searrow \psi \in \mathcal{M}^+$. Then

$$\left(\mathcal{E}^1(X, \omega, \psi_k), d \right) \xrightarrow{cp-GH} \left(\mathcal{E}^1(X, \omega, \psi), d \right).$$

Proof. For any $j \in \mathbb{N}$ let \mathcal{K}_j be the strongly compact set in $\mathcal{E}^1(X, \omega)$ containing all ω -psh functions with bounded entropy by j (see Theorem 3.4.12). Thus, defining for any $j \in \mathbb{N}$ and for any $k \in \mathbb{N}$, $K_{j,k} := P_\omega[\psi_k](\mathcal{K}_j)$ and $K_j := P_\omega[\psi](\mathcal{K}_j)$, the theorem immediately follows from Lemma 3.4.9 and Proposition 3.4.18. \square

The maps $P_{k,j} : P_\omega[\psi_j](\cdot) : (\mathcal{E}^1(X, \omega, \psi_k), d) \rightarrow (\mathcal{E}^1(X, \omega, \psi_j), d)$ for $k \leq j$ are morphisms in the category of metric spaces (see Lemma 3.4.4 and Proposition 3.4.5). Moreover $\{P_{k,j}\}_{j \leq k, (k,j) \in \mathbb{N}}$ produces a direct system again by Lemma 3.4.4, and $\langle (\mathcal{E}^1(X, \omega, \psi), d), P_k \rangle$ is a target of this direct system where $P_k := P_\omega[\psi](\cdot) : (\mathcal{E}^1(X, \omega, \psi_k), d) \rightarrow (\mathcal{E}^1(X, \omega, \psi), d)$.

We recall that a target $\langle (X, d_X), f_{X,n} \rangle$ of a direct system of metric spaces $\langle (X_n, d_n), f_{n,m} \rangle$ is a metric space (X, d_X) with 1-Lipschitz maps $f_{X,n} : (X_n, d_n) \rightarrow (X, d_X)$ such that $f_{X,n} = f_{X,m} \circ f_{m,n}$ for any $n \leq m$.

Therefore since by the universal property the direct limit is the initial target, we immediately find out that the direct system $\langle (\mathcal{E}^1(X, \omega, \psi_j), P_{k,j})$ admits a direct limit (recall that some direct systems in the category of metric spaces do not admit any not-trivial target like, for instance, the direct system $\langle (X_n, d_n), f_{n,m} \rangle := \langle (\mathbb{R}, \frac{1}{n} d_{eucl}), \text{Id} \rangle$). We denote with $\mathbf{m} - \lim_{\rightarrow}$ the direct limit in the category of metric spaces.

Theorem D. There is an isometric embedding

$$\mathbf{m} - \lim_{\rightarrow} \langle (\mathcal{E}^1(X, \omega, \psi_i), d), P_{i,j} \rangle \hookrightarrow (\mathcal{E}^1(X, \omega, \psi), d)$$

with dense image. More precisely the direct limit in the category of metric spaces is isometric to $\left(\bigcup_{k \in \mathbb{N}} P_\omega[\psi](\mathcal{E}^1(X, \omega, \psi_k)), d \right)$.

Proof. As a consequence of Lemma 3.4.9 the set $T := \bigcup_{k \in \mathbb{N}} P_\omega[\psi](\mathcal{E}^1(X, \omega, \psi_k))$ is dense in $(\mathcal{E}^1(X, \omega, \psi), d)$. Then, since as stated before $\langle (T, d), P_k \rangle$ is a target of the direct system considered, to conclude the proof it is enough to show that for any other target $\langle (Y, d_Y), P_{Y,k} \rangle$ there exists a 1-Lipschitz map $P_{Y,T} : T \rightarrow Y$ such that $P_{Y,T} \circ P_k = P_{Y,k}$ for any $k \in \mathbb{N}$.

Therefore, letting $\langle (Y, d_Y), P_{Y,k} \rangle$ a target, for any $u \in T$ we denote with $k_u \in \mathbb{N}$ the minimum natural number k such that $u \in P_\omega[\psi](\mathcal{E}^1(X, \omega, \psi_k))$ and we fix a function $\varphi_u \in \mathcal{E}^1(X, \omega, \psi_{k_u})$ such that $P_\omega[\psi](\varphi_u) = u$. Next we define $P_{Y,T} : T \rightarrow Y$ as

$$P_{Y,T}(u) := P_{Y,k_u}(\varphi_u),$$

i.e. it is defined so that $P_{Y,T} \circ P_k = P_{Y,k}$ for any $k \in \mathbb{N}$. Note that the definition does not depend on representatives since $P_{Y,k_1}(\varphi_1) = P_{Y,k_2}(\varphi_2)$ for $\varphi_1 \in \mathcal{E}^1(X, \omega, \psi_{k_1}), \varphi_2 \in \mathcal{E}^1(X, \omega, \psi_{k_2})$ if $P_{k_1}(\varphi_1) = P_{k_2}(\varphi_2)$. Indeed

$$\begin{aligned} d_Y(P_{Y,k_1}(\varphi_1), P_{Y,k_2}(\varphi_2)) &= d_Y(P_{Y,j} \circ P_{j,k_1}(\varphi_1), P_{Y,j} \circ P_{j,k_2}(\varphi_2)) \leq \\ &\leq d(P_{j,k_1}(\varphi_1), P_{j,k_2}(\varphi_2)) \rightarrow d(P_{k_1}(\varphi_1), P_{k_2}(\varphi_2)) = 0 \end{aligned}$$

as $j \rightarrow \infty$ by Proposition 3.4.7.

To finish the proof it remains to check that $P_{Y,T}$ is 1-Lipschitz. Fixed $u, v \in T$, we have for any $j \in \mathbb{N}$ big enough

$$\begin{aligned} d_Y(P_{Y,T}(u), P_{Y,T}(v)) &= d_Y(P_{Y,j} \circ P_{j,k_u}(\varphi_u), P_{Y,j} \circ P_{j,k_v}(\varphi_v)) \leq \\ &\leq d(P_{j,k_u}(\varphi_u), P_{j,k_v}(\varphi_v)), \end{aligned}$$

where $P_{k_u}(\varphi_u) = u, P_{k_v}(\varphi_v) = v$. Hence $d_Y(P_{Y,T}(u), P_{Y,T}(v)) \leq d(u, v)$ letting $j \rightarrow +\infty$. \square

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PAPER III

The strong topology of ω -plurisubharmonic functions

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arxiv preprint

Chapter 4

The strong topology of ω -plurisubharmonic functions

Abstract

On (X, ω) compact Kähler manifold, given a model type envelope $\psi \in PSH(X, \omega)$ (i.e. a singularity type) we prove that the Monge-Ampère operator is an homeomorphism between the set of ψ -relative finite energy potentials and the set of ψ -relative energy measures endowed with their strong topologies given as the coarsest refinements of the weak topologies such that the relative energies become continuous. Moreover, given a totally ordered family \mathcal{A} of model type envelopes with positive total mass representing different singularities types, the sets $X_{\mathcal{A}}, Y_{\mathcal{A}}$ given respectively as the union of all ψ -relative finite energy potentials and of all ψ -relative finite energy measures varying $\psi \in \overline{\mathcal{A}}$ have two natural strong topologies which extends the strong topologies on each component of the unions. We show that the Monge-Ampère operator produces an homeomorphism between $X_{\mathcal{A}}$ and $Y_{\mathcal{A}}$.

As an application we also prove the strong stability of a sequence of solutions of complex Monge-Ampère equations when the measures have uniformly L^p -bounded densities for $p > 1$ and the prescribed singularities are totally ordered.

Keywords: Complex Monge-Ampère equations, compact Kähler manifolds, quasi-psh functions.

2020 Mathematics subject classification: 32W20 (primary); 32U05, 32Q15 (secondary).

4.1 Introduction

Let (X, ω) be a compact Kähler manifold where ω is a fixed Kähler form, and let \mathcal{H}_ω denote the set of all Kähler potentials, i.e. all $\varphi \in C^\infty$ such that $\omega + dd^c\varphi$ is a Kähler form, the pioneering work of Yau ([Yau78]) shows that the Monge-Ampère operator

$$MA_\omega : \mathcal{H}_{\omega, norm} \longrightarrow \left\{ dV \text{ volume form} : \int_X dV = \int_X \omega^n \right\}, \quad (4.1)$$

$MA_\omega(\varphi) := (\omega + dd^c\varphi)^n$ is a bijection, where for any subset $A \subset PSH(X, \omega)$ of all ω -plurisubharmonic functions we use the notation $A_{norm} := \{u \in A : \sup_X u = 0\}$. Note that the assumption on the total mass of the volume forms in (4.1) is necessary since $\mathcal{H}_{\omega, norm}$ represents all Kähler forms in the cohomology class $\{\omega\}$ and the quantity $\int_X \omega^n$ is cohomological.

In [BEGZ10] the authors extended the Monge-Ampère operator using the *non-pluripolar product* and the bijection (4.1) to

$$MA_\omega : \mathcal{E}_{norm}(X, \omega) \longrightarrow \left\{ \mu \text{ non-pluripolar positive measure} : \mu(X) = \int_X \omega^n \right\} \quad (4.2)$$

where $\mathcal{E}(X, \omega) := \{u \in PSH(X, \omega) : \int_X MA_\omega(u) = \int_X MA_\omega(0)\}$ is the set of all ω -psh functions with full mass.

The set $PSH(X, \omega)$ is naturally endowed with the L^1 -topology which we will call *weak*, but the Monge-Ampère operator in (4.2) is not continuous even if the set of measures is endowed with the weak topology. Thus in [BBEGZ19], setting $V_0 := \int_X MA_\omega(0)$, two strong topologies were respectively introduced for

$$\begin{aligned} \mathcal{E}^1(X, \omega) &:= \{u \in \mathcal{E}(X, \omega) : E(u) > -\infty\} \\ \mathcal{M}^1(X, \omega) &:= \left\{ V_0 \mu : \mu \text{ is a probability measure satisfying } E^*(\mu) < +\infty \right\} \end{aligned}$$

as the coarsest refinements of the weak topologies such that respectively the Monge-Ampère energy $E(u)$ ([Aub84], [BB10], [BEGZ10]) and the energy for probability measures E^* ([BBGZ13], [BBEGZ19]) becomes continuous. The map

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega), strong) \longrightarrow (\mathcal{M}^1(X, \omega), strong) \quad (4.3)$$

is then an homeomorphism. Later Darvas ([Dar15]) showed that $(\mathcal{E}^1(X, \omega), strong)$ actually coincides with the metric closure of \mathcal{H}_ω endowed with the Finsler metric $|f|_{1, \varphi} := \int_X |f| MA_\omega(\varphi)$, $\varphi \in \mathcal{H}_\omega$, $f \in T_\varphi \mathcal{H}_\omega \simeq C^\infty(X)$ and associated distance

$$d(u, v) := E(u) + E(v) - E(P_\omega(u, v))$$

where $P_\omega(u, v)$ is the rooftop envelope given basically as the largest ω -psh function bounded above by $\min(u, v)$ ([RWN14]). This metric topology has played an important role in the last decade to characterize the existence of special metrics ([DR15],

[BDL16], [CC17], [CC18a], [CC18b]).

It is also important and natural to solve complex Monge-Ampère equations requiring that the solutions have some prescribed behavior, for instance along a divisor. We first need to recall that on $PSH(X, \omega)$ there is a natural partial order \preceq given as $u \preceq v$ if $u \leq v + O(1)$, and the total mass through the Monge-Ampère operator respects such partial order, i.e. $V_u := \int_X MA_\omega(u) \leq V_v$ if $u \preceq v$ ([BEGZ10], [WN17]). Thus in [DDNL17] the authors introduced the ψ -relative analogs of the sets $\mathcal{E}(X, \omega)$, $\mathcal{E}^1(X, \omega)$ for $\psi \in PSH(X, \omega)$ fixed as

$$\begin{aligned}\mathcal{E}(X, \omega, \psi) &:= \{u \in PSH(X, \omega) : u \preceq \psi \text{ and } V_u = V_\psi\} \\ \mathcal{E}^1(X, \omega, \psi) &:= \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}\end{aligned}$$

where E_ψ is the ψ -relative energy, and they proved that

$$MA_\omega : \mathcal{E}_{norm}(X, \omega, \psi) \longrightarrow \left\{ \mu \text{ non-pluripolar positive measure} : \mu(X) = V_\psi \right\} \quad (4.4)$$

is a bijection if and only if ψ , up to a bounded function, is a *model type envelope*, i.e. $\psi = (\lim_{C \rightarrow +\infty} P(\psi + C, 0))^*$, satisfying $V_\psi > 0$ (the star is for the upper semicontinuous regularization). There are plenty of these functions, for instance to any ω -psh function ψ with analytic singularities is associated an unique model type envelope. We denote with \mathcal{M} the set of all model type envelopes and with \mathcal{M}^+ those elements ψ such that $V_\psi > 0$.

Letting $\psi \in \mathcal{M}^+$, in [Tru19], we proved that $\mathcal{E}^1(X, \omega, \psi)$ can be endowed with a natural metric topology given by the complete distance $d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$.

Analogously to E^* there is a natural ψ -relative energy for probability measures E_ψ^* , thus the set

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E_\psi^*(\mu) < +\infty\}$$

can be endowed with its strong topology given as the coarsest refinement of the weak topology such that E_ψ^* becomes continuous.

Theorem A. *Let $\psi \in \mathcal{M}^+$. Then*

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega, \psi), d) \rightarrow (\mathcal{M}^1(X, \omega, \psi), \text{strong}) \quad (4.5)$$

is an homeomorphism.

Then it is natural to wonder if one can extend the bijections (4.2), (4.4) to bigger subsets of $PSH(X, \omega)$.

Given $\psi_1, \psi_2 \in \mathcal{M}^+$ such that $\psi_1 \neq \psi_2$ the sets $\mathcal{E}(X, \omega, \psi_1)$, $\mathcal{E}(X, \omega, \psi_2)$ are disjoint (Theorem 1.3 [DDNL17] quoted below as Theorem 4.2.1) but it may happen that $V_{\psi_1} = V_{\psi_2}$. So in that case, as an easy consequence of (4.4) one cannot consider a

set containing both $\mathcal{E}(X, \omega, \psi_1)$ and $\mathcal{E}(X, \omega, \psi_2)$. But given a totally ordered family $\mathcal{A} \subset \mathcal{M}^+$ of model type envelopes, the map $\mathcal{A} \ni \psi \rightarrow V_\psi$ is injective (again by Theorem 1.3 [DDNL17]), i.e.

$$MA_\omega : \bigsqcup_{\psi \in \mathcal{A}} \mathcal{E}(X, \omega, \psi)/\mathbb{R} \longrightarrow \left\{ \mu \text{ non-pluripolar positive measure} : \right. \\ \left. \mu(X) = V_\psi \text{ for } \psi \in \mathcal{A} \right\}$$

is a bijection.

In [Tru19] we introduced a complete distance $d_{\mathcal{A}}$ on

$$X_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

where $\overline{\mathcal{A}} \subset \mathcal{M}$ is the weak closure of \mathcal{A} and where we set $\mathcal{E}^1(X, \omega, \psi_{\min}) = P_{\psi_{\min}}$ if $\psi \in \mathcal{M} \setminus \mathcal{M}^+$ (since in such case $E_\psi \equiv 0$). Here ψ_{\min} is given as the smallest element in $\overline{\mathcal{A}}$, observing that the Monge-Ampère operator $MA_\omega : \overline{\mathcal{A}} \rightarrow MA_\omega(\overline{\mathcal{A}})$ is an homeomorphism when the range is endowed with the weak topology (Lemma 4.3.12). We call strong topology on $X_{\mathcal{A}}$ the metric topology given by $d_{\mathcal{A}}$ since $d_{\mathcal{A}}|_{\mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi)} = d$. The precise definition of $d_{\mathcal{A}}$ is quite technical (in section §5.2 we will recall many of its properties) but the strong topology is natural since it is the coarsest refinement of the weak topology such that $E(\cdot)$ becomes continuous as Theorem 4.6.2 shows. In particular the strong topology is independent on the set \mathcal{A} chosen.

Also the set

$$Y_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi)$$

has a natural strong topology given as the coarsest refinement of the weak topology such that $E^*(\cdot)$ becomes continuous.

Theorem B. *The Monge-Ampère map*

$$MA_\omega : (X_{\mathcal{A}, \text{norm}}, d_{\mathcal{A}}) \rightarrow (Y_{\mathcal{A}}, \text{strong})$$

is an homeomorphism.

Obviously in Theorem B we define $MA_\omega(P_{\psi_{\min}}) := 0$ if $V_{\psi_{\min}} = 0$.

Note that by Hartogs' Lemma and Theorem 4.6.2 the metric subspace $X_{\mathcal{A}, \text{norm}}$ is complete and it represents the set of all closed and positive $(1, 1)$ -currents $T = \omega + dd^c u$ such that $u \in X_{\mathcal{A}}$, where $P_{\psi_{\min}}$ encases all currents whose potentials u are more singular than ψ_{\min} if $V_{\psi_{\min}} = 0$.

Finally, as an application of Theorem B we study an example of the stability of solutions of complex Monge-Ampère equations. Other important situations will be dealt in a future work.

Theorem C. *Let $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be totally ordered, and let $\{f_k\}_{k \in \mathbb{N}} \subset L^1 \setminus \{0\}$ a sequence of non-negative functions such that $f_k \rightarrow f \in L^1 \setminus \{0\}$ and such that $\int_X f_k \omega^n = V_{\psi_k}$ for any $k \in \mathbb{N}$. Assume also that there exists $p > 1$ such that $\|f_k\|_{L^p}, \|f\|_{L^p}$ are uniformly bounded. Then $\psi_k \rightarrow \psi \in \mathcal{M}^+$ weakly, the sequence $\{u_k\}_{k \in \mathbb{N}}$ of solutions of*

$$\begin{cases} MA_\omega(u_k) = f_k \omega^n \\ u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k) \end{cases} \quad (4.6)$$

converges strongly to $u \in X_{\mathcal{A}}$ (i.e. $d_{\mathcal{A}}(u_k, u) \rightarrow 0$), which is the unique solution of

$$\begin{cases} MA_\omega(u) = f \omega^n \\ u \in \mathcal{E}_{norm}^1(X, \omega, \psi). \end{cases}$$

In particular $u_k \rightarrow u$ in capacity.

The existence of the solutions of (4.6) follows by Theorem A in [DDNL18], while the fact that the strong convergence implies the convergence in capacity is our Theorem 4.6.3. Note also that the convergence in capacity of Theorem C was already obtained in [DDNL19] (see Remark 4.7.1).

4.1.1 Structure of the paper

Section §5.2 is dedicated to introduce some preliminaries, and in particular all necessary results presented in [Tru19]. In section §4.3 we extend some known uniform estimates for $\mathcal{E}^1(X, \omega)$ to the relative setting, and we prove the key upper-semicontinuity of the relative energy functional $E(\cdot)$ in $X_{\mathcal{A}}$. Section §4.4 regards the properties of the action of measures on $PSH(X, \omega)$ and in particular their continuity. Then Section §4.5 is dedicated to prove Theorem A. We use a variational approach to show the bijection, then we need some further important properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ to conclude the proof. Section §4.6 is the heart of the article where we extends the results proved in the previous section to $X_{\mathcal{A}}$ and we present our main Theorem B. Finally in the last Section §4.7 we show Theorem C.

4.1.2 Future developments

As said above, in a future work we will present some strong stability results of more general solutions of complex Monge-Ampère equations with prescribed singularities than Theorem C, starting the study of a kind of *continuity method* when also the singularities will vary. As an application we will study the existence of (log) Kähler-Einstein metrics with prescribed singularities with a particular focus on the relationships among them varying the singularities.

4.1.3 Acknowledgments

I want to thank David Witt Nyström and Stefano Trapani for their suggestions and comments. I am also grateful to Hoang-Chinh Lu to have pointed me out a minor mistake in the previous version.

4.2 Preliminaries

We recall that given (X, ω) a Kähler complex compact manifold, the set $PSH(X, \omega)$ is the set of all ω -plurisubharmonic (ω -psh), i.e. all $u \in L^1$ given locally as sum of a smooth function and of a plurisubharmonic function such that $\omega + dd^c u \geq 0$ as $(1, 1)$ -current. Here $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$. For any couple of ω -psh functions u, v the function

$$P_\omega[u](v) := \left(\lim_{C \rightarrow \infty} P_\omega(u + C, v) \right)^* = \left(\sup\{w \in PSH(X, \omega) : w \preceq u, w \leq v\} \right)^*$$

is ω -psh where the star is for the upper semicontinuous regularization and $P_\omega(u, v) := \left(\sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\} \right)^*$. Then the set of all model type envelopes is defined as

$$\mathcal{M} := \{\psi \in PSH(X, \omega) : \psi = P_\omega[\psi](0)\}.$$

We also recall that \mathcal{M}^+ denotes the elements $\psi \in \mathcal{M}$ such that $V_\psi > 0$ where, as said in the Introduction, $V_\psi := \int_X MA_\omega(\psi)$.

The class of ψ -relative full mass functions $\mathcal{E}(X, \omega, \psi)$ complies the following characterization in terms of \mathcal{M} .

Theorem 4.2.1 (Theorem 1.3, [DDNL17]). *Suppose $v \in PSH(X, \omega)$ such that $V_v > 0$ and $u \in PSH(X, \omega)$ more singular than v . The followings are equivalent:*

- (i) $u \in \mathcal{E}(X, \omega, v)$;
- (ii) $P_\omega[u](v) = v$;
- (iii) $P_\omega[u](0) = P_\omega[v](0)$.

The clear inclusion $\mathcal{E}(X, \omega, v) \subset \mathcal{E}(X, \omega, P_\omega[v](0))$ may be strict, and it seems more natural in many cases to consider only functions $\psi \in \mathcal{M}$. For instance as showed in [DDNL17] ψ being a model type envelope is a necessary assumption to make the equation

$$\begin{cases} MA_\omega(u) = \mu \\ u \in \mathcal{E}(X, \omega, \psi) \end{cases}$$

always solvable where μ is a non-pluripolar measure such that $\mu(X) = V_\psi$. It is also worth to recall that there are plenty of elements in \mathcal{M} since $P_\omega[P_\omega[\psi]] = P_\omega[\psi]$. Indeed $v \rightarrow P_\omega[v]$ may be thought as a projection from the set of ω -psh functions to \mathcal{M} .

We also retrieve the following useful result.

Theorem 4.2.2 (Theorem 3.8, [DDNL17]) . *Let $u, \psi \in PSH(X, \omega)$ such that $u \succ \psi$. Then*

$$MA_\omega(P_\omega[\psi](u)) \leq \mathbb{1}_{\{P_\omega[\psi](u)=u\}} MA_\omega(u).$$

In particular if $\psi \in \mathcal{M}$ then $MA_\omega(\psi) \leq \mathbb{1}_{\{\psi=0\}} MA_\omega(0)$.

Note also that in Theorem 4.2.2 the equality holds if u is continuous with bounded distributional laplacian with respect to ω as a consequence of [DNT19]. In particular $MA_\omega(\psi) = \mathbb{1}_{\{\psi=0\}} MA_\omega(0)$ for any $\psi \in \mathcal{M}$.

4.2.1 The metric space $(\mathcal{E}^1(X, \omega, \psi), d)$.

In this subsection we assume $\psi \in \mathcal{M}^+$ where $\mathcal{M}^+ := \{\psi \in \mathcal{M} : V_\psi > 0\}$.

As in [DDNL17] we also denote with $PSH(X, \omega, \psi)$ the set of all ω -psh functions which are more singular than ψ , and we recall that a function $u \in PSH(X, \omega, \psi)$ has ψ -relative minimal singularities if $|u - \psi|$ is globally bounded on X . We also use the notation

$$MA_\omega(u_1^{j_1}, \dots, u_l^{j_l}) := (\omega + dd^c u_1)^{j_1} \wedge \dots \wedge (\omega + dd^c u_l)^{j_l}$$

for $u_1, \dots, u_l \in PSH(X, \omega)$ where $j_1, \dots, j_l \in \mathbb{N}$ such that $j_1 + \dots + j_l = n$.

Definition 4.2.3 ([DDNL17]) . *The ψ -relative energy functional $E_\psi : PSH(X, \omega, \psi) \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as*

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X (u - \psi) MA_\omega(u^j, \psi^{n-j})$$

if u has ψ -relative minimal singularities, and as

$$E_\psi(u) := \inf\{E_\psi(v) : v \in \mathcal{E}(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities, } v \geq u\}$$

otherwise. The subset $\mathcal{E}^1(X, \omega, \psi) \subset \mathcal{E}(X, \omega, \psi)$ is defined as

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}.$$

When $\psi = 0$ the ψ -relative energy functional is the *Aubin-Mabuchi energy functional*, also called *Monge-Ampère energy* (see [Aub84], [Mab86]).

Proposition 4.2.4 ([DDNL17]) . *The following properties hold:*

- (i) E_ψ is non decreasing;
- (ii) $E_\psi(u) = \lim_{j \rightarrow \infty} E_\psi(\max(u, \psi - j))$;
- (iii) E_ψ is continuous along decreasing sequences;
- (iv) E_ψ is concave along affine curves;
- (v) $u \in \mathcal{E}^1(X, \omega, \psi)$ if and only if $u \in \mathcal{E}(X, \omega, \psi)$ and $\int_X (u - \psi) MA_\omega(u) > -\infty$;

- (vi) $E_\psi(u) \geq \limsup_{k \rightarrow \infty} E_\psi(u_k)$ if $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$ and $u_k \rightarrow u$ with respect to the weak topology;
- (vii) letting $u \in \mathcal{E}^1(X, \omega, \psi)$, $\chi \in \mathcal{C}^0(X)$ and $u_t := \sup\{v \in PSH(X, \omega) \mid v \leq u + t\chi\}^*$ for any $t > 0$, then $t \rightarrow E_\psi(u_t)$ is differentiable and its derivative is given by

$$\frac{d}{dt} E_\psi(u_t) = \int_X \chi MA_\omega(u_t);$$

- (viii) if $u, v \in \mathcal{E}^1(X, \omega, \psi)$ then

$$E_\psi(u) - E_\psi(v) = \sum_{j=0}^{n+1} \int_X (u - v) MA_\omega(u^j, v^{n-j})$$

and the function $\mathbb{N} \ni j \rightarrow \int_X (u - v) MA_\omega(u^j, v^{n-j})$ is decreasing. In particular

$$\int_X (u - v) MA_\omega(u) \leq E_\psi(u) - E_\psi(v) \leq \int_X (u - v) MA_\omega(v);$$

- (ix) if $u \leq v$ then $E_\psi(u) - E_\psi(v) \leq \frac{1}{n+1} \int_X (u - v) MA_\omega(u)$.

Remark 4.2.5. All the properties in Proposition 4.2.4 are showed in [DDNL17] when the authors worked assuming ψ having *small unbounded locus*, but the general integration by parts formula proved in [X19a] allows to extend these properties to the general case.

Recalling that for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ the function $P_\omega(u, v) = \sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\}^*$ belongs to $\mathcal{E}^1(X, \omega, \psi)$ (see Proposition 2.10. in [Tru19]), the function $d : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R}_{\geq 0}$ defined as

$$d(u, v) = E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$$

assumes finite values. Moreover it is a complete distance as the next result shows.

Theorem 4.2.6 (Theorem A, [Tru19]) . $(\mathcal{E}^1(X, \omega, \psi), d)$ is a complete metric space.

We call *strong topology* on $\mathcal{E}^1(X, \omega, \psi)$ the metric topology given by the distance d . Note that by construction $d(u_k, u) \rightarrow 0$ as $k \rightarrow \infty$ if $u_k \searrow u$, and that $d(u, v) = d(u, w) + d(w, v)$ if $u \leq w \leq v$ (see Lemma 3.1 in [Tru19]). Moreover as a consequence of Proposition 4.2.4 it follows that for any $C \in \mathbb{R}_{>0}$ the set

$$\mathcal{E}_C^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u \leq C \text{ and } E_\psi(u) \geq -C\}$$

is a weakly compact convex set.

Remark 4.2.7. As described in [Tru19], if $\psi \in \mathcal{M} \setminus \mathcal{M}^+$ then $\mathcal{E}^1(X, \omega, \psi) = PSH(X, \omega, \psi)$ since $E_\psi \equiv 0$ by definition. In particular $d \equiv 0$ and it is natural to identify $(\mathcal{E}^1(X, \omega, \psi), d)$ with a point P_ψ . Moreover we recall that $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \emptyset$ if $\psi_1, \psi_2 \in \mathcal{M}$, $\psi_1 \neq \psi_2$ and $V_{\psi_2} > 0$.

4.2.2 The space $(X_{\mathcal{A}}, d_{\mathcal{A}})$.

From now on we assume $\mathcal{A} \subset \mathcal{M}^+$ to be a totally ordered set of model type envelopes, and we denote with $\overline{\mathcal{A}}$ its closure as subset of $PSH(X, \omega)$ endowed with the weak topology. Note that $\overline{\mathcal{A}} \subset PSH(X, \omega)$ is compact by Lemma 4.2 in [Tru19]. Indeed we will prove in Lemma 4.3.12 that actually $\overline{\mathcal{A}}$ is homeomorphic to its image through the Monge-Ampère operator MA_{ω} when the set of measure is endowed with the weak topology, which yields that $\overline{\mathcal{A}}$ is also homeomorphic to a closed set contained in $[0, \int_X \omega^n]$ through the map $\psi \rightarrow V_{\psi}$.

Definition 4.2.8. *We define the set*

$$X_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$$

if $\psi_{\min} := \inf \mathcal{A}$ satisfies $V_{\psi_{\min}} > 0$, and

$$X_{\mathcal{A}} := P_{\psi_{\min}} \sqcup \bigsqcup_{\psi' \in \overline{\mathcal{A}}, \psi' \neq \psi_{\min}} \mathcal{E}^1(X, \omega, \psi')$$

if $V_{\psi_{\min}} = 0$, where $P_{\psi_{\min}}$ is a singleton.

$X_{\mathcal{A}}$ can be endowed with a natural metric structure as section 4 in [Tru19] shows.

Theorem 4.2.9 (Theorem B, [Tru19]). *$(X_{\mathcal{A}}, d_{\mathcal{A}})$ is a complete metric space such that $d_{\mathcal{A}}|_{\mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi)} = d$ for any $\psi \in \overline{\mathcal{A}} \cap \mathcal{M}^+$.*

We call *strong topology* on $X_{\mathcal{A}}$ the metric topology given by the distance $d_{\mathcal{A}}$. Note that the denomination is coherent with that of subsection 5.2.1 since the induced topology on $\mathcal{E}^1(X, \omega, \psi) \subset X_{\mathcal{A}}$ coincides with the strong topology given by d .

We will also need the following contraction property which is the starting point to construct $d_{\mathcal{A}}$.

Proposition 4.2.10 (Lemma 4.4., Proposition 4.5., [Tru19]). *Let $\psi_1, \psi_2, \psi_3 \in \mathcal{M}$ such that $\psi_1 \preceq \psi_2 \preceq \psi_3$. Then $P_{\omega}[\psi_1](P_{\omega}[\psi_2](u)) = P_{\omega}[\psi_1](u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi_3)$ and $|P_{\omega}[\psi_1](u) - \psi_1| \leq C$ if $|u - \psi_3| \leq C$. Moreover the map*

$$P_{\omega}[\psi_1](\cdot) : \mathcal{E}^1(X, \omega, \psi_2) \rightarrow PSH(X, \omega, \psi_1)$$

has image in $\mathcal{E}^1(X, \omega, \psi_1)$ and it is a Lipschitz map of constant 1 when the sets $\mathcal{E}^1(X, \omega, \psi_i)$, $i = 1, 2$, are endowed with the d distances, i.e.

$$d(P_{\omega}[\psi_1](u), P_{\omega}[\psi_1](v)) \leq d(u, v)$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi_2)$.

Here we report some properties of the distance $d_{\mathcal{A}}$ and some consequences which will be useful in the sequel.

Proposition 4.2.11 ([Tru19]). *The following properties hold:*

i) if $u \in \mathcal{E}^1(X, \omega, \psi_1), v \in \mathcal{E}^1(X, \omega, \psi_2)$ for $\psi_1, \psi_2 \in \overline{\mathcal{A}}, \psi_1 \succcurlyeq \psi_2$ then

$$d_{\mathcal{A}}(u, v) \geq d(P_{\omega}[\psi_2](u), v);$$

ii) if $\{\psi_k\}_{k \in \mathbb{N}}, \psi \in \mathcal{M}$ with $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ a.e.), $u_k \searrow u, v_k \searrow v$ (resp. $u_k \nearrow u$ a.e., $v_k \nearrow v$ a.e.) for $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k), u, v \in \mathcal{E}^1(X, \omega, \psi)$ and $|u_k - v_k|$ is uniformly bounded, then

$$d(u_k, v_k) \rightarrow d(u, v);$$

iii) if $\psi_k, \psi \in \mathcal{M}$ such that $\psi_k \rightarrow \psi$ monotonically a.e., then for any $\psi' \in \mathcal{M}$ such that $\psi' \succcurlyeq \psi_k$ for any $k \gg 1$ big enough, and for any strongly compact set $K \subset (\mathcal{E}^1(X, \omega, \psi'), d)$,

$$d(P_{\omega}[\psi_k](\varphi_1), P_{\omega}[\psi_k](\varphi_2)) \rightarrow d(P_{\omega}[\psi](\varphi_1), P_{\omega}[\psi](\varphi_2))$$

uniformly on $K \times K$, i.e. varying $(\varphi_1, \varphi_2) \in K \times K$. In particular if $\psi_k, \psi \in \overline{\mathcal{A}}$ then

$$\begin{aligned} d_{\mathcal{A}}(P_{\omega}[\psi](u), P_{\omega}[\psi_k](u)) &\rightarrow 0 \\ d(P_{\omega}[\psi_k](u), P_{\omega}[\psi_k](v)) &\rightarrow d(P_{\omega}[\psi](u), P_{\omega}[\psi](v)) \end{aligned}$$

monotonically for any $(u, v) \in \mathcal{E}^1(X, \omega, \psi') \times \mathcal{E}^1(X, \omega, \psi')$;

iv) $d_{\mathcal{A}}(u_1, u_2) \geq |V_{\psi_1} - V_{\psi_2}|$ if $u_1 \in \mathcal{E}^1(X, \omega, \psi_1), u_2 \in \mathcal{E}^1(X, \omega, \psi_2)$ and the equality holds if $u_1 = \psi_1, u_2 = \psi_2$.

The following Lemma is a special case of Theorem 2.2 in [X19a] (see also Lemma 4.1. in [DDNL17]).

Lemma 4.2.12 (Lemma 4.3, [Tru19]). *Let $\psi_k, \psi \in \mathcal{M}$ such that $\psi_k \rightarrow \psi$ monotonically almost everywhere. Let also $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k)$ converging in capacity respectively to $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then for any $j = 0, \dots, n$*

$$MA_{\omega}(u_k^j, v_k^{n-j}) \rightarrow MA_{\omega}(u^j, v^{n-j})$$

weakly. Moreover if $|u_k - v_k|$ is uniformly bounded, then for any $j = 0, \dots, n$

$$(u_k - v_k)MA_{\omega}(u_k^j, v_k^{n-j}) \rightarrow (u - v)MA_{\omega}(u^j, v^{n-j})$$

weakly.

It is well-known that the set of Kähler potentials $\mathcal{H}_{\omega} := \{\varphi \in PSH(X, \omega) \cap C^{\infty}(X) : \omega + dd^c \varphi > 0\}$ is dense into $(\mathcal{E}^1(X, \omega), d)$. The same holds for $P_{\omega}[\psi](\mathcal{H}_{\omega})$ into $(\mathcal{E}^1(X, \omega, \psi), d)$.

Lemma 4.2.13 (Lemma 4.9, [Tru19]). *The set $\mathcal{P}_{\mathcal{H}_{\omega}}(X, \omega, \psi) := P_{\omega}[\psi](\mathcal{H}) \subset \mathcal{P}(X, \omega, \psi)$ is dense in $(\mathcal{E}^1(X, \omega, \psi), d)$.*

4.3 Tools.

In this section we collect some uniform estimates on $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$, we recall the ψ -relative capacity and we will prove the upper semicontinuity of $E(\cdot)$ on $X_{\mathcal{A}}$.

4.3.1 Uniform estimates.

Let $\psi \in \mathcal{M}^+$.

We first define in the ψ -relative setting the analogous of some well-known functionals of the variational approach (see [BBGZ13] and reference therein).

We introduce respectively the ψ -relative I -functional and the ψ -relative J -functional (see also [Aub84]) $I_{\psi}, J_{\psi} : \mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R}$ where $\psi \in \mathcal{M}^+$ as

$$I_{\psi}(u, v) := \int_X (u - v)(MA_{\omega}(v) - MA_{\omega}(u)),$$

$$J_{\psi}(u, v) := J_u^{\psi}(v) := E_{\psi}(u) - E_{\psi}(v) + \int_X (v - u)MA_{\omega}(u).$$

They assume non-negative values by Proposition 4.2.4, I_{ψ} is clearly symmetric while J_{ψ} is convex again by Proposition 4.2.4. Moreover the ψ -relative I and J functionals are related each other by the following result.

Lemma 4.3.1. *Let $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then*

- (i) $\frac{1}{n+1}I_{\psi}(u, v) \leq J_u^{\psi}(v) \leq \frac{n}{n+1}I_{\psi}(u, v);$
- (ii) $\frac{1}{n}J_u^{\psi}(v) \leq J_v^{\psi}(u) \leq nJ_u^{\psi}(v).$

In particular

$$d(\psi, u) \leq nJ_u^{\psi}(\psi) + (\|\psi\|_{L^1} + \|u\|_{L^1})$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \leq \psi$.

Proof. By Proposition 4.2.4 it follows that

$$\begin{aligned} n \int_X (u - v)MA_{\omega}(u) + \int_X (u - v)MA_{\omega}(v) &\leq \\ &\leq (n+1)(E_{\psi}(u) - E_{\psi}(v)) \leq \int_X (u - v)MA_{\omega}(u) + n \int_X (u - v)MA_{\omega}(v) \end{aligned}$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$, which yields (i) and (ii).

Next considering $v = \psi$ and assuming $u \leq \psi$ from the second inequality in (ii) we obtain

$$d(u, \psi) = -E_{\psi}(u) \leq nJ_u^{\psi}(\psi) + \int_X (\psi - u)MA_{\omega}(\psi),$$

which implies the assertion since $MA_{\omega}(\psi) \leq MA_{\omega}(0)$ by Theorem 4.2.2. \square

We can now proceed showing the uniform estimates, adapting some results in [BBGZ13].

Lemma 4.3.2 (Lemma 3.8, [Tru19]). *Let $\psi \in \mathcal{M}^+$. Then there exists positive constants $A > 1$, $B > 0$ depending only on n, ω such that*

$$-d(\psi, u) \leq V_\psi \sup_X (u - \psi) = V_\psi \sup_X u \leq Ad(\psi, u) + B$$

Remark 4.3.3. As a consequence of Lemma 4.3.2 if $d(\psi, u) \leq C$ then $\sup_X u \leq (AC + B)/V_\psi$ while $-E_\psi(u) = d(\psi + (AC + B)/V_\psi, u) - (AC + B) \leq d(\psi, u) \leq C$, i.e. $u \in \mathcal{E}_D^1(X, \omega, \psi)$ where $D := \max(C, (AC + B)/V_\psi)$. Vice versa it is easy to check that $d(u, \psi) \leq C(2V_\psi + 1)$ for any $u \in \mathcal{E}_C^1(X, \omega, \psi)$ using the definitions and the triangle inequality.

Proposition 4.3.4. *Let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_C : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*

$$\left| \int_X (u - v)(MA_\omega(\varphi_1) - MA_\omega(\varphi_2)) \right| \leq f_C(d(u, v)) \quad (4.7)$$

for any $u, v, \varphi_1, \varphi_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u, \psi), d(v, \psi), d(\varphi_1, \psi), d(\varphi_2, \psi) \leq C$.

Proof. As said in Remark 4.3.3 if $w \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, w) \leq C$ then $\tilde{w} := w - (AC + B)/V_\psi$ satisfies $\sup_X \tilde{w} \leq 0$ and

$$-E_\psi(\tilde{w}) = d(\psi, \tilde{w}) \leq d(\psi, w) + d(w, \tilde{w}) \leq C + AC + B =: D.$$

Therefore setting $\tilde{u} := u - (AC + B)/V_\psi, \tilde{v} := v - (AC + B)/V_\psi$ we can proceed exactly as in Lemma 5.8 in [BBGZ13] using the integration by parts formula in [X19a] (see also Theorem 1.14 in [BEGZ10]) to get

$$\left| \int_X (\tilde{u} - \tilde{v})(MA_\omega(\varphi_1) - MA_\omega(\varphi_2)) \right| \leq I_\psi(\tilde{u}, \tilde{v}) + h_D(I_\psi(\tilde{u}, \tilde{v})) \quad (4.8)$$

where $h_D : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is an increasing continuous function depending only on D such that $h_D(0) = 0$. Furthermore, by definition

$$d(\psi, P_\omega(\tilde{u}, \tilde{v})) \leq d(\psi, \tilde{u}) + d(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) \leq d(\psi, \tilde{u}) + d(\tilde{u}, \tilde{v}) \leq 3D,$$

so, by the triangle inequality and (4.8), we have

$$\begin{aligned} \left| \int_X (u - v)(MA_\omega(\varphi_1) - MA_\omega(\varphi_2)) \right| &\leq I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v})) + \\ &+ I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v})) + h_{3D}(I_\psi(\tilde{u}, P_\omega(\tilde{u}, \tilde{v}))) + h_{3D}(I_\psi(\tilde{v}, P_\omega(\tilde{u}, \tilde{v}))). \end{aligned} \quad (4.9)$$

On the other hand, if $w_1, w_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $w_1 \geq w_2$ then by Proposition 4.2.4

$$I_\psi(w_1, w_2) \leq \int_X (w_1 - w_2) MA_\omega(w_2) \leq (n+1)d(w_1, w_2).$$

Hence from (4.9) it is sufficient to set $f_C(x) := (n+1)x + 2h_{3D}((n+1)x)$ to conclude the proof since clearly $d(\tilde{u}, \tilde{v}) = d(u, v)$. \square

Corollary 4.3.5. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing functions $f_C : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ depending only on C, ω, n with $f_C(0) = 0$ such that*

$$\int_X |u - v| MA_\omega(\varphi) \leq f_C(d(u, v))$$

for any $u, v, \varphi \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, u), d(\psi, v), d(\psi, \varphi) \leq C$.

Proof. Since $d(\psi, P_\omega(u, v)) \leq 3C$, letting $g_{3C} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ be the map (4.7) of Proposition 4.3.4, it follows that

$$\begin{aligned} \int_X (u - P_\omega(u, v)) MA_\omega(\varphi) &\leq \int_X (u - P_\omega(u, v)) MA_\omega(P_\omega(u, v)) + \\ &\quad + g_{3C}(d(u, P_\omega(u, v))) \leq (n+1)d(u, P_\omega(u, v)) + g_{3C}(d(u, v)), \end{aligned}$$

where in the last inequality we used Proposition 4.2.4. Hence by the triangle inequality we get

$$\begin{aligned} \int_X |u - v| MA_\omega(\varphi) &\leq (n+1)d(u, P_\omega(u, v)) + (n+1)d(v, P_\omega(u, v)) + \\ &\quad + 2g_{3C}(d(u, v)) = (n+1)d(u, v) + 2g_{3C}(d(u, v)). \end{aligned}$$

Defining $f_C(x) := (n+1)x + 2g_{3C}(x)$ concludes the proof. \square

As first important consequence we obtain that the strong convergence in $\mathcal{E}^1(X, \omega, \psi)$ implies the weak convergence.

Proposition 4.3.6. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a continuous increasing function $f_{C, \psi} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ depending on C, ω, n, ψ with $f_{C, \psi}(0) = 0$ such that*

$$\|u - v\|_{L^1} \leq f_{C, \psi}(d(u, v))$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi)$ with $d(\psi, u), d(\psi, v) \leq C$. In particular $u_k \rightarrow u$ weakly if $u_k \rightarrow u$ strongly.

Proof. Theorem A in [DDNL18] (see also Theorem 1.4 in [DDNL17]) implies that there exists $\phi \in \mathcal{E}^1(X, \omega, \psi)$ with $\sup_X \phi = 0$ such that

$$MA_\omega(\phi) = cMA_\omega(0)$$

where $c := V_\psi/V_0 > 0$. Therefore it follows that

$$\|u - v\|_{L^1} \leq \frac{1}{c} g_{\hat{C}}(d(u, v))$$

where $\hat{C} := \max(d(\psi, \phi), C)$ and $g_{\hat{C}}$ is the continuous increasing function with $g_{\hat{C}}(0) = 0$ given by Corollary 4.3.5. Setting $f_{C, \psi} := \frac{1}{c} g_{\hat{C}}$ concludes the proof. \square

Finally we also get the following useful estimate.

Proposition 4.3.7. *Let $\psi \in \mathcal{M}^+$ and let $C \in \mathbb{R}_{>0}$. Then there exists a constant \tilde{C} depending only on C, ω, n such that*

$$\left| \int_X (u - v)(MA_\omega(\varphi_1) - MA_\omega(\varphi_2)) \right| \leq \tilde{C} I_\psi(\varphi_1, \varphi_2)^{\frac{1}{2}} \quad (4.10)$$

for any $u, v, \varphi_1, \varphi_2 \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u, \psi), d(v, \psi), d(\varphi_1, \psi), d(\varphi_2, \psi) \leq C$.

Proof. As seen during the proof of Proposition 4.3.4 and with the same notations, the function $\tilde{u} := u - (AC + B)/V_\psi$ satisfy $\sup_X u \leq 0$ (by Lemma 4.3.2) and $-E_\psi(u) \leq C + AC + B =: D$ (and similarly for v, φ_1, φ_2). Therefore by integration by parts and using Lemma 4.3.8 below, it follows exactly as in Lemma 3.13 in [BBGZ13] that there exists a constant \tilde{C} depending only on D, n such that

$$\left| \int_X (\tilde{u} - \tilde{v})(MA_\omega(\tilde{\varphi}_1) - MA_\omega(\tilde{\varphi}_2)) \right| \leq \tilde{C} I_\psi(\tilde{\varphi}_1, \tilde{\varphi}_2)^{\frac{1}{2}},$$

which clearly implies (4.10). \square

Lemma 4.3.8. *Let $C \in \mathbb{R}_{>0}$. Then there exists a constant \tilde{C} depending only on C, ω, n such that*

$$\int_X |u_0 - \psi|(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n) \leq \tilde{C}$$

for any $u_0, \dots, u_n \in \mathcal{E}^1(X, \omega, \psi)$ with $d(u_j, \psi) \leq C$ for any $j = 0, \dots, n$.

Proof. As in Proposition 4.3.4 and with the same notations $v_j := u_j - (AC + B)/V_\psi$ satisfies $\sup_X v_j \leq 0$, and setting $v := \frac{1}{n+1}(v_0 + \cdots + v_n)$ we obtain $\psi - u_0 \leq (n+1)(\psi - v)$. Thus by Proposition 4.2.4 it follows that

$$\begin{aligned} \int_X (\psi - v_0) MA_\omega(v) &\leq (n+1) \int_X (\psi - v) MA_\omega(v) \leq (n+1)^2 |E_\psi(v)| \leq \\ &\leq (n+1) \sum_{j=0}^n |E_\psi(v_j)| \leq (n+1) \sum_{j=0}^n (d(\psi, u_j) + D) \leq (n+1)^2 (C + D) \end{aligned}$$

where $D := AC + B$. On the other hand $MA_\omega(v) \geq E(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n)$ where the constant E depends only on n . Finally we get

$$\begin{aligned} \int_X |u_0 - \psi|(\omega + dd^c u_1) \wedge \cdots \wedge (\omega + dd^c u_n) &\leq \\ &\leq D + \frac{1}{E} \int_X (\psi - v_0) MA_\omega(v) \leq D + \frac{(n+1)^2 (C + D)}{E}, \end{aligned}$$

which concludes the proof. \square

4.3.2 ψ -relative Monge-Ampère capacity.

Definition 4.3.9 ([DDNL17], [DDNL18]). *Let $B \subset X$ be a Borel set, and let $\psi \in \mathcal{M}^+$. Then its ψ -relative Monge-Ampère capacity is defined as*

$$\text{Cap}_\psi(B) := \sup \left\{ \int_B MA_\omega(u) : u \in PSH(X, \omega), \psi - 1 \leq u \leq \psi \right\}.$$

In the absolute setting the Monge-Ampère capacity is very useful to study the existence and the regularity of solutions of degenerate complex Monge-Ampère equation ([Kol98]), and analog holds in the relative setting ([DDNL17], [DDNL18]). We refer to these articles just cited to many properties of the Monge-Ampère capacity. Here, for any constant A we introduce the let $\mathcal{C}_{A,\psi}$ be the set of all probability measures μ on X such that

$$\mu(B) \leq A \text{Cap}_\psi(B)$$

for any Borel set $B \subset X$ ([DDNL17]).

Proposition 4.3.10. *Let $u \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities. Then $MA_\omega(u)/V_\psi \in \mathcal{C}_{A,\psi}$ for a constant $A > 0$.*

Proof. Let $j \in \mathbb{R}$ such that $u \geq \psi - j$ and assume without loss of generality that $u \leq \psi$ and that $j \geq 1$. Then the function $v := j^{-1}u + (1 - j^{-1})\psi$ is a candidate in the definition of Cap_ψ , which implies that $MA_\omega(v) \leq \text{Cap}_\psi$. Hence, since $MA_\omega(u) \leq j^n MA_\omega(v)$ we get that $MA_\omega(u) \in \mathcal{C}_{A,\psi}$ for $A = j^n$ and the result follows. \square

We also need to quote the following result.

Lemma 4.3.11 (Lemma 4.18, [DDNL17]). *If $\mu \in \mathcal{C}_{A,\psi}$ then there is a constant $B > 0$ depending only on A, n such that*

$$\int_X (u - \psi)^2 \mu \leq B(|E_\psi(u)| + 1)$$

for any $u \in PSH(X, \omega, \psi)$ such that $\sup_X u = 0$.

Similarly to the case $\psi = 0$ (see [GZ17]), we say that a sequence $u_k \in PSH(X, \omega)$ converges to $u \in PSH(X, \omega)$ in ψ -relative capacity for $\psi \in \mathcal{M}$ if

$$\text{Cap}_\psi(\{|u_k - u| \geq \delta\}) \rightarrow 0$$

as $k \rightarrow \infty$ for any $\delta > 0$.

By Theorem 10.37 in [GZ17] (see also Theorem 5.7 in [BBGZ13]) the convergence in $(\mathcal{E}^1(X, \omega), d)$ implies the convergence in capacity. The analogous holds for $\psi \in \mathcal{M}$, i.e. that the strong convergence in $\mathcal{E}^1(X, \omega, \psi)$ implies the convergence in ψ -relative capacity. Indeed in Proposition 4.5.7 we will prove the the strong convergence implies the convergence in ψ' -relative capacity for any $\psi' \in \mathcal{M}^+$.

4.3.3 (Weak) Upper Semicontinuity of $u \rightarrow E_{P_\omega[u]}(u)$ over $X_{\mathcal{A}}$.

One of the main feature of E_ψ for $\psi \in \mathcal{M}$ is its upper semicontinuity with respect to the weak topology. Here we prove the analogous for $E(\cdot)$ over $X_{\mathcal{A}}$.

Lemma 4.3.12. *The map $MA_\omega : \bar{\mathcal{A}} \rightarrow MA_\omega(\bar{\mathcal{A}}) \subset \{\mu \text{ positive measure on } X\}$ is an homeomorphism considering the weak topologies. In particular $\bar{\mathcal{A}}$ is homeomorphic to a closed set contained in $[0, \int_X MA_\omega(0)]$ through the map $\psi \rightarrow V_\psi$.*

Proof. The map is well-defined and continuous by Lemma 4.2 in [Tru19]. Moreover the injectivity follows from the fact that $V_{\psi_1} = V_{\psi_2}$ for $\psi_1, \psi_2 \in \bar{\mathcal{A}}$ implies $\psi_1 = \psi_2$ using Theorem 4.2.1 and the fact that $\mathcal{A} \subset \mathcal{M}^+$.

Finally to conclude the proof it is enough to prove that $\psi_k \rightarrow \psi$ weakly assuming $V_{\psi_k} \rightarrow V_\psi$ and it is clearly sufficient to show that any subsequence of $\{\psi_k\}_{k \in \mathbb{N}}$ admits a subsequence weakly convergent to ψ . Moreover since $\bar{\mathcal{A}}$ is totally ordered and \succcurlyeq coincides with \geq on \mathcal{M} , we may assume $\{\psi_k\}_{k \in \mathbb{N}}$ monotonic sequence. Then, up to considering a further subsequence, ψ_k converges almost everywhere to an element $\psi' \in \bar{\mathcal{A}}$ by compactness, and Lemma 5.2.5 implies that $V_{\psi'} = V_\psi$, i.e $\psi = \psi'$. \square

In the case $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$, we say that $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ converges weakly to $P_{\psi_{\min}}$ where $\psi_{\min} \in \mathcal{M} \setminus \mathcal{M}^+$ if $|\sup_X u_k| \leq C$ for any $k \in \mathbb{N}$ and any weak accumulation point u of $\{u_k\}_{k \in \mathbb{N}}$ satisfies $u \preccurlyeq \psi_{\min}$. This definition is the most natural since $PSH(X, \omega, \psi) = \mathcal{E}^1(X, \omega, \psi_{\min})$.

Lemma 4.3.13. *Let $\{u_k\}_{k \in \mathbb{N}} \subset X_{\mathcal{A}}$ be a sequence converging weakly to $u \in X_{\mathcal{A}}$. If $E_{P_\omega[u_k]}(u_k) \geq C$ uniformly, then $P_\omega[u_k] \rightarrow P_\omega[u]$ weakly.*

Proof. By Lemma 4.3.12 the convergence requested is equivalent to $V_{\psi_k} \rightarrow V_\psi$, where we set $\psi_k := P_\omega[u_k], \psi := P_\omega[u]$.

Moreover by a simple contradiction argument it is enough to show that any subsequence $\{\psi_{k_h}\}_{h \in \mathbb{N}}$ admits a subsequence $\{\psi_{k_{h_j}}\}_{j \in \mathbb{N}}$ such that $V_{\psi_{k_{h_j}}} \rightarrow V_\psi$. Thus up to considering a subsequence, by abuse of notations and by the lower semicontinuity $\liminf_{k \rightarrow \infty} V_{\psi_k} \geq V_\psi$ of Theorem 2.3. in [DDNL17], we may suppose by contradiction that $\psi_k \searrow \psi'$ for $\psi' \in \mathcal{M}$ such that $V_{\psi'} > V_\psi$. In particular $V_{\psi'} > 0$ and $\psi' \succcurlyeq \psi$. Then by Proposition 5.2.2 and Remark 4.3.3 the sequence $\{P_\omega[\psi'](u_k)\}_{k \in \mathbb{N}}$ is bounded in $(\mathcal{E}^1(X, \omega, \psi'), d)$ and it belongs to $\mathcal{E}_{C'}^1(X, \omega, \psi')$ for some $C' \in \mathbb{R}$. Therefore, up to considering a subsequence, we have that $\{u_k\}_{k \in \mathbb{N}}$ converges weakly to an element $v \in \mathcal{E}^1(X, \omega, \psi)$ (which is the element u itself when $u \neq P_{\psi_{\min}}$) while the sequence $P_\omega[\psi'](u_k)$ converges weakly to an element $w \in \mathcal{E}^1(X, \omega, \psi')$. Thus the contradiction follows from $w \leq v$ since $\psi' \succcurlyeq \psi$, $V_{\psi'} > 0$ and $\mathcal{E}^1(X, \omega, \psi') \cap \mathcal{E}^1(X, \omega, \psi) = \emptyset$. \square

Proposition 4.3.14. *Let $\{u_k\}_{k \in \mathbb{N}} \subset X_{\mathcal{A}}$ be a sequence converging weakly to $u \in X_{\mathcal{A}}$. Then*

$$\limsup_{k \rightarrow \infty} E_{P_\omega[u_k]}(u_k) \leq E_{P_\omega[u]}(u). \quad (4.11)$$

Proof. Let $\psi_k := P_\omega[u_k], \psi := P_\omega[u] \in \overline{\mathcal{A}}$. We may clearly assume $\psi_k \neq \psi_{\min}$ for any $k \in \mathbb{N}$ if $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$.

Moreover we can also suppose that $E_{\psi_k}(u_k)$ is bounded from below, which implies that $u_k \in \mathcal{E}_C^1(X, \omega, \psi_k)$ for an uniform constant C and that $\psi_k \rightarrow \psi$ weakly by Lemma 4.3.13. Thus since $E_{\psi_k}(u_k) = E_{\psi_k}(u_k - C) + CV_{\psi_k}$ for any $k \in \mathbb{N}$, Lemma 4.3.12 implies that we may assume that $\sup_X u_k \leq 0$. Furthermore since \mathcal{A} is totally ordered, it is enough to show (4.11) when $\psi_k \rightarrow \psi$ a.e. monotonically.

If $\psi_k \searrow \psi$, setting $v_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi_k)$, we easily have

$$\limsup_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq \limsup_{k \rightarrow \infty} E_{\psi_k}(v_k) \leq \limsup_{k \rightarrow \infty} E_\psi(P_\omega[\psi](v_k))$$

using the monotonicity of E_{ψ_k} and Proposition 5.2.2. Hence if $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$ then $E_\psi(P_\omega[\psi](v_k)) = 0 = E_\psi(u)$, while otherwise the conclusion follows from Proposition 4.2.4 since $P_\omega[\psi](v_k) \searrow u$ by construction.

If instead $\psi_k \nearrow \psi$, fix $\epsilon > 0$ and for any $k \in \mathbb{N}$ let $j_k \geq k$ such that

$$\sup_{j \geq k} E_{\psi_j}(u_j) \leq E_{\psi_{j_k}}(u_{j_k}) + \epsilon.$$

Thus again by Proposition 5.2.2, $E_{\psi_{j_k}}(u_{j_k}) \leq E_{\psi_l}(P_\omega[\psi_l](u_{j_k}))$ for any $l \leq j_k$.

Moreover, assuming $E_{\psi_{j_k}}(u_{j_k})$ bounded from below, $-E_{\psi_l}(P_\omega[\psi_l](u_{j_k})) = d(\psi_l, P_\omega[\psi_l](u_{j_k}))$ is uniformly bounded in l, k , which implies that $\sup_X P_\omega[\psi_l](u_{j_k})$ is uniformly bounded by Remark 4.3.3 since $V_{\psi_{j_k}} \geq a > 0$ for $k \gg 0$ big enough. By compactness, up to considering a subsequence, we obtain $P_\omega[\psi_l](u_{j_k}) \rightarrow v_l$ weakly where $v_l \in \mathcal{E}^1(X, \omega, \psi_l)$ by the upper semicontinuity of $E_{\psi_l}(\cdot)$ on $\mathcal{E}^1(X, \omega, \psi_l)$. Hence

$$\limsup_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq \limsup_{k \rightarrow \infty} E_{\psi_l}(P_\omega[\psi_l](u_{j_k})) + \epsilon = E_{\psi_l}(v_l) + \epsilon$$

for any $l \in \mathbb{N}$. Moreover by construction $v_l \leq P_\omega[\psi_l](u)$ since $P_\omega[\psi_l](u_{j_k}) \leq u_{j_k}$ for any k such that $j_k \geq l$ and $u_{j_k} \rightarrow u$ weakly. Therefore by the monotonicity of $E_{\psi_l}(\cdot)$ and by Proposition 4.2.11 .(ii) we conclude that

$$\limsup_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq \lim_{l \rightarrow \infty} E_{\psi_l}(P_\omega[\psi_l](u)) + \epsilon = E_\psi(u) + \epsilon$$

letting $l \rightarrow \infty$. □

As a consequence, defining

$$X_{\mathcal{A}, C} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}_C^1(X, \omega, \psi),$$

we get the following compactness result.

Proposition 4.3.15. *Let $C, a \in \mathbb{R}_{>0}$. The set*

$$X_{\mathcal{A},C}^a := X_{\mathcal{A},C} \cap \left(\bigsqcup_{\psi \in \bar{\mathcal{A}} : V_\psi \geq a} \mathcal{E}^1(X, \omega, \psi) \right)$$

is compact with respect to the weak topology.

Proof. It follows directly from the definition that

$$X_{\mathcal{A},C}^a \subset \left\{ u \in PSH(X, \omega) : |\sup_X u| \leq C' \right\}$$

where $C' := \max(C, C/a)$. Therefore by Theorem 8.5 in [GZ17], $X_{\mathcal{A},C}^a$ is weakly relatively compact. Finally Proposition 4.3.14 and Hartogs' Lemma imply that $X_{\mathcal{A},C}^a$ is also closed with respect to the weak topology, concluding the proof. \square

Remark 4.3.16. The whole set $X_{\mathcal{A},C}$ may not be weakly compact. Indeed assuming $V_{\psi_{\min}} = 0$ and letting $\psi_k \in \bar{\mathcal{A}}$ such that $\psi_k \searrow \psi_{\min}$, the functions $u_k := \psi_k - 1/\sqrt{V_{\psi_k}}$ belong to $X_{\mathcal{A},V}$ for $V = \int_X MA_\omega(0)$ since $E_{\psi_k}(u_k) = -\sqrt{V_{\psi_k}}$ but $\sup_X u_k = -1/\sqrt{V_{\psi_k}} \rightarrow -\infty$.

4.4 The action of measures on $PSH(X, \omega)$.

In this section we want to replace the action on $PSH(X, \omega)$ defined in [BBGZ13] given by a probability measure μ with an action which assume finite values on elements $u \in PSH(X, \omega)$ with ψ -relative minimal singularities where $\psi = P_\omega[u]$ for almost all $\psi \in \mathcal{M}$. On the other hand for any $\psi \in \mathcal{M}$ we want that there exists many measures μ whose action over $\{u \in PSH(X, \omega) : P_\omega[u] = \psi\}$ is well-defined. The problem is that μ varies among *all* probability measures while ψ among *all* model type envelopes. So it may happen that μ takes mass on non-pluripolar sets and that the unbounded locus of $\psi \in \mathcal{M}$ is very nasty.

Definition 4.4.1. *Let μ be a probability measure on X . Then μ acts on $PSH(X, \omega)$ through the functional $L_\mu : PSH(X, \omega) \rightarrow \mathbb{R} \cup \{-\infty\}$ defined as $L_\mu(u) = -\infty$ if μ charges $\{P_\omega[u] = -\infty\}$, as*

$$L_\mu(u) := \int_X (u - P_\omega[u]) \mu$$

if u has $P_\omega[u]$ -relative minimal singularities and μ does not charge $\{P_\omega[u] = -\infty\}$ and as

$$L_\mu(u) := \inf \{L_\mu(v) : v \in PSH(X, \omega) \text{ with } P_\omega[u]\text{-relative minimal sing., } v \geq u\}$$

otherwise.

Proposition 4.4.2. *The following properties hold:*

- (i) L_μ is affine, i.e. it satisfies the scaling property $L_\mu(u+c) = L_\mu(u) + c$ for any $c \in \mathbb{R}$, $u \in PSH(X, \omega)$;
- (ii) L_μ is non-decreasing on $\{u \in PSH(X, \omega) : P_\omega[u] = \psi\}$ for any $\psi \in \mathcal{M}$;
- (iii) $L_\mu(u) = \lim_{j \rightarrow \infty} L_\mu(\max(u, P_\omega[u] - j))$ for any $u \in PSH(X, \omega)$;
- (iv) if μ is non-pluripolar then L_μ is convex;
- (v) if μ is non-pluripolar and $u_k \rightarrow u$ and $P_\omega[u_k] \rightarrow P_\omega[u]$ weakly as $k \rightarrow \infty$ then $L_\mu(u) \geq \limsup_{k \rightarrow \infty} L_\mu(u_k)$;
- (vi) if $u \in \mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ then $L_{MA_\omega(u)/V_\psi}$ is finite on $\mathcal{E}^1(X, \omega, \psi)$.

Proof. The first two points follow by definition.

For the third point, setting $\psi := P_\omega[u]$, clearly $L_\mu(u) \leq \lim_{j \rightarrow \infty} L_\mu(\max(u, \psi - j))$. Vice versa for any $v \geq u$ with ψ -relative minimal singularities $v \geq \max(u, \psi - j)$ for $j \gg 0$ big enough, hence by (ii) it follows that $L_\mu(v) \geq \lim_{j \rightarrow \infty} L_\mu(\max(u, \psi - j))$ which implies (iii) by definition.

Next, we prove (iv). Let $v = \sum_{l=1}^m a_l u_l$ be a convex combination of elements $u_l \in PSH(X, \omega)$, and without loss of generality we may assume $\sup_X v, \sup_X u_l \leq 0$. In particular we have $L_\mu(v), L_\mu(u_l) \leq 0$.

Suppose $L_\mu(v) > -\infty$ (otherwise it is trivial) and let $\psi := P_\omega[v]$, $\psi_l := P_\omega[u_l]$. Then for any $C \in \mathbb{R}_{>0}$ it is easy to see that

$$\sum_{l=1}^m a_l P_\omega(u_l + C, 0) \leq P_\omega(v + C, 0) \leq \psi,$$

which leads to $\sum_{l=1}^m a_l \psi_l \leq \psi$ letting $C \rightarrow \infty$. Hence (iii) yields

$$-\infty < L_\mu(v) = \int_X (v - \psi) \mu \leq \sum_{l=1}^n a_l \int_X (u_l - \psi_l) \mu = \sum_{l=1}^n a_l L_\mu(u_l).$$

The point (v) is an easy consequence of $\limsup_{k \rightarrow \infty} \max(u_k, P_\omega[u_k] - j) \leq \max(u, P_\omega[u] - j)$ and (iii), while the last point is a consequence of Lemma 4.3.8. \square

Next, since for any $t \in [0, 1]$ and any $u, v \in \mathcal{E}^1(X, \omega, \psi)$

$$\begin{aligned} & \int_X (u - v) MA_\omega(tu + (1-t)v) = \\ &= (1-t)^n \int_X (u - v) MA_\omega(v) + \sum_{j=1}^n \binom{n}{j} t^j (1-t)^{n-j} \int_X (u - v) MA_\omega(u^j, v^{n-j}) \geq \\ & \geq (1-t)^n \int_X (u - v) MA_\omega(v) + (1 - (1-t)^n) \int_X (u - v) MA_\omega(u), \end{aligned}$$

we can proceed exactly as in Proposition 3.4 in [BBGZ13] (see also Lemma 2.11. in [GZ07]), replacing V_θ with ψ , to get the following result.

Proposition 4.4.3. *Let $A \subset PSH(X, \omega)$ and let $L : A \rightarrow \mathbb{R} \cup \{-\infty\}$ be a convex and non-decreasing function satisfying the scaling property $L(u + c) = L(u) + c$ for any $c \in \mathbb{R}$. Then*

- (i) *if L is finite valued on a weakly compact convex set $K \subset A$, then $L(K)$ is bounded;*
- (ii) *if $\mathcal{E}^1(X, \omega, \psi) \subset A$ and L assumes finite values on the set $\mathcal{E}^1(X, \omega, \psi)$ then $\sup_{\{u \in \mathcal{E}_C^1(X, \omega, \psi) : \sup_X u \leq 0\}} |L| \in O(C^{1/2})$ as $C \rightarrow \infty$.*

4.4.1 When is L_μ continuous?

The continuity of L_μ is an hard problem. However we can characterize its continuity on some weakly compact sets as the next Theorem shows.

Theorem 4.4.4. *Let μ be a non-pluripolar probability measure, and let $K \subset PSH(X, \omega)$ be a compact convex set such that L_μ is finite on K , the set $\{P_\omega[u] : u \in K\} \subset \mathcal{M}$ is totally ordered and its closure in $PSH(X, \omega)$ has at most one element in $\mathcal{M} \setminus \mathcal{M}^+$. Suppose also that there exists $C \in \mathbb{R}$ such that $|E_{P_\omega[u]}(u)| \leq C$ for any $u \in K$. Then the following properties are equivalent:*

- (i) *L_μ is continuous on K ;*
- (ii) *the map $\tau : K \rightarrow L^1(\mu)$, $\tau(u) := u - P_\omega[u]$ is continuous;*
- (iii) *the set $\tau(K) \subset L^1(\mu)$ is uniformly integrable, i.e.*

$$\int_{t=m}^{\infty} \mu\{u \leq P_\omega[u] - t\} \rightarrow 0$$

as $m \rightarrow \infty$, uniformly for $u \in K$.

Proof. We first observe that if $u_k \in K$ converges to $u \in K$ then by Lemma 4.3.13 $\psi_k \rightarrow \psi$ where we set $\psi_k := P_\omega[u_k]$, $\psi := P_\omega[u]$.

Then we can proceed exactly as in Theorem 3.10 in [BBGZ13] to get the equivalence between (i) and (ii), (ii) \Rightarrow (iii) and the fact that the graph of τ is closed. It is important to underline that (iii) is equivalent to say that $\tau(K)$ is *weakly* relative compact by Dunford-Pettis Theorem, i.e. with respect to the weak topology on $L^1(\mu)$ induced by $L^\infty(\mu) = L^1(\mu)^*$.

Finally assuming that (iii) holds, it remains to prove (i). So, letting $u_k, u \in K$ such that $u_k \rightarrow u$, we have to show that $\int_X \tau(u_k) \mu \rightarrow \int_X \tau(u) \mu$. Since $\tau(K) \subset L^1(\mu)$ is bounded, unless considering a subsequence, we may suppose $\int_X \tau(u_k) \rightarrow L \in \mathbb{R}$. By Fatou's Lemma,

$$L = \lim_{k \rightarrow \infty} \int_X \tau(u_k) \mu \leq \int_X \tau(u) \mu. \quad (4.12)$$

Then for any $k \in \mathbb{N}$ the closed convex envelope

$$C_k := \overline{\text{Conv}\{\tau(u_j) : j \geq k\}},$$

is weakly closed in $L^1(\mu)$ by Hahn-Banach Theorem, which implies that C_k is weakly compact since it is contained in $\tau(K)$. Thus since C_k is a decreasing sequence of non-empty weakly compact sets, there exists $f \in \bigcap_{k \geq 1} C_k$ and there exist elements $v_k \in \text{Conv}(u_j : j \geq k)$ given as finite convex combination such that $\tau(v_k) \rightarrow f$ in $L^1(\mu)$. Moreover by the closed graph property $f = \tau(u)$ since $v_k \rightarrow u$ as a consequence of $u_k \rightarrow u$. On the other hand by Proposition 4.4.2 .(iv) we get

$$\int_X \tau(v_k) \mu \leq \sum_{l=1}^{m_k} a_{l,k} \int_X \tau(u_{k_l}) \mu$$

if $v_k = \sum_{l=1}^{m_k} a_{l,k} u_{k_l}$. Hence $L \geq \int_X \tau(u) \mu$, which together (4.12) implies $L = \int_X \tau(u) \mu$ and concludes the proof. \square

Corollary 4.4.5. *Let $\psi \in \mathcal{M}^+$ and $\mu \in \mathcal{C}_{A,\psi}$. Then L_μ is continuous on $\mathcal{E}_C^1(X, \omega, \psi)$ for any $C \in \mathbb{R}_{>0}$. In particular if $\mu = MA_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities then L_μ is continuous on $\mathcal{E}_C^1(X, \omega, \psi)$ for any $C \in \mathbb{R}_{>0}$.*

Proof. With the notations of Theorem 4.4.4, $\tau(\mathcal{E}_C^1(X, \omega, \psi))$ is bounded in $L^2(\mu)$ by Lemma 4.3.11. Hence by Holder's inequality $\tau(\mathcal{E}_C^1(X, \omega, \psi))$ is uniformly integrable and Theorem 4.4.4 yields the continuity of L_μ on $\mathcal{E}_C^1(X, \omega, \psi)$ for any $C \in \mathbb{R}_{>0}$. The last assertion follows directly from Proposition 4.3.10. \square

The following Lemma will be essential to prove Theorem A, Theorem B.

Lemma 4.4.6. *Let $\varphi \in \mathcal{H}_\omega$ and let $\mathcal{A} \subset \mathcal{M}$ be a totally ordered subset. Set also $v_\psi := P_\omega[\psi](\varphi)$ for any $\psi \in \mathcal{A}$. Then the actions $\{V_\psi L_{MA_\omega(v_\psi)/V_\psi}\}_{\psi \in \mathcal{A}}$ take finite values and they are equicontinuous on any compact set $K \subset PSH(X, \omega)$ such that $\{P_\omega[u] : u \in K\}$ is a totally ordered set whose closure in $PSH(X, \omega)$ has at most one element in $\mathcal{M} \setminus \mathcal{M}^+$ and such that $|E_{P_\omega[u]}(u)| \leq C$ uniformly for any $u \in K$. If $\psi \in \mathcal{M} \setminus \mathcal{M}^+$, for the action $V_\psi L_{MA_\omega(v_\psi)/V_\psi}$ we mean the null action. In particular if $\psi_k \rightarrow \psi$ monotonically almost everywhere and $\{u_k\}_{k \in \mathbb{N}} \subset K$ converges weakly to $u \in K$, then*

$$\int_X (u_k - P_\omega[u_k]) MA_\omega(v_{\psi_k}) \rightarrow \int_X (u - P_\omega[u]) MA_\omega(v_\psi). \quad (4.13)$$

Proof. By Theorem 4.2.2, $|V_\psi L_{MA_\omega(v_\psi)/V_\psi}(u)| \leq \int_X |u - P_\omega[u]| MA_\omega(\varphi)$ for any $u \in PSH(X, \omega)$ and any $\psi \in \mathcal{A}$, so the actions in the statement assume finite values. Then the equicontinuity on any weak compact set $K \subset PSH(X, \omega)$ satisfying the assumptions of the Lemma follows from

$$V_\psi |L_{MA_\omega(v_\psi)/V_\psi}(w_1) - L_{MA_\omega(v_\psi)/V_\psi}(w_2)| \leq \int_X |w_1 - P_\omega[w_1] - w_2 + P_\omega[w_2]| MA_\omega(\varphi)$$

for any $w_1, w_2 \in PSH(X, \omega)$ since $MA_\omega(\varphi)$ is a volume form on X and $P_\omega[w_k] \rightarrow P_\omega[w]$ if $\{w_k\}_{k \in \mathbb{N}} \subset K$ converges to $w \in K$ under our hypothesis by Lemma 4.3.13. For the second assertion, if $\psi_k \searrow \psi$ (resp. $\psi_k \nearrow \psi$ almost everywhere), letting $f_k, f \in L^\infty$ such that $MA_\omega(v_{\psi_k}) = f_k MA_\omega(\varphi)$ and $MA_\omega(v_\psi) = f MA_\omega(\varphi)$ (Theorem 4.2.2), we have $0 \leq f_k \leq 1$, $0 \leq f \leq 1$ and $\{f_k\}_{k \in \mathbb{N}}$ is a monotone sequence. Therefore $f_k \rightarrow f$ in L^p for any $p > 1$ as $k \rightarrow \infty$ which implies

$$\int_X (u - P_\omega[u]) MA_\omega(v_{\psi_k}) \rightarrow \int_X (u - P_\omega[u]) MA_\omega(v_\psi)$$

as $k \rightarrow \infty$ since $MA_\omega(\varphi)$ is a volume form. Hence (4.13) follows since by the first part of the proof

$$\int_X (u_k - P_\omega[u_k] - u + P_\omega[u]) MA_\omega(v_{\psi_k}) \rightarrow 0.$$

□

4.5 Theorem A

In this section we fix $\psi \in \mathcal{M}^+$ and using a variational approach we first prove the bijectivity of the Monge-Ampère operator between $\mathcal{E}_{norm}^1(X, \omega, \psi)$ and $\mathcal{M}^1(X, \omega, \psi)$, and then we prove that it is actually an homeomorphism considering the strong topologies.

4.5.1 Degenerate complex Monge-Ampère equations.

Letting μ be a probability measure and $\psi \in \mathcal{M}$, we define the functional $F_{\mu, \psi} : \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R} \cup \{-\infty\}$ as

$$F_{\mu, \psi}(u) := (E_\psi - V_\psi L_\mu)(u)$$

where we recall that $L_\mu(u) = \lim_{j \rightarrow \infty} L_\mu(\max(u, \psi - j)) = \lim_{j \rightarrow \infty} \int_X (\max(u, \psi - j) - \psi) \mu$ (see section 4.4). $F_{\mu, \psi}$ is clearly a translation invariant functional and $F_{\mu, \psi} \equiv 0$ for any μ if $V_\psi = 0$.

Proposition 4.5.1. *Let μ be a probability measure, $\psi \in \mathcal{M}^+$ and let $F := F_{\mu, \psi}$. If L_μ is continuous then F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$. Moreover if L_μ is finite valued on $\mathcal{E}^1(X, \omega, \psi)$ then there exist $A, B > 0$ such that*

$$F(v) \leq -Ad(\psi, v) + B$$

for any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, i.e. F is d -coercive. In particular F is upper semicontinuous on $\mathcal{E}^1(X, \omega, \psi)$ and d -coercive on $\mathcal{E}_{norm}^1(X, \omega, \psi)$ if $\mu = MA_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$.

Proof. If L_μ is continuous then F is easily upper semicontinuous by Proposition 4.2.4.

Then, since $d(\psi, v) = -E_\psi(v)$ on $\mathcal{E}_{norm}^1(X, \omega, \psi)$, it is easy to check that the coercivity requested is equivalent to

$$\sup_{\mathcal{E}_C^1(X, \omega, \psi) \cap \mathcal{E}_{norm}^1(X, \omega, \psi)} |L_\mu| \leq \frac{(1-A)}{V_\psi} C + O(1),$$

which holds by Proposition 4.4.3.(ii).

Next assuming $\mu = MA_\omega(u)/V_\psi$ it is sufficient to check the continuity of L_μ since L_μ is finite valued on $\mathcal{E}^1(X, \omega, \psi)$ by Proposition 4.4.2. We may suppose without loss of generality that $u \leq \psi$. By Proposition 4.3.7 and Remark 4.3.3, for any $C \in \mathbb{R}_{>0}$, L_μ restricted to $\mathcal{E}_C^1(X, \omega, \psi)$ is the uniform limit of L_{μ_j} , where $\mu_j := MA_\omega(\max(u, \psi - j))$, since $I_\psi(\max(u, \psi - j), u) \rightarrow 0$ as $j \rightarrow \infty$. Therefore L_μ is continuous on $\mathcal{E}_C^1(X, \omega, \psi)$ since uniform limit of continuous functionals L_{μ_j} (Corollary 4.4.5). \square

As a consequence of the concavity of E_ψ if $\mu = MA_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ where $V_\psi > 0$ then

$$J_u^\psi(\psi) = F_{\mu, \psi}(u) = \sup_{\mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi},$$

i.e. u is a maximizer for $F_{\mu, \psi}$. The vice versa also holds as the next result shows.

Proposition 4.5.2. *Let $\psi \in \mathcal{M}^+$ and let μ be a probability measure such that L_μ is finite valued on $\mathcal{E}^1(X, \omega, \psi)$. Then $\mu = MA_\omega(u)/V_\psi$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ if and only if u is a maximizer of $F_{\mu, \psi}$.*

Proof. As said before, it is clear that $\mu = MA_\omega(u)/V_\psi$ implies that u is a maximizer for $F_{\mu, \psi}$. Vice versa if u is a maximizer of $F_{\mu, \psi}$ then by Theorem 4.22 in [DDNL17] $\mu = MA_\omega(u)/V_\psi$. \square

Similarly to [BBGZ13] we, thus, define the ψ -relative energy for $\psi \in \mathcal{M}$ of a probability measure μ as

$$E_\psi^*(\mu) := \sup_{u \in \mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi}(u)$$

i.e. essentially as the Legendre transform of E_ψ . It takes non-negative values ($F_{\mu, \psi}(\psi) = 0$) and it is easy to check that E_ψ^* is a convex function.

Moreover defining

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E_\psi^*(\mu) < \infty\},$$

we note that $\mathcal{M}^1(X, \omega, \psi)$ consists only of the null measure if $V_\psi = 0$ while in $V_\psi > 0$ any probability measure μ such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ is non-pluripolar as the next Lemma shows.

Lemma 4.5.3. *Let $A \subset X$ be a (locally) pluripolar set. Then there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. In particular if $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ then μ is non-pluripolar.*

Proof. By Corollary 2.11 in [BBGZ13] there exists $\varphi \in \mathcal{E}^1(X, \omega)$ such that $A \subset \{\varphi = -\infty\}$. Therefore setting $u := P_\omega[\psi](\varphi)$ proves the first part.

Next let $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ and μ probability measure and assume by contradiction that μ takes mass on a pluripolar set A . Then by the first part of the proof there exists $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $A \subset \{u = -\infty\}$. On the other hand, since $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$ by definition μ does not charge $\{\psi = -\infty\}$. Thus by Proposition 4.4.2 (iii) we obtain $L_\mu(u) = -\infty$, which is a contradiction. \square

We can now prove that the Monge-Ampère operation is a bijection between $\mathcal{E}^1(X, \omega, \psi)$ and $\mathcal{M}^1(X, \omega, \psi)$.

Lemma 4.5.4. *Let $\psi \in \mathcal{M}^+$ and let $\mu \in \mathcal{C}_{A, \psi}$ where $A \in \mathbb{R}$. Then there exists $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ maximizing $F_{\mu, \psi}$.*

Proof. By Lemma 4.3.11 L_μ is finite valued on $\mathcal{E}^1(X, \omega, \psi)$, and it is continuous on $\mathcal{E}_C^1(X, \omega, \psi)$ for any $C \in \mathbb{R}$ thanks to Corollary 4.4.5. Therefore it follows from Proposition 4.5.1 that $F_{\mu, \psi}$ is upper semicontinuous and d -coercive on $\mathcal{E}_{norm}^1(X, \omega, \psi)$. Hence $F_{\mu, \psi}$ admits a maximizer $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ as easy consequence of the weak compactness of $\mathcal{E}_C^1(X, \omega, \psi)$. \square

Proposition 4.5.5. *Let $\psi \in \mathcal{M}^+$. Then the Monge-Ampère map $MA : \mathcal{E}_{norm}^1(X, \omega, \psi) \rightarrow \mathcal{M}^1(X, \omega, \psi)$, $u \rightarrow MA(u)$ is bijective. Furthermore if $V_\psi \mu = MA_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ then any maximizing sequence $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ for $F_{\mu, \psi}$ necessarily converges weakly to u .*

Proof. The proof is inspired by Theorem 4.7 in [BBGZ13].

The map is well-defined as a consequence of Proposition 4.5.1, i.e. $MA_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$. Moreover the injectivity follows from Theorem 4.8 in [DDNL18].

Let $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ be a sequence such that $F_{\mu, \psi}(u_k) \nearrow \sup_{\mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi}$ where $\mu = MA_\omega(u)/V_\psi$ is a probability measure and $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. Up to considering a subsequence, we may also assume that $u_k \rightarrow v \in PSH(X, \omega)$. Then, by the upper semicontinuity and the d -coercivity of $F_{\mu, \psi}$ (Proposition 4.5.1) it follows that $v \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and $F_{\mu, \psi}(v) = \sup_{\mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi}$. Thus by Proposition 4.5.2 we get $\mu = MA_\omega(v)/V_\psi$. Hence $v = u$ since $\sup_X v = \sup_X u = 0$.

Then let μ be a probability measure such that $V_\psi \mu \in \mathcal{M}^1(X, \omega, \psi)$. Again by Proposition 4.5.2, to prove the existence of $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $\mu = MA_\omega(u)/V_\psi$ it is sufficient to check that $F_{\mu, \psi}$ admits a maximum over $\mathcal{E}_{norm}^1(X, \omega, \psi)$. Moreover by Proposition 4.5.1 we also know that $F_{\mu, \psi}$ is d -coercive on $\mathcal{E}_{norm}^1(X, \omega, \psi)$. Thus if there exists a constant $A > 0$ such that $\mu \in \mathcal{C}_{A, \psi}$ then Corollary 4.4.5 leads to the upper semicontinuity of $F_{\mu, \psi}$ which clearly implies that $V_\psi \mu = MA_\omega(u)$ for $u \in \mathcal{E}^1(X, \omega, \psi)$ since $\mathcal{E}_C^1(X, \omega, \psi) \subset PSH(X, \omega)$ is compact for any $C \in \mathbb{R}_{>0}$.

In the general case by Lemma 4.26 in [DDNL17] (see also [Ceg98]) μ is absolutely continuous with respect to $\nu \in \mathcal{C}_{1, \psi}$ using also that μ is a non-pluripolar measure

(Lemma 4.5.3). Therefore letting $f \in L^1(\nu)$ such that $\mu = f\nu$, we define for any $k \in \mathbb{N}$

$$\mu_k := (1 + \epsilon_k) \min(f, k)\nu$$

where $\epsilon_k > 0$ are chosen so that μ_k is a probability measure, noting that $(1 + \epsilon_k) \min(f, k) \rightarrow f$ in $L^1(\nu)$. Then by Lemma 4.5.4 it follows that $\mu_k = MA_\omega(u_k)/V_\psi$ for $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi)$.

Moreover by weak compactness, without loss of generality, we may also assume that $u_k \rightarrow u \in PSH(X, \omega)$. Note that $u \leq \psi$ since $u_k \leq \psi$ for any $k \in \mathbb{N}$. Then by Lemma 2.8 in [DDNL18] we obtain

$$MA_\omega(u) \geq V_\psi f\nu = V_\psi \mu,$$

which implies $MA_\omega(u) = V_\psi \mu$ by [WN17] since u is more singular than ψ and μ is a probability measure. It remains to prove that $u \in \mathcal{E}^1(X, \omega, \psi)$.

It is not difficult to see that $\mu_k \leq 2\mu$ for $k \gg 0$, thus Proposition 4.4.3 implies that there exists a constant $B > 0$ such that

$$\sup_{\mathcal{E}_C^1(X, \omega, \psi)} |L_{\mu_k}| \leq 2 \sup_{\mathcal{E}_C^1(X, \omega, \psi)} |L_\mu| \leq 2B(1 + C^{1/2})$$

for any $C \in \mathbb{R}_{>0}$. Therefore

$$J_{u_k}^\psi(\psi) = E_\psi(u_k) + V_\psi |L_{\mu_k}(u_k)| \leq \sup_{C>0} (2V_\psi B(1 + C^{1/2}) - C)$$

and Lemma 4.3.1 yields $d(\psi, u_k) \leq D$ for an uniform constant D , i.e. $u_k \in \mathcal{E}_{D'}^1(X, \omega, \psi)$ for any $k \in \mathbb{N}$ for an uniform constant D' (Remark 4.3.3). Hence since $\mathcal{E}_{D'}^1(X, \omega, \psi)$ is weakly compact we obtain $u \in \mathcal{E}_{D'}^1(X, \omega, \psi)$. \square

4.5.2 Proof of Theorem A.

We first need to explore further the properties of the strong topology on $\mathcal{E}^1(X, \omega, \psi)$.

By Proposition 4.3.6 the strong convergence implies the weak convergence. Moreover the strong topology is the coarsest refinement of the weak topology such that $E_\psi(\cdot)$ becomes continuous.

Proposition 4.5.6. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$. Then $u_k \rightarrow u$ strongly if and only if $u_k \rightarrow u$ weakly and $E_\psi(u_k) \rightarrow E_\psi(u)$.*

Proof. Assume that $u_k \rightarrow u$ weakly and that $E_\psi(u_k) \rightarrow E_\psi(u)$. Then $w_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ and it decreases to u . Thus by Proposition 4.2.4 $E_\psi(w_k) \rightarrow E_\psi(u)$ and

$$d(u_k, u) \leq d(u_k, w_k) + d(w_k, u) = 2E_\psi(w_k) - E_\psi(u_k) - E_\psi(u) \rightarrow 0.$$

Vice versa, assuming that $d(u_k, u) \rightarrow 0$, we immediately get that $u_k \rightarrow u$ weakly as said above (Proposition 4.3.6). Moreover $\sup_X u_k, \sup_X u \leq A$ uniformly for a constant $A \in \mathbb{R}$. Thus

$$|E_\psi(u_k) - E_\psi(u)| = |d(\psi + A, u_k) - d(\psi + A, u)| \leq d(u_k, u) \rightarrow 0,$$

which concludes the proof. \square

Then we also observe that the strong convergence implies the convergence in ψ' -capacity for any $\psi' \in \mathcal{M}^+$.

Proposition 4.5.7. *Let $\psi \in \mathcal{M}^+$ and $u_k, u \in \mathcal{E}^1(X, \omega, \psi)$ such that $d(u_k, u) \rightarrow 0$. Then there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ such that $w_j := (\sup\{u_{k_h} : h \geq j\})^*$, $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $\mathcal{E}^1(X, \omega, \psi)$ and converge monotonically almost everywhere to u . In particular $u_k \rightarrow u$ in ψ' -capacity for any $\psi' \in \mathcal{M}^+$ and $MA_\omega(u_k^j, \psi^{n-j}) \rightarrow MA_\omega(u^j, \psi^{n-j})$ weakly for any $j = 0, \dots, n$.*

Proof. Since the strong convergence implies the weak convergence by Proposition 4.5.6 it is clear that $w_k \in \mathcal{E}^1(X, \omega, \psi)$ and that it decreases to u . In particular up to considering a subsequence we may assume that $d(u_k, w_k) \leq 1/2^k$ for any $k \in \mathbb{N}$. Next for any $j \geq k$ we set $v_{k,j} := P_\omega(u_k, \dots, u_j) \in \mathcal{E}^1(X, \omega, \psi)$ and $v_{k,j}^u := P_\omega(v_{k,j}, u) \in \mathcal{E}^1(X, \omega, \psi)$. Then it follows from Proposition 4.2.4 and Lemma 3.7 in [DDNL17] that

$$\begin{aligned} d(u, v_{k,j}^u) &\leq \int_X (u - v_{k,j}^u) MA_\omega(v_{k,j}^u) \leq \int_{\{v_{k,j}^u = v_{k,j}\}} (u - v_{k,j}) MA_\omega(v_{k,j}) \leq \\ &\leq \sum_{s=k}^j \int_X (w_s - u_s) MA_\omega(u_s) \leq (n+1) \sum_{s=k}^j d(w_s, u_s) \leq \frac{(n+1)}{2^{k-1}}. \end{aligned}$$

Therefore by Proposition 4.3.15 $v_{k,j}^u$ decreases (hence converges strongly) to a function $\phi_k \in \mathcal{E}^1(X, \omega, \psi)$ as $j \rightarrow \infty$. Similarly we also observe that

$$d(v_{k,j}, v_{k,j}^u) \leq \int_{\{v_{k,j}^u = u\}} (v_{k,j} - u) MA_\omega(u) \leq \int_X |v_{k,1} - u| MA_\omega(u) \leq C$$

uniformly in j by Corollary 4.3.5. Hence by definition $d(u, v_{k,j}) \leq C + \frac{(n+1)}{2^{k-1}}$, i.e. $v_{k,j}$ decreases and converges strongly as $j \rightarrow \infty$ to the function $v_k = P_\omega(u_k, u_{k+1}, \dots) \in \mathcal{E}^1(X, \omega, \psi)$ again by Proposition 4.3.15. Moreover by construction $u_k \geq v_k \geq \phi_k$ since $v_k \leq v_{k,j} \leq u_k$ for any $j \geq k$. Hence

$$d(u, v_k) \leq d(u, \phi_k) \leq \frac{(n+1)}{2^{k-1}} \rightarrow 0$$

as $k \rightarrow \infty$, i.e. $v_k \nearrow u$ strongly.

The convergence in ψ' -capacity for $\psi' \in \mathcal{M}^+$ is now clearly an immediate consequence. Indeed by an easy contradiction argument it is enough to prove that any

arbitrary subsequence, which we will keep denoting with $\{u_k\}_{k \in \mathbb{N}}$ for the sake of simplicity, admits a further subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ converging in ψ' -capacity to u . Thus taking the subsequence satisfying $v_j \leq u_{k_j} \leq w_j$ where v_j, w_j are the monotonic sequence of the first part of the Proposition, the convergence in ψ' -capacity follows from the inclusions

$$\{|u - u_{k_j}| > \delta\} = \{u - u_{k_j} > \delta\} \cup \{u_k - u > \delta\} \subset \{u - v_j > \delta\} \cup \{w_j - u > \delta\}$$

for any $\delta > 0$. Finally Lemma 5.2.5 gives the weak convergence of the measures. \square

We can now endow the set $\mathcal{M}^1(X, \omega, \psi) = \{V_\psi \mu : \mu \text{ is a probability measure satisfying } E_\psi^*(\mu) < +\infty\}$ (subsection 4.5.1) with its natural strong topology given as the coarsest refinement of the weak topology such that $E_\psi^*(\cdot)$ becomes continuous, and prove our Theorem A.

Theorem A. *Let $\psi \in \mathcal{M}^+$. Then*

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega, \psi), d) \rightarrow (\mathcal{M}^1(X, \omega, \psi), \text{strong})$$

is an homeomorphism.

Proof. The map is bijective as immediate consequence of Proposition 4.5.5. Next, letting $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ converging strongly to $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, Proposition 4.5.7 gives the weak convergence of $MA_\omega(u_k) \rightarrow MA_\omega(u)$ as $k \rightarrow \infty$. Moreover since $E_\psi^*(MA_\omega(v)/V_\psi) = J_v^\psi(\psi)$ for any $v \in \mathcal{E}^1(X, \omega, \psi)$, we get

$$\begin{aligned} & \left| E_\psi^*(MA_\omega(u_k)/V_\psi) - E_\psi^*(MA_\omega(u)/V_\psi) \right| \leq \\ & \leq |E_\psi(u_k) - E_\psi(u)| + \left| \int_X (\psi_k - u_k) MA_\omega(u_k) - \int_X (\psi_k - u_k) MA_\omega(u) \right| \leq \\ & \leq |E_\psi(u_k) - E_\psi(u)| + \left| \int_X (\psi_k - u_k) (MA_\omega(u_k) - MA_\omega(u)) \right| + \int_X |u_k - u| MA_\omega(u). \end{aligned} \tag{4.14}$$

Hence $MA_\omega(u_k) \rightarrow MA_\omega(u)$ strongly in $\mathcal{M}^1(X, \omega, \psi)$ since each term on the right side hand of 4.14 goes to 0 as $k \rightarrow +\infty$ combining Proposition 4.5.6, Proposition 4.3.7 and Corollary 4.3.5 recalling that by Proposition 4.3.4 $I_\psi(u_k, u) \rightarrow 0$ as $k \rightarrow \infty$. Vice versa suppose that $MA_\omega(u_k) \rightarrow MA_\omega(u)$ strongly in $\mathcal{M}^1(X, \omega, \psi)$ where $u_k, u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. Then, letting $\{\varphi_j\}_{j \in \mathbb{N}} \subset \mathcal{H}_\omega$ such that $\varphi_j \searrow u$ ([BK07]) and setting $v_j := P_\omega[\psi](\varphi_j)$, by Lemma 4.3.1

$$\begin{aligned} (n+1)I_\psi(u_k, v_j) & \leq E_\psi(u_k) - E_\psi(v_j) + \int_X (v_j - u_k) MA_\omega(u_k) = \\ & = E_\psi^*(MA_\omega(u_k)/V_\psi) - E_\psi^*(MA_\omega(v_j)/V_\psi) + \int_X (v_j - \psi) (MA_\omega(u_k) - MA_\omega(v_j)). \end{aligned} \tag{4.15}$$

By construction and the first part of the proof, it follows that $E_\psi^*(MA_\omega(u_k)/V_\psi) - E_\psi^*(MA_\omega(v_j)/V_\psi) \rightarrow 0$ as $k, j \rightarrow \infty$. While setting $f_j := v_j - \psi$ we want to prove that

$$\limsup_{k \rightarrow \infty} \int_X f_j MA_\omega(u_k) = \int_X f_j MA_\omega(u),$$

which would imply $\limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} I_\psi(u_k, v_j) = 0$ since $\int_X f_j (MA_\omega(u) - MA_\omega(v_j)) \rightarrow 0$ as a consequence of Propositions 4.3.7 and 4.3.4.

We observe that $\|f_j\|_{L^\infty} \leq \|\varphi_j\|_{L^\infty}$ by Proposition 5.2.2 and we denote with $\{f_j^s\}_{s \in \mathbb{N}} \subset C^\infty$ a sequence of smooth functions converging in capacity to f_j such that $\|f_j^s\|_{L^\infty} \leq 2\|f_j\|_{L^\infty}$. We recall here briefly how to construct such sequence. Let $\{g_j^s\}_{s \in \mathbb{N}}$ be the sequence of bounded functions converging in capacity to f_j defined as $g_j^s := \max(v_j, -s) - \max(\psi, -s)$. We have that $\|g_j^s\|_{L^\infty} \leq \|f_j\|_{L^\infty}$ and that $\max(v_j, -s), \max(\psi, -s) \in PSH(X, \omega)$. Therefore by a regularization process (see for instance [BK07]) and a diagonal argument we can now construct a sequence $\{f_j^s\}_{j \in \mathbb{N}} \subset C^\infty$ converging in capacity to f_j such that $\|f_j^s\|_{L^\infty} \leq 2\|g_j^s\| \leq 2\|f_j\|_{L^\infty}$ where $f_j^s = v_j^s - \psi^s$ with v_j^s, ψ^s quasi-psh functions decreasing respectively to v_j, ψ . Then letting $\delta > 0$ we have

$$\begin{aligned} \int_X (f_j - f_j^s) MA_\omega(u_k) &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty} \int_{\{f_j - f_j^s > \delta\}} MA_\omega(u_k) \leq \\ &\leq \delta V_\psi + 3\|\varphi_j\|_{L^\infty} \int_{\{\psi^s - \psi > \delta\}} MA_\omega(u_k) \end{aligned}$$

from the trivial inclusion $\{f_j - f_j^s > \delta\} \subset \{\psi^s - \psi > \delta\}$. Therefore

$$\begin{aligned} \limsup_{s \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X (f_j - f_j^s) MA_\omega(u_k) &\leq \delta V_\psi + \\ + \limsup_{s \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_{\{\psi^s - \psi \geq \delta\}} MA_\omega(u_k) &\leq \delta V_\psi + \limsup_{s \rightarrow \infty} \int_{\{\psi^s - \psi \geq \delta\}} MA_\omega(u) = \delta V_\psi, \end{aligned}$$

where we used that $\{\psi^s - \psi \geq \delta\}$ is a closed set in the plurifine topology. Hence since $f_j^s \in C^\infty$ we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_X f_j MA_\omega(u_k) &= \\ = \limsup_{s \rightarrow \infty} \limsup_{k \rightarrow \infty} \left(\int_X (f_j - f_j^s) MA_\omega(u_k) + \int_X f_j^s MA_\omega(u_k) \right) &\leq \\ \leq \limsup_{s \rightarrow \infty} \int_X f_j^s MA_\omega(u) = \int_X f_j MA_\omega(u), \end{aligned}$$

which as said above implies $I_\psi(u_k, v_j) \rightarrow 0$ letting $k, j \rightarrow \infty$ in this order.

Next, again by Lemma 4.3.1, we obtain $u_k \in \mathcal{E}_C^1(X, \omega, \psi)$ for some $C \in \mathbb{N}$ big enough since $J_{u_k}^\psi(\psi) = E_\psi^*(MA_\omega(u_k)/V_\psi)$. In particular, up to considering a subsequence,

$u_k \rightarrow w \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ weakly by Proposition 4.3.15. Observe also that by Proposition 4.3.7

$$\left| \int_X (\psi - u_k)(MA_\omega(v_j) - MA_\omega(u_k)) \right| \rightarrow 0 \quad (4.16)$$

as $k, j \rightarrow \infty$ in this order. Moreover by Proposition 4.3.14 and Lemma 4.4.6

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left(E_\psi^*(MA_\omega(u_k)/V_\psi) + \int_X (\psi - u_k)(MA_\omega(v_j) - MA_\omega(u_k)) \right) = \\ & = \limsup_{k \rightarrow \infty} \left(E_\psi(u_k) + \int_X (\psi - u_k)MA_\omega(v_j) \right) \leq E_\psi(w) + \int_X (\psi - w)MA_\omega(v_j). \end{aligned} \quad (4.17)$$

Therefore combining (4.16) and (4.17) with the strong convergence of v_j to u we obtain

$$\begin{aligned} E_\psi(u) + \int_X (\psi - u)MA_\omega(u) &= \lim_{k \rightarrow \infty} E_\psi^*(MA_\omega(u_k)/V_\psi) \leq \\ &\leq \limsup_{j \rightarrow \infty} \left(E_\psi(w) + \int_X (\psi - w)MA_\omega(v_j) \right) = E_\psi(w) + \int_X (\psi - w)MA_\omega(u), \end{aligned}$$

i.e. w is a maximizer of $F_{MA_\omega(u)/V_\psi, \psi}$. Hence $w = u$ (Proposition 4.5.5), i.e. $u_k \rightarrow u$ weakly. Furthermore again by Lemma 4.3.1 and Lemma 4.4.6

$$\begin{aligned} & \limsup_{k \rightarrow \infty} (E_\psi(v_j) - E_\psi(u_k)) \leq \\ & \leq \limsup_{k \rightarrow \infty} \left(\frac{n}{n+1} I_\psi(u_k, v_j) + \left| \int_X (u_k - v_j)MA_\omega(v_j) \right| \right) \leq \\ & \leq \left| \int_X (u - v_j)MA_\omega(v_j) \right| + \limsup_{k \rightarrow \infty} \frac{n}{n+1} I_\psi(u_k, v_j). \end{aligned} \quad (4.18)$$

Finally letting $j \rightarrow \infty$, since $v_j \searrow u$ strongly, we obtain $\liminf_{j \rightarrow \infty} E_\psi(u_k) \geq \lim_{j \rightarrow \infty} E_\psi(v_j) = E_\psi(u)$ which implies that $E_\psi(u_k) \rightarrow E_\psi(u)$ and that $u_k \rightarrow u$ strongly by Proposition 4.5.6. \square

The main difference between the proof of Theorem A with respect to the same result in the absolute setting, i.e. when $\psi = 0$, is that for fixed $u \in \mathcal{E}^1(X, \omega, \psi)$ the action $\mathcal{M}^1(X, \omega, \psi) \ni MA_\omega(v) \rightarrow \int_X (u - \psi)MA_\omega(v)$ is not a priori continuous with respect to the weak topologies of measures even if we restrict the action on $\mathcal{M}_C^1(X, \omega, \psi) := \{V_\psi \mu : E_\psi^*(\mu) \leq C\}$ for $C \in \mathbb{R}$ while in the absolute setting this is given by Proposition 1.7. in [BBEGZ19] where the authors used the fact that any $u \in \mathcal{E}^1(X, \omega)$ can be approximated inside the class $\mathcal{E}^1(X, \omega)$ by a sequence of continuous functions.

4.6 Strong Topologies.

In this section we investigate the strong topology on $X_{\mathcal{A}}$ in detail, proving that it is the coarsest refinement of the weak topology such that $E(\cdot)$ becomes continuous (Theorem 4.6.2) and proving that the strong convergence implies the convergence in ψ -capacity for any $\psi \in \mathcal{M}^+$ (Theorem 4.6.3), i.e. we extend all the typical properties of the L^1 -metric geometry to the bigger space $X_{\mathcal{A}}$, justifying further the construction of the distance $d_{\mathcal{A}}$ ([Tru19]) and its naturality. Moreover we define the set $Y_{\mathcal{A}}$, and we prove Theorem B.

4.6.1 About $(X_{\mathcal{A}}, d_{\mathcal{A}})$.

First we prove that the strong convergence in $X_{\mathcal{A}}$ implies the weak convergence, recalling that for weak convergence of $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ to $P_{\psi_{\min}}$ where $\psi_{\min} \in \mathcal{M}$ with $V_{\psi_{\min}} = 0$ we mean that $|\sup_X u_k| \leq C$ and that any weak accumulation point of $\{u_k\}_{k \in \mathbb{N}}$ is more singular than ψ_{\min} .

Proposition 4.6.1. *Let $u_k, u \in X_{\mathcal{A}}$ such that $u_k \rightarrow u$ strongly. If $u \neq P_{\psi_{\min}}$ then $u_k \rightarrow u$ weakly. If instead $u = P_{\min}$ the following dichotomy holds:*

- (i) $u_k \rightarrow P_{\min}$ weakly;
- (ii) $\limsup_{k \rightarrow \infty} |\sup_X u_k| = \infty$.

Proof. The dichotomy for the case $u = P_{\psi_{\min}}$ follows by definition. Indeed if $|\sup_X u_k| \leq C$ and $d_{\mathcal{A}}(u_k, u) \rightarrow 0$ as $k \rightarrow \infty$, then $V_{\psi_k} \rightarrow V_{\psi_{\min}} = 0$ by Proposition 4.2.11.(iv) which implies that $\psi_k \rightarrow \psi_{\min}$ by Lemma 4.3.12. Hence any weak accumulation point u of $\{u_k\}_{k \in \mathbb{N}}$ satisfies $u \leq \psi_{\min} + C$.

Thus, let $\psi_k, \psi \in \mathcal{A}$ such that $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $u \in \mathcal{E}^1(X, \omega, \psi)$ where $\psi \in \mathcal{M}^+$. Observe that

$$d(u_k, \psi_k) \leq d_{\mathcal{A}}(u_k, u) + d(u, \psi) + d_{\mathcal{A}}(\psi, \psi_k) \leq A$$

for an uniform constant $A > 0$ by Proposition 4.2.11.(iv)

On the other hand for any $j \in \mathbb{N}$ by [BK07] there exists $h_j \in \mathcal{H}_{\omega}$ such that $h_j \geq u$, $\|h_j - u\|_{L^1} \leq 1/j$ and $d(u, P_{\omega}[\psi](h_j)) \leq 1/j$. In particular by the triangle inequality and Proposition 4.2.11 we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} d(P_{\omega}[\psi_k](h_j), \psi_k) &\leq \\ &\leq \limsup_{k \rightarrow \infty} \left(d_{\mathcal{A}}(P_{\omega}[\psi_k](h_j), P_{\omega}[\psi](h_j)) + \frac{1}{j} + d(u, \psi) + d(\psi, \psi_k) \right) \leq d(u, \psi) + \frac{1}{j}, \end{aligned} \tag{4.19}$$

Similarly again by the triangle inequality and Proposition 4.2.11

$$\begin{aligned} \limsup_{k \rightarrow \infty} d(u_k, P_\omega[\psi_k](h_j)) &\leq \\ &\leq \limsup_{k \rightarrow \infty} \left(d_{\mathcal{A}}(P_\omega[\psi_k](h_j), P_\omega[\psi](h_j)) + \frac{1}{j} + d_{\mathcal{A}}(u, u_k) \right) \leq \frac{1}{j} \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|u_k - u\|_{L^1} &\leq \\ \limsup_{k \rightarrow \infty} \left(\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} + \|P_\omega[\psi_k](h_j) - P_\omega[\psi](h_j)\|_{L^1} + \|P_\omega[\psi](h_j) - u\|_{L^1} \right) &\leq \\ &\leq \frac{1}{j} + \limsup_{k \rightarrow \infty} \|u_k - P_\omega[\psi_k](h_j)\|_{L^1}. \end{aligned} \quad (4.21)$$

In particular from (4.19) and (4.20) we deduce that $d(\psi_k, P_\omega[\psi_k](h_j)), d(\psi_k, u_k) \leq C$ for an uniform constant $C \in \mathbb{R}$. Next let $\phi_k \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ the unique solution of $MA_\omega(\phi_k) = \frac{V_{\psi_k}}{V_0} MA_\omega(0)$ and observe that by Proposition 4.2.4

$$\begin{aligned} d(\psi_k, \phi_k) &= -E_{\psi_k}(\phi_k) \leq \int_X (\psi_k - \phi_k) MA_\omega(\phi_k) \leq \\ &\leq \frac{V_{\psi_k}}{V_0} \int_X |\phi_k| MA_\omega(0) \leq \|\phi_k\|_{L^1} \leq C' \end{aligned}$$

since ϕ_k belongs to a compact (hence bounded) subset of $PSH(X, \omega) \subset L^1$. Therefore, since $V_{\psi_k} \geq a > 0$ for $k \gg 0$ big enough, by Proposition 4.3.6 it follows that there exists a continuous increasing function $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ with $f(0) = 0$ such that

$$\|u_k - P_\omega[\psi_k](h_j)\|_{L^1} \leq f(d(u_k, P_\omega[\psi_k](h_j)))$$

for any k, j big enough. Hence combining (4.20) and (4.21) the convergence requested follows letting $k, j \rightarrow +\infty$ in this order. \square

We can now prove the important characterization of the strong convergence as the coarsest refinement of the weak topology such that $E(\cdot)$ becomes continuous.

Theorem 4.6.2. *Let $u_k \in \mathcal{E}^1(X, \omega, \psi_k), u \in \mathcal{E}^1(X, \omega, \psi)$ for $\{\psi_k\}_{k \in \mathbb{N}}, \psi \in \overline{\mathcal{A}}$. If $\psi \neq \psi_{\min}$ or $V_{\psi_{\min}} > 0$ then the followings are equivalent:*

i) $u_k \rightarrow u$ strongly;

ii) $u_k \rightarrow u$ weakly and $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$.

In the case $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$, if $u_k \rightarrow P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \rightarrow 0$ then $u_k \rightarrow P_{\psi_{\min}}$ strongly. Finally if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \rightarrow 0$ as $k \rightarrow \infty$, then the following dichotomy holds:

- a) $u_k \rightarrow P_{\psi_{\min}}$ weakly and $E_{\psi_k}(u_k) \rightarrow 0$;
 b) $\limsup_{k \rightarrow \infty} |\sup_X u_k| = \infty$.

Proof. **Implication (ii) \Rightarrow (i).**

Assume that (ii) holds where we include the case $u = P_{\psi_{\min}}$ setting $E_{\psi}(P_{\psi_{\min}}) := 0$. Clearly it is enough to prove that any subsequence of $\{u_k\}_{k \in \mathbb{N}}$ admits a subsequence which is d_A -convergent to u . For the sake of simplicity we denote with $\{u_k\}_{k \in \mathbb{N}}$ the arbitrary initial subsequence, and since A is totally ordered by Lemma 4.3.13 we may also assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. In particular even if $u = P_{\psi_{\min}}$ we may suppose that u_k converges weakly to a proper element $v \in \mathcal{E}^1(X, \omega, \psi)$ up to considering a further subsequence by definition of weak convergence to the point $P_{\psi_{\min}}$. In this case by abuse of notation we denote the function v , which depends on the subsequence chosen, with u . Note also that by Hartogs' Lemma we have $u_k \leq \psi_k + A, u \leq \psi + A$ for an uniform constant $A \in \mathbb{R}_{\geq 0}$ since $|\sup_X u_k| \leq A$. In the case $\psi_k \searrow \psi$, $v_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi_k)$ decreases to u . Thus $w_k := P_{\omega}[\psi](v_k) \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u , which implies $d(u, w_k) \rightarrow 0$ as $k \rightarrow \infty$ (if $u = P_{\psi_{\min}}$ we immediately have $w_k = P_{\psi_{\min}}$).

Moreover by Propositions 4.2.4 and 5.2.2 it follows that

$$\begin{aligned} E_{\psi}(u) &= \lim_{k \rightarrow \infty} E_{\psi}(w_k) = AV_{\psi} - \lim_{k \rightarrow \infty} d(\psi + A, w_k) \geq \\ &\geq \lim_{k \rightarrow \infty} (AV_{\psi_k} - d(\psi_k + A, v_k)) = \limsup_{k \rightarrow \infty} E_{\psi_k}(v_k) \geq \lim_{k \rightarrow \infty} E_{\psi_k}(u_k) = E_{\psi}(u) \end{aligned}$$

since $\psi_k + A = P_{\omega}[\psi_k](A)$. Hence $\limsup_{k \rightarrow \infty} d(v_k, u_k) = \limsup_{k \rightarrow \infty} d(\psi_k + A, u_k) - d(v_k, \psi_k + A) = \lim_{k \rightarrow \infty} E_{\psi_k}(v_k) - E_{\psi_k}(u_k) = 0$. Thus by the triangle inequality it is sufficient to show that $\limsup_{k \rightarrow \infty} d_A(u, v_k) = 0$.

Next for any $C \in \mathbb{R}$ we set $v_k^C := \max(v_k, \psi_k - C)$, $u^C := \max(u, \psi - C)$ and we observe that $d(\psi_k + A, v_k^C) \rightarrow d(\psi + A, u^C)$ by Proposition 4.2.11 since $v_k^C \searrow u^C$. This implies that

$$\begin{aligned} d(v_k, v_k^C) &= d(\psi_k + A, v_k) - d(\psi_k + A, v_k^C) = AV_{\psi_k} - E_{\psi_k}(v_k) - d(\psi_k + A, v_k^C) \longrightarrow \\ &\longrightarrow AV_{\psi} - E_{\psi}(u) - d(\psi + A, u^C) = d(\psi + A, u) - d(\psi + A, u^C) = d(u, u^C). \end{aligned}$$

Thus, since $u^C \rightarrow u$ strongly, again by the triangle inequality it remains to estimate $d_A(u, v_k^C)$. Fix $\epsilon > 0$ and $\phi_{\epsilon} \in \mathcal{P}_{\mathcal{H}\omega}(X, \omega, \psi)$ such that $d(\phi_{\epsilon}, u) \leq \epsilon$ (by Lemma 4.2.13). Then letting $\varphi \in \mathcal{H}_{\omega}$ such that $\phi_{\epsilon} = P_{\omega}[\psi](\varphi)$ and setting $\phi_{\epsilon, k} := P_{\omega}[\psi_k](\varphi)$ by Proposition 4.2.11 we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} d_A(u, v_k^C) &\leq \limsup_{k \rightarrow \infty} (d(u, \phi_{\epsilon}) + d_A(\phi_{\epsilon}, \phi_{\epsilon, k}) + d(\phi_{\epsilon, k}, v_k^C)) \leq \\ &\leq \epsilon + d(\phi_{\epsilon}, u^C) \leq 2\epsilon + d(u, u^C), \end{aligned}$$

which concludes the first case of (ii) \Rightarrow (i) by the arbitrariness of ϵ since $u^C \rightarrow u$ strongly in $\mathcal{E}^1(X, \omega, \psi)$.

Next assume that $\psi_k \nearrow \psi$ almost everywhere. In this case we clearly may assume $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$. Then $v_k := (\sup\{u_j : j \geq k\})^* \in \mathcal{E}^1(X, \omega, \psi)$ decreases to u . Moreover setting $w_k := P_\omega[\psi_k](v_k) \in \mathcal{E}^1(X, \omega, \psi_k)$ and combining the monotonicity of $E_{\psi_k}(\cdot)$, the upper semicontinuity of $E(\cdot)$ (Proposition 4.3.14) and the contraction property of Proposition 5.2.2 we obtain

$$\begin{aligned} E_\psi(u) &= \lim_{k \rightarrow \infty} E_\psi(v_k) = AV_\psi - \lim_{k \rightarrow \infty} d(v_k, \psi + A) \leq \\ &\leq \liminf_{k \rightarrow \infty} (AV_{\psi_k} - d(w_k, \psi_k + A)) = \liminf_{k \rightarrow \infty} E_{\psi_k}(w_k) \leq \limsup_{k \rightarrow \infty} E_{\psi_k}(w_k) \leq E_\psi(u), \end{aligned}$$

i.e. $E_{\psi_k}(w_k) \rightarrow E_\psi(u)$ as $k \rightarrow \infty$. As a easy consequence we also get $d(w_k, u_k) = E_{\psi_k}(w_k) - E_{\psi_k}(u_k) \rightarrow 0$, thus it is sufficient to prove that

$$\limsup_{k \rightarrow \infty} d_{\mathcal{A}}(u, w_k) = 0.$$

Similarly to the previous case, fix $\epsilon > 0$ and let $\phi_\epsilon = P_\omega[\psi](\phi_\epsilon)$ for $\varphi \in \mathcal{H}_\omega$ such that $d(u, \phi_\epsilon) \leq \epsilon$. Again Proposition 5.2.2 and Proposition 4.2.11 yield

$$\begin{aligned} \limsup_{k \rightarrow \infty} d_{\mathcal{A}}(u, w_k) &\leq \epsilon + \limsup_{k \rightarrow \infty} (d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(P_\omega[\psi_k](\phi_\epsilon), w_k)) \leq \\ &\leq \epsilon + \limsup_{k \rightarrow \infty} (d_{\mathcal{A}}(\phi_\epsilon, P_\omega[\psi_k](\phi_\epsilon)) + d(\phi_\epsilon, v_k)) \leq 2\epsilon, \end{aligned}$$

which concludes the first part.

Implication (i) \Rightarrow (ii) if $u \neq P^{\psi_{\min}}$ while (i) implies the dichotomy if $u = P^{\psi_{\min}}$.

If $u \neq P^{\psi_{\min}}$, Proposition 4.6.1 implies that $u_k \rightarrow u$ weakly and in particular that $|\sup_X u_k| \leq A$. Thus it remains to prove that $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$.

If $u = P^{\psi_{\min}}$ then again by Proposition 4.6.1 it remains to show that $E_{\psi_k}(u_k) \rightarrow 0$ assuming $u_{k_h} \rightarrow P^{\psi_{\min}}$ strongly and weakly. Note that we also have $|\sup_X u_k| \leq A$ for an uniform constant $A \in \mathbb{R}$ by definition of weak convergence to $P^{\psi_{\min}}$.

So, since by an easy contradiction argument it is enough to prove that any subsequence of $\{u_k\}_{k \in \mathbb{N}}$ admits a further subsequence such that the convergence of the energies holds, without loss of generality we may assume that $u_k \rightarrow u \in \mathcal{E}^1(X, \omega, \psi)$ weakly even in the case $V_\psi = 0$ (i.e. when, with abuse of notation, $u = P^{\psi_{\min}}$).

Therefore we want to show the existence of a further subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ such that $E_{\psi_{k_h}}(u_{k_h}) \rightarrow E_\psi(u)$ (note that if $V_\psi = 0$ then $E_\psi(u) = 0$). It easily follows that

$$\begin{aligned} |E_{\psi_k}(u_k) - E_\psi(u)| &\leq |d(\psi_k + A, u_k) - d(\psi + A, u)| + A|V_{\psi_k} - V_\psi| \leq \\ &\leq d_{\mathcal{A}}(u, u_k) + d(\psi_k + A, \psi + A) + A|V_{\psi_k} - V_\psi|, \end{aligned}$$

and this leads to $\lim_{k \rightarrow \infty} E_{\psi_k}(u_k) = E_\psi(u)$ by Proposition 4.2.11 since $\psi_k + A = P_\omega[\psi_k](A)$ and $\psi + A = P_\omega[\psi](A)$. Hence $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$ as requested. \square

Note that in Theorem 4.6.2 the case (b) may happen (Remark 5.4.13) but obviously one can consider

$$X_{\mathcal{A}, norm} = \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}_{norm}^1(X, \omega, \psi)$$

to exclude such pathology.

The strong convergence also implies the convergence in ψ' -capacity for any $\psi' \in \mathcal{M}^+$ as our next result shows.

Theorem 4.6.3. *Let $\psi_k, \psi \in \mathcal{A}$, and let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ strongly converging to $u \in \mathcal{E}^1(X, \omega, \psi)$. Assuming also that $V_\psi > 0$. Then there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ such that the sequences $w_j := (\sup\{u_{k_s} : s \geq j\})^*$, $v_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ belong to $X_{\mathcal{A}}$, satisfy $v_j \leq u_{k_j} \leq w_j$ and converge strongly and monotonically to u . In particular $u_k \rightarrow u$ in ψ' -capacity for any $\psi' \in \mathcal{M}^+$ and $MA_\omega(u_k^j, \psi_k^{n-j}) \rightarrow MA_\omega(u^k, \psi^{n-j})$ weakly for any $j \in \{0, \dots, n\}$.*

Proof. We first observe that by Theorem 4.6.2 $u_k \rightarrow u$ weakly and $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$. In particular $\sup_X u_k$ is uniformly bounded and the sequence of ω -psh $w_k := (\sup\{u_j : j \geq k\})^*$ decreases to u .

Up to considering a subsequence we may assume either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere. We treat the two cases separately.

Assume first that $\psi_k \searrow \psi$. Since clearly $w_k \in \mathcal{E}^1(X, \omega, \psi_k)$ and $E_{\psi_k}(w_k) \geq E_{\psi_k}(u_k)$, Theorem 4.6.2 and Proposition 4.3.14 yields

$$E_\psi(u) = \lim_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq \limsup_{k \rightarrow \infty} E_{\psi_k}(w_k) \leq E_\psi(u),$$

i.e. $w_k \rightarrow u$ strongly. Thus up to considering a further subsequence we can suppose that $d(u_k, w_k) \leq 1/2^k$ for any $k \in \mathbb{N}$.

Next similarly as during the proof of Proposition 4.5.7 we define $v_{j,l} := P_\omega(u_j, \dots, u_{j+l})$ for any $j, l \in \mathbb{N}$, observing that $v_{j,l} \in \mathcal{E}^1(X, \omega, \psi_{j+l})$. Thus the function $v_{j,l}^u := P_\omega(u, v_{j,l}) \in \mathcal{E}^1(X, \omega, \psi)$ satisfies

$$\begin{aligned} d(u, v_{j,l}^u) &\leq \int_X (u - v_{j,l}^u) MA_\omega(v_{j,l}^u) \leq \int_{\{v_{j,l}^u = v_{j,l}\}} (u - v_{j,l}) MA_\omega(v_{j,l}) \leq \\ &\leq \sum_{s=j}^{j+l} \int_X (w_s - u_s) MA_\omega(u_s) \leq (n+1) \sum_{s=j}^{j+l} d(w_s, u_s) \leq \frac{(n+1)}{2^{j-1}}, \end{aligned} \quad (4.22)$$

where we combined Proposition 4.2.4 and Lemma 3.7. in [DDNL17]. Therefore by Proposition 4.3.15 $v_{j,l}^u$ converges decreasingly and strongly in $\mathcal{E}^1(X, \omega, \psi)$ to a function ϕ_j which satisfies $\phi_j \leq u$.

Similarly $\int_{\{P_\omega(u, v_{j,l}^u) = u\}} (v_{j,l}^u - u) MA_\omega(u) \leq \int_X |v_{j,l}^u - u| MA_\omega(u) < \infty$ by Corollary 4.3.5, which implies that $v_{j,l}$ converges decreasingly to $v_j \in \mathcal{E}^1(X, \omega, \psi)$ such that $u \geq v_j \geq \phi_j$ since $v_j \leq u_s$ for any $s \geq j$ and $v_{j,l} \geq v_{j,l}^u$. Hence from (4.22) we obtain

$$d(u, v_j) \leq d(u, \phi_j) = \lim_{l \rightarrow \infty} d(u, v_{j,l}^u) \leq \frac{(n+1)}{2^{j-1}},$$

i.e. v_j converges increasingly and strongly to u as $j \rightarrow \infty$.

Next assume $\psi_k \nearrow \psi$ almost everywhere. In this case $w_k \in \mathcal{E}^1(X, \omega, \psi)$ for any $k \in \mathbb{N}$, and clearly w_k converges strongly and decreasingly to u . On the other hand, letting $w_{k,k} := P_\omega[\psi_k](w_k)$ we observe that $w_{k,k} \rightarrow u$ weakly since $w_k \geq w_{k,k} \geq u_k$ and

$$E_\psi(u) = \lim_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq \limsup_{k \rightarrow \infty} E_{\psi_k}(w_{k,k}) \leq E_\psi(u)$$

by Theorem 4.6.2 and Proposition 4.3.14, i.e. $w_{k,k} \rightarrow u$ strongly again by Theorem 4.6.2. Thus, similarly to the previous case, we may assume that $d(u_k, w_{k,k}) \leq 1/2^k$ up to considering a further subsequence. Therefore setting $v_{j,l} := P_\omega(u_j, \dots, u_{j+l}) \in \mathcal{E}^1(X, \omega, \psi_j)$, $u^j := P_\omega[\psi_j](u)$ and $v_{j,l}^{u^j} := P_\omega(v_{j,l}, u^j)$ we obtain

$$d(u^j, v_{j,l}^{u^j}) \leq \int_X (u^j - v_{j,l}^{u^j}) MA_\omega(v_{j,l}^{u^j}) \leq \sum_{s=j}^{j+l} \int_X (w_{s,s} - u_s) MA_\omega(u_s) \leq \frac{(n+1)}{2^{j-1}} \quad (4.23)$$

proceeding similarly as before. This implies that $v_{j,l}^{u^j}$ and $v_{j,l}$ converge decreasingly and strongly respectively to functions $\phi_j, v_j \in \mathcal{E}^1(X, \omega, \psi_j)$ as $l \rightarrow +\infty$ which satisfy $\phi_j \leq v_j \leq u^j$. Therefore combining (4.23), Proposition 4.2.11 and the triangle inequality we get

$$\begin{aligned} \limsup_{j \rightarrow \infty} d_{\mathcal{A}}(u, v_j) &\leq \limsup_{j \rightarrow \infty} \left(d_{\mathcal{A}}(u, u^j) + d(u^j, \phi_j) \right) \leq \\ &\leq \limsup_{j \rightarrow \infty} \left(d_{\mathcal{A}}(u, u^j) + \frac{(n+1)}{2^{j-1}} \right) = 0. \end{aligned}$$

Hence v_j converges strongly and increasingly to u , so $v_j \nearrow u$ almost everywhere (Proposition 4.6.1) and the first part of the proof is concluded.

The convergence in ψ' -capacity and the weak convergence of the mixed Monge-Ampère measures follow exactly as seen during the proof of Proposition 4.5.7. \square

We observe that the assumption $u \neq P_{\psi_{\min}}$ if $V_{\psi_{\min}} = 0$ in Theorem 4.6.3 is obviously necessary as the counterexample of Remark 5.4.13 shows. On the other hand if $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \rightarrow 0$ then trivially $MA_\omega(u_k^j, \psi_k^{n-j}) \rightarrow 0$ weakly as $k \rightarrow \infty$ for any $j \in \{0, \dots, n\}$ as a consequence of $V_{\psi_k} \searrow 0$.

4.6.2 Proof of Theorem B

Definition 4.6.4. We define $Y_{\mathcal{A}}$ as

$$Y_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi),$$

and we endow it with its natural strong topology given as the coarsest refinement of the weak topology such that E^* becomes continuous, i.e. $V_{\psi_k}\mu_k$ converges strongly to $V_\psi\mu$ if and only if $V_{\psi_k}\mu_k \rightarrow V_\psi\mu$ weakly and $E_{\psi_k}^*(\mu_k) \rightarrow E_\psi^*(\mu)$ as $k \rightarrow \infty$.

Observe that $Y_{\mathcal{A}} \subset \{\text{non-pluripolar measures of total mass belonging to } [V_{\psi_{\min}}, V_{\psi_{\max}}]\}$ where clearly $\psi_{\max} := \sup \mathcal{A}$. As stated in the Introduction, the denomination is coherent with [BBEGZ19] since if $\psi = 0 \in \overline{\mathcal{A}}$ then the induced topology on $\mathcal{M}^1(X, \omega)$ coincides with the strong topology as defined in [BBEGZ19]. We also recall that

$$X_{\mathcal{A}, \text{norm}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}_{\text{norm}}^1(X, \omega, \psi)$$

where $\mathcal{E}_{\text{norm}}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) \text{ such that } \sup_X u = 0\}$ (if $V_{\psi_{\min}} = 0$ then we clearly assume $P_{\psi_{\min}} \in X_{\mathcal{A}, \text{norm}}$).

Theorem B. *The Monge-Ampère map*

$$MA_\omega : (X_{\mathcal{A}, \text{norm}}, d_{\mathcal{A}}) \rightarrow (Y_{\mathcal{A}}, \text{strong})$$

is an homeomorphism.

Proof. The map is a bijection as a consequence of Lemma 4.3.12 and Proposition 4.5.5 defining clearly $MA_\omega(P_{\psi_{\min}}) := 0$, i.e. to be the null measure.

Step 1: Continuity. Assume first that $V_{\psi_{\min}} = 0$ and that $d_{\mathcal{A}}(u_k, P_{\psi_{\min}}) \rightarrow 0$ as $k \rightarrow \infty$. Then easily $MA_\omega(u_k) \rightarrow 0$ weakly. Moreover, assuming $u_k \neq P_{\psi_{\min}}$ for any k , it follows from Proposition 4.2.4 that

$$\begin{aligned} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) &= E_{\psi_k}(u_k) + \int_X (\psi_k - u_k) MA_\omega(u_k) \leq \\ &\leq \frac{n}{n+1} \int_X (\psi_k - u_k) MA_\omega(u_k) \leq -nE_{\psi_k}(u_k) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$ where the convergence is given by Theorem 4.6.2. Hence $MA_\omega(u_k) \rightarrow 0$ strongly in $Y_{\mathcal{A}}$.

We can now assume that $u \neq P_{\psi_{\min}}$.

Theorem 4.6.3 immediately gives the weak convergence of $MA_\omega(u_k)$ to $MA_\omega(u)$. Fix $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to u such that $d(u, P_\omega[\psi](\varphi_j)) \leq 1/j$ for any $j \in \mathbb{N}$ ([BK07]) and set $v_{k,j} := P_\omega[\psi_k](\varphi_j)$ and $v_j := P_\omega[\psi](\varphi_j)$. Observe also that as a consequence of Proposition 4.2.11 and Theorem 4.6.2, for any $j \in \mathbb{N}$ there exists $k_j \gg 0$ big enough such that $d(\psi_k, v_{k,j}) \leq d_{\mathcal{A}}(\psi_k, \psi) + d(\psi, v_j) + d_{\mathcal{A}}(v_j, v_{k,j}) \leq d(\psi, v_j) + 1 \leq C$ for any $k \geq k_j$, where C is an uniform constant independent on $j \in \mathbb{N}$. Therefore combining again Theorem 4.6.2 with Lemma 4.4.6 and Proposition

4.3.7 we obtain

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left| E_{\psi_k}^* (MA_\omega(u_k)/V_{\psi_k}) - E_{\psi_k}^* (MA_\omega(v_{k,j})/V_{\psi_k}) \right| \leq \\ & \leq \limsup_{k \rightarrow \infty} \left(\left| E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j}) \right| + \left| \int_X (\psi_k - u_k)(MA_\omega(u_k) - MA_\omega(v_{k,j})) \right| + \right. \\ & \quad \left. + \left| \int_X (v_{k,j} - u_k) MA_\omega(v_{k,j}) \right| \right) \leq |E_\psi(u) - E_\psi(v_j)| + \\ & \quad + \limsup_{k \rightarrow \infty} CI_{\psi_k}(u_k, v_{k,j})^{1/2} + \int_X (v_j - u) MA_\omega(v_j) \quad (4.24) \end{aligned}$$

since clearly we may assume that either $\psi_k \searrow \psi$ or $\psi_k \nearrow \psi$ almost everywhere, up to considering a subsequence. On the other hand, if $k \geq k_j$, Proposition 4.3.4 implies $I_{\psi_k}(u_k, v_{k,j}) \leq 2f_{\tilde{C}}(d(u_k, v_{k,j}))$ where \tilde{C} is a uniform constant independent of j, k and $f_{\tilde{C}} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is a continuous increasing function such that $f_{\tilde{C}}(0) = 0$. Hence continuing the estimates in (4.24) we get

$$(4.24) \leq |E_\psi(u) - E_\psi(v_j)| + 2Cf_{\tilde{C}}(d(u, v_j)) + d(v_j, u) \quad (4.25)$$

using also Propositions 4.2.4 and 4.2.11. Letting $j \rightarrow \infty$ in (4.25), it follows that

$$\limsup_{j \rightarrow \infty} \limsup_{k \rightarrow \infty} \left| E_{\psi_k}^* (MA_\omega(u_k)/V_{\psi_k}) - E_{\psi_k}^* (MA_\omega(v_{k,j})/V_{\psi_k}) \right| = 0$$

since $v_j \searrow u$. Furthermore it is easy to check that $E_{\psi_k}^*(MA_\omega(v_{k,j})/V_{\psi_k}) \rightarrow E_\psi^*(MA_\omega(v_j)/V_\psi)$ as $k \rightarrow \infty$ for j fixed by Lemma 4.4.6 and Proposition 4.2.11. Therefore the convergence

$$E_\psi^*(MA_\omega(v_j)/V_\psi) \rightarrow E_\psi^*(MA_\omega(u)/V_\psi) \quad (4.26)$$

as $j \rightarrow \infty$ given by Theorem A concludes this step.

Step 2: Continuity of the inverse. Assume $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$, $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $MA_\omega(u_k) \rightarrow MA_\omega(u)$ strongly. Note that when $\psi = \psi_{\min}$ and $V_{\psi_{\min}} = 0$ the assumption does not depend on the function u chosen. Clearly this implies $V_{\psi_k} \rightarrow V_\psi$ which leads to $\psi_k \rightarrow \psi$ as $k \rightarrow \infty$ by Lemma 4.3.12 since $\mathcal{A} \subset \mathcal{M}^+$ is totally ordered. Hence, up to considering a subsequence, we may assume that $\psi_k \rightarrow \psi$ monotonically almost everywhere. We keep the same notations of the previous step for $v_{k,j}, v_j$. We may also suppose that $V_{\psi_k} > 0$ for any $k \in \mathbb{N}$ big enough otherwise it would be trivial.

The strategy is to proceed similarly as during the proof of Theorem A, i.e. we want first to prove that $I_{\psi_k}(u_k, v_{k,j}) \rightarrow 0$ as $k, j \rightarrow \infty$ in this order. Then we want to use this to prove that the unique weak accumulation point of $\{u_k\}_{k \in \mathbb{N}}$ is u . Finally we will deduce also the convergence of the ψ_k -relative energies to conclude that $u_k \rightarrow u$ strongly thanks to Theorem 4.6.2.

By Lemma 4.3.1

$$\begin{aligned} (n+1)^{-1}I_{\psi_k}(u_k, v_{k,j}) &\leq E_{\psi_k}(u_k) - E_{\psi_k}(v_{k,j}) + \int_X (v_{k,j} - u_k)MA_\omega(u_k) = \\ &= E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) - E_{\psi_k}^*(MA_\omega(v_{k,j})/V_{\psi_k}) + \\ &\quad + \int_X (v_{k,j} - \psi_k)(MA_\omega(u_k) - MA_\omega(v_{k,j})) \end{aligned} \quad (4.27)$$

for any j, k . Moreover by Step 1 and Proposition 4.2.11 $E_{\psi_k}^*(MA_\omega(v_{k,j})/V_{\psi_k})$ converges, as $k \rightarrow +\infty$, respectively to 0 if $V_\psi = 0$ and to $E_\psi^*(MA_\omega(v_j)/V_\psi)$ if $V_\psi > 0$. Next by Lemma 4.4.6

$$\int_X (v_{k,j} - \psi_k)MA_\omega(v_{k,j}) \rightarrow \int_X (v_j - \psi)MA_\omega(v_j)$$

letting $k \rightarrow \infty$. So if $V_\psi = 0$ then from $\lim_{k \rightarrow \infty} \sup_X (v_{k,j} - \psi_k) = \sup_X (v_j - \psi) = \sup_X v_j$ we easily get $\limsup_{k \rightarrow \infty} I_{\psi_k}(u_k, v_{k,j}) = 0$. Thus we may assume $V_\psi > 0$ and it remains to estimate $\int_X (v_{k,j} - \psi_k)MA_\omega(u_k)$ from above.

We set $f_{k,j} := v_{k,j} - \psi_k$ and analogously to the proof of Theorem A we construct a sequence of smooth functions $f_j^s := v_j^s - \psi^s$ converging in capacity to $f_j := v_j - \psi$ and satisfying $\|f_j^s\|_{L^\infty} \leq 2\|f_j\|_{L^\infty} \leq 2\|\varphi_j\|_{L^\infty}$. Here v_j^s, ψ^s are sequences of ω -psh functions decreasing respectively to v_j, ψ . Then we write

$$\int_X f_{k,j}MA_\omega(u_k) = \int_X (f_{k,j} - f_j^s)MA_\omega(u_k) + \int_X f_j^sMA_\omega(u_k) \quad (4.28)$$

and we observe that $\limsup_{s \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X f_j^sMA_\omega(u_k) = \int_X f_jMA_\omega(u)$ since $MA_\omega(u_k) \rightarrow MA_\omega(u)$ weakly, $f_j^s \in C^\infty$, f_j^s converges to f_j in capacity and $\|f_j^s\|_{L^\infty} \leq 2\|f_j\|_{L^\infty}$. While we claim that the first term on the right hand side of (4.28) goes to 0 letting $k, s \rightarrow \infty$ in this order. Indeed for any $\delta > 0$

$$\begin{aligned} \int_X (f_{k,j} - f_j)MA_\omega(u_k) &\leq \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{f_{k,j} - f_j > \delta\}} MA_\omega(u_k) \leq \\ &\leq \delta V_{\psi_k} + 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{k,j} - h_j| > \delta\}} MA_\omega(u_k) \end{aligned} \quad (4.29)$$

where we set $h_{k,j} := v_{k,j}, h_j := v_j$ if $\psi_k \searrow \psi$ and $h_{k,j} := \psi_k, h_j := \psi$ if instead $\psi_k \nearrow \psi$ almost everywhere. Moreover since $\{|h_{k,j} - h_j| > \delta\} \subset \{|h_{l,j} - h_j| > \delta\}$ for any $l \leq k$, from (4.29) we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \int_X (f_{k,j} - f_j)MA_\omega(u_k) &\leq \\ &\leq \delta V_\psi + \limsup_{l \rightarrow \infty} \limsup_{k \rightarrow \infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j} - h_j| \geq \delta\}} MA_\omega(u_k) \leq \\ &\leq \delta V_\psi + \limsup_{l \rightarrow \infty} 2\|\varphi_j\|_{L^\infty} \int_{\{|h_{l,j} - h_j| \geq \delta\}} MA_\omega(u) = \delta V_\psi \end{aligned}$$

where we also used that $\{|h_{l,j} - h_j| \geq \delta\}$ is a closed set in the plurifine topology since it is equal to $\{\psi_l - v_j \geq \delta\}$ if $\psi_l \searrow \psi$ and to $\{\psi - \psi_l \geq \delta\}$ if $\psi_l \nearrow \psi$ almost everywhere. Hence $\limsup_{k \rightarrow \infty} \int_X (f_{k,j} - f_j) MA_\omega(u_k) \leq 0$. Similarly we also get $\limsup_{s \rightarrow \infty} \limsup_{k \rightarrow \infty} \int_X (f_j - f_j^s) MA_\omega(u_k) \leq 0$. (see also the proof of Theorem A).

Summarizing from (4.27), we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} (n+1)^{-1} I_{\psi_k}(u_k, v_{k,j}) &\leq E_\psi^*(MA_\omega(u)/V_\psi) - E_\psi^*(MA_\omega(v_j)/V_\psi) + \\ &+ \int_X (v_j - \psi) MA_\omega(u) - \int_X (v_j - \psi) MA_\omega(v_j) =: F_j, \end{aligned} \quad (4.30)$$

and $F_j \rightarrow 0$ as $j \rightarrow \infty$ by Step 1 and Proposition 4.3.7 since $\mathcal{E}^1(X, \omega, \psi) \ni v_j \searrow u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, hence strongly.

Next by Lemma 4.3.1 $u_k \in X_{A,C}$ for $C \gg 1$ since $E^*(MA_\omega(u_k)/V_{\psi_k}) = J_{u_k}^\psi(\psi)$ and $\sup_X u_k = 0$, thus up to considering a further subsequence $u_k \rightarrow w \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ weakly where $d(w, \psi) \leq C$. Indeed if $V_\psi > 0$ this follows from Proposition 4.3.15 while it is trivial if $V_\psi = 0$. In particular by Lemma 4.4.6

$$\int_X (\psi_k - u_k) MA_\omega(v_{k,j}) \rightarrow \int_X (\psi - w) MA_\omega(v_j) \quad (4.31)$$

$$\int_X (v_{k,j} - u_k) MA_\omega(v_{k,j}) \rightarrow \int_X (v_j - w) MA_\omega(v_j) \quad (4.32)$$

as $j \rightarrow \infty$. Therefore if $V_\psi = 0$ then combining $I_{\psi_k}(u_k, v_{k,j}) \rightarrow 0$ as $k \rightarrow \infty$ with (4.32) and Lemma 4.3.1, we obtain

$$\begin{aligned} \limsup_{k \rightarrow \infty} \left(-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j}) \right) &\leq \\ &\leq \limsup_{k \rightarrow \infty} \left(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \left| \int_X (v_{k,j} - u_k) MA_\omega(v_{k,j}) \right| \right) = 0. \end{aligned}$$

This implies that $d(\psi_k, u_k) = -E_{\psi_k}(u_k) \rightarrow 0$ as $k \rightarrow \infty$, i.e. that $d_A(P_{\psi_{\min}}, u_k) \rightarrow 0$ using Theorem 4.6.2. Thus we may assume from now until the end of the proof that $V_\psi > 0$.

By (4.31) and Proposition 4.3.14 it follows that

$$\begin{aligned} &\limsup_{k \rightarrow \infty} \left(E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) + \int_X (\psi_k - u_k) (MA_\omega(v_{k,j}) - MA_\omega(u_k)) \right) = \\ &= \limsup_{k \rightarrow \infty} \left(E_{\psi_k}(u_k) + \int_X (\psi_k - u_k) MA_\omega(v_{k,j}) \right) \leq E_\psi(w) + \int_X (\psi - w) MA_\omega(v_j). \end{aligned} \quad (4.33)$$

On the other hand by Proposition 4.3.7 and (4.30),

$$\limsup_{k \rightarrow \infty} \left| \int_X (\psi_k - u_k) (MA_\omega(v_{k,j}) - MA_\omega(u_k)) \right| \leq CF_j^{1/2}. \quad (4.34)$$

In conclusion by the triangle inequality combining (4.33) and (4.34) we get

$$\begin{aligned} E_\psi(u) + \int_X (\psi - u) MA_\omega(u) &= \lim_{k \rightarrow \infty} E^*(MA_\omega/(u_k)/V_{\psi_k}) \leq \\ &\leq \limsup_{j \rightarrow \infty} \left(E_\psi(w) + \int_X (\psi - w) MA_\omega(v_j) + CF_j^{1/2} \right) = E_\omega(w) + \int_X (\psi - w) MA_\omega(u) \end{aligned}$$

since $F_j \rightarrow 0$, i.e. $w \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ is a maximizer of $F_{MA_\omega(u)/V_\psi, \psi}$. Hence $w = u$ (Proposition 4.5.5), i.e. $u_k \rightarrow u$ weakly. Furthermore, similarly to the case $V_\psi = 0$, Lemma 4.3.1 and (4.32) imply

$$\begin{aligned} E_\psi(v_j) - \liminf_{k \rightarrow \infty} E_{\psi_k}(u_k) &= \limsup_{k \rightarrow \infty} \left(-E_{\psi_k}(u_k) + E_{\psi_k}(v_{k,j}) \right) \leq \\ &\leq \limsup_{k \rightarrow \infty} \left(\frac{n}{n+1} I_{\psi_k}(u_k, v_{k,j}) + \left| \int_X (u_k - v_{k,j}) MA_\omega(v_{k,j}) \right| \right) \leq \\ &\leq \frac{n}{n+1} F_j + \left| \int_X (u - v_j) MA_\omega(v_j) \right| \end{aligned}$$

Finally letting $j \rightarrow \infty$, since $v_j \rightarrow u$ strongly, we obtain $\liminf_{k \rightarrow \infty} E_{\psi_k}(u_k) \geq \lim_{j \rightarrow \infty} E_\psi(v_j) = E_\psi(u)$. Hence $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$ by Proposition 4.3.14 which implies $d_A(u_k, u) \rightarrow 0$ by Theorem 4.6.2 and concludes the proof. \square

4.7 Stability of Complex Monge-Ampère equations.

As stated in the Introduction, we want to use the homeomorphism of Theorem B to deduce the strong stability of solutions of complex Monge-Ampère equations with prescribed singularities when the measures have uniformly bounded L^p density for $p > 1$.

Theorem C. *Let $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be totally ordered, and let $\{f_k\}_{k \in \mathbb{N}} \subset L^1$ a sequence of non-negative functions such that $f_k \rightarrow f \in L^1 \setminus \{0\}$ and such that $\int_X f_k \omega^n = V_{\psi_k}$ for any $k \in \mathbb{N}$. Assume also that there exists $p > 1$ such that $\|f_k\|_{L^p}, \|f\|_{L^p}$ are uniformly bounded. Then $\psi_k \rightarrow \psi \in \overline{\mathcal{A}} \subset \mathcal{M}^+$, and the sequence of solutions of*

$$\begin{cases} MA_\omega(u_k) = f_k \omega^n \\ u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k) \end{cases} \quad (4.35)$$

converges strongly to $u \in X_{\mathcal{A}}$ which is the unique solution of

$$\begin{cases} MA_\omega(u) = f \omega^n \\ u \in \mathcal{E}_{norm}^1(X, \omega, \psi). \end{cases} \quad (4.36)$$

In particular $u_k \rightarrow u$ in capacity.

Proof. We first observe that the existence of the unique solutions of (4.35) follows by Theorem A in [DDNL18].

Moreover letting u any weak accumulation point for $\{u_k\}_{k \in \mathbb{N}}$ (there exists at least one by compactness), Lemma 2.8 in [DDNL18] yields $MA_\omega(u) \geq f\omega^n$ and by the convergence of f_k to f we also obtain $\int_X f\omega^n = \lim_{k \rightarrow \infty} \int_X f_k \omega^n$. Moreover since $u_k \leq \psi_k$ for any $k \in \mathbb{N}$, by [WN17] we obtain $\int_X MA_\omega(u) \leq \lim_{k \rightarrow \infty} \int_X \psi_k \omega^n$. Hence $MA_\omega(u) = f\omega^n$ which in particular means that there is an unique weak accumulation point for $\{u_k\}_{k \in \mathbb{N}}$ and that $\psi_k \rightarrow \psi$ as $k \rightarrow \infty$ since $V_{\psi_k} \rightarrow V_\psi$ (by Lemma 4.3.12). Then it easily follows combining Fatou's Lemma with Proposition 5.2.2 and Lemma 5.2.5 that for any $\varphi \in \mathcal{H}_\omega$

$$\begin{aligned} \liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) &\geq \\ &\geq \liminf_{k \rightarrow \infty} \left(E_{\psi_k}(P_\omega[\psi_k](\varphi)) + \int_X (\psi_k - P_\omega[\psi_k](\varphi)) f_k \omega^n \right) \geq \\ &\geq E_\psi(P_\omega[\psi](\varphi)) + \int_X (\psi - P_\omega[\psi](\varphi)) f \omega^n \end{aligned} \quad (4.37)$$

since $(\psi_k - P_\omega[\psi_k](\varphi)) f_k \rightarrow (\psi - P_\omega[\psi](\varphi)) f$ almost everywhere. Thus, for any $v \in \mathcal{E}^1(X, \omega, \psi)$ letting $\varphi_j \in \mathcal{H}_\omega$ be a decreasing sequence converging to v ([BK07]), from the inequality (5.12) we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) &\geq \\ &\geq \limsup_{j \rightarrow \infty} \left(E_\psi(P_\omega[\psi](\varphi_j)) + \int_X (\psi - P_\omega[\psi](\varphi_j)) f \omega^n \right) = \\ &= E_\psi(v) + \int_X (\psi - v) f \omega^n \end{aligned}$$

using Proposition 4.2.4 and the Monotone Converge Theorem. Hence by definition

$$\liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) \geq E_\psi^*(f\omega^n/V_\psi). \quad (4.38)$$

On the other hand since $\|f_k\|_{L^p}, \|f\|_{L^p}$ are uniformly bounded where $p > 1$ and $u_k \rightarrow u, \psi_k \rightarrow \psi$ in L^q for any $q \in [1, +\infty)$ (see Theorem 1.48 in [GZ17]), we also have

$$\int_X (\psi_k - u_k) f_k \omega^n \rightarrow \int_X (\psi - u) f \omega^n < +\infty,$$

which implies that $\int_X (\psi - u) MA_\omega(u) < +\infty$, i.e. $u \in \mathcal{E}^1(X, \omega, \psi)$ by Proposition 4.2.4. Moreover by Proposition 4.3.14 we also get

$$\limsup_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) \leq E_\psi^*(MA_\omega(u)/V_\psi),$$

which together with (5.14) leads to $MA_\omega(u_k) \rightarrow MA_\omega(u)$ strongly in Y_A . Hence $u_k \rightarrow u$ strongly by Theorem B while the convergence in capacity follows from Theorem 4.6.3. \square

Remark 4.7.1. As said in the Introduction, the convergence in capacity of Theorem C was already obtained in Theorem 1.4 in [DDNL19]. Indeed under the hypothesis of Theorem C it follows from Lemma 5.2.5 and Lemma 3.4 [DDNL19] that $d_S(\psi_k, \psi) \rightarrow 0$ where d_S is the pseudometric on $\{[u] : u \in PSH(X, \omega)\}$ introduced in [DDNL19] where the class $[u]$ is given by the partial order \preceq .

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PAPER IV

Continuity method with movable singularities
for classical complex Monge-Ampère equations.

Antonio Trusiani

arxiv preprint

Chapter 5

Continuity method with movable singularities for classical complex Monge-Ampère equations.

Abstract

On a compact Kähler manifold (X, ω) , we study the strong continuity of solutions with prescribed singularities of complex Monge-Ampère equations with convergent integrable Lebesgue densities. Then we address the strong continuity of solutions when the right hand sides are modified to includes all (log-)Kähler Einstein metrics with prescribed singularities. This leads to the closedness of a new continuity method when the densities are modified together with the prescribed singularities setting. For Monge-Ampère equations of Fano type, we also prove an openness result when the singularities decrease. Finally we deduce a strong stability result for (log-)Kähler Einstein metrics on semi-Kähler classes given as modifications of $\{\omega\}$.

5.1 Introduction.

Let (X, ω) be a compact Kähler manifold endowed with a Kähler form. This article concerns the study of (degenerate) complex Monge-Ampère equations of the type

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in PSH(X, \omega) \end{cases} \quad (5.1)$$

where $PSH(X, \omega)$ denotes the set of all ω -plurisubharmonic functions on X , $MA_\omega(u) = (\omega + dd^c u)^n$ in the sense of the non-pluripolar product ([BEGZ10]), $\lambda \in \mathbb{R}$ and $f \in L^1 \setminus \{0\}$. Here $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{1}{\pi} \partial \bar{\partial}$.

The study of equations like (5.1) plays a principal role in several questions in Kähler geometry, like the search of (log) Kähler-Einstein metrics ([Yau78], [Tian]). The classical way to approach the existence of solutions of these equations is through a *continuity method*. Namely approximating $g_1(v) := e^{\lambda v} f$ with a family $\{g_t\}_{t \in [0,1]}$ and proving that the set $t \in [0,1]$ such that $MA_\omega(u) = g_t(u) \omega^n$ admits a solution is not-empty, open and closed. It is also important to underline that along this continuity method one requires that the set of solutions u_t have *enough regularity*, and in particular that they have finite Monge-Ampère energy $E(u_t) := \lim_{k \rightarrow \infty} \frac{1}{n+1} \sum_{j=0}^n \int_X \max(u, -k) (\omega + dd^c \max(u, -k))^j \wedge \omega^{n-j}$ which means $u \in \mathcal{E}^1(X, \omega)$ ([BEGZ10]). Usually the hard part of the continuity method relies on the closedness, i.e. if a sequence of solutions $u_{t_k} \in \mathcal{E}^1(X, \omega)$ of $MA_\omega(u) = g_{t_k}(u) \omega^n$ converges as $t_k \rightarrow t_0$ to a solution u_{t_0} of $MA_\omega(u) = g_{t_0}(u) \omega^n$. The type of convergence required depends on the family of equations considered and on the kind of regularity one wants to achieve on the solutions.

In this paper we want to study the closedness of some new continuity methods with *movable singularities*, i.e. we allow the solutions to have some prescribed singularities and we require a certain *strong convergence*.

More precisely for $\lambda \in \mathbb{R}$, letting $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of non-negative L^1 functions converging to f in L^1 , we assume to have a family of solutions $\{u_k\}_{k \in \mathbb{N}}$ of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f_k \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi_k) \end{cases} \quad (5.2)$$

and we want to give necessary conditions for a strong convergence of u_k to a solution u of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (5.3)$$

Here $\psi_k, \psi \in PSH(X, \omega)$ represents the prescribed singularities. Indeed the set

$$\mathcal{E}^1(X, \omega, \psi) := \left\{ u \in PSH(X, \omega) : u \leq \psi + C, V_u := \int_X MA_\omega(u) = V_\psi \text{ and } E_\psi(u) > -\infty \right\}$$

was introduced in [DDNL18b] as the ψ -relative version of the set $\mathcal{E}^1(X, \omega)$ where E_ψ is the natural generalization of the Monge-Ampère energy (see section §5.2). Note that by [WN19] the total mass of the Monge-Ampère operator respects the partial order \preceq given as $u \preceq v$ if $u \leq v + C$ for a constant $C \in \mathbb{R}$, i.e. $V_u \leq V_v$ if $u \preceq v$. Hence $\mathcal{E}^1(X, \omega, \psi)$ represents all functions more singular than ψ which have the same ψ -relative full mass (i.e. $V_u = V_v$) and finite ψ -relative energy E_ψ . The set of all ψ -relative full mass is denoted by $\mathcal{E}(X, \omega, \psi)$.

In [DDNL18b] the authors also proved that there is a natural assumption to add on ψ so that $MA_\omega(u) = \mu$ is solvable in the class $\mathcal{E}(X, \omega, \psi)$ for any μ non-pluripolar measure with the right total mass, i.e. ψ must be a *model type envelope* (see section §5.2). We denote with \mathcal{M}^+ the set of all model type envelopes ψ such that $V_\psi > 0$.

The most interesting case to study is when the singularities are increasing or decreasing, so we suppose to have a totally ordered sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ converging weakly (i.e. in L^1) to an element $\psi \in \mathcal{M}^+$. In this case we can work with the *strong convergence* of solutions in the sense of [Tru19] and [Tru20a], i.e. $u_k \rightarrow u$ strongly if $u_k \rightarrow u$ weakly (i.e. in the usual L^1 -topology) and $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$. In fact this convergence is equivalent to $d_{\mathcal{A}}(u_k, u) \rightarrow 0$ as $k \rightarrow \infty$, where $d_{\mathcal{A}}$ is the complete distance on $X_{\mathcal{A}} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$ introduced in [Tru19] for $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}}$. Note that it is a very natural which implies the convergence in capacity ([Tru20a]).

To state the results, we need to distinguish three different cases based on the different sign of λ .

If $\lambda = 0$, we obviously must add the necessary assumption $\int_X f \omega^n = V_\psi$ on (5.3). In this case by Proposition C in [Tru20a] the equation is solvable if and only if $f \omega^n \in \mathcal{M}^1(X, \omega, \psi)$ (and a solution is unique modulo translation by constants).

Theorem A. *Assume*

- (i) $f_k, f \in L^1 \setminus \{0\}$ non-negative such that $f_k \rightarrow f$ as $k \rightarrow \infty$;
- (ii) $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered such that $V_{\psi_k} = \int_X f_k \omega^n$ for any $k \in \mathbb{N}$ and such that $\psi_k \rightarrow \psi \in \mathcal{M}^+$ in L^1 ;
- (iii) $f_k \omega^n \in \mathcal{M}^1(X, \omega, \psi_k)$ for any $k \in \mathbb{N}$ and denote with $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$ the unique solution of (5.2) with $\sup_X u = 0$, for $\lambda = 0$.

Then, letting u be a weak accumulation point of $\{u_k\}_{k \in \mathbb{N}}$, $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, $u_k \rightarrow u$ strongly and $MA_\omega(u) = f \omega^n$ if and only if $E_{\psi_k}(u_k) \geq -C$ for an uniform constant $C \geq 0$ and

$$\limsup_{k \rightarrow \infty} \int_X (\psi_k - u_k) f_k \omega^n \leq \int_X (\psi - u) f \omega^n. \quad (5.4)$$

With obvious notations, $\mathcal{E}_{norm}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$.

Note that by compactness in L^1 , there always exists a weak accumulation point u for $\{u_k\}_{k \in \mathbb{N}}$ as in the statement (and it is actually unique, see subsection 5.3.1).

Moreover the not-trivial condition (iii) is satisfied if $f_k \in L^{p_k}$ for $p_k > 1$. Since (5.4) and the uniform boundedness on the energies may be difficult to detect, in Remark 5.3.2 we collect some particular easier cases. Finally we stress that if $f \in L^1$ but $f \notin L^p$ for any $p > 1$, it is often difficult to find out if the unique $u \in \mathcal{E}_{norm}(X, \omega, \psi)$ satisfying $MA_\omega(u) = f\omega^n$ belongs to $\mathcal{E}_{norm}^1(X, \omega, \psi)$, which is essentially a regularity condition. Thus Theorem A gives a new tool to study the regularity of u .

If $\lambda < 0$ then (5.3) admits an unique solution by Theorem 4.23 in [DDNL18b] since the latter can be generalized to the case ψ with not small unbounded locus thank to [X19a]. In this case there are no obstruction to the strong convergence.

Theorem B. *Assume*

- (i) $\lambda < 0$;
- (ii) $f_k, f \in L^1 \setminus \{0\}$ non-negative such that $f_k \rightarrow f$ in L^1 ;
- (iii) $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered such that $\psi_k \rightarrow \psi$ weakly.

Let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$, $u \in \mathcal{E}^1(X, \omega, \psi)$ be the unique solutions respectively of

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k), \end{cases} \quad \begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (5.5)$$

Then $u_k \rightarrow u$ strongly.

Finally the case $\lambda > 0$ is much more complicated. For instance, if $\psi = 0$, $\{\omega\} = -K_X$ and $f = 0$ then a solution of (5.3) corresponds to a Kähler-Einstein metrics on a Fano manifold, whose existence is characterized by an algebrico-geometrical stability (see [CDS15]) and the uniqueness depends on the identity component of the automorphism group $\text{Aut}(X)^\circ$ (see [Bern15]).

However through a variational approach, our next Theorem corresponds to an openness result on the continuity method when the singularities decrease and the densities are constant. Indeed we first introduce a functional $F_{f, \psi, \lambda}$ which generalizes the Ding functional (see [Ding88]) and whose maximizers solve $MA_\omega(u) = e^{-\lambda u} f \omega^n$. Then we prove that its coercivity on $\mathcal{E}^1(X, \omega, \psi)$ expressed in terms of a ψ -relative J -functional (or in terms of the distance $d_{A|\mathcal{E}_{norm}^1(X, \omega, \psi) \times \mathcal{E}_{norm}^1(X, \omega, \psi)}$) implies the coercivity for any $\psi' \in \mathcal{M}^+$ slightly less singular than ψ .

Theorem C. *Let $\psi \in \mathcal{M}^+$, $\lambda > 0$ and $f \in L^p$ for $p \in (1, \infty]$. Assume also that $c(\psi) > \frac{\lambda p}{p-1}$ where $\frac{\lambda p}{p-1} = \lambda$ if $p = \infty$. If the functional $F_{f, \psi, \lambda}$ is coercive then the complex Monge-Ampère equation*

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi') \end{cases} \quad (5.6)$$

admits a solution for any $\psi' \in \mathcal{M}^+$, $\psi' \succcurlyeq \psi$ such that $V_{\psi'} < AV_\psi$ where $A > 1$.

In Theorem C, the constant $A > 1$ depends uniquely on the coefficient of the coercivity of $F_{f,\psi,\lambda}$, i.e. its slope at infinity. Moreover with $c(\psi)$ we indicate the classical *complex singularity exponent* (see for instance [DK01]). In particular the more is higher p and the more is lower λ , the more ψ can be singular. In the limit case $p = \infty$ the condition $c(\psi) > \lambda$ becomes necessary to solve the Monge-Ampère equation as a consequence of the resolution of the strong openness conjecture ([GZ15]). The reason of this bound on the complex singularity exponent is because it leads to the upper-semicontinuity of $F_{f,\psi,\lambda}$, hence to the fact that the coercivity of $F_{f,\psi,\lambda}$ implies the existence of a maximizer.

About the continuity of solutions, i.e. the closedness of the continuity method in the case $\lambda > 0$, we prove the following result.

Theorem D. *Let $\lambda > 0$, $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered sequence such that $\psi_k \preceq \psi_{k+1}$ for any $k \in \mathbb{N}$ which converges to $\psi \in \mathcal{M}^+$, and $f_k, f \geq 0$ such that $f_k \rightarrow f$ in L^p as $k \rightarrow \infty$ for $p \in (1, \infty]$. Assume also the following conditions:*

(i) $c(\psi) > \frac{\lambda p}{p-1}$;

(ii) *the complex Monge-Ampère equations*

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k); \end{cases}$$

admit solutions u_k given as maximizers of $F_{f_k, \psi_k, \lambda}$;

(iii) $\sup_X u_k \leq C$ for an uniform constant C .

Then there exists a subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ which converges strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$ solution of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases}$$

Since in many settings we expect that any solution of (5.2) maximizes $F_{f,\psi,\lambda}$ (see for instance [Tru20b] for the Fano case), and since the assumption (i) is satisfied for many $\psi \in \mathcal{M}^+$ and as said before it becomes necessary when $p = \infty$, the unique real big obstacle is the uniform estimate in (iii) as in the other classical continuity methods. This assumption is necessary even when $f_k \equiv f$ for any $k \in \mathbb{N}$ as Example 5.4.16 shows.

In the second part of the paper we give a definition of $(D, [\psi])$ -log Kähler-Einstein metrics, Namely given ω Kähler form, $\psi \in \mathcal{M}^+$, D \mathbb{Q} -divisor, we say that $\omega + dd^c u$ is a $(D, [\psi])$ -log KE metric if

$$Ric(\omega + dd^c u) - [D] = \lambda(\omega + dd^c u)$$

for $\lambda \in \mathbb{Q}$ and $u \in \mathcal{E}^1(X, \omega, \psi)$. This abuse of language is due to the fact that the current $\omega + dd^c u$ actually defines a (class of) singular log KE metric. The extension

of the Ricci form to the singular setting ([BBJ15]) and a generalization of these metrics when D is a \mathbb{R} -divisor and $\lambda \in \mathbb{R}$ are provided in section §6.4.

We need also to recall that the definition of log KE metrics extends when ω is *semi-Kähler*, i.e. ω is smooth semipositive and $\int_X \omega^n > 0$.

Then we introduce \mathcal{M}_{an}^+ as the set of all model type envelopes ψ such that $\psi - \varphi$ is globally bounded for a ω -psh function φ with *analytic singularities*. The elements in \mathcal{M}_{an}^+ are said to have *analytical singularities type*. Using these model type envelopes and the log-resolutions of their ideal sheaves we define a map

$$\Phi : \mathcal{M}_{an}^+ \rightarrow \left\{ (Y, \eta) : \eta \text{ semi-Kähler with } \omega \geq p_*\eta, \text{ and } p : Y \rightarrow X \right. \\ \left. \text{given by a sequence of blow-ups} \right\} / \sim \quad (5.7)$$

where $(Y, \eta) \sim (Y', \eta')$ if there exists an another element $(Z, \tilde{\eta})$ which dominates (Y, η) , (Y', η') in the usual way. We denote with $\mathcal{K}_{(X, \omega)}$ the image of this map. $\mathcal{K}_{(X, \omega)}$ inherits a partial order (we say *smaller*, *bigger* in correspondence of \preceq , \succeq), a notion of convergence, and it is possible to define a *log-KE metric* in $\alpha \in \mathcal{K}_{(X, \omega)}$ as a class of log-KE metrics on any representative (Y, η) of α . Moreover, when $\{\alpha_k\}_{k \in \mathbb{N}}$ is a totally ordered sequence, there is a natural *strong convergence* for a sequence of log-KE metrics in $\{\alpha_k\}_{k \in \mathbb{N}}$ which obviously comes from the strong convergence defined above on $PSH(X, \omega)$ through the map (5.7) (see section §6.4 for more details). When α_k, α have representatives on the same compact Kähler manifold Y , the strong convergence of log-KE metrics in α_k implies in particular the weak convergence of the log-KE metrics on Y .

Theorem E. *Let ω be a Kähler form such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ for $\lambda \in \mathbb{R}$ and (X, D) klt where D is a \mathbb{R} -divisor. If $\psi \in \mathcal{M}_{an}^+$ and $\omega + dd^c u$ is a $(D, [\psi])$ -log KE metric, then $u \in C^\infty(X \setminus A)$ where A is a closed analytic set. Moreover the followings holds.*

- (i) *Suppose $\lambda \leq 0$. Then any element in $\mathcal{K}_{(X, \omega)}$ admits an unique log-KE metric and such log-KE metrics are stable with respect to the strong convergence, i.e. if $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{K}_{(X, \omega)}$ is a totally ordered sequence converging to $\alpha \in \mathcal{K}_{(X, \omega)}$, then the sequence of log-KE metrics converge strongly to the log-KE metric on α .*
- (ii) *Suppose $\lambda > 0$ and let $\alpha \in \mathcal{K}_{(X, \omega)}$. If the log-Ding functional associated to (Y, η) , representative of α , is coercive, then any $\alpha' \in \mathcal{K}_{(X, \omega)}$ slightly bigger than α admits a log-KE metric.*
- (iii) *Suppose $\lambda > 0$. If $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{K}_{(X, \omega)}$ is an increasing sequence converging to $\alpha \in \mathcal{K}_{(X, \omega)}^P$, and the sequence α_k admits a subsequence of log-KE metrics which is uniformly bounded from above, then there exists a subsequence which converges strongly to a log-KE metric in α .*

Some comments about Theorem E.

The topological assumption $c_1(X) - \{[D]\} = \lambda\{\omega\}$ is a necessary hypothesis to

the existence of log-KE metrics while the assumption on the singularity of (X, D) (i.e. (X, D) klt) is necessary when $\lambda \geq 0$. In particular there are no obstruction to the case $\lambda = 0$, while we do not investigate the case $\lambda < 0$ with (X, D) not necessarily klt since it goes beyond the purpose of this paper. We have a precise estimate about the openness result of the second point in terms of the volumes of α (which is defined as $\int_Y \eta^n$ for any (Y, η) representative), and of the *slope* at infinity of the log-Ding functional which is independent on the representative chosen. This point is clearly a consequence of Theorem C, but it is worth to underline that there is not assumptions on the class α (while in Theorem C we restricted to $[\psi]$ satisfying $c(\psi) > \frac{\lambda p}{p-1}$). Finally in the last point the restriction to $\mathcal{K}_{(X, \omega)}^D$ and the assumption on the uniform boundedness correspond respectively to the assumptions (i) and (iii) of Theorem D and we refer to section §6.4 for precise definitions.

5.1.1 Structure of the paper.

After recalling some preliminaries in section §5.2, section §5.3 is the core of the paper where in three different subsections based on the sign of λ we prove Theorems A, B, C and D. Finally in section §6.4 we introduce the notion of $(D, [\psi])$ -log KE metrics and we prove Theorem E connecting these metrics to the more classical log-KE metrics when $\psi \in \mathcal{M}_{an}^+$.

5.1.2 Acknowledgments.

I would like to thank my advisors Stefano Trapani and David Witt Nyström for their comments.

5.2 Preliminaries.

The set of all model type envelopes is defined as

$$\mathcal{M} := \{\psi \in PSH(X, \omega) : \psi = P_\omega[\psi](0)\}.$$

where for any couple of ω -psh functions u, v

$$\begin{aligned} P_\omega[u](v) &:= \left(\lim_{C \rightarrow \infty} P_\omega(u + C, v) \right)^* = \\ &= \left(\sup\{w \in PSH(X, \omega) : w \preceq u, w \leq v\} \right)^* \in PSH(X, \omega). \end{aligned}$$

Here the star is for the upper semicontinuous regularization and $P_\omega(u, v) := \left(\sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\} \right)^*$ ([RWN14]). We set $P_\omega[\psi] := P_\omega[\psi](0)$ for simplicity. As stated in the Introduction, $|\psi - P_\omega[\psi]|$ bounded is a necessary assumption to make the equation

$$\begin{cases} MA_\omega(u) = \mu \\ u \in \mathcal{E}(X, \omega, \psi) \end{cases}$$

always solvable where μ is a non-pluripolar measure such that $\mu(X) = V_\psi$ ([DDNL18b]). So without loss of generality we may assume ψ be a model type envelope. It is also worth to recall that there are plenty of elements in \mathcal{M} since $P_\omega[P_\omega[\psi]] = P_\omega[\psi]$, i.e. $v \rightarrow P_\omega[v]$ may be thought as a projection from the set of ω -psh functions to \mathcal{M} . We denote with \mathcal{M}^+ the elements $\psi \in \mathcal{M}$ such that $V_\psi := \int_X MA_\omega(\psi) > 0$ ([Tru19]). We also recall that if $u \in \mathcal{E}(X, \omega, \psi)$ and $\psi \in \mathcal{M}^+$ then $P_\omega[u] = \psi$ (Theorem 1.3 in [DDNL18b]).

5.2.1 The metric space (X_A, d_A) .

A function $u \in PSH(X, \omega, \psi) := \{v \in PSH(X, \omega) : v \preceq \psi\}$ is said to have *ψ -relative minimal singularities* if $|u - \psi|$ is globally bounded on X .

Definition 5.2.1 ([DDNL18b]). *The ψ -relative energy functional $E_\psi : PSH(X, \omega, \psi) \rightarrow \mathbb{R} \cup \{-\infty\}$ is defined as*

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X (u - \psi)(\omega + dd^c u)^j \wedge (\omega + dd^c \psi)^{n-j}$$

if u has ψ -relative minimal singularities, and as

$$E_\psi(u) := \inf \{E_\psi(v) : v \in \mathcal{E}(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities, } v \geq u\}$$

otherwise. The subset $\mathcal{E}^1(X, \omega, \psi) \subset \mathcal{E}(X, \omega, \psi)$ is defined as

$$\mathcal{E}^1(X, \omega, \psi) := \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}.$$

Note that the 0-relative energy functional is the *Aubin-Mabuchi energy functional*, also called *Monge-Ampère energy* (see [Aub84], [Mab86]). As shown in [DDNL18b], E_ψ is non-decreasing, continuous along decreasing sequences and the convergence $E_\psi(u) = \lim_{k \rightarrow \infty} E_\psi(\max(u, \psi - k))$ holds. It is worth to underline that the authors in [DDNL18b] assumed ψ to have *small unbounded locus*, but all these properties extend to the general setting as an immediate consequence of the integration by parts formula proved in [X19a] (see also [Lu20]).

We also recall that $(\mathcal{E}^1(X, \omega, \psi), d)$ for $\psi \in \mathcal{M}^+$ is a complete metric space where

$$d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$$

by Theorem A in [Tru19]. A key feature of this distance, which is the starting point to glue together spaces associated to different model type envelopes, is the following contraction property.

Proposition 5.2.2 (Lemma 4.4., Proposition 4.5., [Tru19]). *Let $\psi_1, \psi_2, \psi_3 \in \mathcal{M}$ such that $\psi_1 \preceq \psi_2 \preceq \psi_3$. Then $P_\omega[\psi_1](P_\omega[\psi_2](u)) = P_\omega[\psi_1](u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi_3)$ and $|P_\omega[\psi_1](u) - \psi_1| \leq C$ if $|u - \psi_3| \leq C$. Moreover the map*

$$P_\omega[\psi_1](\cdot) : \mathcal{E}^1(X, \omega, \psi_2) \rightarrow PSH(X, \omega, \psi_1)$$

has image in $\mathcal{E}^1(X, \omega, \psi_1)$ and it is a Lipschitz map of constant 1 when the sets $\mathcal{E}^1(X, \omega, \psi_i)$, $i = 1, 2$, are endowed with the d distances, i.e.

$$d(P_\omega[\psi_1](u), P_\omega[\psi_1](v)) \leq d(u, v)$$

for any $u, v \in \mathcal{E}^1(X, \omega, \psi_2)$.

Next, assuming $\mathcal{A} \subset \mathcal{M}^+$ to be a totally ordered set of model type envelopes, its closure $\bar{\mathcal{A}}$ as subset of $PSH(X, \omega)$ (i.e. the weak closure) belongs to \mathcal{M} ([Tru19]). Moreover by Lemma 3.14 the Monge-Ampère operator becomes an homeomorphism when restricted to $\bar{\mathcal{A}}$ and when one considers the weak topologies.

Assuming from now on that $\bar{\mathcal{A}} \subset \mathcal{M}^+$ (see [Tru19], [Tru20a] for the general case) and observing that $\mathcal{E}^1(X, \omega, \psi_1) \cap \mathcal{E}^1(X, \omega, \psi_2) = \emptyset$ if $\psi_1, \psi_2 \in \mathcal{M}^+$, $\psi_1 \not\preceq \psi_2$, we have the following theorem.

Theorem 5.2.3 ([Tru19], Theorem B). *The set $X_{\mathcal{A}} := \bigsqcup_{\psi \in \bar{\mathcal{A}}} \mathcal{E}^1(X, \omega, \psi)$ can be endowed of a complete distance $d_{\mathcal{A}}$ such that $d_{\mathcal{A}|_{\mathcal{E}^1(X, \omega, \psi) \times \mathcal{E}^1(X, \omega, \psi)}} = d$ for any $\psi \in \bar{\mathcal{A}}$.*

We call *strong topology* the metric topology on $X_{\mathcal{A}}$ given by the distance $d_{\mathcal{A}}$. This topology is the most natural on $X_{\mathcal{A}}$ as the next result shows (see also [BBEGZ19]).

Proposition 5.2.4 ([Tru20a], Theorems 6.2, 6.3). *The strong topology on $X_{\mathcal{A}}$ is the coarsest refinement of the weak topology such that $E(\cdot)$ becomes continuous, i.e. given $\{u_k\}_{k \in \mathbb{N}}, u \in X_{\mathcal{A}}$ then the followings are equivalent:*

- i) $u_k \rightarrow u$ strongly;
- ii) $u_k \rightarrow u$ weakly and $E_{P_\omega[u_k]}(u_k) \rightarrow E_{P_\omega[u]}(u)$.

Moreover if $u_k \rightarrow u$ strongly, then there exists a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ such that $v_j := (\sup_{h \geq j} u_{k_h})^*$, $w_j := P_\omega(u_{k_j}, u_{k_{j+1}}, \dots)$ converge monotonically almost everywhere to u . In particular the strong topology implies the convergence in capacity.

Here we obvious notations $P_\omega(u_{k_j}, u_{k_{j+1}}, \dots) := \sup\{w \in PSH(X, \omega) : w \leq u_{k_h} \text{ for any } h \geq j\}$. We also recall that a sequence $\{u_k\}_{k \in \mathbb{N}} \subset PSH(X, \omega)$ is said to converge in capacity to $u \in PSH(X, \omega)$ if for any $\delta > 0$

$$\text{Cap}(\{|u_k - u| \geq \delta\}) \rightarrow 0$$

as $k \rightarrow \infty$ where for any $B \subset X$ Borel set

$$\text{Cap}(B) := \sup \left\{ \int_B MA_\omega(u) : u \in PSH(X, \omega), -1 \leq u \leq 0 \right\} \quad (5.8)$$

(see [Kol98], [GZ17] and reference therein).

Note also that as an immediate consequence of Proposition 5.2.4 the strong convergence does not depend on the choice of the set \mathcal{A} .

Next, since ω is Kähler, by [BK07] any element $u \in PSH(X, \omega)$ can be approximated by a decreasing sequence of Kähler potentials, i.e. elements in $\mathcal{H} := \{\varphi \in PSH(X, \omega) \cap C^\infty(X) : \omega + dd^c \varphi > 0\}$. Thus we will use several times that $E_{\psi_k}(P_\omega[\psi_k](\varphi)) \rightarrow E_\psi(P_\omega[\psi](\varphi))$ if $\psi_k, \psi \in \mathcal{M}^+$, $\psi_k \rightarrow \psi$ weakly and $\varphi \in \mathcal{H}$, which is an easy consequence of Proposition 5.2.2 and the following result (see also Theorem 2.2 in [X19a], and Lemma 4.1. in [DDNL18b]).

Lemma 5.2.5 (Lemma 4.3, [Tru19]). *Let $\psi_k, \psi \in \mathcal{M}$ such that $\psi_k \rightarrow \psi$ monotonically almost everywhere. Let also $u_k, v_k \in \mathcal{E}^1(X, \omega, \psi_k)$ converging in capacity respectively to $u, v \in \mathcal{E}^1(X, \omega, \psi)$. Then for any $j = 0, \dots, n$*

$$(\omega + dd^c u_k)^j \wedge (\omega + dd^c v_k)^{n-j} \rightarrow (\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$$

weakly. Moreover if $|u_k - v_k|$ is uniformly bounded, then for any $j = 0, \dots, n$

$$(u_k - v_k)(\omega + dd^c u_k)^j \wedge (\omega + dd^c v_k)^{n-j} \rightarrow (u - v)(\omega + dd^c u)^j \wedge (\omega + dd^c v)^{n-j}$$

weakly.

Finally we need to recall the following essential property of the energy $E(\cdot)$ in $X_{\mathcal{A}}$ and its consequent compactness result.

Proposition 5.2.6 ([Tru20a], Lemma 3.13, Propositions 3.14, 3.15). *Let $\mathcal{A} \subset \mathcal{M}^+$ be a totally ordered family such that $\overline{\mathcal{A}} \subset \mathcal{M}^+$, and let $\{u_k\}_{k \in \mathbb{N}} \subset X_{\mathcal{A}}$ converging weakly to $u \in X_{\mathcal{A}}$. Then*

$$\limsup_{k \rightarrow \infty} E_{P_\omega[u_k]}(u_k) \leq E_{P_\omega[u]}(u).$$

Moreover if $E_{P_\omega[u_k]}(u_k) \geq -C$ uniformly, then $P_\omega[u_k] \rightarrow P_\omega[u]$ weakly. In particular for any $C \in \mathbb{N}$ the set

$$X_{\mathcal{A}, C} := \{u \in X_{\mathcal{A}} : \sup_X u \leq C \text{ and } E_{P_\omega[u]}(u) \geq -C\}$$

is weakly compact.

5.2.2 The space $(Y_{\mathcal{A}}, \text{strong})$.

On the set of all probability measures the counterpart of the ψ -relative energy $E_\psi(\cdot)$ and of the correspondent set $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ are respectively the ψ -relative energy E_ψ^* and the set $\mathcal{M}^1(X, \omega, \psi)$.

For μ positive probability measure, the first one is defined as

$$E_\psi^*(\mu) := \sup_{\mathcal{E}^1(X, \omega, \psi)} F_{\mu, \psi} := \sup_{u \in \mathcal{E}^1(X, \omega, \psi)} \left(E_\psi(u) - V_\psi L_\mu(u) \right) \in [0, \infty]$$

where $V_\psi := \int_X MA_\omega(\psi) > 0$ and where $L_\mu(u) := \lim_{k \rightarrow \infty} \int_X (\max(u, \psi - k) - \psi) \mu$ if μ does not charge $\{\psi = -\infty\}$ and $L_\mu \equiv -\infty$ otherwise (see [Tru20a]). The

maximizers of the translation invariant functional $F_{\mu,\psi}$ solve the Monge-Ampère equation $MA_\omega(u) = V_\psi\mu$ (Proposition 5.2 in [Tru20a]) and, defining

$$\mathcal{M}^1(X, \omega, \psi) := \{V_\psi\mu : \mu \text{ probability measure such that } E_\psi^*(\mu) < \infty\},$$

and $\mathcal{E}_{norm}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$, we have the following correspondence.

Theorem 5.2.7 ([Tru20a], Theorem A). *Endowing $\mathcal{M}^1(X, \omega, \psi)$ with its natural strong topology defined as the coarsest refinement of the weak topology such that E_ψ^* becomes continuous, the Monge-Ampère operator $MA_\omega : (\mathcal{E}_{norm}^1(X, \omega, \psi), d) \rightarrow (\mathcal{M}^1(X, \omega, \psi), \text{strong})$ is an homeomorphism. Moreover $E_\psi^*(\mu) = F_{\mu,\psi}(u)$ for any $V_\psi\mu = MA_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$.*

More generally, given $\mathcal{A} \subset \mathcal{M}^+$ totally ordered such that $\overline{\mathcal{A}} \subset \mathcal{M}^+$ and endowing the set

$$Y_{\mathcal{A}} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{M}^1(X, \omega, \psi)$$

with the *strong topology* given as the coarsest refinement of the weak topology of measures such that $E^*(\cdot)$ becomes continuous, we get the following Theorem.

Theorem 5.2.8 ([Tru20a], Theorem B). *The Monge-Ampère map*

$$MA_\omega : (X_{\mathcal{A}, norm}, d_{\mathcal{A}}) \rightarrow (Y_{\mathcal{A}}, \text{strong})$$

is an homeomorphism where $X_{\mathcal{A}, norm} := \bigsqcup_{\psi \in \overline{\mathcal{A}}} \mathcal{E}_{norm}^1(X, \omega, \psi)$.

5.3 Strong continuity of solutions.

As stated in the Introduction given a totally ordered sequence $\psi_k \in \mathcal{M}^+$ converging weakly to $\psi \in \mathcal{M}^+$, and given $f_k \in L^1 \setminus \{0\}$ non-negative functions L^1 -converging to $f \in L^1 \setminus \{0\}$ we want to give necessary conditions so that a sequence of solutions $\{u_k\}_{k \in \mathbb{N}}$ of

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k) \end{cases} \quad (5.9)$$

converges strongly in $X_{\mathcal{A}}$ for $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}}$ to a solution u of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (5.10)$$

We have three very different cases based on the sign of $\lambda \in \mathbb{R}$.

5.3.1 Case $\lambda = 0$.

In this subsection $\lambda = 0$.

In addition to the setting described above, we must assume $V_{\psi_k} = \int_X f_k \omega^n$ for any $k \in \mathbb{N}$. Moreover we normalize the solution u_k of (5.9) to have $\sup_X u_k = 0$, i.e. $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$. Note that by Theorem 5.2.8 the existence of such u_k is equivalent to ask $f_k \omega^n \in \mathcal{M}^1(X, \omega, \psi_k)$, which is a non trivial condition. However if $f_k \in L^{p_k}$ for $p_k > 1$ then by Theorem A in [DDNL18d] there exists a unique solution $u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$ for (5.9) and it has ψ_k -relative minimal singularities. Then letting $u \in PSH(X, \omega)$ be a (weak) accumulation point for $\{u_k\}_{k \in \mathbb{N}}$, Lemma 2.8 in [DDNL18d] gives $MA_\omega(u) \geq f \omega^n$. Therefore since $u \leq \psi$ by Hartogs' Lemma, we immediately get

$$MA_\omega(u) = f \omega^n$$

as a consequence of [WN19]. In particular there is exactly one weak accumulation point for $\{u_k\}_{k \in \mathbb{N}}$. But a priori u may not belong to $\mathcal{E}^1(X, \omega, \psi)$ which is essentially a ψ -regularity condition. Moreover we want to characterize when u is actually the strong limit of u_k , which in particular would imply that the convergence is in capacity (Proposition 5.2.4).

Theorem A. *Let $f_k, f \in L^1$, $\psi_k, \psi \in \mathcal{M}^+$, $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$, and $u \in PSH(X, \omega)$ as in the setting described above. Then $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and $u_k \rightarrow u$ strongly if and only if $E_{\psi_k}(u_k) \geq -C$ for an uniform constant $C \geq 0$ and*

$$\limsup_{k \rightarrow \infty} \int_X (\psi_k - u_k) f_k \omega^n \leq \int_X (\psi - u) f \omega^n. \quad (5.11)$$

Proof. Set $\mathcal{A} := \{\psi_k\}_{k \in \mathbb{N}}$.

As said before, by Lemma 2.8 in [DDNL18d] and [WN19], $MA_\omega(u) = f \omega^n$ since $u \leq \psi$.

Then assuming $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and $d_{\mathcal{A}}(u_k, u) \rightarrow 0$, we immediately obtain $u_k \rightarrow u$ weakly and $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$ as $k \rightarrow \infty$ (Proposition 5.2.4). Thus $d(\psi_k, u_k) = -E_{\psi_k}(u_k)$ is uniformly bounded. Moreover by Theorem 5.2.7 $E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) \rightarrow E_\psi^*(MA_\omega(u)/V_\psi)$, which implies that $\int_X (\psi_k - u_k) f_k \omega^n \rightarrow \int_X (\psi - u) f \omega^n$ and concludes one implication.

Vice versa suppose that $d(\psi_k, u_k) \leq C$ for an uniform constant $C \in \mathbb{R}$ and that $\limsup_{k \rightarrow \infty} \int_X (\psi_k - u_k) f_k \omega^n \leq \int_X (\psi - u) f \omega^n$. Next, combining Fatou's Lemma with Proposition 5.2.2, Lemma 5.2.5 and Theorem 5.2.7, it follows that for any $\varphi \in \mathcal{H}$

$$\begin{aligned} \liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) &\geq \\ &\geq \liminf_{k \rightarrow \infty} \left(E_{\psi_k}(P_\omega[\psi_k](\varphi)) + \int_X (\psi_k - P_\omega[\psi_k](\varphi)) f_k \omega^n \right) \geq \\ &\geq E_\psi(P_\omega[\psi](\varphi)) + \int_X (\psi - P_\omega[\psi](\varphi)) f \omega^n \end{aligned} \quad (5.12)$$

since $(\psi_k - P_\omega[\psi_k](\varphi))f_k \rightarrow (\psi - P_\omega[\psi](\varphi))f$ almost everywhere. Thus, for any $v \in \mathcal{E}^1(X, \omega, \psi)$ letting $\varphi_j \in \mathcal{H}$ be a decreasing sequence converging to v ([BK07]), from the inequality (5.12) we get

$$\begin{aligned} \liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) &\geq \\ &\geq \limsup_{j \rightarrow \infty} \left(E_\psi(P_\omega[\psi](\varphi_j)) + \int_X (\psi - P_\omega[\psi](\varphi_j))f\omega^n \right) = \\ &= E_\psi(v) + \int_X (\psi - v)f\omega^n \end{aligned} \quad (5.13)$$

using also the continuity of $E_\psi(\cdot)$ along decreasing sequences and the Monotone Convergence Theorem. Therefore by definition

$$\liminf_{k \rightarrow \infty} E_{\psi_k}^*(MA_\omega(u_k)/V_{\psi_k}) \geq E_\psi^*(f\omega^n/V_\psi) = E_\psi^*(MA_\omega(u)/V_\psi), \quad (5.14)$$

which together with $\int_X (\psi_k - u_k)f_k\omega^n \rightarrow \int_X (\psi - u)f\omega^n$ (by Fatou's Lemma and the assumption (5.11)) and the upper semicontinuity of $E_\psi(\cdot)$ (Proposition 6.2.1) imply $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$. Hence $u_k \rightarrow u$ strongly as consequence of Proposition 5.2.4. \square

Remark 5.3.1. It is clear from the proof of Theorem A that to prove that $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ it is enough to show that $E_{\psi_k}(u_k) \geq -C$. Moreover we observe that the assumption (5.11) can be replaced with the uniform integrability of $\{(\psi_k - u_k)f_k\}_{k \in \mathbb{N}}$ in the measure-theoretical sense, i.e. for every $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$ such that $\sup_{k \in \mathbb{N}} \int_E (\psi_k - u_k)f_k\omega^n < \epsilon$ for any measurable set E such that $\omega^n(E) \leq \delta$. Indeed since $(\psi_k - u_k)f_k \rightarrow (\psi - u)f$ almost everywhere and since all that we need is that $\int_X (\psi_k - u_k)f_k\omega^n \rightarrow \int_X (\psi - u)f\omega^n$, the equivalence between this two hypothesis follows from the Vitali Convergence Theorem and Fatou's Lemma.

Remark 5.3.2. In some cases the assumptions on the boundedness of the energy and (5.11) in Theorem A are easily satisfied.

For instance if there exists $h \in L^1$ such that $(\psi_k - u_k)f_k \leq h$ almost everywhere for any $k \in \mathbb{N}$ then (5.11) trivially holds, while by Theorem 4.10 in [DDNL18b] $-E_{\psi_k}(u_k) \leq \int_X (\psi_k - u_k)f_k\omega^n \leq \|h\|_{L^1}$.

Similarly if $\|f_k\|_{L^p}, \|f\|_{L^p}$ are uniformly bounded for $p > 1$, then the boundedness of the energy and (5.11) are consequences of $\psi_k - u_k \rightarrow \psi - u$ in L^r for any $r \in [1, \infty)$ (see Theorem 1.48 in [GZ17]). In particular Theorem A extends Theorem C in [Tru20a].

Finally if $f_k = c_k g_k$ for $g_k \searrow f$, then we claim that the assumption (5.11) can be substituted with $\int_B f\omega^n \leq A \text{Cap}_{\psi_k}(B)$ for any Borel set $B \subset X$ and for any $k \gg 1$ big enough where $A > 0$ is a fixed constant. Here Cap_ψ denotes the *relative Monge-Ampère capacity* introduced in [DDNL18b] (see also [DDNL18d]) whose definition is similar to (5.8) asking $\psi - 1 \leq u \leq \psi$. Indeed combining Lemma 4.18 in [DDNL18b] and Theorem 4.4 in [Tru20a] we would easily have

$$\limsup_{k \rightarrow \infty} \int_X (\psi_k - u_k)f_k\omega^n \leq \limsup_{k \rightarrow \infty} c_k \int_X (\psi_k - u_k)f\omega^n = \int_X (\psi - u)f\omega^n.$$

5.3.2 Case $\lambda < 0$.

Here we deal with the case $\lambda < 0$.

Letting $f \in L^1 \setminus \{0\}$ non negative, we first assume that $\lambda \in \mathbb{R} \setminus \{0\}$ to introduce the functional $L_{f,\lambda} : PSH(X, \omega) \rightarrow \overline{\mathbb{R}}$ as

$$L_{f,\lambda}(u) := \frac{-1}{\lambda} \log \int_X e^{-\lambda u} f \omega^n.$$

Thus, for $\psi \in \mathcal{M}$, we define the functional $F_{f,\psi,\lambda} : \mathcal{E}^1(X, \omega, \psi) \rightarrow \overline{\mathbb{R}}$ as $F_{f,\psi,\lambda}(u) := (E_\psi - V_\psi L_{f,\lambda})(u)$. We hope that this functional do not lead to confusion with the functional $F_{\mu,\psi}$ defined in section 5.2. It is easy to see that $F_{f,\psi,\lambda}$ is invariant by translation, i.e. it descends to the space of currents. Moreover its maximizers solve the complex Monge-Ampère equation (5.15) as the next result recalls.

Theorem 5.3.3 ([DDNL18b], Theorem 4.22). *Let $f \in L^1 \setminus \{0\}$ non negative and $\lambda \neq 0$. If $u \in \mathcal{E}^1(X, \omega, \psi)$ maximizes $F_{f,\psi,\lambda}$ then u solves*

$$\begin{cases} MA_\omega(v) = e^{-\lambda v + C} \mu \\ v \in \mathcal{E}^1(X, \omega, \psi) \end{cases} \quad (5.15)$$

for a constant $C \in \mathbb{R}$.

From now on until the end of the subsection we will assume $\lambda < 0$.

Theorem 5.3.4 (Theorem 4.23 - Lemma 4.24., [DDNL18b]). *Let $\lambda < 0$ and $f \in L^1 \setminus \{0\}$ non negative. Then the complex Monge-Ampère equation (5.15) admits an unique solution and it maximizes $F_{f,\psi,\lambda}$ over $\mathcal{E}^1(X, \omega, \psi)$.*

A key Lemma of the proof of the Theorem just recalled is the following Lemma.

Lemma 5.3.5. *Let μ be a non-pluripolar measure, $g_k, g \in L^1$ non-negative functions such that $g_k \rightarrow g$ in L^1 , and let $u, \{u_k\}_{k \in \mathbb{N}} \subset PSH(X, \omega')$ such that $u_k \rightarrow u$ weakly where ω' is a Kähler form on X . Then*

$$\int_X e^{u_k} g_k \omega^n \rightarrow \int_X e^u g \omega^n$$

as $k \rightarrow \infty$

Proof. By an easy calculation we have

$$\int_X e^{u_k} g_k \omega^n \leq e^{\sup_X u_k} \int_X |g_k - g| \omega^n + \int_X e^{u_k} g \omega^n$$

and the result follows from $|\sup_X u_k| \leq C$ and Lemma 11.5 in [GZ17]. \square

We can now prove Theorem B.

Theorem B. *Assume*

- (i) $\lambda < 0$;
 - (ii) $f_k, f \in L^1 \setminus \{0\}$ non-negative functions such that $f_k \rightarrow f$ in L^1 ;
 - (iii) $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered such that ψ_k converges weakly to $\psi \in \mathcal{M}^+$.
- Let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$, $u \in \mathcal{E}^1(X, \omega, \psi)$ be the unique solutions respectively of

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k), \end{cases} \quad \begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (5.16)$$

Then $u_k \rightarrow u$ strongly.

Proof. We assume $\lambda = -1$ for simplicity and we observe that by an easy contradiction argument it is enough to check that any subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ admits a further subsequence $\{u_{k_{j_h}}\}_{h \in \mathbb{N}}$ converging strongly to u . So without loss of generality we may assume u_{k_j} to be the whole sequence and we set $F_k := F_{f_k, \psi_k, -1}$, $F := F_{f, \psi, -1}$. Observe that by Theorem 5.3.4 u_k maximizes F_k for any $k \in \mathbb{N}$ while u maximizes F .

Therefore, letting $\varphi \in \mathcal{H}$, it follows that

$$\liminf_{k \rightarrow \infty} F_k(u_k) \geq \liminf_{k \rightarrow \infty} F_k(P_\omega[\psi_k](\varphi)) = F(P_\omega[\psi](\varphi))$$

by Lemmas 5.2.5 and 5.3.5. Thus, passing to the supremum over \mathcal{H} , combining [BK07], the continuity of E_ψ along decreasing sequences and Lemma 5.3.5 we get

$$\liminf_{k \rightarrow \infty} F_k(u_k) \geq F(u). \quad (5.17)$$

Moreover, up to considering a subsequence, the sequence $v_k := u_k - \sup_X u_k$ converges weakly to $v \in PSH(X, \omega)$, $v \leq \psi$ and

$$a_k := \int_X e^{v_k} f_k \omega^n \rightarrow \int_X e^v f \omega^n \in (0, \|f\|_{L^1}],$$

again by Lemma 5.3.5. Thus, using the complex Monge-Ampère equations,

$$\sup_X u_k = \log V_{\psi_k} - \log a_k$$

is uniformly bounded and $\{u_k\}_{k \in \mathbb{N}}$ admits a subsequence $\{u_{k_j}\}_{j \in \mathbb{N}}$ converging weakly to $\tilde{u} \in PSH(X, \omega)$, $\tilde{u} \leq \psi$. Without loss of generality we will assume $\{u_{k_j}\}_{j \in \mathbb{N}}$ to be the whole sequence $\{u_k\}_{k \in \mathbb{N}}$. On the other hand from (5.17) and the triangle inequality, since $\sup_X u_k$ is uniformly bounded and $f_k \rightarrow f$, we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} d(\psi_k, u_k) \leq \\ & \leq 2AV_\psi - \liminf_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq 2AV_\psi - F(u) - \limsup_{k \rightarrow \infty} V_{\psi_k} \int_X e^{u_k} f_k \omega^n \leq -F(u) + C \end{aligned}$$

if $A \geq \sup_X u_k$ for any $k \in \mathbb{N}$. Therefore $\tilde{u} \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ by Proposition 6.2.1 since $\{u_k\}_{k \in \mathbb{N}} \subset X_{\mathcal{A}, C'}$ for an uniform constant C' . Furthermore by Proposition 6.2.1 and Lemma 5.3.5 we obtain

$$\limsup_{k \rightarrow \infty} F_k(u_k) \leq F(\tilde{u}),$$

which implies necessarily that $\tilde{u} = u + B$ for a constant $B \in \mathbb{R}$ by (5.17) and Theorem 5.3.4. But from the Monge-Ampère equations it follows that

$$e^B \int_X e^u f \omega^n = \int_X e^{\tilde{u}} f \omega^n = \lim_{k \rightarrow \infty} \int_X e^{u_k} f_k \omega^n = \lim_{k \rightarrow \infty} V_{\psi_k} = V_\psi = \int_X e^u f \omega^n,$$

i.e. $B = 0$. In conclusion we have proved that $u_k \rightarrow u$ weakly, that $F_k(u_k) \rightarrow F(u)$ and $\int_X e^{u_k} f_k \omega^n \rightarrow \int_X e^u f \omega^n$. Hence $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$, which by Proposition 5.2.4 implies $d_{\mathcal{A}}(u_k, u) \rightarrow 0$ and concludes the proof. \square

Remark 5.3.6. It is easy to observe that Theorem B generalizes to the case when ω^n is replaced by a non-pluripolar measure μ and $f_k \rightarrow f \in L^1(\mu)$ since analogs of Theorem 5.3.4 and of Lemma 5.3.5 hold in this setting.

5.3.3 Case $\lambda > 0$.

If $\lambda > 0$ then the study of (5.10) is much more complicated than the case $\lambda \leq 0$ even in the absolute setting $\psi = 0$. As stated in the Introduction, for instance, if $\{\omega\} = -K_X$, i.e. X is a *Fano manifold*, and $f \equiv 1$, the existence of a solution for (5.10) is characterized by an algebrico-geometric notion called *K-stability* (see [CDS15]). The uniqueness of solutions of (5.10) is an hard problem as well (see [Bern15]). Note that in this case $F_{1,0,1}$ coincides with the Ding functional ([Ding88]) where we recall that $F_{f,\psi,\lambda} := E_\psi - V_\psi L_{f,\lambda}$ for $f \in L^1$, $\lambda \in \mathbb{R} \setminus \{0\}$, $\psi \in \mathcal{M}$ is the functional introduced in the previous subsection. We refer to the companion paper [Tru20b] where we analyze the case when $\{\omega\} = -K_X$ more in detail. To prove Theorems C and D we need first to set

$$J_\psi(u) := -E_\psi(u) + \int_X (u - \psi) M A_\omega(\psi)$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ where $\psi \in \mathcal{M}^+$ (see [Tru20a] where the notation is slightly different). It is immediate to check that J_ψ is non-negative and translation invariant. Indeed it represents the translation invariant version of the distance d as the following key lemma shows.

Lemma 5.3.7. *Let $\psi \in \mathcal{M}^+$. Then there exists $C \in \mathbb{R}_{\geq 0}$ depending only on (X, ω) such that*

$$d(u, \psi) - C \leq J_\psi(u) \leq d(u, \psi)$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$.

Proof. From the definitions it immediately follows that $J_\psi(u) \leq d(\psi, u)$ for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. Vice versa it is enough to observe that on $\mathcal{E}_{norm}^1(X, \omega, \psi)$ we have

$$\int_X (\psi - u) MA_\omega(\psi) \leq \int_X |u| MA_\omega(0) = \|u\|_{L^1} \leq C$$

as immediate consequence of Theorem 3.8. in [DDNL18b] and of the weak compactness of $\{u \in PSH(X, \omega) : \sup_X u = 0\}$. \square

Similarly to the case $\lambda < 0$, since the ψ -relative energy is upper semicontinuous with respect to the weak topology (Proposition 6.2.1), the continuity properties of $L_{f, \lambda}$ varying also f play a key role to the variational approach, and hence to prove Theorems C, D. This is the reason to recall the following well-known and important quantity (see [DK01]).

Definition 5.3.8. *Let $u \in PSH(X, \omega)$. The quantity*

$$c(u) := \sup\{p \geq 0 : \int_X e^{-pu} \omega^n < \infty\}$$

is called the complex singularity exponent of u .

By the resolution of the strong openness conjecture the supremum in the definition is never achieved, i.e. $e^{-c(u)u} \notin L^1$. Clearly $c(\cdot)$ increases when the singularities decreases and it is a lower semicontinuous function with respect to the weak topology as the Main Theorem in [DK01] shows. Moreover the next result proves that $c(\cdot)$ is constant on any set $\mathcal{E}(X, \omega, \psi)$, $\psi \in \mathcal{M}^+$.

We first need to recall the definition of the *Lelong numbers* and of the *multiplier ideal sheaves*.

Given $u \in PSH(X, \omega)$ and $x \in X$ the Lelong number $\nu(u, x)$ of u at x is given as

$$\nu(u, x) := \sup\{\gamma \geq 0 : u(z) \leq \gamma \log \|z - x\|^2 + O(1) \text{ on } U\}$$

where $x \in U \subset X$ is an holomorphic chart. It does not depend on the chart chosen. The Lelong number measures the logarithmic singularity of an ω -psh function at a point x .

The multiplier ideal sheaf $\mathcal{I}(tu)$, $t \geq 0$, of $u \in PSH(X, \omega)$ is the analytic coherent sheaf whose germs are given by

$$\mathcal{I}(tu, x) := \left\{ f \in \mathcal{O}_{X, x} : \int_V |f|^2 e^{-tu} \omega^n < \infty \text{ for some open set } x \in V \subset X \right\}.$$

Proposition 5.3.9. *Let $u \in PSH(X, \omega)$ and $\psi := P_\omega[u]$. Then*

$$\nu(u, x) = \nu(\psi, x) \text{ and } \mathcal{I}(tu, x) = \mathcal{I}(t\psi, x) \text{ for any } t > 0, x \in X. \quad (5.18)$$

In particular $c(u) = c(\psi)$ and $\alpha(\cdot)$ is constant on any $\mathcal{E}(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$.

Proof. We first observe that $c(u) := \sup \{p \geq 0 : \mathcal{I}(pu) = \mathcal{O}_X\}$, thus (5.18) implies immediately $c(u) = c(\psi)$. Moreover by Theorems 1.2, 1.3 in [DDNL18b] if $\psi \in \mathcal{M}^+$ then $P_\omega[u] = \psi$ if and only if $u \in \mathcal{E}(X, \omega, \psi)$ and the last assertion follows. Next, we claim that $P_\omega[u](\psi) = \psi$. Indeed clearly $P_\omega[u](\psi) \leq \psi$, while vice versa for any $C \in \mathbb{R}_{\geq 0}$,

$$P_\omega[u](\psi) \geq P_\omega(u + C, P_\omega[u]) \geq P_\omega(u + C, P_\omega(u + C, 0)) = P_\omega(u + C, 0),$$

which implies $P_\omega[u](\psi) \geq P_\omega[u] = \psi$ since $P_\omega(u + C, 0) \nearrow P_\omega[u]$.

Then the proof of (5.18) is similar to that of Theorem 1.1.(i) in [DDNL18a], but we write the details for the courtesy of the reader.

Trivially $\gamma := \nu(u, x) \geq \nu(\psi, x)$. Assume by contradiction that $\gamma > \nu(\psi, x)$ for $x \in X$, and fix holomorphic coordinates centered at x such that the unit ball $\mathbb{B} \subset \mathbb{C}^n$ is contained in the chart. By definition $u(z) \leq \gamma \log |z|^2 + O(1)$ locally in such coordinates. Let also g be a smooth potential of ω such that $u + g, \psi + g \leq 0$ in \mathbb{B} . Thus, locally

$$g + \psi = g + P_\omega[u](\psi) \leq \sup\{v \in PSH(\mathbb{B}) : v \leq 0, v \leq \gamma \log |z|^2 + O(1)\}$$

where the inequality follows considering $P_\omega(u + C, \psi)$ for $C \rightarrow +\infty$ instead of $P_\omega[u](\psi)$ and noting that the right hand is upper semicontinuous since it coincides with the pluricomplex Green function $G_{\mathbb{B}}(z, 0)$ of \mathbb{B} with a logarithmic pole at 0 of order γ . Hence, by Proposition 6.1 in [Kli91] we get the contradiction $\nu(\psi, x) \geq \gamma$ since $G_{\mathbb{B}}(z, 0) \sim \gamma \log |z|^2 + O(1)$.

For the second equality, letting $x \in X$ fixed, we observe that $\mathcal{I}(t\psi, x) = \mathcal{I}(tP_\omega(u + C, \psi), x)$ for $C \gg 0$ big enough as a consequence of the resolution of the strong openness conjecture ([GZ15], see also Theorem 1.1 in [Lemp17]) since $P_\omega(u + C, \psi) \nearrow \psi$ for $C \rightarrow +\infty$. Therefore to conclude the proof it is sufficient to note that $\mathcal{I}(tu, x) = \mathcal{I}(tP_\omega(u + C, \psi), x)$ for any $t, C > 0, x \in X$ since ψ is less singular than u . \square

It is also possible to estimate the complex singularity exponent of ψ in terms of the Lelong numbers by the following classical result.

Proposition 5.3.10 ([Sko72]). *Let $\psi \in \mathcal{M}$ and set $\nu(\psi) := \sup_{x \in X} \nu(\psi, x)$. Then*

$$\frac{2}{\nu(\psi)} \leq c(\psi) \leq \frac{2n}{\nu(\psi)}.$$

We can now introduce an integrability condition which will be sufficient for the purposes of this paper.

Definition 5.3.11. *Given $\psi \in \mathcal{M}$, $\lambda > 0$ and $p \in (1, \infty]$. We say that $[\psi]$ satisfies the Strong Integrability Condition (SIC) with respect to λ, p if*

$$c(\psi) > \frac{\lambda p}{p-1},$$

where we mean $c(\psi) > \lambda$ if $p = \infty$.

Observe that when $p = \infty$ the SIC $\alpha(\psi) > \lambda$ is a necessary condition to solve the Monge-Ampère equation $MA_\omega(u) = e^{-\lambda u} f \omega^n$ in the class $\mathcal{E}(X, \omega, \psi)$. In general if $\psi \in \mathcal{M}^+$ then as a consequence of Proposition 6.3.1 the SIC gives $e^{-\lambda u} f \in L^1$ for any $u \in \mathcal{E}(X, \omega, \psi)$ through a clear Hölder's pairing.

Proposition 5.3.12. *Let $u_k, u \in PSH(X, \omega)$ such that $u_k \rightarrow u$ weakly, $\lambda > 0$ and let $f_k, f \in L^p$ for $p \in (1, \infty]$ non-negative functions such that $f_k \rightarrow f$ in L^p . Letting $\psi := P_\omega[u]$ and $\psi_k := P_\omega[u_k]$, assume also that $c(\psi), c(\psi_k) > \frac{\lambda p}{p-1}$ for any $k \gg 1$, where $\frac{\lambda p}{p-1} = \lambda$ if $p = \infty$. Then*

$$e^{-\lambda u_k} f_k \rightarrow e^{-\lambda u} f$$

in L^1 as $k \rightarrow \infty$.

Proof. We set $g_k := e^{-\lambda u_k} f_k$, $g := e^{-\lambda u} f$ and $q := p/(p-1)$ for the Sobolev conjugate of p . Note also that by Hartogs' Lemma we may suppose $\sup_X u_k \leq 0$ for any $k \in \mathbb{N}$. By the triangle inequality

$$\|g_k - g\|_{L^1} \leq \|e^{-\lambda u_k} (f_k - f)\|_{L^1} + \|(e^{-\lambda u_k} - e^{-\lambda u})f\|_{L^1} \quad (5.19)$$

and the strategy is to prove that both terms in the right hand goes to 0 as $k \rightarrow \infty$. As immediate consequence of the Hölder's inequality we obtain

$$\|e^{-\lambda u_k} (f_k - f)\|_{L^1} \leq \|e^{-u_k}\|_{L^{\lambda q}}^\lambda \|f_k - f\|_p$$

which converges to 0 since $f_k \rightarrow f$ in L^p by assumption and $\|e^{-u_k}\|_{L^{\lambda q}}$ is uniformly bounded by Lemma 5.3.13. In fact $c(u_k) = c(\psi_k) > \lambda q$, $c(u) = c(\psi) > \lambda q$ (see also Proposition 6.3.1).

For the second term in (5.19) again by Hölder's inequality it follows that it is enough to prove that

$$e^{-\lambda q u_k} \rightarrow e^{-\lambda q u} \quad (5.20)$$

in L^1 . But since $c(u) > \lambda q$, (5.20) is a consequence of the Main Theorem in [DK01]. \square

Lemma 5.3.13. *Let $K \subset PSH(X, \omega)$ and $p > 0$ such that $c(u) > p$ for any $u \in K$. Then there exists a constant $C = C_{K,p}$ such that*

$$\sup_{u \in K} \int_X e^{-pu} \omega^n \leq C.$$

Proof. Let us assume by contradiction that there exists a sequence $\{u_j\}_{j \in \mathbb{N}} \subset K$ such that

$$\int_X e^{-pu_j} \omega^n \geq j \quad (5.21)$$

for any $j \in \mathbb{N}$. Up to considering a subsequence we may also assume that $u_j \rightarrow u \in K$ weakly. In particular $\int_X e^{-pu} \omega^n < \infty$. Thus by the Main Theorem in [DK01] $e^{-pu_j} \rightarrow e^{-pu}$ in L^1 , which contradicts (5.35). \square

We can now prove Theorem C, which as said in the Introduction represents an openness result for a new continuity method where the singularities are movable (see also [Tru20b]).

Theorem C. *Let $\psi \in \mathcal{M}^+$, $\lambda > 0$ and let $0 \leq f \in L^p \setminus \{0\}$ for $p \in (1, \infty]$. Assume also that $c(\psi) > \frac{\lambda p}{p-1}$. If there exist $A > 0, B \geq 0$ such that*

$$F_\psi(u) := F_{f,\psi,\lambda} \leq -Ad(u, \psi) + B$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, then there exists an uniform constant $C = C(V_\psi, B, X, \omega) \geq 0$ such that for any $\psi' \in \mathcal{M}^+$, $\psi' \succcurlyeq \psi$

$$F_{\psi'}(v) \leq -\left(1 - \frac{V_{\psi'}}{V_\psi}(1 - A)\right)d(v, \psi') + C$$

for any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi')$. In particular for any $\psi' \succcurlyeq \psi$ such that $V_{\psi'} < V_\psi/(1 - A)$, $F_{\psi'}$ is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi')$ and the complex Monge-Ampère equation

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi') \end{cases} \quad (5.22)$$

admits a solution.

Remark 5.3.14. It is easy to check that the constant $A > 0$ in the d -coercivity of F_ψ cannot be larger than 1. Indeed it easily follows from $(A - 1)E_\psi(u) + B \geq \frac{V_\psi}{\lambda} \log \int_X e^{-\lambda u} f \omega^n \geq \frac{V_\psi}{\lambda} \log \|f\|_{L^1}$ for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and the fact that $\sup_{u \in \mathcal{E}_{norm}^1(X, \omega, \psi)} |E_\psi(u)| = +\infty$ for any $\psi \in \mathcal{M}^+$.

Proof. We divide the proof in two parts. We first prove that the d -coercivity of $F_{\psi'}$ implies the existence of a solution of (5.22) for a fixed $\psi' \succcurlyeq \psi$, then we show that the d -coercivity of F_ψ implies the d -coercivity of $F_{\psi'}$ for any $\psi' \succcurlyeq \psi$ such that $V_{\psi'} < V_\psi/(1 - A)$.

Let $\psi' \succcurlyeq \psi$ and assume that $F_{\psi'}$ is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ with respect to constants $A > 0, B \geq 0$. Then letting $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{norm}^1(X, \omega, \psi')$ be a maximizing sequence for $F_{\psi'}$, i.e. $F_{\psi'}(u_k) \nearrow \sup_{\mathcal{E}_{norm}^1(X, \omega, \psi')} F_{\psi'}$, by the coercivity we immediately have

$$d(\psi', u_k) \leq D$$

for a constant $D \in \mathbb{R}_{\geq 0}$. Therefore by Proposition 6.2.1, up to considering a subsequence, $u_k \rightarrow u \in \mathcal{E}_{norm}^1(X, \omega, \psi')$ weakly. Thus Lemma 5.3.12 and again Proposition 6.2.1 give

$$\sup_{\mathcal{E}_{norm}^1(X, \omega, \psi)} F_{\psi'} = \lim_{k \rightarrow \infty} F_{\psi'}(u_k) \leq F_{\psi'}(u),$$

i.e. u is a maximizer of $F_{\psi'}$ over $\mathcal{E}_{norm}^1(X, \omega, \psi')$. Hence since $F_{\psi'}$ is translation invariant, by Theorem 5.3.3 there exists a constant A such that $u + A$ solves (5.22)

which finishes the first part of the proof.

Next the d -coercivity of F_ψ implies that for any $u \in \mathcal{E}^1(X, \omega, \psi)$

$$\begin{aligned} F_\psi(u) &= F_\psi(u - \sup_X u) \leq -Ad(\psi, u - \sup_X u) + B \leq \\ &\leq -AJ_\psi(u - \sup_X u) + B = -AJ_\psi(u) + B \end{aligned}$$

by Lemma 5.3.7, which is equivalent to

$$\frac{V_\psi}{\lambda} \log \int_X e^{-\lambda u} f \omega^n \leq (1-A)J_\psi(u) + B + \int_X (\psi - u)MA_\omega(\psi) \quad (5.23)$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$. In particular letting $\varphi \in \mathcal{H}$ such that $\sup_X P_\omega[\psi'](\varphi) = 0$ where $\psi' \succcurlyeq \psi$ and setting $v := P_\omega[\psi](\varphi)$, $v' := P_\omega[\psi'](\varphi)$, we have

$$\begin{aligned} F_{\psi'}(v') &\leq E_{\psi'}(v') + \frac{V_{\psi'}}{\lambda} \log \int_X e^{-\lambda v} f \omega^n \leq \\ &\leq E_{\psi'}(v') + \frac{V_{\psi'}}{V_\psi} \left((1-A)J_\psi(v) + B + \int_X (\psi - v)MA_\omega(\psi) \right) \quad (5.24) \end{aligned}$$

combining the inequality $v' \geq v$ with (5.23). Next by [DNT19] $MA_\omega(\psi) = \mathbb{1}_{\{\psi=0\}}MA_\omega(0)$ and similarly for ψ' . Thus

$$\begin{aligned} \int_X (v - \psi)MA_\omega(\psi) &\leq \int_X (v' - \psi')MA_\omega(\psi') + \int_{\{\psi'=0\} \setminus \{\psi=0\}} (-v')MA_\omega(u) \leq \\ &\leq \int_X (v' - \psi')MA_\omega(\psi') + C' \end{aligned}$$

for an uniform constant $C' = C'(X, \omega) > 0$ since $\sup_X v' = 0$. Hence as a consequence of Proposition 5.2.2 we get

$$J_\psi(v) \leq J_{\psi'}(v') + C',$$

which together with (5.24) and again [DNT19] (by Remark 5.3.14 $A \leq 1$), implies

$$\begin{aligned} F_{\psi'}(v') &\leq \left(\frac{V_{\psi'}(1-A)}{V_\psi} - 1 \right) J_{\psi'}(v') + \frac{V_{\psi'}}{V_\psi} (B + C') + \int_X (v' - \psi')MA_\omega(\psi') + \\ &+ \frac{V_{\psi'}}{V_\psi} \int_X (\psi - v)MA_\omega(\psi) \leq \left(\frac{V_{\psi'}(1-A)}{V_\psi} - 1 \right) J_{\psi'}(v') + \frac{V_0}{V_\psi} (B + 2C'). \end{aligned}$$

Therefore since $J_{\psi'}(\cdot)$ and of $F_{\psi'}(\cdot)$ are translation invariant and continuous along decreasing sequences in $\mathcal{E}^1(X, \omega, \psi)$, combining [BK07] and Lemma 5.3.7 we finally obtain

$$F_{\psi'}(u) \leq \left(\frac{V_{\psi'}(1-A)}{V_\psi} - 1 \right) J_{\psi'}(u) + \frac{V_0}{V_\psi} (B + 2C') \leq \left(\frac{V_{\psi'}(1-A)}{V_\psi} - 1 \right) d(\psi', u) + C$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi')$, which concludes the proof. \square

Finally we can give necessary conditions to have the strong continuity of a sequence of solutions of $MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n$ with prescribed singularities, i.e. Theorem D.

Theorem D. *Let $\lambda > 0$, $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered sequence such that $\psi_k \preceq \psi_{k+1}$ for any $k \in \mathbb{N}$ which converges to $\psi \in \mathcal{M}^+$, and $f_k, f \geq 0$ not trivial such that $f_k \rightarrow f$ in L^p as $k \rightarrow \infty$ for $p \in (1, \infty]$. Assume also the following conditions:*

- (i) $c(\psi) > \frac{\lambda p}{p-1}$;
- (ii) the complex Monge-Ampère equations

$$\begin{cases} MA_\omega(u_k) = e^{-\lambda u_k} f_k \omega^n \\ u_k \in \mathcal{E}^1(X, \omega, \psi_k); \end{cases}$$

admit solutions u_k given as maximizers of $F_{f_k, \psi_k, \lambda}$;

- (iii) $\sup_X u_k \leq C$ for an uniform constant C .

Then there exists a subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ which converges strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$ solution of

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} f \omega^n \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases}$$

Proof. We first observe that $c(\psi_k) > \frac{\lambda p}{p-1}$ if $k \gg 1$ big enough since $\psi_k \nearrow \psi$ a.e. and $c(\cdot)$ is lower semicontinuous with respect to the weak topology as said before (it is the Main Theorem in [DK01]).

Then we set $F_k := F_{f_k, \psi_k, \lambda}$ for any $k \in \mathbb{N}$, $F := F_{f, \psi, \lambda}$ and $v_k := u_k - \sup_X u_k \in \mathcal{E}_{norm}^1(X, \omega, \psi_k)$. In particular $MA_\omega(v_k) = e^{-\lambda(v_k + \sup_X u_k)} f_k \omega^n$ for any $k \in \mathbb{N}$ and up to considering a subsequence we may assume that v_k converges weakly to a function $v \in PSH(X, \omega)$. Then by an easy calculation we obtain

$$\begin{aligned} C_1 &\leq \frac{V_{\psi_k}}{\lambda} \log \|f_k\|_{L^1} \leq \frac{V_{\psi_k}}{\lambda} \log \int_X e^{-\lambda \psi_k} f_k \omega^n = F_k(\psi_k) \leq \\ &\leq F_k(v_k) = E_{\psi}(v_k) + \frac{V_{\psi_k}}{\lambda} \log V_{\psi_k} + V_{\psi_k} \sup_X u_k \leq E_{\psi}(v_k) + C_2. \end{aligned}$$

for two uniform constants C_1, C_2 . Therefore by Proposition 6.2.1 we obtain $v \in \mathcal{E}^1(X, \omega, \psi)$ and $\limsup_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq E_{\psi}(u)$. Thus since Lemma 5.3.12 gives $\int_X e^{-\lambda v_k} f_k \omega^n \rightarrow \int_X e^{-\lambda v} f \omega^n$, it follows that

$$\limsup_{k \rightarrow \infty} F_k(v_k) \leq F(v).$$

On the other hand similarly to the proof of Theorem B, letting $\varphi \in \mathcal{H}$ we obtain

$$\liminf_{k \rightarrow \infty} F_k(v_k) \geq \liminf_{k \rightarrow \infty} F_k(P_\omega[\psi_k](\varphi)) = F(P_\omega[\psi](\varphi))$$

combining Lemma 5.2.5 and Lemma 5.3.12, which together with [BK07] and the continuity of F along decreasing sequences implies

$$\liminf_{k \rightarrow \infty} F_k(v_k) \geq \sup_{\mathcal{E}^1(X, \omega, \psi)} F.$$

Hence v is a maximizer of F over $\mathcal{E}^1(X, \omega, \psi)$ and $F_k(v_k) \rightarrow F(v)$. In particular there exists a constant $C \in \mathbb{R}$ such that $MA_\omega(v) = e^{-\lambda(v+C)} f \omega^n$ (Theorem 5.3.3) and $E_{\psi_k}(v_k) \rightarrow E_\psi(v)$ which leads to $v_k \rightarrow v$ strongly by Proposition 5.2.4. Next setting $C_k := \sup_X u_k$ we observe that

$$V_{\psi_k} = \int_X e^{-\lambda u_k} f_k \omega^n = e^{-\lambda C_k} \int_X e^{-\lambda v_k} f_k \omega^n,$$

i.e. $C_k \rightarrow \frac{1}{\lambda} (\log \int_X e^{-\lambda v} f \omega^n - \log V_\psi) = C$. Hence $u_k = v_k + C_k$ converges weakly to $u = v + C$ and $MA_\omega(u) = e^{-\lambda u} f \omega^n$. Finally thanks to Proposition 5.2.4, to conclude the proof it is enough to observe that

$$E_{\psi_k}(u_k) = E_{\psi_k}(v_k) + C_k V_{\psi_k} \rightarrow E_\psi(v) + C V_\psi = E_\psi(u).$$

□

Remark 5.3.15. Observe that the assumption (i) in Theorem D is satisfied if all the Lelong numbers of ψ_k are small enough (Proposition 6.4.27), while (ii) is a natural hypothesis when all the solutions are given as maximizers (see also [Tru20b]). As stated in the Introduction the real big obstacle is the bound in (iii), which is necessary even when $f_k \equiv f$ (Example 5.4.16, see also [Tru20b] for a deeper discussion regarding (iii) in the Fano case).

5.4 Log semi-Kähler Einstein metrics with prescribed singularities.

We recall that on a line bundle $L \rightarrow X$ any (smooth) hermitian metric h can be described by its *weight* $\phi = \{\phi_\alpha\}_{\alpha \in I}$ defined locally for a trivializing local section s_α of L on an open set U_α as $\phi_\alpha := -\log |s_\alpha|_h^2$. Observe that the current $dd^c \phi$ is globally well-defined and represents the curvature of h . In this section we identify the hermitian metrics with their weights, and we say for simplicity just *metric*. Given a \mathbb{Q} -divisor D on X we have the following key definition.

Definition 5.4.1 (Definition 3.1, [BBEGZ19]). *Let ϕ be a metric on $-r(K_X + D)$ where $r \in \mathbb{N}$ such that rD is a divisor. The adapted measure μ_ϕ is locally defined by choosing a nowhere zero section σ of $r(K_X + D)$ over a small open set U and setting*

$$\mu_\phi := (i^{rn^2} \sigma \wedge \bar{\sigma})^{1/r} / |\sigma|_\phi^{2/r}.$$

We observe that μ_ϕ is globally defined since the definition does not depend on the choice of σ . Moreover $\mu_{\phi_1} = \mu_{\phi_2}$ if ϕ_i are metric on $-r_i(K_X + D)$ such that $r_2\phi_1 = r_1\phi_2$. This property allows to enlarge the definition of adapted measures to metrics of the \mathbb{Q} -line bundle $-(K_X + D)$ where we say that ϕ is a metric on $-(K_X + D)$ if there exists $r \in \mathbb{N}$ divisible enough such that $r\phi$ is a metric on $-r(K_X + D)$.

Note that if $D = 0$ and ϕ is a metric on $-K_X$, then locally

$$\mu_\phi = e^{-\phi} i^{n^2} dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n.$$

More generally by the natural identification of $-(K_X + D)$ with $-K_X$ on the complement of the support of the divisor D , if ϕ is a metric on $-(K_X + D)$ then locally on $X \setminus \text{Supp}(D)$

$$\mu_\phi = e^{-(\phi + \frac{1}{r} \log |s_{rD_+}|^2 - \frac{1}{r} \log |s_{rD_-}|^2)} i^{n^2} \Omega \wedge \bar{\Omega}$$

for s_{rD_+}, s_{rD_-} holomorphic sections cutting respectively the effective divisors rD_+, rD_- where $D = D_+ - D_-$, and Ω is a nowhere zero local holomorphic section of K_X (see also [Berm16], [BBJ15]). Furthermore the adapted measures are compatible under blow-ups of smooth centers. Indeed if $p: Y \rightarrow X$ is a morphism given by a sequence of blow-ups of smooth centers, letting D' such that $p^*(K_X + D) = K_Y + D'$, $\mu_{p^*\phi}$ coincides with the *lift* of μ_ϕ (usually denoted by $\tilde{\mu}_\phi$), i.e. with the trivial extension of the push-forward by p^{-1} of μ_ϕ over the Zariski open set where p is an isomorphism. Vice versa $p_*\mu_{p^*\phi} = \mu_\phi$.

Next, it is well-known that smooth positive volume forms μ are in one-one correspondence with metrics on the canonical line bundle K_X and the relationship is given by

$$\mu = e^{-f} i^{n^2} \Omega \wedge \bar{\Omega} \tag{5.25}$$

where $f := \log |\Omega|_\phi^2$ for any nowhere zero local holomorphic section Ω of K_X . Thus, as in [BBJ15], being aware that our definition of d^c differs from theirs of a multiplicative factor equal to 2, we say that a positive measure μ on X is said to have *well-defined Ricci curvature* if it corresponds to a *singular metric* on K_X in the sense of Demailly ([Dem90]), i.e. if locally it is of the form (6.1) with $f \in L_{loc}^2$, and in this case $\text{Ric}(\mu) := dd^c f$. Observe that if μ_ϕ is the adapted measure of Definition 5.4.1 then $\text{Ric}(\mu_\phi) = \omega + [D]$ where ω is the curvature form of ϕ .

Then, letting η be a *semi-Kähler* form, i.e. a closed smooth semipositive $(1, 1)$ -form such that $\eta^n > 0$ (see [EGZ09]), we set, for $u \in PSH(X, \eta)$, $\text{Ric}(\eta + dd^c u) := \text{Ric}(MA_\eta(u))$ so that $\text{Ric}(\eta) := \text{Ric}(\eta^n)$ is the usual Ricci curvature when η is actually Kähler.

Definition 5.4.2. *Let D be a \mathbb{Q} -divisor and η a (semi-)Kähler form. A D -log (semi-)Kähler Einstein metric on X in the cohomology class $\{\eta\}$ is a positive current*

$\eta_u := \eta + dd^c u$ with well-defined Ricci curvature such that

$$\text{Ric}(\eta_u) - [D] = \lambda \eta_u$$

for $\lambda \in \mathbb{R}$ where $[D]$ is the current of integration along the divisor D . Furthermore, when η is Kähler, if η_u is a D -log (semi-)KE metric and $u \in \mathcal{E}^1(X, \eta, \psi)$ for $\psi \in \mathcal{M}$, then we say that η_u is $(D, [\psi])$ -log (semi-)KE metric.

Note that when η is Kähler a $(D, [0])$ -log KE metric in [BBJ15] is called $[D]$ -twisted KE, and that the abuse of language is due to the fact that $(D, [\psi])$ -log KE metrics define (class of) singular D -log KE metrics.

When $D = 0$ one obtains the definition of *Kähler-Einstein metrics* (which coincides with the usual definition of Kähler-Einstein metrics under the additional request on the regularity).

It is immediate to see that there is the topological obstruction

$$c_1(X) - \{[D]\} = \lambda\{\eta\} \quad (5.26)$$

to the existence of D -log semi-KE metrics. However under the assumption (5.26), we recall the following pluripotential description of D -log semi-KE currents.

Lemma 5.4.3. *Let D be a \mathbb{Q} -divisor such that (5.26) holds for $\lambda \in \mathbb{Q}$ and η semi-Kähler form. Let ϕ be a metric on $\lambda\{\eta\}$ with curvature $\lambda\eta$, and let $u \in \text{PSH}(X, \eta)$. Then η_u is a D -log semi-KE metric if and only if*

$$MA_\eta(u) = e^{-\lambda u + C} \mu_\phi \quad (5.27)$$

for a constant $C \in \mathbb{R}$ where μ_ϕ is the adapted measure associated to ϕ .

Proof. The proof is similar to that in Lemma 2.2 in [BBJ15], but for the courtesy of the reader we report it here.

If $u \in \text{PSH}(X, \omega)$ solves (5.27) then η_u has well-defined Ricci curvature and

$$\text{Ric}(\eta_u) = \lambda dd^c u + \text{Ric}(\mu_\phi) = \lambda dd^c u + \lambda \eta + [D] = \lambda \eta_u + [D].$$

Vice versa assume that η_u has well-defined Ricci curvature and $\text{Ric}(\eta_u) - [D] = \lambda \eta_u$. Then, letting $D = \sum_{j=1}^N a_j D_j$ for D_j prime divisors, $\{s_j\}_{j=1}^N$ holomorphic sections cutting the divisors $\{D_j\}_{j=1}^N$ and letting $\{\phi_j\}_{j=1}^N$ metrics on the associated line bundles, we obtain locally on $X \setminus \text{Supp}(D)$

$$\mu_\phi = e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2} e^{-\tilde{\phi}_i \cdot n^2} \Omega \wedge \bar{\Omega}$$

where $\tilde{\phi} := \phi + \sum_{j=1}^N a_j \phi_j$ is a metric on $-K_X$. In particular we have $\mu_\phi = e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2} dV$ for a volume form dV . Therefore by definition there exists $f \in L^1$ such that $MA_\eta(u) = e^{-f} dV$, which implies

$$\text{Ric}(\eta_u) = dd^c f + \text{Ric}(dV) = dd^c f + \lambda \eta + [D] - \sum_{j=1}^N a_j dd^c \log |s_j|_{\phi_j}^2.$$

Next since $Ric(\eta_u) = \lambda\eta_u + [D]$, the function $f - \lambda u - \sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2$ is pluriharmonic. Hence there exists a constant $C \in \mathbb{R}$ such that

$$MA_\eta(u) = e^{-\lambda u + C} e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2} dV = e^{-\lambda u + C} \mu_\phi,$$

which concludes the proof. \square

Remark 5.4.4. If (5.26) holds for $\lambda \in \mathbb{R}$ and a \mathbb{R} -divisor D then it is possible to enlarge the definition of D -log (semi-)KE metrics to the class $\{\eta\}$ thanks to the pluripotential description of Lemma 5.4.3. Indeed in this case $\lambda\eta$ can be thought as the curvature of a metric ϕ on a \mathbb{R} -line bundle, i.e. on a formal real combination of line bundles. More precisely if $\{\lambda\eta\} = \{\sum_{k=1}^m b_k L_k\}$ where $b_k \in \mathbb{R}$ and L_k line bundles, then there exist metrics ϕ'_k on L_k such that $\phi := \sum_{k=1}^m b_k \phi'_k$ satisfies $dd^c \phi = \lambda\eta$. Next if $D = \sum_{j=1}^N a_j D_j$ for D_j prime divisors, we fix $\{s_j\}_{j=1}^N$ holomorphic sections cutting the divisors D_j and metrics ϕ_j on the associated line bundle. Thus setting $\tilde{\phi} := \phi + \sum_{j=1}^N a_j \phi_j$ the local volume forms $e^{-\tilde{\phi}_i n^2} \Omega \wedge \bar{\Omega}$ glue together to give a global volume form dV . Set $\mu_\phi := e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2} dV$, where we mean the trivial extension to 0 of the measure of the right hand side restricted to $X \setminus \text{Supp}(D)$. We say that $\eta + dd^c u$ is a D -log (semi-)KE metric if $MA_\eta(u) = e^{-\lambda u + C} \mu_\phi$ for a constant $C \in \mathbb{R}$, and if η is Kähler we say that $\eta + dd^c u$ is a $(D, [\psi])$ -log KE metric if we further have $u \in \mathcal{E}^1(X, \eta, \psi)$. Note that this definition of D -log KE metrics does not depend on the choice done on the metrics. Moreover if $p : Y \rightarrow X$ is given by a sequence of blow-ups of smooth centers $\tilde{\mu}_\phi = \mu_{p^* \phi}$ and $p_* \mu_{p^* \phi} = \mu_\phi$.

It is not difficult to check that the adapted measure μ_ϕ has finite total mass if and only if D is *klt* (see [Kol13]), which reads as $a_j < 1$ if $D = \sum_{j=1}^N a_j D_j$ for prime divisors D_j when D is assumed to have simple normal crossing. A similar condition holds when one considers $(D, [\psi])$ -log KE currents. Indeed letting $\{s_j\}_{j=1}^N$, $\{\phi_j\}_{j=1}^N$ and dV as in proof of Lemma 5.4.3, i.e.

$$\mu_\phi = e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2} dV,$$

we obtain the following necessary condition to the existence of $(D, [\psi])$ -log semi-KE metrics in terms of multiplier ideal sheaves.

Corollary 5.4.5. *Let η be a Kähler form such that (5.26) holds for D \mathbb{R} -divisor and $\lambda \in \mathbb{R}$. If η_u is a $(D, [\psi])$ -log semi-KE current, then*

$$\mathcal{I}\left(\lambda\psi + \sum_{\{j: a_j > 0\}} a_j \log |s_j|_{\phi_j}^2\right) = \mathcal{O}_X \quad \text{if } \lambda > 0, \quad (5.28)$$

$$\mathcal{I}\left(\sum_{\{j: a_j > 0\}} a_j \log |s_j|_{\phi_j}^2\right) = \mathcal{O}_X \quad \text{if } \lambda \leq 0. \quad (5.29)$$

If $\lambda > 0$ (resp. $\lambda \leq 0$) we will say that $(D, [\psi])$ (resp. D) is *klt* when (5.28) (resp. (5.29)) holds. The definition does not depends on the metrics ϕ_j chosen and it is coherent with the usual definition (see for instance Proposition 8.2 in [Kol96]).

5.4.1 Analytical Singularities.

In this subsection ω Kähler and $\psi := P_\omega[\varphi] \in \mathcal{M}^+$ where $\varphi \in PSH(X, \omega)$ has *analytical singularities*, i.e. locally $\varphi|_U := g + c \log(|f_1|^2 + \cdots + |f_k|^2)$ where $c \in \mathbb{R}_{\geq 0}$, $g \in C^\infty$, and $\{f_j\}_j^k$ are local holomorphic functions. The coherent ideal sheaf \mathcal{I} generated by these functions has integral closure globally defined, hence the singularities of φ are formally encoded in (\mathcal{I}, c) . It is well-known in this case that there exists a smooth resolution $p : Y \rightarrow X$ given by a sequence of blow-ups of smooth centers such that $p^*\mathcal{I} = \mathcal{O}_Y(-D)$ for an effective divisor D . Moreover the Siu Decomposition ([Siu74]) of $p^*(\omega_\varphi)$ is given by

$$p^*(\omega_\varphi) = \eta + c[D]$$

where η is a smooth semipositive $(1, 1)$ -form on Y , which becomes semi-Kähler if $\int_X \eta^n > 0$. In such case it is possible to define the sets $\mathcal{E}(Y, \eta)$ and $\mathcal{E}^1(Y, \eta)$ similarly to the Kähler case (see [BEGZ10]).

Lemma 5.4.6. *In the setting just described $\int_X \eta^n = \int_X MA_\omega(\varphi)$ and there is a bijective map $f : PSH(X, \omega, \psi) \rightarrow PSH(X, \eta)$ such that $f(\mathcal{E}(X, \omega, \psi)) = \mathcal{E}(Y, \eta)$ and $f(\mathcal{E}^1(X, \omega, \psi)) = \mathcal{E}^1(Y, \eta)$.*

Proof. By Remark 4.6 in [RWN14] $\psi - \varphi$ is globally bounded, so for any $u \in PSH(X, \omega, \psi)$ we have $u \preceq \varphi$ which implies that $p^*(\omega_u) - c[D]$ is a closed and positive current on Y with cohomology class $\{\eta\}$. Therefore there exists a unique $\tilde{u} \in PSH(Y, \eta)$ such that $\sup_Y \tilde{u} = \sup_X (u - \varphi)$ and

$$p^*(\omega_u) = \eta_{\tilde{u}} + c[D].$$

Thus we define $f : PSH(X, \omega, \psi) \rightarrow PSH(Y, \eta)$ as $f(u) := \tilde{u}$. By Proposition 1.2.7.(ii) in [BouTh] f is a bijection. It is also easy to check that $\tilde{u} - (u - \varphi) \circ p$ is pluriharmonic on Y , which leads to $f(u) = \tilde{u} = (u - \varphi) \circ p$.

Next, since p is an isomorphism over $Y \setminus p^{-1}V(\mathcal{I})$ and $[D]$ has support in a pluripolar set, it is not difficult to check that

$$p_* MA_\eta(\tilde{u}) = MA_\omega(u) \tag{5.30}$$

using the definition of non-pluripolar product. Then (5.30) gives $f(\mathcal{E}(X, \omega, \psi)) = \mathcal{E}(Y, \eta)$. Hence to conclude the proof it is enough to observe that the equalities

$$\begin{aligned} \int_Y \tilde{u} MA_\eta(\tilde{u}) &= \int_Y p^* p_* ((u - \varphi) \circ p MA_\eta(\tilde{u})) = \\ &= \int_Y p^* ((u - \varphi) MA_\omega(u)) = \int_X (u - \psi) MA_\omega(u) + \int_X (\varphi - \psi) MA_\omega(u) \end{aligned}$$

imply $f(\mathcal{E}^1(X, \omega, \psi)) = \mathcal{E}^1(Y, \eta)$ thanks to Theorem 4.10 in [DDNL18b], Proposition 2.11 in [BEGZ10] and the fact that $\left| \int_X (\varphi - \psi) MA_\omega(u) \right| \leq C$ uniformly for any $u \in PSH(X, \omega, \psi)$. \square

For completeness we also prove that in this setting the metric space $(\mathcal{E}^1(X, \omega, \psi), d)$ is isometric to the metric space $(\mathcal{E}^1(Y, \eta), d)$ studied in [DDNL18c] where

$$d(u, v) = E(u) + E(v) - 2E(P_\eta(u, v))$$

for any $u, v \in \mathcal{E}^1(Y, \eta)$ recalling that $P_\eta(\cdot, \cdot), E(\cdot)$ are defined in the same way as in the Kähler case, i.e. for instance $E(u) = \frac{1}{n+1} \sum_{j=0}^n \int_X u(\eta + dd^c u)^j \wedge \eta^{n-j}$ if u has minimal singularities (remember that η is semipositive).

Proposition 5.4.7. *The metric space $(\mathcal{E}^1(X, \omega, \psi), d)$ is isometric to $(\mathcal{E}^1(Y, \eta), d)$ through the map of Lemma 5.4.6*

Proof. With the same notation of Lemma 5.4.6 we have $\tilde{u} := f(u) = (u - \varphi) \circ p$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$. Moreover similarly as in the proof on Lemma 5.4.6 we can show that $p_*(\eta_{\tilde{u}_1}^k \wedge \eta_{\tilde{u}_2}^{n-k}) = \omega_{u_1}^k \wedge \omega_{u_2}^{n-k}$ for any $k = 0, \dots, n$, and that these equalities lead to $E(\tilde{u}) = E_\psi(u) - E_\psi(\varphi)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$. Hence to conclude the proof it is enough to prove that $f(P_\omega(u_1, u_2)) = P_\omega(\tilde{u}_1, \tilde{u}_2)$. By construction we easily have $\tilde{u}_1 \leq \tilde{u}_2$ if and only if $u_1 \leq u_2$. Therefore we get $f(P_\omega(u_1, u_2)) \leq P_\omega(\tilde{u}_1, \tilde{u}_2)$ from $P_\omega(u_1, u_2) \leq u_1, u_2$, while letting $\phi \in \mathcal{E}^1(X, \omega, \psi)$ such that $\tilde{\phi} = P_\omega(\tilde{u}_1, \tilde{u}_2)$ we have $\phi \leq u_1, u_2$, i.e. $\phi \leq P_\omega(u_1, u_2)$ which conclude the proof by composing with f . \square

We can now relate the $(D, [\psi])$ -log KE metrics on X with the D' -log semi-KE metrics on Y . More precisely, let D be a klt \mathbb{R} -divisor on X such that

$$c_1(X) - \{[D]\} = \lambda\{\omega\}$$

for $\lambda \in \mathbb{R}$ and ω Kähler form. Let $\psi \in \mathcal{M}^+$ given as $P_\omega[\varphi]$ for a function $\varphi \in PSH(X, \omega)$ with analytic singularities encoded in (\mathcal{I}, c) , and let $p : Y \rightarrow X$ be a smooth resolution of \mathcal{I} . Then $p^*\mathcal{I} = \mathcal{O}_Y(-D_1)$ for an effective divisor D_1 and $p^*(K_X + D) = K_Y + D_2$ for a \mathbb{R} -divisor D_2 . We denote with η the semi-Kähler part of the Siu Decomposition $p^*(\omega_\varphi) = \eta + c[D_1]$.

Proposition 5.4.8. *In the setting described above, there is a bijection between the set of all $(D, [\psi])$ -log KE metrics on X in the cohomology class $\{\omega\}$ and the set of all D' -log semi-KE metrics on Y in the cohomology class $\{\eta\}$ where $D' := \lambda c[D_1] + [D_2]$. More precisely letting ϕ_ω and ϕ_η be metrics respectively on the \mathbb{R} -line bundles $-(K_X + D), -(K_Y + D_2 + \lambda cD_1)$ with curvatures $\lambda\omega$ and $\lambda\eta$, a function $u \in \mathcal{E}^1(X, \omega, \psi)$ solves $MA_\omega(u) = e^{-\lambda u} \mu_{\phi_\omega}$ if and only if $\tilde{u} = (u - \varphi) \circ p \in \mathcal{E}^1(Y, \eta)$ solves $MA_\eta(\tilde{u}) = e^{-\lambda \tilde{u}} \mu_{\phi_\eta}$.*

Proof. Let ϕ_ω, ϕ_η as in the statement. Set also $\phi := p^*\phi_\omega - \phi_\eta$ metric on λcD_1 with curvature $\theta := dd^c \phi$. Then for $r_1 = \frac{1}{\lambda c} \in \mathbb{R}_{>0}$, $r_1 \lambda cD_1 = D_1$ is an effective divisor and there exists an holomorphic section s_1 on the associate line bundle such

that $r_1\theta + dd^c \log |s_1|_{r_1\phi}^2 = r_1\lambda c[D_1]$. Thus, since by construction $\lambda\eta + \theta = p^*\lambda\omega$, it follows that

$$dd^c \frac{1}{r_1} \log |s_1|_{r_1\phi}^2 = dd^c \lambda\varphi \circ p,$$

i.e. $\lambda\varphi \circ p = \frac{1}{r_1} \log |s_1|_{r_1\phi}^2 + C$ for a constant $C \in \mathbb{R}$ which without loss of generality we may suppose to be equal to 0. Therefore the lift of the measure $e^{-\lambda u} \mu_{\phi_\omega} = e^{-\lambda(u-\varphi)} e^{-\lambda\varphi} \mu_{\phi_\omega}$ becomes

$$e^{-\lambda\tilde{u} - \frac{1}{r_1} \log |s_1|_{r_1\phi}^2} \mu_{p^*\phi_\omega}$$

where $\tilde{u} = (u - \varphi) \circ p$. Next for $\{a_j\}_{j=1}^{N_1}, \{b_j\}_{j=1}^{N_2} \subset \mathbb{R}_{>0}$ and prime divisors $\{D_{2,+j}\}_{j=1}^{N_1}, \{D_{2,-j}\}_{j=1}^{N_2}$, we have $D_2 = \sum_{j=1}^{N_1} a_j D_{2,+j} - \sum_{j=1}^{N_2} b_j D_{2,-j}$ as the difference of two effective \mathbb{R} -divisors. Thus locally on $Y \setminus (\text{Supp}(D_1) \cup \text{Supp}(D_2))$ by definition there exists Ω nowhere zero local holomorphic section of K_Y such that

$$\mu_{p^*\phi_\omega} = e^{-(p^*\phi_\omega + \sum_{j=1}^{N_1} a_j \log |s_{2,+j}|^2 - \sum_{j=1}^{N_2} b_j \log |s_{2,-j}|^2)} i^{n^2} \Omega \wedge \bar{\Omega}$$

where $\{s_{2,+j}\}_{j=1}^{N_1}, \{s_{2,-j}\}_{j=1}^{N_2}$ are holomorphic sections cutting respectively $\{D_{2,+j}\}_{j=1}^{N_1}, \{D_{2,-j}\}_{j=1}^{N_2}$. For simplicity of notations we define $\varphi_{2,+} := \sum_{j=1}^{N_1} a_j \log |s_{2,+j}|^2$ and similarly for $\varphi_{2,-}$. Therefore locally on $Y \setminus (\text{Supp}(D_1) \cup \text{Supp}(D_2))$

$$\begin{aligned} e^{-\frac{1}{r_1} \log |s_1|_{r_1\phi}^2} \mu_{p^*\phi_\omega} &= e^{-(\phi + \frac{1}{r_1} \log |s_1|_{r_1\phi}^2 + \phi_\eta + \varphi_{2,+} - \varphi_{2,-})} i^{n^2} \Omega \wedge \bar{\Omega} = \\ &= e^{-(\phi_\eta + \frac{1}{r_1} \log |s_1|^2 + \varphi_{2,+} - \varphi_{2,-})} i^{n^2} \Omega \wedge \bar{\Omega} = \mu_{\phi_\eta}. \end{aligned}$$

In conclusion for any $u \in \mathcal{E}^1(X, \omega, \psi)$ and the measures $e^{-\lambda u} \mu_{\phi_\omega}$ and $e^{-\lambda\tilde{u}} \mu_{\phi_\eta}$ are related by lifting and by push-forward through p_* . Hence the Proposition follows since the same correspondence holds for $MA_\omega(u)$ and $MA_\eta(\tilde{u})$ as seen during the proof of Lemma 5.4.6. \square

We can prove the following regularity result on $(D, [\psi])$ -log semi-KE metrics in this case, which is the first part of Theorem E.

Theorem 5.4.9. *Let ω_u be a $(D, [\psi])$ -log KE metric where D is a \mathbb{R} -divisor and $\psi = P_\omega[\varphi] \in \mathcal{M}^+$ for φ with analytic singularities formally encoded in (\mathcal{I}, c) . Then $u \in C^\infty(X \setminus A)$ where $A = V(\mathcal{I}) \cup \text{Supp}(D)$.*

Proof. By Proposition 5.4.8 and $\tilde{u} := (u - \varphi) \circ p$ is a solution of

$$\begin{cases} MA_\eta(\tilde{u}) = e^{-\lambda\tilde{u}} \mu_{\phi_\eta} \\ \tilde{u} \in \mathcal{E}^1(Y, \eta) \end{cases}$$

where η is semi-Kähler form. Moreover writing $\mu_{\phi_\eta} = e^{v_1 - v_2} dV$ where $v_1, v_2 \in PSH(Y, \omega')$ for ω' Kähler form and dV volume form on Y , by the Monge-Ampère

equation and the resolution of the openness conjecture ([GZ15]) we immediately obtain $e^{-\lambda\tilde{u}+v_1-v_2} \in L^p$ for $p > 1$ (see also Corollary 5.4.5). Now the proof is standard.

Indeed by Theorem C in [EGZ11] we get that \tilde{u} is bounded on X and continuous on $\text{Amp}(\{\eta\})$ (see also [Kol98]), where the latter is the *ample locus* of η ([Bou04]). Then, assuming first $\lambda > 0$, let $C > 0$ big enough such that $\sup_X v_1 \leq C$, $C\omega' + dd^c v_1 \geq 0$, $C\omega' + dd^c(v_1 + \lambda\tilde{u}) \geq 0$ and $\|e^{-\lambda\tilde{u}-v_2}\|_{L^p} \leq C$. Thus by Theorem 10.1 in [BBEGZ19] for any relatively compact open set $U \Subset \text{Amp}(\{\eta\})$ there exists $A > 0$ depending on C, η, p, U such that

$$0 \leq \eta + dd^c \tilde{u} \leq Ae^{-\lambda\tilde{u}-v_2}\omega'.$$

Similarly if $\lambda \leq 0$, letting $C > 0$ big enough such that $\sup_X (v_1 - \lambda\tilde{u}) \leq C$, $C\omega' + dd^c(v_1 - \lambda\tilde{u}) \geq 0$, $C\omega' + dd^c v_2 \geq 0$ and $\|e^{-v_2}\|_{L^p} \leq C$, we obtain

$$0 \leq \eta + dd^c \tilde{u} \leq Ae^{-v_2}\omega'$$

for any relatively compact open set $U \Subset \text{Amp}(\{\eta\})$.

Moreover by construction v_1, v_2 are smooth outside the union of the supports of the divisors D_1, D_2 (with the notations used in Proposition 5.4.8). So, since \tilde{u} is globally bounded it immediately follows that $\Delta_{\omega'}\tilde{u}$ is locally bounded over $\text{Amp}(\{\eta\}) \cap (Y \setminus (\text{Supp}(D_1) \cup \text{Supp}(D_2)))$. By the Evans-Krylov Theorem and a classical bootstrap argument this also implies that \tilde{u} is smooth over $\text{Amp}(\{\eta\}) \cap (Y \setminus (\text{Supp}(D_1) \cup \text{Supp}(D_2)))$. Then the ample locus is a not-empty Zariski open set ($\{\eta\}$ is big, see [Bou04]) and it includes $Y \setminus (\text{Supp}(D_1) \cup \text{Supp}(D_2))$ since $\{\omega\}$ is Kähler and the support of the exceptional locus of $p : Y \rightarrow X$ is contained in the union of the supports of D_1, D_2 . Hence since $\tilde{u} = (u - \varphi) \circ p$, we get that $u \in \mathcal{C}^\infty(X \setminus B)$ where $B = p_*(\text{Supp}(D_1) \cup \text{Supp}(D_2)) \subset V(\mathcal{I}) \cup \text{Supp}(D)$ which concludes the proof. \square

Remark 5.4.10. In Theorem 5.4.9, if there exists a resolution of \mathcal{I} such that η is Kähler and $\Delta := \lambda cD_1 + D_2$ is effective, then the solution \tilde{u} has *conic singularities along Δ* as proved in [GP16].

5.4.2 Theorem E.

In the subsection we conclude the proof of Theorem E.

As shown in the previous subsection if $\psi \in \mathcal{M}^+$ has analytic singularities type, i.e. $\psi = P_\omega[\varphi]$ for φ with analytic singularities formally encoded in (\mathcal{I}, c) where \mathcal{I} is an integrally closed coherent ideal sheaf and $c \in \mathbb{R}_{>0}$, then taking $p : Y \rightarrow X$ a resolution of \mathcal{I} there exists a semi-Kähler form η on Y such that $p^*(\omega_\varphi) = \eta + c[D]$ where $p^*\mathcal{I} = \mathcal{O}_X(-D)$ and D is an effective divisor. Thus, we first set

$\mathcal{M}_{an}^+ := \{\psi \in \mathcal{M}^+ \text{ with analytic singularities type } \}$ and we fix for any $\psi \in \mathcal{M}_{an}^+$ an element φ with analytic singularities such that $\sup_X \varphi = 0$ and $\psi = P_\omega[\varphi]$ (i.e. $\psi - \varphi$ globally bounded). Then setting $\mathcal{K}_{(X,\omega)}^t := \{(Y, \eta) : \omega - p_*\eta = [D] \text{ for an effective } \mathbb{R}\text{-divisor } D \text{ where } \eta \text{ is semi-Kähler and } p : Y \rightarrow X \text{ is given by a sequence of blow-ups}\}$ the construction described above leads to a natural map

$$\Phi : \mathcal{M}_{an}^+ \longrightarrow \mathcal{K}_{(X,\omega)}^t / \sim$$

where $(Y, \eta) \sim (Y', \eta')$ on $\mathcal{K}_{(X,\omega)}^t$ if there exists $(Z, \tilde{\eta}) \in \mathcal{K}_{(X,\omega)}^t$ such that Z dominates Y, Y' through morphism $q : Z \rightarrow Y$, $q' : Z \rightarrow Y'$ and $\tilde{\eta} = q^*\eta = q'^*\eta'$. Note that for a different choice of the elements φ with analytic singularities, the forms η in the representatives in $\mathcal{K}_{(X,\omega)}^t$ may change but their cohomology classes $\{\eta\}$ would remain unchanged.

We also claim that Φ is injective. Indeed letting $\psi_1, \psi_2 \in \mathcal{M}_{an}^+$ and letting $(Y, \eta_1), (Y, \eta_2)$ be representatives on the same manifold Y (taking a common resolution), if $\Phi(\psi_1) = \Phi(\psi_2)$ then $\eta_1 = \eta_2$. Thus, denoting with φ_1, φ_2 the fixed functions with analytic singularities, $\eta_1 = \eta_2$ and cohomological reasons imply that $(\varphi_1 - \varphi_2) \circ p$ is pluri-harmonic, hence $\varphi_1 = \varphi_2 + C$ which clearly gives $\psi_1 = \psi_2$.

We can now define

$$\mathcal{K}_{(X,\omega)} := \text{Im}(\Phi).$$

It is worth to underline that for any small perturbation $\{\mu_N^*\omega - a_1[E_1] - a_2[E_2] + \dots - a_N[E_N]\}$ where $\mu_N : Y \rightarrow X$ is the blow-up of X at N distinct points, E_i the exceptional divisors and $a_i > 0$ small enough, there exists a smooth semi-Kähler form η such that $[(Y, \eta)] \in \mathcal{K}_{(X,\omega)}$.

As an immediate consequence of the construction the set $\mathcal{K}_{(X,\omega)}$ inherits a partial order and a notion of convergence given by the set \mathcal{M}_{an}^+ . In particular for any $\alpha, \alpha' \in \mathcal{K}_{(X,\omega)}$ with associated model type envelopes $\psi, \psi' \in \mathcal{M}_{an}^+$, we will say that α is *smaller* (resp. *bigger*) than α' if $\psi \preceq \psi'$ (resp. $\psi \succeq \psi'$). Note that if α is smaller than α' then taken representatives $(Z, \tilde{\eta}), (Z, \tilde{\eta}')$ on the same compact Kähler manifold Z we have $\tilde{\eta}' - \tilde{\eta} = [F]$ for an effective \mathbb{R} -divisor F . The notion of volume $\text{Vol}(\alpha)$ is also well-defined for $\alpha \in \mathcal{K}_{(X,\omega)}$ since for any $(Y, \eta) \sim (Y', \eta')$, $\int_Y \eta^n = \int_{Y'} \eta'^n$, and in particular $\text{Vol}(\alpha) = V_\psi$ where $\Phi(\psi) = \alpha$ (see also Lemma 5.4.6).

Next, it is possible to talk about log-KE currents for a class in $\mathcal{K}_{(X,\omega)}$ thanks to Proposition 5.4.8 since for two different representatives $(Y, \eta), (Y', \eta')$ of a same class in $\mathcal{K}_{(X,\omega)}$ the sets of log-KE currents are in bijection. Indeed the bijection is a level of quasi-psh functions, i.e. we identify two log-KE currents $\eta + dd^c \tilde{u}$, $\eta' + dd^c \tilde{u}'$ respectively on $(Y, \eta), (Y', \eta')$ representative of the same class in $\mathcal{K}_{(X,\omega)}$ if $\tilde{u} = (u - \varphi) \circ p$, $\tilde{u}' = (u - \varphi) \circ p'$ for the same function $u \in \mathcal{E}^1(X, \omega, \psi)$. Thus a *log-KE current for a class* in $\mathcal{K}_{(X,\omega)}$ is a family of log-KE currents which are related through the bijection just described. We can then define a *strong convergence* on sequences of log-KE currents for totally ordered sequences in $\mathcal{K}_{(X,\omega)}$ after a suitable normalization. Namely when $\lambda = 0$, in accord with Theorem B, for any log-KE

current $\eta + dd^c \tilde{u}$ on (Y, η) representative of a class in $\mathcal{K}_{(X, \omega)}$ the function \tilde{u} will be normalized so that the corresponding ω -psh function u through Lemma 5.4.6 satisfies $\sup_X u = 0$. When instead $\lambda \neq 0$, we will normalize \tilde{u} so that $MA_\omega(u) = e^{-\lambda u} \mu_{\phi_\omega}$ where we fix ϕ_ω metric on $-(K_X + D)$ with curvature $\lambda\omega$ once and for all (see again Proposition 5.4.8). In conclusion, given a totally ordered sequence $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{K}_{(X, \omega)}$ converging to $\alpha \in \mathcal{K}_{(X, \omega)}$, we will say that a sequence of log-KE currents $\eta_k + dd^c \tilde{u}_k$ converges strongly to a log-KE current $\eta + dd^c \tilde{u}$ if $u_k \rightarrow u$ strongly. When there exists a common compact Kähler manifold Z such that $(Y_k, \eta_k) \sim (Z, \theta_k)$ and $(Y, \eta) \sim (Z, \theta)$, the strong convergence implies in particular that the associated sequence of log-KE currents $\theta_k + dd^c v_k$ converges weakly to the log-KE current $\theta + dd^c v$.

We can now prove the second part of Theorem E.

Theorem 5.4.11. *Let ω be a Kähler form such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ holds for $\lambda \in \mathbb{R}, \lambda \leq 0$ and let D be a klt \mathbb{R} -divisor. Then any class in $\mathcal{K}_{(X, \omega)}$ admits an unique log-KE current and such log-KE currents are stable with respect to the strong convergence, i.e. if $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{K}_{(X, \omega)}$ is a totally ordered sequence converging to $\alpha \in \mathcal{K}_{(X, \omega)}$, then the sequence of log-KE currents converges strongly to the log-KE current on α .*

Proof. By Proposition 5.4.8 and by definition to find a log-KE metric on $\alpha \in \mathcal{K}_{(X, \omega)}$ is equivalent to solve

$$\begin{cases} MA_\omega(u) = e^{-\lambda u} \mu_{\phi_\omega}, \\ u \in \mathcal{E}^1(X, \omega, \psi) \end{cases} \quad (5.31)$$

where $\psi \in \mathcal{M}^+$ is the model type envelope with analytic singularities associated to α . Moreover by the resolution of the openness conjecture ([GZ15]) since D is klt we have $\mu_{\phi_\omega} = f dV$ for $f \in L^p$ for $p > 1$. Therefore the theorem follows from Theorems A, B. \square

Next it remains to treat the case $\lambda > 0$.

We first note that in the case of $(D, [\psi])$ -log KE currents the density $f_D \in L^1 \setminus \{0\}$ of the corresponding Monge-Ampère equation $MA_\omega(u) = e^{-\lambda u} f_D \omega^n$ is given as

$$f_D = e^{-\sum_{j=1}^N a_j \log |s_j|_{\phi_j}^2 + g} \quad (5.32)$$

where g is a smooth function, and as usual we fixed $\{s_j\}_{j=1}^N$ holomorphic sections cutting the prime divisors D_j and metrics ϕ_j on the associated line bundle where $D = \sum_{j=1}^N a_j D_j$.

We then observe that in Theorem C we used the assumption $\alpha(\psi) > \frac{\lambda p}{p-1}$ only on the first part of the proof to prove that the d -coercivity of $F_{f_D, \psi, \lambda}$ implies the existence of a maximizer. Such hypothesis will not be necessary for the study of log-KE currents in $\mathcal{K}_{(X, \omega)}$ as consequence of the following result.

Lemma 5.4.12. *Let ω be a Kähler form such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ holds for $\lambda > 0$ and let D be an \mathbb{R} -divisor. Let also $\psi \in \mathcal{M}_{an}^+$ and assume that $(D, [\psi])$ is klt. Then the d -coercivity of $F_{f_D, \psi, \lambda}$ over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ implies the existence of a maximizer.*

Proof. Fix (Y, η) representative of $\alpha = \Phi(\psi) \in \mathcal{K}_{(X, \omega)}$.

Then using the same notation of Proposition 5.4.8, for any $v \in \mathcal{E}^1(X, \omega, \psi)$ by construction the lift of $e^{-\lambda v} f_D \omega^n = e^{-\lambda v} \mu_{\phi_\omega}$ to Y is $e^{-\lambda \tilde{v}} \mu_{\phi_\eta}$ where $\tilde{v} = (v - \varphi) \circ p$. Therefore using also Proposition 5.4.7 it follows that

$$F_{f_D, \psi, \lambda}(v) - E_\psi(\varphi) = E(\tilde{v}) + \frac{V_\psi}{\lambda} \log \int_X e^{-\lambda \tilde{v}} \mu_{\phi_\eta} =: D_\eta(\tilde{v}) \quad (5.33)$$

for any $v \in \mathcal{E}^1(X, \omega, \psi)$. Observe that, up to rescaling the class ω , since $V_\psi = \int_X \eta^n$, the functional D_η coincides with the (opposite of the) *log-Ding functional* in the class $\{\eta\}$ as described in [BBEGZ19]. Hence since again by Proposition 5.4.7 the map $(\mathcal{E}^1(X, \omega, \psi), d) \ni u \rightarrow \tilde{u} \in (\mathcal{E}^1(Y, \eta), d)$ is an isometry and by assumption $F_{f_D, \psi, \lambda}$ is d -coercive, we obtain that D_η is d -coercive over $\mathcal{E}_{norm}^1(Y, \eta)$. Thus let $\{\tilde{v}_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{norm}^1(Y, \eta)$ be a maximizing sequence, which without loss of generality by the compactness of $\{\tilde{v} \in PSH(Y, \eta) : \sup_Y \tilde{v} = 0\}$ we may assume to be weakly convergent to $\tilde{v} \in \mathcal{E}_{norm}^1(Y, \eta)$. Then writing $\mu_{\phi_\eta} = g dV$ where $g \in L^p$ for $p > 1$ and dV is a smooth volume form, by applying twice the Holder's inequality we have

$$\begin{aligned} \int_X |e^{-\lambda \tilde{v}_k} - e^{-\lambda \tilde{v}}| d\mu_{\phi_\eta} &\leq \lambda \int_X e^{-\lambda(\tilde{v}_k + \tilde{v})} |\tilde{v}_k - \tilde{v}| d\mu_{\phi_\eta} \leq \\ &\leq \lambda \|e^{-\lambda(\tilde{v}_k + \tilde{v})}\|_{L^q} \|(\tilde{v}_k - \tilde{v})f\|_{L^{p/2}} \leq \\ &\leq \lambda \|e^{-\lambda(\tilde{v}_k + \tilde{v})}\|_{L^q} \|f\|_{L^p} \|\tilde{v}_k - \tilde{v}\|_{L^p} \end{aligned} \quad (5.34)$$

where $1 < q < \infty$ is the Sobolev conjugate of $p/2$. Therefore since any element in $\mathcal{E}^1(Y, \eta)$ has vanishing Lelong numbers, by [Zer01] the first factor in the right side in (5.34) is uniformly bounded, and we obtain $e^{-\lambda \tilde{v}_k} \rightarrow e^{-\lambda \tilde{v}}$ in $L^1(\mu_{\phi_\eta})$ as a consequence of $\tilde{v}_k \rightarrow \tilde{v}$ in L^p . Hence by the upper semicontinuity of $E(\cdot)$ in $\mathcal{E}^1(Y, \eta)$ with respect to the weak topology ([BBEGZ19]) we obtain

$$\sup_{\mathcal{E}^1(Y, \eta)} D_\eta = \lim_{k \rightarrow \infty} D_\eta(\tilde{v}_k) \leq D_\eta(\tilde{v}),$$

i.e. \tilde{v} is a maximizer of D_η . Hence from (5.33) the corresponding function $v \in \mathcal{E}^1(X, \omega, \psi)$ (Lemma 5.4.6) is a maximizer of $F_{f_D, \psi, \lambda}$. \square

Remark 5.4.13. As seen during the proof of Lemma 5.4.12, the d -coercivity of $F_{f_D, \psi, \lambda}$ over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ with respect to coefficients $A > 0, B \geq 0$ (i.e. $F_{f_D, \psi, \lambda}(u) \leq -Ad(\psi, u) + B$ for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$) is equivalent to the d -coercivity of the log-Ding functional D_η over $\mathcal{E}_{norm}^1(Y, \eta)$ with respect coefficients $A > 0, B_\eta \geq 0$ for any (Y, η) representative of the class $\Phi(\psi) \in \mathcal{K}_{(X, \omega)}$. In particular $F_{f_D, \psi, \lambda}$ and D_η have the same *slope* at infinity (i.e. the coefficient A of the d -coercivity).

We can now state the third part of Theorem E.

Theorem 5.4.14. *Let ω be a Kähler form such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ holds for $\lambda > 0$ and let D be a klt \mathbb{R} -divisor. If the log-Ding functional associated to a representative (Y, η) of $\alpha \in \mathcal{K}_{(X, \omega)}$ is d -coercive over $\mathcal{E}_{norm}^1(Y, \eta)$ with slope $1 > A > 0$, then any $\alpha' \in \mathcal{K}_{(X, \omega)}$ bigger than α satisfying $\text{Vol}(\alpha') < \text{Vol}(\alpha)/(1 - A)$ admits a log-KE current.*

Proof. It follows directly from Theorem C thanks to Lemma 5.4.12 and Remark 5.4.13. □

Finally to apply Theorem D to the class $\mathcal{K}_{(X, \omega)}$, we fix a klt \mathbb{R} -divisor D such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ for $\lambda > 0$, with associated function f_D as in (5.32). Then it is a classical fact that $f_D \in L^p$ for any $1 < p < \text{lct}(X, D) := \sup\{t > 0 : (X, tD) \text{ is klt}\}$ (the log canonical threshold) and $f_D \notin L^{\text{lct}(X, D)}$ (see for instance Proposition 3.20 in [Kol96]). Therefore we define

$$\mathcal{K}_{(X, \omega)}^D := \left\{ \Phi(\psi) \in \mathcal{K}_{(X, \omega)} : \psi \in \mathcal{M}_{an}^+ \text{ such that } \alpha(\psi) > \frac{\lambda \text{lct}(X, D)}{\text{lct}(X, D) - 1} \right\}.$$

Theorem 5.4.15. *Let ω be a Kähler form such that $c_1(X) - \{[D]\} = \lambda\{\omega\}$ holds for $\lambda > 0$ and let D be a klt \mathbb{R} -divisor. Assume that*

- (i) $\{\alpha_k\}_{k \in \mathbb{N}} \subset \mathcal{K}_{(X, \omega)}$ is an increasing sequence converging to $\alpha \in \mathcal{K}_{(X, \omega)}^D$;
- (ii) $\eta_k + dd^c \tilde{u}_k$ are representatives of a sequence of log-KE currents in α_k such that $\sup_X u_k \leq C$ uniformly.

Then the sequence of log-KE currents of (ii) converges strongly to a log-KE current in α .

Observe that by the definition of the normalization, the assumption (ii) in Theorem 5.4.15 is independent on the representatives $\eta_k + dd^c \tilde{u}_k$ chosen.

We conclude the paper with the following example which shows that the assumption (iii) in Theorem D is necessary.

Example 5.4.16. Let ω be a Kähler form on a Fano manifold X representative of the anticanonical class, and let D be a smooth divisor \mathbb{Q} -linearly equivalent to $-K_X$, i.e. $D \in |-rK_X|$ for $r \in \mathbb{N}$. Next, letting $\varphi_D \in \text{PSH}(X, \omega)$ such that $\omega + dd^c \varphi_D = \frac{1}{r}[D]$ and $\psi_t := P_\omega[t\varphi_D]$ for any $t \in [0, 1)$, by Proposition 5.4.8 the set of all solutions of

$$\begin{cases} MA_\omega(u_t) = e^{-u_t} \mu_{\phi_\omega} \\ u_t \in \mathcal{E}^1(X, \omega, \psi_t), \end{cases} \quad (5.35)$$

is in bijection with the set of all $\frac{t}{r}D$ -log KE currents in the cohomology class $\{(1 - t)\omega\}$, i.e. with all solutions of

$$\begin{cases} MA_{(1-t)\omega}(v_t) = e^{-v_t - \frac{t}{r}\varphi_D} \mu_{\phi_\omega} \\ v_t \in \mathcal{E}^1(X, (1 - t)\omega). \end{cases}$$

where the correspondence is clearly given by $u_t = v_t + \frac{t}{r}\varphi_D$. Thus setting $w_t := \frac{1}{(1-t)}v_t \in PSH(X, \omega)$ we have

$$\begin{cases} MA_\omega(w_t) = (1-t)^{-n} e^{-(1-t)w_t - \frac{t}{r}\varphi_D} \mu_{\phi_\omega} \\ w_t \in \mathcal{E}^1(X, \omega), \end{cases}$$

which is equivalent to the renowned path

$$Ric(\omega_{v_t}) = (1-t)\omega_{v_t} + \frac{t}{r}[D]. \quad (5.36)$$

considered in [CDS15]. Thus the set $S := \{t \in [0, 1) : (5.35) \text{ admits a solution}\}$ is not empty ([Berm13],[JMR16]) and open (by the implicit function theorem, see [Aub84]). Moreover it is well-known that when X does not admit a KE metric (for instance $X = \text{Bl}_p \mathbb{P}^2$) then there exists $t_0 \in (0, 1)$ such that $\liminf_{t \searrow t_0} \sup_X w_t = +\infty$, which clearly implies $\liminf_{t \searrow t_0} \sup_X u_t = +\infty$.

Hence since the assumption (i) in Theorem D is satisfied for any $t \in [0, 1)$ and since (ii) follows from (5.33) in Lemma 5.4.12, it follows that (iii) in Theorem D is a necessary hypothesis.

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PAPER V

Kähler-Einstein metrics with prescribed
singularities on Fano manifolds.

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arxiv preprint

Chapter 6

Kähler-Einstein metrics with prescribed singularities on Fano manifolds.

Abstract

Given a Fano manifold (X, ω) we characterize analytically the existence of Kähler-Einstein metrics with prescribed singularities through a variational approach when the singularities can be approximated algebraically and are concentrated in a complete closed pluripolar set.

Moreover we define an increasing function α_ω on the set of all prescribed singularities which generalizes Tian's α -invariant, showing that its upper contour set $\{\alpha_\omega(\cdot) > \frac{n}{n+1}\}$ produces a subset of the Kähler-Einstein locus, i.e. of the locus given by all prescribed singularities which admits a Kähler-Einstein metric. In particular we prove that many K -stable manifolds admits all possible Kähler-Einstein metrics with prescribed singularities while vice versa lower bounds of the α -invariant function at not trivial prescribed singularities imply lower bounds on the classical α -invariant and consequently the existence of genuine Kähler-Einstein metrics.

Through a continuity method we also prove the strong continuity of Kähler-Einstein metrics on curves of totally ordered prescribed singularities when the relative automorphism groups are discrete.

6.1 Introduction

A Fano manifold X admits a Kähler-Einstein (KE) metric if and only if $(X, -K_X)$ is *K-stable* ([CDS15]). This is the famous solution to the Yau-Tian-Donaldson conjecture for the anticanonical polarization, and it connects a differential-geometric notion to a GIT-like algebrico-geometric notion as predicted by S.T. Yau ([Yau93]). There are now two natural possible singular versions of this correspondence: when X is singular or when the metric has some *prescribed singularities*. In this article we will deal with the second problem.

Letting ω be a Kähler form with cohomology class $c_1(-K_X)$, since any KE metric corresponds to a function $u \in PSH(X, \omega) \cap C^\infty(X)$ such that

$$\text{Ric}(\omega + dd^c u) = \omega + dd^c u \quad (6.1)$$

where $d^c := \frac{i}{2\pi}(\bar{\partial} - \partial)$ so that $dd^c = \frac{1}{\pi}\partial\bar{\partial}$, the most natural extension to the prescribed singularities setting is to fix $\psi \in PSH(X, \omega)$ and to look for $u \in PSH(X, \omega)$ which satisfies (6.1) in a singular sense and *behaves* as ψ . We refer to section §6.4 for the precise definition of a *Kähler-Einstein metric with prescribed singularities* [ψ] ($[\psi]$ -KE metric), here we underline its characterization in terms of Monge-Ampère equations: by abuse of language $\omega + dd^c u$ is a $[\psi]$ -KE metric if and only if u solves

$$\begin{cases} MA_\omega(u) = e^{-u+C} \mu \\ u \in \mathcal{E}^1(X, \omega, \psi). \end{cases} \quad (6.2)$$

for $C \in \mathbb{R}$. The measure μ in (6.2) is the usual smooth volume form on X given as $\mu = e^{-\rho} \omega^n$ for ρ Ricci potential, $MA_\omega(u) := \langle (\omega + dd^c u)^n \rangle$ is the Monge-Ampère measure of u in terms of the *non-pluripolar product* (see [BEGZ10]) while $\mathcal{E}^1(X, \omega, \psi)$ is the set of all $u \in PSH(X, \omega)$ more singular than ψ , i.e. $u \leq \psi + C$ for $C \in \mathbb{R}$, such that the ψ -relative energy $E_\psi(u)$ is finite (see [DDNL18b], [Tru19], [Tru20a]). Note that the set $\mathcal{E}^1(X, \omega, \psi)$ contains all u such that $u - \psi$ is globally bounded. Recalling that $PSH(X, \omega)$ is naturally endowed with a partial order $u \preceq v$ if $u \leq v + C$, the following conditions on ψ are necessary to solve (6.2):

- i) $\psi = P_\omega[\psi] := \left(\sup\{u \in PSH(X, \omega) : u \preceq \psi, u \leq 0\} \right)^*$ where the star is for the upper semicontinuous regularization;
- ii) $V_\psi := \int_X MA_\omega(\psi) > 0$;
- iii) $\mathcal{I}(\psi) = \mathcal{O}_X$ where $\mathcal{I}(\psi)$ is the multiplier ideal sheaf attached to ψ .

The first condition means that ψ is a *model type envelope*, $\psi \in \mathcal{M}$ (it is shown to be necessary in [DDNL18b]), while we will say that (X, ψ) is *Kawamata Log Terminal (klt)* when (iii) holds. Note that this notion immediately extends to any quasi-psh function. With obvious notations we denote respectively with $\mathcal{M}^+, \mathcal{M}_{klt}^+$ the set of all model type envelopes which satisfies (ii), resp. (ii) and (iii). Thus \mathcal{M}_{klt}^+ can be

thought as the set of all admissible prescribed singularities and it is natural to define the *Kähler-Einstein locus* as

$$\mathcal{M}_{KE} := \{\psi \in \mathcal{M}_{klt}^+ : \text{there exists a } [\psi]\text{-KE metric}\}.$$

Then, observing also that a $[0]$ -KE metric is a genuine KE metric, it is natural to wonder the following questions.

Question A. *Let (X, ω) be a Fano manifold. Is it possible to characterize \mathcal{M}_{KE} ? When $\mathcal{M}_{KE} = \mathcal{M}_{klt}^+$? Is there some not-trivial locus on \mathcal{M}_{klt}^+ whose intersection with \mathcal{M}_{KE} implies that $0 \in \mathcal{M}_{KE}$?*

To start addressing Question A, we define a function $\mathcal{M} \ni \psi \rightarrow \alpha_\omega(\psi) \in (0, +\infty)$,

$$\alpha_\omega(\psi) := \sup \left\{ \alpha \geq 0 : \sup_{u \preceq \psi, \sup_X u = 0} \int_X e^{-\alpha u} d\mu < \infty \right\}. \quad (6.3)$$

which generalizes to the ψ -relative setting the classical Tian's α -invariant ([Tian87]), and we prove the following result.

Theorem A. *Let (X, ω) be a Fano manifold. Then*

$$\left\{ \psi \in \mathcal{M}_{klt}^+ : \alpha_\omega(\psi) > \frac{n}{n+1} \right\} \subset \mathcal{M}_{KE}.$$

Moreover (i) \Rightarrow (ii) \Rightarrow (iii) in the following conditions:

i) *there exists $\psi \in \mathcal{M}$, $t \in (0, 1]$ such that*

$$\alpha_\omega(\psi_t) > \frac{n}{(n+1)t};$$

for $\psi_t := P_\omega[(1-t)\psi]$;

ii) $\alpha_\omega(0) > \frac{n}{n+1}$;

iii) $\mathcal{M}_{KE} = \mathcal{M}_{klt}^+$.

Furthermore if $\psi \in \mathcal{M}_{klt}^+$ satisfies $\text{lct}(X, 0, \psi) := \sup \{p > 1 : (X, p\psi) \text{ is klt}\} \geq \frac{n^2+1}{n^2-n}$ then

$$\alpha_\omega(\psi) > \frac{n^2+1}{n+1} \implies 0 \in \mathcal{M}_{KE}. \quad (6.4)$$

We refer to section §6.5 for a sharper estimate in (6.4) which also holds for more general $\psi \in \mathcal{M}_{klt}^+$.

Let us stress that the advantage of the relative setting is that we can choose $\psi \in \mathcal{M}_{klt}^+$ and that the computation of the ψ -relative α -invariant is easier than the computation of the usual α -invariant as immediately follows from the definition. See for instance subsection §6.5.2 where, for $\psi \in \mathcal{M}_{klt}^+$ having isolated singularities at N

points, we produce lower bounds for the ψ -relative α -invariant in terms of *multi-point Seshadri constants* and *pseudoeffective thresholds*.

The assumption (ii) in Theorem A cannot be replaced with $\alpha_\omega(0) \geq \frac{n}{n+1}$ as $X = \mathbb{P}^2$ shows (Example 6.5.4), which suggests the following conjecture.

Conjecture A. *Let (X, ω) be a Fano manifold such that $\text{Aut}(X)^\circ = \{Id\}$. Then*

$$0 \in \mathcal{M}_{KE} \iff \mathcal{M}_{KE} = \mathcal{M}_{klt}^+.$$

With $\text{Aut}(X)^\circ$ we denoted the connected component of the identity map.

The upshot of Theorem A is that the value of the function $\alpha_\omega(\cdot)$ at singular model type envelopes may help to understand if X admits a genuine KE metric. Moreover the implication (ii) \Rightarrow (iii) in Theorem A (as Conjecture A) implies the existence of many log-KE metrics for weak log Fano pairs (Y, Δ) given by resolutions of integrally closed coherent analytic sheaves. Indeed among \mathcal{M}_{klt}^+ there are particular model type associated to analytic singularities. Namely we say that ψ has *analytic singularities type* if $\psi = P_\omega[\varphi]$ for $\varphi \in PSH(X, \omega)$ with analytic singularities formally encoded in (\mathcal{J}, c) . In this case, taking $p: Y \rightarrow X$ resolution of \mathcal{J} , the set of $[\psi]$ -KE metrics is in correspondence with the set of log-KE metrics for the log pair (Y, Δ) where $\Delta := cD - K_{Y/X}$ and $p^*\mathcal{J} = \mathcal{O}_Y(-D)$ (see [Tru20b]). Furthermore note that if the singularities are *algebraic* (i.e. $c \in \mathbb{Q}$) then

$$\alpha_\omega(\psi) \geq \min \left\{ 1, \alpha(Y, \Delta) \right\},$$

i.e. $\alpha_\omega(\psi)$ is a finer invariant than the usual log α -invariant $\alpha(Y, \Delta)$ (see Proposition 6.4.13 and Example 6.5.11).

In particular if D is a smooth divisor in $|-rK_X|$, $\varphi_D \in PSH(X, \omega)$ such that $\omega + dd^c \varphi_D = \frac{1}{r}[D]$ and $\psi_t := P_\omega[t\varphi_D]$ for any $t \in [0, 1]$, finding a $[\psi_t]$ -KE metric is equivalent to find a KE metric $\omega_{u_t} := \omega + dd^c u_t$ with conic singularities along D of angle $2\pi(1-t)/r$, i.e. $u_t \in PSH(X, \omega)$ locally bounded such that

$$\text{Ric}(\omega_{u_t}) = t\omega_{u_t} + \frac{(1-t)}{r}[D].$$

This is the path considered in [CDS15] to solve the Yau-Tian-Donaldson Conjecture and it is well-known that there exists $t_0 \in (0, 1]$ such that $\psi_t \in \mathcal{M}_{KE}$ for any $t \in (0, t_0)$ ([Berm13], [JMR16], see also Remark 6.4.4 for more details) and

$$\{\psi_t\}_{t \in (0, 1]} \subset \mathcal{M}_{KE} \iff 0 \in \mathcal{M}_{KE}. \quad (6.5)$$

Therefore condition (i) in Theorem A gives a valuative criterion to detect if the curve $\{\psi_t\}_{t \in (0, 1]}$ is entirely contained in \mathcal{M}_{KE} .

Since \mathcal{M}_{klt}^+ is a star domain with respect to $0 \in \mathcal{M}_{klt}^+$ (Lemma 6.4.1), it is then natural to wonder if it is possible to perform a continuity method for the weakly continuous curve $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}_{klt}^+$ given as $\psi_t = P_\omega[(1-t)\psi]$, $\psi \in \mathcal{M}_{klt}^+$.

In the companion paper [Tru20b] we introduced a continuity method with *movable singularities* based on the *strong topology* of ω -psh functions given as the coarsest refinement of the weak topology such that the energy $E(\cdot)$ becomes continuous ([Tru19], [Tru20a]). Thus, denoting with \mathcal{M}_D the subset of \mathcal{M} of all model type envelopes which are approximable by a decreasing sequence of model type envelopes with algebraic singularities (see section 6.2), we can state our next result.

Theorem B. *Let (X, ω) be a Fano manifold and let $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}_{klt}^+$ be a weakly continuous segment such that*

- i) $\psi_0 \in \mathcal{M}_{KE}$;
- ii) ψ_0 has small unbounded locus;
- iii) $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}_D$;
- iv) $\psi_t \preceq \psi_s$ if $t \leq s$;
- v) $\text{Aut}(X, [\psi_t])^\circ = \{Id\}$ for any $t \in [0, 1]$.

Then the set

$$S := \{t \in [0, 1] : \psi_t \in \mathcal{M}_{KE}\}$$

is open, the unique family of $[\psi_t]$ -KE currents $\{\omega_{u_t}\}_{t \in S}$ is weakly continuous and the family of potentials $\{u_t\}_{t \in S}$ can be chosen so that the curve $S \ni t \rightarrow u_t \in \mathcal{E}^1(X, \omega, \psi_t)$ is strongly continuous.

In Theorem B having *small unbounded locus* is a technical assumption which means locally boundedness on the complement of a closed complete pluripolar set, while $\text{Aut}(X, [\psi])^\circ := \text{Aut}(X)^\circ \cap \text{Aut}(X, [\psi])$ where $\text{Aut}(X, [\psi])$ is the set of all automorphisms $F : X \rightarrow X$ such that $F^*\psi - \psi$ is globally bounded and $\text{Aut}(X)^\circ$ is the connected component of the identity map. (v) is a necessary hypothesis for the uniqueness of $[\psi]$ -KE metrics as explained below in Theorem C.

The set \mathcal{M}_D contains plenty of model type envelopes, but in general $\mathcal{M}_D \subsetneq \mathcal{M}$ (see Example 6.3.6). Anyway it is worth to underline that if $\psi \in \mathcal{M}_D^+ := \mathcal{M}_D \cap \mathcal{M}^+$ then $\psi_t = (1-t)\psi \in \mathcal{M}_D^+$ for any $t \in [0, 1]$ (Proposition 6.3.7), thus Theorem B includes these particular paths discussed above.

To prove Theorems A, B we develop a variational approach to study the existence of $[\psi]$ -KE metrics for a fixed $\psi \in \mathcal{M}_{klt}^+$ similar to [BBGZ13], [DR17]. Namely we define two translation invariant functionals D_ψ , M_ψ , called respectively the ψ -relative Ding and Mabuchi functional, which generalize the well-known functionals to the ψ -relative setting as our next result shows.

Theorem C. *Let $\psi \in \mathcal{M}_{D,klt}^+ := \mathcal{M}_{klt}^+ \cap \mathcal{M}_D$ with small unbounded locus and let $u \in \mathcal{E}^1(X, \omega, \psi)$. Then the following statements are equivalent:*

- i) $\omega_u := \omega + dd^c u$ is a $[\psi]$ -KE metric;
- ii) $D_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi$;
- iii) $M_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} M_\psi$.

Moreover if ω_u is a $[\psi]$ -KE metric then u has ψ -relative minimal singularities (i.e. $u - \psi$ globally bounded) and if ω_v is another $[\psi]$ -KE metric then there exists $F \in \text{Aut}(X, [\psi])^\circ$ such that $F^* \omega_v = \omega_u$.

Next when $\text{Aut}(X, [\psi])^\circ = \{\text{Id}\}$, it is natural to wonder if the existence of the unique $[\psi]$ -KE metric is equivalent to the *coercivity* of the ψ -relative Ding and Mabuchi functionals similarly to the absolute setting. We recall that the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ is a metric topology given by a complete distance d which generalizes to the ψ -relative setting the distance introduced by T. Darvas ([Dar15]) as proved in our previous works [Tru19],[Tru20a].

Theorem D. *Let $\psi \in \mathcal{M}_{D, klt}^+$ with small unbounded locus. Assume also $\text{Aut}(X, [\psi])^\circ = \{\text{Id}\}$. Then the following conditions are equivalent:*

- i) *the ψ -relative Ding functional is d -coercive over $\mathcal{E}_{\text{norm}}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$;*
- ii) *the ψ -relative Mabuchi functional is d -coercive over $\mathcal{E}_{\text{norm}}^1(X, \omega, \psi)$;*
- iii) *there exists an unique $[\psi]$ -KE metric.*

6.1.1 About the assumptions.

Proving the linearity of the ψ -relative energy along weak geodesic segment for $\psi \in \mathcal{M}^+$, i.e. extends Theorem 6.3.11, would remove the assumption on \mathcal{M}_D in Theorems B, C and D.

Similarly, if the Berndtsson's convexity result (Theorem 6.4.17) holds for $\psi \in \mathcal{M}_{klt}^+$ then the hypothesis on the small unbounded locus in Theorems B, C and D would become unnecessary.

In Theorem B if we replace (v) with $\text{Aut}(X, [\psi_t])^\circ = \{\text{Id}\}$ for any $t \in [0, 1]$, which may be useful when $\text{Aut}(X)$ is not discrete, then the openness and the strong continuity result hold in $[0, 1]$. Anyway in this situation is unclear if it may happen that $S = [0, 1]$ but the family of KE metrics $\{\omega_{u_t}\}_{t \in [0, 1]}$ does not converge to a $[\psi_1]$ -KE metric. Indeed the *closedness* of the continuity method depends on an uniform bounds on the supremum of the potentials appropriately chosen (as in other more classical continuity methods), and in the proof of Theorem B the bound is basically a consequence of an uniform coercivity.

Finally note that on Theorem A there are no assumptions on \mathcal{M}_D and/or on small unbounded loci. Indeed this follows from the fact that the arrow $(i) \Rightarrow (iii)$ in Theorem D holds even when $\psi \in \mathcal{M}_{klt}^+$.

6.1.2 Related Works

During the last period of the preparation of this article, T. Darvas and M. Xia in ([DX20]) defined the same set \mathcal{M}_D , exploring deeply its properties and its relations with the algebraic approximations of geodesic rays in $(\mathcal{E}^1(X, \omega), d)$ where $\{\omega\} = c_1(L)$ for L ample line bundle.

6.1.3 Structure of the paper

In the next two sections we work with a general compact Kähler manifold (X, ω) , i.e. ω is not necessarily integral. In Section §6.2 we collect some preliminaries on model type envelopes and on the strong topologies, while in Section §6.3 we define the set \mathcal{M}_D , characterizing it through a version of the Demailly's regularization Theorem (Theorem 6.3.3). Moreover in the same section we prove the linearity of the Monge-Ampère energy $E_\psi(\cdot)$ along weak geodesic segments for $\psi \in \mathcal{M}_D^+$, showing also that $(\mathcal{E}^1(X, \omega, \psi), d)$ is a geodesic metric space.

In Section §6.4 we assume $\{\omega\} = c_1(X)$ and we develop the variational approach to study Kähler-Einstein metrics with prescribed singularities. We prove Theorems C and D. Furthermore, since the ψ -relative α -invariant is an important tool to show these two theorems, Subsection §6.4.1 is dedicated to define and explore some of the properties of the function $\mathcal{M} \ni \psi \rightarrow \alpha_\omega(\psi)$.

Finally Section §6.5 includes the proof of Theorems A, B.

6.1.4 Acknowledgments

I would like to thank my advisors Stefano Trapani and David Witt Nyström for their comments.

6.2 Preliminaries

Letting (X, ω) be a compact Kähler manifold endowed with a Kähler form ω , we denote with $PSH(X, \omega)$ the set of all ω -plurisubharmonic (ω -psh) functions u , i.e. all upper semicontinuous function $u \in L^1$ such that $\omega + dd^c u \geq 0$ in the sense of $(1, 1)$ -currents. Here $d^c := \frac{i}{2\pi}(\partial - \bar{\partial})$ so that $dd^c = \frac{i}{\pi}\partial\bar{\partial}$.

The maximum of two ω -psh functions u, v still belongs to $PSH(X, \omega)$ but $\min(u, v)$ may not be ω -psh. This is one reason to introduce the function

$$P_\omega(u, v) := \left(\sup\{w \in PSH(X, \omega) : w \leq \min(u, v)\} \right)^*$$

(the star is for the upper semicontinuous regularization), which is ω -psh. It is clearly the largest ω -psh function which is smaller than u, v . But sometimes we may want to find the largest function $w \in PSH(X, \omega)$ which is bounded above by $v \in PSH(X, \omega)$ and that is *more singular* than $u \in PSH(X, \omega)$, where w is more singular than u if

$w \leq u + C$ for a constant $C \in \mathbb{R}$ (we denote such partial order with \preceq). Thus we recall the following envelope ([RWN14]):

$$P_\omega[u](v) := \left(\lim_{C \rightarrow +\infty} P_\omega(u + C, v) \right)^*.$$

If now we take $v = 0$ we obtain a projection map $P_\omega[\cdot] := P_\omega[\cdot](0) : PSH(X, \omega) \rightarrow PSH(X, \omega)$. The image of this map is denoted with \mathcal{M} ([Tru19]) and its elements are called *model type envelopes*. It is an easy exercise to check that on \mathcal{M} the two partial orders \leq, \preceq coincides. The definition of the set \mathcal{M} is essential when one tries to solve complex Monge-Ampère equations with prescribed singularities ([DDNL18b]), i.e. equations as

$$\begin{cases} MA_\omega(u) = \nu \\ [u] = [\psi] \end{cases} \quad (6.6)$$

where we set $[u]$ for the equivalence class of $u \in PSH(X, \omega)$ under the partial order \preceq , i.e. $[u] = [\psi]$ for $\psi \in PSH(X, \omega)$ is equivalent to say $u - \psi$ uniformly bounded, ν is a measure on X and

$$MA_\omega(u) := \langle (\omega + dd^c u)^n \rangle$$

is n -times the non-pluripolar product of the closed and positive current $\omega + dd^c u$ (see [BEGZ10]). We also need to recall that the *total mass* of the Monge-Ampère operator well-behaves with respect to the partial order \preceq by [WN19], i.e.

$$u \preceq v \implies \int_X MA_\omega(u) \leq \int_X MA_\omega(v).$$

Given $\psi \in PSH(X, \omega)$, $\mathcal{E}(X, \omega, \psi) := \{u \preceq \psi : \int_X MA_\omega(u) = \int_X MA_\omega(\psi)\}$ is the set of all ω -psh functions with ψ -relative full mass.

Finally we underline that $PSH(X, \omega)$ is naturally endowed with a *weak* topology given by the inclusion $PSH(X, \omega) \subset L^1$ (i.e. the L^1 -topology), and that \mathcal{M} is weakly closed. Moreover, setting $\mathcal{M}^+ := \{\psi \in \mathcal{M} : V_\psi > 0\}$ and given a totally ordered family $\mathcal{A} := \{\psi_i\}_{i \in I} \subset \mathcal{M}^+$, the Monge-Ampère operator produces an homeomorphism between \mathcal{A} and its image endowed with the weak topology of measures (Lemma 3.12 in [Tru20a]).

6.2.1 Strong topologies

The Monge-Ampère operator may not be continuous with respect to the weak topology on $PSH(X, \omega)$. Here we recall briefly a strengthened of the weak topology for some particular subsets of $PSH(X, \omega)$ which is more efficient when one wants to study complex Monge-Ampère equations. See our previous works [Tru19], [Tru20a] and references therein.

Given $\psi \in \mathcal{M}$, the sets $\mathcal{E}^1(X, \omega, \psi) \subset PSH(X, \omega)$ and $\mathcal{M}^1(X, \omega, \psi) \subset \mathcal{P}(X) := \{\text{probability measures on } X\}$ are defined respectively as

$$\begin{aligned}\mathcal{E}^1(X, \omega, \psi) &:= \{u \in \mathcal{E}(X, \omega, \psi) : E_\psi(u) > -\infty\}, \\ \mathcal{M}^1(X, \omega, \psi) &:= \{V_\psi \nu : \nu \in \mathcal{P}(X) \text{ satisfies } E_\psi^*(\mu) < +\infty\}\end{aligned}$$

where E_ψ, E_ψ^* are the ψ -relative energies. More precisely

$$E_\psi(u) := \frac{1}{n+1} \sum_{j=0}^n \int_X (u - \psi) \langle (\omega + dd^c u)^j \wedge (\omega + dd^c \psi)^{n-j} \rangle$$

if u has ψ -relative minimal singularities, i.e. $[u] = [\psi]$, and as $E_\psi(u) := \lim_{k \rightarrow \infty} E_\psi(\max(u, \psi - k))$ otherwise. See [DDNL18b], [Tru19] for many of its properties, here we recall the following upper semicontinuity.

Proposition 6.2.1 ([Tru20a], Lemma 3.13, Propositions 3.14, 3.15). *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be a totally ordered sequence of model type envelopes, and let $\{u_k\}_{k \in \mathbb{N}} \subset PSH(X, \omega)$ such that $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ for any $k \in \mathbb{N}$. If $u_k \rightarrow u$ weakly. Then*

$$\limsup_{k \rightarrow \infty} E_{\psi_k}(u_k) \leq E_{P_\omega[u]}(u).$$

Moreover if $E_{\psi_k}(u_k) \geq -C$ uniformly, then $\psi_k \rightarrow P_\omega[u]$ weakly. In particular for any $C \in \mathbb{N}, \psi \in \mathcal{M}^+$ the set

$$\mathcal{E}_C^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u \leq C \text{ and } E_\psi(u) \geq -C\}$$

is weakly compact.

The ψ -relative energy E_ψ^* ([Tru20a]) is instead defined as

$$E_\psi^*(\nu) := \sup_{u \in \mathcal{E}^1(X, \omega, \psi)} \left(E_\psi(u) - V_\psi L_\nu(u) \right) \in [0, +\infty]$$

where $L_\nu(u) := \lim_{k \rightarrow \infty} \int_X (\max\{u, \psi - k\} - \psi) d\nu$ if ν does not charge $\{\psi = -\infty\}$ and as $L_\nu \equiv -\infty$ otherwise. We refer to [Tru20a] for its properties.

It is then natural to endow these sets with *strong topologies* given as the coarsest refinements of the weak topologies such that the ψ -relative energies become continuous. Then we have the following summarized result.

Theorem 6.2.2 ([Tru19], [Tru20a]). *Let $\psi \in \mathcal{M}^+$. Then:*

- i) *the strong topology on $\mathcal{E}^1(X, \omega, \psi)$ is a metric topology given by the complete distance $d(u, v) := E_\psi(u) + E_\psi(v) - 2E_\psi(P_\omega(u, v))$;*
- ii) *the Monge-Ampère operator $MA_\omega(\cdot)$ produces an homeomorphism*

$$MA_\omega : (\mathcal{E}_{norm}^1(X, \omega, \psi), d) \rightarrow (\mathcal{M}^1(X, \omega, \psi), \text{strong}) \quad (6.7)$$

where we set $\mathcal{E}_{norm}^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : \sup_X u = 0\}$;

iii) for any $V_\psi \nu = MA_\omega(u) \in \mathcal{M}^1(X, \omega, \psi)$ the equality $E_\psi^*(\nu) = E_\psi(u) - \int_X (u - \psi) MA_\omega(u)$ holds.

Furthermore it is possible to extend the strong topology of $\mathcal{E}^1(X, \omega, \psi)$ considering different model type envelopes. For $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ totally ordered set, we say that a sequence $\{u_k\}_{k \in \mathbb{N}}$ such that $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ converges strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$ weak limit of ψ_k if $u_k \rightarrow u$ weakly and $E_{\psi_k}(u_k) \rightarrow E_\psi(u)$.

Proposition 6.2.3 ([Tru20a]). *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}^+$ be a totally ordered sequence converging weakly to $\psi \in \mathcal{M}^+$, and let $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ be a sequence converging strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$. Then there exists a subsequence $\{u_{k_h}\}_{h \in \mathbb{N}}$ and two sequences $v_h \geq u_{k_h} \geq w_h$ of ω -psh functions such that $v_h \searrow u$, $w_h \nearrow u$ almost everywhere, and in particular $u_k \rightarrow u$ in capacity.*

6.2.2 Case with analytical singularities.

In this subsection we assume $\psi := P_\omega[\varphi] \in \mathcal{M}^+$ where $\varphi \in PSH(X, \omega)$ has *analytical singularities*, i.e. locally $\varphi|_U = g + c \log(|f_1|^2 + \dots + |f_k|^2)$ where $c \in \mathbb{R}_{\geq 0}$, $g \in C^\infty$, and $\{f_j\}_{j=1}^k$ are local holomorphic functions. The coherent ideal sheaf \mathcal{I} generated by these functions has integral closure globally defined, hence the singularities of φ are formally encoded in (\mathcal{I}, c) . We also recall that φ has ψ -relative minimal singularities (see Proposition 4.36 in [DDNL18b]).

It is well-known in this case that there exists a smooth resolution $p : Y \rightarrow X$ given by a sequence of blow-ups of smooth centers such that $p^*\mathcal{I} = \mathcal{O}_Y(-D)$ for an effective divisor D . Moreover the Siu Decomposition ([Siu74]) of $p^*(\omega + dd^c\varphi)$ is given by

$$p^*(\omega + dd^c\varphi) = \eta + c[D]$$

where η is a big and semipositive smooth $(1, 1)$ -form on Y . We also recall that it is possible to define the sets $\mathcal{E}(Y, \eta)$ and $\mathcal{E}^1(Y, \eta)$ similarly to the Kähler case (see [BEGZ10]) and that the latter becomes a complete metric space where endowed with the distance

$$d(u, v) := E(u) + E(v) - 2E(P_\eta(u, v)).$$

The quantities $P_\eta(\cdot, \cdot), E(\cdot)$ are defined in the same way as in the Kähler case.

Proposition 6.2.4 (Lemma 4.6, Proposition 4.7 in [Tru20b]). *The metric spaces $(\mathcal{E}^1(X, \omega, \psi), d)$, $(\mathcal{E}^1(Y, \eta), d)$ are isometric through the map $f : u \rightarrow \tilde{u} := (u - \varphi) \circ p$, and the two energies $E_\psi(\cdot)$ and $E(\cdot)$ respectively on $\mathcal{E}^1(X, \omega, \psi)$ and on $\mathcal{E}^1(Y, \eta)$ satisfy $E_\psi(u) - E_\psi(\varphi) = E(\tilde{u})$. Moreover f extends to a bijection $f : \{u \in PSH(X, \omega) : u \preceq \psi\} \rightarrow PSH(Y, \eta)$.*

6.3 Some particular model type envelopes.

In this section $\nu(u, x)$ will indicate the *Lelong number* of $u \in PSH(X, \omega)$ at $x \in X$, i.e., fixing an holomorphic chart $x \in U \subset X$,

$$\nu(u, x) := \sup\{\gamma \geq 0 : u(z) \leq \gamma \log \|z - x\|^2 + O(1) \text{ on } U\}.$$

We also recall that the *multiplier ideal sheaf* $\mathcal{I}(tu)$, $t \geq 0$, of $u \in PSH(X, \omega)$ is the analytic coherent and integrally closed ideal sheaf whose germs are given by

$$\mathcal{I}(tu, x) := \left\{ f \in \mathcal{O}_{X, x} \text{ such that } \int_V |f|^2 e^{-tu} MA_\omega(0) < \infty \text{ for some open set } x \in V \subset X \right\}.$$

From now on, we call *singularity data* associated to a function u the data given by all the Lelong numbers and all the germs of the multiplier ideal sheaves tu for $t > 0$.

Proposition 6.3.1 ([Tru20b], Proposition 3.9) . *Let $u \in PSH(X, \omega)$. Then u and $\psi := P_\omega[u]$ have the same singularity data, i.e.*

$$\nu(u, x) = \nu(\psi, x) \text{ and } \mathcal{I}(tu, x) = \mathcal{I}(t\psi, x) \text{ for any } t > 0, x \in X.$$

By Theorem 1.2. in [DDNL18b] if $P_\omega[u] = \psi$ then $u \in \mathcal{E}(X, \omega, \psi)$. The reverse arrow also holds when $\psi \in \mathcal{M}^+$ by Theorem 1.3 in [DDNL18b].

Definition 6.3.2. *We define the subset $\mathcal{M}_D \subset \mathcal{M}$ of all model type envelopes $\psi \in \mathcal{M}$ such that $\psi \succ \psi'$ for any $\psi' \in \mathcal{M}$ with the same singularity data of ψ .*

Observe that by definition \mathcal{M}_D includes any $\psi \in \mathcal{M}$ with analytical singularities type, i.e. $\psi = P_\omega[u]$ for u with analytical singularities. Indeed any other $\psi' \in \mathcal{M}$ with the same singularity data of ψ corresponds to a η -psh function for η as in subsection §6.2.2 and Proposition 6.2.4 gives the claim.

In the next subsection we will prove that for any $\psi \in \mathcal{M}$ there exists an unique $\psi' \in \mathcal{M}_D$ with the same singularity data of ψ (see Corollary 6.3.4).

6.3.1 A regularization process

The following key result is a consequence of the well-known Demailly's regularization Theorem ([Dem92]).

Theorem 6.3.3. *Let $\psi \in \mathcal{M}$. Then there exists a decreasing sequence $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ such that to any $u \in PSH(X, \omega)$ having the same singularity data of ψ can be associated a sequence $\{u_k\}_{k \in \mathbb{N}}$ with the following properties:*

- i) *for any $k \in \mathbb{N}$, $u_k \in \mathcal{E}(X, \omega, \psi_k)$, u_k has algebraic singularities and u_k has ψ_k -relative minimal singularities;*
- ii) *u_k converges to u in capacity.*

If $|u_1 - u_2|$ is bounded over X then $|u_{1,k} - u_{2,k}|$ is uniformly bounded over X .
Moreover $\psi_k \searrow \tilde{\psi} \in \mathcal{M}_D$ where $\tilde{\psi}$ has the same singularity data of ψ .

A function $u \in PSH(X, \omega)$ with analytic singularities formally encoded in (\mathcal{I}, c) is said to have *algebraic* singularities when $c \in \mathbb{Q}$.

Proof. Step 1: a Demailly's regularization.

As described in [Dem92], for $\{W_\nu\}_{\nu \in \Lambda}$ fixed finite covering of open coordinate sets, it is possible to choose a finite open covering $\{\Omega_j\}_{j \in J}$ of coordinate balls of radius 2δ (if δ is small enough) such that any Ω_j is contained in at least one W_ν and such that the set of all coordinate balls of radius δ produces another open covering $\{\Omega'_j\}_{j \in J}$. Then, letting $\epsilon(\delta)$ be a continuous function such that $\epsilon(\delta) \rightarrow 0$ for $\delta \rightarrow 0$ and such that $\omega_{x'} - \omega_x \leq \epsilon(\delta)\omega_x/2$ for all $x, x' \in \Omega_j$, it follows that $0 \leq -\omega - \tau_j^* \gamma_j \leq 2\epsilon(\delta)\omega$ on Ω_j where γ_j is a $(1, 1)$ -form with constant coefficients on $\tau_j(\Omega_j) = B_{2\delta}(a_j)$ such that $-\omega - \epsilon(\delta)\omega = \tau_j^* \gamma_j$ at $\tau_j^{-1}(a_j)$. We denote by $\tilde{\gamma}_j$ the homogeneous quadratic function in $z - a_j$ such that $dd^c \tilde{\gamma}_j = \gamma_j$. Thus for any $j \in J$, $m \in \mathbb{N}$ and $\phi \in PSH(X, \omega)$ we define locally on Ω_j

$$\hat{\varphi}_{j,m} := \frac{1}{m} \log \left(\sum_l |\sigma_{j,m,l}|^2 \right)$$

where $\{\sigma_{j,m,l}\}_{l \in \mathbb{N}}$ is an orthonormal base of the Hilbert space $\mathcal{H}_{\Omega_j}(m\tilde{\varphi}_j) := \{f \in \mathcal{O}_{\Omega_j}(\Omega_j) : \|f\|_{m\tilde{\varphi}_j, \Omega_j}^2 := \int_{\Omega_j} |f|^2 e^{-m\tilde{\varphi}_j} < \infty\}$ for $\tilde{\varphi}_j := \phi - \tilde{\gamma}_j \circ \tau_j$.

Moreover as proved in Theorem 2.2.1.(Step 3) in [DPS01] we also get

$$\hat{\varphi}_{j,m_1+m_2} \leq \frac{A_1}{m_1+m_2} + \frac{m_1}{m_1+m_2} \hat{\varphi}_{j,m_1} + \frac{m_2}{m_1+m_2} \hat{\varphi}_{j,m_2}$$

for any $m_1, m_2 \in \mathbb{N}$ where A_1 depends only on n . Therefore in the gluing process described in [Dem92] (in particular Lemma 3.5), considering $\varphi_{j,k} := \hat{\varphi}_{j,2^k} + \frac{A_1}{2^k}$ instead of $\hat{\varphi}_{j,2^k}$ we get (if $\delta = \delta_{2^k}$ goes to 0 very slowly) a decreasing sequence of almost psh function φ_k such that $\omega + dd^c \varphi_k \geq -\epsilon_k \omega$ for $\epsilon_k \searrow 0$. Next we claim that φ_k has *logarithmic poles* along $(\mathcal{I}(2^k \varphi), \frac{1}{2^k})$, i.e. locally $\varphi_k|_U = \frac{1}{2^k} \log(|f_1|^2 + \dots + |f_N|^2) + g$ where $\{f_j\}_{j=1}^N$ are local holomorphic functions which locally generates $\mathcal{I}(2^k \varphi)$ and where g is a bounded function. Indeed by the gluing process and Lemma 3.6 in [Dem92] for any k, j

$$\varphi_k = g_{j,k} + \frac{1}{2^k} \log \left(\sum_l |\sigma_{j,2^k,l}|^2 \right)$$

over Ω'_j where $g_{j,k}$ are bounded functions. Thus the claim now follows from [Nad89] (see also Proposition 5.7 in [Dem12]). Then by a regularization argument of Richberg ([Ric68], see also Lemma 2.15 in [Dem92]) we approximate φ_k with a smooth almost-psh function ϕ_k on $X \setminus V(\mathcal{I}(2^k \phi))$ such that $|\phi_k - \varphi_k| \leq 1/k$ and such that it extends to a almost-psh function on X with

$$\omega + dd^c \phi_k \geq -2\epsilon_k \omega.$$

Thus, since ϕ_k has the same singularity of φ_k , we get that ϕ_k has analytical singularities.

Step 2: the regularization for elements with fixed singularity data.

Next assuming u such that $\psi = P_\omega[u]$, we apply the regularization just described to the ω -psh function $\tilde{u} := u - \sup_X u - 1$, obtaining a sequence \tilde{u}_k . Then we define

$$u_k := \frac{1}{1 + 2\epsilon_k} \tilde{u}_k + \sup_X u + 1.$$

By construction $u_k \in PSH(X, \omega)$, u_k has algebraic singularities assuming without loss of generality that $\{\epsilon_k\}_{k \in \mathbb{N}} \in \mathbb{Q}$ and, as a consequence of Proposition 6.3.1, the singularity type $[u_k]$ is constant varying u which satisfies $P_\omega[u] = \psi$. Therefore defining $\psi_k := P_\omega[u_k]$ the first point follows.

About the convergence in capacity, clearly we may assume $\sup_X u = -1$. Then we denote with $v_k \in \mathcal{E}(X, \omega, \psi)$ the decreasing sequence of almost-psh function with logarithmic poles converging to u obtained by the process described above (i.e. the φ_k 's of before). By Hartogs' Lemma (see Proposition 8.4 in [GZ17]) $\sup_X v_k \rightarrow -1$ and it is immediate to check that $\frac{v_k}{1+2\epsilon_k}$ becomes a decreasing sequence converging to u when $\sup_X v_k \leq 0$. Thus we get that $\frac{v_k}{1+2\epsilon_k} \rightarrow u$ in capacity. Next we note that for any $\delta > 0$

$$\left\{ |u_k - u| \geq \delta \right\} \subset \left\{ \left| \frac{v_k}{1 + 2\epsilon_k} - u \right| \geq \delta - \frac{1}{k(1 + 2\epsilon_k)} \right\}$$

since $|\tilde{u}_k - v_k| \leq 1/k$ by construction. Hence taking $k = k_\delta \gg 0$ big enough we get that

$$\left\{ |u_k - u| \geq \delta \right\} \subset \left\{ \left| \frac{v_k}{1 + 2\epsilon_k} - u \right| \geq \frac{\delta}{2} \right\},$$

which implies that $u_k \rightarrow u$ in capacity.

Assuming $|u_1 - u_2| \leq C$, to prove that $u_{1,k} - u_{2,k}$ is uniformly bounded it is clearly enough to check that $|v_{1,k} - v_{2,k}|$ is uniformly bounded, where as before we denote with $v_{i,k}$ the sequence of almost-psh function with logarithmic poles which decreases to u_i for $i = 1, 2$ (i.e. in the process described above we replace $\phi, \varphi_k, \hat{\varphi}_{j,m}, \tilde{\varphi}_j, \varphi_{j,k}$ respectively with $u_i, v_{i,k}, \hat{v}_{i,j,m}, \tilde{v}_{i,j}, v_{i,j,k}$). Thus if $u_1 \leq u_2 + C$ and assuming without loss of generality that $\sup_X u_1 = \sup_X u_2 = -1$ then $\hat{v}_{1,j,2^k} \leq \hat{v}_{2,j,2^k} + C$ for any $j \in J$ and any $k \in \mathbb{N}$ since

$$\hat{v}_{1,j,2^k} = \sup_{f \in B(1)} \frac{2}{2^k} \log |f|$$

where $B(1)$ is the unit ball in $\mathcal{H}_{\Omega_j}(2^k \tilde{v}_{1,j})$ and similarly for u_2 . Hence we get that $|v_{1,j,k} - v_{2,j,k}|$ is uniformly bounded in j, k and by the gluing process described in [Dem92] it follows that also $|v_{1,k} - v_{2,k}|$ is uniformly bounded in k .

Step 3: the singularity data of ψ .

For this last step, we first observe that clearly $\psi_k \succcurlyeq \psi_{k+1}$ which is equivalent to

$\psi_k \geq \psi_{k+1}$. Thus $\tilde{\psi} := \lim_{k \rightarrow \infty} \psi_k \in \mathcal{M}$ and $\tilde{\psi} \geq \psi$ by construction since $\psi_k \geq \psi$ for any $k \in \mathbb{N}$. Moreover, letting $u \in \mathcal{E}(X, \omega, \psi)$ fixed, by the estimates in [Dem92]

$$\nu(\psi_k, x) = \nu(u_k, x) \rightarrow \nu(u, x) = \nu(\psi, x)$$

for any $x \in X$, which implies $\nu(\tilde{\psi}, x) \geq \lim_{k \rightarrow \infty} \nu(\psi_k, x) = \nu(\psi, x) \geq \nu(\tilde{\psi}, x)$ since $\psi \preceq \tilde{\psi} \preceq \psi_k$ for any $k \in \mathbb{N}$. Hence $\nu(\tilde{\psi}, x) = \nu(\psi, x)$ for any $x \in X$.

Next fix $t > 0$, and set $u := \psi - 1$. Since $\tilde{\psi} \geq \psi$ we immediately have $\mathcal{I}(t\psi) \subset \mathcal{I}(t\tilde{\psi})$. Viceversa we claim that

$$\mathcal{I}(t\psi) \supset \mathcal{I}((1 + \tau_k)t\tilde{u}_k) = \mathcal{I}((1 + \tau_k)(1 + 2\epsilon_k)tu_k) \quad (6.8)$$

for $\tau_k = \frac{t}{2^k - t}$ if $k \gg 1$ such that $2^k > t$ where $\tilde{u}_k = (1 + 2\epsilon_k)u_k$ is the almost psh function with analytic singularities formally encoded in $(\mathcal{I}(2^k\psi), \frac{1}{2^k})$ constructed in Step 1. The inclusion in (6.8) would imply

$$\mathcal{I}(t\psi) \supset \mathcal{I}((1 + \tau_k)(1 + 2\epsilon_k)t\tilde{\psi})$$

since $u_k \succcurlyeq \tilde{\psi}$ for any $k \in \mathbb{N}$. Thus since by the resolution of the strong openness conjecture (see [GZ15]) $\mathcal{I}((1 + \epsilon)t\tilde{\psi}) = \mathcal{I}(t\tilde{\psi})$ if $0 < \epsilon \ll 1$ small enough, we would get

$$\mathcal{I}(t\psi) \supset \mathcal{I}(t\tilde{\psi})$$

letting $k \rightarrow \infty$. Hence $\mathcal{I}(t\psi) = \mathcal{I}(t\tilde{\psi})$.

To prove the inclusion in (6.8) we first note that for any $U \subset X$ open set and any holomorphic function f over U , we have

$$\begin{aligned} \int_U |f|^2 e^{-tu} \omega^n &= \\ &= \int_{U \cap \{u \geq (1+\tau_k)\tilde{u}_k\}} |f|^2 e^{-tu} \omega^n + \int_{U \cap \{u < (1+\tau_k)\tilde{u}_k\}} |f|^2 e^{2^k(\tilde{u}_k - u)} e^{(2^k - t)u - 2^k\tilde{u}_k} \omega^n \leq \\ &\leq \int_U |f|^2 e^{-(1+\tau_k)t\tilde{u}_k} \omega^n + \int_U e^{2^k(\tilde{u}_k - u)} \omega^n \end{aligned}$$

since $(2^k - t)u - 2^k\tilde{u}_k < 0$ over $\{u < (1 + \tau_k)\tilde{u}_k\}$ by the choice of $\tau_k = \frac{t}{2^k - t}$. Moreover $e^{2^k(\tilde{u}_k - u)} \in L^1_{loc}$ since \tilde{u}_k has analytic singularities formally encoded in $(\mathcal{I}(2^k u), \frac{1}{2^k})$. Therefore the inclusion in (6.8) follows.

Finally since by construction $\tilde{\psi} \geq \psi'$ for any $\psi' \in \mathcal{M}$ with the same singularity data of ψ (simply switching ψ with ψ'), $\tilde{\psi}$ is a maximal element in \mathcal{M} for fixed singularity data, i.e. $\tilde{\psi} \in \mathcal{M}_D$ which concludes the proof. \square

We say that $\psi \in \mathcal{M}$ has *analytic (resp. algebraic) singularities type* if $\psi = P_\omega[\varphi]$ for $\varphi \in PSH(X, \omega)$ with analytic (resp. algebraic) singularities.

Corollary 6.3.4. *For any $\psi \in \mathcal{M}$ there exists a unique $\psi' \in \mathcal{M}_D$ having the same singularity data of ψ . Moreover if $\psi_1, \psi_2 \in \mathcal{M}_D$ and the singularity data of ψ_1 are worse than the singularity data of ψ_2 (i.e. $\nu(\psi_1, x) \geq \nu(\psi_2, x)$ and $\mathcal{I}(t\psi_1, x) \subset \mathcal{I}(t\psi_2, x)$ for any $x \in X, t > 0$), then $\psi_1 \preceq \psi_2$.*

Proof. The first statement is a trivial consequence of Theorem 6.3.3.

Next if $\psi_{1,k}, \psi_{2,k} \in \mathcal{M}_D$ are the sequences with algebraic singularities type converging respectively to ψ_1, ψ_2 given by Theorem 6.3.3 with respect to the same Demailly's regularization, then we have $\psi_{1,k} \leq \psi_{2,k}$ if the singularity data of ψ_1 are worse than the singularity data of ψ_2 , which concludes the proof. \square

Theorem 6.3.3 implies that the elements in \mathcal{M}_D can be approximated by a decreasing sequence of model type envelopes with algebraic singularities type. This property defines the set \mathcal{M}_D as immediate consequence of the following result.

Proposition 6.3.5. *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}_D$ be a decreasing sequence converging to ψ . Then $\psi \in \mathcal{M}_D$.*

Proof. Let $\psi' \in \mathcal{M}_D$ having the same singularity data of ψ . Then by Corollary 6.3.4 $\psi_k \succcurlyeq \psi'$ for any $k \in \mathbb{N}$, which is equivalent to $\psi_k \geq \psi'$ since we are considering model type envelopes. Hence $\psi \geq \psi'$, which implies $\psi = \psi'$ and concludes the proof. \square

The following example shows that \mathcal{M}_D is a proper subset of \mathcal{M} .

Example 6.3.6. Let $K \subset \mathbb{P}^1$ be a polar Cantor set, $\omega = \omega_{FS}$ be the Fubini-Study metric on \mathbb{P}^1 , and μ_K be the measure on \mathbb{C} associated to K . Then the potential $u(z) := \int_{\mathbb{C}} \log|z - w| d\mu_K(w)$ is a subharmonic function on \mathbb{C} , harmonic on $\mathbb{C} \setminus \text{Supp}(\mu_K) = \mathbb{C} \setminus K$ and $u(z) = \mu_K(\mathbb{C}) \log|z| + O(|z|^{-1})$ as $z \rightarrow \infty$ (see Theorem 3.1.2 in [Rans]). Thus, up to rescaling the Fubini-Study metric, u extends to an ω_{FS} -psh function, i.e. $u \in PSH(\mathbb{P}^1, \omega_{FS})$. Moreover since μ_K has no atoms, $\nu(u, z) = 0$ for any $z \in \mathbb{P}^1$, which by Skoda's Integrability Theorem ([Sko72], see also Theorem 6.4.10 below) implies that u has trivial singularity data. Therefore by Proposition 6.3.1, the family of model type envelopes $\{\psi_t := P_\omega[tu]\}_{t \in [0,1]} \subset \mathcal{M}$ has constant singularity data, but $V_{\psi_t} = \int_X MA_\omega(tu) = (1-t) \int_X \omega + t \int_X MA_\omega(u) = (1-t) \int_X \omega$ since $MA_\omega(u)$ is concentrated on K which is polar. Hence clearly $\{\psi_t\}_{t \in (0,1]} \subset \mathcal{M} \setminus \mathcal{M}_D$.

Finally it is remarkable to observe that \mathcal{M}_D^+ is a star domain with respect to $0 \in \mathcal{M}_D^+$ as our next result shows.

Proposition 6.3.7. *Let $\psi \in \mathcal{M}_D^+$ and $t \in [0, 1]$. Then $t\psi \in \mathcal{M}_D^+$.*

Proof. Define $\psi_t := P_\omega[t\psi] \in \mathcal{M}^+$ for any $t \in [0, 1]$. We want to prove that $\psi_t \in \mathcal{M}_D$ and that $\psi_t = t\psi$.

Since $\psi \in \mathcal{M}_D$ by Theorem 6.3.3 there exists a decreasing sequence $\{\psi_k = P_\omega[\varphi_k]\}_{k \in \mathbb{N}} \subset$

\mathcal{M}_D^+ of model type envelopes with algebraic singularity type converging to ψ . We indicated with φ_k the ω -psh functions with algebraic singularities. Then, for any $t \in [0, 1)$ the sequence $\{\psi_{k,t} := P_\omega[t\varphi_k]\}$ is clearly a decreasing sequence with analytic singularity type, which implies that $\psi'_t := \lim_{k \rightarrow \infty} \psi_{k,t} \in \mathcal{M}_D$ by Proposition 6.3.5. Moreover since φ_k has ψ_k -relative minimal singularities we have $\psi_{k,t} = P_\omega[t\psi_k]$, and by construction

$$V_{\psi_{k,t}} = \int_X MA_\omega(t\psi_k) = \sum_{j=0}^n t^{n-j} (1-t)^j \int_X \langle \omega^j \wedge (\omega + dd^c \psi_k)^{n-j} \rangle.$$

Observe also that $V_{\psi_k} \searrow V_\psi$ by what said in section §6.2 since ψ_k are model type envelopes decreasing to ψ . More generally we have

$$\int_X \langle \omega^j \wedge (\omega + dd^c \psi_k)^{n-j} \rangle \rightarrow \int_X \langle \omega^j \wedge (\omega + dd^c \psi)^{n-j} \rangle$$

for any $j = 0, \dots, n$ by Proposition 4.8 in [DDNL19] since we are assuming $V_\psi > 0$. Hence $V_{\psi_{k,t}} \rightarrow V_{\psi'_t} = V_{\psi_t} > 0$, which implies that $\psi_t = \psi'_t$ by Theorem 1.3. in [DDNL18b] since by construction ψ_t is more singular than ψ'_t , i.e. $\psi_t \in \mathcal{M}_D^+$ for any $t \in [0, 1]$.

Next since for any $k \in \mathbb{N}$, $t\varphi_k$ has $\psi_{k,t}$ -relative minimal singularities, we get that $\psi_{k,t} + (1-t)\psi_k$ is more singular than ψ_k , i.e.

$$\psi_{k,t} \leq t\psi_k.$$

Letting $k \rightarrow \infty$ we obtain $\psi_t \leq t\psi$, which implies $\psi_t = t\psi$ and concludes the proof. \square

6.3.2 Geodesic segments.

Definition 6.3.8. Let $S := \{t \in \mathbb{C} : 0 < \operatorname{Re} t < 1\}$ be the open strip and let $u_0, u_1 \in \operatorname{PSH}(X, \omega)$. The elements $U' \in \operatorname{PSH}(X \times S, \pi_X^* \omega)$ such that $\limsup_{t \rightarrow 0^+} U'(\cdot, t) \leq u_0$ and $\limsup_{t \rightarrow 1^-} U'(\cdot, t) \leq u_1$ are called weak subgeodesics of u_0, u_1 , and if there exists at least one of these subgeodesics then the $\pi_X^* \omega$ -psh function

$$u_t(p) := U(p, t) := \left(\sup \{U' \in \operatorname{PSH}(X \times S, \pi_X^* \omega) : U' \text{ subgeodesic of } u_0, u_1\} \right)^*$$

is called weak geodesic joining u_0, u_1 .

The next Proposition explores the properties of weak geodesics segments joining potentials in $\mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}$. We denote with $\mathcal{H}_\omega := \{u \in \operatorname{PSH}(X, \omega) : \omega + dd^c u \text{ is Kähler}\}$ the set of all Kähler potentials.

Proposition 6.3.9. Let $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$ for $\psi \in \mathcal{M}^+$. Then the followings holds:

- i) there exists the weak geodesic $u_t(p) = U(t, p) \in PSH(X \times S, \pi_X^* \omega)$ and it only depends on $\operatorname{Re} t$ in the t -variable;
- ii) $u_t \in \mathcal{E}^1(X, \omega, \psi)$ for any $t \in [0, 1]$;
- iii) letting $\{u_0^k\}_{k \in \mathbb{N}}, \{u_1^k\}_{k \in \mathbb{N}} \subset \mathcal{H}_\omega$ be decreasing sequences such that $u_0^k \searrow u_0, u_1^k \searrow u_1$ and letting u_t^k the weak geodesic joining u_0^k, u_1^k , the convergence $u_t^k \searrow u_t$ holds.

Moreover if u_0, u_1 have ψ -relative minimal singularities, then

- iv) $u_t \rightarrow u_0, u_t \rightarrow u_1$ in capacity;
- v) u_t has ψ -relative minimal singularities for any $t \in S$;
- vi) $|u_t - u_s| \leq C|\operatorname{Re} t - \operatorname{Re} s|$ for any $t, s \in S$ where $C := \|u_1 - u_0\|_\infty$.

The existence of the approximations of point (iii) is contained in [BK07] while the existence of the weak geodesics joining elements in \mathcal{H}_ω is shown in [Chen00].

Proof. By Proposition 2.10 in [Tru19] $P_\omega(u_0, u_1) \in \mathcal{E}^1(X, \omega, \psi)$, thus (i) and (iii) follows directly from Theorem 5.(i) in [Dar17]. Then by the $\operatorname{Re} t$ -convexity we obtain $P_\omega(u_0, u_1) \leq u_t \leq (\operatorname{Re} t)u_1 + (1 - \operatorname{Re} t)u_0$, hence (ii) is given by the monotonicity of E_ψ .

Next, assuming u_0, u_1 with ψ -relative minimal singularities, (iv) is a consequence of the second part of Theorem 5 in [Dar17] since by definition it is immediate to check that $P_\omega[u_0](u_1) = u_1$ and similarly by symmetry. Then, letting $C := \|u_0 - u_1\|_{L^\infty}$, from

$$\max \{u_0 - C\operatorname{Re} t, u_1 + C(\operatorname{Re} t - 1)\} \leq u_t \leq (\operatorname{Re} t)u_1 + (1 - \operatorname{Re} t)u_0$$

we obtain that $\|u_0 - u_t\|_{L^\infty} \leq C$, which implies that u_t has uniformly bounded in ψ -relative minimal singularities and in particular (v) follows. Moreover (i) and the $\operatorname{Re} t$ -convexity of u_t yield that the t -derivative of u_t is increasing. On the other hand the inequality

$$\max \{-C\operatorname{Re} t, u_1 - u_0 + C(\operatorname{Re} t - 1)\} \leq u_t - u_0$$

implies that the one-side derivative at 0 of u_t lie between $-C$ and C . Similarly for the one-side derivative at 1 of u_t . Hence it follows that all the t -derivatives are bounded between $-C, C$, which gives (vi) concluding the proof. \square

Since the weak geodesic joining two elements $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$ depends only on $\operatorname{Re} t$ in the t -variable, with weak geodesic segment we will mean the path $[0, 1] \ni t \rightarrow u_t$.

When $\psi \in \mathcal{M}^+$ has algebraic singularities type it is possible to relate the weak geodesics in $\mathcal{E}^1(X, \omega, \psi)$ in terms of the weak geodesics in $\mathcal{E}^1(Y, \eta)$, keeping the same notation of subsection 6.2.2.

Proposition 6.3.10. *Let $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$ and let $\tilde{u}_0 = (u_0 - \varphi) \circ p, \tilde{u}_1 = (u_1 - \varphi) \circ p \in \mathcal{E}^1(Y, \eta)$. Then the weak geodesic U joining u_0, u_1 is given by*

$$U = (p \times Id)_* \tilde{U} + \Phi$$

where \tilde{U} is the weak geodesic joining \tilde{u}_0, \tilde{u}_1 and $\Phi \in PSH(X \times S, \pi_X^* \omega)$ is the constant weak geodesic at φ , i.e. $\Phi(\cdot, t) = \varphi(\cdot)$ for any $t \in S$

Proof. By Proposition 6.3.9 there exists the geodesic U joining u_0, u_1 and $u_t \in \mathcal{E}^1(X, \omega, \psi)$ for any t . Then by Proposition 6.2.4 the function

$$\tilde{U} := (U - \Phi) \circ (p \times Id)$$

satisfies $\tilde{u}_t \in \mathcal{E}^1(Y, \omega)$ for any $t \in S$. Moreover it depends only on $\operatorname{Re} t$ on the t -variable, and it is not difficult to check that it is upper semicontinuous and regular enough to consider the $(1, 1)$ -current $\pi_Y^* \eta + dd_{w,t}^c \tilde{U}$, which satisfies

$$(p \times Id)^* (\pi_X^* \omega + dd_{z,t}^c U) = \pi_Y^* \eta + dd_{w,t}^c \tilde{U} + c\pi_Y^* [D] \quad (6.9)$$

where we are using the same notations of subsection 6.2.2. Therefore since $\pi_Y^* \eta + dd_{w,t}^c \tilde{U}$ is positive on each fiber and since $U \in PSH(X \times S, \pi_X^* \omega)$ from (6.9) we get that $\pi_Y^* \eta + dd_{w,t}^c \tilde{U} \geq 0$, i.e. \tilde{U} is a weak subgeodesic joining \tilde{u}_0, \tilde{u}_1 .

On the other hand letting $\tilde{V} \in PSH(Y \times S, \pi_Y^* \eta)$ be the weak geodesic joining \tilde{u}_0, \tilde{u}_1 , we obtain that

$$V := (p \times Id)_* \tilde{V} + \Phi$$

is a weak subgeodesic joining u_0, u_1 from the equality (6.9) for V, \tilde{V} . Moreover $V \geq U$ by construction since \tilde{V} is the weak geodesic, which implies that $V = U$. Hence $\tilde{V} = \tilde{U}$, i.e. \tilde{U} is the weak geodesic joining \tilde{u}_0, \tilde{u}_1 and the proof is concluded. \square

The reason of considering $\psi \in \mathcal{M}_D^+$ in Theorems C, D is because we can prove that the space $(\mathcal{E}^1(X, \omega, \psi), d)$ is *geodesic* showing also that any weak geodesic is a metric geodesic. Moreover along these geodesics the ψ -relative energy becomes linear.

Theorem 6.3.11. *Let $\psi \in \mathcal{M}_D^+$ and let U be the weak geodesic joining $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$. Then E_ψ is linear along $[0, 1] \ni t \rightarrow u_t := U(t, \cdot) \in \mathcal{E}^1(X, \omega, \psi)$ which is also a geodesic segment in $(\mathcal{E}^1(X, \omega, \psi), d)$, i.e.*

$$d(u_t, u_s) = |t - s|d(u_0, u_1).$$

Proof. We set $u_{0,k} := \max(u_0, \psi - k), u_{1,k} := \max(u_1, \psi - k)$ observing that by construction the sequence of weak geodesic segments U_k joining $u_{0,k}, u_{1,k}$ decreases to U (see Proposition 6.3.9). In particular since the ψ -relative energy E_ψ and the distance d are continuous along decreasing sequences in $\mathcal{E}^1(X, \omega, \psi)$ we may assume that u_0, u_1 have ψ -relative minimal singularities.

Moreover if $\psi = P_\omega[\varphi]$ for φ with analytical singularities, then the results required follow combining Proposition 6.2.4 with Proposition 6.3.10. Indeed, keeping the

same notation of subsection 6.2.2, by Theorem 3.12 in [DDNL18a] the energy $E(\cdot)$ for $\mathcal{E}^1(Y, \eta)$ is linear along weak geodesic segments, which are metric geodesic in $(\mathcal{E}^1(X, \eta), d)$ by Proposition 3.13 in [DDNL18c].

For general $\psi \in \mathcal{M}_D^+$, by Theorem 6.3.3 there exist $\{\psi_k\} \in \mathcal{M}_D^+$, decreasing sequence converging to ψ of model type envelopes with algebraic singularity type, and $u_0^k, u_1^k \in \mathcal{E}^1(X, \omega, \psi_k)$ two sequences decreasing respectively to u_0, u_1 . Observe that $\|u_0^k - u_1^k\|_{L^\infty}$ is uniformly bounded since we are assuming u_0, u_1 with ψ -relative minimal singularities. Moreover by the first part of the proof the weak geodesic segment $[0, 1] \ni t \rightarrow u_t^k \in \mathcal{E}^1(X, \omega, \psi_k)$ joining u_0^k, u_1^k is a metric geodesic in $(\mathcal{E}^1(X, \omega, \psi_k), d)$ and E_{ψ_k} is linear along it. Furthermore for any $t, s \in [0, 1]$, $\|u_t^k - u_s^k\|_{L^\infty} \leq C$ for an uniform constant C and u_t^k decreases to u_t by Proposition 6.3.9 as $k \rightarrow \infty$. Hence the results required follow from the convergences

$$\begin{aligned} d(u_t^k, u_s^k) &\rightarrow d(u_t, u_s), \\ E_{\psi_k}(u_t^k) &\rightarrow E_\psi(u_t) \end{aligned}$$

as $k \rightarrow \infty$, given by Lemma 4.3 in [Tru19]. □

6.4 $[\psi]$ -KE metrics with prescribed singularities.

From now on we will assume $\{\omega\} = c_1(X)$, i.e. X is a Fano manifold and ω is a Kähler form in the anticanonical class.

With \mathcal{M}_{klt} we denote the set of all the model type envelopes ψ such that (X, ψ) is *klt*, i.e. as said in the Introduction $\mathcal{J}(\psi) = \mathcal{O}_X$. Note that by the resolution of the openness conjecture ([GZ15]) $\psi \in \mathcal{M}_{klt}$ if and only if there exists $p > 1$ such that $e^{-\psi} \in L^p$. Moreover for a pair (X, ψ) being *klt* is independent on the Kähler form chosen, i.e. it holds for quasi-psh functions. We will also use the notation $\mathcal{M}_{klt}^+ := \mathcal{M}_{klt} \cap \mathcal{M}^+$ and similarly for $\mathcal{M}_{D, klt}^+$.

Proposition 6.4.1. \mathcal{M}^+ and \mathcal{M}_{klt}^+ are star domains with respect to 0 as subset of $PSH(X, \omega)$.

Proof. It is clearly enough to prove the result for \mathcal{M}^+ . Letting $\psi \in \mathcal{M}^+$, we define $[0, 1] \ni t \rightarrow \psi_t := P_\omega[t\psi]$, and we want to prove its weakly continuity. Thus, letting $t_k \nearrow \bar{t} \in [0, 1]$ (resp. $t_k \searrow \bar{t} \in [0, 1]$), we observe that the sequence ψ_{t_k} converges weakly and monotonically to a model type envelope $\psi'_{\bar{t}}$ which is more singular (resp. less singular) than $\psi_{\bar{t}}$. But by construction it follows that

$$V_{\psi_{t_k}} = \int_X t_k^{n-j} (1 - t_k)^j \langle \omega^j \wedge \omega_{\psi}^j \rangle \rightarrow V_{\psi_{\bar{t}}},$$

which by Theorem 1.3 in [DDNL18b] implies that $\psi'_{\bar{t}} = \psi_{\bar{t}}$ since we are assuming $V_\psi > 0$. □

Recall that a positive measure μ on X is said to have *well-defined Ricci curvature* if it corresponds to a singular metric on $-K_X$, i.e. locally

$$\mu = e^{-f} i^{n^2} \Omega \wedge \bar{\Omega}$$

where $f \in L^2_{loc}$ and Ω is a nowhere zero local holomorphic section of $-K_X$, and in such case $Ric(\mu) := dd^c f$ (see [Berm16], [BBJ15]). We also set $Ric(\omega + dd^c u) := Ric(MA_\omega(u))$ for any $u \in PSH(X, \omega)$ so that it coincides with the usual definition of the Ricci curvature when $u \in C^\infty$.

Definition 6.4.2 ([Tru20b]). *Let $u \in PSH(X, \omega)$. Then $\omega_u := \omega + dd^c u$ is said to be a Kähler-Einstein metric with prescribed singularities $[\psi]$ ($[\psi]$ -KE metric) if it has well-defined Ricci curvature,*

$$Ric(\omega_u) = \omega_u \tag{6.10}$$

and $u \in \mathcal{E}^1(X, \omega, \psi)$.

The abuse of language comes from the fact that actually ω_u is the curvature of a (class of) singular metric on $-K_X$ which is Kähler-Einstein metric in the weak sense of (6.10).

Similarly to the absolute case $\psi = 0$, ω_u is a $[\psi]$ -KE metric if and only u solves the complex Monge-Ampère equation

$$\begin{cases} MA_\omega(u) = e^{-u+C} \mu \\ u \in \mathcal{E}^1(X, \omega, \psi) \end{cases} \tag{6.11}$$

for $C \in \mathbb{R}$ where μ is a suitable volume form on X such that $Ric(\mu) = \omega$ (Lemma 4.3 in [Tru20b]), i.e. $\mu = e^{-f} \omega^n$ for f Ricci potential. Note that combining the resolution of the openness conjecture, Proposition 6.3.1 and Theorem A in [DDNL18d], any solution u of (6.11) has ψ -relative minimal singularities.

Definition 6.4.3. *We define the Kähler-Einstein (KE) locus of \mathcal{M} as*

$$\mathcal{M}_{KE} := \{\psi \in \mathcal{M}^+ : \text{there exists a } [\psi]\text{-KE metric}\}.$$

Clearly $\mathcal{M}_{KE} \subset \mathcal{M}_{klt}^+$ since the assumption (X, ψ) klt is necessary to the existence of a solution of (6.11).

Remark 6.4.4. \mathcal{M}_{KE} is not empty. Indeed letting D smooth divisor in $| -rK_X |$ for $r \in \mathbb{N}$, and letting $\varphi_D \in PSH(X, \omega)$ such that $\omega + dd^c \varphi_D = \frac{1}{r}[D]$, then finding a $[\psi_t]$ -KE metric for $\psi_t := P_\omega[t\varphi_D]$ and $t \in [0, 1)$ is equivalent to solve

$$Ric(\eta_v) = \eta_v + \frac{t}{r}[D]$$

where $\eta = (1 - t)\omega$. Thus rescaling we get the renowned path

$$\text{Ric}(\omega_w) = (1 - t)\omega_w + \frac{t}{r}[D] \quad (6.12)$$

used for instance in [CDS15]. It is then well-known ([Berm13], [JMR16]) that (6.12) admits a solution for $0 \ll t < 1$ close to 1. Hence there exists a $[\psi_t]$ -KE metric for $0 \ll t < 1$ close to 1.

The set of all $[\psi]$ -KE metrics varying $\psi \in \mathcal{M}_{klt}^+$ includes all possible log KE metrics with respect to (X, D) where D varies among all effective klt \mathbb{R} -divisors such that $-(K_X + D)$ is ample. But clearly the set of all $\psi \in \mathcal{M}_{klt}^+$ with analytic singularities type is much bigger than the one associated to pairs (X, D) as above. However, considering a resolution of the ideal \mathcal{I} associated to ψ , it is still possible to describe the set of all $[\psi]$ -KE metrics in a more classical way.

Proposition 6.4.5 ([Tru20b], Proposition 4.8 and Theorem 4.9). *Let $\psi = P_\omega[\varphi] \in \mathcal{M}_{klt}^+$ with analytic singularities type formally encoded in (\mathcal{I}, c) . Then any $[\psi]$ -KE metric is smooth outside $V(\mathcal{I})$. Moreover, letting $p : Y \rightarrow X$ be a resolution of the ideal \mathcal{I} and letting η be the big and semipositive $(1, 1)$ -form such that $p^*(\omega + dd^c\varphi) = \eta + c[D]$, the set of all $[\psi]$ -KE metrics is in bijection with the set of all log-KE metrics in the class $\{\eta\}$ with respect to the weak log Fano pair (Y, Δ) for $\Delta := cD - K_{Y/X}$.*

In Proposition 6.4.5 with $K_{Y/X}$ we indicate the *relative canonical divisor* of $p : Y \rightarrow X$, i.e. $K_{Y/X} = K_Y - p^*K_X$. Note that the divisor $\Delta = cD - K_{Y/X}$ is neither necessarily effective nor necessarily antieffective. Indeed when $p = Id$ as described above Δ is clearly effective, while considering $\psi = P_\omega[\varphi]$ for φ with analytic singularities along one point $x \in X$ such that $\delta := \nu(\varphi, x) < n - 1$ it follows that, for $p = Bl_x X : Y \rightarrow X$, $\Delta = -(n - 1 - \delta)E$ where E is the exceptional divisor. Observe that when η is Kähler and Δ is effective then any log KE metric in the class $\{\eta\}$ has *conic singularities* along D on its simple normal crossing locus by Theorem 6.2 in [GP16].

6.4.1 ψ -relative alpha invariant.

We introduce the key concept of ψ -relative α -invariant which generalizes to the relative setting the renowned Tian's α -invariant ([Tian87]).

Definition 6.4.6. *Let $\psi \in \mathcal{M}$. We define the ψ -relative α -invariant $\alpha_\omega(\psi)$ as*

$$\alpha_\omega(\psi) := \sup \left\{ \alpha \geq 0 : \sup_{\{u \leq \psi : \sup_X u = 0\}} \int_X e^{-\alpha u} d\mu < \infty \right\},$$

It is often more useful to use the following equivalent form of $\alpha_\omega(\psi)$ in terms of the *complex singularity exponents* (see also [DK01]).

Lemma 6.4.7. *Let $\psi \in \mathcal{M}$ and define, for any $u \in PSH(X, \omega)$, $c(u) := \sup\{\alpha \geq 0 : \int_X e^{-\alpha u} d\mu < \infty\}$. Then*

$$\alpha_\omega(\psi) = \inf_{u \preceq \psi} c(u).$$

In the absolute setting $\psi = 0$ this characterization of the α -invariant was proved by Demailly (see for instance Proposition 8.1 in [Tos12]). The proof for the ψ -relative setting is similar but we report it here for the courtesy of the reader.

Proof. By definition clearly $\alpha_\omega(\psi) \leq c(u)$ for any $u \preceq \psi$ with $\sup_X u = 0$. So the first inequality immediately follows observing that $c(u) = c(u - \sup_X u)$ for any $u \preceq \psi$.

Next assume by contradiction that there exists $\alpha > 0$ such that $\alpha_\omega(\psi) < \alpha < \inf_{u \preceq \psi} c(u)$. Then we can find a sequence $\{u_j\}_{j \in \mathbb{N}} \subset PSH_{\text{norm}}(X, \omega, \psi) := \{u \preceq \psi : \sup_X u = 0\}$ such that

$$\int_X e^{-\alpha u_j} d\mu \geq j \quad (6.13)$$

for any $j \in \mathbb{N}$. Moreover by weak compactness of $\{u \in PSH(X, \omega) : \sup_X u = 0\}$ we may also assume that $u_j \rightarrow u \in PSH_{\text{norm}}(X, \omega, \psi)$ weakly. In particular $\int_X e^{-\alpha u} d\mu < \infty$ since $\alpha < c(u)$. Hence by Theorem 6.4.8 quoted below $e^{-\alpha u_j} \rightarrow e^{-\alpha u}$ in L^1 , which contradicts (6.13) and concludes the proof. \square

Theorem 6.4.8 ([DK01]). *Let (X, ω) be a compact Kähler manifold. Then $PSH(X, \omega) \ni u \rightarrow c(u)$ is lower semicontinuous with respect to the weak topology. Moreover if $\{u_k\}_{k \in \mathbb{N}} \subset PSH(X, \omega)$ converges weakly to $u \in PSH(X, \omega)$, then*

$$e^{-\alpha u_k} \rightarrow e^{-\alpha u}$$

in L^1 for any $\alpha < c(u)$.

We can now study more in detail the function

$$\mathcal{M} \ni \psi \rightarrow \alpha_\omega(\psi).$$

Proposition 6.4.9. *The following properties hold:*

- i) $(\mathcal{M}, \preceq) \ni \psi \rightarrow \alpha_\omega(\psi) \in (0, +\infty)$ is decreasing and right-continuous;
- ii) letting $\psi_t := P_\omega[t\psi_0 + (1-t)\psi_1] \in \mathcal{M}$ for $t \in [0, 1]$ where $\psi_0, \psi_1 \in \mathcal{M}$ such that $\psi_0 \succcurlyeq \psi_1$, then for any $t, s \in [0, 1], t \geq s$

$$t\alpha_\omega(\psi_t) \geq s\alpha_\omega(\psi_s),$$

i.e. $[0, 1] \ni t \rightarrow t\alpha_\omega(\psi_t)$ is increasing.

Proof. As immediate consequence of Lemma 6.4.7 $\alpha_\omega(\cdot)$ is clearly decreasing, i.e. it decreases when the singularities decreases, and $\alpha_\omega(0) > 0$ by the uniform version of the Skoda's Integrability Theorem recalled below (Theorem 6.4.10). Then letting $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}$ decreasing to $\psi \in \mathcal{M}$ we want to prove that $\alpha_\omega(\psi_k) \rightarrow \alpha_\omega(\psi)$ as $k \rightarrow \infty$. By monotonicity, we may assume by contradiction that there exists $\alpha > 0$ such that $\alpha_\omega(\psi_k) < \alpha < \alpha_\omega(\psi)$ for any $k \in \mathbb{N}$. This implies that for any $k \in \mathbb{N}$ there exists an element $u_k \in PSH_{norm}(X, \omega, \psi_k) := \{u \in PSH(X, \omega) : u \preceq \psi_k, \sup_X u = 0\}$ such that

$$\int_X e^{-\alpha u_k} d\mu \geq k. \quad (6.14)$$

By weak compactness we may also suppose $u_k \rightarrow u \in PSH(X, \omega)$ weakly. Thus since $u_k \leq \psi_k$ by construction and $\psi_k \searrow \psi$, we obtain $u \in PSH_{norm}(X, \omega, \psi)$. In particular

$$\int_X e^{-\alpha u} d\mu < \infty$$

since by assumption $\alpha < \alpha_\omega(\psi)$. Finally Theorem 6.4.8 implies that $e^{-\alpha u_k} \rightarrow e^{-\alpha u}$ in L^1 which contradicts (6.14) and concludes the proof of (i).

Next suppose $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}$ as in (ii) and let $s, t \in (0, 1]$ such that $t \geq s$. Then for any $u \in PSH(X, \omega, \psi_t)$ we claim that the ω -function

$$v := \frac{s}{t}u + \frac{t-s}{t}\psi_1 \quad (6.15)$$

belongs to $PSH(X, \omega, \psi_s)$. Indeed v is clearly more singular than $\frac{s}{t}\psi_t + \frac{t-s}{t}\psi_1$, and for any $C > 0$ the function $\psi_{t,C} := P_\omega(t\psi_0 + (1-t)\psi_1 + C, 0)$ is more singular than $t\psi_0 + (1-t)\psi_1$, i.e.

$$\frac{s}{t}\psi_{t,C} + \frac{t-s}{t}\psi_1 \preceq \frac{s}{t}(t\psi_0 + (1-t)\psi_1) + \frac{t-s}{t}\psi_1 = s\psi_0 + (1-s)\psi_1,$$

which implies that $\frac{s}{t}\psi_{t,C} + \frac{t-s}{t}\psi_1 \leq \psi_s$ since $\psi_{t,C}, \psi_0, \psi_1 \leq 0$. Thus letting $C \rightarrow \infty$ and taking the upper semicontinuity regularization we get $\frac{s}{t}\psi_t + \frac{t-s}{t}\psi_1 \leq \psi_s$ which concludes the claim. Then for any $0 < \alpha < \alpha_\omega(\psi_s)$, letting $u \in PSH(X, \omega, \psi_t)$ and $v \in PSH(X, \omega, \psi_s)$ as in (6.15) the inequality

$$\int_X e^{-\frac{s}{t}\alpha u} d\mu = \int_X e^{-\alpha v} e^{\frac{\alpha(t-s)}{t}\psi_1} d\mu \leq \int_X e^{-\alpha v} d\mu$$

implies $tc(u) \geq s\alpha$. Hence Lemma 6.4.7 concludes the proof. \square

Theorem 6.4.10 ([Zer01]). *Let $K \subset PSH(X, \omega)$ be a weakly compact set such that $\sup_{x \in X} \sup_{u \in K} \nu(u, x) < 2$. Then*

$$\sup_{u \in K} \int_X e^{-u} \omega^n < +\infty.$$

Since when ψ has algebraic singularities type, finding a $[\psi]$ -KE metric is equivalent to find a log-KE metric on a resolution (Proposition 6.4.5), it is natural to wonder if it is possible to express $\alpha_\omega(\psi)$ algebraically and what is the connection between this new invariant and the usual *log α -invariant*. Letting (Y, Δ) be a weak log Fano pair, i.e. Y be a projective variety (which for our purpose we can assume smooth) and Δ be a \mathbb{Q} -divisor such that (Y, Δ) is klt and $-(K_Y + \Delta) =: L$ is big and semipositive, the log α -invariant of the pair (Y, Δ) is defined as

$$\alpha(Y, \Delta) := \sup\{\alpha \in \mathbb{Q}_{\geq 0} : (Y, \Delta + \alpha F) \text{ is klt for any } F \geq 0 \text{ } \mathbb{Q}\text{-divisor such that } F \sim_{\mathbb{Q}} L\} = \inf_{F \sim_{\mathbb{Q}} L, F \geq 0} \text{lct}(Y, \Delta, F) \quad (6.16)$$

where $\sim_{\mathbb{Q}}$ is the linear equivalence extended under rescaling, i.e. there exists $r \in \mathbb{N}$ such that $rF \in |rL|$, and where $\text{lct}(Y, \Delta, F) := \sup\{\alpha \in \mathbb{Q}_{\geq 0} : (Y, \Delta + \alpha F) \text{ is klt}\}$ is the *log canonical threshold* of F with respect to (Y, Δ) . We refer to [Kol13] and [Kol96] for the theory of singularities of pairs (X, D) , here we just need to recall the following analytical description (see Proposition 3.20 in [Kol96]): a pair $(Y, \Delta + \alpha F)$ is klt over a projective manifold Y if and only if $e^{-\alpha v_F} \nu_{(Y, \Delta)} \in L^1$ where if $F = \sum_{j=1}^m a_j F_j$ for prime divisors F_j then $v_F = \sum_{j=1}^m a_j \log |s_j|_{h_j}^2$ for s_j are holomorphic sections cutting F_j and h_j are smooth metrics on $\mathcal{O}_Y(F_j)$. Here $\nu_{(Y, \Delta)}$ is an *adapted measure* associated to the pair (Y, Δ) , in particular letting D_j prime divisors such that $\Delta = \sum_{k=1}^l b_k D_k$, t_k holomorphic sections cutting the divisors D_k and \tilde{h}_k smooth metric on $\mathcal{O}_Y(D_k)$, then $\nu_{(Y, \Delta)} = e^{-\sum_{k=1}^l b_k \log |s_{D_k}|_{\tilde{h}_k}^2} dV$ for a suitable volume form dV on Y (see [BBJ15]).

Remark 6.4.11. Considering the example in Remark 6.4.4, with the same notations, it is easy to see from the definitions that, letting $\nu_t := \nu_{(X, \frac{t}{r}D)}$, we have

$$\alpha_\omega(\psi_t) = \sup\left\{\alpha > 0 : \sup_{u \in PSH(X, \omega), \sup_X u = 0} \int_X e^{-\alpha u} e^{(1-\alpha)\varphi_D} d\nu_t < \infty\right\},$$

$$\alpha\left(X, \frac{t}{r}D\right) = \sup\left\{\alpha > 0 : \sup_{u \in PSH(X, \omega), \sup_X u = 0} \int_X e^{-\alpha u} d\nu_t < \infty\right\}.$$

Hence in particular $\alpha_\omega(\psi_t) \geq \min\{1, \alpha(X, \frac{t}{r}D)\}$, i.e. $\alpha_\omega(\psi_t)$ is a *finer* invariant than $\alpha(X, \frac{t}{r}D)$ since what matter is understanding when these quantities are larger than $n/(n+1)$ (by the general theory for the log setting, i.e. to get the existence of log KE metrics).

Lemma 6.4.12. *Let (Y, Δ) be a weak Fano pair and η be a smooth $(1, 1)$ -form representative of $c_1(L)$ where $L := -(K_Y + \Delta)$. Let also D be an effective \mathbb{Q} -divisor on Y , and θ smooth $(1, 1)$ -form on $\{[D]\}$. Then*

$$\inf_{F \sim_{\mathbb{Q}} L, F \geq 0} \text{lct}(Y, \Delta, F + D) = \inf_{v \in PSH(Y, \eta)} \text{lct}(Y, \Delta, v + D) \quad (6.17)$$

where we set $\text{lct}(Y, \Delta, v + D) := \sup\{\alpha \in \mathbb{Q}_{\geq 0} : \int_Y e^{-\alpha(v+v_D)} d\nu_{(Y, \Delta)} < \infty\}$ for v_D quasi-psh function such that $\theta + dd^c v_D = [D]$.

Proof. One inequality in (6.17) follows immediately from the fact that to any effective \mathbb{Q} -divisor $F \sim_{\mathbb{Q}} L$ it is associated a function $v_F \in \text{PSH}(Y, \eta)$ such that $\eta + dd^c v_F = [F]$ (obviously v_F is defined up to an additive constant), and $\text{lct}(Y, \Delta, F + D) = \text{lct}(Y, \Delta, v_F + D)$ by what said above.

For the reverse inequality, letting ω' be a fixed Kähler form, we first assume $v \in \text{PSH}(X, \eta)$ such that $\eta + dd^c v \geq \epsilon \omega'$ (i.e. a Kähler current). Then fix $r \in \mathbb{N}$ such that rL is a line bundle and denote with h the singular hermitian metric on rL associated to $r\eta + dd^c rv$. It is then well-known that for any $k \in \mathbb{N}$ the set $H^0(Y, L^{kr} \otimes \mathcal{I}(krv))$ of holomorphic sections $\sigma \in H^0(Y, L^{kr})$ such that $\int_Y |\sigma|_{h^k}^2 \omega'^n < \infty$ is a not-empty finite-dimensional Hilbert space. Choose for any $k \in \mathbb{N}$ an element $\sigma_k \in H^0(Y, L^{kr} \otimes \mathcal{I}(krv))$ of norm 1 and define

$$w_k := v + \frac{1}{kr} \log |\sigma|_{h^k}^2 \in \text{PSH}(Y, \eta).$$

Then since (Y, Δ) is klt there exists $f \in L^{p'}$ for $p' > 1$ such that $\nu_{(Y, \Delta)} = f \omega'^n$. We denote with q' the Sobolev conjugate exponent of p' . For $k \in \mathbb{N}$ and $\alpha < \text{lct}(Y, \Delta, w_k + D)$ fixed, we also set $c := \frac{\alpha q'}{r}$, $p := 1 + \frac{k}{c}$ and $q := 1 + \frac{c}{k}$. Clearly p, q are Sobolev conjugate exponents. Then by construction and using Hölder's inequality twice, we obtain

$$\begin{aligned} \int_Y e^{-\frac{rk}{pq'}(v+v_D)} d\nu_{(Y, \Delta)} &= \int_Y \left(e^{\frac{rk}{pq'}(w_k-v)} e^{-\frac{rk}{pq'}(w_k+v_D)} \right) d\nu_{(Y, \Delta)} \leq \\ &\leq \left(\int_Y e^{\frac{rk}{q'}(w_k-v)} f \omega'^n \right)^{\frac{1}{p}} \left(\int_Y e^{-\frac{rkq}{pq'}(w_k+v_D)} d\nu_{(Y, \Delta)} \right)^{\frac{1}{q}} \leq \\ &\leq \|f\|_{L^{p'}(\omega'^n)}^{1/p} \left(\int_X e^{-rk(w_k-v)} \omega'^n \right)^{\frac{1}{p}} \left(\int_Y e^{-\frac{rkq}{pq'}(w_k+v_D)} d\nu_{(Y, \Delta)} \right) \leq \\ &\leq \|f\|_{L^{p'}(\omega'^n)}^{1/p} \left(\int_Y e^{-\alpha(w_k+v_D)} d\nu_{(Y, \Delta)} \right)^{1/q} < +\infty. \end{aligned} \quad (6.18)$$

Thus $\text{lct}(Y, \Delta, v + D) \geq \frac{rk}{q'(1+\frac{k}{c})}$, i.e.

$$\text{lct}(Y, \Delta, v + D) \geq \frac{rk}{rk + q' \text{lct}(Y, \Delta, w_k + D)} \text{lct}(Y, \Delta, w_k + D)$$

by the arbitrariness of $\alpha < \text{lct}(Y, \Delta, w_k + D)$. Therefore since $\eta + dd^c w_k = [F_k]$ for

a \mathbb{Q} -effective divisor F_k by construction, it follows that

$$\begin{aligned} lct(Y, \Delta, v + D) &\geq \liminf_{k \rightarrow \infty} \left(\frac{rk}{rk + q'lct(Y, \Delta, F_k + D)} lct(Y, \Delta, F_k + D) \right) \geq \\ &\geq \inf_{F \sim_{\mathbb{Q}} L, F \geq 0} lct(Y, \Delta, F + D) \liminf_{k \rightarrow \infty} \frac{rk}{rk + q'lct(Y, \Delta, F_k + D)} = \\ &= \inf_{F \sim_{\mathbb{Q}} L, F \geq 0} lct(Y, \Delta, F + D) \end{aligned} \quad (6.19)$$

using also $lct(Y, \Delta, F_k + D) \leq lct(Y, \Delta, D) < +\infty$.

Since L is big, we can now fix $v \in PSH(Y, \eta)$ such that $\eta + dd^c v \geq \epsilon \omega'$ (i.e. a Kähler current) and note that for any $w \in PSH(Y, \eta)$ and for any $t \in [0, 1)$, $w_t := tw + (1-t)v \in PSH(Y, \eta)$ is a Kähler current since $\eta_{w_t} \geq (1-t)\epsilon \omega'$. Moreover

$$\begin{aligned} lct(Y, \Delta, w_t + D) &= \sup \left\{ \alpha \in \mathbb{Q}_{\geq 0} : \int_Y e^{-\alpha(w_t + v_D)} d\nu_{(Y, \Delta)} < +\infty \right\} \leq \\ &\leq \sup \left\{ \alpha \in \mathbb{Q}_{\geq 0} : \int_Y e^{-\alpha t(w + v_D)} d\nu_{(Y, \Delta)} < +\infty \right\} = \frac{1}{t} lct(Y, \Delta, w + D) \end{aligned} \quad (6.20)$$

since $w_t + v_D$ is more singular than $t(w + v_D)$ for $t \in [0, 1)$. Hence combining (6.19) and (6.20) the conclusion follows letting $t \rightarrow 1$. \square

Proposition 6.4.13. *Assume $\psi = P_\omega[\varphi] \in \mathcal{M}_{klt}$ with algebraic singularities type formally encoded in (\mathcal{J}, c) . Let $p : Y \rightarrow X$, η , D and Δ as in Proposition 6.4.5, and let L be the \mathbb{Q} -line bundle on Y such that $c_1(L) = \{\eta\}$. Then*

$$\min\{1, \alpha(Y, \Delta)\} \leq \alpha_\omega(\psi) \leq 1 + \inf_{F \sim_{\mathbb{Q}} L, F \geq 0} lct(Y, \Delta, F + cD). \quad (6.21)$$

Proof. Since $p^*(\omega) = \eta + \theta$ for θ smooth $(1, 1)$ -form and $p^*(\omega + dd^c \varphi) = \eta + c[D]$, letting v_D such that $\theta + dd^c v_D = c[D]$, it follows from pluriharmonicity that $\varphi \circ p = v_D + a$ for a constant a , which we may assume to be equal to 0 up to replace v_D . Moreover as proved in Proposition 4.8 in [Tru20b] it is not difficult to check that over the open Zariski set Ω where p is an isomorphism we have

$$p_*^{-1}(e^{-\varphi} \mu) = \nu_{(Y, \Delta)} \quad (6.22)$$

where $\nu_{(Y, \Delta)}$ is an adapted measure of the pair (Y, Δ) . Thus since p is an isomorphism outside a pluripolar set, we can extend to 0 the measure $\nu_{(Y, \Delta)}$ and (6.22) means that the lift of $e^{-\varphi} \mu$ is equal to $\nu_{(Y, \Delta)}$. Therefore for any $\alpha \geq 0$ and for any $u \preceq \psi$ we obtain that $e^{-\alpha u} \mu$ lifts to $e^{-\alpha \tilde{u} - (\alpha-1)v_D} \nu_{(Y, \Delta)}$ using also Proposition 6.2.4 (and its notations). It follows that

$$\alpha_\omega(\psi) = \inf_{u \preceq \psi} c(u) = \inf_{\tilde{u} \in PSH(Y, \eta)} \sup \left\{ \alpha \geq 0 : \int_X e^{-\alpha \tilde{u}} e^{-(\alpha-1)v_D} d\nu_{(Y, \Delta)} < +\infty \right\}. \quad (6.23)$$

and assuming $\alpha_\omega(\psi) \leq 1$ the left inequality in (6.21) is an easy consequence of Lemma 6.4.7 since clearly $v_D \leq C$. Similarly from Lemma 6.4.7 and the fact that $c(u) = c(u - \sup_X u)$ we obtain, supposing $\alpha \geq 1$,

$$\alpha_\omega(\psi) \leq \inf_{\tilde{u} \in PSH(Y, \eta)} lct(Y, \Delta, \tilde{u} + cD) + 1$$

and Lemma 6.4.12 concludes the proof. \square

Remark 6.4.14. Set $\beta_\omega(\psi) := \sup\{\beta \geq 0 : \sup_{\{u \preceq \psi : \sup_X u = 0\}} \int_X e^{-\beta u} e^{-\psi} d\mu < +\infty\}$ for any $\psi \in \mathcal{M}_{klt}$, and observe that the analog of Lemma 6.4.7 holds for this new invariant. Then by Holder's inequality it is not hard to check that

$$\frac{\alpha_\omega(\psi)}{q} \leq \beta_\omega(\psi) \leq \alpha_\omega(\psi) \quad (6.24)$$

where q is the Sobolev conjugate exponent of $lct(X, 0, \psi)$. Moreover if ψ has algebraic singularities, with the usual notations, Lemma 6.4.12 yields

$$\beta_\omega(\psi) = \inf_{F \sim_{\mathbb{Q}} L, F \geq 0} lct(Y, \Delta, F + cD)$$

proceeding as in Proposition 6.4.13. In particular (6.24) often produces a better algebraic upper bound for $\alpha_\omega(\psi)$ than the right inequality in (6.21).

6.4.2 Ding functional and uniqueness.

Similarly to the companion paper [Tru20b], we define for $\psi \in \mathcal{M}_{klt}^+$ the functional $D_\psi : \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R}$ as

$$D_\psi := V_\psi L_\mu - E_\psi$$

where $L_\mu(u) := -\log \int_X e^{-u} d\mu$. It is translation invariant and it assumes finite values by Proposition 6.3.1 since we are assuming $\psi \in \mathcal{M}_{klt}^+$. We call it ψ -relative *Ding functional* since it coincides with the renewed Ding functional in the case $\psi = 0$.

Remark 6.4.15. When $\psi = P_\omega[\varphi]$ has analytic singularities type, then with the same notations of Propositions 6.2.4, 6.4.5,

$$D_\psi(u) + E_\psi(\varphi) = -V_\eta \log \int_X e^{-\tilde{u}} d\nu - E(\tilde{u}) =: D_\eta(\tilde{u})$$

where ν is a suitable non-pluripolar measure associated to the log-setting. Indeed $D_\eta(\tilde{u})$ is the usual log-Ding functional associated to the pair (Y, Δ) .

Proposition 6.4.16. *Let $\psi \in \mathcal{M}_{klt}^+$. Then D_ψ is continuous on $(\mathcal{E}^1(X, \omega, \psi), d)$ and it is lower semicontinuous with respect to the weak topology.*

Proof. The continuity of $\mathcal{E}^1(X, \omega, \psi) \ni u \rightarrow L_\mu(u)$ with respect to the weak topology is given by Theorem 6.4.8. Therefore the result follows observing that E_ψ is upper semicontinuous in $\mathcal{E}^1(X, \omega, \psi)$ with respect to the weak topology (Proposition 6.2.1) while it is strongly continuous by definition. \square

In the absolute setting, a key property of Ding functional is its convexity along weak geodesic segments, which is the starting point to study the uniqueness of KE metrics. The analog holds in the relative setting if ψ belongs to $\mathcal{M}_{D, klt}^+$ and it has *small unbounded locus*, i.e. it is locally bounded on the complement of a closed complete pluripolar set.

Theorem 6.4.17 ([Bern09], [Bern11], [Bern15]). *Assume that $\psi \in \mathcal{M}_{klt}$ has small unbounded locus. Let $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities and let u_t be the weak geodesic joining them. Then $\mathcal{F}(t) := L_\mu(u_t)$ is subharmonic on S . Moreover if \mathcal{F} is affine over the real segment, then there is an holomorphic vector field V with flow F_s such that $F_s^*(\omega + dd^c u_s) = \omega + dd^c u_0$ for any $s \in [0, 1]$.*

Note that \mathcal{F} can be thought as a function on $[0, 1]$ since u_t does not depend on $\text{Im } t$.

Proof. With the same notations and terminology of the references quoted, we first observe that replacing the potentials ψ, u_t with the corresponding *metrics*, the functional L_μ becomes $-\log \int_X e^{-u}$. Therefore the subharmonicity of $\mathcal{F}(t)$ is a consequence of [Bern09], [Bern11].

Next writing $u_t := \psi + (u_t - \psi)$, we are in the situation described in section § 6.1 of [Bern15] thanks to Proposition 6.3.9. Thus Theorem 6.1 in [Bern15] concludes the proof. \square

Since by Theorem 6.3.11 the ψ -relative energy E_ψ is linear along weak geodesic segments if $\psi \in \mathcal{M}_D^+$, Theorem 6.4.17 gives the convexity of D_ψ requested.

Corollary 6.4.18. *Assume $\psi \in \mathcal{M}_{D, klt}^+$ with small unbounded locus. Then the ψ -relative Ding functional D_ψ is convex along any weak geodesic segment $[0, 1] \ni t \rightarrow u_t \in \mathcal{E}^1(X, \omega, \psi)$ joining two potentials $u_0, u_1 \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities.*

Next, to prove the first part of Theorem C we need to introduce the set

$$\text{Aut}(X, [\psi]) := \{F \in \text{Aut}(X) : [F^*\psi] = [\psi]\}$$

of all automorphisms which preserve the singularity type $[\psi]$, where we recall that $[u] = [v]$ is equivalent to $\|u - v\|_\infty < \infty$. Observe that $\text{Aut}(X, [\psi])$ is a linear algebraic group since it is a subgroup of $\text{Aut}(X)$. We denote with $\text{Aut}(X, [\psi])^\circ := \text{Aut}(X, [\psi]) \cap \text{Aut}(X)^\circ$ where $\text{Aut}(X)^\circ$ is the connected component of the identity map.

Theorem 6.4.19. *Assume $\psi \in \mathcal{M}_{D, klt}^+$ with small unbounded locus and let $u \in \mathcal{E}^1(X, \omega, \psi)$. Then the following statement are equivalent:*

i) $\omega_u := \omega + dd^c u$ is a $[\psi]$ -KE metric;

ii) $D_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi$.

Furthermore if ω_u, ω_v are $[\psi]$ -KE metrics, then there exists $F \in \text{Aut}(X, [\psi])^\circ$ such that $F^*(\omega_u) = \omega_v$.

Proof. The implication (ii) \Rightarrow (i) follows from Theorem 4.22 in [DDNL18b]. Vice versa the proof of (i) \Rightarrow (ii) is the ψ -relative version of that of Theorem 6.6 in [BBGZ13].

We want to prove that $D_\psi(u) \leq D_\psi(v)$ for any $v \in \mathcal{E}^1(X, \omega, \psi)$ and by the continuity of D_ψ along decreasing sequences in $\mathcal{E}^1(X, \omega, \psi)$ we may suppose v to have ψ -relative minimal singularities. Moreover without loss of generality we can assume $\int_X e^{-u} d\mu = V_\psi$, i.e. $C = 0$ in the Monge-Ampère equation (6.11). Recall also that any solution of the same equation has ψ -relative minimal singularities. Then, letting u_t be the weak geodesic joining $u_0 := u$ and $u_1 := v$, Corollary 6.4.18 implies that $t \rightarrow D_\psi(u_t)$ is a convex function. Therefore it will be enough to prove that

$$\frac{d}{dt}(D_\psi(u_t))|_{t=0+} \geq 0. \quad (6.25)$$

By Proposition 6.3.9 the function $w_t := (u_t - u)/t$ is uniformly bounded and converges almost everywhere to a bounded function w . Moreover by the concavity of the ψ -relative energy ([DDNL18b])

$$\frac{E_\psi(u_t) - E_\psi(u)}{t} \leq \int_X w_t M A_\omega(u) = \int_X w_t e^{-u} d\mu,$$

which implies

$$\frac{d}{dt}(E_\psi(u_t))|_{t=0+} \leq \int_X w e^{-u} d\mu. \quad (6.26)$$

On the other hand

$$\frac{\int_X (e^{-u_t} - e^{-u}) d\mu}{t} = - \int_X w_t f(u_t - u) e^{-u} d\mu$$

where $f(x) := (1 - e^{-x})/x$ is a continuous function. Thus $f(u_t - u)$ is uniformly bounded since $\|u_t - u\|_\infty \leq C$ for any $t \in [0, 1]$, and by Dominated Convergence Theorem it follows that

$$\frac{d}{dt} \left(\int_X e^{-u_t} d\mu \right) |_{t=0+} = - \int_X w e^{-u} d\mu \quad (6.27)$$

since $f(u_t - u) \rightarrow 1$ as $t \rightarrow 0^+$ again by Proposition 6.3.9. Therefore the inequality in (6.25) follows combining (6.26), (6.27) and using the chain rule of derivation. \square

Remark 6.4.20. The implication (ii) \Rightarrow (i) in Theorem 6.4.19 holds as soon as $\psi \in \mathcal{M}_{klt}^+$. When $\psi = 0$, Theorem 6.4.19 was proved in [BBGZ13].

6.4.3 Mabuchi functional and Theorem C.

In this subsection we keep assuming $\psi \in \mathcal{M}_{klt}^+$.

Before defining the ψ -relative Mabuchi functional we need to recall the ψ -relative I, J -functionals:

$$J_\psi(u) := \int_X (u - \psi) MA_\omega(\psi) - E_\psi(u),$$

$$I_\psi(u) := \int_X (u - \psi) (MA_\omega(\psi) - MA_\omega(u))$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$. These functionals are translation invariant and strongly continuous, i.e. in $(\mathcal{E}^1(X, \omega, \psi), d)$, as a consequence of Corollary 3.5 and Proposition 3.6 in [Tru20a]. Moreover they satisfy the following important properties.

Proposition 6.4.21 ([Tru20a], [Tru20b]). *Let $u \in \mathcal{E}^1(X, \omega, \psi)$. Then*

- i) $\frac{1}{n+1} I_\psi(u) \leq J_\psi(u) \leq \frac{n}{n+1} I_\psi(u)$;
- ii) *there exists a constant $C > 0$ depending uniquely on (X, ω) such that*

$$d(\psi, u) - C \leq J_\psi(u) \leq d(\psi, u) \quad (6.28)$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$.

We recall that with $\mathcal{E}_{norm}^1(X, \omega, \psi)$ we denote all elements $u \in \mathcal{E}^1(X, \omega, \psi)$ such that $\sup_X u = 0$.

Next it is also necessary to retrieve the definition of the *entropy*.

Definition 6.4.22. *Let $\nu_1, \nu_2 \in \mathcal{P}(X)$, i.e. two probability measures on X . The relative entropy $H_{\nu_1}(\nu_2) \in [0, +\infty]$ of ν_2 with respect to ν_1 is defined as follows. If ν_2 is absolutely continuous with respect to ν_1 with density $f := \frac{d\nu_2}{d\nu_1}$ satisfying $f \log f \in L^1(\nu_1)$ then*

$$H_{\nu_1}(\nu_2) := \int_X f \log f d\nu_1 = \int_X \log f d\nu_2.$$

Otherwise we set $H_{\nu_1}(\nu_2) := +\infty$.

Then we set $\psi' := \psi + a$ where $a := \log \int_X e^{-\psi} d\mu$, so that $e^{-\psi'} \mu$ is a probability measure.

Definition 6.4.23. *The ψ -relative Mabuchi functional $M_\psi : \mathcal{E}^1(X, \omega, \psi) \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as*

$$M_\psi(u) := V_\psi H_{e^{-\psi'} \mu}(MA_\omega(u)/V_\psi) + J_\psi(u) - I_\psi(u).$$

Observe that it is clearly a translation invariant functional and that in the absolute setting $\psi = 0$ it coincides with the usual Mabuchi functional (see [Mab86] and the Tian's formula in [Chen00], [Tian] for the Fano case). Moreover it is lower semicontinuous with respect to the strong topology since J_ψ, I_ψ are continuous and the entropy is lower semicontinuous with respect to the weak topology. Furthermore by definition and Theorem 6.2.2 we have

$$M_\psi(u) = (V_\psi H_{e^{-\psi'} \mu} - E_\psi^*)(MA_\omega(u)/V_\psi).$$

See subsection §6.2.1 for the definition of the energy E^* . The Mabuchi functional dominates the Ding functional as the next result shows.

Proposition 6.4.24. *Let $a := \log \int_X e^{-\psi} d\mu$. Then $D_\psi(u) + aV_\psi \leq M_\psi(u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$ with the equality if and only if ω_u is a $[\psi]$ -KE metric.*

Proof. We may assume $H_{e^{-\psi'} \mu}(MA_\omega(u)/V_\psi) < +\infty$. Observe that this implies $H_{\mu_u}(MA_\omega(u)/V_\psi) < +\infty$ where $\mu_u := \frac{e^{-u} \mu}{\int_X e^{-u} \mu}$ since $u \preceq \psi'$. Moreover by an immediate calculation

$$(M_\psi - D_\psi)(u) = \int_X \log \left(\frac{MA_\omega(u)/V_\psi}{e^{-\psi'} d\mu} \right) MA_\omega(u) + \int_X (u - \psi) MA_\omega(u) - V_\psi L_\mu(u).$$

Thus, since $\mu_u = e^{-u+L_\mu(u)} \mu$, we get

$$V_\mu L_\mu(u) = \int_X L_\mu(u) MA_\omega(u) = \int_X \left(\log \left(\frac{d\mu_u}{e^{-\psi'} d\mu} \right) + (u - \psi') \right) MA_\omega(u),$$

which implies

$$(M_\psi - D_\psi)(u) = \int_X \log \left(\frac{MA_\omega(u)/V_\psi}{d\mu_u} \right) MA_\omega(u) + aV_\psi.$$

Therefore Proposition 2.10.(ii) in [BBEGZ19] concludes the proof. \square

We can now finish to prove Theorem C using the following two Lemmas.

Lemma 6.4.25 ([BBEGZ19], Lemma 2.11.). *For any lower semicontinuous function g on X and any $\nu_1 \in \mathcal{P}(X)$,*

$$\log \int_X e^g d\nu_1 = \sup_{\nu_2 \in \mathcal{P}(X)} \left(\int_X g d\nu_2 - H_{\nu_1}(\nu_2) \right).$$

Lemma 6.4.26. *For any $u \in \mathcal{E}^1(X, \omega, \psi)$,*

$$\begin{aligned} V_\psi L_\mu(u) &= \inf_{v \in \mathcal{E}^1(X, \omega, \psi)} \left(V_\psi H_{e^{-\psi'} \mu}(MA_\omega(v)/V_\psi) + \int_X (u - \psi') MA_\omega(v) \right), \\ E_\psi(u) &= \inf_{v \in \mathcal{E}^1(X, \omega, \psi)} \left(E_\psi^*(MA_\omega(v)/V_\psi) + \int_X (u - \psi) MA_\omega(v) \right). \end{aligned}$$

Proof. The second equality follows easily from the concavity of E_ψ ([DDNL18b]) since

$$E_\psi(u) \leq E_\psi(v) + \int_X (u - v) MA_\omega(v) = E_\psi^*(MA_\omega(v)/V_\psi) + \int_X (u - \psi) MA_\omega(v)$$

with the equality when $v = u$.

For the first equality we can clearly restrict to consider $v \in \mathcal{E}^1(X, \omega, \psi)$ such that $H_{e^{-\psi'}\mu}(MA_\omega(v)/V_\psi) < +\infty$. Then, setting $b := -\log \mu(X)$, we observe that

$$\begin{aligned} -\infty &< \int_X (\psi' - u) MA_\omega(v) - V_\psi H_{e^{-\psi'}\mu}(MA_\omega(v)/V_\psi) = \\ &= \int_X \left(-u - \log \left(\frac{MA_\omega(v)/V_\psi}{e^b d\mu} \right) \right) MA_\omega(v) - b V_\psi \leq V_\psi \log \int_X e^{-u+b} d\mu - b V_\psi = -V_\psi L_\mu(u) \end{aligned}$$

where the last inequality is a consequence of Lemma 6.4.25. Hence

$$V_\psi L_\mu(u) \leq \inf_{v \in \mathcal{E}^1(X, \omega, \psi)} \left(V_\psi H_{e^{-\psi'}\mu}(MA_\omega(v)/V_\psi) + \int_X (u - \psi') MA_\omega(v) \right).$$

To prove the equality we set $\mu_u := e^{-u}\mu / \int_X e^{-u}\mu$ and we claim that $H_{e^{-\psi'}\mu}(\mu_u) < +\infty$. Indeed by the resolution of the openness conjecture there exists $p > 1$ such that $e^{-u} \in L^p$, thus by Theorem 1.4.(i) in [DDNL18b] there exists $v \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities such that $MA_\omega(v) = V_\psi \mu_u$. Therefore since μ_u is clearly absolutely continuous with respect to $e^{-\psi'}\mu$ with density equal to $f := e^{-u+\psi'}/\int_X e^{-u}\mu$, the claim follows by definition since

$$\int_X f \log(f) e^{-\psi'} d\mu = \int_X (\psi' - u) d\mu_u = \int_X (\psi' - u) MA_\omega(v) < +\infty.$$

Next, since $MA_\omega(v)/V_\psi = \mu_u = e^{-u+L_\mu(u)}\mu$, by an easy calculation we obtain

$$\begin{aligned} V_\psi L_\mu(u) &= \int_X L_\mu(u) MA_\omega(v) = \int_X \left(\log \left(\frac{MA_\omega(v)/V_\psi}{e^{-\psi'}\mu} \right) + (u - \psi') \right) MA_\omega(v) = \\ &= V_\psi H_{e^{-\psi'}\mu}(MA_\omega(v)/V_\psi) + \int_X (u - \psi') MA_\omega(v), \end{aligned}$$

which concludes the proof. \square

Theorem C. Assume $\psi \in \mathcal{M}_{D, \text{klt}}^+$ with small unbounded locus and let $u \in \mathcal{E}^1(X, \omega, \psi)$. Then the following statements are equivalent:

- i) $\omega_u = \omega + dd^c u$ is a $[\psi]$ -KE metric;
- ii) $D_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi$;
- iii) $M_\psi(u) = \inf_{\mathcal{E}^1(X, \omega, \psi)} M_\psi$.

Moreover if ω_u is a $[\psi]$ -KE metric then u has ψ -relative minimal singularities and if ω_v is another $[\psi]$ -KE metric then there exists $F \in \text{Aut}(X, [\psi])^\circ$ such that $F^*\omega_v = \omega_u$.

Proof. As said in the beginning of this section if $\omega + dd^c u$ is a $[\psi]$ -KE metric then by the complex Monge-Ampère equation (6.11) it follows that u has ψ -relative minimal singularities. Moreover the uniqueness modulo $\text{Aut}(X, [\psi])^\circ$ was already stated in Theorem 6.4.19 where we also proved the equivalence between (i) and (ii). Furthermore if (ii) holds, then (iii) is given by (i) and Proposition 6.4.24. Thus it remains to prove that (iii) implies (ii).

Set $m := M_\psi(u)$. Then for any $v \in \mathcal{E}^1(X, \omega, \psi)$ by Lemma 6.4.26 we obtain

$$\begin{aligned} & V_\psi H_{e^{-\psi'}_\mu}(MA_\omega(v)/V_\psi) + \int_X (u - \psi') MA_\omega(v) - E_\psi(u) \geq \\ & \geq V_\psi H_{e^{-\psi'}_\mu}(MA_\omega(v)/V_\psi) - E_\psi^*(MA_\omega(v)/V_\psi) - aV_\psi = M_\psi(v) - aV_\psi \geq m - aV_\psi \end{aligned}$$

Hence taking the infimum among all $v \in \mathcal{E}^1(X, \omega, \psi)$ again by Lemma 6.4.26 we get

$$\inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi \geq m - aV_\psi.$$

So to conclude the proof it is enough to observe that by Proposition 6.4.24 $D_\psi(u) = m - aV_\psi$. \square

6.4.4 Proof of Theorem D.

In this subsection we assume $\psi \in \mathcal{M}_{klt}^+$.

We first prove that the ψ -relative α -invariant controls the level sets of the entropy in terms of the ψ -relative energy E_ψ^* , and hence in the metric space $(\mathcal{E}^1(X, \omega, \psi), d)$ thanks to Theorem 6.2.2. In particular probability measures with ψ -relative finite entropy are in the range of the Monge-Ampère operator restricted to $\mathcal{E}^1(X, \omega, \psi)$.

Proposition 6.4.27. *Let $0 < \alpha < \alpha_\omega(\psi)$. Then there exists $C \geq 0$ such that*

$$H_{e^{-\psi'}_\mu}(\nu) \geq \frac{\alpha}{V_\psi} E_\psi^*(\nu) - C \quad (6.29)$$

for any ν probability measure. In particular if $H_{e^{-\psi'}_\mu}(\nu) < +\infty$ then there exists $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $V_\psi \nu = MA_\omega(u)$ and

$$H_{e^{-\psi'}_\mu}(MA_\omega(u)/V_\psi) \geq \frac{\alpha}{V_\psi} I_\psi(u) - C$$

for an uniform constant $C \geq 0$.

Proof. By definition there exists $A > 0$ such that

$$\log \int_X e^{-\alpha u} d\mu \leq -\alpha \sup_X u + A$$

for any $u \in PSH(X, \omega)$ such that $u \preceq \psi$. Then since $\sup_X u = \sup_X (u - \psi)$ (Lemma 3.7. in [Tru19] quoted in Lemma 6.5.6 below) and clearly $E_\psi(u) \leq V_\psi \sup_X (u - \psi)$ we obtain

$$-\log \int_X e^{-\alpha u} d\mu \geq \frac{\alpha}{V_\psi} E_\psi(u) - A. \quad (6.30)$$

Next we fix a positive probability measure ν and we define $\psi_k := \max(\psi, -k)$ and $\psi'_k := \psi_k + a_k$ where $a_k \in \mathbb{R}$ such that $e^{-\psi'_k} \mu$ is a probability measure. Obviously $\psi'_k \rightarrow \psi' = \psi + a$ where as usual $\psi' = \psi + a$ for $a = \log \int_X e^{-\psi} d\mu$. Combining (6.30) with Lemma 6.4.25, for any $k \in \mathbb{N}$ fixed, it follows that

$$\begin{aligned} H_{e^{-\psi'_k} \mu}(\nu) &= H_\mu(\nu) + \int_X \psi'_k d\nu \geq \frac{\alpha}{V_\psi} \left(E_\psi(u) - \int_X (u - \psi'_k) V_\psi d\nu \right) - A \geq \\ &\geq \frac{\alpha}{V_\psi} \left(E_\psi(u) - \int_X (u - \psi) V_\psi d\nu \right) - A + a_k V_\psi. \end{aligned} \quad (6.31)$$

Then supposing ν such that $H_{e^{-\psi'} \mu}(\nu) < +\infty$, there exists $f \in L^1(e^{-\psi'} \mu)$ such that $\nu = f e^{-\psi'} \mu$ and we define for any $j \in \mathbb{N}$

$$\nu_j := c_j \min(fe^{-\psi'}, j) \mu =: c_j f_j \mu$$

where $c_j \geq 1$ such that $\nu_j \in \mathcal{P}(X)$. Thus by definition $H_{e^{-\psi'_k} \mu}(\nu_j) < +\infty$ and from (6.31) we get

$$\int_X (\log(c_j f_j) + \psi'_k) c_j f_j d\mu \geq \frac{\alpha}{V_\psi} \left(E_\psi(u) - \int_X (u - \psi) V_\psi c_j f_j d\mu \right) - A + a_k V_\psi, \quad (6.32)$$

and letting $k \rightarrow \infty$ the left side hand converges to $\int_X (\log(c_j f_j) + \psi') c_j f_j d\mu$ since $|\sup_X \psi'_k| \leq C$ by construction. Then moving $j \rightarrow \infty$ we obtain $\int_X \log(f) d\nu = H_{e^{-\psi'} \mu}(\nu)$ by Monotone Convergence Theorem since $c_j \searrow 1$ while $f_j \nearrow f e^{-\psi'}$. On the other hand the right side in (6.32) is invariant under translation on u , thus assuming $u \leq \psi$, again by Monotone Convergence Theorem it converges to $\frac{\alpha}{V_\psi} \left(E_\psi(u) - \int_X (u - \psi) V_\psi d\nu \right) - A + a V_\psi$. Summarizing, letting $k, j \rightarrow \infty$ in this order in (6.32), it follows that

$$H_{e^{-\psi'} \mu}(\nu) \geq \frac{\alpha}{V_\psi} \left(E_\psi(u) - \int_X (u - \psi) V_\psi d\nu \right) - C$$

setting $C := \max(A - a V_\psi, 0)$. Taking the supremum over all $u \in \mathcal{E}^1(X, \omega, \psi)$, we obtain (6.29). We also deduce that $V_\psi \nu \in \mathcal{M}^1(X, \omega, \psi)$ for any $\nu \in \mathcal{P}(X)$

such that $H_{e^{-\psi'}\mu}(\nu) < +\infty$. Hence by Theorem 6.2.2 there exists an unique $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $MA_\omega(u) = V_\psi \nu$ and similarly to before we get

$$\begin{aligned} H_{e^{-\psi'}\mu}(\nu) &\geq \alpha \sup_X (u - \psi) - \alpha \int_X u d\nu - A \geq \\ &\geq \frac{\alpha}{V_\psi} \left(\int_X (u - \psi) MA_\omega(\psi) - \int_X (u - \psi) V_\psi d\nu \right) - A = \frac{\alpha}{V_\psi} I_\psi(u) - A, \end{aligned}$$

which concludes the proof. \square

We recall the definition of d -coercivity.

Definition 6.4.28. *Let $F : \mathcal{E}^1(X, \omega, \psi) \rightarrow \overline{\mathbb{R}}$ be a translation invariant functional. Then F is said to be d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ if there exist $A > 0, B \geq 0$ such that*

$$F(u) \geq Ad(u, \psi) - B$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$.

Note that, for any translation invariant functional F , as an easy consequence of Proposition 6.4.21 the d -coercivity over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ is equivalent to the J_ψ -coercivity over $\mathcal{E}^1(X, \omega, \psi)$, i.e.

$$F(u) \geq AJ_\psi(u) - B$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ where $A > 0, B \geq 0$.

The d -coercivity of the ψ -relative Ding functional and of the ψ -relative Mabuchi functional are both equivalent to a ψ -relative version of a Mose-Trudinger type inequality as our next result shows.

Proposition 6.4.29. *The followings are equivalent:*

- i) *the ψ -relative Ding functional is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$;*
- ii) *the ψ -relative Mabuchi functional is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$;*
- iii) *there exist $p > 1, C > 0$ such that*

$$\|e^{\psi-u}\|_{L^p(e^{-\psi'}\mu)} \leq Ce^{-E_\psi(u)/V_\psi} \quad (6.33)$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$.

Proof. The implication (i) \Rightarrow (ii) follows from Proposition 6.4.24. Then let assume (ii) to hold, i.e. there exists $A > 0, B \geq 0$ such that

$$M_\psi(u) \geq Ad(u, \psi) - B$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. Since $d(u - \sup_X u, \psi) \geq J_\psi(u - \sup_X u) = J_\psi(u)$ for any $u \in \mathcal{E}^1(X, \omega, \psi)$ (Proposition 6.4.21) and since M_ψ is translation invariant, we get

$$M_\psi(u) \geq AJ_\psi(u) - B = A\left(\frac{n+1}{n}J_\psi(u) - \frac{1}{n}J_\psi(u)\right) - B \geq \frac{A}{n}E_\psi^*(MA_\omega(u)/V_\psi) - B \quad (6.34)$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$, where we used again Proposition 6.4.21 for the last inequality. The equation (6.34) is equivalent to

$$V_\psi H_{e^{-\psi'}_\mu}(MA_\omega(u)/V_\psi) \geq pE_\psi^*(MA_\omega(u)/V_\psi) - B$$

where $p = 1 + A/n > 1$, which implies $V_\psi H_{e^{-\psi'}_\mu}(\nu) \geq pE_\psi^*(\nu) - B$ for any $\nu \in \mathcal{P}(X)$ by Proposition 6.4.27. Next we observe that considering the approximants $u_k := \max(u, \psi - k)$ by Monotone Convergence Theorem and by the continuity of E_ψ along decreasing sequences it is enough to prove (6.33) for $u \in \mathcal{E}^1(X, \omega, \psi)$ with ψ -relative minimal singularities. Thus by Lemma 6.4.25, letting $b := -\log \int_X e^{(p-1)\psi} d\mu$, we have

$$\log \int_X e^{-pu} e^{(p-1)\psi+b} d\mu = \sup_{\nu \in \mathcal{P}(X)} \left\{ \int_X (-pu) d\nu - H_{e^{(p-1)\psi+b}_\mu}(\nu) \right\}.$$

So for any $\epsilon > 0$ fixed there exists $\nu_{u,\epsilon} \in \mathcal{P}(X)$ such that $H_{e^{(p-1)\psi+b}_\mu}(\nu_{u,\epsilon}) < +\infty$ and

$$\log \int_X e^{-pu} e^{(p-1)\psi+b} d\mu \leq \epsilon - \int_X (pu) d\nu_{u,\epsilon} - H_{e^{(p-1)\psi+b}_\mu}(\nu_{u,\epsilon}).$$

Then since $H_{e^{-\psi'}_\mu}(\nu_{u,\epsilon}) = H_{e^{(p-1)\psi+b}_\mu}(\nu_{u,\epsilon}) + p \int_X \psi d\nu_{u,\epsilon} + a + b$ (where as usual $\psi' = \psi + a$ for $a = \log \int_X e^{-\psi} d\mu$), by an easy calculation we get

$$\begin{aligned} V_\psi \log \int_X e^{-pu} e^{(p-1)\psi+b} d\mu &\leq V_\psi \epsilon + \int_X p(\psi - u) V_\psi d\nu_{u,\epsilon} + a V_\psi + b V_\psi - V_\psi H_{e^{-\psi'}_\mu}(\nu_{u,\epsilon}) \leq \\ &\leq V_\psi(\epsilon + a + b) + B + p \left(\int_X (\psi - u) V_\psi d\nu_{u,\epsilon} - E_\psi^*(\nu_{u,\epsilon}) \right) \leq \\ &\leq V_\psi(\epsilon + a + b) + B - p \inf_{\nu \in \mathcal{P}(X)} \left\{ E_\psi^*(\nu) + \int_X (u - \psi) V_\psi d\nu \right\} = V_\psi(\epsilon + a + b) + B - pE_\psi(u) \end{aligned}$$

where in the equality we used Lemma 6.4.26. Hence by the arbitrariness of ϵ we obtain

$$V_\psi \log \int_X e^{p(\psi-u)} e^{-\psi'} d\mu \leq -pE_\psi(u) + B,$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ which is equivalent to (6.33) setting $C := e^{\frac{B}{pV_\psi}}$.

Finally supposing (iii) to hold it remains to prove the d -coercivity of D_ψ . Fix $\epsilon \in (0, 1)$ small enough such that $p := 1 + \epsilon$ satisfies (6.33). Then for any $u \in$

$\mathcal{E}_{norm}^1(X, \omega, \psi)$ combining the equality $(u - \psi) = (1 + \epsilon)(1 - \epsilon)(u - \psi) + \epsilon^2(u - \psi)$ with the convexity of $f \rightarrow \log \int_X e^{-f} d\nu$ for any $\nu \in \mathcal{P}(X)$ we get

$$\log \int_X e^{-(u-\psi)} e^{-\psi'} d\mu \leq (1-\epsilon) \log \int_X e^{-(1+\epsilon)(u-\psi)} e^{-\psi'} d\mu + \epsilon \log \int_X e^{-\epsilon(u-\psi)} e^{-\psi'} d\mu,$$

and the first term in the right side is dominated by $(1 - \epsilon) \left(-\frac{(1+\epsilon)}{V_\psi} E_\psi(u) + D \right)$ for a constant D . For the second term,

$$\int_X e^{-\epsilon(u-\psi)} e^{-\psi'} d\mu \leq \int_X e^{-\epsilon u} e^{-\psi'} d\mu,$$

where the right hand side is uniformly bounded if $\epsilon \ll 1$ small enough combining Holder's inequality with the klt assumption and Theorem 6.4.10 (indeed it is enough that $\epsilon < \beta_\omega(\psi)$, see Remark 6.4.14). Therefore it follows that

$$V_\psi \log \int_X e^{-u} d\mu = a + V_\psi \log \int_X e^{-(u-\psi)} e^{-\psi'} d\mu \leq -(1 - \epsilon^2) E_\psi(u) + B$$

for an constant B . Hence

$$D_\psi(u) \geq (1 - \epsilon^2) E_\psi(u) + B - E_\psi(u) = \epsilon^2 d(\psi, u) - B,$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$, which concludes the proof. \square

We can now prove our second main result which partly generalizes to the relative setting Theorem 2.4 in [DR17].

Theorem D. *Let $\psi \in \mathcal{M}_{D, klt}^+$ with small unbounded locus. Assume also $\text{Aut}(X, [\psi])^\circ = \{Id\}$. Then the following conditions are equivalent:*

- i) *the ψ -relative Ding functional is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$;*
- ii) *the ψ -relative Mabuchi functional is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$;*
- iii) *there exists an unique $[\psi]$ -KE metric.*

Proof. The equivalence between (i) and (ii) is part of the content in Proposition 6.4.29, and the implication (i) \Rightarrow (iii) follows from Theorem C in [Tru20b] but we recall briefly here the proof for the courtesy to the reader. Let $A > 0, B \geq 0$ such that $D_\psi(u) \geq Ad(\psi, u) - B$ for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and let $\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $D_\psi(u_k) \searrow \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi \geq -B$. Then from the coercivity there exists $C > 0$ such that

$$\{u_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_C^1(X, \omega, \psi) := \{u \in \mathcal{E}^1(X, \omega, \psi) : E_\psi(u) \geq -C, \sup_X u \leq C\},$$

which is weakly compact by Proposition 6.2.1. Therefore up to considering a subsequence we may suppose $u_k \rightarrow u \in \mathcal{E}_C^1(X, \omega, \psi)$ weakly. Moreover by Hartogs'

Lemma $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. Finally by the lower semicontinuity of D_ψ (Proposition 6.4.16) it follows that

$$D_\psi(u) \leq \liminf_{k \rightarrow +\infty} D_\psi(u_k) = \inf_{\mathcal{E}^1(X, \omega, \psi)} D_\psi,$$

i.e. $\omega + dd^c u$ is the unique KE metric with prescribed singularities $[\psi]$ by Theorem 6.4.19.

Finally we want to prove that (iii) \Rightarrow (i). Letting $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that ω_u is the unique KE metric with prescribed singularities $[\psi]$, we define

$$A := \inf \left\{ \frac{D_\psi(v) - D_\psi(u)}{d(u, v)} : v \in \mathcal{E}_{norm}^1(X, \omega, \psi) \text{ with } \psi\text{-relative minimal singularities} \right. \\ \left. \text{such that } d(u, v) \geq 1 \right\},$$

and we claim that it is enough to prove $A > 0$. Indeed setting $B' := A \sup\{d(v, \psi) : d(v, u) \leq 1\} - D_\psi(u) \leq A + Ad(u, \psi) - D_\psi(u)$ we clearly have

$$D_\psi(v) \geq Ad(v, \psi) - B'$$

for any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ such that $d(u, v) \leq 1$. Thus

$$D_\psi(v) \geq Ad(v, \psi) - \max\{B', -D_\psi(u)\} \quad (6.35)$$

for any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ with ψ -relative minimal singularities. And by the strong continuity of D_ψ (Proposition 6.4.16) the inequality (6.35) would extend to any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ considering the sequence $v_k := \max(v, \psi - k)$.

Therefore it remains to prove that $A > 0$. Assume by contradiction $A = 0$. Then there exists a sequence $\{v^k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{norm}^1(X, \omega, \psi)$ of potentials with ψ -relative minimal singularities such that $d(v^k, u) \geq 1$ and

$$\frac{D_\psi(v^k) - D_\psi(u)}{d(v^k, u)} \rightarrow 0$$

as $k \rightarrow \infty$. Thus letting $[0, d(v^k, u)] \ni t \rightarrow v_t^k$ be the unit speed weak geodesic segment joining u and v^k , the function $t \rightarrow D_\psi(v_t^k)$ is convex by Corollary 6.4.18. Hence defining $w_k := v_1^k$ we have $d(w_k, u) = 1$ and

$$0 \leq D_\psi(w_k) - D_\psi(u) \leq \frac{D_\psi(v^k) - D_\psi(u)}{d(u, v^k)} \rightarrow 0 \quad (6.36)$$

as $k \rightarrow \infty$. Moreover since by the triangle inequality

$$\{w_k\}_{k \in \mathbb{N}} \subset \mathcal{E}_{1+d(u, \psi)}^1(X, \omega, \psi) := \{w \in \mathcal{E}^1(X, \omega, \psi) : E_\psi(w) \geq -1 - d(u, \psi), \\ \sup_X w \leq 1 + d(u, \psi)\}$$

which is weakly compact by the upper semicontinuity of E_ψ , up to considering a subsequence we may assume that $w_k \rightarrow w$ weakly for $w \in \mathcal{E}_{norm}^1(X, \omega, \psi)$. But from (6.36) and the lower-semicontinuity of D_ψ with respect to the weak topology (Proposition 6.4.16) we get $D_\psi(w) \leq D_\psi(u)$ which by Theorem 6.4.19 implies $w = u$. In particular $\liminf_{k \rightarrow \infty} D_\psi(w_k) = D_\psi(w)$ which implies that $E_\psi(w_k) \rightarrow E_\psi(w)$ because L_μ is continuous with respect to the weak topology (Theorem 6.4.8). Hence $w_k \rightarrow u$ in $(\mathcal{E}^1(X, \omega, \psi), d)$ as $k \rightarrow \infty$ by Theorem 6.2.2, and letting $k, j \rightarrow +\infty$, in this order, in the inequality

$$d(u, w_j) \geq d(u, w_k) - d(w_k, w_j) = 1 - d(w_k, w_j)$$

we find out the contradiction $0 \geq 1$, which concludes the proof. \square

Remark 6.4.30. As seen during the proof of Theorem D the d -coercivity of the ψ -relative Ding functional implies the existence of a $[\psi]$ -KE metric as soon as $\psi \in \mathcal{M}_{klt}^+$.

6.5 Why the prescribed singularities setting?

As stated in the Introduction, there are two main reasons to study these KE metrics with prescribed singularities: it is natural to look for canonical metrics which have prescribed singularities, and the following questions.

Question A. *Let (X, ω) be a Fano manifold. Is it possible to characterize the KE locus \mathcal{M}_{KE} ? When $\mathcal{M}_{KE} = \mathcal{M}_{klt}^+$? Is there some not-trivial locus on \mathcal{M}_{klt}^+ whose intersection with \mathcal{M}_{KE} implies that $0 \in \mathcal{M}_{KE}$?*

Theorem A, which gives a first answer to Question A, will be a consequence of the following two results and of Proposition 6.4.9.

Theorem 6.5.1 ([Tru20b], Theorem C). *Let $\psi \in \mathcal{M}_{klt}^+$. If D_ψ is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ with slope $1 > A > 0$, i.e. such that $D_\psi(u) \geq Ad(\psi, u) - B$ for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ where $B \geq 0$, then for any $\psi' \in \mathcal{M}_{klt}^+, \psi' \succ \psi$*

$$D_{\psi'}(u) \geq \left(1 - \frac{V_{\psi'}}{V_\psi}(1 - A)\right)d(\psi', u) - C$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi')$ where $C = C(B, V_\psi, X, \omega)$. In particular $D_{\psi'}$ is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi')$ for any $\psi' \succ \psi$ such that $V_{\psi'} < V_\psi/(1 - A)$.

Proposition 6.5.2. *Let $\psi \in \mathcal{M}_{klt}^+$. If $\alpha_\omega(\psi) > \frac{n}{n+1}$ then D_ψ, M_ψ are d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$. More precisely*

$$M_\psi(u) \geq \left(\frac{n+1}{n}\alpha - 1\right)d(u, \psi) - B_\alpha,$$

for any $\alpha \in (\frac{n}{n+1}, \alpha_\omega(\psi))$, while

$$D_\psi(u) \geq \left(\frac{n+1}{n^2}\alpha - \frac{1}{n}\right)^2 d(u, \psi) - B'_\alpha \quad (6.37)$$

for any $u \in \mathcal{E}_{norm}^1(X, \omega, \psi)$ and any $\frac{n}{n+1} < \alpha < \min\left\{\alpha_\omega(\psi), \frac{n^2 \min\{\beta_\omega(\psi), 1\} + n}{n+1}\right\}$ where $B, B'_\alpha \geq 0$ and $\beta_\omega(\psi)$ was defined in Remark 6.4.14. Moreover when $\alpha_\omega(\psi) > 1$ the slopes of the coercivity can be improved respectively to $\frac{n+1}{n}\alpha - \frac{1}{n}$ for any $\alpha \in (1, \alpha_\omega(\psi))$ and to $\left(\frac{n+1}{n^2}\alpha - \frac{1}{n^2}\right)^2$ for any $1 < \alpha < \min\left\{\alpha_\omega(\psi), \frac{n^2 \min\{\beta_\omega(\psi), 1\} + 1}{n+1}\right\}$.

Proof. By Proposition 6.4.27 there exists $C \geq 0$ such that

$$M_\psi(u) \geq (\alpha - 1)I_\psi(u) + J_\psi(u) - C \quad (6.38)$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$ and for any $\alpha < \alpha_\omega(\psi)$. Then by Proposition 6.4.21, if $\alpha \in (\frac{n}{n+1}, \alpha_\omega(\psi))$ we easily obtain

$$M_\psi(u) \geq \left(\alpha - 1 + \frac{1}{n+1}\right)I_\psi(u) - C \geq \frac{n+1}{n}\left(\alpha - \frac{n}{n+1}\right)J_\psi(u) - C$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$, which is equivalent to the requested d -coercivity on $\mathcal{E}_{norm}^1(X, \omega, \psi)$. Furthermore if $\alpha \geq 1$ then from (6.38) and again by Proposition 6.4.21 it follows that

$$M_\psi(u) \geq \left(\frac{n+1}{n}\alpha - \frac{n+1}{n} + 1\right)J_\psi(u) - C = \left(\frac{n+1}{n}\alpha - \frac{1}{n}\right)J_\psi(u),$$

which concludes the statements about the ψ -relative Mabuchi functional.

Next letting $A := (\frac{n+1}{n}\alpha - 1)$ for $\frac{n}{n+1} < \alpha < \min\left\{\alpha_\omega(\psi), \frac{n^2 \min\{\beta_\omega(\psi), 1\} + n}{n+1}\right\}$ we want to prove the d -coercivity of the ψ -relative Ding functional. We observe that by the proof of Proposition 6.4.29 the estimate (6.33) holds for $p = 1 + A/n$. Therefore in the implication (iii) \Rightarrow (i) the functional D_ψ is coercive with slope ϵ^2 for any

$$0 < \epsilon < \min\left\{\frac{A}{n}, \beta_\omega(\psi), 1\right\}.$$

But the assumption on α leads to $A \leq n \min\{\beta_\omega(\psi), 1\}$ and (6.37) follows. The case $\alpha_\omega(\psi) > 1$ is similar. \square

Theorem A. *Let (X, ω) be a Fano manifold. Then*

$$\left\{\psi \in \mathcal{M}_{ktt}^+ : \alpha_\omega(\psi) > \frac{n}{n+1}\right\} \subset \mathcal{M}_{KE}.$$

Moreover (i) \Rightarrow (ii) \Rightarrow (iii) in the following conditions:

i) there exists $\psi \in \mathcal{M}$, $t \in (0, 1]$ such that

$$\alpha_\omega(\psi_t) > \frac{n}{(n+1)t};$$

for $\psi_t := P_\omega[(1-t)\psi]$;

ii) $\alpha_\omega(0) > \frac{n}{n+1}$;

iii) $\mathcal{M}_{KE} = \mathcal{M}_{klt}^+$.

Furthermore $0 \in \mathcal{M}_{KE}$ if there exists $\psi \in \mathcal{M}_{klt}^+$ such that

$$\alpha_\omega(\psi) > \max \left\{ \frac{lct(X, 0, \psi)}{lct(X, 0, \psi) - 1}, \frac{n^2}{n+1} \left(\left(\frac{V_0 - V_\psi}{V_0} \right)^{1/2} + \frac{1}{n^2} \right) \right\} \quad (6.39)$$

where $lct(X, 0, \psi) = \sup\{p > 0 : (X, p\psi) \text{ is } klt\}$.

Proof. Suppose $\psi \in \mathcal{M}_{klt}^+$ such that $\alpha_\omega(\psi) > \frac{n}{n+1}$. Then the ψ -relative Ding functional is d -coercive over $\mathcal{E}_{norm}^1(X, \omega, \psi)$ as immediate consequence of Proposition 6.5.2. Therefore by its lower-semicontinuity with respect to the weak topology (Proposition 6.4.16) there exists a minimizer, which produces a $[\psi]$ -KE metric (see Theorem D and Remark 6.4.30). In particular the implication (ii) \Rightarrow (iii) follows from the monotonicity of $\alpha_\omega(\cdot)$.

Next if the assumption (i) holds then by Proposition 6.4.9 $\alpha_\omega(0) > \frac{n}{n+1}$.

For the last statement we suppose $\psi \in \mathcal{M}_{klt}^+$ and we give a more precise estimate than (6.39) in terms of the following quantity:

$$\gamma_\omega(\psi) := \min \left\{ \alpha_\omega(\psi), \frac{n^2 \min\{\beta_\omega(\psi), 1\} + n}{n+1} \right\}$$

where $\beta_\omega(\psi)$ is defined in Remark 6.4.14. Indeed by an easy computation from Proposition 6.5.2 we obtain that

$$\gamma_\omega(\psi) > \frac{n^2}{n+1} \left(\left(\frac{V_0 - V_\psi}{V_0} \right)^{1/2} + \frac{1}{n} \right) \quad (6.40)$$

implies that the ψ -relative Ding functional is d -coercive with slope

$$\left(\frac{n+1}{n^2} \gamma - \frac{1}{n} \right)^2$$

for any $\gamma < \gamma_\omega(\psi)$. Thus by Theorem 6.5.1 if (6.40) holds and γ is close to $\gamma_\omega(\psi)$ then we deduce that the usual Ding functional is d -coercive over $\mathcal{E}^1(X, \omega)$, hence there exists a KE metric. Note also that when $\alpha_\omega(\psi) \geq 1$ we can replace (6.40) with

$$\gamma_\omega(\psi) > \frac{n^2}{n+1} \left(\left(\frac{V_0 - V_\psi}{V_0} \right)^{1/2} + \frac{1}{n^2} \right). \quad (6.41)$$

Next if $\alpha_\omega(\psi) \geq \frac{lct(X, 0, \psi)}{lct(X, 0, \psi) - 1}$ then Remark 6.4.14 leads to

$$\min\{\beta_\omega(\psi), 1\} = 1$$

and (6.39) follows easily from (6.41). □

Corollary 6.5.3. *Let X be a Fano manifold and denote with $\mathcal{M}_{klt,strict}^+ := \{\psi \in \mathcal{M}_{klt}^+ : lct(X, 0, \psi) \geq \frac{n^2+1}{n^2-n}\}$. Then*

$$\sup_{\psi \in \mathcal{M}_{klt,strict}^+} \alpha_\omega(\psi) > \frac{n^2+1}{n+1} \implies 0 \in \mathcal{M}_{KE}.$$

Proof. It is an immediate consequence of Theorem A since $lct(X, 0, \psi) \geq \frac{n^2+1}{n^2-n}$ implies that the maximum on the right side in (6.39) is smaller than $\frac{n^2+1}{n+1}$. \square

The monotonicity of $\alpha_\omega(\cdot)$ and Theorem A may suggest that

$$0 \in \mathcal{M}_{KE} \implies \mathcal{M}_{KE} = \mathcal{M}_{klt}^+ \quad (6.42)$$

but when $\text{Aut}(X)$ is not finite, this is false as the following easy example shows. It also proves that (ii) cannot be replaced with $\alpha_\omega(0) \geq \frac{n}{n+1}$ and that $\alpha_\omega(\cdot)$ may be constant on some no trivial segment $\psi_t = t\psi$ for $\psi \in \mathcal{M}_{klt}^+$, $t \in [0, 1]$.

Example 6.5.4. Let $X = \mathbb{P}^2$, $\omega = \omega_{FS}$ and let $Z := \{p_1, p_2\} \in X$ be two distinct points. Then it is well-known that there exists a function $\varphi \in PSH(X, \omega)$ with analytic singularities formally encoded in $(\mathcal{I}_Z, 1)$. So, letting $\psi := P_\omega[\varphi] \in \mathcal{M}_{klt}^+$, by Proposition 6.4.5 and with the same notations, the set of $[\psi]$ -KE metrics is in bijection with the set of log KE metrics in the class $\{\eta\}$ for the weak log Fano pair (Y, Δ) where $Y = \text{Bl}_Z X$. But $\{\eta\} = -K_Y$ and $\Delta = 0$, so since Y does not have any KE metric we necessarily have $\psi \notin \mathcal{M}_{KE}$. Therefore by Theorem A and the monotonicity of $\alpha_\omega(\cdot)$, we necessarily have $\alpha_\omega(t\psi) = \frac{2}{3}$ for any $t \in [0, 1]$. Note that $\alpha_\omega(\psi)$ could also easily be computed explicitly since given $p_3 \in X$ not collinear to p_1, p_2 , there exists a function $\varphi \in PSH(X, \omega)$ such that $\nu(\varphi, p_3) = 3$ which implies $\alpha_\omega(\psi) \leq \frac{2}{3}$, while the lower bound $\alpha_\omega(\psi) \geq \frac{2}{3}$ follows from Theorem 6.4.10 (see also subsection 6.5.2).

However we think that the existence of no trivial holomorphic vector fields is the unique obstruction to (6.42), i.e. we pose the following conjecture.

Conjecture A. *Let (X, ω) be a Fano manifold such that $\text{Aut}(X)^\circ = \{Id\}$. Then*

$$0 \in \mathcal{M}_{KE} \implies \mathcal{M}_{KE} = \mathcal{M}_{klt}^+.$$

6.5.1 Strong continuity in \mathcal{M}_{KE} .

Here we prove our Theorem B.

Theorem B. *Let X be a Fano manifold and let $\{\psi_t\}_{t \in [0,1]} \subset \mathcal{M}_{klt}^+$ be a weakly continuous segment such that*

$$i) \ \psi_0 \in \mathcal{M}_{KE};$$

- ii) ψ_0 has small unbounded locus;
- iii) $\text{Aut}(X, [\psi_t])^\circ = \{\text{Id}\}$ for any $t \in [0, 1]$;
- iv) $\psi_t \preceq \psi_s$ if $t \leq s$;
- v) $\{\psi_t\}_{t \in [0, 1]} \subset \mathcal{M}_D$.

Then the set

$$S := \{t \in [0, 1] : \psi_t \in \mathcal{M}_{KE}\}$$

is open, the unique family of $[\psi_t]$ -KE metrics $\{\omega_{u_t}\}_{t \in S}$ is weakly continuous and the family of potentials $\{u_t\}_{t \in S}$ can be chosen so that the curve $S \ni t \rightarrow u_t \in \mathcal{E}^1(X, \omega, \psi_t)$ is strongly continuous.

Observe that by Proposition 6.3.7 the assumptions (iv), (v) are automatically satisfied for the segment $\psi_t := (1 - t)\psi$ if $\psi \in \mathcal{M}_{D, klt}^+$. We also recall that the strong convergence means that $u_t \rightarrow u$ weakly and $E_{\psi_t}(u_t) \rightarrow E_\psi(u)$ (section §6.2).

The strategy to prove Theorem B is to use the new continuity method introduced in the companion paper [Tru20b] where one *moves the prescribed singularities* instead of the density of the Monge-Ampère equation (see [Tru20b] for a mixed continuity method). Theorem 6.5.1 represents an openness result, which combined with the following closedness result will give the tools to prove Theorem B.

Theorem 6.5.5 ([Tru20b], Theorem D). *Let $\{\psi_k\}_{k \in \mathbb{N}} \subset \mathcal{M}_{klt}^+$ be a increasing sequence of model type envelopes converging weakly to $\psi \in \mathcal{M}_{klt}^+$. Assume that*

- i) ω_{u_k} is a sequence of $[\psi_k]$ -KE metrics where $u_k \in \mathcal{E}^1(X, \omega, \psi_k)$ minimizes D_{ψ_k} and it is normalized so that satisfies $MA_\omega(u_k) = e^{-u_k} \mu$ for any $k \in \mathbb{N}$;
- ii) the sequence $\{u_k\}_{k \in \mathbb{N}}$ is uniformly bounded from above, i.e. there exists $C \in \mathbb{R}$ such that $\sup_X u_k \leq C$ for any $k \in \mathbb{N}$.

Then there exists a subsequence u_{k_h} which converges strongly to $u \in \mathcal{E}^1(X, \omega, \psi)$ solution of $MA_\omega(u) = e^{-u} \mu$.

If $\psi_k \in \mathcal{M}_{D, klt}^+$ with small unbounded locus then by Theorem C the potential of any $[\psi_k]$ -KE metric maximizes D_{ψ_k} , so in this case (i) is part of the setting, and as noted in subsection §6.1.1 Theorem C extends to more general $\psi \in \mathcal{M}_{klt}^+$ as soon as one proves the linearity of the Monge-Ampère energy $E_\psi(\cdot)$ along weak geodesic segments (i.e. Theorem 6.3.11). Instead the assumption (ii) is the real obstruction to the closedness and it is also a necessary hypothesis as the curve considered in Remark 6.4.4 shows.

To prove Theorem B we will also use the following properties of the distances d and of the operator $P_\omega\cdot$.

Lemma 6.5.6 ([Tru19], Lemmas 3.7, 4.4 and Proposition 4.5). *Let $\psi_1, \psi_2 \in \mathcal{M}^+$ such that $\psi_2 \preceq \psi_1$. Then*

- i) for any $u, v \in \mathcal{E}^1(X, \omega, \psi_1)$ such that $u - v$ is globally bounded, $\|P_\omega[\psi_2](u) - P_\omega[\psi_2](v)\|_{L^\infty} \leq \|u - v\|_{L^\infty}$;
- ii) for any $u, v \in \mathcal{E}^1(X, \omega, \psi_1)$, $d(P_\omega[\psi_2](u), P_\omega[\psi_2](v)) \leq d(u, v)$;
- iii) there are two constants $A > 1, B > 0$ depending uniquely on n, X, ω such that

$$-d(u, \psi_1) \leq V_{\psi_1} \sup_X u = V_{\psi_1} \sup_X (u - \psi_1) \leq Ad(u, \psi_1) + B$$

for any $u \in \mathcal{E}^1(X, \omega, \psi)$.

Proof. of Theorem B

Step 1: Openness with respect to $\mathcal{T} := \{[a, b]\}_{a < b}$.

We first note that since $[0, 1] \ni t \rightarrow \psi_t \in \mathcal{M}_{klt}^+$ is weakly continuous and $\psi_t \preceq \psi_s$ if $t \leq s$ then

$$[0, 1] \ni t \rightarrow V_{\psi_t} = \int_X MA_\omega(\psi_t)$$

is continuous by what said in Section §6.2 (it follows from Lemma 3.12 in [Tru20a]). Thus combining Theorem D and Theorem 6.5.1 it immediately follows that S is open with respect to the induced topology given by the generating open sets $[a, b)$.

Step 2: Conclusion of the openness.

As a consequence of Step 1 to prove that S is open it is sufficient to show that given $t_0 \in S$, $t_0 > 0$ there exists $0 < \epsilon \ll 1$ small enough such that $(t_0 - \epsilon, t_0] \subset S$.

By Theorem D the ψ_{t_0} -Ding functional is coercive and there exists a unique $[\psi_{t_0}]$ -KE metric. We denote with $u_{t_0} \in \mathcal{E}^1(X, \omega, \psi_{t_0})$ its potential given as solution of the Monge-Ampère equation

$$\begin{cases} MA_\omega(u_{t_0}) = e^{-u_{t_0}} \mu \\ u_{t_0} \in \mathcal{E}^1(X, \omega, \psi_{t_0}). \end{cases}$$

Then we assume by contradiction that there exists a sequence $t_k \nearrow t_0$ such that $t_k \notin S$ for any $k \in \mathbb{N}$. By Theorem C this means that $D_{\psi_{t_k}}$ does not admit a minimizer for any $k \in \mathbb{N}$. Recalling that $D_{\psi_{t_k}}$ is translation invariant and lower-semicontinuous with respect to the weak topology (Proposition 6.4.16), we get that any minimizer sequence $\{u_{k,h}\}_{h \in \mathbb{N}} \subset \mathcal{E}_{norm}^1(X, \omega, \psi_{t_k})$, for $k \in \mathbb{N}$ fixed, necessarily satisfies $d(u_{k,h}, \psi_{t_k}) \rightarrow +\infty$ as $h \rightarrow \infty$. Indeed if $d(\psi_{t_k}, u_{k,h}) \leq C$ then

$$\{u_{k,h}\}_{h \in \mathbb{N}} \subset \mathcal{E}_C^1(X, \omega, \psi_{t_k}) := \{u \in \mathcal{E}^1(X, \omega, \psi_{t_k}) : \sup_X u \leq C, E_{\psi_{t_k}}(u) > -k\}$$

which is weakly compact (Proposition 6.2.1). Hence, up to considering a subsequence, $u_{k,h} \rightarrow u_k \in \mathcal{E}_C^1(X, \omega, \psi_{t_k})$ and

$$D_{\psi_{t_k}}(u_k) \leq \liminf_{h \rightarrow \infty} D_{\psi_{t_k}}(u_{k,h}) = \inf_{\mathcal{E}^1(X, \omega, \psi_{t_k})} D_{\psi_{t_k}},$$

which would lead to $t_k \in S$ by Theorem C.

Therefore we can fix a sequence $\{u_k\}_{k \in \mathbb{N}}$ such that $u_k \in \mathcal{E}^1(X, \omega, \psi_{t_k})$ for any $k \in \mathbb{N}$,

$d(u_k, \psi_{t_k}) \rightarrow +\infty$, $E_{\psi_{t_k}}(u_k) = 0$ and

$$D_{\psi_{t_k}}(u_k) < D_{\psi_{t_k}}(v_k)$$

where we set $v_k := P_\omega[\psi_{t_k}](u_{t_0}) - c_k$ for $c_k = V_{\psi_{t_k}} E_{\psi_{t_k}}(P_\omega[\psi_{t_k}](u_{t_0}))$. By continuity of the ψ -relative Ding functional with respect to decreasing sequences we may also assume without loss of generality that u_k has ψ_{t_k} -relative minimal singularities. We now claim that $v_k \rightarrow u_{t_0}^N := u_{t_0} - V_{\psi_{t_0}} E_{\psi_{t_0}}(u_{t_0})$ strongly, noting that by definition it is enough to prove that $\tilde{v}_k := P_\omega[\psi_{t_k}](u_{t_0})$ converges strongly to u_{t_0} . But u_{t_0} has ψ_{t_0} -relative minimal singularities, thus $\|\tilde{v}_k - \psi_k\|_{L^\infty} \leq \|u_{t_0} - \psi_{t_0}\|_{L^\infty} \leq C$ by Lemma 6.5.6. Therefore $E_{\psi_{t_k}}(\tilde{v}_k) \rightarrow E_{\psi_{t_0}}(u_{t_0})$ as consequence of Lemma 4.3 in [Tru19], and the claim follows.

In particular since $\int_X e^{-v_k} d\mu \rightarrow \int_X e^{-u_{t_0}^N} d\mu$ by Theorem 6.4.8 we also have

$$D_{\psi_{t_k}}(v_k) \rightarrow D_{\psi_{t_0}}(u_{t_0})$$

as $k \rightarrow \infty$. Next for $C = d(\psi_{t_0}, u_{t_0}^N) + 1$ fixed and $k \gg 1$ big enough we denote with $w_k \in \mathcal{E}^1(X, \omega, \psi_{t_k})$ the element on the weak geodesic segment joining v_k and u_k such that $d(\psi_{t_k}, w_k) = C$. Note that such sequence w_k exists since $d(\psi_{t_k}, v_k) \leq d(\psi_{t_0}, u_{t_0}^N)$ by Lemma 6.5.6. Moreover $E_{\psi_{t_k}}(w_k) = 0$ by linearity of the Monge-Ampère energy along weak geodesic segments (Theorem 6.3.11). Then by convexity of the ψ_{t_k} -Ding functional it follows that

$$D_{\psi_{t_k}}(w_k) < D_{\psi_{t_k}}(v_k)$$

for any $k \in \mathbb{N}$. Furthermore by Lemma 6.5.6 $|\sup_X w_k| \leq A$ uniformly since $d(\psi_{t_k}, w_k) = C$ and $V_{\psi_{t_k}} \geq V_{\psi_{t_0}} > 0$. Hence by compactness, up to considering a subsequence, $w_k \rightarrow w$ weakly where $w \in \mathcal{E}^1(X, \omega, \psi_{t_0})$ by Proposition 6.2.1 which also yields $E_{\psi_{t_0}}(w) \geq 0$. Thus since by Theorem 6.4.8 $\int_X e^{-w_k} d\mu \rightarrow \int_X e^{-w} d\mu$ we obtain

$$D_{\psi_{t_0}}(w) \leq \liminf_{k \rightarrow \infty} D_{\psi_{t_k}}(w_k) \leq \lim_{k \rightarrow \infty} D_{\psi_{t_k}}(v_k) = D_{\psi_{t_0}}(u_{t_0}) = \inf_{\mathcal{E}^1(X, \omega, \psi_{t_0})} D_{\psi_{t_0}} \leq D_{\psi_{t_0}}(w).$$

Therefore $D_{\psi_{t_k}}(w_k) \rightarrow D_{\psi_{t_0}}(w)$ which reads as $w_k \rightarrow w$ strongly. Moreover since $E_{\psi_{t_0}}(w) = E_{\psi_{t_0}}(u_{t_0}^N) = 0$ the uniqueness of solutions (Theorem C) implies $w = u_{t_0}$. Finally the contradiction is given by

$$d(\psi_{t_k}, w_k) - d(\psi_{t_k}, v_k) \leq d(v_k, w_k).$$

since, as $k \rightarrow +\infty$, the left hand side converges to 1 ($v_k \rightarrow u_{t_0}^N$ strongly) while the right hand side goes to 0.

Step 3: Strong Continuity.

Suppose $\{t_k\}_{k \in \mathbb{N}} \subset S$ be a converging sequence to $t_0 \in S$ and denote with $u_k \in \mathcal{E}^1(X, \omega, \psi_{t_k})$ the unique potential of the corresponding KE metric such that

$$\begin{cases} MA_\omega(u_k) = e^{-u_k} d\mu \\ u_k \in \mathcal{E}^1(X, \omega, \psi_{t_k}), \end{cases}$$

and similarly for $u \in \mathcal{E}^1(X, \omega, \psi_{t_0})$ potential for the $[\psi_{t_0}]$ -KE metric. Then to prove that $u_k \rightarrow u$ strongly, since $\{\psi_t\}_{t \in [0,1]}$ is totally ordered, it is enough to consider the two monotonically cases $t_k \nearrow t_0, t_k \searrow t_0$ and prove the result for a subsequence. In the case $t_k \searrow t_0$, Theorem 6.5.1 implies that there exist uniform coefficients for the coercivity if $k \gg 1$ big enough, i.e. there exists $A > 0, B \geq 0$ such that

$$D_{\psi_{t_k}}(v) \geq Ad(\psi_{t_k}, v) - B$$

for any $v \in \mathcal{E}_{norm}^1(X, \omega, \psi_{t_k})$. Thus since clearly $D_{\psi_{t_k}}(u_k) \leq D_{\psi_{t_k}}(\psi_{t_k}) \leq C_1$ uniformly, we obtain $d(u_k, \psi_{t_k}) \leq C_2$ uniformly. Hence by Lemma 6.5.6 we also have $|\sup_X u_k| \leq C_3$ uniformly and Theorem 6.5.5 concludes this case. If instead $t_k \nearrow t_0$ we first replace u_k, u respectively with $u_k^N := u_k - V_{\psi_{t_k}} E_{\psi_{t_k}}(u_k)$, $u^N := u - V_{\psi_{t_0}} E_{\psi_{t_0}}(u)$ so that they have null relative energies. Then proceeding as in Step 2 we necessarily have $d(\psi_{t_k}, u_k^N) \leq C_4$ uniformly, which again by Lemma 6.5.6 implies

$$\sup_X u_k^N \leq C_5. \quad (6.43)$$

Therefore by weak compactness and Proposition 6.2.1, up to considering a subsequence, it follows that $u_k^N \rightarrow \tilde{u} \in \mathcal{E}^1(X, \omega, \psi_{t_0})$. On the other hand the Monge-Ampère equations yields

$$V_{\psi_{t_k}} E_{\psi_{t_k}}(u_k) = \log V_{\psi_{t_k}} - \log \int_X e^{-u_k^N} d\mu,$$

and by Theorem 6.4.8 we deduce $E_{\psi_{t_k}}(u_k) \leq C_6$ uniformly. Hence (6.43) implies $\sup_X u_k \leq C_7$ and Theorem 6.5.5 concludes the proof. \square

Remark 6.5.7. Observe that when $\psi \in \mathcal{M}_{KE}$ does not belong to \mathcal{M}_D but its ψ -relative Ding functional is coercive with slope $A > 0$, then a natural way to connect it with other $\tilde{\psi} \in \mathcal{M}_D, \tilde{\psi} \succcurlyeq \psi$ as in Theorem B is to pass through the model type envelope $\psi' \in \mathcal{M}_D$ having the same singularity data of ψ . Indeed if

$$V_{\psi'} < V_{\psi}/(1 - A)$$

then by Theorem 6.5.1 the coefficients of the d -coercivity of D_{ψ_s} for $\psi_s := s\psi' + (1-s)\psi$ are uniformly bounded. Thus, proceeding as in the proof of Theorem B, appropriate potentials for the KE metrics are uniformly bounded from above and the strong continuity of Theorem B holds for this path as a consequence of Theorem 6.5.5.

6.5.2 0-dimensional equisingularities

As an consequence of Theorem A an estimate on ψ -relative α -invariants gives an estimate on the α -invariant in the absolute setting, which is often useful to detect if a Fano manifold admits a KE metric (see also Question A and Conjecture A).

Moreover by definition it is easier to produce lower bounds for $\alpha_\omega(\psi)$ with respect to finding lower bounds for the usual α -invariant (i.e. $\alpha_\omega(0)$).

Moreover Remark 6.4.4 and Corollary 6.5.3 suggest that the most natural case to consider is when the model type envelope has isolated singularities at N points with the same *weight* (i.e. with the same Lelong numbers at these points). Indeed in this way the singularities can have weight arbitrarily small (which clearly implies $\text{lct}(X, 0, \psi)$ arbitrarily big), the locus of the singularities is always 0-dimensional and the total mass V_ψ may basically be chosen arbitrary and independent on the weight of the singularities. In particular, roughly speaking, the set $\{\psi' \in \mathcal{M} : \psi' \preceq \psi\}$ have *few* elements if V_ψ is small enough and this makes the computation of $\alpha_\omega(\psi)$ easier as underlined before.

Furthermore intuitively we expect that, assuming $\text{Aut}(X)^\circ = \{\text{Id}\}$, if there exists a KE metric then it should be recovered by these KE metrics with 0-dimensional equisingularities when N moves to $+\infty$, and vice versa we expect that if the sequence of these KE metrics diverges then X should not admit a KE metric. This *point process* will be subject of study in future works.

To be more precise we first recall that given an nef line bundle L and $R := \{p_1, \dots, p_N\}$ a set of N distinct points on Y compact Kähler manifold, the *multipoint Seshadri constant* at p_1, \dots, p_N is defined as

$$\epsilon(L; p_1, \dots, p_N) := \sup\{a > 0 : f_N^* L - a\mathbb{E}_N \text{ is nef}\}$$

where $f_N : Z \rightarrow Y$ is the blow-up along R and $\mathbb{E}_N := \sum_{j=1}^N E_j$ the sum of the exceptional divisors (see [Dem90], [Laz04], [BDRH⁺09]). The definition extends to \mathbb{Q} -line bundle by rescaling and to \mathbb{R} -line bundle by continuity. Moreover $\epsilon(L; \cdot)$ is lower-semicontinuous and its supremum is reached outside a countable union of proper subvarieties, i.e. when the points are in *very general position*. In this case will indicate $\epsilon(L; N)$ for simplicity.

The characterization of multipoint Seshadri constants in terms of jets implies that given $N \in \mathbb{N}, \delta > 0$ there exists a ω -psh function $\varphi_{N,\delta}$ with analytic singularities formally encoded in (\mathcal{I}_Z, δ) if and only if $\delta < \epsilon(L; N)$. In particular, letting $\psi_{N,\delta} := P_\omega[\varphi_{N,\delta}]$ and $\eta_{N,\delta}$ the big and semipositive form given by $f^* \omega_{\varphi_{N,\delta}} = \eta_{N,\delta} + \delta[\mathbb{E}_N]$,

$$V_{\psi_{N,\delta}} = \text{Vol}_Z(\{\eta_{N,\delta}\}) = \text{Vol}_Y(L) - N\delta^n$$

where we indicated with $\text{Vol}_Y(L) = \int_X \omega^n$, $\text{Vol}_Z(\{\eta_{N,\delta}\}) = \int_Z \eta_{N,\delta}^n$. Observe also that $\psi_{N,\delta} \in \mathcal{M}_{\text{plt}}^+$ if and only if $\delta < n$ and that $\delta = \nu(\psi_{N,\delta}) := \sup_{y \in Y} \nu(\psi_{N,\delta}, y)$.

Then letting $\Theta \in H^{1,1}(Y, \mathbb{R})$ be a pseudoeffective cohomology class on a compact manifold Y , we call the quantity

$$\sigma(\Theta, y) := \sup\{a \geq 0 : f^* \Theta - aE \text{ is pseudoeffective}\}$$

the *pseudoeffective threshold of Θ at y* , where we denoted with $f : Z \rightarrow Y$ the blow-up at y and with E the exceptional divisor. When Θ is associated to a \mathbb{R} -line

bundle L , we will also replace Θ with L in the definitions since they are clearly cohomological.

Lemma 6.5.8. *Let $\Theta \in H^{1,1}(Y, \mathbb{R})$ be a pseudoeffective cohomology class on a compact manifold Y and let η be a smooth closed $(1,1)$ -form representative of Θ . Then*

$$\sup_{u \in PSH(Y, \eta)} \nu(u, y) = \sigma(\Theta, y) \text{ for any } y \in Y,$$

Proof. For any $u \in PSH(Y, \eta)$ and any $y \in Y$,

$$g^*(\eta_u) - \nu(u, y)E$$

is a closed and positive $(1,1)$ -current, where $g : Z \rightarrow Y$ is the blow-up at y and E the exceptional divisor. Thus

$$\sup_{u \in PSH(Y, \eta)} \nu(u, y) \leq \sigma(\Theta; y).$$

Vice versa if $g^*\Theta - aE$ is pseudoeffective there exists a positive and closed $(1,1)$ -current T representative of $g^*\Theta - aE$. Therefore the current $T + aE$ is closed and positive with cohomology class $g^*\Theta$. But this implies that there exists a closed and positive current S such that $f^*S = T + aE$ (see for instance Proposition 1.2.7.(ii) in [BouTh]). Thus by the $\partial\bar{\partial}$ -Lemma $S = \eta + dd^c u$ for $u \in PSH(Y, \eta)$, and by construction $\nu(u, y) = \inf_{z \in E} \nu(T + aE, z) \geq a$ where we recall that the Lelong number of a closed and positive $(1,1)$ -current is defined as the Lelong number of its potential once that a smooth form is fixed. Hence $\sup_{u \in PSH(Y, \eta)} \nu(u, y) \geq \sigma(\Theta; y)$ which concludes the proof. \square

We can now state the following final estimate for the $\psi_{N, \delta}$ -relative α -invariant.

Proposition 6.5.9. *Let $0 < \delta < \epsilon(-K_X; N)$ and let $\psi \in \mathcal{M}^+$ the model type envelopes with analytic singularity types formally encoded in (\mathcal{I}_S, δ) where $S = \{p_1, \dots, p_N\}$ is the set of points. We also set $L := f^*(-K_X) - \delta\mathbb{E}$ for the corresponding ample \mathbb{R} -line bundle, where with obvious notations $f : Y \rightarrow X$ is the blow-up at S and $\mathbb{E} := \sum_{j=1}^N E_j$ the sum of the exceptional divisors. Then letting*

$$\sigma_{exc}(L) := \sup_{y \in \mathbb{E}} \sigma(L, y),$$

$$\sigma_{gen}(L) := \sup_{y \in Y \setminus \mathbb{E}} \sigma(L, y),$$

$$\epsilon_{exc}(L) := \inf_{y \in \mathbb{E}} \epsilon(L, y),$$

$$\epsilon_{gen}(L) := \inf_{y \in Y \setminus \mathbb{E}} \epsilon(L, y),$$

we have

$$\alpha_\omega(\psi) \geq \frac{2}{\max\{\delta + \sigma_{exc}(L), \sigma_{gen}(L)\}}, \quad (6.44)$$

$$\alpha_\omega(\psi) \geq \min\left\{\frac{2}{\delta + \frac{V - \delta^n N}{\epsilon_{exc}(L)^{n-1}}}, \frac{2\epsilon_{gen}(L)^{n-1}}{V - \delta^n N}\right\} \quad (6.45)$$

where we set $V := \text{Vol}_X(-K_X) = (-K_X)^n$

Proof. Proposition 6.2.4 yields a bijection between $PSH(X, \omega, \psi) := \{u \in PSH(X, \omega) : u \preceq \psi\}$ and $PSH(Y, \eta)$ where η is a smooth closed $(1, 1)$ -form with cohomology class $c_1(L)$. Moreover denoting with $\tilde{u} \in PSH(X, \eta)$ the function corresponding to $u \in PSH(X, \omega, \psi)$, it follows by construction that

$$\nu(u; p_j) = \delta + \inf_{y \in E_j} \nu(\tilde{u}; y),$$

while $\nu(u, x) = \nu(\tilde{u}, f^{-1}x)$ if $x \notin S$. Thus we get

$$\delta + \sup_{y \in \mathbb{E}} \sup_{\tilde{u} \in PSH(Y, \eta)} \nu(\tilde{u}, y) \geq \sup_{x \in S} \sup_{u \preceq \psi} \nu(u, x), \quad (6.46)$$

$$\sup_{y \in Y \setminus \mathbb{E}} \sup_{\tilde{u} \in PSH(Y, \eta)} \nu(\tilde{u}, y) = \sup_{x \in X \setminus S} \sup_{u \preceq \psi} \nu(u, x). \quad (6.47)$$

Then by Lemma 6.5.8 the left hand sides in (6.46) and in (6.47) are equal respectively to $\delta + \sigma_{exc}(L)$ and $\sigma_{gen}(L)$. Therefore

$$\frac{2}{\max\{\delta + \sigma_{exc}(L), \sigma_{gen}(L)\}} \leq \frac{2}{\sup_{x \in X} \sup_{u \preceq \psi} \nu(u, x)},$$

which implies (6.44). Indeed combining Theorem 6.4.10 and Lemma 6.4.7 $\alpha_\omega(\psi) \geq \alpha$ for any $\alpha > 0$ such that

$$\alpha < \frac{2}{\sup_{x \in X} \sup_{u \preceq \psi} \nu(u, x)}.$$

Next (6.45) is a consequence of (6.44) since

$$\sigma(L; y) \epsilon(L; y)^{n-1} \leq \text{Vol}_X(L).$$

holds for any $y \in Y$. One easy way to check this last inequality is through the convexity of the Okounkov body of L at y with respect to an *infinitesimal flag* (see [LM09], [KL17]). \square

Remark 6.5.10. As seen during the proof of Proposition 6.5.9 the lower bound in terms of the pseudoeffective thresholds is sharper than the one given by the Seshadri constants. Anyway giving upper bounds for the pseudoeffective threshold is usually harder than finding lower bounds for the Seshadri constant. Moreover the latter is much more studied in the literature since it is related to different famous problems in Algebraic Geometry (see [BDRH⁺09]).

We conclude the article with the following easy example of a K -unstable Fano manifold which admits a $[\psi]$ -KE metric with isolated singularities at N points of weight δ using Proposition 6.5.9.

Example 6.5.11. Let $S_1 = \text{Bl}_p \mathbb{P}^2$ endowed with a Kähler form ω . Since $\epsilon(-K_{S_1}; 6) > 1$, we consider $\psi \in \mathcal{M}^+$ with isolated singularities respectively at 6 points in very general position of weight $\delta = 1$. Thus letting $f : S_7 \rightarrow S_1$ the blow-up at these points, the line bundle $L = f_6^*(-K_{S_1}) - \mathbb{E}_6$ coincides with the anticanonical bundle $-K_{S_7}$.

Then one way to produce a $[\psi]$ -KE current is through Proposition 6.4.5 since S_7 admits a KE metric. Anyway here we want to show that there exists a $[\psi]$ -KE current producing a lower bound for $\alpha_\omega(\psi)$ by Proposition 6.5.9 and using Theorem A. Indeed, since $\epsilon(-K_{S_7}; y) = 4/3$ if $y \in S_7$ is general and $\epsilon(-K_{S_7}; y) = 1$ otherwise ([Bro06]), from Proposition 6.5.9 we easily have

$$\alpha_\omega(\psi) \geq \frac{4}{5}, \quad (6.48)$$

and Theorem A gives $\psi \in \mathcal{M}_{KE}$ since (X, ψ) is klt. Observe also that the estimate in (6.48) is better than $\alpha_\omega(\psi) > \frac{3}{4}$, which seems to be the known lower bound for the usual α -invariant $\alpha(S_7, 0)$ (see [Chel08]).

Moreover considering $\psi_t := (1-t)\psi$ for $t \in [0, 1]$, by the right continuity of $\alpha_\omega(\cdot)$ (Proposition 6.4.9) it follows that $\alpha_\omega(\psi_t) > \frac{2}{3}$ for any $0 \leq t \ll 1$ small enough. Hence Theorem A produces the existence of a $[\psi_t]$ -KE current for any $0 \leq t \ll 1$ big enough. Note that using Proposition 6.4.9 .(ii), which *has not* restriction on \mathcal{M}_{klt} , it is possible to produce good estimate on the largest $t \in (0, 1]$ such that $\alpha_\omega(\psi_t) > \frac{2}{3}$, i.e. on $S := \{t \in [0, 1] : \psi_t \in \mathcal{M}_{KE}\}$. Obviously $S \neq [0, 1]$ since X does not admit a KE metric.

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