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The two-mass contribution to the three-loop polarized gluonic operator matrix element $A_{gg,Q}^{(3)}$

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Abstract

We compute the two-mass contributions to the polarized massive operator matrix element $A_{gg,Q}^{(3)}$ at third order in the strong coupling constant α_s in Quantum Chromodynamics analytically. These corrections are important ingredients for the matching relations in the variable flavor number scheme and for the calculation of Wilson coefficients in deep–inelastic scattering in the asymptotic regime $Q^2 \gg m_c^2, m_b^2$. The analytic result is expressed in terms of nested harmonic, generalized harmonic, cyclotomic and binomial sums in N-space and by iterated integrals involving square-root valued arguments in z space, as functions of the mass ratio. Numerical results are presented. New two–scale iterative integrals are calculated.

1. Introduction

The massive operator matrix elements (OMEs) of twist–2 operators emerge in the asymptotic representation of the massive Wilson coefficients in deeply inelastic scattering [1] in the region

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$Q^2 \gg m_c^2, m_b^2$. Here $m_i = m_{c(b)}$ are the charm and bottom quark masses[1] and Q^2 denotes the virtuality of the deeply inelastic process. They also appear as the matching coefficients in the variable flavor number scheme (VFNS). Various of these OMEs have been calculated already in the unpolarized and polarized case in Refs. [1–22]. Furthermore, two–mass corrections contribute from the two–loop corrections onward, [23], with one–particle irreducible contributions starting at the three–loop level [20,24–28]. At 3–loop order also the contributing anomalous dimensions are obtained as a by-product of the calculation of the massive OMEs [2,9,29,30], confirming the results obtained in massless calculations [31–33].

In performing the different calculation steps we use the algorithms encoded in the packages `Sigma` [34,35], `HarmonicSums` [36–43], `EvaluateMultiSums` and `SumProduction` [44]. The computation follows the methods used in the unpolarized case [25].

In the present paper we calculate the polarized three–loop massive operator matrix element $A_{gg,Q}^{(3)}$ using dimensional regularization in $d = 4 + \varepsilon$ dimensions. The computation is performed in the Larin scheme [45].[2] The renormalization procedure is the same as in the unpolarized case, cf. [25,26]. We perform the calculation of the massive OME first in Mellin N space and switch then to momentum fraction z space by performing an inverse Mellin transform analytically. Various calculation techniques applied are described in Ref. [46]. In both spaces the results are expressed by special functions introduced in Refs. [36–40,47,48]. Starting from the result in N space, the inverse Mellin transform to z space is performed by finding a recurrence relation satisfied by the various sums, and by using the properties of the inverse Mellin transform to build a differential equation that the Mellin inverse must satisfy, see Refs. [49–51]. The differential equation is then solved. These methods are implemented in `HarmonicSums`.

The paper is organized as follows. In Section 2 we summarize the details of the calculation. In many technical aspects we will refer to the calculation in the unpolarized case [25] and to Ref. [26]. In Section 3, fixed moments are calculated for the non-pole part of the unrenormalized massive OME by a different method to have an étalon to check the present results in Mellin N space. The result in Mellin N space is derived in Section 4. In Section 5 we present the transformation to z space, and in Section 6 we derive numerical results comparing the size of the two–mass effects to the complete contributions of $O(C_F T_F^2)$. Section 7 contains the conclusions. In Appendix A, a series of new iterative integrals are presented and relations between different iterated integrals are listed in Appendix B beyond those given in Ref. [25] before.

2. Details of the calculation

As it is well–known from Ref. [25], the new contribution is the non-pole part of the unrenormalized polarized two mass OME $\hat{A}_{gg,Q}^{(3)}$, since the pole contributions are predicted by the renormalization group equations for the given quantity and are thus determined by lower order terms, which are already known. We first compute the irreducible contributions to the unrenormalized polarized OME $\hat{A}_{gg,Q}^{(3)}$. The reducible contributions were given in Ref. [52]. There are seventy different diagrams contributing. Since many of them have the same value, one may group them into eleven equivalence classes, see Table 1. The number is smaller than in the unpolarized

[1] We will use the on–shell scheme for the mass renormalization. The transformation to the $\overline{\mathrm{MS}}$ scheme is straightforward, cf. [2].

[2] For the anomalous dimensions in the Larin scheme see [30,33].

Table 1
Symmetry properties under $m_1 \leftrightarrow m_2$ and multiplicity of the eleven diagrams of Fig. 1. The multiplicity refers to the sum of the two mass assignments.

Diagram	Multiplicity	Diagram	Multiplicity
1	3	7	4
2	2	8	4
3	1	9	2
4	4	10	1
5	2	11	4
6	8		

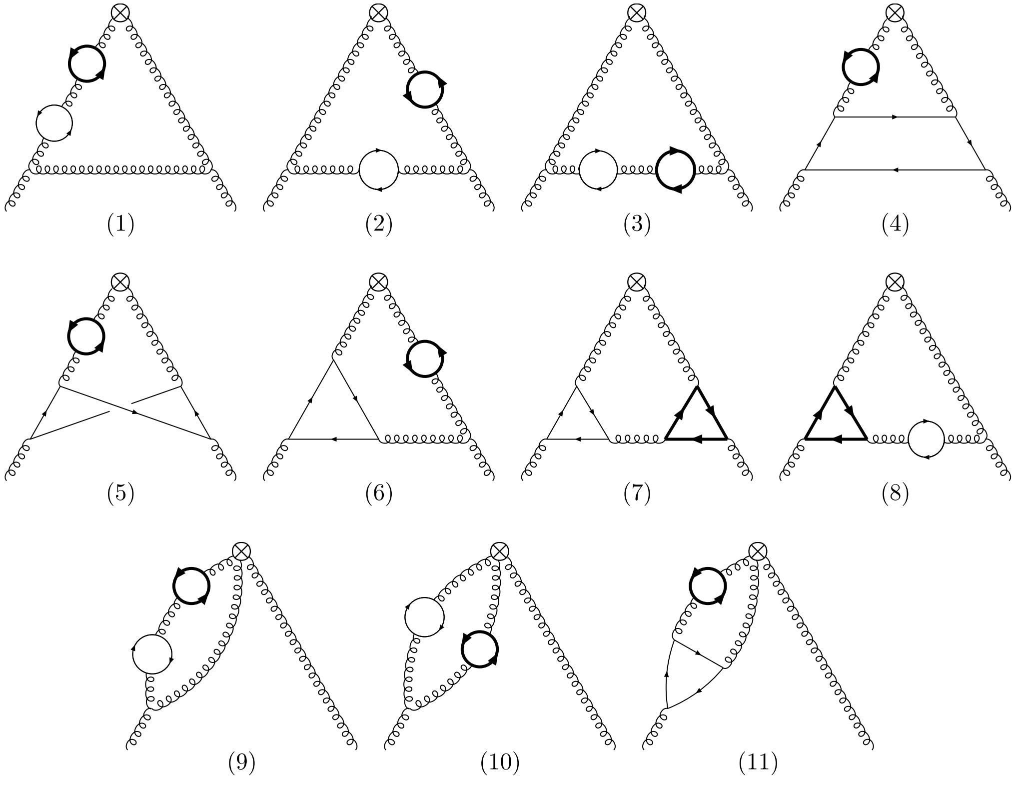

Fig. 1. The 11 different topologies for $\tilde{A}_{gg,Q}^{(3)}$. Curly lines: gluons; thin arrow lines: lighter massive quark; thick arrow lines: heavier massive quark; the symbol $\otimes$ represents the corresponding local operator insertion, cf. [55] for the related Feynman rules.

case, because some of the diagrams being present there vanish in the polarized case. One example graph for each class is depicted in Fig. 1.

We use the notation

$$\eta = \frac{m_2^2}{m_1^2} < 1, \qquad L_1 = \log\left(\frac{m_1^2}{\mu^2}\right), \qquad L_2 = \log\left(\frac{m_2^2}{\mu^2}\right), \tag{2.1}$$

where m_i are the unrenormalized quark masses, and μ the renormalization and factorization scale. The total result for $\hat{A}_{gg,Q}^{(3)}$ is symmetric interchanging the masses $m_1 \leftrightarrow m_2$. For diagrams which are not symmetric, it is always possible to recover one mass assignment from the other using the simultaneous exchange

$$m_1 \leftrightarrow m_2, \qquad \eta \to \frac{1}{\eta}. \tag{2.2}$$

However, as a cross–check on the computation, for all asymmetric diagrams, except diagram 8 and 11, we choose to compute the two mass assignments independently, using a different Mellin–Barnes decomposition for the two cases, and we explicitly check that the results are related by Eq. (2.2).

The Feynman diagrams are generated using `QGRAF` [53] and the Dirac algebra is performed by using `FORM` [54]. The Feynman diagrams contain local operator insertions, cf. [26], and the corresponding Feynman rules were given in [55,56]; see also [30]. We work in the Larin scheme [45]. The unrenormalized OME is obtained by applying to the Green's function for the eleven irreducible diagrams the projector [57,58]

$$A_{gg,Q} = \frac{\delta^{ab}}{N_c^2-1}\frac{1}{(d-2)(d-3)}(\Delta\cdot p)^{-N-1}\varepsilon^{\mu\nu\rho\sigma}\Delta\hat{G}^{ab}_{Q,\mu\nu}\Delta_\rho p_\sigma\,, \tag{2.3}$$

where p is the external momentum, with $p^2=0$, Δ is a lightlike momentum, N_c is the number of colours, and $\varepsilon^{\alpha_1\alpha_2\alpha_3\alpha_4}$ denotes the Levi–Civita tensor. The one–particle reducible terms have still to be added, cf. [26]. Ghost diagrams do not contribute to the amplitudes.

The renormalized expression for the OME is given by [25]

$$\begin{aligned}
\tilde{A}_{gg,Q}^{(3),\overline{\mathsf{MS}}} = {} & \left\{\frac{25}{24}\beta_{0,Q}^2\gamma_{gg}^{(0)} + \frac{25}{12}\beta_0\beta_{0,Q}^2 + \frac{9}{2}\beta_{0,Q}^3 + \frac{23}{96}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)}\right\}\left(L_1^3+L_2^3\right) \\
& + \left\{\frac{1}{8}\hat{\gamma}_{qg}^{(0)}\beta_{0,Q}\gamma_{gq}^{(0)} + \beta_{0,Q}^2\gamma_{gg}^{(0)} + 2\beta_0\beta_{0,Q}^2 + 6\beta_{0,Q}^3\right\}\left(L_1^2L_2+L_2^2L_1\right) \\
& + \left\{-\frac{1}{4}\beta_{1,Q}\beta_{0,Q} + \frac{13}{16}\beta_{0,Q}\hat{\gamma}_{gg}^{(1)} + \frac{29}{4}\delta m_1^{(-1)}\beta_{0,Q}^2 - \frac{1}{64}\hat{\gamma}_{qg}^{(0)}\hat{\gamma}_{gq}^{(1)}\right\}\left(L_1^2+L_2^2\right) \\
& + 8L_2L_1\delta m_1^{(-1)}\beta_{0,Q}^2 + \left\{\frac{9}{4}\beta_0\beta_{0,Q}^2\zeta_2 + \frac{27}{2}\beta_{0,Q}^3\zeta_2 - 3\beta_{0,Q}\tilde{\delta}m_2^{(-1)}\right. \\
& \left. + \frac{9}{8}\zeta_2\beta_{0,Q}^2\gamma_{gg}^{(0)} + 12\delta m_1^{(0)}\beta_{0,Q}^2 + \frac{3}{32}\beta_{0,Q}\zeta_2\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} + 6\beta_{0,Q}a_{gg,Q}^{(2)}\right\}(L_2+L_1) \\
& - \frac{1}{32}\hat{\gamma}_{qg}^{(0)}\zeta_2\hat{\gamma}_{gq}^{(1)} + \frac{1}{8}\beta_{0,Q}\zeta_2\hat{\gamma}_{gg}^{(1)} + \frac{1}{3}\beta_0\beta_{0,Q}^2\zeta_3 + 12\beta_{0,Q}\overline{a}_{gg,Q}^{(2)} \\
& + 6\beta_{0,Q}^3\zeta_3 + 16\delta m_1^{(1)}\beta_{0,Q}^2 + \frac{1}{6}\beta_{0,Q}^2\zeta_3\gamma_{gg}^{(0)} - 2\beta_{0,Q}\left(\tilde{\delta}m_2{}^{1,(0)} + \tilde{\delta}m_2{}^{2,(0)}\right) \\
& + \frac{9}{2}\delta m_1^{(-1)}\beta_{0,Q}^2\zeta_2 - \frac{1}{24}\zeta_3\beta_{0,Q}\gamma_{gq}^{(0)}\hat{\gamma}_{qg}^{(0)} - \frac{1}{2}\zeta_2\beta_{0,Q}\beta_{1,Q} + \tilde{a}_{gg,Q}^{(3)}\left(m_1^2,m_2^2,\mu^2\right).
\end{aligned} \tag{2.4}$$

Here, β_i and $\beta_{Q,i}$ are expansion coefficients of the $\overline{\mathsf{MS}}$ scheme and background field β functions, $\gamma_{ij}^{(k)}$ are anomalous dimensions, $\delta m_k^{(l)}$ are expansion parameters of the unrenormalized quark masses and $a_{ij}^{(k)}$ and $\overline{a}_{ij}^{(k)}$ are lower loop expansion coefficients of massive OMEs. $\zeta_k,\ k \geq 2$ denotes the Riemann ζ function at integer arguments [59], see Refs. [25,26] for details.

Because the number of classes of diagrams is small, we choose to derive one Feynman parameterization per diagram and we do not cancel numerator and denominator structures, nor do we reduce to master integrals. This allows us to deal with a uniform denominator structure which we find to be simpler to treat. The additional complexity potentially induced on the numerators by this choice still turns out to be manageable. For quark bubble insertions, we use the relation

$$\Pi_{ab}^{\mu\nu}(k) = -i\frac{8T_F g^2}{(4\pi)^{d/2}}\delta_{ab}(k^2 g^{\mu\nu} - k^\mu k^\nu)\int_0^1 \mathrm{d}x \frac{\Gamma(2-d/2)\,(x(1-x))^{d/2-1}}{\left(-k^2 + \frac{m^2}{x(1-x)}\right)^{2-d/2}}. \tag{2.5}$$

Once the Feynman parameterization has been obtained for the whole diagram, the integrals over the loop momenta are performed one after another, using the tensor identities

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k_{\mu_1} k_{\mu_2} f(k^2) = \frac{g_{\mu_1\mu_2}}{d}\int \frac{\mathrm{d}^d k}{(2\pi)^d} k^2 f(k^2), \tag{2.6}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4} f(k^2) = \frac{S_{\mu_1\mu_2\mu_3\mu_4}}{d(d+2)}\int \frac{\mathrm{d}^d k}{(2\pi)^d} (k^2)^2 f(k^2), \tag{2.7}$$

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} k_{\mu_1} k_{\mu_2} k_{\mu_3} k_{\mu_4} k_{\mu_5} k_{\mu_6} f(k^2) = \frac{S_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}}{d(d+2)(d+4)}\int \frac{\mathrm{d}^d k}{(2\pi)^d} (k^2)^3 f(k^2), \tag{2.8}$$

with the symmetric tensors

$$S_{\mu_1\mu_2\mu_3\mu_4} = g_{\mu_1\mu_2}g_{\mu_3\mu_4} + g_{\mu_1\mu_3}g_{\mu_2\mu_4} + g_{\mu_1\mu_4}g_{\mu_2\mu_3} \tag{2.9}$$

$$\begin{aligned} S_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} = {} & g_{\mu_1\mu_2}\left[g_{\mu_3\mu_4}g_{\mu_5\mu_6} + g_{\mu_3\mu_5}g_{\mu_4\mu_6} + g_{\mu_3\mu_6}g_{\mu_4\mu_5}\right] \\ & + g_{\mu_1\mu_3}\left[g_{\mu_2\mu_4}g_{\mu_5\mu_6} + g_{\mu_2\mu_5}g_{\mu_4\mu_6} + g_{\mu_2\mu_6}g_{\mu_4\mu_5}\right] \\ & + g_{\mu_1\mu_4}\left[g_{\mu_2\mu_3}g_{\mu_5\mu_6} + g_{\mu_2\mu_5}g_{\mu_3\mu_6} + g_{\mu_2\mu_6}g_{\mu_3\mu_5}\right] \\ & + g_{\mu_1\mu_5}\left[g_{\mu_2\mu_3}g_{\mu_4\mu_6} + g_{\mu_2\mu_4}g_{\mu_3\mu_6} + g_{\mu_2\mu_6}g_{\mu_3\mu_4}\right] \\ & + g_{\mu_1\mu_6}\left[g_{\mu_2\mu_3}g_{\mu_4\mu_5} + g_{\mu_2\mu_4}g_{\mu_3\mu_5} + g_{\mu_2\mu_5}g_{\mu_3\mu_4}\right]. \end{aligned} \tag{2.10}$$

The scalar integrals can then be performed using the relation

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d}\frac{(k^2)^m}{(k^2+R^2)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(n-m-d/2)}{\Gamma(n)}\frac{\Gamma(m+d/2)}{\Gamma(d/2)}\left(R^2\right)^{m-n+d/2}. \tag{2.11}$$

After this step, only integrations over the Feynman parameters remain. They can always be cast into the form

$$\prod_{i=1}^{j}\int_0^1 \mathrm{d}x_i\ x_i^{a_i}(1-x_i)^{b_i}\ R_0^N\ \left[R_1\, m_1^2 + R_2\, m_2^2\right]^{-s}, \tag{2.12}$$

where R_0 is a polynomial in the Feynman parameters x_i, and R_1 and R_2 are rational functions in x_i. At this point the polynomial R_0 can be treated by applying the binomial theorem (multiple times if necessary)

$$(A+B)^N = \sum_{i=0}^{N} \binom{N}{i} A^i B^{N-i} \tag{2.13}$$

and a Mellin–Barnes decomposition [60–64] is applied to the factor $\left[R_1\, m_1^2 + R_2\, m_2^2\right]^{-s}$,

$$\frac{1}{(A+B)^s} = \frac{1}{2\pi i}\frac{1}{\Gamma(s)} B^{-s} \int_{-i\infty}^{+i\infty} d\sigma \left(\frac{A}{B}\right)^{\sigma} \Gamma(-\sigma)\Gamma(\sigma+s), \tag{2.14}$$

where the integration contour must separate the ascending from the descending poles in the Γ–functions. After appropriately closing the contour at infinity, this integral is turned into a number of infinite sums using the residue theorem. At this point the integrals over x_i can always be expressed in terms of Beta functions.

The effect of these steps is to transform expressions having the form (2.12), i.e. integrals over Feynman parameters, into nested sums. This sum representation can be handled by `Sigma`, `EvaluateMultiSums`, `SumProduction`, assisted by `HarmonicSums`.

We checked our computation using the packages `MB` [65] and `MBResolve` [66] to resolve the singularity structure of the Mellin–Barnes integrals numerically and to compute the ε-poles of the various diagrams for fixed values of N, and compared them to the general N results obtained using `Sigma`.

The target function space for the result is that of harmonic sums [36,37]

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{(\text{sign}(b))^k}{k^{|b|}} S_{\vec{a}}(k),\, a_i, b \in \mathbb{Z}\backslash\{0\},\, N \in \mathbb{N},\, S_{\emptyset} = 1, \tag{2.15}$$

generalized harmonic sums [38,47,67]

$$S_{b,\vec{a}}(d,\vec{c})(N) = \sum_{k=1}^{N} \frac{d^k}{k^{|b|}} S_{\vec{a}}(\vec{c})(k),\, a_i, b \in \mathbb{N}\backslash\{0\},\, c_i, d \in \mathbb{C}(\sqrt{\eta})\backslash\{0\},\, N \in \mathbb{N},\, S_{\emptyset} = 1, \tag{2.16}$$

cyclotomic sums [39] and binomial sums [40]. Here the index d denotes a function of $\sqrt{\eta}$. Harmonic polylogarithms [48] will also appear in the result. The procedure adopted here is the same as used in Ref. [25], where the corresponding unpolarized OME was computed. There, the reader can find further details and also explicit examples about the computational method.

The renormalization procedure is the same as in the unpolarized case, including the removal of collinear singularities, cf. [26].

All structures of the massive OME are predicted by the renormalization or are available from lower loop results, except for the non-pole term in the two–mass case, $\tilde{a}_{gg,Q}^{(3)}$, of the unrenormalized three–loop OME.

First we turn to the calculation of fixed moments of this quantity.

3. Fixed moments of $\tilde{a}_{gg,Q}^{(3)}$

Here we present the results for the fixed moments $N = 3$ and 5 of the massive OME including the reducible contributions. They were computed using the packages `Q2E/EXP` [68,69] up to $\mathcal{O}(\eta^2)$ and $\mathcal{O}(\eta)$, respectively. In what follows we use the abbreviation

$$L_\eta = \ln(\eta). \tag{3.1}$$

The moments are given by

$$\begin{aligned}
&\tilde{a}_{gg,Q}^{(3)}(N=3)\\
&= C_F T_F^2 \Biggl\{ -\frac{444341}{11664} + \Biggl(-\frac{1031}{18} + \frac{25L_1}{9} + \frac{25L_2}{9} \Biggr)\zeta_2 - \frac{220}{81}\zeta_3 - \frac{22121}{648}L_1 \\
&\quad -\frac{5635}{108}L_1^2 + \frac{155}{81}L_1^3 - \frac{76411}{1944}L_2 - \frac{1822}{27}L_1L_2 + \frac{10}{27}L_1^2L_2 - \frac{5635}{108}L_2^2 \\
&\quad +\frac{50}{27}L_1L_2^2 + \frac{115}{81}L_2^3 + \eta\Biggl(\frac{5528}{675} + \frac{112}{45}L_\eta\Biggr) + \eta^2\Biggl(-\frac{6202718}{1157625} + \frac{32804}{11025}L_\eta \\
&\quad -\frac{116}{105}L_\eta^2\Biggr)\Biggr\} + C_A T_F^2 \Biggl\{ -\frac{2146957}{8748} + \Biggl(-\frac{3070}{81} - \frac{560}{9}L_1 - \frac{560}{9}L_2 \Biggr)\zeta_2 \\
&\quad +\frac{320}{81}\zeta_3 - \frac{94403}{486}L_1 - \frac{3005}{81}L_1^2 - \frac{2320}{81}L_1^3 - \frac{16781}{162}L_2 - \frac{3200}{81}L_1L_2 \\
&\quad -\frac{800}{27}L_1^2L_2 - \frac{3005}{81}L_2^2 - \frac{1120}{27}L_1L_2^2 - \frac{2000}{81}L_2^3 + \eta\Biggl(-\frac{1869176}{30375} \\
&\quad +\frac{61346}{2025}L_\eta - \frac{313}{135}L_\eta^2\Biggr) + \eta^2\Biggl(-\frac{94073888}{10418625} + \frac{486314}{99225}L_\eta - \frac{1126}{945}L_\eta^2\Biggr)\Biggr\} \\
&\quad +T_F^3\Biggl\{\Biggl(32L_1 + 32L_2\Biggr)\zeta_2 + \frac{128}{9}\zeta_3 + \frac{32}{3}L_1^3 + \frac{64}{3}L_1^2L_2 + \frac{64}{3}L_1L_2^2 + \frac{32}{3}L_2^3\Biggr\} \\
&\quad +\mathcal{O}\Bigl(\eta^3 L_\eta^3\Bigr),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\tilde{a}_{gg,Q}^{(3)}(N=5)\\
&= C_F T_F^2 \Biggl\{ -\frac{20217949}{455625} + \Biggl(-\frac{12976}{225} + \frac{56}{45}L_1 + \frac{56}{45}L_2 \Biggr)\zeta_2 - \frac{2464}{2025}\zeta_3 \\
&\quad -\frac{1202098}{30375}L_1 - \frac{177256}{3375}L_1^2 + \frac{1736}{2025}L_1^3 - \frac{411926}{10125}L_2 - \frac{229408}{3375}L_1L_2 \\
&\quad +\frac{112}{675}L_1^2L_2 - \frac{177256}{3375}L_2^2 + \frac{112}{135}L_1L_2^2 + \frac{1288}{2025}L_2^3 + \eta\Biggl(\frac{52768}{10125} + \frac{1984}{675}L_\eta\Biggr)\Biggr\} \\
&\quad +C_A T_F^2 \Biggl\{ -\frac{466789273}{1366875} + \Biggl(-\frac{125176}{2025} - \frac{3724}{45}L_1 - \frac{3724}{45}L_2 \Biggr)\zeta_2 + \frac{2128}{405}\zeta_3 \\
&\quad -\frac{2733284}{10125}L_1 - \frac{120092}{2025}L_1^2 - \frac{15428}{405}L_1^3 - \frac{4547582}{30375}L_2 - \frac{135344}{2025}L_1L_2 \\
&\quad -\frac{1064}{27}L_1^2L_2 - \frac{120092}{2025}L_2^2 - \frac{7448}{135}L_1L_2^2 - \frac{2660}{81}L_2^3 + \eta\Biggl(-\frac{21339914}{253125} \\
&\quad +\frac{73016}{1875}L_\eta - \frac{11146}{3375}L_\eta^2\Biggr)\Biggr\} + T_F^3\Biggl\{(32L_1 + 32L_2)\zeta_2 + \frac{128}{9}\zeta_3 + \frac{32}{3}L_1^3 \\
&\quad +\frac{64}{3}L_1^2L_2 + \frac{64}{3}L_1L_2^2 + \frac{32}{3}L_2^3\Biggr\} + \mathcal{O}\Bigl(\eta^2 L_\eta^2\Bigr),
\end{aligned} \tag{3.3}$$

where $C_F = (N_c^2 - 1)/(2N_c)$, $T_F = 1/2$, $C_A = N_c$ for $SU(N_c)$ and $N_c = 3$ in the case of QCD. The calculational methods used in the packages Q2E/EXP are very different of those of the present calculation. Therefore, these results, in form of an expansion in η, can be used for comparison.

We have compared also the fixed moments of each diagram to their calculation at general values of N. For diagrams 1–8 we also calculated the first moment.

4. The result in Mellin space

In Mellin N space the renormalized massive OME $\tilde{A}_{gg,Q}^{(3),\overline{\rm MS}}$ is given by

$$
\begin{aligned}
\tilde{A}_{gg,Q}^{(3),\overline{\rm MS}} = & \left[-\frac{184}{9} C_F T_F^2 h_1 + \frac{400}{27} C_A T_F^2 h_2 - \frac{32}{3} T_F^3\right]\left(L_1^3 + L_2^3\right) \\
& + \left[-\frac{32}{3} C_F T_F^2 h_1 + \frac{128}{9} C_A T_F^2 h_2 - \frac{128}{9} T_F^3\right]\left(L_1^2 L_2 + L_1 L_2^2\right) \\
& + \frac{256}{3} C_F T_F^2 L_1 L_2 + \Biggl[C_A T_F^2\left(-\frac{4T_2}{27N^2(N+1)^2} + \frac{1040}{27} S_1\right) \\
& + C_F T_F^2 \left(\frac{8T_4}{9N^3(N+1)^3} - \frac{16}{3} h_1 S_1\right)\Biggr]\left(L_1^2 + L_2^2\right) \\
& + \Biggl[C_A T_F^2\left(-\frac{16T_5}{27N^3(N+1)^3} + \frac{32(56N+47)}{27(N+1)} S_1 + \frac{112}{3}\zeta_2 h_2\right) \\
& + C_F T_F^2\left(-\frac{32T_6}{3N^4(N+1)^4} - 40\zeta_2 h_1\right) - 32 T_F^3 \zeta_2\Biggr](L_1 + L_2) \\
& + C_F T_F^2 \Biggl\{\frac{4T_8}{9N^5(N+1)^5\eta} + \frac{16T_9\zeta_2}{9N^3(N+1)^3\eta^2} \\
& - \frac{64}{3}\left(\sqrt{\eta}+1\right)^2\left(\eta - \sqrt{\eta} + 1\right)\left[\frac{1}{\eta^2} {\rm H}_{-1,0}\left(\frac{1}{\sqrt{\eta}}\right) + {\rm H}_{-1,0}\left(\sqrt{\eta}\right)\right] \\
& + \frac{64}{3}\left(\sqrt{\eta}-1\right)^2\left(\eta + \sqrt{\eta} + 1\right)\left[\frac{1}{\eta^2} {\rm H}_{1,0}\left(\frac{1}{\sqrt{\eta}}\right) + {\rm H}_{1,0}\left(\sqrt{\eta}\right)\right] \\
& - \frac{32}{3} h_1\left(\zeta_2 S_1 + \frac{5}{3}\zeta_3\right) + \frac{32(\eta^2-1)}{3\eta}\ln(\eta) + \frac{16\left(\eta^4+1\right)}{3\eta^2}\ln^2(\eta)\Biggr\} \\
& + C_A T_F^2 \Biggl\{-\frac{16T_7}{81N^4(N+1)^4} + \left[\frac{32T_1}{81(N+1)^2} + \frac{1120}{27}\zeta_2\right] S_1 \\
& + \frac{16S_1^2}{3(N+1)} - \frac{16(2N+1)}{3(N+1)} S_2 - \frac{8T_3\zeta_2}{27N^2(N+1)^2} + \frac{448}{27}\zeta_3 h_2\Biggr\} \\
& - \frac{128}{9} T_F^3 \zeta_3 + \tilde{a}_{gg,Q}^{(3)}\left(m_1^2, m_2^2, \mu^2\right),
\end{aligned}
\tag{4.1}
$$

where h_1 and h_2 are given by

$$
h_1 = \frac{(N-1)(N+2)}{N^2(N+1)^2}, \tag{4.2}
$$

$$
h_2 = S_1 - \frac{2}{N(N+1)}, \tag{4.3}
$$

and the polynomials T_i read

$$T_1 = 328N^2 + 584N + 283, \tag{4.4}$$
$$T_2 = 171N^4 + 342N^3 + 847N^2 + 676N - 156, \tag{4.5}$$
$$T_3 = 99N^4 + 198N^3 + 463N^2 + 364N - 84, \tag{4.6}$$
$$T_4 = 66N^6 + 198N^5 + 169N^4 + 80N^3 + 140N^2 - 47N - 78, \tag{4.7}$$
$$T_5 = 15N^6 + 45N^5 + 374N^4 + 601N^3 + 161N^2 - 24N + 36, \tag{4.8}$$
$$T_6 = 2N^8 + 8N^7 + 18N^6 + 20N^5 + 2N^4 - 3N^2 - 9N - 6, \tag{4.9}$$
$$\begin{aligned} T_7 = {} & 3N^8 + 12N^7 + 2080N^6 + 5568N^5 + 4602N^4 + 1138N^3 \\ & -3N^2 - 36N - 108, \end{aligned} \tag{4.10}$$
$$\begin{aligned} T_8 = {} & -144\eta + 72\eta^2N^{10} + 247\eta N^{10} + 72N^{10} + 360\eta^2N^9 + 1235\eta N^9 \\ & +360N^9 + 720\eta^2N^8 + 2182\eta N^8 + 720N^8 + 720\eta^2N^7 + 1606\eta N^7 \\ & +720N^7 + 360\eta^2N^6 + 875\eta N^6 + 360N^6 + 72\eta^2N^5 + 1183\eta N^5 \\ & +72N^5 + 1152\eta N^4 + 216\eta N^3 - 288\eta N^2 - 360\eta N, \end{aligned} \tag{4.11}$$
$$\begin{aligned} T_9 = {} & -42\eta^2 - 36\eta^{7/2}N^6 - 36\eta^{5/2}N^6 - 36\eta^{3/2}N^6 + 12\eta^4N^6 + 57\eta^2N^6 \\ & -36\sqrt{\eta}N^6 + 12N^6 - 108\eta^{7/2}N^5 - 108\eta^{5/2}N^5 - 108\eta^{3/2}N^5 \\ & +36\eta^4N^5 + 171\eta^2N^5 - 108\sqrt{\eta}N^5 + 36N^5 - 108\eta^{7/2}N^4 \\ & -108\eta^{5/2}N^4 - 108\eta^{3/2}N^4 + 36\eta^4N^4 + 160\eta^2N^4 - 108\sqrt{\eta}N^4 \\ & +36N^4 - 36\eta^{7/2}N^3 - 36\eta^{5/2}N^3 - 36\eta^{3/2}N^3 + 12\eta^4N^3 + 71\eta^2N^3 \\ & -36\sqrt{\eta}N^3 + 12N^3 + 68\eta^2N^2 - 29\eta^2N. \end{aligned} \tag{4.12}$$

Here, the functions $\mathrm{H}_{\vec{a}}$ denote the harmonic polylogarithms [48]

$$\mathrm{H}_{b,\vec{a}}(x) = \int_0^x dz f_b(z)\mathrm{H}_{\vec{a}}(z), \quad f_b(z) \in \left\{ f_0 = \frac{1}{z}, f_1 = \frac{1}{1-z}, f_{-1} = \frac{1}{1+z} \right\}, \quad H_\emptyset = 1. \tag{4.13}$$

Later also cyclotomic harmonic polylogarithms [39] will contribute. To account for their letters, a two–index structure is needed, e.g.

$$f_{4,0} = \frac{1}{1+x^2}, \qquad f_{4,1} = \frac{x}{1+x^2}. \tag{4.14}$$

The denominators are cyclotomic polynomials with $p_3 = 1 + x + x^2$, $p_4 = 1 + x^2$, $p_5 = 1 + x + x^2 + x^3 + x^4$, $p_6 = 1 - x + x^2$ etc. The maximal degree of the numerator for the kth cyclotomic denominator polynomial is given by $l = \varphi(k)$, where φ denotes Euler's totient function.

All contributions except of the non-pole part of the unrenormalized OME, $\tilde{a}_{gg,Q}^{(3)}(m_1^2, m_2^2, \mu^2)$, are predicted by the renormalization procedure, which provides a check of the present calculation.

The Mellin–Barnes and binomial representations of the individual Feynman diagrams imply sum representations which can be summed using the technologies provided by the packages `Sigma`, `EvaluateMultisums` and `SumProduction`. Here we distinguish the cases of the real parameter η and $1/\eta$. The computation time for the calculation amounted to several months. The solution is given in terms of harmonic sums, generalized harmonic sums and finite binomial and inverse binomial sums [40] over generalized sums.

In the following, whenever the argument of a harmonic polylogarithm is omitted, it is implied to be η, and whenever the argument of a harmonic sum is omitted, it is implied to be N.

One obtains

$$\begin{aligned}
\tilde{a}_{gg,Q}^{(3)} = &\frac{1}{2}\left(1-(-1)^N\right)\Biggl\{C_F T_F^2\Biggl\{-\frac{L_1 L_2 P_{39}}{12\eta N^3(1+N)^3}-\frac{8(L_1+L_2)P_{51}}{27N^4(1+N)^4}\\
&+\frac{P_{59}}{243\eta N^5(1+N)^5(-3+2N)(-1+2N)}-\frac{5(\eta^2-1)(L_1-L_2)}{2\eta}\\
&+\frac{16(N-1)(2+N)\left(L_1^2 L_2+L_1 L_2^2\right)}{N^2(1+N)^2}+\frac{24(N-1)(2+N)\left(L_1^3+L_2^3\right)}{N^2(1+N)^2}\\
&+\frac{N-1}{N^3(1+N)^2(-3+2N)(-1+2N)}\Biggl[\Biggl[\frac{8(2+N)S_2(1-\eta,N)P_{10}}{3\eta}-\frac{4\mathrm{H}_0^2 P_{21}}{3\eta}\\
&-\frac{8\mathrm{H}_0 S_1(1-\eta,N)P_{21}}{3\eta}-\frac{8S_{1,1}(1-\eta,1,N)P_{21}}{3\eta}\Biggr]\left(\frac{1}{1-\eta}\right)^N\\
&+\Biggl[\frac{8}{3}(2+N)S_2\left(\frac{\eta-1}{\eta},N\right)P_8-\frac{4}{3}\mathrm{H}_0^2 P_{15}+\frac{8}{3}\mathrm{H}_0 S_1\left(\frac{\eta-1}{\eta},N\right)P_{15}\\
&-\frac{8}{3}S_{1,1}\left(\frac{\eta-1}{\eta},1,N\right)P_{15}\Biggr]\left(\frac{\eta}{1-\eta}\right)^N\Biggr]+\frac{(N-1)(2+N)}{N^2(1+N)^2}\Biggl[\frac{128}{3}\Bigl(\mathrm{H}_0\mathrm{H}_{0,1}\\
&-\mathrm{H}_{0,0,1}\Bigr)-\frac{32}{3}\left[S_1\left(\frac{1}{1-\eta},N\right)+S_1\left(\frac{\eta}{\eta-1},N\right)\right]\mathrm{H}_0^2-\frac{32}{9}\mathrm{H}_0^3\\
&-\frac{64}{3}\mathrm{H}_0^2\mathrm{H}_1+\frac{32}{27}S_1^3-\frac{704}{27}S_3+\frac{128}{3}S_{2,1}\\
&-\frac{64}{3}\mathrm{H}_0\left[S_{1,1}\left(\frac{1}{1-\eta},1-\eta,N\right)-S_{1,1}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)\right]\\
&+\frac{64}{3}\left[S_{1,2}\left(\frac{1}{1-\eta},1-\eta,N\right)+S_{1,2}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\right)\right]\\
&-\frac{64}{3}\left[S_{1,1,1}\left(\frac{1}{1-\eta},1-\eta,1,N\right)+S_{1,1,1}\left(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},1,N\right)\right]\\
&-\frac{352}{9}\zeta_3\Biggr]+\left(L_1^2+L_2^2\right)\left(\frac{5}{8\eta}+\frac{5\eta}{8}-\frac{P_{36}}{12N^3(1+N)^3}\right)\\
&+\frac{N-1}{\eta N(1+N)^2(-3+2N)(-1+2N)4^N}\binom{2N}{N}\\
&\times\Biggl[-\frac{4}{3}(2+N)\sum_{i=1}^N\frac{4^i\left(\frac{1}{1-\eta}\right)^i S_2(1-\eta,i)}{i\binom{2i}{i}}P_4-\frac{4}{3}\sum_{i=1}^N\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i S_{1,1}\left(\frac{\eta-1}{\eta},1,i\right)}{i\binom{2i}{i}}P_7\\
&-\frac{2}{3}\mathrm{H}_0^2\sum_{i=1}^N\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i}{i\binom{2i}{i}}P_7-\frac{16}{3}(\eta-1)\mathrm{H}_0\sum_{i=1}^N\frac{4^i}{i^2\binom{2i}{i}}P_9-\frac{8}{3}\sum_{i=1}^N\frac{4^i S_1(i)}{i^2\binom{2i}{i}}P_{12}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{8}{3}\sum_{i=1}^{N}\frac{4^i}{i^3\binom{2i}{i}}P_{12}+\frac{2}{3}\mathrm{H}_0^2\sum_{i=1}^{N}\frac{4^i\left(\frac{1}{1-\eta}\right)^i}{i\binom{2i}{i}}P_{13}\\
&+\frac{4}{3}\Biggl((2+N)\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i S_2\left(\frac{\eta-1}{\eta},i\right)}{i\binom{2i}{i}}P_1+\mathrm{H}_0\sum_{i=1}^{N}\frac{4^i\left(\frac{\eta}{\eta-1}\right)^i S_1\left(\frac{\eta-1}{\eta},i\right)}{i\binom{2i}{i}}P_7\\
&+\mathrm{H}_0\sum_{i=1}^{N}\frac{4^i\left(\frac{1}{1-\eta}\right)^i S_1(1-\eta,i)}{i\binom{2i}{i}}P_{13}+\sum_{i=1}^{N}\frac{4^i\left(\frac{1}{1-\eta}\right)^i S_{1,1}(1-\eta,1,i)}{i\binom{2i}{i}}P_{13}\Biggr)\Biggr]\\
&+\frac{(1+\eta)(N-1)(2+N)P_2}{3\eta^{3/2}N(1+N)^2(-3+2N)(-1+2N)4^N}\binom{2N}{N}\\
&\times\Bigl[-16\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\\
&+8\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\mathrm{H}_0-2\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\mathrm{H}_0^2\Bigr]\\
&+\frac{8(5+8N)P_{24}}{9\eta N(1+N)^2(-3+2N)(-1+2N)(1+2N)}\mathrm{H}_0\\
&+\frac{(1+\eta)\bigl(5-2\eta+5\eta^2\bigr)}{\eta^{3/2}}\Bigl[-\frac{1}{4}(L_1-L_2)\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\\
&-\frac{1}{2}\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)-\frac{1}{16}(L_1-L_2)^2\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\Bigr]\\
&-\frac{8P_{55}}{9\eta N^3(1+N)^3(-3+2N)(-1+2N)(1+2N)}\mathrm{H}_0\\
&+\frac{32(N-1)(2+N)\bigl(-6-8N+N^2\bigr)(L_1-L_2)}{3N^3(1+N)^3}\mathrm{H}_0\\
&+\frac{48(N-1)(2+N)\bigl(L_1^2-L_2^2\bigr)}{N^2(1+N)^2}\mathrm{H}_0+\frac{32(N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{3N^3(1+N)^3}\mathrm{H}_0^2\\
&+\frac{48(N-1)(2+N)(L_1+L_2)}{N^2(1+N)^2}\mathrm{H}_0^2\\
&+\Biggl[-\frac{16(N-1)P_{49}}{81\eta N^4(1+N)^4(-3+2N)(-1+2N)}\\
&+\frac{32(N-1)(2+N)\bigl(-6-8N+N^2\bigr)(L_1+L_2)}{9N^3(1+N)^3}\\
&+\frac{16(N-1)(2+N)\bigl(L_1^2+L_2^2\bigr)}{N^2(1+N)^2}\\
&+\frac{(N-1)(2+N)}{N^2(1+N)^2}\Bigl(32(L_1-L_2)\mathrm{H}_0+32\mathrm{H}_0^2-\frac{32}{3}S_2\Bigr)\Biggr]S_1\\
&+\frac{32(N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{27N^3(1+N)^3}S_1^2+\frac{16(N-1)(2+N)(L_1+L_2)}{3N^2(1+N)^2}S_1^2\\
&-\frac{32(N-1)(2+N)\bigl(-6-8N+N^2\bigr)}{9N^3(1+N)^3}S_2-\frac{16(N-1)(2+N)(L_1+L_2)}{N^2(1+N)^2}S_2
\end{aligned}
$$

$$
+\Biggl[-\frac{8P_{33}}{9N^3(1+N)^3}+\frac{40(N-1)(2+N)(L_1+L_2)}{N^2(1+N)^2}
+\frac{32(N-1)(2+N)S_1}{3N^2(1+N)^2}\Biggr]\zeta_2\Biggr\}
$$

$$
+C_A T_F^2\Biggl\{-\frac{8L_1^2L_2\bigl(-8+N+N^2\bigr)}{3N(1+N)}-\frac{16L_1L_2^2\bigl(-4+N+N^2\bigr)}{3N(1+N)}
+\frac{40L_2^3\bigl(12+N+N^2\bigr)}{9N(1+N)}+\frac{32L_1^3\bigl(15+N+N^2\bigr)}{9N(1+N)}-\frac{L_1L_2P_{25}}{27\eta N^2(1+N)^2}
+\frac{2(L_1+L_2)P_{34}}{27N^3(1+N)^3}+\frac{P_{50}}{18\eta^3(1+N)^4(2+N)^4(-1+2N)(1+2N)}
+\frac{P_{61}}{7290\eta^3N^4(1+N)^4(2+N)^4(-3+2N)(-1+2N)(1+2N)}
-\frac{8(\eta^2-1)(L_1-L_2)}{3\eta}+\frac{1}{54}\bigl(L_1^2+L_2^2\bigr)\biggl(\frac{36}{\eta}+36\eta+\frac{P_{18}}{N^2(1+N)^2}\biggr)
+\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\biggl[\frac{(1+\eta)P_5}{90N(1+N)}\frac{\mathrm{H}_0}{\eta^{3/2}}
-\frac{1}{3}(1+\eta)\bigl(4+11\eta+4\eta^2\bigr)(L_1-L_2)\frac{1}{\eta^{3/2}}\biggr]
$$

$$
+\Biggl[\frac{1}{90}\Biggl[\frac{P_{44}}{\eta N(1+N)^2(-3+2N)(-1+2N)}\binom{2N}{N}\mathrm{H}_0\sum_{i=1}^{N}\frac{4^i\bigl(\frac{1}{1-\eta}\bigr)^iS_1(1-\eta,i)}{i\binom{2i}{i}}
+\frac{P_{42}}{\eta N(1+N)^2(-3+2N)(-1+2N)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\bigl(\frac{1}{1-\eta}\bigr)^iS_2(1-\eta,i)}{i\binom{2i}{i}}
+\frac{P_{44}}{\eta N(1+N)^2(-3+2N)(-1+2N)}\binom{2N}{N}\sum_{i=1}^{N}\frac{4^i\bigl(\frac{1}{1-\eta}\bigr)^iS_{1,1}(1-\eta,1,i)}{i\binom{2i}{i}}\Biggr]
+\Biggl[-\frac{8P_{28}}{135\eta^2N(1+N)^2(-1+2N)}-\frac{2(\eta-1)P_{27}}{81\eta^2(1+N)(-1+2N)}\mathrm{H}_0\Biggr]2^{2N}
+\frac{1}{\eta^2N(1+N)^2(-3+2N)(-1+2N)}\Biggl[\frac{1}{180}\mathrm{H}_0^2\sum_{i=1}^{N}\frac{4^i\bigl(\frac{\eta}{\eta-1}\bigr)^i}{i\binom{2i}{i}}P_{37}
+\frac{1}{90}\Biggl(\mathrm{H}_0\sum_{i=1}^{N}\frac{4^i\bigl(\frac{\eta}{\eta-1}\bigr)^iS_1\bigl(\frac{\eta-1}{\eta},i\bigr)}{i\binom{2i}{i}}P_{30}+\sum_{i=1}^{N}\frac{4^i\bigl(\frac{\eta}{\eta-1}\bigr)^iS_2\bigl(\frac{\eta-1}{\eta},i\bigr)}{i\binom{2i}{i}}P_{30}
+\sum_{i=1}^{N}\frac{4^i\bigl(\frac{\eta}{\eta-1}\bigr)^iS_{1,1}\bigl(\frac{\eta-1}{\eta},1,i\bigr)}{i\binom{2i}{i}}P_{37}+\sum_{i=1}^{N}\frac{4^iS_1(i)}{i^2\binom{2i}{i}}P_{45}+\mathrm{H}_0\sum_{i=1}^{N}\frac{4^i}{i^2\binom{2i}{i}}P_{46}
+\sum_{i=1}^{N}\frac{4^i}{i^3\binom{2i}{i}}P_{47}\Biggr)\Biggr]\binom{2N}{N}
$$

$$
+\Biggl[\frac{4(1+\eta)P_{41}}{45N(1+N)^2(-3+2N)(-1+2N)}\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}
$$

$$
\begin{aligned}
&-\frac{2(1+\eta)P_{41}}{45N(1+N)^2(-3+2N)(-1+2N)}\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{\mathrm{H}_0}{\eta^{3/2}}\\
&+\frac{(1+\eta)P_{41}}{90N(1+N)^2(-3+2N)(-1+2N)}\Big(\mathrm{H}_{1}\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{\mathrm{H}_0^2}{\eta^{3/2}}\Bigg]\binom{2N}{N}\\
&+\frac{P_{44}}{180\eta N(1+N)^2(-3+2N)(-1+2N)}\binom{2N}{N}\mathrm{H}_0^2\sum_{i=1}^{N}\frac{4^i\left(\frac{1}{1-\eta}\right)^i}{i\binom{2i}{i}}\Bigg]2^{-2N}\\
&+\Bigg[\frac{1}{\eta N^2(1+N)^2(-3+2N)(-1+2N)}\bigg(\frac{1}{45}\mathrm{H}_0 S_1(1-\eta,N)P_{38}\\
&+\frac{1}{45}S_{1,1}(1-\eta,1,N)P_{38}+\frac{1}{45}S_2(1-\eta,N)P_{40}+\frac{\mathrm{H}_0^2P_{56}}{810(-1+\eta)}\bigg)\\
&+\frac{P_{43}}{810(-1+\eta)\eta N(1+N)^2(-1+2N)}\mathrm{H}_0^2\Bigg]\left(\frac{1}{1-\eta}\right)^N\\
&-\frac{(1+\eta)P_6}{45N(1+N)}\Big(\mathrm{H}_{0,0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\\
&+\Bigg[\frac{1}{\eta N^2(1+N)^2(-3+2N)(-1+2N)}\bigg(\frac{1}{45}\mathrm{H}_0 S_1\Big(\frac{\eta-1}{\eta},N\Big)P_{31}\\
&+\frac{1}{45}S_2\Big(\frac{\eta-1}{\eta},N\Big)P_{31}+\frac{1}{45}S_{1,1}\Big(\frac{\eta-1}{\eta},1,N\Big)P_{35}+\frac{\mathrm{H}_0^2P_{52}}{810(-1+\eta)}\bigg)\\
&+\frac{P_{32}}{810(-1+\eta)\eta N(1+N)^2(-1+2N)}\mathrm{H}_0^2\Bigg]\left(\frac{\eta}{1-\eta}\right)^N+\frac{4(L_1-L_2)P_{16}}{9N^2(1+N)^2}\mathrm{H}_0\\
&+\frac{P_{48}}{405\eta^2N(1+N)^3(-1+2N)(1+2N)}\mathrm{H}_0\\
&+\frac{P_{58}}{405\eta^2N^2(1+N)^3(-3+2N)(-1+2N)(1+2N)}\mathrm{H}_0\\
&+\frac{4\big(16+3N+3N^2\big)\big(L_1^2-L_2^2\big)}{N(1+N)}\mathrm{H}_0+\Bigg[-\frac{(1+\eta)P_5}{360N(1+N)}\frac{\mathrm{H}_{-1}\big(\sqrt{\eta}\big)}{\eta^{3/2}}\\
&+\frac{P_{26}}{180\eta N^2(1+N)^2}+\frac{8\big(16+3N+3N^2\big)(L_1+L_2)}{3N(1+N)}\Bigg]\mathrm{H}_0^2\\
&-\frac{4\big(16+3N+3N^2\big)}{27N(1+N)}\mathrm{H}_0^3\\
&+\Bigg[\frac{8}{3}\big(L_1^2-2L_1L_2+L_2^2\big)-\frac{8\big(16+3N+3N^2\big)}{9N(1+N)}\mathrm{H}_0^2\Bigg]\mathrm{H}_1\\
&+\Bigg[-\frac{(1+\eta)}{360N(1+N)}\frac{\mathrm{H}_0^2}{\eta^{3/2}}P_5-\frac{1}{12}(1+\eta)\big(4+11\eta+4\eta^2\big)(L_1-L_2)^2\frac{1}{\eta^{3/2}}\Bigg]\\
&\times\mathrm{H}_1\big(\sqrt{\eta}\big)-\frac{1}{12}(1+\eta)\big(4+11\eta+4\eta^2\big)(L_1-L_2)^2\frac{\mathrm{H}_{-1}\big(\sqrt{\eta}\big)}{\eta^{3/2}}\\
&+\Bigg[\frac{16}{3}(L_1-L_2)+\frac{16\big(16+3N+3N^2\big)}{9N(1+N)}\mathrm{H}_0\Bigg]\mathrm{H}_{0,1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{256\mathrm{H}_{0,0,1}}{9N(1+N)}+\frac{2P_{19}}{135\eta^2N(1+N)^3(-1+2N)}S_1\\
&+\Biggl[-\frac{640L_1L_2}{27}+\frac{8\mathrm{H}_0\zeta_2P_{22}}{45\eta N^2(1+N)^2}+\frac{8\mathrm{H}_0P_{23}}{45\eta N^2(1+N)^2}-\frac{8(L_1+L_2)P_{17}}{27N^2(1+N)^2}\\
&+\frac{2P_{57}}{3645\eta^2N^3(1+N)^3(-3+2N)(-1+2N)}-\frac{32}{3}L_1L_2(L_1+L_2)\\
&-\frac{1360}{27}\bigl(L_1^2+L_2^2\bigr)-\frac{80}{3}\bigl(L_1^3+L_2^3\bigr)\\
&-\frac{8}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&-\frac{8\bigl(-93+37N+46N^2\bigr)(L_1-L_2)}{9N(1+N)}\mathrm{H}_0-32\bigl(L_1^2-L_2^2\bigr)\mathrm{H}_0\\
&+\frac{4}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{\mathrm{H}_0}{\eta^{3/2}}+\Biggl[\frac{2P_3}{9\eta N(1+N)}\\
&-\frac{64}{3}(L_1+L_2)-\frac{1}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\frac{\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)}{\eta^{3/2}}\Biggr]\mathrm{H}_0^2+\frac{32}{27}\mathrm{H}_0^3\\
&+\frac{64}{9}\mathrm{H}_0^2\mathrm{H}_1-\frac{1}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\frac{\mathrm{H}_0^2\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)}{\eta^{3/2}}-\frac{128}{9}\mathrm{H}_0\mathrm{H}_{0,1}\\
&+\frac{128}{9}\mathrm{H}_{0,0,1}-\frac{64L_2}{3}\zeta_2+\frac{2P_{53}}{3645\eta^2N^3(1+N)^3(-3+2N)(-1+2N)}\zeta_2\Biggr]S_1\\
&+\Biggl[-\frac{4P_{11}}{135\eta N^2(1+N)^2}-\frac{16}{3}(L_1-L_2)\mathrm{H}_0-\frac{16}{3}\mathrm{H}_0^2\Biggr]S_1^2\\
&+\Biggl[-\frac{4P_{20}}{135\eta N^2(1+N)^2}+\frac{16}{15\eta(1+N)}\bigl(-8+8\eta^2-4N+4\eta^2N\bigr)\mathrm{H}_0\\
&-16(L_1-L_2)\mathrm{H}_0-16\mathrm{H}_0^2\Biggr]S_2-\frac{64}{15\eta(1+N)}\bigl(2+2\eta^2+N+\eta^2N\bigr)S_3\\
&+\frac{64}{15\eta(1+N)}\bigl(2+2\eta^2+N+\eta^2N\bigr)S_{2,1}-\frac{32(2\eta+\eta N)}{15(1+N)}\mathrm{H}_0^2S_1\Bigl(\frac{1}{1-\eta},N\Bigr)\\
&-\frac{32(2+N)\mathrm{H}_0^2S_1\bigl(\frac{\eta}{\eta-1},N\bigr)}{15\eta(1+N)}-\frac{64(2\eta+\eta N)}{15(1+N)}\mathrm{H}_0S_{1,1}\Bigl(\frac{1}{1-\eta},1-\eta,N\Bigr)\\
&+\frac{64(2+N)\mathrm{H}_0S_{1,1}\bigl(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\bigr)}{15\eta(1+N)}+\frac{64(2\eta+\eta N)}{15(1+N)}S_{1,2}\Bigl(\frac{1}{1-\eta},1-\eta,N\Bigr)\\
&+\frac{64(2+N)S_{1,2}\bigl(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},N\bigr)}{15\eta(1+N)}-\frac{64(2\eta+\eta N)}{15(1+N)}S_{1,1,1}\Bigl(\frac{1}{1-\eta},1-\eta,1,N\Bigr)\\
&-\frac{64(2+N)S_{1,1,1}\bigl(\frac{\eta}{\eta-1},\frac{\eta-1}{\eta},1,N\bigr)}{15\eta(1+N)}\\
&+\frac{8L_2P_{29}}{3N(1+N)(-3+2N)(-1+2N)(1+2N)}\zeta_2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{P_{60}}{7290\eta^3 N^4(1+N)^4(2+N)^4(-3+2N)(-1+2N)(1+2N)}\zeta_2 \\
&+\Biggl[\frac{224 L_1}{3N(1+N)}-\frac{4P_{14}}{27N^2(1+N)^2}-\frac{8L_2\bigl(20+15N+7N^2\bigr)}{3N(1+N)} \\
&+\frac{4\bigl(48+15N+7N^2\bigr)}{3N(1+N)}\mathrm{H}_0 \\
&+\biggl[-\frac{112L_1}{3}-16L_2-\frac{16\bigl(-69+38N+47N^2\bigr)}{27N(1+N)}-\frac{32}{3}\mathrm{H}_0\biggr]S_1\Biggr]\zeta_2 \\
&+\frac{P_{54}}{405\eta^2N^2(1+N)^3(-3+2N)(-1+2N)(1+2N)}\mathrm{H}_0\zeta_2 \\
&+\biggl[-\frac{128}{27N(1+N)}+\frac{64}{27}S_1\biggr]\zeta_3\Biggr\} \\
&+T_F^3\Biggl\{\frac{32}{3}\bigl(L_1^3+2L_1^2L_2+2L_1L_2^2+L_2^3\bigr)+32(L_1+L_2)\zeta_2+\frac{128}{9}\zeta_3\Biggr\}\Biggr\}, \qquad (4.15)
\end{aligned}
$$

where the polynomials P_i are

$$P_1=-27-36N^2-36N(-2+\eta)+54\eta+5\eta^2 \qquad (4.16)$$

$$P_2=5-2\bigl(-11-18N+18N^2\bigr)\eta+5\eta^2 \qquad (4.17)$$

$$P_3=372\eta+N^2\bigl(5-102\eta+5\eta^2\bigr)+N\bigl(5-66\eta+5\eta^2\bigr) \qquad (4.18)$$

$$P_4=-5+18(-3+2N)\eta+9(-3+2N)(-1+2N)\eta^2 \qquad (4.19)$$

$$
\begin{aligned}
P_5=&-80\bigl(5+22\eta+5\eta^2\bigr)+3N^2\bigl(71-46\eta+71\eta^2\bigr) \\
&+3N\bigl(167+18\eta+167\eta^2\bigr) \qquad (4.20)
\end{aligned}
$$

$$
\begin{aligned}
P_6=&-80\bigl(5+22\eta+5\eta^2\bigr)+3N^2\bigl(111+64\eta+111\eta^2\bigr) \\
&+3N\bigl(207+128\eta+207\eta^2\bigr) \qquad (4.21)
\end{aligned}
$$

$$P_7=-(2+N)\Bigl[27+36N^2+36N(-2+\eta)-54\eta-5\eta^2\Bigr] \qquad (4.22)$$

$$P_8=24-48N-18N^3+5N^2(9+\eta) \qquad (4.23)$$

$$P_9=(2+N)(-4+3N)(-2+3N)(1+\eta) \qquad (4.24)$$

$$P_{10}=-24\eta+48N\eta+18N^3\eta-5N^2(1+9\eta) \qquad (4.25)$$

$$P_{11}=9+160\eta+50N^2\eta+140N^3\eta+9\eta^2-9N\bigl(7+10\eta+7\eta^2\bigr) \qquad (4.26)$$

$$P_{12}=(2+N)\Bigl[11-54\eta+11\eta^2+18N^2\bigl(1+\eta^2\bigr)-36N\bigl(1-\eta+\eta^2\bigr)\Bigr] \qquad (4.27)$$

$$P_{13}=(2+N)\Bigl[-5+18(-3+2N)\eta+9(-3+2N)(-1+2N)\eta^2\Bigr] \qquad (4.28)$$

$$P_{14}=168-536N-407N^2-278N^3-167N^4 \qquad (4.29)$$

$$P_{15}=-(2+N)\Bigl[-24+48N+18N^3-5N^2(9+\eta)\Bigr] \qquad (4.30)$$

$$P_{16}=-96+370N+277N^2+12N^3+9N^4 \qquad (4.31)$$

$$P_{17}=48-27N+263N^2+634N^3+344N^4 \qquad (4.32)$$

$$P_{18} = -1632 + 7072N + 8611N^2 + 3078N^3 + 1539N^4 \tag{4.33}$$

$$\begin{aligned} P_{19} = & -20N^4(10+81\eta) - 4N^3\big(41+855\eta+135\eta^2\big) \\ & +N\big(41+855\eta-765\eta^2-87\eta^3\big) - 3\big(17+60\eta-135\eta^2+2\eta^3\big) \\ & +N^2\big(254+1125\eta+180\eta^2+45\eta^3\big) \end{aligned} \tag{4.34}$$

$$P_{20} = 50N^2\eta - 660N^3\eta - 440N^4\eta - 3\big(3+160\eta+3\eta^2\big) + 9N\big(7+30\eta+7\eta^2\big) \tag{4.35}$$

$$P_{21} = (2+N)\Big[-24\eta+48N\eta+18N^3\eta-5N^2(1+9\eta)\Big] \tag{4.36}$$

$$P_{22} = 60N^2(1+N)^2\eta \tag{4.37}$$

$$P_{23} = \big(3-21N+25N^2+50N^3+25N^4\big)(-1+\eta)(1+\eta) \tag{4.38}$$

$$\begin{aligned} P_{24} = & -36N^3(-1+\eta)\eta + 36N^4\eta^2 + N(1+15\eta)(-5+21\eta) - 4\big(5+27\eta^2\big) \\ & -N^2\big(5+18\eta+189\eta^2\big) \end{aligned} \tag{4.39}$$

$$\begin{aligned} P_{25} = & \, 384\eta - 1664N\eta + 18N^3\big(4-93\eta+4\eta^2\big) + 9N^4\big(4-93\eta+4\eta^2\big) \\ & +N^2\big(36-2501\eta+36\eta^2\big) \end{aligned} \tag{4.40}$$

$$\begin{aligned} P_{26} = & -48\big(1+160\eta+\eta^2\big) - 16N\big(4-1405\eta+4\eta^2\big) + 3N^4\big(71+134\eta+71\eta^2\big) \\ & +N^2\big(101+15234\eta+101\eta^2\big) + 2N^3\big(357+418\eta+357\eta^2\big) \end{aligned} \tag{4.41}$$

$$\begin{aligned} P_{27} = & \, 56N^4(1+\eta)\big(1+\eta^2\big) - 6(1+\eta)\big(5+86\eta+5\eta^2\big) \\ & -N(1+\eta)\big(41+390\eta+41\eta^2\big) \\ & +4N^3(1+\eta)\big(41+405\eta+41\eta^2\big) + 2N^2(1+\eta)\big(53+972\eta+53\eta^2\big) \end{aligned} \tag{4.42}$$

$$\begin{aligned} P_{28} = & \, 3\big(17+62\eta-270\eta^2+62\eta^3+17\eta^4\big) + 20N^4\big(10+81\eta+81\eta^3+10\eta^4\big) \\ & -N\big(41+768\eta-1530\eta^2+768\eta^3+41\eta^4\big) \\ & +4N^3\big(41+855\eta+270\eta^2+855\eta^3+41\eta^4\big) \\ & -2N^2\big(127+585\eta+180\eta^2+585\eta^3+127\eta^4\big) \end{aligned} \tag{4.43}$$

$$P_{29} = 144 - 51N - 585N^2 + 190N^3 + 36N^4 + 56N^5 \tag{4.44}$$

$$\begin{aligned} P_{30} = & -800N^6 - 8N^5(169+270\eta) + 4N^4\big(599-645\eta+30\eta^2\big) \\ & -2N^3\big(-1565-3810\eta+225\eta^2+6\eta^3\big) \\ & +3N^2\big(-355+2255\eta+335\eta^2+53\eta^3\big) \\ & +3\big(43+705\eta+405\eta^2+175\eta^3\big) - 2N\big(349+4530\eta+855\eta^2+342\eta^3\big) \end{aligned} \tag{4.45}$$

$$\begin{aligned} P_{31} = & -400N^6 + 24\eta(80+3\eta) - 4N^5(119+128\eta) + N^3\big(847+2314\eta-513\eta^2\big) \\ & +N\big(129-4882\eta-171\eta^2\big) - 4N^4\big(-359+172\eta+3\eta^2\big) \\ & +4N^2\big(-239+572\eta+189\eta^2\big) \end{aligned} \tag{4.46}$$

$$\begin{aligned} P_{32} = & -280N^6 - 100N^5(11+81\eta) + N^3\big(13967+41445\eta-1755\eta^2-2325\eta^3\big) \\ & +27\big(-61-160\eta-45\eta^2+2\eta^3\big) - 90N^4\big(-65-72\eta+27\eta^2+20\eta^3\big) \\ & +N^2\big(5143+15660\eta-405\eta^2+174\eta^3\big) \\ & +3N\big(-1141-3285\eta+1125\eta^2+377\eta^3\big) \end{aligned} \tag{4.47}$$

$$P_{33} = 2\big(-42 - 29N + 68N^2 + 47N^3 + 88N^4 + 99N^5 + 33N^6\big) \tag{4.48}$$

$$P_{34} = 528 - 224N + 2008N^2 + 7149N^3 + 4239N^4 + 279N^5 + 93N^6 \tag{4.49}$$

$$\begin{aligned} P_{35} = {} & 400N^6 - 24\eta(80 + 3\eta) + 4N^5(119 + 128\eta) + 4N^4\big(-359 + 172\eta + 3\eta^2\big) \\ & +N\big(-129 + 4882\eta + 171\eta^2\big) - 4N^2\big(-239 + 572\eta + 189\eta^2\big) \\ & +N^3\big(-847 - 2314\eta + 513\eta^2\big) \end{aligned} \tag{4.50}$$

$$P_{36} = -1088 - 800N + 1664N^2 + 1081N^3 + 1675N^4 + 1899N^5 + 633N^6 \tag{4.51}$$

$$\begin{aligned} P_{37} = {} & 800N^6 + 8N^5(169 + 270\eta) - 4N^4\big(599 - 645\eta + 30\eta^2\big) \\ & +2N^3\big(-1565 - 3810\eta + 225\eta^2 + 6\eta^3\big) \\ & -3N^2\big(-355 + 2255\eta + 335\eta^2 + 53\eta^3\big) \\ & -3\big(43 + 705\eta + 405\eta^2 + 175\eta^3\big) + 2N\big(349 + 4530\eta + 855\eta^2 + 342\eta^3\big) \end{aligned} \tag{4.52}$$

$$\begin{aligned} P_{38} = {} & -400N^6\eta^2 + 24(3 + 80\eta) - 4N^5\eta(128 + 119\eta) + N\big(-171 - 4882\eta + 129\eta^2\big) \\ & -4N^2\big(-189 - 572\eta + 239\eta^2\big) + 4N^4\big(-3 - 172\eta + 359\eta^2\big) \\ & +N^3\big(-513 + 2314\eta + 847\eta^2\big) \end{aligned} \tag{4.53}$$

$$\begin{aligned} P_{39} = {} & -512\eta - 256N\eta + 1024N^2\eta + 3N^3\big(5 + 282\eta + 5\eta^2\big) + 9N^5\big(5 + 282\eta + 5\eta^2\big) \\ & +3N^6\big(5 + 282\eta + 5\eta^2\big) + N^4\big(45 + 2282\eta + 45\eta^2\big) \end{aligned} \tag{4.54}$$

$$\begin{aligned} P_{40} = {} & 400N^6\eta^2 - 24(3 + 80\eta) + 4N^5\eta(128 + 119\eta) + N^3\big(513 - 2314\eta - 847\eta^2\big) \\ & +N\big(171 + 4882\eta - 129\eta^2\big) + 4N^2\big(-189 - 572\eta + 239\eta^2\big) \\ & -4N^4\big(-3 - 172\eta + 359\eta^2\big) \end{aligned} \tag{4.55}$$

$$\begin{aligned} P_{41} = {} & 400N^6\big(1 - \eta + \eta^2\big) - 3\big(109 + 446\eta + 109\eta^2\big) + 3N^2\big(151 - 1446\eta + 151\eta^2\big) \\ & +4N^5\big(169 + 101\eta + 169\eta^2\big) - 2N^4\big(599 - 1214\eta + 599\eta^2\big) \\ & +N\big(691 + 4694\eta + 691\eta^2\big) - N^3\big(1559 + 2026\eta + 1559\eta^2\big) \end{aligned} \tag{4.56}$$

$$\begin{aligned} P_{42} = {} & -800N^6\eta^3 - 8N^5\eta^2(270 + 169\eta) + 4N^4\eta\big(30 - 645\eta + 599\eta^2\big) \\ & +3N^2\big(53 + 335\eta + 2255\eta^2 - 355\eta^3\big) + 3\big(175 + 405\eta + 705\eta^2 + 43\eta^3\big) \\ & -2N\big(342 + 855\eta + 4530\eta^2 + 349\eta^3\big) \\ & +2N^3\big(-6 - 225\eta + 3810\eta^2 + 1565\eta^3\big) \end{aligned} \tag{4.57}$$

$$\begin{aligned} P_{43} = {} & -280N^6\eta^3 - 100N^5\eta^2(81 + 11\eta) - 27\big(-2 + 45\eta + 160\eta^2 + 61\eta^3\big) \\ & +90N^4\big(-20 - 27\eta + 72\eta^2 + 65\eta^3\big) - 3N\big(-377 - 1125\eta + 3285\eta^2 + 1141\eta^3\big) \\ & +N^2\big(174 - 405\eta + 15660\eta^2 + 5143\eta^3\big) \\ & +N^3\big(-2325 - 1755\eta + 41445\eta^2 + 13967\eta^3\big) \end{aligned} \tag{4.58}$$

$$\begin{aligned} P_{44} = {} & 800N^6\eta^3 + 8N^5\eta^2(270 + 169\eta) - 4N^4\eta\big(30 - 645\eta + 599\eta^2\big) \\ & -3\big(175 + 405\eta + 705\eta^2 + 43\eta^3\big) + 2N\big(342 + 855\eta + 4530\eta^2 + 349\eta^3\big) \\ & +3N^2\big(-53 - 335\eta - 2255\eta^2 + 355\eta^3\big) \\ & -2N^3\big(-6 - 225\eta + 3810\eta^2 + 1565\eta^3\big) \end{aligned} \tag{4.59}$$

$$\begin{aligned} P_{45} = {} & -800N^6\big(1 + \eta^4\big) + 3\big(43 + 880\eta + 810\eta^2 + 880\eta^3 + 43\eta^4\big) \\ & +N^3\big(3130 + 7608\eta - 900\eta^2 + 7608\eta^3 + 3130\eta^4\big) \end{aligned}$$

$$
\begin{aligned}
& -8N^5\big(169+270\eta+270\eta^3+169\eta^4\big) \\
& -2N\big(349+4872\eta+1710\eta^2+4872\eta^3+349\eta^4\big) \\
& -3N^2\big(355-2308\eta-670\eta^2-2308\eta^3+355\eta^4\big) \\
& +4N^4\big(599-645\eta+60\eta^2-645\eta^3+599\eta^4\big)
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
P_{46} = & -800N^6(-1+\eta)(1+\eta)\big(1+\eta^2\big)-698N(-1+\eta)(1+\eta)\big(1+12\eta+\eta^2\big) \\
& +3(-1+\eta)(1+\eta)\big(43+530\eta+43\eta^2\big) \\
& -8N^5(-1+\eta)(1+\eta)\big(169+270\eta+169\eta^2\big) \\
& -3N^2(-1+\eta)(1+\eta)\big(355-2202\eta+355\eta^2\big) \\
& +4N^4(-1+\eta)(1+\eta)\big(599-645\eta+599\eta^2\big) \\
& +2N^3(-1+\eta)(1+\eta)\big(1565+3816\eta+1565\eta^2\big)
\end{aligned}
\tag{4.61}
$$

$$
\begin{aligned}
P_{47} = & 800N^6\big(1+\eta^4\big)-3\big(43+880\eta+810\eta^2+880\eta^3+43\eta^4\big) \\
& +N\big(698+9744\eta+3420\eta^2+9744\eta^3+698\eta^4\big) \\
& +8N^5\big(169+270\eta+270\eta^3+169\eta^4\big) \\
& +3N^2\big(355-2308\eta-670\eta^2-2308\eta^3+355\eta^4\big) \\
& -4N^4\big(599-645\eta+60\eta^2-645\eta^3+599\eta^4\big) \\
& -2N^3\big(1565+3804\eta-450\eta^2+3804\eta^3+1565\eta^4\big)
\end{aligned}
\tag{4.62}
$$

$$
\begin{aligned}
P_{48} = & -560N^7-40N^6(62+405\eta)+20N^5\big(-190-2187\eta+729\eta^2\big) \\
& +N^2\big(1417+12690\eta-5535\eta^2-240\eta^3\big)-9\big(17+60\eta-135\eta^2+2\eta^3\big) \\
& +10N^4\big(-260-3807\eta+2430\eta^2+15\eta^3\big) \\
& -N^3\big(107+9045\eta-4455\eta^2+75\eta^3\big) \\
& -3N\big(-91-1665\eta+1575\eta^2+137\eta^3\big)
\end{aligned}
\tag{4.63}
$$

$$
\begin{aligned}
P_{49} = & -(2+N)\Big[-216\eta-144N\eta-696N^2\eta+148N^6\eta-30N^4\big(9-16\eta+9\eta^2\big) \\
& -9N^5\big(15-38\eta+15\eta^2\big)-N^3\big(135+1022\eta+135\eta^2\big)\Big]
\end{aligned}
\tag{4.64}
$$

$$
\begin{aligned}
P_{50} = & 225+1630\eta+3456\eta^2+2466\eta^3+415\eta^4 \\
& +N\big(930+5372\eta+6912\eta^2+2820\eta^3+350\eta^4\big) \\
& -152N^6\eta(1+\eta)\big(5+22\eta+5\eta^2\big)-16N^7\eta(1+\eta)\big(5+22\eta+5\eta^2\big) \\
& -192N^3\big(-5+3\eta+135\eta^2+157\eta^3+30\eta^4\big) \\
& -72N^4\big(-5+53\eta+405\eta^2+427\eta^3+80\eta^4\big) \\
& -12N^5\big(-5+218\eta+1296\eta^2+1318\eta^3+245\eta^4\big)-6N^2\big(-225-830\eta \\
& +864\eta^2+1854\eta^3+385\eta^4\big)
\end{aligned}
\tag{4.65}
$$

$$
\begin{aligned}
P_{51} = & 396+690N+518N^2+240N^3-289N^4-432N^5+494N^6 \\
& +588N^7+147N^8
\end{aligned}
\tag{4.66}
$$

$$
\begin{aligned}
P_{52} = & 560N^8-216(-1+\eta)\eta(80+3\eta)+40N^7(34+405\eta) \\
& +N^4\big(44539+73899\eta+1629\eta^2-7215\eta^3\big)
\end{aligned}
$$

$$+60N^6\big(-310-561\eta+81\eta^2+60\eta^3\big)$$
$$+9N\big(-420-6451\eta+4306\eta^2+189\eta^3\big)$$
$$-2N^5\big(7334+31887\eta-414\eta^2+375\eta^3\big)$$
$$-9N^2\big(1731-919\eta+137\eta^2+391\eta^3\big)$$
$$+3N^3\big(9966+26631\eta-11136\eta^2+959\eta^3\big) \tag{4.67}$$
$$P_{53}=-1080N^2(1+N)^2\eta^2\big(207-456N+89N^2-56N^3+92N^4\big) \tag{4.68}$$
$$P_{54}=-540N(1+N)^2\eta^2\big(144-51N-585N^2+190N^3+36N^4+56N^5\big) \tag{4.69}$$
$$P_{55}=N^2(1+N)\Big[288N^5\eta^2-36N^4\eta(-8+3\eta)-160\big(1+3\eta^2\big)$$
$$-4N^3\big(-5-9\eta+438\eta^2\big)$$
$$+N\big(-275-270\eta+801\eta^2\big)+N^2\big(25-522\eta+1485\eta^2\big)\Big] \tag{4.70}$$
$$P_{56}=560N^8\eta^3+40N^7\eta^2(405+34\eta)+216(-1+\eta)(3+80\eta)$$
$$-60N^6\big(-60-81\eta+561\eta^2+310\eta^3\big)$$
$$-9N\big(-189-4306\eta+6451\eta^2+420\eta^3\big)$$
$$-9N^2\big(391+137\eta-919\eta^2+1731\eta^3\big)$$
$$-2N^5\big(375-414\eta+31887\eta^2+7334\eta^3\big)$$
$$+3N^3\big(959-11136\eta+26631\eta^2+9966\eta^3\big)$$
$$+N^4\big(-7215+1629\eta+73899\eta^2+44539\eta^3\big) \tag{4.71}$$
$$P_{57}=-51840\eta^2+103680N\eta^2+80N^8\eta\big(405-10412\eta+405\eta^2\big)$$
$$-4N^7\big(-2700-20331\eta+165688\eta^2+1539\eta^3\big)$$
$$-36N^6\big(204+378\eta-49156\eta^2+1863\eta^3\big)$$
$$-9N^2\big(459+5184\eta+20987\eta^2+3618\eta^3\big)$$
$$-2N^5\big(13500+171477\eta-942766\eta^2+3807\eta^3\big)$$
$$-3N^3\big(-2025-24489\eta+185329\eta^2+4077\eta^3\big)$$
$$+N^4\big(18360+94851\eta+70490\eta^2+58239\eta^3\big) \tag{4.72}$$
$$P_{58}=N\Big[2240N^9(-1+\eta)(1+\eta)\big(1+\eta^2\big)+N^4\big(6264+156159\eta+63990\eta^2$$
$$-251679\eta^3-13850\eta^4\big)$$
$$+N^3\big(-6175-104082\eta+24435\eta^2+51822\eta^3+3020\eta^4\big)$$
$$+N^2\big(-525-58899\eta-7155\eta^2+87081\eta^3+4230\eta^4\big)$$
$$+27\big(-17-96\eta+135\eta^2+34\eta^3\big)+160N^8\big(-48-405\eta+405\eta^3+55\eta^4\big)$$
$$+8N^7\big(-450-13887\eta+17937\eta^3+860\eta^4\big)$$
$$+9N\big(25-482\eta-1845\eta^2+2134\eta^3+100\eta^4\big)$$
$$-10N^5\big(-2214-30615\eta+486\eta^2+36153\eta^3+2834\eta^4\big)$$
$$-4N^6\big(-3660-21279\eta+7290\eta^2+11559\eta^3+3620\eta^4\big)\Big] \tag{4.73}$$

$$
\begin{aligned}
P_{59} = {} & 36288\eta - 19872N\eta - 220032N^2\eta - 252192N^3\eta \\
& + N^5\big(-61155 + 394298\eta - 61155\eta^2\big) \\
& + N^6\big(-17415 + 597938\eta - 17415\eta^2\big) - 320N^4\big(81 + 526\eta + 81\eta^2\big) \\
& + 36N^{11}\big(405 - 3766\eta + 405\eta^2\big) + 12N^{12}\big(405 - 3766\eta + 405\eta^2\big) \\
& - 5N^9\big(3483 - 56554\eta + 3483\eta^2\big) + N^{10}\big(3645 + 44314\eta + 3645\eta^2\big) \\
& + 10N^7\big(4455 - 2386\eta + 4455\eta^2\big) + 2N^8\big(7695 - 64514\eta + 7695\eta^2\big)
\end{aligned} \tag{4.74}
$$

$$
\begin{aligned}
P_{60} = {} & 1080N^3(1+N)^2(2+N)^4\eta^3\big(576 + 1173N - 2988N^2 - 4835N^3 \\
& + 2674N^4 + 572N^5 + 248N^6\big)
\end{aligned} \tag{4.75}
$$

$$
\begin{aligned}
P_{61} = {} & -5806080\eta^3 - 7464960N\eta^3 + 39628800N^2\eta^3 \\
& + N^4\big(273375 - 2809566\eta + 5935680\eta^2 + 522717086\eta^3 + 2240865\eta^4 \\
& - 4790016\eta^5\big) \\
& + 864N^{15}\eta\big(400 + 3639\eta + 589\eta^2 + 3639\eta^3 + 400\eta^4\big) \\
& - 768N^3\eta\big(1377 + 4293\eta - 244034\eta^2 + 4293\eta^3 + 1377\eta^4\big) \\
& + 432N^{14}\eta\big(7856 + 80679\eta + 16689\eta^2 + 80679\eta^3 + 7856\eta^4\big) \\
& - 16N^{11}\eta\big(450738 - 6962355\eta - 48003782\eta^2 - 6819795\eta^3 + 483138\eta^4\big) \\
& + 8N^{13}\eta\big(1570752 + 18898191\eta + 7445585\eta^2 + 18898191\eta^3 + 1570752\eta^4\big) \\
& + 4N^{12}\eta\big(4566888 + 73693719\eta + 72807653\eta^2 + 73622439\eta^3 + 4550688\eta^4\big) \\
& + 4N^5\big(236925 + 975942\eta + 27088992\eta^2 + 117199930\eta^3 + 25711587\eta^4 \\
& - 325728\eta^5\big) \\
& - 12N^8\big(28350 + 912078\eta + 59340789\eta^2 + 250807007\eta^3 + 58332339\eta^4 \\
& + 564588\eta^5\big) \\
& - 36N^9\big(6075 + 1870314\eta + 30831858\eta^2 + 19135996\eta^3 + 30932703\eta^4 \\
& + 1867884\eta^5\big) \\
& - 8N^{10}\big(6075 + 7288137\eta + 73658484\eta^2 - 103483664\eta^3 + 74427579\eta^4 \\
& + 7437582\eta^5\big) \\
& + 4N^7\big(18225 + 8623746\eta + 9768924\eta^2 - 816997534\eta^3 + 15310539\eta^4 \\
& + 9807156\eta^5\big) \\
& + 2N^6\big(443475 + 12987162\eta + 132854256\eta^2 - 575608138\eta^3 \\
& + 137257821\eta^4 + 12137472\eta^5\big).
\end{aligned} \tag{4.76}
$$

In the expression of $\tilde{a}_{gg,Q}^{(3)}$, denominators with poles at $N = 1/2$ and $N = 3/2$ contribute. We have checked using the algorithms of `HarmonicSums` that there are no singularities at these points. The proof of this can either be performed in Mellin N space or in z space. Even using the technologies available in `HarmonicSums` it is far from being trivial, since various new iterative integrals depending on η emerge, the cancellation of which have to be shown analytically. In general, it is necessary to reduce these integrals to higher functions known, cf. [46]. This is not always simple because of a proliferation of letters for the corresponding integrals, see (5.1). In the expansion around $N = 1/2,\ 3/2$ letters occur, which do not belong to the class of those emerging in the N- or z space results. Examples for the associated iterative integrals are

$$G\left(\left\{-\frac{\sqrt{2}-\sqrt{z(1+z)}}{1-z}, \frac{\sqrt{1-z^2}}{z}, \frac{1}{z}\right\}; 1\right) \tag{4.77}$$

$$G\left(\left\{-\frac{1-\sqrt{(2-z)z}}{1-z}, \sqrt{(1-z)(2-z)}, \frac{1}{1-z}\right\}; 1\right) \tag{4.78}$$

$$G\left(\left\{\frac{z}{2-z^2}, \frac{1}{1+z}, \frac{1}{1-z}\right\}; 1\right) \tag{4.79}$$

$$G\left(\left\{\frac{\sqrt{1-z^2}}{z}, \frac{1}{1-z}, -\frac{1}{\sqrt{2}+\sqrt{1+z}}\right\}; 1\right). \tag{4.80}$$

Fortunately, these functions cancel in the physical expressions and have not to be simplified.

For $N = 3, 5$ we can compare to the fixed moments calculated in Section 3 and find agreement for the terms within the accuracy the η expansion has been performed to.

5. The transformation to z space

From the Mellin space result we compute the inverse Mellin transform to z space using the methods of [49–51], which are implemented in the package `HarmonicSums`. The idea is to find the difference equation satisfied by the sums appearing in N space, and convert them into differential equations to be satisfied by the inverse Mellin transform, and to solve them. The result is then given in the form of harmonic polylogarithms and more general iterated integrals of the type

$$G\left[\left\{g(x), \vec{h}(x)\right\}, z\right] = \int_0^z dy\, g(y) G\left[\left\{\vec{h}(x)\right\}, y\right]. \tag{5.1}$$

In what follows we define

$$n = N - 1 \tag{5.2}$$

$$\mathbf{M}[g(z)] = \int_0^1 dz\, z^n g(z) = g(n) \tag{5.3}$$

$$\mathbf{M}^{-1}[g(n)] = g(z). \tag{5.4}$$

In general, the result of the inverse Mellin transform of a function D will contain different pieces

$$D(z) \propto \frac{1-(-1)^N}{2}\left\{D^\delta \delta(1-z) + D^+(z) + D^{\rm reg}(z) + \sum_a \mathbf{M}^{-1}\left[n^a g_a(n)\right](z)\right\}. \tag{5.5}$$

One distinguishes between the distribution D^δ, D^+, and $D^{\rm reg}$, the regular part, where D^δ is a function of η and D^+ is a $+$-distribution,

$$\int_0^1 dz\, g(z)\big(f(z)\big)_+ = \int_0^1 dz\left(g(z) - g(1)\right) f(z), \tag{5.6}$$

the regular part in z, $D^{\rm reg}$. At times, it is necessary to absorb from the output of `HarmonicSums` a rational pre–factor in n after applying a partial fractioning of the result, using the relations

$$\frac{1}{(n+a)^i}\int_0^1 {\rm d}z\, z^n\, f(z) = \int_0^1 {\rm d}z\, z^n \left\{\int_z^1 {\rm d}y\, (-1)^{i-1}\left(\frac{y}{z}\right)^a \left[{\rm H}_0\left(\frac{y}{z}\right)\right]^{i-1} f(y)\right\} \tag{5.7}$$

$$n\int_0^1 {\rm d}z\, z^n\, f(z) = (z^n-1)zf(z)\Big|_0^1 - \int_0^1 (z^n-1)\frac{d}{dz}\left(zf(z)\right). \tag{5.8}$$

We denote the Mellin convolution by $\otimes$

$$f(z)\otimes g(z) = \int_0^1 {\rm d}z_1 \int_0^1 {\rm d}z_2\, \delta(z-z_1z_2) f(z_1) g(z_2) \tag{5.9}$$

for regular functions and, cf. [40],

$$[f(z)]_+ \otimes g(z) = \int_z^1 dy\, f(y)\left[\frac{1}{y}g\left(\frac{z}{y}\right) - g(z)\right] - g(z)\int_0^z dy\, f(y) \tag{5.10}$$

for the Mellin convolution of a $+$-distribution and a regular function.

Now we turn to $a_{gg,Q}^{(3)}(z)$, which we represent in terms of the three contributing parts combining to

$$\tilde{a}_{gg,Q}^{(3)}(N) = \int_0^1 {\rm d}z\, z^{N-1}\, \delta(1-z)\, \tilde{a}_{gg,Q}^{(3),\delta}(z) + \int_0^1 {\rm d}z\, \left(z^{N-1}-1\right)\, \tilde{a}_{gg,Q}^{(3),+}(z)$$
$$+\int_0^1 {\rm d}z\, z^{N-1}\, \tilde{a}_{gg,Q}^{(3),{\rm reg}}(z)\,. \tag{5.11}$$

The result is expressed in terms of the iterated integrals G_i and the constants K_i defined in Appendix D of [25]. In the following the argument of the functions G_i is implied to be z in the formulas for $\tilde{a}_{gg,Q}^{(3),\delta}(z)$, $\tilde{a}_{gg,Q}^{(3),+}(z)$ and $\tilde{a}_{gg,Q}^{(3),{\rm reg}}(z)$, and it is implied to be y in the functions Φ_i which follow. Such arguments are omitted in the interest of brevity. One finds

$$\tilde{a}_{gg,Q}^{(3),\delta}(z) = T_F^3\left\{\frac{32}{3}(L_1^3+L_2^3) + \frac{64}{3}(L_1^2L_2+L_1L_2^2) + 32\zeta_2(L_1+L_2) + \frac{128}{9}\zeta_3\right\}$$
$$+C_FT_F^2\left\{\frac{405-3766\eta+405\eta^2}{81\eta} + L_1L_2\left[\frac{-5-282\eta-5\eta^2}{4\eta}\right.\right.$$
$$\left.+\frac{1}{8}(1+\eta)\left(5-2\eta+5\eta^2\right)\Big({\rm H}_1\big(\sqrt{\eta}\big)+{\rm H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\right]$$
$$+(L_1^2+L_2^2)\left[\frac{5-422\eta+5\eta^2}{8\eta}\right.$$
$$\left.-\frac{1}{16}(1+\eta)\left(5-2\eta+5\eta^2\right)\Big({\rm H}_1\big(\sqrt{\eta}\big)+{\rm H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\right]$$

$$
\begin{aligned}
&+L_1\Bigg[\frac{45-784\eta-45\eta^2}{18\eta}\\
&-\frac{1}{4}(1+\eta)\big(5-2\eta+5\eta^2\big)\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\Bigg]\\
&+L_2\Bigg[\frac{-45-784\eta+45\eta^2}{18\eta}\\
&+\frac{1}{4}(1+\eta)\big(5-2\eta+5\eta^2\big)\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\Bigg]\\
&-\frac{1}{2}(1+\eta)\big(5-2\eta+5\eta^2\big)\Big(\mathrm{H}_{0,0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}-\frac{176}{3}\zeta_2\Bigg\}\\
&+C_AT_F^2\Bigg\{\frac{38\big(21+31\eta+21\eta^2\big)}{135\eta}+\frac{32L_1^3}{9}-\frac{8L_1^2L_2}{3}-\frac{16L_1L_2^2}{3}+\frac{40L_2^3}{9}\\
&+L_1L_2\Bigg[\frac{-4+93\eta-4\eta^2}{3\eta}\\
&+\frac{1}{6}(1+\eta)\big(4+11\eta+4\eta^2\big)\Big(\mathrm{H}_1\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}-\frac{16}{3}\mathrm{H}_1(\eta)\Bigg]\\
&+L_2^2\Bigg[\frac{4+171\eta+4\eta^2}{6\eta}-\frac{1}{12}(1+\eta)\big(4+11\eta+4\eta^2\big)\\
&\times\Big(\mathrm{H}_1\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}-12\mathrm{H}_0(\eta)+\frac{8}{3}\mathrm{H}_1(\eta)\Bigg]\\
&+L_1^2\Bigg[\frac{4+171\eta+4\eta^2}{6\eta}-\frac{1}{12}(1+\eta)\big(4+11\eta+4\eta^2\big)\\
&\times\Big(\mathrm{H}_1\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}+12\mathrm{H}_0(\eta)+\frac{8}{3}\mathrm{H}_1(\eta)\Bigg]\\
&+L_2\Bigg[\frac{2\big(-12+31\eta+12\eta^2\big)}{9\eta}\\
&+\frac{1}{3}(1+\eta)\big(4+11\eta+4\eta^2\big)\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}-4\mathrm{H}_0(\eta)\\
&+8\mathrm{H}_0^2(\eta)-\frac{16}{3}\mathrm{H}_{0,1}(\eta)\Bigg]+L_1\Bigg[-\frac{2\big(-12-31\eta+12\eta^2\big)}{9\eta}\\
&-\frac{1}{3}(1+\eta)\big(4+11\eta+4\eta^2\big)\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\\
&+4\mathrm{H}_0(\eta)+8\mathrm{H}_0^2(\eta)+\frac{16}{3}\mathrm{H}_{0,1}(\eta)\Bigg]\\
&-\frac{1}{15}(1+\eta)\big(111+64\eta+111\eta^2\big)\Big(\mathrm{H}_{0,0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}}\\
&+\frac{13(-1+\eta)(1+\eta)\mathrm{H}_0(\eta)}{45\eta}\\
&+\frac{1}{30}(1+\eta)\big(71-46\eta+71\eta^2\big)\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{\mathrm{H}_0(\eta)}{\eta^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl[\frac{71+134\eta+71\eta^2}{60\eta}-\frac{1}{120}(1+\eta)\bigl(71-46\eta+71\eta^2\bigr)\frac{\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)}{\eta^{3/2}}\Biggr]\mathrm{H}_0^2(\eta)\\
&-\frac{4}{9}\mathrm{H}_0^3(\eta)-\frac{8}{3}\mathrm{H}_0^2(\eta)\mathrm{H}_1(\eta)-\frac{1}{120}(1+\eta)\bigl(71-46\eta+71\eta^2\bigr)\frac{\mathrm{H}_0^2(\eta)\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)}{\eta^{3/2}}\\
&+\frac{16}{3}\mathrm{H}_0(\eta)\mathrm{H}_{0,1}(\eta)+\frac{88}{3}\zeta_2\Biggr\}
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3),+}(z)={}&C_AT_F^2\Biggl\{\frac{1}{1-z}\Biggl\{\frac{2848L_1L_2}{27}+\frac{8Q_1}{729\eta^2}+\frac{64}{3}\bigl(L_1^2L_2+L_1L_2^2\bigr)\\
&+(L_1+L_2)\biggl(\frac{2752}{27}+\frac{112\zeta_2}{3}\biggr)+\frac{256}{27}\bigl(L_1^2+L_2^2\bigr)+16\bigl(L_1^3+L_2^3\bigr)\\
&+(1-\eta)\biggl[\frac{64}{15}\eta(G_{10}+G_{11}-K_{11}-K_{12})-\frac{64(G_8+G_9-K_8-K_9)}{15\eta}\biggr]\\
&+(L_1-L_2)^2\biggl(\frac{32}{3}\mathrm{H}_0(z)-\frac{32}{3}\mathrm{H}_1(z)\biggr)\\
&+\frac{8}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&+\Biggl[(1-\eta)\biggl[\frac{64(G_2-K_2)}{15\eta}+\frac{64}{15}\eta(G_3-K_5)\biggr]\\
&+(1-\eta^2)\biggl[\frac{20Q_3}{9\eta^2(1-z+\eta z)(-\eta-z+\eta z)}-\frac{64\mathrm{H}_0(z)}{15\eta}\biggr]\\
&-\frac{4}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&+\frac{128}{9}\mathrm{H}_{0,1}(\eta)\Biggr]\mathrm{H}_0(\eta)\\
&+\Biggl[-\frac{2\bigl(5-102\eta+5\eta^2\bigr)}{9\eta}+\frac{1}{9}(1+\eta)\bigl(5+22\eta+5\eta^2\bigr)\\
&\times\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&-\frac{32}{3}\mathrm{H}_0(z)-\frac{64}{9}\mathrm{H}_1(\eta)+\frac{32}{3}\mathrm{H}_1(z)\Biggr]\mathrm{H}_0^2(\eta)-\frac{32}{27}\mathrm{H}_0^3(\eta)\\
&-\frac{4Q_4}{27\eta^2(1-z+\eta z)(-\eta-z+\eta z)}\mathrm{H}_0(z)+\frac{32\bigl(1+\eta^2\bigr)\mathrm{H}_0^2(z)}{15\eta}\\
&+\frac{64\bigl(1+\eta^2\bigr)\mathrm{H}_{0,1}(z)}{15\eta}-\frac{128}{9}\mathrm{H}_{0,0,1}(\eta)-\frac{32\bigl(18-175\eta+18\eta^2\bigr)\zeta_2}{135\eta}-\frac{64}{27}\zeta_3\Biggr\}\\
&+\Biggl\{\frac{100(1+\eta)^2\bigl(1-\eta+\eta^2\bigr)}{27\eta^2\pi}-\frac{80K_4(K_8+K_9)}{9(-1+\eta)\pi}\\
&+\frac{40(1+\eta)^2\bigl(1-\eta+\eta^2\bigr)}{9\eta^2}\biggl[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\eta)^2\Biggl[-\frac{80\bigl(1+\eta+\eta^2\bigr)G_1}{9\eta^2}+\frac{5\bigl(1+\eta+\eta^2\bigr)\pi}{9\eta^2}\\
&-\frac{10\eta}{9}\Biggl[G_{12}+G_{13}-K_{13}-K_{14}+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Biggr]\\
&+\frac{10}{9\eta^2}\Biggl[G_{14}+G_{15}-K_{16}-K_{17}+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Biggr]\Biggr]\\
&-\frac{40(1+\eta)\bigl(1-\eta+\eta^2\bigr)}{9\pi}\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&+\Biggl[\frac{80K_2K_4}{9(-1+\eta)\pi}-\frac{40(1+\eta)\bigl(1-\eta+\eta^2\bigr)\bigl(1+\eta+\eta^2\bigr)}{27(-1+\eta)\eta^2\pi}\\
&+(1-\eta^2)\Biggl[-\frac{40\bigl(1+\eta+\eta^2\bigr)G_1}{9\eta^2}+\frac{5\bigl(1+\eta+\eta^2\bigr)\pi}{18\eta^2}\Biggr]\\
&+(1-\eta)^2\Biggl[-\frac{80\eta K_{15}}{9\pi}-\frac{10\bigl(G_5-K_7+\frac{8K_{18}}{\pi}\bigr)}{9\eta^2}-\frac{10}{9}\eta(G_4-K_6)\Biggr]\\
&+\frac{20(1+\eta)\bigl(1-\eta+\eta^2\bigr)}{9\pi}\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\Biggr]\mathrm{H}_0(\eta)\\
&+\Biggl[-\frac{40K_4}{9(-1+\eta)^2\pi}+\frac{10(1+\eta)^2\bigl(1-\eta+\eta^2\bigr)}{9(-1+\eta)^2\eta\pi}\\
&-\frac{5(1+\eta)\bigl(1-\eta+\eta^2\bigr)}{9\pi}\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\Biggr]\mathrm{H}_0^2(\eta)\Biggr\}\frac{1}{(1-z)^{3/2}\sqrt{z}}\\
&+\frac{20Q_2}{9\eta^2(1-z+\eta z)(-\eta-z+\eta z)}\mathrm{H}_1(z)\Biggr\}, \qquad (5.13)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a}_{gg,Q}^{(3),\mathrm{reg}}(z)=C_FT_F^2\Biggl\{&\frac{110432}{243}(1-z)\\
&+\frac{16\bigl(1+\eta^2\bigr)\bigl(-1+\sqrt{z}\bigr)\bigl(-14-14\sqrt{z}-14z+261z^{3/2}+261z^2\bigr)}{135\eta z^{3/2}}\\
&+L_1L_2\Biggl[(1+z)\biggl(\frac{448}{3}\mathrm{H}_0^2(z)+128\mathrm{H}_{0,1}(z)-128\zeta_2\biggr)\\
&-\frac{64}{3}(-31+17z)\mathrm{H}_0(z)+\frac{64}{3}(1-z)\bigl(31+15\mathrm{H}_1(z)\bigr)\Biggr]\\
&+\bigl(L_1^2+L_2^2\bigr)\Biggl[(1-z)\Bigl(-72-72\mathrm{H}_0(z)-80\mathrm{H}_1(z)\Bigr)\\
&+(1+z)\Bigl(-\frac{56}{3}\mathrm{H}_0^2(z)-32\mathrm{H}_{0,1}(z)+32\zeta_2\Bigr)\Biggr]\\
&+(1+\eta)\Biggl\{-\frac{32(8G_1-\pi)Q_5}{9\pi}\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl[\frac{32Q_8}{135\pi}\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}z^{3/2}}\\
&-\frac{16Q_8}{135\pi}\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{\mathrm{H}_0(\eta)}{\eta^{3/2}z^{3/2}}\\
&+\frac{4Q_8}{135\pi}\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{\mathrm{H}_0^2(\eta)}{\eta^{3/2}z^{3/2}}\Biggr]\sqrt{1-z}\Biggr\}\\
&+\bigl(L_1^2L_2+L_1L_2^2\bigr)\Bigl[80(1-z)+32(1+z)\mathrm{H}_0(z)\Bigr]\\
&+\bigl(L_1^3+L_2^3\bigr)\Bigl[120(1-z)+48(1+z)\mathrm{H}_0(z)\Bigr]\\
&+(1-z)\Biggl[-\frac{320}{3}(1-\eta)\eta K_3(K_8+K_9)+\frac{160}{27}\mathrm{H}_1^2(z)+\frac{160}{27}\mathrm{H}_1^3(z)\\
&-\frac{640}{27}\mathrm{H}_{0,1}(z)-\frac{640}{3}\mathrm{H}_{0,0,1}(\eta)-\frac{640}{9}\mathrm{H}_{0,0,1}(z)+\frac{3200}{9}\mathrm{H}_{0,1,1}(z)\Biggr]\\
&+(L_1+L_2)\Biggl\{(1+z)\Biggl(\frac{176}{9}\mathrm{H}_0^3(z)-\frac{128}{3}\mathrm{H}_{0,0,1}(z)+\frac{64}{3}\mathrm{H}_{0,1,1}(z)+\frac{64}{3}\zeta_3\Biggr)\\
&+\Biggl[\frac{16}{27}(859+217z)+\frac{320}{3}(1-z)\mathrm{H}_1(z)+\frac{128}{3}(1+z)\mathrm{H}_{0,1}(z)\Biggr]\mathrm{H}_0(z)\\
&-\frac{8}{9}(-149+73z)\mathrm{H}_0^2(z)+(1-z)\Biggl(\frac{22816}{27}+\frac{832}{9}\mathrm{H}_1(z)+\frac{80}{3}\mathrm{H}_1^2(z)\Biggr)\\
&+\frac{64}{9}(-2+7z)\mathrm{H}_{0,1}(z)+\Biggl[-\frac{8}{9}(-121+161z)+\frac{112}{3}(1+z)\mathrm{H}_0(z)\Biggr]\zeta_2\Biggr\}\\
&+\Biggl\{(1-\eta^2)\Biggl(\frac{440}{27\eta}-\frac{232z}{15\eta}-\frac{112}{135\eta z^{3/2}}\Biggr)\\
&+(1-z)\Biggl[\frac{320}{3}(1-\eta)\eta K_2K_3+\frac{640}{3}\mathrm{H}_{0,1}(\eta)\Biggr]\\
&+\frac{16(1+\eta)(8G_1-\pi)Q_5}{9\pi}\Bigl(\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&+\Biggl[-\frac{16}{9\eta}\bigl(-5+5\eta^2-3z+3\eta^2z\bigr)\\
&+(1+z)\Biggl[\frac{128}{3}(1-\eta)\eta K_2K_3+\frac{256}{3}\mathrm{H}_{0,1}(\eta)\Biggr]\Biggr]\mathrm{H}_0(z)\Biggr\}\mathrm{H}_0(\eta)\\
&+\Biggl\{(1-z)\Biggl(\frac{160}{3}+\frac{160\eta K_3}{3}-\frac{320}{3}\mathrm{H}_1(\eta)+160\mathrm{H}_1(z)\Biggr)\\
&-\frac{4(1+\eta)(8G_1-\pi)Q_5}{9\pi}\Bigl(\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\frac{1}{\eta^{3/2}}\\
&+\Biggl[-\frac{32}{3}(-11+7z)+(1+z)\Biggl(\frac{64\eta K_3}{3}-\frac{128}{3}\mathrm{H}_1(\eta)+64\mathrm{H}_1(z)\Biggr)\Biggr]\mathrm{H}_0(z)\\
&+32(1+z)\mathrm{H}_0^2(z)\Biggr\}\mathrm{H}_0^2(\eta)+\Biggl[-\frac{160}{9}(1-z)-\frac{64}{9}(1+z)\mathrm{H}_0(z)\Biggr]\mathrm{H}_0^3(\eta)
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl\{-\frac{32Q_7}{243\eta}+(1+z)\Biggl[-\frac{128}{3}(1-\eta)\eta K_3(K_8+K_9)+\frac{64}{27}\mathrm{H}_1^3(z) \\
&-\frac{256}{3}\mathrm{H}_{0,0,1}(\eta)-\frac{256}{9}\mathrm{H}_{0,0,1}(z)+\frac{1280}{9}\mathrm{H}_{0,1,1}(z)\Biggr] \\
&+\Biggl[\frac{16Q_6}{81\eta}-\frac{256}{3}(1+z)\mathrm{H}_{0,1}(z)\Biggr]\mathrm{H}_1(z) \\
&-\frac{32}{27}(-41+37z)\mathrm{H}_1^2(z)-\frac{512}{27}(-1+2z)\mathrm{H}_{0,1}(z)\Biggr\}\mathrm{H}_0(z) \\
&+\Biggl[\frac{16}{81}(577+379z)+(1+z)\biggl(\frac{160}{9}\mathrm{H}_1^2(z)+\frac{128}{9}\mathrm{H}_{0,1}(z)\biggr) \\
&-\frac{160}{27}(-5+z)\mathrm{H}_1(z)\Biggr]\mathrm{H}_0^2(z) \\
&+\Biggl[-\frac{16}{81}(-67+23z)+\frac{64}{27}(1+z)\mathrm{H}_1(z)\Biggr]\mathrm{H}_0^3(z)+\frac{40}{27}(1+z)\mathrm{H}_0^4(z) \\
&+\Biggl[\frac{5264}{81}(1-z) \\
&-\frac{8\bigl(1+\eta^2\bigr)\bigl(-1+\sqrt{z}\bigr)\bigl(-14-14\sqrt{z}-14z+261z^{3/2}+261z^2\bigr)}{135\eta z^{3/2}} \\
&-\frac{640}{3}(1-z)\mathrm{H}_{0,1}(z)\Biggr]\mathrm{H}_1(z) \\
&+\Biggl\{\Biggl[-\frac{16}{9}(-67+23z)+\frac{704}{9}(1+z)\mathrm{H}_1(z)\Biggr]\mathrm{H}_0(z) \\
&+\frac{80}{3}(1+z)\mathrm{H}_0^2(z)+\frac{880}{9}(1-z)\Bigl(1+2\mathrm{H}_1(z)\Bigr)\Biggr\}\zeta_2+\Biggl[-\frac{1760}{9}(1-z) \\
&-\frac{704}{9}(1+z)\mathrm{H}_0(z)\Biggr]\zeta_3+\int_z^1 \mathrm{d}y\Biggl[\Phi_1^{C_F}(z,y)\mathrm{H}_0^3\left(\frac{z}{y}\right)+\Phi_2^{C_F}(z,y)\mathrm{H}_0^2\left(\frac{z}{y}\right) \\
&+\Phi_3^{C_F}(y)\frac{z}{y^2}\mathrm{H}_0\left(\frac{z}{y}\right)+\Phi_4^{C_F}(y)\frac{1}{y^2}\mathrm{H}_0\left(\frac{z}{y}\right)+\Phi_5^{C_F}(z,y)\frac{z}{y^2}+\Phi_6^{C_F}(z,y)\frac{1}{y} \\
&+\Phi_7^{C_F}(y)\frac{\sqrt{y}}{z^{3/2}}\Biggr]+\int_0^z \mathrm{d}y\ \Phi_8^{C_F}(y)\Biggr\} \\
&+C_A T_F^2\Biggl\{-\frac{4\mathrm{H}_1^2(z)Q_{16}}{135\eta}+\frac{8\mathrm{H}_{0,1}(z)Q_{17}}{135\eta}-\frac{4Q_{20}}{18225\eta^2} \\
&+L_1L_2\Biggl[-\frac{16}{27}(-817+1004z)+\frac{64}{9}(17+5z)\mathrm{H}_0(z) \\
&-\frac{16}{3}(-27+28z)\mathrm{H}_1(z)\Biggr]-\frac{64}{15}(1-\eta)K_3z(K_8+K_9) \\
&+(1+\eta)\Biggl[\Bigl(\mathrm{H}_{0,0,1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,0,-1}\bigl(\sqrt{\eta}\bigr)\Bigr)\biggl(\frac{32G_1Q_{14}}{45\pi\eta^{3/2}}-\frac{4}{45}\frac{1}{\eta^{3/2}}Q_{18}\biggr)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{4Q_{26}}{675\pi}\Big(\mathrm{H}_{0,0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}\sqrt{1-z}z^{3/2}}\Bigg]\\
&+\big(L_1^2+L_2^2\big)\Bigg[\frac{8}{27}(-229+206z)\\
&+\frac{16}{9}(-13+11z)\mathrm{H}_0(z)+\frac{8}{3}(-27+28z)\mathrm{H}_1(z)\Bigg]\\
&+\big(1+\eta^2\big)\Bigg(-\frac{196}{675\eta z^{3/2}}-\frac{64z\mathrm{H}_{0,0,1}(z)}{15\eta}+\frac{128z\mathrm{H}_{0,1,1}(z)}{15\eta}\Bigg)\\
&+(L_1+L_2)\Bigg[-\frac{688}{27}(-13+17z)+(1+z)\Bigg(\frac{176}{9}\mathrm{H}_0^2(z)+\frac{128}{9}\mathrm{H}_{0,1}(z)\Bigg)\\
&-\frac{8}{27}(-452+49z)\mathrm{H}_0(z)-\frac{8}{9}(-41+23z)\mathrm{H}_1(z)-\frac{16}{9}(-13+50z)\zeta_2\Bigg]\\
&+(1-2z)\Bigg[\frac{64}{3}\big(L_1^2L_2+L_1L_2^2\big)+16\big(L_1^3+L_2^3\big)-\frac{32}{27}\mathrm{H}_0^3(\eta)\\
&-\frac{128}{9}\mathrm{H}_{0,0,1}(\eta)-\frac{64}{27}\zeta_3\Bigg]+\Bigg\{\Bigg\{-\frac{Q_{10}}{90\eta^2\pi}+\frac{8(73+90\eta)K_4(K_8+K_9)}{15(-1+\eta)\pi}\\
&-\frac{4(1+\eta)^2\big(73+17\eta+73\eta^2\big)}{15\eta^2}\Bigg[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\Bigg]\\
&+(1-\eta)^2\Bigg[\frac{8\big(73+163\eta+73\eta^2\big)G_1}{15\eta^2}-\frac{\big(73+163\eta+73\eta^2\big)\pi}{30\eta^2}\\
&+\frac{1}{15}(90+73\eta)\Bigg[G_{12}+G_{13}-K_{13}-K_{14}+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Bigg]\\
&-\frac{73+90\eta}{15\eta^2}\Bigg[G_{14}+G_{15}-K_{16}-K_{17}+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg]\Bigg\}\frac{1}{\sqrt{z}}\\
&+\Bigg\{-\frac{2(1+\eta)Q_{26}}{675\pi}\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}z^{3/2}}\\
&+\Bigg[-\frac{4(-1+\eta)(1+\eta)\big(73+163\eta+73\eta^2\big)G_1}{15\eta^2}-\frac{8(73+90\eta)K_2K_4}{15(-1+\eta)\pi}\\
&+\frac{(-1+\eta)(1+\eta)\big(73+163\eta+73\eta^2\big)\pi}{60\eta^2}+\frac{(1+\eta)Q_9}{180(-1+\eta)\eta^2\pi}\\
&+(1-\eta)^2\Bigg[\frac{1}{15}(90+73\eta)\Bigg(G_4-K_6+\frac{8K_{15}}{\pi}\Bigg)\\
&+\frac{(73+90\eta)\big(G_5-K_7+\frac{8K_{18}}{\pi}\big)}{15\eta^2}\Bigg]\Bigg]\frac{1}{\sqrt{z}}\Bigg\}\mathrm{H}_0(\eta)\\
&+\Bigg[\frac{(1+\eta)Q_{26}}{1350\pi}\Big(\mathrm{H}_1\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\frac{1}{\eta^{3/2}z^{3/2}}+\Bigg[\frac{4(73+90\eta)K_4}{15(-1+\eta)^2\pi}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{(1+\eta)^2\big(73+17\eta+73\eta^2\big)}{15(-1+\eta)^2\eta\pi}\Bigg]\frac{1}{\sqrt{z}}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-z}} \\
&+\Bigg\{\frac{64}{15}(1-\eta)K_2K_3z+(1+\eta)\Bigg[\Big(\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)\Big)\Bigg(-\frac{16G_1Q_{14}}{45\pi\eta^{3/2}} \\
&+\frac{2}{45}\frac{1}{\eta^{3/2}}Q_{18}\Bigg)\Bigg]-(-1+\eta)(1+\eta)\Bigg[-\frac{2Q_{23}}{675\eta^2(1-z+\eta z)(-\eta-z+\eta z)} \\
&-\frac{2Q_{21}}{675\eta^2(1-z+\eta z)(-\eta-z+\eta z)}\frac{1}{z^{3/2}}-\frac{64z\mathrm{H}_{0,1}(z)}{15\eta}\Bigg] \\
&+\Bigg[\frac{4}{45\eta}\big(-33+33\eta^2-76z+76\eta^2z\big) \\
&-\frac{8}{15\eta}\big(-1+\eta^2-16z+16\eta^2z\big)\mathrm{H}_1(z)\Bigg]\mathrm{H}_0(z) \\
&+\frac{24}{5\eta}\big(1-\eta^2-z+\eta^2z\big)\mathrm{H}_1(z)+\frac{128}{9}(1-2z)\mathrm{H}_{0,1}(\eta)\Bigg\}\mathrm{H}_0(\eta) \\
&+\Bigg[\frac{32K_3z}{15}+\frac{4\mathrm{H}_0(z)Q_{11}}{15\eta}+\frac{2Q_{12}}{45\eta} \\
&+(1+\eta)\Bigg[\Big(\mathrm{H}_1\big(\sqrt{\eta}\big)+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\Big)\Bigg(\frac{4G_1Q_{14}}{45\pi\eta^{3/2}}-\frac{1}{90}\frac{1}{\eta^{3/2}}Q_{18}\Bigg)\Bigg] \\
&-\frac{64}{9}(1-2z)\mathrm{H}_1(\eta)-\frac{8}{3}(-27+28z)\mathrm{H}_1(z)\Bigg]\mathrm{H}_0^2(\eta) \\
&+\Bigg[\frac{4\mathrm{H}_1^2(z)Q_{15}}{135\eta}-\frac{2\mathrm{H}_1(z)Q_{19}}{405\eta}+\frac{2Q_{24}}{405\eta^2(1-z+\eta z)(-\eta-z+\eta z)} \\
&-\frac{8}{135\eta}\big(9+320\eta+9\eta^2+320\eta z\big)\mathrm{H}_{0,1}(z)\Bigg]\mathrm{H}_0(z) \\
&+\Bigg[-\frac{56}{81}(-47+4z)+\frac{8\mathrm{H}_1(z)Q_{13}}{135\eta}\Bigg]\mathrm{H}_0^2(z)+\frac{224}{81}(1+z)\mathrm{H}_0^3(z) \\
&+\Bigg[\frac{2Q_{25}}{2025\eta^2(1-z+\eta z)(-\eta-z+\eta z)} \\
&+\frac{2Q_{22}}{2025\eta^2(1-z+\eta z)(-\eta-z+\eta z)}\frac{1}{z^{3/2}}-\frac{64\big(1+\eta^2\big)z\mathrm{H}_{0,1}(z)}{15\eta}\Bigg]\mathrm{H}_1(z) \\
&+\Bigg[-\frac{224}{27}(-14+19z)+\frac{224}{9}(1+z)\mathrm{H}_0(z)+\frac{64\big(1+\eta^2\big)z\mathrm{H}_1(z)}{15\eta}\Bigg]\zeta_2 \\
&+\int_z^1 \mathrm{d}y\Bigg[\Phi_2^{C_A}(z,y)\mathrm{H}_0^2\left(\frac{z}{y}\right)+\Phi_3^{C_A}(y)\frac{z}{y^2}\mathrm{H}_0\left(\frac{z}{y}\right)+\Phi_4^{C_A}(y)\frac{1}{y^2}\mathrm{H}_0\left(\frac{z}{y}\right) \\
&+\Phi_5^{C_A}(z,y)\frac{z}{y^2}+\Phi_6^{C_A}(z,y)\frac{1}{y}+\Phi_7^{C_A}(y)\frac{\sqrt{y}}{z^{3/2}}\Bigg]+\int_0^z \mathrm{d}y\;\Phi_8^{C_A}(y)\Bigg\}, \qquad (5.14)
\end{aligned}
$$

with the polynomials

$$Q_1 = -405 - 405\eta + 10412\eta^2 - 405\eta^3 - 405\eta^4 + 405z(-1+\eta)^2\big(1+\eta+\eta^2\big) \quad (5.15)$$
$$Q_2 = -z(-1+\eta)^2\big(1+\eta^4\big) - 2\eta\big(1+\eta^4\big) + z^2(-1+\eta)^2(1+\eta)^2\big(1-\eta+\eta^2\big) \quad (5.16)$$
$$Q_3 = 2\eta\big(1-\eta+\eta^2\big) + z^3(-1+\eta)^2\big(1+\eta+\eta^2\big) + z\big(1-6\eta+6\eta^2-6\eta^3+\eta^4\big) \\ -z^2(-1+\eta)^2\big(2-\eta+2\eta^2\big) \quad (5.17)$$
$$Q_4 = 30\eta + 88\eta^3 + 30\eta^5 + z\big(15-60\eta+103\eta^2-176\eta^3+103\eta^4-60\eta^5+15\eta^6\big) \\ +15z^3(-1+\eta)^2(1+\eta)^2\big(1-\eta+\eta^2\big) \\ -z^2(-1+\eta)^2\big(30+15\eta+88\eta^2+15\eta^3+30\eta^4\big) \quad (5.18)$$
$$Q_5 = 5 + 22\eta + 5\eta^2 + 2z\big(1-10\eta+\eta^2\big) \quad (5.19)$$
$$Q_6 = 45 + 302\eta + 45\eta^2 + z\big(27-10\eta+27\eta^2\big) \quad (5.20)$$
$$Q_7 = 135 - 3436\eta + 135\eta^2 + z\big(81+596\eta+81\eta^2\big) \quad (5.21)$$
$$Q_8 = z^2\big(-287+62\eta-287\eta^2\big) + 120z^4\big(1-10\eta+\eta^2\big) - 7\big(1-\eta+\eta^2\big) \\ +240z^3\big(1+8\eta+\eta^2\big) - 6z\big(11+64\eta+11\eta^2\big) \quad (5.22)$$
$$Q_9 = 16\big(73+90\eta+163\eta^2+90\eta^3+73\eta^4\big) \quad (5.23)$$
$$Q_{10} = 20(1+\eta)^2\big(73+17\eta+73\eta^2\big) \quad (5.24)$$
$$Q_{11} = 1 - 70\eta + \eta^2 + 8z\big(1+40\eta+\eta^2\big) \quad (5.25)$$
$$Q_{12} = 29 + 2540\eta + 29\eta^2 + 2z\big(16-1523\eta+16\eta^2\big) \quad (5.26)$$
$$Q_{13} = 9 + 400\eta + 9\eta^2 + 4z\big(9+100\eta+9\eta^2\big) \quad (5.27)$$
$$Q_{14} = 109 + 446\eta + 109\eta^2 + 64z\big(1+14\eta+\eta^2\big) \quad (5.28)$$
$$Q_{15} = 9 + 160\eta + 9\eta^2 + 8z\big(9+20\eta+9\eta^2\big) \quad (5.29)$$
$$Q_{16} = -81 - 410\eta - 81\eta^2 + z\big(81+550\eta+81\eta^2\big) \quad (5.30)$$
$$Q_{17} = -81 - 820\eta - 81\eta^2 + 3z\big(27+260\eta+27\eta^2\big) \quad (5.31)$$
$$Q_{18} = 59 + 226\eta + 59\eta^2 + 4z\big(59+346\eta+59\eta^2\big) \quad (5.32)$$
$$Q_{19} = 9 - 11246\eta + 9\eta^2 + 8z\big(261+1568\eta+261\eta^2\big) \quad (5.33)$$
$$Q_{20} = -5\big(17739+24192\eta+397052\eta^2+24192\eta^3+17739\eta^4\big) \\ +z\big(88695+567\eta+2287160\eta^2+567\eta^3+88695\eta^4\big) \quad (5.34)$$
$$Q_{21} = -49\eta\Big[-z(-1+\eta)^2 + z^2(-1+\eta)^2 - \eta\Big] \quad (5.35)$$
$$Q_{22} = 147\eta\big(1+\eta^2\big)\Big[-z(-1+\eta)^2 + z^2(-1+\eta)^2 - \eta\Big] \quad (5.36)$$
$$Q_{23} = 5\eta\big(807+574\eta+807\eta^2\big) + z\big(3285-6985\eta-7249\eta^2-6985\eta^3+3285\eta^4\big) \\ +3z^3(-1+\eta)^2\big(1095+2693\eta+1095\eta^2\big) \\ -z^2(-1+\eta)^2\big(6570+11699\eta+6570\eta^2\big) \quad (5.37)$$
$$Q_{24} = \eta\big(2421+792\eta-33824\eta^2+792\eta^3+2421\eta^4\big) \\ +z\big(1971-5121\eta-40574\eta^2+67548\eta^3-40574\eta^4-5121\eta^5+1971\eta^6\big) \\ +z^3(-1+\eta)^2\big(1971+7137\eta+1684\eta^2+7137\eta^3+1971\eta^4\big) \\ -z^2(-1+\eta)^2\big(3942+8379\eta-32140\eta^2+8379\eta^3+3942\eta^4\big) \quad (5.38)$$

$$
\begin{aligned}
Q_{25} = & z\big(9855 - 4695\eta - 27488\eta^2 - 123060\eta^3 - 27488\eta^4 - 4695\eta^5 + 9855\eta^6\big) \\
& +5\eta\big(2421 + 4974\eta + 8324\eta^2 + 4974\eta^3 + 2421\eta^4\big) \\
& +z^3(-1+\eta)^2\big(9855 + 29223\eta + 89560\eta^2 + 29223\eta^3 + 9855\eta^4\big) \\
& -z^2(-1+\eta)^2\big(19710 + 56343\eta + 131180\eta^2 + 56343\eta^3 + 19710\eta^4\big) \qquad (5.39)
\end{aligned}
$$

$$
\begin{aligned}
Q_{26} = & 49\big(1 - \eta + \eta^2\big) + 3840z^5\big(1 + 14\eta + \eta^2\big) + 60z^4\big(13 - 898\eta + 13\eta^2\big) \\
& -58z^3\big(173 + 1102\eta + 173\eta^2\big) + z\big(1463 + 412\eta + 1463\eta^2\big) \\
& +z^2\big(7187 + 64438\eta + 7187\eta^2\big). \qquad (5.40)
\end{aligned}
$$

The functions $\Phi_1, \ldots, \Phi_8$, which appear as arguments of a further integral, are

$$
\Phi_1^{C_F}(z,y) = -\frac{64}{27(1-y)y} - \frac{64z}{27(1-y)y^2} \qquad (5.41)
$$

$$
\begin{aligned}
\Phi_2^{C_F}(z,y) = & \frac{z}{y^2}\left[\frac{224}{27(1-y)} - \frac{128\mathrm{H}_0(y)}{9(1-y)} - \frac{64\mathrm{H}_1(y)}{9(1-y)}\right] \\
& +\frac{1}{y}\left[-\frac{352}{27(1-y)} - \frac{128\mathrm{H}_0(y)}{9(1-y)} - \frac{64\mathrm{H}_1(y)}{9(1-y)}\right] \qquad (5.42)
\end{aligned}
$$

$$
\begin{aligned}
\Phi_3^{C_F}(y) = & \frac{1}{1-y}\Biggl\{-\frac{16R_3}{81\eta^2} \\
& -\frac{128}{3}(1-\eta)\big(-G_{10} - G_{11} + G_8 + G_9 + K_{11} + K_{12} - K_8 - K_9\big) \\
& -64\mathrm{H}_0^2(\eta) + \left[-\frac{8R_5}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)} - \frac{256}{9}\mathrm{H}_1(y)\right]\mathrm{H}_0(y) \\
& +\frac{128}{9}\mathrm{H}_0^2(y) - \frac{8R_6}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y) - \frac{64}{9}\mathrm{H}_1^2(y) \\
& +\frac{256}{3}\mathrm{H}_{0,1}(y) - \frac{704}{9}\zeta_2\Biggr\} \\
& +\Biggl\{\frac{2R_1}{9\eta^2\pi} + \frac{32\eta\big(-27+18\eta+\eta^2\big)K_4(K_8+K_9)}{3(-1+\eta)\pi} \\
& -\frac{16(1+\eta)^2\big(1-10\eta+\eta^2\big)}{3\eta^2}\left[G_6 + G_7 - \frac{8(K_{19}+K_{20})}{\pi}\right] \\
& +(1-\eta)^2\Biggl[-\frac{32\big(1+46\eta+\eta^2\big)G_1}{3\eta^2} + \frac{2\big(1+46\eta+\eta^2\big)\pi}{3\eta^2} \\
& -\frac{4\big(-1-18\eta+27\eta^2\big)}{3\eta^2}\biggl[G_{12} + G_{13} - K_{13} - K_{14} \\
& +\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\biggr] \\
& -\frac{4\big(-27+18\eta+\eta^2\big)}{3\eta}\biggl[G_{14} + G_{15} - K_{16} - K_{17} \\
& +\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\biggr]\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl[-\frac{32\eta\bigl(-27+18\eta+\eta^2\bigr)K_2K_4}{3(-1+\eta)\pi}-\frac{(-1+\eta)(1+\eta)\bigl(1+46\eta+\eta^2\bigr)\pi}{3\eta^2} \\
&+(1+\eta)\Biggl[-\frac{R_2}{9(-1+\eta)\eta^2\pi}+\frac{16(-1+\eta)\bigl(1+46\eta+\eta^2\bigr)G_1}{3\eta^2}\Biggr] \\
&+(1-\eta)^2\Biggl[-\frac{4\bigl(-1-18\eta+27\eta^2\bigr)}{3\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right) \\
&+\frac{4\bigl(-27+18\eta+\eta^2\bigr)}{3\eta}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Biggr]\Biggr]\mathrm{H}_0(\eta) \\
&+\Biggl[-\frac{4(1+\eta)^2\bigl(1-10\eta+\eta^2\bigr)}{3(-1+\eta)^2\eta\pi}+\frac{16\eta\bigl(-27+18\eta+\eta^2\bigr)K_4}{3(-1+\eta)^2\pi}\Biggr]\mathrm{H}_0^2(\eta)\Biggr\} \\
&\times\frac{1}{\sqrt{1-y}\sqrt{y}}+\Biggl[-\frac{8(-1+\eta)(1+\eta)R_4}{3\eta^2(1-y+\eta y)(-\eta-y+\eta y)} \\
&-\frac{128(-1+\eta)(G_2+G_3-K_2-K_5)}{3(1-y)}\Biggr]\mathrm{H}_0(\eta)
\end{aligned}
\tag{5.43}
$$

$$
\begin{aligned}
\Phi_4^{C_F}(y)=&\frac{1}{1-y}\Biggl\{-\frac{16yR_{10}}{81\eta^2}+\frac{128}{3}(-1+\eta)y\bigl(-G_{10}-G_{11}+G_8+G_9 \\
&+K_{11}+K_{12}-K_8-K_9\bigr) \\
&-64y\mathrm{H}_0^2(\eta)+\Biggl[-\frac{8R_{12}}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}-\frac{256}{9}y\mathrm{H}_1(y)\Biggr]\mathrm{H}_0(y) \\
&+\frac{128}{9}y\mathrm{H}_0^2(y)-\frac{8R_{11}}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)-\frac{64}{9}y\mathrm{H}_1^2(y) \\
&+\frac{256}{3}y\mathrm{H}_{0,1}(y)-\frac{704}{9}y\zeta_2\Biggr\} \\
&+\Biggl\{\frac{2R_7}{27\eta^2\pi}+\frac{32\eta\bigl(-27+54\eta+5\eta^2\bigr)K_4(K_8+K_9)}{9(-1+\eta)\pi} \\
&-\frac{16(1+\eta)^2\bigl(5+22\eta+5\eta^2\bigr)}{9\eta^2}\left[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\right] \\
&+(1-\eta)^2\Biggl[-\frac{32\bigl(5+86\eta+5\eta^2\bigr)G_1}{9\eta^2}+\frac{2\bigl(5+86\eta+5\eta^2\bigr)\pi}{9\eta^2} \\
&-\frac{4\bigl(-5-54\eta+27\eta^2\bigr)}{9\eta^2}\Biggl[G_{12}+G_{13}-K_{13}-K_{14} \\
&+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Biggr] \\
&-\frac{4\bigl(-27+54\eta+5\eta^2\bigr)}{9\eta}\Biggl[G_{14}+G_{15}-K_{16}-K_{17} \\
&+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Biggr]\Biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+\Biggl[-\frac{32\eta\bigl(-27+54\eta+5\eta^2\bigr)K_2K_4}{9(-1+\eta)\pi}-\frac{(-1+\eta)(1+\eta)\bigl(5+86\eta+5\eta^2\bigr)\pi}{9\eta^2}\\
&+(1+\eta)\biggl[-\frac{R_8}{27(-1+\eta)\eta^2\pi}+\frac{16(-1+\eta)\bigl(5+86\eta+5\eta^2\bigr)G_1}{9\eta^2}\biggr]\\
&+(1-\eta)^2\biggl[-\frac{4\bigl(-5-54\eta+27\eta^2\bigr)}{9\eta^2}\biggl(G_4-K_6+\frac{8K_{15}}{\pi}\biggr)\\
&+\frac{4\bigl(-27+54\eta+5\eta^2\bigr)}{9\eta}\biggl(G_5-K_7+\frac{8K_{18}}{\pi}\biggr)\biggr]\Biggr]\mathrm{H}_0(\eta)\\
&+\biggl[-\frac{4(1+\eta)^2\bigl(5+22\eta+5\eta^2\bigr)}{9(-1+\eta)^2\eta\pi}+\frac{16\eta\bigl(-27+54\eta+5\eta^2\bigr)K_4}{9(-1+\eta)^2\pi}\biggr]\mathrm{H}_0^2(\eta)\Biggr\}\\
&\times\frac{\sqrt{y}}{\sqrt{1-y}}+\biggl[-\frac{8(-1+\eta)(1+\eta)y^2R_9}{9\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&-\frac{128(-1+\eta)y(G_2+G_3-K_2-K_5)}{3(1-y)}\biggr]\mathrm{H}_0(\eta)
\end{aligned}
\tag{5.44}
$$

$$
\begin{aligned}
\Phi_5^{C_F}(z,y)=&-\frac{8R_{15}}{405\eta^2}-\frac{320}{3}(-1+\eta)(1+y)\bigl(-G_{10}-G_{11}+G_8+G_9\\
&+K_{11}+K_{12}-K_8-K_9\bigr)+\frac{1}{1-y}\Biggl\{\biggl[\frac{16\bigl(27+110\eta+27\eta^2\bigr)y^2}{81\eta}\\
&-\frac{128}{3}(1-\eta)y^2\bigl(G_{10}+G_{11}-G_8-G_9-K_{11}-K_{12}+K_8+K_9\bigr)\\
&-\frac{448}{27}y^2\mathrm{H}_1(y)+\frac{64}{9}y^2\mathrm{H}_1^2(y)-\frac{256}{3}y^2\mathrm{H}_{0,1}(y)+\frac{704}{9}y^2\zeta_2\biggr]\mathrm{H}_0(z)\\
&+\biggl[-\frac{224y^2}{27}+\frac{64}{9}y^2\mathrm{H}_1(y)\biggr]\mathrm{H}_0^2(z)+\frac{64}{27}y^2\mathrm{H}_0^3(z)\Biggr\}\\
&+\Biggl\{-\frac{R_{13}}{45\eta^2\pi}-\frac{16\eta\bigl(-495+450\eta+29\eta^2\bigr)K_4(K_8+K_9)}{15(-1+\eta)\pi}\\
&+\frac{8(1+\eta)^2\bigl(29-74\eta+29\eta^2\bigr)}{15\eta^2}\biggl[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\biggr]\\
&+(1-\eta)^2\Biggl[\frac{16\bigl(29+974\eta+29\eta^2\bigr)G_1}{15\eta^2}-\frac{\bigl(29+974\eta+29\eta^2\bigr)\pi}{15\eta^2}\\
&+\frac{2\bigl(-29-450\eta+495\eta^2\bigr)}{15\eta^2}\biggl[G_{12}+G_{13}-K_{13}-K_{14}\\
&+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\biggr]\\
&+\frac{2\bigl(-495+450\eta+29\eta^2\bigr)}{15\eta}\biggl[G_{14}+G_{15}-K_{16}-K_{17}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg]\\
&+\Bigg[-\frac{8(-1+\eta)(1+\eta)\big(29+974\eta+29\eta^2\big)G_1}{15\eta^2}\\
&+\frac{16\eta\big(-495+450\eta+29\eta^2\big)K_2K_4}{15(-1+\eta)\pi}\\
&+\frac{(-1+\eta)(1+\eta)\big(29+974\eta+29\eta^2\big)\pi}{30\eta^2}+\frac{(1+\eta)R_{14}}{90(-1+\eta)\eta^2\pi}\\
&+(1-\eta)^2\Bigg[\frac{2\big(-29-450\eta+495\eta^2\big)}{15\eta^2}\Bigg(G_4-K_6+\frac{8K_{15}}{\pi}\Bigg)\\
&-\frac{2\big(-495+450\eta+29\eta^2\big)}{15\eta}\Bigg(G_5-K_7+\frac{8K_{18}}{\pi}\Bigg)\Bigg]\Bigg]\mathrm{H}_0(\eta)\\
&+\Bigg[\frac{2(1+\eta)^2\big(29-74\eta+29\eta^2\big)}{15(-1+\eta)^2\eta\pi}\\
&-\frac{8\eta\big(-495+450\eta+29\eta^2\big)K_4}{15(-1+\eta)^2\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}\\
&+\Bigg[\frac{4(-1+\eta)R_{16}}{15\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&+\frac{320}{3}(-1+\eta)(1+y)(G_2+G_3-K_2-K_5)\\
&-\frac{128(1-\eta)y^2(G_2+G_3-K_2-K_5)}{3(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(\eta)\\
&+\Bigg[160(1+y)+\frac{64y^2\mathrm{H}_0(z)}{1-y}\Bigg]\mathrm{H}_0^2(\eta)\\
&+\Bigg\{-\frac{4R_{17}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&+\frac{1}{1-y}\Bigg[\Bigg(-\frac{896y^2}{27}+\frac{256}{9}y^2\mathrm{H}_1(y)\Bigg)\mathrm{H}_0(z)+\frac{128}{9}y^2\mathrm{H}_0^2(z)\Bigg]\\
&+\frac{640}{9}(1+y)\mathrm{H}_1(y)\Bigg\}\mathrm{H}_0(y)+\Bigg[-\frac{320}{9}(1+y)-\frac{128y^2\mathrm{H}_0(z)}{9(1-y)}\Bigg]\mathrm{H}_0^2(y)\\
&-\frac{4R_{18}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)\\
&+\frac{160}{9}(1+y)\mathrm{H}_1^2(y)-\frac{640}{3}(1+y)\mathrm{H}_{0,1}(y)+\frac{1760}{9}(1+y)\zeta_2
\end{aligned}
\tag{5.45}
$$

$$
\begin{aligned}
\Phi_6^{C_F}(z,y)=&\frac{8R_{21}}{81\eta^2}-\frac{320}{3}(-1+\eta)\big(G_{10}+G_{11}-G_8-G_9-K_{11}-K_{12}+K_8+K_9\big)\\
&+\frac{1}{1-y}\Bigg\{\Bigg[\frac{16\big(45+182\eta+45\eta^2\big)y}{81\eta}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{128}{3}(1-\eta)y\big(G_{10}+G_{11}-G_8-G_9-K_{11}-K_{12}+K_8+K_9\big)\\
&+\frac{704}{27}y\mathrm{H}_1(y)+\frac{64}{9}y\mathrm{H}_1^2(y)-\frac{256}{3}y\mathrm{H}_{0,1}(y)+\frac{704}{9}y\zeta_2\Bigg]\mathrm{H}_0(z)\\
&+\left(\frac{352y}{27}+\frac{64}{9}y\mathrm{H}_1(y)\right)\mathrm{H}_0^2(z)+\frac{64}{27}y\mathrm{H}_0^3(z)\Bigg\}\\
&+\Bigg\{\frac{R_{19}}{81\eta^2\pi}+\frac{(1-\eta)^2\big(55+1594\eta+55\eta^2\big)\pi}{27\eta^2}\\
&-\frac{16(1-\eta)^2\big(-55-810\eta+729\eta^2\big)(K_{21}+K_{22}+K_{23}+K_{24})}{27\eta^2\pi}\\
&+\frac{16\eta\big(-729+810\eta+55\eta^2\big)K_4(K_8+K_9)}{27(-1+\eta)\pi}\\
&-\frac{8(1+\eta)^2\big(55+26\eta+55\eta^2\big)}{27\eta^2}\left[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\right]\\
&-\frac{2(1-\eta)^2\big(-729+810\eta+55\eta^2\big)}{27\eta}\Bigg[G_{14}+G_{15}-K_{16}-K_{17}\\
&+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\\
&+(1-\eta)^2\Bigg[-\frac{16\big(55+1594\eta+55\eta^2\big)G_1}{27\eta^2}\\
&-\frac{2\big(-55-810\eta+729\eta^2\big)(G_{12}+G_{13}-K_{13}-K_{14})}{27\eta^2}\Bigg]\\
&+\Bigg[\frac{8(-1+\eta)(1+\eta)\big(55+1594\eta+55\eta^2\big)G_1}{27\eta^2}\\
&-\frac{16\eta\big(-729+810\eta+55\eta^2\big)K_2K_4}{27(-1+\eta)\pi}\\
&-\frac{(-1+\eta)(1+\eta)\big(55+1594\eta+55\eta^2\big)\pi}{54\eta^2}-\frac{(1+\eta)R_{20}}{162(-1+\eta)\eta^2\pi}\\
&-\frac{2(1-\eta)^2\big(-55-810\eta+729\eta^2\big)}{27\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right)\\
&+\frac{2(1-\eta)^2\big(-729+810\eta+55\eta^2\big)}{27\eta}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Bigg]\mathrm{H}_0(\eta)\\
&+\Bigg[-\frac{2(1+\eta)^2\big(55+26\eta+55\eta^2\big)}{27(-1+\eta)^2\eta\pi}\\
&+\frac{8\eta\big(-729+810\eta+55\eta^2\big)K_4}{27(-1+\eta)^2\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}\\
&+\Bigg[-\frac{4(-1+\eta)R_{22}}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{320}{3}(-1+\eta)(G_2+G_3-K_2-K_5)\\
&-\frac{128(1-\eta)y(G_2+G_3-K_2-K_5)}{3(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(\eta)\\
&+\Bigg(-160+\frac{64y\mathrm{H}_0(z)}{1-y}\Bigg)\mathrm{H}_0^2(\eta)\\
&+\Bigg\{\frac{4R_{23}}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&+\frac{1}{1-y}\Bigg[\Bigg(\frac{1408y}{27}+\frac{256}{9}y\mathrm{H}_1(y)\Bigg)\mathrm{H}_0(z)\\
&+\frac{128}{9}y\mathrm{H}_0^2(z)\Bigg]-\frac{640}{9}\mathrm{H}_1(y)\Bigg\}\mathrm{H}_0(y)+\Bigg(\frac{320}{9}-\frac{128y\mathrm{H}_0(z)}{9(1-y)}\Bigg)\mathrm{H}_0^2(y)\\
&+\frac{4R_{24}}{27\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)-\frac{160}{9}\mathrm{H}_1^2(y)\\
&+\frac{640}{3}\mathrm{H}_{0,1}(y)-\frac{1760}{9}\zeta_2
\end{aligned}
\tag{5.46}
$$

$$
\begin{aligned}
\Phi_7^{C_F}(y)=&-\frac{56\big(\mathrm{H}_0(y)+\mathrm{H}_1(y)\big)R_{27}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&-\frac{112}{135\eta^2\big(1+\sqrt{y}\big)}\Big(\eta+\eta^3+y-\eta y-\eta^3y+\eta^4y+\frac{1}{\sqrt{y}}R_{25}\Big)\\
&+\Bigg\{+\frac{56(1+\eta)^2\big(1-\eta+\eta^2\big)}{81\eta^2\pi}-\frac{14(1-\eta)^2\big(1+\eta+\eta^2\big)\pi}{135\eta^2}\\
&-\frac{224(1-\eta)^2(K_{21}+K_{22}+K_{23}+K_{24})}{135\eta^2\pi}-\frac{224\eta^3K_4(K_8+K_9)}{135(-1+\eta)\pi}\\
&+\frac{112(1+\eta)^2\big(1-\eta+\eta^2\big)}{135\eta^2}\Bigg[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\Bigg]\\
&+\frac{28}{135}(1-\eta)^2\eta\Bigg[G_{14}+G_{15}-K_{16}-K_{17}+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\\
&+(1-\eta)^2\Bigg[\frac{224\big(1+\eta+\eta^2\big)G_1}{135\eta^2}-\frac{28(G_{12}+G_{13}-K_{13}-K_{14})}{135\eta^2}\Bigg]\\
&+\Bigg[\frac{224\eta^3K_2K_4}{135(-1+\eta)\pi}-\frac{112(-1+\eta)(1+\eta)\big(1+\eta+\eta^2\big)G_1}{135\eta^2}\\
&-\frac{112(1+\eta)\big(1-\eta+\eta^2\big)\big(1+\eta+\eta^2\big)}{405(-1+\eta)\eta^2\pi}+\frac{7(-1+\eta)(1+\eta)\big(1+\eta+\eta^2\big)\pi}{135\eta^2}\\
&-\frac{28(1-\eta)^2}{135\eta^2}\Bigg(G_4-K_6+\frac{8K_{15}}{\pi}\Bigg)\\
&-\frac{28}{135}(1-\eta)^2\eta\Bigg(G_5-K_7+\frac{8K_{18}}{\pi}\Bigg)\Bigg]\mathrm{H}_0(\eta)
\end{aligned}
$$

$$+\Bigg[-\frac{112\eta^3 K_4}{135(-1+\eta)^2\pi}+\frac{28(1+\eta)^2\big(1-\eta+\eta^2\big)}{135(-1+\eta)^2\eta\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}$$

$$+\frac{56(-1+\eta)R_{26}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(\eta) \tag{5.47}$$

$$\Phi_8^{C_F}(y)=-\frac{64(-1+\eta)}{3(-1+y)}\Bigg\{\Big[G_{10}+G_{11}-G_8-G_9-K_{11}-K_{12}+K_8+K_9$$

$$+(G_2+G_3-K_2-K_5)\mathrm{H}_0(\eta)\Big]\Big[5-5z+2(1+z)\mathrm{H}_0(z)\Big]\Bigg\} \tag{5.48}$$

$$\Phi_2^{C_A}(z,y)=\frac{128(y+z)}{27(-1+y)y^2} \tag{5.49}$$

$$\Phi_3^{C_A}(y)=\frac{1}{1-y}\Bigg\{\frac{4R_{30}}{405\eta^2}-\frac{2(-1+\eta)(1+\eta)R_{31}}{45\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(\eta)$$

$$+\frac{2R_{32}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(y)$$

$$+\frac{2R_{33}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)\Bigg\}$$

$$+\Bigg\{\frac{16(-1+\eta)^4G_1}{\eta^2}-\frac{(-1+\eta)^4\pi}{\eta^2}$$

$$-\frac{R_{29}}{9\eta^2\pi}-\frac{8\big(-7+165\eta+75\eta^2+23\eta^3\big)K_4(K_8+K_9)}{15(-1+\eta)\pi}$$

$$+\frac{64(1+\eta)^2\big(1+14\eta+\eta^2\big)}{15\eta^2}\Bigg[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\Bigg]$$

$$+(1-\eta)^2\Bigg[\frac{-23-75\eta-165\eta^2+7\eta^3}{15\eta^2}\Bigg[G_{12}+G_{13}-K_{13}-K_{14}$$

$$+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Bigg]$$

$$+\frac{-7+165\eta+75\eta^2+23\eta^3}{15\eta^2}\Bigg[G_{14}+G_{15}-K_{16}-K_{17}$$

$$+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg]$$

$$+\Bigg[-\frac{8(-1+\eta)^3(1+\eta)G_1}{\eta^2}+\frac{(-1+\eta)^3(1+\eta)\pi}{2\eta^2}$$

$$+\frac{8\big(-7+165\eta+75\eta^2+23\eta^3\big)K_2K_4}{15(-1+\eta)\pi}+\frac{(1+\eta)R_{28}}{90(-1+\eta)\eta^2\pi}$$

$$+(1-\eta)^2\Bigg[\frac{\big(-23-75\eta-165\eta^2+7\eta^3\big)}{15\eta^2}\Bigg(G_4-K_6+\frac{8K_{15}}{\pi}\Bigg)$$

$$-\frac{\big(-7+165\eta+75\eta^2+23\eta^3\big)}{15\eta^2}\Bigg(G_5-K_7+\frac{8K_{18}}{\pi}\Bigg)\Bigg]\Bigg]\mathrm{H}_0(\eta)$$

$$+\left[\frac{16(1+\eta)^2\big(1+14\eta+\eta^2\big)}{15(-1+\eta)^2\eta\pi}\right.$$
$$\left.-\frac{4\big(-7+165\eta+75\eta^2+23\eta^3\big)K_4}{15(-1+\eta)^2\pi}\right]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}} \tag{5.50}$$

$$\Phi_4^{C_A}(y)=\frac{1}{1-y}\Bigg\{\frac{yR_{36}}{405\eta^2}-\frac{(-1+\eta)(1+\eta)y^2R_{37}}{90\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(\eta)$$
$$-\frac{R_{39}}{270\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(y)$$
$$-\frac{R_{38}}{270\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)\Bigg\}$$
$$+\Bigg\{-\frac{R_{34}}{270\eta^2\pi}-\frac{2\big(43+705\eta+405\eta^2+175\eta^3\big)K_4(K_8+K_9)}{45(-1+\eta)\pi}$$
$$+\frac{2(1+\eta)^2\big(109+446\eta+109\eta^2\big)}{45\eta^2}\left[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\right]$$
$$+(1-\eta)^2\Bigg[\frac{8\big(11-14\eta+11\eta^2\big)G_1}{15\eta^2}-\frac{\big(11-14\eta+11\eta^2\big)\pi}{30\eta^2}$$
$$-\frac{175+405\eta+705\eta^2+43\eta^3}{180\eta^2}\Bigg[G_{12}+G_{13}-K_{13}-K_{14}$$
$$+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Bigg]$$
$$+\frac{43+705\eta+405\eta^2+175\eta^3}{180\eta^2}\Bigg[G_{14}+G_{15}-K_{16}-K_{17}$$
$$+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg]$$
$$+\Bigg[-\frac{4(-1+\eta)(1+\eta)\big(11-14\eta+11\eta^2\big)G_1}{15\eta^2}$$
$$+\frac{2\big(43+705\eta+405\eta^2+175\eta^3\big)K_2K_4}{45(-1+\eta)\pi}$$
$$+\frac{(-1+\eta)(1+\eta)\big(11-14\eta+11\eta^2\big)\pi}{60\eta^2}+\frac{(1+\eta)R_{35}}{540(-1+\eta)\eta^2\pi}$$
$$+(1-\eta)^2\Bigg[-\frac{\big(175+405\eta+705\eta^2+43\eta^3\big)}{180\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right)$$
$$-\frac{\big(43+705\eta+405\eta^2+175\eta^3\big)}{180\eta^2}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Bigg]\Bigg]\mathrm{H}_0(\eta)$$
$$+\Bigg[\frac{(1+\eta)^2\big(109+446\eta+109\eta^2\big)}{90(-1+\eta)^2\eta\pi}$$

$$
-\frac{\big(43+705\eta+405\eta^2+175\eta^3\big)K_4}{45(-1+\eta)^2\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{\sqrt{y}}{\sqrt{1-y}} \tag{5.51}
$$

$$
\begin{aligned}
\Phi_5^{C_A}(z,y) = {} & \frac{2R_{42}}{2025\eta^2}+\frac{64(-1+\eta)(1+y)(G_8+G_9-K_8-K_9)}{15\eta} \\
& -\frac{64}{15}(-1+\eta)(G_{10}+G_{11}-K_{11}-K_{12})(\eta+\eta y) \\
& +\frac{1}{1-y}\Bigg\{\Bigg[-\frac{8\big(279+796\eta+279\eta^2\big)y^2}{405\eta} \\
& +\frac{64\big(9+20\eta+9\eta^2\big)y^2}{135\eta}\mathrm{H}_1(y)\Bigg]\mathrm{H}_0(z)+\frac{128}{27}y^2\mathrm{H}_0^2(z)\Bigg\} \\
& +\Bigg\{\frac{R_{40}}{1350\eta^2\pi}+\frac{4\big(-865+11775\eta+2625\eta^2+1137\eta^3\big)K_4(K_8+K_9)}{225(-1+\eta)\pi} \\
& -\frac{32(1+\eta)^2\big(17+883\eta+17\eta^2\big)}{225\eta^2}\Bigg[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\Bigg] \\
& +(1-\eta)^2\Bigg[-\frac{8\big(1001-3574\eta+1001\eta^2\big)G_1}{225\eta^2} \\
& +\frac{\big(1001-3574\eta+1001\eta^2\big)\pi}{450\eta^2} \\
& -\frac{-1137-2625\eta-11775\eta^2+865\eta^3}{450\eta^2}\Bigg[G_{12}+G_{13}-K_{13}-K_{14} \\
& +\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Bigg] \\
& -\frac{-865+11775\eta+2625\eta^2+1137\eta^3}{450\eta^2}\Bigg[G_{14}+G_{15}-K_{16}-K_{17} \\
& +\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg] \\
& +\Bigg[\frac{4(-1+\eta)(1+\eta)\big(1001-3574\eta+1001\eta^2\big)G_1}{225\eta^2} \\
& -\frac{4\big(-865+11775\eta+2625\eta^2+1137\eta^3\big)K_2K_4}{225(-1+\eta)\pi} \\
& -\frac{(-1+\eta)(1+\eta)\big(1001-3574\eta+1001\eta^2\big)\pi}{900\eta^2}-\frac{(1+\eta)R_{41}}{2700(-1+\eta)\eta^2\pi} \\
& +(1-\eta)^2\Bigg[-\frac{\big(-1137-2625\eta-11775\eta^2+865\eta^3\big)}{450\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right) \\
& +\frac{\big(-865+11775\eta+2625\eta^2+1137\eta^3\big)}{450\eta^2}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Bigg]\Bigg]\mathrm{H}_0(\eta)
\end{aligned}
$$

$$+\Bigg[-\frac{8(1+\eta)^2\big(17+883\eta+17\eta^2\big)}{225(-1+\eta)^2\eta\pi}$$
$$+\frac{2\big(-865+11775\eta+2625\eta^2+1137\eta^3\big)K_4}{225(-1+\eta)^2\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}$$
$$+\Bigg[-\frac{(-1+\eta)R_{43}}{225\eta^2(1-y+\eta y)(-\eta-y+\eta y)}-\frac{64(-1+\eta)(1+y)(G_2-K_2)}{15\eta}$$
$$-\frac{64}{15}(-1+\eta)(G_3-K_5)(\eta+\eta y)+\frac{64}{15\eta}\big(-1+\eta^2-y+\eta^2 y\big)\mathrm{H}_0(y)$$
$$-\frac{64(-1+\eta)(1+\eta)y^2}{15\eta(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(\eta)+\frac{224}{3}(1+y)\mathrm{H}_0^2(\eta)$$
$$+\Bigg[\frac{R_{44}}{225\eta^2(1-y+\eta y)(-\eta-y+\eta y)}+\frac{64\big(9+40\eta+9\eta^2\big)y^2}{135\eta(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(y)$$
$$+\frac{32\big(1+\eta^2+y+\eta^2 y\big)}{15\eta}\mathrm{H}_0^2(y)+\frac{R_{45}}{675\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)$$
$$+\frac{64\big(1+\eta^2+y+\eta^2 y\big)}{15\eta}\mathrm{H}_{0,1}(y)-\frac{64\big(1+\eta^2+y+\eta^2 y\big)}{15\eta}\zeta_2 \tag{5.52}$$

$$\Phi_6^{C_A}(z,y)=-\frac{2R_{48}}{405\eta^2}+\frac{1}{1-y}\Bigg\{\Bigg[-\frac{2\big(981-1406\eta+981\eta^2\big)y}{405\eta}$$
$$+\frac{8\big(9+160\eta+9\eta^2\big)y\mathrm{H}_1(y)}{135\eta}\Bigg]\mathrm{H}_0(z)+\frac{128}{27}y\mathrm{H}_0^2(z)\Bigg\}$$
$$+\Bigg\{-\frac{R_{46}}{1620\eta^2\pi}-\frac{4\big(435+5940\eta+1665\eta^2+692\eta^3\big)K_4(K_8+K_9)}{135(-1+\eta)\pi}$$
$$+\frac{2(1+\eta)^2\big(1127+6478\eta+1127\eta^2\big)}{135\eta^2}\Bigg[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\Bigg]$$
$$+(1-\eta)^2\Bigg[\frac{4\big(257-4018\eta+257\eta^2\big)G_1}{135\eta^2}-\frac{\big(257-4018\eta+257\eta^2\big)\pi}{540\eta^2}$$
$$-\frac{692+1665\eta+5940\eta^2+435\eta^3}{270\eta^2}\Bigg[G_{12}+G_{13}-K_{13}-K_{14}$$
$$+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\Bigg]$$
$$+\frac{435+5940\eta+1665\eta^2+692\eta^3}{270\eta^2}\Bigg[G_{14}+G_{15}-K_{16}-K_{17}$$
$$+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\Bigg]\Bigg]$$
$$+\Bigg[-\frac{2(-1+\eta)(1+\eta)\big(257-4018\eta+257\eta^2\big)G_1}{135\eta^2}$$

$$
\begin{aligned}
&+\frac{4\big(435+5940\eta+1665\eta^2+692\eta^3\big)K_2K_4}{135(-1+\eta)\pi}\\
&+\frac{(-1+\eta)(1+\eta)\big(257-4018\eta+257\eta^2\big)\pi}{1080\eta^2}+\frac{(1+\eta)R_{47}}{3240(-1+\eta)\eta^2\pi}\\
&+(1-\eta)^2\Bigg[-\frac{\big(692+1665\eta+5940\eta^2+435\eta^3\big)}{270\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right)\\
&-\frac{\big(435+5940\eta+1665\eta^2+692\eta^3\big)}{270\eta^2}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Bigg]\Bigg]\mathrm{H}_0(\eta)\\
&+\Bigg[\frac{(1+\eta)^2\big(1127+6478\eta+1127\eta^2\big)}{270(-1+\eta)^2\eta\pi}\\
&-\frac{2\big(435+5940\eta+1665\eta^2+692\eta^3\big)K_4}{135(-1+\eta)^2\pi}\Bigg]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}\\
&+\Bigg[\frac{(-1+\eta)R_{49}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}-\frac{8(-1+\eta)(1+\eta)y}{15\eta(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(\eta)\\
&-72\mathrm{H}_0^2(\eta)+\Bigg[\frac{R_{50}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&+\frac{8\big(9+320\eta+9\eta^2\big)y}{135\eta(1-y)}\mathrm{H}_0(z)\Bigg]\mathrm{H}_0(y)\\
&+\frac{R_{51}}{135\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_1(y)
\end{aligned}
\tag{5.53}
$$

$$
\begin{aligned}
\Phi_7^{C_A}(y)=&\frac{49\big(\mathrm{H}_0(y)+\mathrm{H}_1(y)\big)R_{54}}{675\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\\
&+\frac{98}{675\eta^2\big(1+\sqrt{y}\big)}\left(\eta+\eta^3+y-\eta y-\eta^3y+\eta^4y+\frac{1}{\sqrt{y}}R_{52}\right)\\
&+\Bigg\{-\frac{49(1+\eta)^2\big(1-\eta+\eta^2\big)}{405\eta^2\pi}+\frac{196\eta^3K_4(K_8+K_9)}{675(-1+\eta)\pi}\\
&-\frac{98(1+\eta)^2\big(1-\eta+\eta^2\big)}{675\eta^2}\left[G_6+G_7-\frac{8(K_{19}+K_{20})}{\pi}\right]\\
&+(1-\eta)^2\Bigg[-\frac{196\big(1+\eta+\eta^2\big)G_1}{675\eta^2}+\frac{49\big(1+\eta+\eta^2\big)\pi}{2700\eta^2}\\
&+\frac{49}{1350\eta^2}\left[G_{12}+G_{13}-K_{13}-K_{14}+\frac{8(K_{21}+K_{22}+K_{23}+K_{24})}{\pi}\right]\\
&-\frac{49\eta}{1350}\left[G_{14}+G_{15}-K_{16}-K_{17}+\frac{8(K_{25}+K_{26}+K_{27}+K_{28})}{\pi}\right]\Bigg]\\
&+\Bigg[-\frac{196\eta^3K_2K_4}{675(-1+\eta)\pi}+\frac{98(-1+\eta)(1+\eta)\big(1+\eta+\eta^2\big)G_1}{675\eta^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{98(1+\eta)\big(1-\eta+\eta^2\big)\big(1+\eta+\eta^2\big)}{2025(-1+\eta)\eta^2\pi}-\frac{49(-1+\eta)(1+\eta)\big(1+\eta+\eta^2\big)\pi}{5400\eta^2}\\
&+(1-\eta)^2\Bigg[\frac{49}{1350\eta^2}\left(G_4-K_6+\frac{8K_{15}}{\pi}\right)\\
&+\frac{49\eta}{1350}\left(G_5-K_7+\frac{8K_{18}}{\pi}\right)\Bigg]\Bigg]\mathrm{H}_0(\eta)\\
&+\left[\frac{98\eta^3K_4}{675(-1+\eta)^2\pi}-\frac{49(1+\eta)^2\big(1-\eta+\eta^2\big)}{1350(-1+\eta)^2\eta\pi}\right]\mathrm{H}_0^2(\eta)\Bigg\}\frac{1}{\sqrt{1-y}\sqrt{y}}\\
&-\frac{49(-1+\eta)R_{53}}{675\eta^2(1-y+\eta y)(-\eta-y+\eta y)}\mathrm{H}_0(\eta)
\end{aligned}
\tag{5.54}
$$

$$
\begin{aligned}
\Phi_8^{C_A}(y)=&-\frac{64(-1+\eta)z}{15\eta(-1+y)}\Big\{-G_8-G_9+K_8+K_9+\eta^2(G_{10}+G_{11}-K_{11}-K_{12})\\
&+\Big[G_2-K_2+\eta^2(G_3-K_5)\Big]\mathrm{H}_0(\eta)\Big\}\,,
\end{aligned}
\tag{5.55}
$$

where the polynomials R_i are:

$$R_1=-20(1+\eta)^2\big(1-10\eta+\eta^2\big) \tag{5.56}$$

$$R_2=-16\big(1-9\eta-8\eta^2-9\eta^3+\eta^4\big) \tag{5.57}$$

$$
\begin{aligned}
R_3=&\,\eta\big(729-862\eta+729\eta^2\big)+27y^2(-1+\eta)^2\big(1+46\eta+\eta^2\big)\\
&-27y\big(1+70\eta-126\eta^2+70\eta^3+\eta^4\big)
\end{aligned}
\tag{5.58}
$$

$$
\begin{aligned}
R_4=&\,18\eta^2+2y\eta\big(13-50\eta+13\eta^2\big)+y^3(-1+\eta)^2\big(1+46\eta+\eta^2\big)\\
&-y^2(-1+\eta)^2\big(1+74\eta+\eta^2\big)
\end{aligned}
\tag{5.59}
$$

$$
\begin{aligned}
R_5=&\,2\eta^2\big(81-34\eta+81\eta^2\big)\\
&+y^2\big(9-576\eta+1391\eta^2-1360\eta^3+1391\eta^4-576\eta^5+9\eta^6\big)\\
&+9y^4(-1+\eta)^2(1+\eta)^2\big(1-10\eta+\eta^2\big)-18y^3(-1+\eta)^2\big(1-21\eta-21\eta^3+\eta^4\big)\\
&+2y\eta\big(126-385\eta+302\eta^2-385\eta^3+126\eta^4\big)
\end{aligned}
\tag{5.60}
$$

$$
\begin{aligned}
R_6=&\,2\eta^2\big(81-62\eta+81\eta^2\big)\\
&+y^2\big(9-576\eta+1447\eta^2-1472\eta^3+1447\eta^4-576\eta^5+9\eta^6\big)\\
&+9y^4(-1+\eta)^2(1+\eta)^2\big(1-10\eta+\eta^2\big)-18y^3(-1+\eta)^2\big(1-21\eta-21\eta^3+\eta^4\big)\\
&+2y\eta\big(126-413\eta+358\eta^2-413\eta^3+126\eta^4\big)
\end{aligned}
\tag{5.61}
$$

$$R_7=-20(1+\eta)^2\big(5+22\eta+5\eta^2\big) \tag{5.62}$$

$$R_8=-16\big(5+27\eta+32\eta^2+27\eta^3+5\eta^4\big) \tag{5.63}$$

$$
\begin{aligned}
R_9=&\,2\eta\big(11-70\eta+11\eta^2\big)+y^2(-1+\eta)^2\big(5+86\eta+5\eta^2\big)\\
&-y(-1+\eta)^2\big(5+118\eta+5\eta^2\big)
\end{aligned}
\tag{5.64}
$$

$$
\begin{aligned}
R_{10}=&\,\eta\big(243-790\eta+243\eta^2\big)+9y^2(-1+\eta)^2\big(5+86\eta+5\eta^2\big)\\
&-9y\big(5+98\eta-270\eta^2+98\eta^3+5\eta^4\big)
\end{aligned}
\tag{5.65}
$$

$$R_{11} = y\Big[-280\eta^3 + y^2\big(15 - 96\eta + 757\eta^2 - 1736\eta^3 + 757\eta^4 - 96\eta^5 + 15\eta^6\big)$$
$$+3y^4(-1+\eta)^2(1+\eta)^2\big(5 + 22\eta + 5\eta^2\big)$$
$$-6y^3(-1+\eta)^2\big(5 + 21\eta + 108\eta^2 + 21\eta^3 + 5\eta^4\big)$$
$$+4y\eta\big(24 - 79\eta + 254\eta^2 - 79\eta^3 + 24\eta^4\big)\Big] \tag{5.66}$$

$$R_{12} = y\Big[-368\eta^3 + y^2\big(15 - 96\eta + 845\eta^2 - 1912\eta^3 + 845\eta^4 - 96\eta^5 + 15\eta^6\big)$$
$$+3y^4(-1+\eta)^2(1+\eta)^2\big(5 + 22\eta + 5\eta^2\big)$$
$$-6y^3(-1+\eta)^2\big(5 + 21\eta + 108\eta^2 + 21\eta^3 + 5\eta^4\big)$$
$$+4y\eta\big(24 - 101\eta + 298\eta^2 - 101\eta^3 + 24\eta^4\big)\Big] \tag{5.67}$$

$$R_{13} = -20(1+\eta)^2\big(29 - 74\eta + 29\eta^2\big) \tag{5.68}$$

$$R_{14} = 16\big(-29 + 45\eta + 16\eta^2 + 45\eta^3 - 29\eta^4\big) \tag{5.69}$$

$$R_{15} = -55\eta\big(243 - 382\eta + 243\eta^2\big) + y\big(783 + 23949\eta - 54320\eta^2 + 23949\eta^3$$
$$+783\eta^4\big) \tag{5.70}$$

$$R_{16} = (1+\eta)\Big[270\eta^2 + 2y\eta\big(233 - 982\eta + 233\eta^2\big) + y^3(-1+\eta)^2\big(29 + 974\eta + 29\eta^2\big)$$
$$-y^2(-1+\eta)^2\big(29 + 1498\eta + 29\eta^2\big)\Big] \tag{5.71}$$

$$R_{17} = -2\eta^2\big(1215 - 2686\eta + 1215\eta^2\big)$$
$$-9y^2(-1+\eta)^2\big(29 - 482\eta + 810\eta^2 - 482\eta^3 + 29\eta^4\big)$$
$$+y^3(-1+\eta)^2\big(261 - 144\eta - 1610\eta^2 - 144\eta^3 + 261\eta^4\big)$$
$$-4y\eta\big(1179 - 3476\eta + 4250\eta^2 - 3476\eta^3 + 1179\eta^4\big) \tag{5.72}$$

$$R_{18} = -2\eta^2\big(1215 - 2486\eta + 1215\eta^2\big)$$
$$-9y^2(-1+\eta)^2\big(29 - 482\eta + 810\eta^2 - 482\eta^3 + 29\eta^4\big)$$
$$+y^3(-1+\eta)^2\big(261 - 144\eta - 1210\eta^2 - 144\eta^3 + 261\eta^4\big)$$
$$-4y\eta\big(1179 - 3376\eta + 4150\eta^2 - 3376\eta^3 + 1179\eta^4\big) \tag{5.73}$$

$$R_{19} = -20(1+\eta)^2\big(55 + 26\eta + 55\eta^2\big) \tag{5.74}$$

$$R_{20} = -16\big(55 + 81\eta + 136\eta^2 + 81\eta^3 + 55\eta^4\big) \tag{5.75}$$

$$R_{21} = \eta\big(-2187 + 4202\eta - 2187\eta^2\big) + 3y(-1+\eta)^2\big(55 + 1594\eta + 55\eta^2\big) \tag{5.76}$$

$$R_{22} = (1+\eta)\Big[324\eta^2 + 2y\eta\big(337 - 1526\eta + 337\eta^2\big) + y^3(-1+\eta)^2\big(55 + 1594\eta + 55\eta^2\big)$$
$$-y^2(-1+\eta)^2\big(55 + 2378\eta + 55\eta^2\big)\Big] \tag{5.77}$$

$$R_{23} = -108(-3+\eta)\eta^2(-1+3\eta) + y^3(-1+\eta)^2(1+\eta)^2\big(55 + 26\eta + 55\eta^2\big)$$
$$-y^2(-1+\eta)^2\big(55 - 538\eta + 1942\eta^2 - 538\eta^3 + 55\eta^4\big)$$
$$-4y\eta\big(196 - 573\eta + 890\eta^2 - 573\eta^3 + 196\eta^4\big) \tag{5.78}$$

$$
\begin{aligned}
R_{24} = & -4\eta^2\big(81-250\eta+81\eta^2\big)+y^3(-1+\eta)^2(1+\eta)^2\big(55+26\eta+55\eta^2\big) \\
& -y^2(-1+\eta)^2\big(55-538\eta+1862\eta^2-538\eta^3+55\eta^4\big) \\
& -4y\eta\big(196-553\eta+850\eta^2-553\eta^3+196\eta^4\big)
\end{aligned}
\tag{5.79}
$$

$$
R_{25} = \eta+\eta^3+y\eta\big(1+\eta^2\big)+y^2(-1+\eta)^2\big(1+\eta+\eta^2\big)
\tag{5.80}
$$

$$
\begin{aligned}
R_{26} = & y(1+\eta)\Big[-y(-1+\eta)^2(1+\eta)^2+y^2(-1+\eta)^2\big(1+\eta+\eta^2\big) \\
& -\eta\big(1+\eta+\eta^2\big)\Big]
\end{aligned}
\tag{5.81}
$$

$$
\begin{aligned}
R_{27} = & 2\eta^3-y\eta(1+\eta)^2\big(1-\eta+\eta^2\big)+y^3(-1+\eta)^2(1+\eta)^2\big(1-\eta+\eta^2\big) \\
& -y^2(-1+\eta)^2\big(1+2\eta+2\eta^3+\eta^4\big)
\end{aligned}
\tag{5.82}
$$

$$
R_{28} = -128\big(1+15\eta+16\eta^2+15\eta^3+\eta^4\big)
\tag{5.83}
$$

$$
R_{29} = -32(1+\eta)^2\big(1+14\eta+\eta^2\big)
\tag{5.84}
$$

$$
\begin{aligned}
R_{30} = & 189+810y^2(-1+\eta)^4-4518\eta-2458\eta^2-4518\eta^3+189\eta^4 \\
& -27y\big(37-308\eta+30\eta^2-308\eta^3+37\eta^4\big)
\end{aligned}
\tag{5.85}
$$

$$
\begin{aligned}
R_{31} = & 90y^4(-1+\eta)^4-2\eta\big(7-158\eta+7\eta^2\big) \\
& +y\big(-21+634\eta-1418\eta^2+634\eta^3-21\eta^4\big) \\
& +4y^2(-1+\eta)^2\big(33-296\eta+33\eta^2\big)-3y^3(-1+\eta)^2\big(67-262\eta+67\eta^2\big)
\end{aligned}
\tag{5.86}
$$

$$
\begin{aligned}
R_{32} = & y\big(63-1488\eta+5429\eta^2+5816\eta^3+5429\eta^4-1488\eta^5+63\eta^6\big) \\
& +144y^4(-1+\eta)^2(1+\eta)^2\big(1+14\eta+\eta^2\big) \\
& -45y^3(-1+\eta)^2\big(5+140\eta+222\eta^2+140\eta^3+5\eta^4\big) \\
& +2\eta\big(21-474\eta-470\eta^2-474\eta^3+21\eta^4\big) \\
& +2y^2\big(9+2640\eta-3217\eta^2-3472\eta^3-3217\eta^4+2640\eta^5+9\eta^6\big)
\end{aligned}
\tag{5.87}
$$

$$
\begin{aligned}
R_{33} = & y\big(63-1488\eta+4789\eta^2+7096\eta^3+4789\eta^4-1488\eta^5+63\eta^6\big) \\
& +144y^4(-1+\eta)^2(1+\eta)^2\big(1+14\eta+\eta^2\big) \\
& -45y^3(-1+\eta)^2\big(5+140\eta+222\eta^2+140\eta^3+5\eta^4\big) \\
& +2\eta\big(21-474\eta-790\eta^2-474\eta^3+21\eta^4\big) \\
& +2y^2\big(9+2640\eta-2897\eta^2-4112\eta^3-2897\eta^4+2640\eta^5+9\eta^6\big)
\end{aligned}
\tag{5.88}
$$

$$
R_{34} = -10(1+\eta)^2\big(109+446\eta+109\eta^2\big)
\tag{5.89}
$$

$$
R_{35} = -8\big(109+555\eta+664\eta^2+555\eta^3+109\eta^4\big)
\tag{5.90}
$$

$$
\begin{aligned}
R_{36} = & -387-5958\eta-10102\eta^2-5958\eta^3-387\eta^4 \\
& +108y^2(-1+\eta)^2\big(11-14\eta+11\eta^2\big) \\
& -9y\big(89-1312\eta-210\eta^2-1312\eta^3+89\eta^4\big)
\end{aligned}
\tag{5.91}
$$

$$
\begin{aligned}
R_{37} = & 43+794\eta-1530\eta^2+794\eta^3+43\eta^4+12y^3(-1+\eta)^2\big(11-14\eta+11\eta^2\big) \\
& -y^2(-1+\eta)^2\big(221-866\eta+221\eta^2\big) \\
& +2y\big(23-835\eta+1576\eta^2-835\eta^3+23\eta^4\big)
\end{aligned}
\tag{5.92}
$$

$$R_{38} = y\Big[5120\eta^3 + y\big(129 + 1332\eta - 4129\eta^2 - 18568\eta^3 - 4129\eta^4 + 1332\eta^5 + 129\eta^6\big)$$
$$-6y^4(-1+\eta)^2(1+\eta)^2\big(109 + 446\eta + 109\eta^2\big)$$
$$+3y^3(-1+\eta)^2\big(479 + 3536\eta + 5250\eta^2 + 3536\eta^3 + 479\eta^4\big)$$
$$-2y^2\big(456 + 3195\eta - 3680\eta^2 - 7910\eta^3 - 3680\eta^4 + 3195\eta^5 + 456\eta^6\big)\Big] \tag{5.93}$$

$$R_{39} = y\Big[2560\eta^3 + y\big(129 + 1332\eta - 6689\eta^2 - 13448\eta^3 - 6689\eta^4 + 1332\eta^5 + 129\eta^6\big)$$
$$-6y^4(-1+\eta)^2(1+\eta)^2\big(109 + 446\eta + 109\eta^2\big)$$
$$+3y^3(-1+\eta)^2\big(479 + 3536\eta + 5250\eta^2 + 3536\eta^3 + 479\eta^4\big)$$
$$-2y^2\big(456 + 3195\eta - 4960\eta^2 - 5350\eta^3 - 4960\eta^4 + 3195\eta^5 + 456\eta^6\big)\Big] \tag{5.94}$$

$$R_{40} = -160(1+\eta)^2\big(17 + 883\eta + 17\eta^2\big) \tag{5.95}$$

$$R_{41} = -128\big(17 + 900\eta + 917\eta^2 + 900\eta^3 + 17\eta^4\big) \tag{5.96}$$

$$R_{42} = -5\big(1557 - 21798\eta + 3602\eta^2 - 21798\eta^3 + 1557\eta^4\big)$$
$$+2y\big(9009 - 53793\eta + 49720\eta^2 - 53793\eta^3 + 9009\eta^4\big) \tag{5.97}$$

$$R_{43} = (1+\eta)\Big[10\eta\big(53 - 690\eta + 53\eta^2\big)$$
$$+y\big(865 - 15172\eta + 30678\eta^2 - 15172\eta^3 + 865\eta^4\big)$$
$$+2y^3(-1+\eta)^2\big(1001 - 3034\eta + 1001\eta^2\big)$$
$$-y^2(-1+\eta)^2\big(2867 - 17786\eta + 2867\eta^2\big)\Big] \tag{5.98}$$

$$R_{44} = 2\eta\big(-265 + 3450\eta + 2022\eta^2 + 3450\eta^3 - 265\eta^4\big)$$
$$+8y^3(-1+\eta)^2\big(34 + 1969\eta + 4900\eta^2 + 1969\eta^3 + 34\eta^4\big)$$
$$+y^2(-1+\eta)^2\big(593 - 27584\eta - 34050\eta^2 - 27584\eta^3 + 593\eta^4\big)$$
$$-y\big(865 - 12898\eta + 35567\eta^2 + 24180\eta^3 + 35567\eta^4 - 12898\eta^5 + 865\eta^6\big) \tag{5.99}$$

$$R_{45} = 2\eta\big(-795 + 10350\eta + 10666\eta^2 + 10350\eta^3 - 795\eta^4\big)$$
$$+8y^3(-1+\eta)^2\big(102 + 5907\eta + 13550\eta^2 + 5907\eta^3 + 102\eta^4\big)$$
$$+3y^2(-1+\eta)^2\big(593 - 27584\eta - 34050\eta^2 - 27584\eta^3 + 593\eta^4\big)$$
$$-y\big(2595 - 38694\eta + 97501\eta^2 + 81740\eta^3 + 97501\eta^4 - 38694\eta^5$$
$$+2595\eta^6\big) \tag{5.100}$$

$$R_{46} = -20(1+\eta)^2\big(1127 + 6478\eta + 1127\eta^2\big) \tag{5.101}$$

$$R_{47} = -16\big(1127 + 7605\eta + 8732\eta^2 + 7605\eta^3 + 1127\eta^4\big) \tag{5.102}$$

$$R_{48} = 1305 + 16902\eta + 1666\eta^2 + 16902\eta^3 + 1305\eta^4$$
$$+3y(-1+\eta)^2\big(257 - 4018\eta + 257\eta^2\big) \tag{5.103}$$

$$R_{49} = (1+\eta)\Big[-3\eta\big(43 + 1046\eta + 43\eta^2\big) + 2y^2(-1+\eta)^2\big(89 + 4957\eta + 89\eta^2\big)$$
$$+y^3(-1+\eta)^2\big(257 - 4018\eta + 257\eta^2\big)$$
$$-y\big(435 + 5633\eta - 15640\eta^2 + 5633\eta^3 + 435\eta^4\big)\Big] \tag{5.104}$$

$$R_{50} = y\big(-435 - 4249\eta + 17803\eta^2 + 8690\eta^3 + 17803\eta^4 - 4249\eta^5 - 435\eta^6\big)$$
$$-y^3(-1+\eta)^2(1+\eta)^2\big(1127 + 6478\eta + 1127\eta^2\big)$$
$$-\eta\big(129 + 3138\eta + 3074\eta^2 + 3138\eta^3 + 129\eta^4\big)$$
$$+2y^2(-1+\eta)^2\big(781 + 7358\eta + 5990\eta^2 + 7358\eta^3 + 781\eta^4\big) \quad (5.105)$$
$$R_{51} = y\big(-435 - 4249\eta + 14523\eta^2 + 15250\eta^3 + 14523\eta^4 - 4249\eta^5 - 435\eta^6\big)$$
$$-y^3(-1+\eta)^2(1+\eta)^2\big(1127 + 6478\eta + 1127\eta^2\big)$$
$$-3\eta\big(43 + 1046\eta + 2118\eta^2 + 1046\eta^3 + 43\eta^4\big)$$
$$+2y^2(-1+\eta)^2\big(781 + 7358\eta + 7630\eta^2 + 7358\eta^3 + 781\eta^4\big) \quad (5.106)$$
$$R_{52} = \eta + \eta^3 + y\eta\big(1+\eta^2\big) + y^2(-1+\eta)^2\big(1+\eta+\eta^2\big) \quad (5.107)$$
$$R_{53} = y(1+\eta)\Big[-y(-1+\eta)^2(1+\eta)^2 + y^2(-1+\eta)^2\big(1+\eta+\eta^2\big)$$
$$-\eta\big(1+\eta+\eta^2\big)\Big] \quad (5.108)$$
$$R_{54} = 2\eta^3 - y\eta(1+\eta)^2\big(1-\eta+\eta^2\big) + y^3(-1+\eta)^2(1+\eta)^2\big(1-\eta+\eta^2\big)$$
$$-y^2(-1+\eta)^2\big(1+2\eta+2\eta^3+\eta^4\big). \quad (5.109)$$

6. Numerical results

In Fig. 2 we illustrate the size of the two–mass contribution in relation to the total contribution of $O(T_F^2)$. The polarized single mass corrections to $A_{gg,Q}^{(3)}$ were calculated in [70]. Also here the N space representation has evanescent poles at $N = 1/2, 3/2$, which can be shown to vanish by performing an analytic expansion.

The corrections are of the size of 20 to 60%, except of the region $z \sim 0.02 - 0.03$, where the $O(T_F^2)$ terms vanish. In wide ranges the correction behaves as constant for fixed virtualities μ^2. The relative size of the correction is of similar size as in the unpolarized case [25]. The two mass corrections are as important as the $O(T_F^2)$ terms and have to be considered in precision analyses at three–loop order.

7. Conclusions

We have calculated the polarized massive OME $A_{gg,Q}^{(2)}$ in analytic form both in Mellin N space and z space in the Larin scheme. In the latter case we made use of a single integral representation, in order to shorten the notation, which would lead to G-functions of higher depth for which no numerical representation is available yet. The mathematical quantities allowing the representation in N space range up to generalized finite binomial and inverse binomial sums depending on the real parameter η. In z space one obtains iterated integrals, based on square–root valued letters, containing the real parameter η. The representations derived allow a fast numerical calculation given the analytic representations of the iterative integrals. The evanescent poles at $N = 1/2$ and $3/2$ present in Mellin N space cancel and the rightmost singularity is obtained at $N = 0$ as expected.

In the present calculation, the method of direct integration turned out to be the most efficient. Therefore, we did not perform an integration-by-parts reduction in this two–scale problem, but applied a minimal Mellin–Barnes representation, followed by the solution of the respective multi–sums using the algorithms encoded in the packages `Sigma, EvaluateMultiSums`

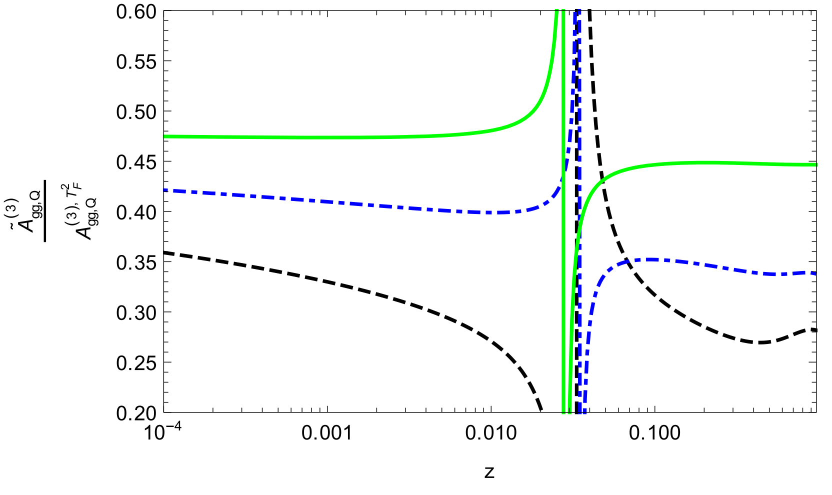

Fig. 2. The ratio of the two mass contribution to the total contribution at $O(T_F^2)$ for the polarized massive OME $A_{gg,Q}^{(3)}$ as a function of the momentum fraction z and the virtuality μ^2. Dashed line: $\mu^2 = 50$ GeV^2; Dash–doted line: $\mu^2 = 100$ GeV^2; Full line: $\mu^2 = 1000$ GeV^2. For the values of m_c and m_b we refer to the on–shell heavy quark masses $m_c = 1.59$ GeV and $m_b = 4.78$ GeV [71,72].

and `SumProduction`. The obtained expressions have been simplified using the methods encoded in the package `HarmonicSums`. Checks have been performed using the package `Q2E/Exp`. In course of the present calculation we have obtained also a series of new iterative integrals, which could be represented in terms of simpler functions. They can be of use in other two–mass calculations.

Comparing to the complete $O(T_F^2)$ corrections to $A_{gg,Q}^{(3)}$, we showed that the two–mass contributions form an important part and it is necessary to consider them in quantitative analyses. In particular, they contribute to the variable flavor number scheme, as well as in other places, at three–loop order. The OME calculated in the present paper can be used both as a building block for structure functions as well as for the transition matrix in the variable flavor number scheme, provided that the Larin scheme is used. There is the possibility also to define the parton distribution functions and the massless Wilson coefficients in this scheme.

With this result only the two–mass contributions for both the unpolarized and polarized massive OME $A_{Qg}^{(3)}$ at three–loop order remain to be calculated, which is work in progress.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Representation of certain iterated integrals

In the following we present a series of integrals and relations which appeared in intermediate steps of the calculation and which may be of further use in similar applications, extending the results given in [25] before. We obtained the following iterative integrals

$$G_{44} = G\left(\left\{\frac{\sqrt{1-x}}{x}\right\}, z\right) = -\mathrm{H}_{-1}\big(u_1\big) - \mathrm{H}_1\big(u_1\big) + 2\ln(2) - 2$$
$$+u_1\Big[\mathrm{H}_0(z) + \mathrm{H}_1\big(u_1\big) - \mathrm{H}_{-1}\big(u_1\big) + 2\Big], \tag{A.1}$$

$$G_{45} = G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta - x\eta + x)}\right\}, z\right)$$
$$= \frac{\pi}{\sqrt{\eta}+\eta} + 2\frac{\arcsin(\sqrt{1-z})}{1-\eta} - 2\frac{\arctan\left(\frac{\sqrt{\eta}\sqrt{1-z}}{\sqrt{z}}\right)}{\sqrt{\eta}(1-\eta)} \tag{A.2}$$

$$G_{46} = G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta - x + 1)}\right\}, z\right)$$
$$= \frac{1}{\eta - 1}\left[\pi\left(\sqrt{\eta}-1\right) - 2\sqrt{\eta}\arctan\left(\frac{u_1}{\sqrt{\eta z}}\right) + 2\arctan\left(\frac{u_1}{\sqrt{z}}\right)\right], \tag{A.3}$$

$$G_{47} = G\left(\left\{\frac{\sqrt{x}}{x\eta - x + 1}\right\}, z\right)$$
$$= \frac{1}{v_1^3}\Big[2\mathrm{H}_{-1}\big(v_1\sqrt{z}\big) + \mathrm{H}_1(z - z\eta)\Big] - \frac{2\sqrt{z}}{v_1^2}, \tag{A.4}$$

$$G_{48} = G\left(\left\{\sqrt{1-x}\sqrt{x\eta - x + 1}\right\}, z\right) = \frac{2-\eta}{4(1-\eta)} - \frac{u_1^3 u_2}{4}$$
$$-\frac{u_1 u_2^3}{4(1-\eta)} + \frac{\eta^2}{4v_1^3}\left[\operatorname{arcsinh}\left(\frac{u_1 v_1}{\sqrt{\eta}}\right) - \operatorname{arcsinh}\left(\frac{v_1}{\sqrt{\eta}}\right)\right], \tag{A.5}$$

$$G_{49} = G\left(\left\{\frac{\sqrt{x\eta - x + 1}}{x}\right\}, z\right) = -\mathrm{H}_1\big(u_2\big) - \mathrm{H}_0(z - z\eta) - \mathrm{H}_{-1}\big(u_2\big)$$
$$+\mathrm{H}_0(z) + 2\ln(2) - 2 + u_2\Big[\mathrm{H}_0(z - z\eta) + 2 + \mathrm{H}_1\big(u_2\big) - \mathrm{H}_{-1}\big(u_2\big)\Big], \tag{A.6}$$

$$G_{50} = G\left(\left\{\frac{\sqrt{1-x}}{x}, \frac{1}{x}\right\}, z\right) = u_1\left[\frac{1}{2}H_0^2(z) - 4\right] + 2\ln^2(2)$$
$$-4\ln(2) + 4 + \big(1 + u_1\big)\left[2\mathrm{H}_{-1}\big(u_1\big) - \frac{1}{2}\mathrm{H}_{-1}^2\big(u_1\big) + \mathrm{H}_{-1,1}\big(u_1\big)\right]$$
$$+\big(1 - u_1\big)\left[2\mathrm{H}_1\big(u_1\big) - \mathrm{H}_{1,-1}\big(u_1\big) + \frac{1}{2}\mathrm{H}_1^2\big(u_1\big)\right] - \zeta_2, \tag{A.7}$$

$$G_{51} = G\left(\left\{\frac{\sqrt{x}}{\eta - x\eta + x}, \frac{1}{x}\right\}, z\right) = \frac{2\sqrt{z}}{v_1^2}\big(\mathrm{H}_0(z) - 2\big)$$
$$+\frac{\sqrt{\eta}}{v_1^3}\left[-2\mathrm{H}_0(z)\mathrm{H}_{\{4,0\}}\left(v_1\sqrt{\frac{z}{\eta}}\right) + 4\mathrm{H}_{\{0,0\},\{4,0\}}\left(v_1\sqrt{\frac{z}{\eta}}\right)\right], \tag{A.8}$$

$$G_{52} = G\left(\left\{\frac{\sqrt{x\eta - x + 1}}{x}, \frac{1}{x}\right\}, z\right) = u_2\big(2\mathrm{H}_0(z) - 4\big) + \frac{1}{2}H_0^2(z)$$
$$+\big(1 + u_2\big)\Big[-\mathrm{H}_{-1}\big(u_2\big)\mathrm{H}_0(z) + \mathrm{H}_{-1,-1}\big(u_2\big) - \mathrm{H}_{1,-1}\big(u_2\big)\Big]$$
$$-2\mathrm{H}_0(z) + \big(1 - u_2\big)\Big[-\mathrm{H}_0(z)\mathrm{H}_1\big(u_2\big) - \mathrm{H}_0(z)\mathrm{H}_0(z - z\eta)$$
$$+\frac{1}{2}\mathrm{H}_0^2(z - z\eta) + \mathrm{H}_{-1,1}\big(u_2\big) - \frac{1}{2}\mathrm{H}_1^2\big(u_2\big)\Big] + 4\mathrm{H}_{-1}\big(u_2\big) - \zeta_2$$
$$+2\ln^2(2) + 2\ln(2)\Big[\mathrm{H}_1\big(u_2\big) - \mathrm{H}_{-1}\big(u_2\big) + \mathrm{H}_0(z)\Big] - 4\ln(2) + 4, \tag{A.9}$$

$$G_{53} = G\left(\left\{\frac{\sqrt{1-x}}{x}, \frac{\sqrt{1-x}}{x}, \frac{1}{x}\right\}, z\right) = z\mathrm{H}_0(z) - \frac{z}{2}H_0^2(z) + \frac{1}{6}H_0^3(z)$$
$$+2u_1\Big[-\frac{1}{2}\mathrm{H}_{-1}^2\big(u_1\big) + \mathrm{H}_{-1,1}\big(u_1\big) - \mathrm{H}_{1,-1}\big(u_1\big) + \frac{1}{2}\mathrm{H}_1^2\big(u_1\big)\Big]$$
$$-(z+2)\mathrm{H}_1\big(u_1\big)\mathrm{H}_{-1}\big(u_1\big) + \frac{z-2}{2}\Big[\mathrm{H}_{-1}^2\big(u_1\big) + \mathrm{H}_1^2\big(u_1\big)\Big]$$
$$+2\mathrm{H}_{-1,1,-1}\big(u_1\big) - 2\mathrm{H}_{-1,1,1}\big(u_1\big) + 2\mathrm{H}_{1,-1,-1}\big(u_1\big) - 2\mathrm{H}_{1,-1,1}\big(u_1\big)$$
$$+\big(\zeta_2 + 4\ln(2) - 2\ln^2(2) + 4u_1\big)\Big[\mathrm{H}_{-1}\big(u_1\big) + \mathrm{H}_1\big(u_1\big)\Big] - \frac{3}{2}\zeta_3$$
$$-(3z+4)\mathrm{H}_{-1}\big(u_1\big) + (3z-4)\mathrm{H}_1\big(u_1\big) + \frac{4}{3}\ln^3(2)$$
$$+\big(8 - 8\ln(2) + 4\ln^2(2) - 2\zeta_2\big)u_1 - 4\ln^2(2) + 8\ln(2) + 6z - 8, \tag{A.10}$$

$$G_{54} = G\left(\left\{\frac{1}{x\eta - x + 1}, \frac{1}{x}, \frac{1}{x}\right\}, z\right) = \frac{1}{1-\eta}\Big[\frac{1}{2}H_0^2(z)\mathrm{H}_1(z - z\eta)$$
$$-\mathrm{H}_0(z)\mathrm{H}_{0,1}(z - z\eta) + \mathrm{H}_{0,0,1}(z - z\eta)\Big], \tag{A.11}$$

$$G_{55} = G\left(\left\{\frac{\sqrt{x\eta - x + 1}}{x}, \frac{1}{x}, \frac{1}{x}\right\}, z\right) = -8 + \frac{4\ln^3(2)}{3}$$
$$+\ln(2)\Big[8 - 4\mathrm{H}_1(\eta) - 2\zeta_2 + \mathrm{H}_1(\eta)^2\Big]$$
$$+u_2\Big\{8 + \frac{1}{2}\Big[-\mathrm{H}_1(u_2) + \mathrm{H}_{-1}(u_2)\Big]^2\Big[2 - \mathrm{H}_1(\eta)\Big]$$
$$-\frac{1}{2}\Big[-\mathrm{H}_1(u_2) + \mathrm{H}_{-1}(u_2)\Big]\Big[-4 + \mathrm{H}_1(\eta)\Big]\mathrm{H}_1(\eta) + \frac{1}{6}\mathrm{H}_0(z)^3 - 4\mathrm{H}_1(\eta)$$

$$-\frac{1}{6}\mathrm{H}_1(\eta)^3-\frac{1}{2}\mathrm{H}_{-1}(u_2)\mathrm{H}_1(u_2)^2+\frac{1}{6}\mathrm{H}_1(u_2)^3-4\mathrm{H}_{-1}(u_2)-\frac{1}{6}\mathrm{H}_{-1}(u_2)^3$$
$$+\mathrm{H}_1(\eta)^2+\frac{1}{2}\mathrm{H}_1(u_2)\Big[8+\mathrm{H}_{-1}(u_2)^2\Big]\Big\}$$
$$+\Big\{4+\frac{1}{2}\mathrm{H}_1(u_2)^2+2\mathrm{H}_{-1}(u_2)-\frac{1}{2}\mathrm{H}_{-1}(u_2)^2-\zeta_2$$
$$+\mathrm{H}_1(u_2)\Big[2-\mathrm{H}_{-1}(u_2)\Big]\Big\}\mathrm{H}_1(\eta)-\mathrm{H}_1(\eta)^2-\frac{1}{2}\Big[\mathrm{H}_1(u_2)+\mathrm{H}_{-1}(u_2)\Big]\mathrm{H}_1(\eta)^2$$
$$+\frac{1}{6}\mathrm{H}_1(\eta)^3-\frac{1}{6}\mathrm{H}_1(u_2)^3-4\mathrm{H}_{-1}(u_2)-\frac{1}{6}\mathrm{H}_{-1}(u_2)^3$$
$$+\Big[-4+2\mathrm{H}_1(\eta)+2\mathrm{H}_{-1}(u_2)\Big]\mathrm{H}_{-1,1}(u_2)-2\mathrm{H}_{-1,1,1}(u_2)-2\mathrm{H}_{-1,-1,1}(u_2)$$
$$+2\zeta_2+2\zeta_3+2\ln^2(2)\Big[-2+\mathrm{H}_1(\eta)\Big]+\frac{1}{2}\mathrm{H}_1(u_2)^2\Big[-2+\mathrm{H}_{-1}(u_2)\Big]+\mathrm{H}_{-1}(u_2)^2$$
$$+\frac{1}{2}\mathrm{H}_1(u_2)\Big[-8+4\mathrm{H}_{-1}(u_2)-\mathrm{H}_{-1}(u_2)^2\Big], \tag{A.12}$$

$$G_{56}=G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x},\frac{1}{x\eta-x+1},\frac{1}{x}\right\},z\right)$$
$$=\frac{u_2}{1-\eta}\Big\{4\mathrm{H}_{-1,0}(u_2)-4\mathrm{H}_{1,0}(u_2)-4\mathrm{H}_{-1,-1,0}(u_2)+4\mathrm{H}_{-1,1,0}(u_2)$$
$$+\Big[4-4\mathrm{H}_0(u_2)+2\mathrm{H}_{-1,0}(u_2)+\mathrm{H}_{0,1}(z-z\eta)-2\mathrm{H}_{1,0}(u_2)-\zeta_2\Big]\mathrm{H}_0(z)$$
$$-4\mathrm{H}_{1,1,0}(u_2)-2\mathrm{H}_{0,0,1}(z-z\eta)+4\mathrm{H}_{1,-1,0}(u_2)+2\zeta_2\mathrm{H}_{-1}(u_2)$$
$$-2\zeta_2\mathrm{H}_1(u_2)-2\zeta_2+2\zeta_3-8\Big\}+\frac{1}{1-\eta}\Big\{-4\mathrm{H}_{-1,-1,0}(u_2)$$
$$+\Big[2\mathrm{H}_{-1,0}(u_2)+2\mathrm{H}_{1,0}(u_2)+3\zeta_2-4\Big]\mathrm{H}_0(z)+8-\frac{7}{2}\zeta_3$$
$$+4\mathrm{H}_{1,1,0}(u_2)-2\big(\zeta_2-4\big)\mathrm{H}_{-1}(u_2)+4\mathrm{H}_1(u_2)\zeta_2-8\ln(2)\Big\}, \tag{A.13}$$

where

$$u_1=\sqrt{1-z}, \tag{A.14}$$
$$u_2=\sqrt{1-z(1-\eta)}, \tag{A.15}$$
$$v_1=\sqrt{1-\eta}. \tag{A.16}$$
$$K_{49}=G\left(\left\{\frac{\sqrt{1-x}}{x}\right\},1\right)=2\ln(2)-2, \tag{A.17}$$
$$K_{50}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)}\right\},1\right)=\frac{\pi}{\eta+\sqrt{\eta}}, \tag{A.18}$$
$$K_{51}=G\left(\left\{\frac{\sqrt{x}}{-\eta+x\eta+1}\right\},1\right)=\frac{2v_1}{\eta^{3/2}}\,\mathrm{H}_{\{4,0\}}\left(\frac{1-2v_1\sqrt{\eta}}{1-2\eta}\right)$$

$$+\frac{1}{\left(1-2v_1\sqrt{\eta}\right)^2}\Biggl[\frac{2\pi+2}{\eta}-2\pi-\frac{\pi}{v_1}\sqrt{\eta}\left(2\eta-4+\frac{3}{2\eta}+\frac{1}{2\eta^2}\right)$$
$$+8-8\eta+v_1\sqrt{\eta}\bigl(16\eta-24\bigr)+\frac{8}{v_1}\sqrt{\eta}\left(2\eta^2-5\eta+4-\frac{1}{\eta}\right)\Biggr], \tag{A.19}$$

$$K_{52}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)}\right\},1\right)=\frac{\pi}{1+\sqrt{\eta}}, \tag{A.20}$$

$$K_{53}=G\left(\left\{\frac{\sqrt{x}}{x\eta-x+1}\right\},1\right)$$
$$=\frac{1}{1-\eta}\Bigl[\mathrm{H}_{-1}(v_1)-\mathrm{H}_1(v_1)-\mathrm{H}_0(\eta)-2\Bigr]+\frac{1}{v_1^3}\Bigl[\mathrm{H}_{-1}(v_1)+\mathrm{H}_1(v_1)\Bigr], \tag{A.21}$$

$$K_{54}=G\left(\left\{\sqrt{1-x}\sqrt{x\eta-x+1}\right\},1\right)=-\frac{2-\eta}{4(\eta-1)}$$
$$+\frac{\eta^2}{8(1-\eta)}\Bigl[\mathrm{H}_1(v_1)-\mathrm{H}_{-1}(v_1)+\mathrm{H}_0(\eta)\Bigr]-\frac{\eta^2}{8v_1^3}\Bigl[\mathrm{H}_{-1}(v_1)+\mathrm{H}_1(v_1)\Bigr], \tag{A.22}$$

$$K_{55}=G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x}\right\},1\right)=2\sqrt{\eta}-2\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)+2\ln(2)-2, \tag{A.23}$$

$$K_{56}=G\left(\left\{\frac{\sqrt{1-x}}{x},\frac{1}{x}\right\},1\right)=2\ln^2(2)-4\ln(2)-\frac{\pi^2}{6}+4, \tag{A.24}$$

$$K_{57}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)},\frac{1}{1-x}\right\},1\right)=\frac{2\pi}{1-\eta}\left[\frac{\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)}{\sqrt{\eta}}-\ln(2)\right], \tag{A.25}$$

$$K_{58}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)},\frac{1}{x}\right\},1\right)$$
$$=\frac{\pi}{(1-\eta)\sqrt{\eta}}\Bigl[-2\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_0(\eta)\Bigr]+\frac{2\ln(2)\pi}{1-\eta}, \tag{A.26}$$

$$K_{59}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)},\sqrt{1-x}\sqrt{x}\right\},1\right)$$
$$=-\frac{1}{8(1-\eta)\sqrt{\eta}}\Bigl[\mathrm{H}_{1,0}(\eta)+4\mathrm{H}_{-1,0}\bigl(\sqrt{\eta}\bigr)\Bigr]+\frac{1}{2(\eta-1)^2}$$
$$+\frac{\pi^2}{16(\eta-1)}-\frac{\mathrm{H}_0\bigl(\sqrt{\eta}\bigr)}{2(\eta-1)}-\frac{(3-\eta)\eta}{4(\eta-1)^3}\mathrm{H}_0(\eta), \tag{A.27}$$

$$K_{60}=G\left(\left\{\frac{\sqrt{x}}{\eta-x\eta+x},\frac{1}{x}\right\},1\right)=-\frac{16C\eta+4}{v_1^2v_2^2}+\frac{16\sqrt{\eta}}{v_1v_2^2}-\frac{16\eta}{v_2^2}$$
$$+\frac{\sqrt{\eta}}{v_1^3v_2^2}\left(2\eta^2-2\eta-\frac{1}{2}\right)\left\{-8C+\pi\Bigl[\ln(2)+\mathrm{H}_1(\eta)+2\mathrm{H}_{\{4,1\}}(v_3)\Bigr]\right\}$$
$$+\frac{16\eta^{3/2}}{v_1^3v_2^2}\bigl(2\eta^2-3\eta+1\bigr)\left[-\frac{\ln(2)}{2}+\mathrm{H}_{-1}(v_3)+\frac{\mathrm{H}_1(\eta)}{2}-\mathrm{H}_{\{4,1\}}(v_3)\right]$$

$$+\frac{4\sqrt{\eta}}{v_1^3}\Bigg[\frac{\ln(2)}{2}\mathrm{H}_{\{4,0\}}(v_3)-\frac{\pi}{4}\mathrm{H}_0(\eta)-\frac{\pi}{4}\mathrm{H}_1(v_3)-\frac{1}{2}\mathrm{H}_1(\eta)\mathrm{H}_{\{4,0\}}(v_3)$$
$$+\mathrm{H}_{1,\{4,0\}}(v_3)-\mathrm{H}_{\{4,0\},-1}(v_3)+\mathrm{H}_{\{4,0\},\{4,1\}}(v_3)+\mathrm{H}_{\{4,1\},\{4,0\}}(v_3)\Bigg]$$
$$+\frac{\eta}{v_1^2v_2^2}\big(16\eta^3-24\eta^2+4\eta+2\big)\Big[\ln(2)-2\mathrm{H}_{-1}(v_3)-\mathrm{H}_1(\eta)$$
$$+2\mathrm{H}_{\{4,1\}}(v_3)\Big]+\frac{2\pi\eta}{v_1^2v_2^2}\Big[\ln(2)+\mathrm{H}_1(\eta)+2\mathrm{H}_{\{4,1\}}(v_3)\Big], \tag{A.28}$$

$$K_{61}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)},\frac{1}{1-x}\right\},1\right)$$
$$=\frac{\pi}{1-\eta}\Big[2\ln(2)-2\sqrt{\eta}\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+\sqrt{\eta}\mathrm{H}_0(\eta)\Big], \tag{A.29}$$

$$K_{62}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)},\frac{1}{x}\right\},1\right)=\frac{2\pi}{1-\eta}\Big[\sqrt{\eta}\mathrm{H}_{-1}\big(\sqrt{\eta}\big)-\ln(2)\Big], \tag{A.30}$$

$$K_{63}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)},\sqrt{1-x}\sqrt{x}\right\},1\right)=\frac{(3-\eta)\eta^2}{4(1-\eta)^3}\mathrm{H}_0(\eta)$$
$$-\frac{\sqrt{\eta}}{2(1-\eta)}\left[\frac{1}{4}\mathrm{H}_{1,0}(\eta)+\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)\right]+\frac{\eta}{2(1-\eta)^2}$$
$$+\frac{\pi^2}{8(1-\eta)^2}\left(\eta^{3/2}-\frac{\eta}{2}-\sqrt{\eta}+\frac{1}{2}\right)+\frac{\eta}{2(1-\eta)}\mathrm{H}_0\big(\sqrt{\eta}\big), \tag{A.31}$$

$$K_{64}=G\left(\left\{\sqrt{1-x}\sqrt{x\eta-x+1},\frac{\sqrt{1-x}}{x}\right\},1\right)$$
$$=\frac{\eta^2\ln(2)}{4(1-\eta)}\big[\mathrm{H}_1(v_1)-\mathrm{H}_{-1}(v_1)+\mathrm{H}_0(\eta)\big]-\frac{\eta^2}{4v_1^3}\ln(2)\big[\mathrm{H}_{-1}(v_1)+\mathrm{H}_1(v_1)\big]$$
$$+\big(4\eta^2-4\eta-1\big)\Bigg\{\frac{\big(3\pi-2\pi^2\big)\eta^2}{16v_1^3v_2^2}-\frac{1}{v_2^2(\eta-1)}\Bigg[\frac{1}{2}\ln(2)(\eta-2)+\frac{\pi^2\eta^2}{48}$$
$$+\frac{1}{1+\sqrt{\eta}}\left(\frac{\eta^{5/2}}{4}-\frac{3}{10}\eta^{3/2}+\frac{\eta^3}{4}+\frac{7}{10}\eta^2-\frac{22}{15}\eta+\frac{8}{15}\sqrt{\eta}+\frac{23}{15}\right)\Bigg]\Bigg\}$$
$$+\frac{\eta^2}{8v_1^3}\Bigg[2\mathrm{H}_{-1}(v_1)+2\mathrm{H}_1(v_1)+6\mathrm{H}_{\{4,0\}}(v_3)+2\mathrm{H}_{-1,0}(v_1)+2\mathrm{H}_{1,0}(v_1)$$
$$-G\left(\left\{\frac{\sqrt{1-x}}{x},\frac{\sqrt{x}}{1-x}\right\},\eta\right)\Bigg]+\frac{\big(3\pi-2\pi^2\big)\eta^{5/2}}{4v_2^2(1-\eta)}+\frac{4\ln(2)(\eta-1)\eta}{\big(1+\sqrt{\eta}\big)v_1^3v_2^2}$$
$$+\frac{\sqrt{\eta}-1}{v_1^3v_2^2}\left(-6\eta^{9/2}-6\eta^4+3\eta^{7/2}+3\eta^3-\frac{32}{15}\eta\right)+\frac{\pi^2}{12v_1v_2^2}\,\eta^{5/2}$$
$$+\frac{1}{v_1v_2^2\big(1+\sqrt{\eta}\big)}\Bigg[\frac{14}{5}\eta^{5/2}+2\ln(2)\sqrt{\eta}\big(\sqrt{\eta}-1\big)\big(\eta+2\sqrt{\eta}+2\big)$$

$$-\frac{2}{15}\sqrt{\eta}\big(45\eta^4+45\eta^{7/2}-30\eta^3-30\eta^{5/2}+9\eta^{3/2}+44\eta-46\big)\Big]$$
$$+\frac{\eta^2}{4(1-\eta)}\Big[\mathrm{H}_{-1}(v_1)+2\mathrm{H}_{-1}\big(\sqrt{\eta}\big)-2\mathrm{H}_0(v_1)-\mathrm{H}_0(\eta)-\mathrm{H}_1(v_1)$$
$$+\mathrm{H}_{-1,0}(v_1)+\frac{1}{2}\mathrm{H}_{0,1}(\eta)-\mathrm{H}_{1,0}(v_1)\Big]+\frac{\eta-2}{2(1-\eta)}\mathrm{H}_{-1}\big(\sqrt{\eta}\big), \tag{A.32}$$
$$K_{65}=G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x},\frac{1}{x}\right\},1\right)=2\mathrm{H}_{-1,1}\big(\sqrt{\eta}\big)-2\mathrm{H}_{-1,-1}\big(\sqrt{\eta}\big)-4\sqrt{\eta}$$
$$+2\big(1+\sqrt{\eta}\big)\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+2\big(1-\sqrt{\eta}\big)\mathrm{H}_1\big(\sqrt{\eta}\big)-2\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta)$$
$$+2\big(-1+\ln(2)+\sqrt{\eta}\big)\mathrm{H}_1(\eta)+2\ln^2(2)-4\ln(2)+4-\frac{\pi^2}{6}, \tag{A.33}$$
$$K_{66}=G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x},\frac{\sqrt{1-x}}{x}\right\},1\right)=-\mathrm{H}_{1,0}(\eta)+\eta\mathrm{H}_0(\eta)$$
$$+4(1-\ln(2))\big[-\sqrt{\eta}+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\big]-\eta\left(1+\frac{1}{v_1}\right)\mathrm{H}_{-1}(v_1)$$
$$+\eta\left(1-\frac{1}{v_1}\right)\mathrm{H}_1(v_1)+2\ln^2(2)-4\ln(2)+2, \tag{A.34}$$
$$K_{67}=G\left(\left\{\frac{\sqrt{1-x}}{x},\frac{1}{x},\frac{1}{x}\right\},1\right)$$
$$=2\zeta_3+\frac{4}{3}\ln^3(2)-4\ln^2(2)+8\ln(2)-\frac{\ln(2)\pi^2}{3}-8+\frac{\pi^2}{3}, \tag{A.35}$$
$$K_{68}=G\left(\left\{\frac{\sqrt{1-x}}{x},\frac{\sqrt{1-x}}{x},\frac{1}{x}\right\},1\right)$$
$$=-\frac{3}{2}\zeta_3+\frac{4}{3}\ln^3(2)-4\ln^2(2)+8\ln(2)-2, \tag{A.36}$$
$$K_{69}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)},\frac{1}{1-x},\sqrt{1-x}\sqrt{x}\right\},1\right)$$
$$=\frac{1}{2(1-\eta)}\Big[-\frac{\pi^2}{16}+\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+\frac{1}{2}\mathrm{H}_0\big(\sqrt{\eta}\big)+\mathrm{H}_1(\eta)-\mathrm{H}_0\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta)$$
$$-\mathrm{H}_0\big(\sqrt{\eta}\big)\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)-\frac{7}{4}\zeta_3\Big]+\frac{\mathrm{H}_1\big(\sqrt{\eta}\big)-\mathrm{H}_{1,0}\big(\sqrt{\eta}\big)}{2\big(\eta+\sqrt{\eta}\big)}$$
$$+\frac{\pi^2}{8(1-\eta)\sqrt{\eta}}\left(\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+\frac{\mathrm{H}_1(\eta)}{2}\right)+\frac{\eta(1+\eta)}{8(\eta-1)^3}\mathrm{H}_0(\eta)$$
$$+\frac{1}{2(1-\eta)\sqrt{\eta}}\Big[-\mathrm{H}_{-1}\big(\sqrt{\eta}\big)-\mathrm{H}_1(\eta)+\frac{3}{8}\mathrm{H}_{1,0}(\eta)+\frac{1}{2}\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)$$
$$+\mathrm{H}_1(\eta)\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)+\frac{1}{4}\mathrm{H}_{1,1,0}(\eta)-\mathrm{H}_{-1,0,1}\big(\sqrt{\eta}\big)-\mathrm{H}_{-1,1,0}\big(\sqrt{\eta}\big)$$
$$+\mathrm{H}_{-1,-1,0}\big(\sqrt{\eta}\big)+\mathrm{H}_{-1,0,-1}\big(\sqrt{\eta}\big)\Big]-\frac{\eta}{4(1-\eta)^2}, \tag{A.37}$$

$$
\begin{aligned}
K_{70} &= G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)}, \frac{1}{x}, \sqrt{1-x}\sqrt{x}\right\}, 1\right) = \frac{2-\eta}{4(\eta-1)^2} \\
&+\frac{1}{2(1-\eta)}\left[\frac{\pi^2}{16} - \mathrm{H}_{-1}\big(\sqrt{\eta}\big) + \frac{3}{4}\mathrm{H}_0\big(\eta\big) - \mathrm{H}_1(\eta) + \mathrm{H}_0\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta)\right. \\
&\left.+\mathrm{H}_0\big(\sqrt{\eta}\big)\mathrm{H}_{-1}\big(\sqrt{\eta}\big) - \mathrm{H}_{0,1}\big(\sqrt{\eta}\big) - \frac{7}{4}\zeta_3\right] + \frac{(5-3\eta)\eta}{8(1-\eta)^3}\mathrm{H}_0(\eta) \\
&+\frac{\mathrm{H}_{1,0}\big(\sqrt{\eta}\big) - \mathrm{H}_1\big(\sqrt{\eta}\big)}{2\big(\eta+\sqrt{\eta}\big)} - \frac{\pi^2}{8(1-\eta)\sqrt{\eta}}\left[\mathrm{H}_{-1}\big(\sqrt{\eta}\big) + \frac{1}{2}\mathrm{H}_1(\eta)\right] \\
&+\frac{1}{2(1-\eta)\sqrt{\eta}}\left[\mathrm{H}_{-1}\big(\sqrt{\eta}\big) + \mathrm{H}_1(\eta) - \frac{3}{8}\mathrm{H}_{1,0}(\eta) - \frac{1}{2}\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)\right. \\
&-\mathrm{H}_1(\eta)\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big) - \frac{1}{4}\mathrm{H}_{1,1,0}(\eta) + \mathrm{H}_{-1,0,1}\big(\sqrt{\eta}\big) + \mathrm{H}_{-1,1,0}\big(\sqrt{\eta}\big) \\
&\left.-\mathrm{H}_{-1,-1,0}\big(\sqrt{\eta}\big) - \mathrm{H}_{-1,0,-1}\big(\sqrt{\eta}\big) - 2\mathrm{H}_{0,-1,0}\big(\sqrt{\eta}\big) - \frac{1}{4}\mathrm{H}_{0,1,0}(\eta)\right],
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
K_{71} &= G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)}, \sqrt{1-x}\sqrt{x}, \frac{1}{1-x}\right\}, 1\right) \\
&= \frac{1}{2(1-\eta)}\left[\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big) - \frac{1}{2}\mathrm{H}_0\big(\sqrt{\eta}\big)\right] + \frac{1}{2(1-\eta)\sqrt{\eta}}\left[-\frac{1}{2}\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)\right. \\
&\left.+\frac{1}{4}\mathrm{H}_{1,1,0}(\eta) - \mathrm{H}_{-1,-1,0}\big(\sqrt{\eta}\big) + \mathrm{H}_{-1,1,0}\big(\sqrt{\eta}\big)\right] + \frac{\mathrm{H}_{1,0}\big(\sqrt{\eta}\big)}{2\big(\eta+\sqrt{\eta}\big)} \\
&+\frac{\pi^2}{24(1-\eta)\sqrt{\eta}}\left[\mathrm{H}_{-1}\big(\sqrt{\eta}\big) + \frac{1}{2}\mathrm{H}_1(\eta)\right] + \frac{1}{1-\eta}\left(\frac{7\zeta_3}{16} - \frac{\pi^2}{8}\ln(2)\right) \\
&+\frac{3}{(1-\eta)^3}\left[\frac{1}{8} - \frac{\eta^2}{8} + \frac{\eta}{24}(\eta+1)\mathrm{H}_0(\eta) - \frac{\pi^2}{48}\left(\frac{1}{6} + \frac{5\eta}{3} - \frac{\eta^2}{2}\right)\right. \\
&\left.+\left(\frac{\eta^2}{12} - \frac{\eta^{3/2}}{16} - \frac{\eta}{4} + \frac{\sqrt{\eta}}{8} - \frac{1}{16\sqrt{\eta}}\right)\mathrm{H}_{1,0}(\eta)\right],
\end{aligned}
\tag{A.39}
$$

$$
\begin{aligned}
K_{72} &= G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(\eta-x\eta+x)}, \sqrt{1-x}\sqrt{x}, \frac{1}{x}\right\}, 1\right) \\
&= -\frac{\mathrm{H}_{1,0}\big(\sqrt{\eta}\big)}{2\big(\eta+\sqrt{\eta}\big)} + \frac{1}{2(1-\eta)}\left[-\frac{3}{4}\mathrm{H}_0(\eta) + \frac{1}{4}\mathrm{H}_0^2(\eta) - \mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)\right] \\
&+\frac{1}{8(1-\eta)\sqrt{\eta}}\left[2\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big) - \mathrm{H}_{1,1,0}(\eta) + 4\mathrm{H}_{-1,-1,0}\big(\sqrt{\eta}\big)\right. \\
&\left.-8\mathrm{H}_{-1,0,0}\big(\sqrt{\eta}\big) - 4\mathrm{H}_{-1,1,0}\big(\sqrt{\eta}\big) - \mathrm{H}_{1,0,0}(\eta)\right] + \frac{\eta(3\eta-5)}{8(1-\eta)^3}\mathrm{H}_0(\eta) \\
&-\frac{\pi^2}{24(1-\eta)\sqrt{\eta}}\left[\mathrm{H}_{-1}\big(\sqrt{\eta}\big) + \frac{1}{2}\mathrm{H}_1(\eta)\right] + \frac{1}{1-\eta}\left(\frac{\pi^2}{8}\ln(2) + \frac{7\zeta_3}{16}\right)
\end{aligned}
$$

$$
+\frac{3}{(1-\eta)^3}\Biggl\{-\frac{3}{8}+\frac{\eta}{2}-\frac{\eta^2}{8}+\frac{\pi^2}{3}\left(-\frac{\eta^2}{32}+\frac{5\eta}{48}+\frac{1}{96}\right)
$$

$$
-\left[\frac{\eta^2}{12}-\frac{\eta^{3/2}}{16}-\frac{\eta}{4}+\frac{\sqrt{\eta}}{8}-\frac{1}{16\sqrt{\eta}}\right]\mathrm{H}_{1,0}(\eta)+\frac{\eta}{24}(3-\eta)\mathrm{H}_0^2(\eta)\Biggr\}, \tag{A.40}
$$

$$
K_{73}=G\left(\left\{\frac{1}{x\eta-x+1},\frac{1}{x},\frac{1}{x}\right\},1\right)=\frac{1}{\eta-1}\Bigl[\mathrm{H}_{1,1,0}(\eta)-\zeta_3+\zeta_2\mathrm{H}_1(\eta)\Bigr], \tag{A.41}
$$

$$
K_{74}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)},\frac{1}{1-x},\sqrt{1-x}\sqrt{x}\right\},1\right)=-\frac{1}{4(\eta-1)^2}
$$

$$
+\frac{\eta}{2(1-\eta)}\Biggl[\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)-\frac{3}{4}\mathrm{H}_0(\eta)+\mathrm{H}_1(\eta)-\frac{1}{2}\mathrm{H}_0(\eta)\mathrm{H}_1(\eta)
$$

$$
-\mathrm{H}_0\bigl(\sqrt{\eta}\bigr)\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)\Biggr]+\frac{\sqrt{\eta}}{2(1-\eta)}\Biggl[-\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)
$$

$$
-\mathrm{H}_1(\eta)+\frac{3}{8}\mathrm{H}_{1,0}(\eta)+\frac{1}{2}\mathrm{H}_{-1,0}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_1(\eta)\mathrm{H}_{-1,0}\bigl(\sqrt{\eta}\bigr)
$$

$$
+\frac{1}{4}\mathrm{H}_{1,1,0}(\eta)+\mathrm{H}_{-1,-1,0}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1,0,-1}\bigl(\sqrt{\eta}\bigr)-\mathrm{H}_{-1,0,1}\bigl(\sqrt{\eta}\bigr)
$$

$$
-\mathrm{H}_{-1,1,0}\bigl(\sqrt{\eta}\bigr)+2\mathrm{H}_{0,-1,0}\bigl(\sqrt{\eta}\bigr)+\frac{1}{4}\mathrm{H}_{0,1,0}(\eta)\Biggr]+\frac{\pi^2\bigl(1-2\sqrt{\eta}\bigr)}{32(1-\eta)}
$$

$$
+\frac{\sqrt{\eta}}{2\bigl(1+\sqrt{\eta}\bigr)}\Bigl[\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)-\mathrm{H}_{1,0}\bigl(\sqrt{\eta}\bigr)\Bigr]+\frac{2\ln(2)\pi^2\sqrt{\eta}+7\zeta_3}{8(1-\eta)}
$$

$$
+\frac{\pi^2\sqrt{\eta}}{8(1-\eta)}\left[\mathrm{H}_0(\eta)-\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)+\frac{\mathrm{H}_1(\eta)}{2}\right]+\frac{\eta^2(3\eta-5)}{8(1-\eta)^3}\mathrm{H}_0(\eta), \tag{A.42}
$$

$$
K_{75}=G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)},\frac{1}{x},\sqrt{1-x}\sqrt{x}\right\},1\right)
$$

$$
=\frac{\eta}{2(1-\eta)}\Biggl[-\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)-\frac{1}{4}\mathrm{H}_0(\eta)-\mathrm{H}_1(\eta)+\frac{1}{2}\mathrm{H}_0(\eta)\mathrm{H}_1(\eta)
$$

$$
+\frac{1}{2}\mathrm{H}_0(\eta)\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)-\mathrm{H}_{0,1}\bigl(\sqrt{\eta}\bigr)\Biggr]+\frac{\sqrt{\eta}}{2(1-\eta)}\Biggl[\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)
$$

$$
+\mathrm{H}_1(\eta)-\frac{3}{8}\mathrm{H}_{1,0}(\eta)-\frac{1}{2}\mathrm{H}_{-1,0}\bigl(\sqrt{\eta}\bigr)-\mathrm{H}_1(\eta)\mathrm{H}_{-1,0}\bigl(\sqrt{\eta}\bigr)
$$

$$
+\mathrm{H}_{-1,1,0}\bigl(\sqrt{\eta}\bigr)-\mathrm{H}_{-1,0,-1}\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{-1,0,1}\bigl(\sqrt{\eta}\bigr)-\frac{1}{4}\mathrm{H}_{1,1,0}(\eta)
$$

$$
-\mathrm{H}_{-1,-1,0}\bigl(\sqrt{\eta}\bigr)\Biggr]+\frac{\sqrt{\eta}}{2\bigl(1+\sqrt{\eta}\bigr)}\Bigl[-\mathrm{H}_1\bigl(\sqrt{\eta}\bigr)+\mathrm{H}_{1,0}\bigl(\sqrt{\eta}\bigr)\Bigr]
$$

$$
+\frac{\pi^2\sqrt{\eta}}{8(1-\eta)}\left[\frac{1}{2}-2\ln(2)+\mathrm{H}_{-1}\bigl(\sqrt{\eta}\bigr)-\frac{1}{2}\mathrm{H}_1(\eta)\right]+\frac{28\zeta_3-\pi^2}{32(1-\eta)}
$$

$$
+\frac{\eta^2(1+\eta)}{8(1-\eta)^3}\mathrm{H}_0(\eta)+\frac{2\eta-1}{4(1-\eta)^2}, \tag{A.43}
$$

$$
\begin{aligned}
K_{76} &= G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)}, \sqrt{1-x}\sqrt{x}, \frac{1}{1-x}\right\}, 1\right) \\
&= \frac{\sqrt{\eta}}{2\left(1+\sqrt{\eta}\right)}\mathrm{H}_{1,0}\left(\sqrt{\eta}\right) + \frac{1}{(1-\eta)^3}\left[\frac{\pi^2}{8}\left(\frac{\eta^{5/2}}{2} - \frac{7\eta^2}{12} - \eta^{3/2} + \frac{\eta}{6} - \frac{1}{4}\right)\right. \\
&\quad + \left(\frac{\eta^3}{4} - \frac{3\eta^{5/2}}{16} - \frac{3\eta^2}{4} + \frac{3\eta^{3/2}}{8} - \frac{3\sqrt{\eta}}{16}\right)\mathrm{H}_{1,0}(\eta) + \frac{\eta^2}{8}(\eta-3)\mathrm{H}_0^2(\eta) \\
&\quad \left. + \frac{\eta^2}{8}(5-3\eta)\mathrm{H}_0(\eta) + \frac{\pi^2}{16}\sqrt{\eta}\right] + \frac{1-2\sqrt{\eta}}{8(1-\eta)}\ln(2)\pi^2 + \frac{3(\eta+1)}{8(\eta-1)^2} \\
&\quad + \frac{\eta}{2(1-\eta)}\left[\frac{3}{4}\mathrm{H}_0(\eta) - \frac{1}{4}H_0^2(\eta) + \mathrm{H}_{-1,0}\left(\sqrt{\eta}\right)\right] + \frac{7\zeta_3}{16(\eta-1)} \\
&\quad + \frac{\sqrt{\eta}}{8(1-\eta)}\left[\frac{\pi^2}{3}\mathrm{H}_{-1}\left(\sqrt{\eta}\right) + \frac{\pi^2}{6}\mathrm{H}_1(\eta) + \mathrm{H}_{1,0,0}(\eta) - 2\mathrm{H}_{-1,0}\left(\sqrt{\eta}\right)\right. \\
&\quad \left. + \mathrm{H}_{1,1,0}(\eta) + 8\mathrm{H}_{-1,0,0}\left(\sqrt{\eta}\right) - 4\mathrm{H}_{-1,-1,0}\left(\sqrt{\eta}\right) + 4\mathrm{H}_{-1,1,0}\left(\sqrt{\eta}\right)\right],
\end{aligned} \tag{A.44}
$$

$$
\begin{aligned}
K_{77} &= G\left(\left\{\frac{\sqrt{1-x}}{\sqrt{x}(x\eta-x+1)}, \sqrt{1-x}\sqrt{x}, \frac{1}{x}\right\}, 1\right) \\
&= \frac{1}{(1-\eta)^3}\left[\frac{3}{8} - \frac{3\eta}{2} + \frac{9\eta^2}{8} - \frac{\pi^2}{8}\left(\frac{\eta^{5/2}}{2} - \frac{7}{12}\eta^2 - \eta^{3/2} + \frac{\eta}{6} - \frac{1}{4}\right)\right. \\
&\quad \left. - \frac{\sqrt{\eta}}{4}\left(\eta^{5/2} - \frac{3\eta^2}{4} - 3\eta^{3/2} + \frac{3\eta}{2} - \frac{3}{4}\right)\mathrm{H}_{1,0}(\eta)\right] + \frac{7\zeta_3}{16(\eta-1)} \\
&\quad - \frac{\sqrt{\eta}}{2\left(1+\sqrt{\eta}\right)}\mathrm{H}_{1,0}\left(\sqrt{\eta}\right) + \frac{\eta}{2(1-\eta)}\left[\frac{1}{2}\mathrm{H}_0\left(\sqrt{\eta}\right) - \mathrm{H}_{-1,0}\left(\sqrt{\eta}\right)\right] \\
&\quad + \frac{\sqrt{\eta}}{2(1-\eta)}\left[\frac{1}{2}\mathrm{H}_{-1,0}\left(\sqrt{\eta}\right) - \frac{1}{4}\mathrm{H}_{1,1,0}(\eta) + \mathrm{H}_{-1,-1,0}\left(\sqrt{\eta}\right)\right. \\
&\quad \left. - \mathrm{H}_{-1,1,0}\left(\sqrt{\eta}\right) - \frac{\pi^2}{12}\mathrm{H}_{-1}\left(\sqrt{\eta}\right) - \frac{\pi^2}{24}\mathrm{H}_1(\eta)\right] - \frac{\pi^2\sqrt{\eta}}{16(1-\eta)^3} \\
&\quad - \frac{\eta^2(\eta+1)}{8(1-\eta)^3}\mathrm{H}_0(\eta) - \frac{1-2\sqrt{\eta}}{8(1-\eta)}\ln(2)\pi^2,
\end{aligned} \tag{A.45}
$$

$$
\begin{aligned}
K_{78} &= G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x}, \frac{1}{x}, \frac{1}{x}\right\}, 1\right) = -\frac{\pi^2}{3}\ln(2) - \frac{\pi^2}{6}\mathrm{H}_1(\eta) \\
&\quad - \mathrm{H}_1(\eta)\mathrm{H}_{-1}^2\left(\sqrt{\eta}\right) + 4\sqrt{\eta}\left[2 - \mathrm{H}_{-1}\left(\sqrt{\eta}\right) + \mathrm{H}_1\left(\sqrt{\eta}\right) - \mathrm{H}_1(\eta)\right. \\
&\quad \left. + \frac{1}{4}\left(\mathrm{H}_1(\eta) + \mathrm{H}_{-1}\left(\sqrt{\eta}\right) - \mathrm{H}_1\left(\sqrt{\eta}\right)\right)^2\right] + \mathrm{H}_{-1}^2\left(\sqrt{\eta}\right) - \mathrm{H}_1^2(\eta) \\
&\quad - 2\mathrm{H}_{-1,1}\left(\sqrt{\eta}\right) + 2\mathrm{H}_1(\eta)\mathrm{H}_{-1,1}\left(\sqrt{\eta}\right) + 2\mathrm{H}_{1,-1}\left(\sqrt{\eta}\right) - \mathrm{H}_1^2\left(\sqrt{\eta}\right) \\
&\quad - 2\mathrm{H}_{-1}\left(\sqrt{\eta}\right)\mathrm{H}_{1,1}(\eta) + 2\mathrm{H}_{-1,1,-1}\left(\sqrt{\eta}\right) + 2\mathrm{H}_{-1,-1,1}\left(\sqrt{\eta}\right) \\
&\quad - \frac{1}{3}\mathrm{H}_{-1}^3\left(\sqrt{\eta}\right) - 2\mathrm{H}_{-1,1,1}\left(\sqrt{\eta}\right) + \ln(2)\left[8 - 4\mathrm{H}_1(\eta) + \mathrm{H}_1^2(\eta)\right]
\end{aligned}
$$

$$+2\zeta_3 - 4\mathrm{H}_{-1}\big(\sqrt{\eta}\big) - 4\mathrm{H}_1\big(\sqrt{\eta}\big) + 4\mathrm{H}_1(\eta) + 2\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta)$$
$$+2\mathrm{H}_1\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta) + 2\ln^2(2)\big[\mathrm{H}_1(\eta) - 2\big] + \frac{4}{3}\ln^3(2) - 8 + \frac{\pi^2}{3}, \tag{A.46}$$

$$K_{79} = G\left(\left\{\frac{\sqrt{x\eta - x + 1}}{x}, \frac{\sqrt{1-x}}{x}, \frac{\sqrt{1-x}}{x}\right\}, 1\right)$$
$$= \frac{\pi^2}{6v_2^2}(3-8\eta)\eta + \frac{16v_1}{v_2^2}(2-3\eta)\eta^2 + \frac{\ln^2(2)}{v_2^2}\big(-20\eta^2 + 40\eta + 9\big)$$
$$-\frac{4\ln(2)}{v_2^2}\big(4\eta^2 + 6\eta + 1\big) + \frac{\pi^2}{6v_2^2}\big(4\eta^3 + 1\big) - \frac{2}{v_2^2}\left[\frac{28\eta^3}{3} - \frac{5\eta^2}{3} - 4\eta - \frac{2}{3}\right]$$
$$+4\sqrt{\eta}\big[\mathrm{H}_1\big(\sqrt{\eta}\big) - \mathrm{H}_{-1}\big(\sqrt{\eta}\big) - \mathrm{H}_1(\eta)\big] + \ln(2)\Big[2\eta\mathrm{H}_0(\eta) + 2\eta\mathrm{H}_1(v_1)$$
$$-2\eta\mathrm{H}_{-1}(v_1) + 8\mathrm{H}_{-1}\big(\sqrt{\eta}\big) - 2\mathrm{H}_{1,0}(\eta) + \frac{628}{3}\mathrm{H}_{-1,0}\left(\frac{1}{\sqrt{2}}\right)$$
$$-\frac{25}{3}\sqrt{2}\mathrm{H}_{-1,0}\left(\frac{1}{\sqrt{2}}\right)\Big] + \eta\Big[2\mathrm{H}_{-1}(v_1) + 4\mathrm{H}_{-1}\big(\sqrt{\eta}\big) - 4\mathrm{H}_0(v_1)$$
$$-2\mathrm{H}_0(\eta) - 2\mathrm{H}_1(v_1) + 2\mathrm{H}_{-1,0}(v_1) + \mathrm{H}_{0,1}(\eta) - 2\mathrm{H}_{1,0}(v_1)\Big]$$
$$+\frac{2\eta}{v_1}\Big\{-\ln(2)\Big[\mathrm{H}_{-1}(v_1) + \mathrm{H}_1(v_1)\Big] - \frac{1}{2}G\left(\left\{\frac{\sqrt{1-x}}{x}, \frac{\sqrt{x}}{1-x}\right\}, \eta\right)$$
$$+\mathrm{H}_{-1}(v_1) + \mathrm{H}_1(v_1) + 3\mathrm{H}_{\{4,0\}}(v_3) + \mathrm{H}_{-1,0}(v_1) + \mathrm{H}_{1,0}(v_1)\Big\}$$
$$+\frac{v_1}{v_2^2}\sqrt{\eta}\left\{\frac{32\sqrt{2}}{3} + \left(\frac{2828}{9} - \frac{4\sqrt{2}}{3}\right)\ln^3(2) + \frac{134\sqrt{2}}{9}\ln^2(2)\right.$$
$$+4\ln(2)\left[\left(\frac{16}{9} + \frac{\sqrt{2}}{3}\right)\pi^2 - 5\sqrt{2}\right] + 8\left[\frac{16\sqrt{2}}{9} + \left(\frac{17\sqrt{2}}{9} - \frac{8}{3}\right)\ln(2)\right.$$
$$\left.+\left(\frac{5\sqrt{2}}{3} - \frac{623}{9}\right)\ln^2(2) - \frac{7}{3}\pi^2 - \frac{2}{3}\right]\ln\big(2+\sqrt{2}\big) - \frac{77}{3}\zeta_3$$
$$\left.+\left[-\frac{68\sqrt{2}}{9} + \left(224 + \frac{16\sqrt{2}}{3}\right)\ln(2)\right]\ln^2\big(2+\sqrt{2}\big) - \frac{37}{9\sqrt{2}}\pi^2\right\}$$
$$+\frac{\sqrt{\eta}\,v_1}{(1+\sqrt{\eta})v_2^2}\Big[-\frac{16}{3} - \frac{32}{3}\eta - \ln^2(2)\big(16\eta + 36\big) + 16\ln(2)\big(1 + 2\eta\big)$$
$$+\frac{2}{3}\pi^2\big(\eta^{3/2} + \eta - 1\big)\Big] + \frac{\eta\,v_1}{(1+\sqrt{\eta})v_2^2}\big[48\ln(2) - 52\ln^2(2)\big]$$
$$+\frac{1}{v_2^2}\big(1 + 4\eta - 4\eta^2\big)\Big[-\frac{8\sqrt{2}}{3} - \frac{67}{9\sqrt{2}}\ln^2(2) - \frac{707}{9}\ln^3(2) + \frac{\sqrt{2}}{3}\ln^3(2)$$
$$+\frac{37\pi^2}{36\sqrt{2}} - \frac{16}{9}\ln(2)\pi^2 + \sqrt{2}\ln(2)\left(5 - \frac{\pi^2}{3}\right) + 2\Big(\frac{2}{3} - \frac{5}{3}\sqrt{2}\ln^2(2)$$

$$
\begin{aligned}
&+\frac{623}{9}\ln^2(2)-\frac{17\sqrt{2}}{9}\ln(2)+\frac{8}{3}\ln(2)+\frac{7}{3}\pi^2-\frac{16\sqrt{2}}{9}\Big)\ln\big(2+\sqrt{2}\big)\\
&+\Big(-\frac{4\sqrt{2}}{3}\ln(2)-56\ln(2)+\frac{17\sqrt{2}}{9}\Big)\ln^2\big(2+\sqrt{2}\big)+\frac{77}{12}\zeta_3\Big]\\
&+\frac{1}{v_1v_2^2}\Big[72\eta^3-4\pi^2\eta^3-48\eta^4+\pi\eta\Big(6\eta^2-6\eta-\frac{3}{2}\Big)\Big]\\
&+\frac{8}{(1+\sqrt{\eta})v_1v_2^2}\Big[\frac{\pi^2}{8}\Big(4\eta^{5/2}+\frac{14}{3}\eta^2+\eta^{3/2}+\frac{\eta}{3}\Big)-\frac{8\eta}{3}-\frac{\eta^2}{3}-3\eta^{5/2}\Big]\\
&+\frac{1}{(1+\sqrt{\eta})v_2^2}\Big[\big[4\ln(2)-2\ln^2(2)\big]\big(8\eta^3+16\eta^{5/2}-6\eta^{3/2}-2\sqrt{\eta}\big)\\
&+4\sqrt{\eta}+\frac{34}{3}\eta^{3/2}-\frac{98}{3}\eta^{5/2}+\frac{80}{3}\eta^{7/2}+8\eta^4+\big(6\pi-4\pi^2\big)\big(\eta^2+\eta^{3/2}\big)\Big]\\
&+\frac{1}{\sqrt{2}}G\left(\left\{\frac{\sqrt{2-x}}{x},\frac{\sqrt{1-x}}{x},\frac{\sqrt{1-x}}{x}\right\},1\right)-4\ln^2(2)\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\\
&-4\mathrm{H}_1\big(\sqrt{\eta}\big)+4\mathrm{H}_1(\eta)-\frac{\pi^2}{3}\mathrm{H}_1(\eta)+2\mathrm{H}_{1,0}(\eta)-4\mathrm{H}_{0,1,-1}\big(\sqrt{\eta}\big)\\
&+10\sqrt{2}\mathrm{H}_{-1,0}\left(\frac{1}{\sqrt{2}}\right)+4\mathrm{H}_1(\eta)\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)-112\mathrm{H}_{-1,0,1}\left(\frac{1}{\sqrt{2}}\right)\\
&-216\mathrm{H}_{0,-1,-1}\left(\frac{1}{\sqrt{2}}\right)+8\mathrm{H}_{0,-1,-1}\big(\sqrt{\eta}\big)-4\mathrm{H}_{0,-1,1}\big(\sqrt{\eta}\big)\\
&-112\mathrm{H}_{-1,0}\left(\frac{1}{\sqrt{2}}\right)\ln\big(2+\sqrt{2}\big),
\end{aligned}
\tag{A.47}
$$

$$
\begin{aligned}
K_{80}&=G\left(\left\{\frac{\sqrt{x\eta-x+1}}{x},\frac{1}{x\eta-x+1},\frac{1}{x}\right\},1\right)\\
&=\frac{4\mathrm{H}_1\big(\sqrt{\eta}\big)}{1+\sqrt{\eta}}+\frac{4\sqrt{\eta}}{1-\eta}\Big[\mathrm{H}_{-1}\big(\sqrt{\eta}\big)+\mathrm{H}_1(\eta)-\mathrm{H}_0\big(\sqrt{\eta}\big)\mathrm{H}_1(\eta)+\mathrm{H}_{0,1}\big(\sqrt{\eta}\big)\\
&-\mathrm{H}_{0,-1}\big(\sqrt{\eta}\big)-2-\frac{\pi^2}{12}\Big]+\frac{4}{1-\eta}\Big[2-2\ln(2)+\Big(1+\frac{\pi^2}{12}\Big)\mathrm{H}_{-1}\big(\sqrt{\eta}\big)\\
&-\mathrm{H}_1(\eta)+\Big(\frac{\pi^2}{6}+\mathrm{H}_{-1,0}\big(\sqrt{\eta}\big)\Big)\mathrm{H}_1(\eta)+\frac{1}{2}\mathrm{H}_{1,1,0}(\eta)+\mathrm{H}_{-1,0,-1}\big(\sqrt{\eta}\big)\\
&-\mathrm{H}_{-1,0,1}\big(\sqrt{\eta}\big)-\frac{7}{8}\zeta_3\Big],
\end{aligned}
\tag{A.48}
$$

where C is Catalan's constant and

$$v_2=1-2\sqrt{(1-\eta)\eta}, \tag{A.49}$$

$$v_3=\frac{v_2}{1-2\eta} \tag{A.50}$$

and

$$G\left(\left\{\frac{\sqrt{1-x}}{x}, \frac{\sqrt{x}}{1-x}\right\}, \eta\right) = \pi^2 - 2\sqrt{(1-\eta)\eta} - 2\arcsin(\sqrt{\eta})$$
$$+4\sqrt{1-\eta}\,\mathrm{arctanh}(\sqrt{\eta}) + 8\arctan\left[-1+\frac{\sqrt{1-\eta}}{\sqrt{\eta}}\right]$$
$$-8\,\mathrm{arctanh}\left[\frac{\sqrt{1-\sqrt{\eta}}}{\sqrt{1+\sqrt{\eta}}}\right]\mathrm{arctanh}(\sqrt{\eta})$$
$$+4\mathrm{Li}_2\left(-\frac{\sqrt{1-\sqrt{\eta}}}{\sqrt{1+\sqrt{\eta}}}\right) - 4\mathrm{Li}_2\left(\frac{\sqrt{1-\sqrt{\eta}}}{\sqrt{1+\sqrt{\eta}}}\right). \tag{A.51}$$

The following constant is calculated numerically

$$G\left(\left\{\frac{\sqrt{2-x}}{x}, \frac{\sqrt{1-x}}{x}, \frac{\sqrt{1-x}}{x}\right\}, 1\right) = 0.41373402691074161495 3\,. \tag{A.52}$$

Appendix B. Relations between certain iterated integrals

$$G_{59} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right\}; x\right)$$
$$= -\frac{1}{1-\eta}\Big[\ln^3(x)\ln(1-(1-\eta)x) + 3\ln^2(x)\mathrm{Li}_2(x(1-\eta)) - 6\ln(x)\mathrm{Li}_3(x(1-\eta))$$
$$+6\mathrm{Li}_4(x(1-\eta))\Big] \tag{B.1}$$

$$G_{60} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}, \frac{1}{1-\tau}, \frac{1}{1-\tau}\right\}; 1-x\right)$$
$$= -\frac{1}{1-\eta}\mathrm{H}_{1/(1-\eta),0,1,1}(1-x) \tag{B.2}$$

$$G_{61} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{1-\tau}\right\}; 1-x\right) = -\frac{1}{1-\eta}\mathrm{H}_{1/(1-\eta),0,0,1}(1-x) \tag{B.3}$$

$$G_{62} = G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau}\right\}; y\right)$$
$$= -g_5(x) - G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.4}$$

$$G_{63} = G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau}\right\}; y\right)$$
$$= -g_5(1-x) - G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.5}$$

$$G_{64} = G\left(\left\{\frac{\tau}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau}\right\}; y\right)$$
$$= g_6(x) - G\left(\left\{\frac{\tau}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.6}$$

$$G_{65} = G\left(\left\{\frac{\tau}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau}\right\}; y\right)$$
$$= -g_6(1-x) - G\left(\left\{\frac{\tau}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.7}$$
$$G_{66} = G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{\tau}{1+\tau^2}\right\}; y\right)$$
$$= g_7(x) - G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.8}$$
$$G_{67} = G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{\tau}{1+\tau^2}\right\}; y\right)$$
$$= g_8(x) - G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1-\tau}\right\}; y\right) \tag{B.9}$$
$$G_{68} = G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau^2}\right\}; y\right) = g_9(x) \tag{B.10}$$
$$G_{69} = G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}, \frac{1}{1+\tau^2}\right\}; y\right) = g_9(1-x) \tag{B.11}$$
$$G_{70} = G\left(\left\{\frac{\tau}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}\right\}; y\right) = g_{10}(x) \tag{B.12}$$
$$G_{71} = G\left(\left\{\frac{\tau}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}\right\}; y\right) = g_{10}(1-x) \tag{B.13}$$
$$G_{72} = G\left(\left\{\frac{1}{1+\eta+2\tau-2\eta\tau+\tau^2+\eta\tau^2}\right\}; y\right) = g_{11}(x) \tag{B.14}$$
$$G_{73} = G\left(\left\{\frac{1}{1+\eta-2\tau+2\eta\tau+\tau^2+\eta\tau^2}\right\}; y\right) = -g_{11}(1-x) \tag{B.15}$$
$$G_{74} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}, \frac{1}{1-\tau}\right\}; x\right) = \frac{1}{1-\eta}\mathrm{H}_{1/(1-\eta),0,1}(x)$$
$$= g_{12}(x) - G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{1-\tau}, \frac{1}{\tau}\right\}; x\right) \tag{B.16}$$
$$G_{75} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{\tau}, \frac{1}{1-\tau}\right\}; 1-x\right) = \frac{1}{1-\eta}\mathrm{H}_{1/(1-\eta),0,1}(1-x)$$
$$= g_{12}(1-x) - G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{1-\tau}, \frac{1}{\tau}\right\}; 1-x\right) \tag{B.17}$$
$$G_{76} = G\left(\left\{\frac{1}{1-\tau+\eta\tau}, \frac{1}{1-\tau}, \frac{1}{\tau}\right\}; x\right) = \frac{1}{1-\eta}\mathrm{H}_{1/(1-\eta),1,0}(x) = g_{13}(x). \tag{B.18}$$

Some of the integrals are generalized harmonic polylogarithms based on three different letters, which are known not to reduce to Nielsen integrals [73] in general. Here one of the letters contains a linear function of the real parameter η.

The functions G_{62} to G_{67} contain the η-dependent letters

$$\frac{1}{(1-t)^2+(1+t)^2\eta}, \tag{B.19}$$

$$\frac{t}{(1-t)^2+(1+t)^2\eta}, \tag{B.20}$$

and their replacement with $\eta \to 1/\eta$. These letters can be partial fractioned as

$$\frac{1}{(t-a)(t-b)} = \frac{1}{a-b}\left[\frac{1}{t-a} - \frac{1}{t-b}\right] \tag{B.21}$$

$$\frac{t}{(t-a)(t-b)} = \frac{1}{a-b}\left[\frac{a}{t-a} - \frac{b}{t-b}\right]. \tag{B.22}$$

Then these functions can be obtained as linear combinations of the following four integrals

$$I_1 = \int_0^y dt \frac{\ln(1+t)}{t-a} = \ln\left[\frac{a-y}{a+1}\right]\ln(1+y) - \text{Li}_2\left[\frac{1}{1+a}\right] + \text{Li}_2\left[\frac{1+y}{1+a}\right] \tag{B.23}$$

$$I_2 = -\int_0^y \frac{\ln(1-t)}{t-a} = -\ln(1-y)\ln\left[\frac{a-y}{a-1}\right] + \text{Li}_2\left[\frac{1}{1-a}\right] - \text{Li}_2\left[\frac{y-1}{a-1}\right] \tag{B.24}$$

$$\begin{aligned} I_3 = \int_0^y \frac{\arctan(t)}{t-a} = -\frac{i}{2}\Biggl\{ & 2i \arctan y \ln(a-y) + \ln(i+a)\ln(1-iy) \\ & -\ln(a-i)\ln(1+iy) - \text{Li}_2\left[-\frac{i}{a-i}\right] + \text{Li}_2\left[\frac{i}{i+a}\right] + \text{Li}_2\left[\frac{y-i}{a-i}\right] \\ & -\text{Li}_2\left[\frac{y+i}{a+i}\right]\Biggr\} \end{aligned} \tag{B.25}$$

$$\begin{aligned} I_4 = \frac{1}{2}\int_0^y \frac{\ln(1+t^2)}{t-a} = \frac{1}{2}\Biggl\{ & -[\ln(a-i)+\ln(a+i)][\ln(-a)-\ln(y-a)] \\ & +\text{Li}_2\left[\frac{a}{a-i}\right] + \text{Li}_2\left[\frac{a}{a+i}\right] - \text{Li}_2\left[\frac{a-y}{a-i}\right] - \text{Li}_2\left[\frac{a-y}{a+i}\right]\Biggr\}. \end{aligned} \tag{B.26}$$

The roots of the quadratic form (B.19)

$$(1-t)^2 + (1+t^2)\eta = c(t-a)(t-b) \tag{B.27}$$

are:

$$a = \frac{1-2i\sqrt{\eta}-\eta}{1+\eta}, \tag{B.28}$$

$$b = \frac{1+2i\sqrt{\eta}-\eta}{1+\eta}, \tag{B.29}$$

with

$$c = 1+\eta. \tag{B.30}$$

We further present representations of a series of functions g_i which are functions of x and η. The symbol y, not to be confused with its meaning in the main text, is defined here as

$$y = \frac{1-2\sqrt{1-x}\sqrt{x}}{1-2x}, \tag{B.31}$$

and the formulas are valid for

$$\begin{aligned} &0 < \eta < 1, \\ &0 < x < 1, \end{aligned} \tag{B.32}$$

$$\begin{aligned} g_1(x) &= \int_0^y dz \frac{\arctan(z)\big(\ln(1-z)-\ln(1+z)\big)}{(1-z)^2+\eta(1+z)^2} \\ &= \frac{(1+y)\arctan(y)}{2(1+\eta)(1-y)}\Big[1+\ln(1-y)-\ln(1+y)\Big] \\ &+\frac{1}{1+\eta}\Bigg\{\frac{1}{12}\Bigg[\pi^2-6\pi\arctan\left(y+\sqrt{1+y^2}\right)+3\ln^2(2) \\ &-6\Big(-1+\ln(2)\Big)\ln(1-y)+3\ln^2(1-y)-3\ln\left(1+y^2\right)\Bigg] \\ &+\frac{1}{2}\mathrm{Li}_2\Big(\frac{1}{2}-\frac{y}{2}\Big)-\frac{1}{4}\mathrm{Li}_2\Big[\frac{1-y}{2}-i\Big(\frac{1+y}{2}\Big)\Big] \\ &-\frac{1}{4}\mathrm{Li}_2\Big[\frac{1-y}{2}+i\Big(\frac{1+y}{2}\Big)\Big]+\frac{1}{4}\mathrm{Li}_2\Big[\frac{1+y}{2}+i\Big(\frac{1-y}{2}\Big)\Big] \\ &+\frac{1}{4}\mathrm{Li}_2\Big[\frac{1+y}{2}-i\Big(\frac{1-y}{2}\Big)\Big]\Bigg\} \end{aligned} \tag{B.33}$$

$$\begin{aligned} g_5(x) &= \int_0^y dz \frac{\ln(1-z)-\ln(1+z)}{(1-z)^2+\eta(1+z)^2} \\ &= \Bigg\{-i\frac{1}{8}\mathrm{Li}_2\Big(-\frac{\eta(1+y)^2}{(1-y)^2}\Big) \\ &+\arctan\left(\frac{\sqrt{\eta}(1+y)}{1-y}\right)\Big[\frac{1}{2}\ln(1-y)-\frac{1}{2}\ln(1+y)\Big] \\ &+\frac{1}{2}i\mathrm{Li}_2\big(i\sqrt{\eta}\big)-\frac{1}{8}i\mathrm{Li}_2(-\eta)+\frac{1}{2}i\mathrm{Li}_2\left(-\frac{i\sqrt{\eta}(1+y)}{1-y}\right)\Bigg\}\frac{1}{\sqrt{\eta}} \end{aligned} \tag{B.34}$$

$$\begin{aligned} g_6(x) &= \int_0^y dz \frac{z\big(-\ln(1-z)+\ln(1+z)\big)}{(1-z)^2+\eta(1+z)^2} \\ &= -\frac{3\pi^2}{8(1+\eta)}-\frac{\pi\arctan\left(\sqrt{\eta}\right)}{2(1+\eta)}+\frac{\pi}{2(1+\eta)}\arctan\left(\frac{2\sqrt{\eta}}{1-\eta}\right) \\ &+\frac{\pi}{1+\eta}\arctan\left(\frac{1-y-\eta(1+y)+\sqrt{(1+\eta)\big((1-y)^2+\eta(1+y)^2\big)}}{2\sqrt{\eta}}\right) \\ &+\left(\frac{1}{2(1+\eta)}+\frac{i(1-\eta)}{4(1+\eta)}\frac{1}{\sqrt{\eta}}\right)\mathrm{Li}_2\left(\frac{1+y}{2}+\frac{i(1-y)}{2\sqrt{\eta}}\right) \end{aligned}$$

$$+\left(\frac{1}{2(1+\eta)}-\frac{i(1-\eta)}{4(1+\eta)}\frac{1}{\sqrt{\eta}}\right)\mathrm{Li}_2\left(\frac{1+y}{2}-\frac{i(1-y)}{2\sqrt{\eta}}\right)$$
$$-\left(\frac{1}{2(1+\eta)}+\frac{i(1-\eta)}{4(1+\eta)}\frac{1}{\sqrt{\eta}}\right)\mathrm{Li}_2\left(\frac{1-y}{2}-\frac{1}{2}i\sqrt{\eta}(1+y)\right)$$
$$-\left(\frac{1}{2(1+\eta)}-\frac{i(1-\eta)}{4(1+\eta)}\frac{1}{\sqrt{\eta}}\right)\mathrm{Li}_2\left(\frac{1-y}{2}+\frac{1}{2}i\sqrt{\eta}(1+y)\right)$$
$$-\frac{1}{1+\eta}\Biggl\{-\frac{1}{8}(1-\eta)\pi\ln\left[(1-y)^2+\eta(1+y)^2\right]$$
$$+(1-\eta)\Biggl[\frac{i\ln^2(2)}{2}-\frac{i\pi^2}{48}-\frac{1}{4}i\pi\arctan\left(\sqrt{\eta}\right)+\frac{1}{2}i\arctan^2\left(\sqrt{\eta}\right)$$
$$+i\frac{1}{2}\mathrm{Li}_2\left(\frac{1}{2}+\frac{i}{2\sqrt{\eta}}\right)+i\frac{1}{2}\mathrm{Li}_2\left(\frac{1}{2}+\frac{i\sqrt{\eta}}{2}\right)+\frac{1}{8}(-4i\ln(2)+\pi)\ln(1+\eta)$$
$$+\frac{1}{8}i\ln^2(1+\eta)\Biggr]\Biggr\}\frac{1}{\sqrt{\eta}}-\Biggl\{-\frac{\ln(2)}{2(1+\eta)}+\frac{1}{4(1+\eta)}\ln\left[(1-y)^2+\eta(1+y)^2\right]$$
$$+\frac{1}{1+\eta}\Biggl\{-\frac{1}{2}(1-\eta)$$
$$\times\arctan\left(\frac{1-y-\eta(1+y)+\sqrt{(1+\eta)\left((-1+y)^2+\eta(1+y)^2\right)}}{2\sqrt{\eta}}\right)$$
$$+(1-\eta)\Biggl[\frac{i\ln(2)}{4}+\frac{\pi}{4}-\frac{1}{4}\arctan\left(\frac{2\sqrt{\eta}}{1-\eta}\right)-\frac{1}{8}i\ln(1+\eta)\Biggr]\Biggr\}\frac{1}{\sqrt{\eta}}\Biggr\}\ln(\eta)$$
$$-\left[-\frac{1}{8(1+\eta)}+\frac{i(1-\eta)}{16(1+\eta)}\frac{1}{\sqrt{\eta}}\right]\ln^2(\eta) \tag{B.35}$$

$$g_7(x)=\int_0^y dz\frac{\left(-2\ln(1-z)+\ln\left(1+z^2\right)\right)}{2\left((1-z)^2+\eta(1+z)^2\right)}$$
$$=\Biggl\{-\frac{i\ln^2(2)}{4}+\frac{7i\pi^2}{96}+\frac{1}{4}i\pi\arctan\left(\sqrt{\eta}\right)-\frac{1}{4}i\arctan^2\left(\sqrt{\eta}\right)$$
$$-i\frac{1}{2}\mathrm{Li}_2\left(\frac{1}{2}+\frac{i}{2\sqrt{\eta}}\right)-\frac{1}{2}\ln(1-y)\arctan\left(\frac{\sqrt{\eta}(1+y)}{1-y}\right)$$
$$+\Biggl[-\arctan\left(\frac{1-y-\eta(1+y)-\sqrt{(1+\eta)\left((1-y)^2+\eta(1+y)^2\right)}}{2\sqrt{\eta}}\right)$$
$$-\arctan\left(\frac{1-y-\eta(1+y)+\sqrt{(1+\eta)\left((1-y)^2+\eta(1+y)^2\right)}}{2\sqrt{\eta}}\right)$$
$$-\arctan\left(\frac{2\sqrt{\eta}}{1-\eta}\right)\Biggr]\left[\frac{\ln(2)}{4}+\frac{1}{4}\ln(1-\eta)-\frac{1}{4}\ln(1+\eta)\right]$$
$$+\left[\ln\left(1-\sqrt{\eta}\right)-\ln\left(1+\sqrt{\eta}\right)\right]\left[\frac{i\ln(2)}{8}-\frac{1}{8}i\ln(1+\eta)\right]$$

$$
\begin{aligned}
&+\arctan\left(\frac{1-\sqrt{\eta}}{1+\sqrt{\eta}}\right)\left[\frac{\ln(2)}{2}-\frac{3i\pi}{8}+\frac{1}{2}\ln(1+\eta)\right]\\
&+\Bigg[-\arctan\left(\frac{1+\sqrt{\eta}(1-y)+y-\sqrt{2}\sqrt{(1+\eta)\big(1+y^2\big)}}{1-y-\sqrt{\eta}(1+y)}\right)\\
&+\arctan\left(\frac{1-\sqrt{\eta}(1-y)+y-\sqrt{2}\sqrt{(1+\eta)\big(1+y^2\big)}}{1-y+\sqrt{\eta}(1+y)}\right)\Bigg]\\
&\times\Bigg[-\frac{\ln(2)}{2}\\
&-\frac{1}{2}i\arctan\left(\frac{1-y-\eta(1+y)-\sqrt{(1+\eta)\big((1-y)^2+\eta(1+y)^2\big)}}{2\sqrt{\eta}}\right)\\
&+\frac{1}{2}i\arctan\left(\frac{1-y-\eta(1+y)+\sqrt{(1+\eta)\big((1-y)^2+\eta(1+y)^2\big)}}{2\sqrt{\eta}}\right)\\
&-\frac{1}{4}\ln(1+\eta)-\frac{1}{4}\ln\left[(1-y)^2+\eta(1+y)^2\right]\Bigg]+\frac{1}{8}i\ln^2\left(1-\sqrt{\eta}\right)\\
&-\frac{1}{8}i\ln^2\left(1+\sqrt{\eta}\right)+\frac{1}{8}\pi\ln(1-\eta)+\left[-\frac{i\ln(2)}{4}+\frac{1}{8}i\ln(1+\eta)\right]\ln(\eta)\\
&-\frac{1}{16}i\ln^2(\eta)+\frac{1}{4}i\Big(\ln(2)+i\pi\Big)\ln(1+\eta)-\frac{1}{16}i\ln^2(1+\eta)+\frac{1}{4}\pi\ln(1-y)\\
&-\frac{1}{4}i\mathrm{Li}_2\left(\frac{1-y}{2}-\frac{i(1-y)}{2\sqrt{\eta}}\right)+\frac{1}{4}i\mathrm{Li}_2\left(\frac{1-y}{2}+\frac{i(1-y)}{2\sqrt{\eta}}\right)\\
&+\frac{1}{4}i\mathrm{Li}_2\left[\frac{1}{2}+i\left(\frac{1}{2}-\frac{1}{1-\sqrt{\eta}}\right)\right]-\frac{1}{4}i\mathrm{Li}_2\left[\frac{1}{2}+i\left(\frac{1}{2}-\frac{1}{1+\sqrt{\eta}}\right)\right]\\
&-\frac{1}{8}i\mathrm{Li}_2\left[-\frac{y}{1-\sqrt{\eta}}+\frac{1+y}{2}-i\left(\frac{1}{1-\sqrt{\eta}}-\frac{1+y}{2}\right)\right]\\
&+\frac{1}{8}i\mathrm{Li}_2\left[-\frac{y}{1-\sqrt{\eta}}+\frac{1+y}{2}+i\left(\frac{1}{1-\sqrt{\eta}}-\frac{1+y}{2}\right)\right]\\
&-\frac{1}{8}i\mathrm{Li}_2\left[-\frac{y}{1+\sqrt{\eta}}+\frac{1+y}{2}+i\left(\frac{1}{1+\sqrt{\eta}}-\frac{1+y}{2}\right)\right]\\
&+\frac{1}{8}i\mathrm{Li}_2\left[-\frac{y}{1+\sqrt{\eta}}+\frac{1+y}{2}-i\left(\frac{1}{1+\sqrt{\eta}}-\frac{1+y}{2}\right)\right]\Bigg\}\frac{1}{\sqrt{\eta}}
\end{aligned}
\tag{B.36}
$$

$$
\begin{aligned}
g_8(x)&=\int_0^y dz\frac{\big(-2\ln(1-z)+\ln\big(1+z^2\big)\big)}{2\big(\eta(1-z)^2+(1+z)^2\big)}\\
&=\frac{i}{8\sqrt{\eta}}\Bigg\{-2\ln^2(2)-\frac{5\pi^2}{12}-2\arctan^2\left(\sqrt{\eta}\right)-4\mathrm{Li}_2\left(\frac{1}{2}+\frac{i\sqrt{\eta}}{2}\right)\\
&-i\big[4\ln(1-y)\big]\arctan\left(\frac{\sqrt{\eta}(1-y)}{1+y}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\arctan\left(\frac{1+y-\eta(1-y)-\sqrt{(1+\eta)\big(\eta(1-y)^2+(1+y)^2\big)}}{2\sqrt{\eta}}\right)\\
&\times\big[2i\ln(2)+2i\ln(1-\eta)-2i\ln(1+\eta)\big]\\
&+\Bigg[\arctan\left(\frac{2\sqrt{\eta}}{1-\eta}\right)\\
&+\arctan\left(\frac{1+y-\eta(1-y)+\sqrt{(1+\eta)\big(\eta(1-y)^2+(1+y)^2\big)}}{2\sqrt{\eta}}\right)\Bigg]\\
&\times\big[-2i\ln(2)-2i\ln(1-\eta)+2i\ln(1+\eta)\big]\\
&+\arctan\left(\frac{1-\sqrt{\eta}}{1+\sqrt{\eta}}\right)\big[4i\ln(2)+3\pi+4i\ln(1+\eta)\big]\\
&+\Bigg[\arctan\left(\frac{1-y+\sqrt{\eta}(1+y)-\sqrt{2}\sqrt{(1+\eta)\big(1+y^2\big)}}{1-\sqrt{\eta}(1-y)+y}\right)\\
&-\arctan\left(\frac{1-y-\sqrt{\eta}(1+y)-\sqrt{2}\sqrt{(1+\eta)\big(1+y^2\big)}}{1+\sqrt{\eta}(1-y)+y}\right)\Bigg]\\
&\times\Bigg[4i\ln(2)+4\arctan\left(\frac{1+y-\eta(1-y)+\sqrt{(1+\eta)\big(\eta(1-y)^2+(1+y)^2\big)}}{2\sqrt{\eta}}\right)\\
&-4\arctan\left(\frac{1+y-\eta(1-y)-\sqrt{(1+\eta)\big(\eta(1-y)^2+(1+y)^2\big)}}{2\sqrt{\eta}}\right)\\
&+2i\ln(1+\eta)+2i\ln\left(\eta(1-y)^2+(1+y)^2\right)\Bigg]\\
&+\Big[-\ln\left(1-\sqrt{\eta}\right)+\ln\left(1+\sqrt{\eta}\right)\Big]\big[\ln(2)-\ln(1+\eta)\big]\\
&-\ln^2\left(1-\sqrt{\eta}\right)+\ln^2\left(1+\sqrt{\eta}\right)+i\pi\ln(1-\eta)+2\big[\ln(2)-i\pi\big]\ln(1+\eta)\\
&-\frac{1}{2}\ln^2(1+\eta)-2\mathrm{Li}_2\Big(\frac{1-y}{2}-\frac{1}{2}i\sqrt{\eta}(1-y)\Big)+2\mathrm{Li}_2\Big(\frac{1-y}{2}+\frac{1}{2}i\sqrt{\eta}(1-y)\Big)\\
&-2\mathrm{Li}_2\Big(\frac{1}{2}+i\Big(\frac{1}{2}-\frac{1}{1-\sqrt{\eta}}\Big)\Big)+2\mathrm{Li}_2\Big(\frac{1}{2}+i\Big(\frac{1}{2}-\frac{1}{1+\sqrt{\eta}}\Big)\Big)\\
&-\mathrm{Li}_2\Big(\frac{1-y}{2}+\frac{y}{1+\sqrt{\eta}}-i\Big(\frac{1}{1+\sqrt{\eta}}-\frac{1-y}{2}\Big)\Big)\\
&+\mathrm{Li}_2\Big(\frac{1-y}{2}+\frac{y}{1-\sqrt{\eta}}-i\Big(\frac{1}{1-\sqrt{\eta}}-\frac{1-y}{2}\Big)\Big)\\
&-\mathrm{Li}_2\Big(\frac{1-y}{2}+\frac{y}{1-\sqrt{\eta}}+i\Big(\frac{1}{1-\sqrt{\eta}}-\frac{1-y}{2}\Big)\Big)\\
&+\mathrm{Li}_2\Big(\frac{1-y}{2}+\frac{y}{1+\sqrt{\eta}}+i\Big(\frac{1}{1+\sqrt{\eta}}-\frac{1-y}{2}\Big)\Big)\Bigg\}
\end{aligned}
\tag{B.37}
$$

$$
\begin{aligned}
g_9(x) &= \int_0^y dz \frac{\arctan(z)}{\eta(1-z)^2+(1+z)^2} \\
&= \Biggl\{ -\frac{\pi^2}{32} + \frac{1}{8}\pi \arctan\left(\frac{1-\sqrt{\eta}}{1+\sqrt{\eta}}\right) - \arctan\left(\frac{1-\sqrt{1+y^2}}{y}\right) \\
&\quad \times \Biggl[-\arctan\left(\frac{1+y+\sqrt{\eta}(1-y)-\sqrt{2}\sqrt{\eta(1-y)^2+(1+y)^2}}{1-\sqrt{\eta}(1-y)+y}\right) \\
&\quad + \arctan\left(\frac{1+y-\sqrt{\eta}(1-y)-\sqrt{2}\sqrt{\eta(1-y)^2+(1+y)^2}}{1+\sqrt{\eta}(1-y)+y}\right)\Biggr] \\
&\quad + \Bigl[-\ln\bigl(1-\sqrt{\eta}\bigr)+\ln\bigl(1+\sqrt{\eta}\bigr)\Bigr]\left[\frac{\ln(2)}{8}-\frac{1}{8}\ln(1+\eta)-\frac{1}{8}\ln\bigl(1+y^2\bigr)\right] \\
&\quad -\frac{1}{8}\ln^2\bigl(1-\sqrt{\eta}\bigr)+\frac{1}{8}\ln^2\bigl(1+\sqrt{\eta}\bigr) \\
&\quad -\frac{1}{8}\mathrm{Li}_2\left[-\frac{y}{1-\sqrt{\eta}}+\frac{1+y}{2}-i\left(\frac{1}{1-\sqrt{\eta}}-\frac{1-y}{2}\right)\right] \\
&\quad +\frac{1}{8}\mathrm{Li}_2\left[-\frac{y}{1+\sqrt{\eta}}+\frac{1+y}{2}-i\left(\frac{1}{1+\sqrt{\eta}}-\frac{1-y}{2}\right)\right] \\
&\quad -\frac{1}{8}\mathrm{Li}_2\left[-\frac{y}{1-\sqrt{\eta}}+\frac{1+y}{2}+i\left(\frac{1}{1-\sqrt{\eta}}-\frac{1-y}{2}\right)\right] \\
&\quad +\frac{1}{8}\mathrm{Li}_2\left[-\frac{y}{1+\sqrt{\eta}}+\frac{1+y}{2}+i\left(\frac{1}{1+\sqrt{\eta}}-\frac{1-y}{2}\right)\right]\Biggr\}\frac{1}{\sqrt{\eta}}
\end{aligned} \tag{B.38}
$$

$$
\begin{aligned}
g_{10}(x) &= \int_0^y dz \frac{z}{(1-z)^2+\eta(1+z)^2} \\
&= \frac{1}{2(1+\eta)}\Bigl[-\ln(1+\eta)+\ln\Bigl((1-y)^2+\eta(1+y)^2\Bigr)\Bigr] \\
&\quad +\frac{1-\eta}{2(1+\eta)}\left\{\arctan\left(\frac{1-\eta}{2\sqrt{\eta}}\right)-\arctan\left(\frac{1-\eta-(1+\eta)y}{2\sqrt{\eta}}\right)\right\}\frac{1}{\sqrt{\eta}}
\end{aligned} \tag{B.39}
$$

$$
\begin{aligned}
g_{11}(x) &= \int_0^y dz \frac{1}{\eta(1-z)^2+(1+z)^2} \\
&= \frac{1}{2\sqrt{\eta}}\left\{-\arctan\left(\frac{1-\eta}{2\sqrt{\eta}}\right)+\arctan\left(\frac{1+y-\eta(1-y)}{2\sqrt{\eta}}\right)\right\}
\end{aligned} \tag{B.40}
$$

$$
\begin{aligned}
g_{12}(x) &= \int_0^x dz \frac{-\ln(1-z)\ln(z)}{1-(1-\eta)z} \\
&= \frac{1}{1-\eta}\Biggl\{-\ln(1-\eta)\ln^2(1-x)+\ln^2(x)\ln\bigl[1+(-1+\eta)x\bigr]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big(-i\pi+\ln(1-\eta)\Big)\mathrm{Li}_2(x)+i\Bigg[\Big[\pi+i\ln(1-\eta)\Big]\mathrm{Li}_2(x-\eta x)\Bigg]\\
&-\mathrm{Li}_3\Big(-\frac{\eta}{(1-\eta)(1-x)}\Big)\\
&-\frac{1}{6}\ln(1-\eta)\Big[\pi^2-3i\pi\ln(1-\eta)+3\ln^2(1-\eta)-3\ln^2\big[1-(1-\eta)x\big]\Big]\\
&+\Bigg[\frac{1}{2}\ln(1-\eta)\Big[\ln(1-\eta)-2\ln\big[1-(1-\eta)x\big]\Big]\\
&+\Big[2\ln(1-\eta)-\ln(x)\Big]\ln(1-x)+\ln(x)\ln\big[1-(1-\eta)x\big]+\mathrm{Li}_2(\eta)\\
&+\mathrm{Li}_2\Big(-\frac{\eta}{(1-\eta)(1-x)}\Big)-\mathrm{Li}_2(x)-\mathrm{Li}_2\Big(-\frac{\eta x}{1-x}\Big)+\mathrm{Li}_2(x-\eta x)\Bigg]\ln(\eta)\\
&+\Bigg[i\Big[\ln(1-\eta)\big[\pi+i\ln(1-\eta)\big]\Big]-\mathrm{Li}_2\Big(-\frac{\eta}{(1-\eta)(1-x)}\Big)\\
&+\ln(1-\eta)\ln(x)-\ln^2(x)+\mathrm{Li}_2(x)+\mathrm{Li}_2\Big(-\frac{\eta x}{1-x}\Big)\\
&-\mathrm{Li}_2(x-\eta x)\Bigg]\ln(1-x)+\frac{1}{6}\ln^3(1-x)\\
&+\Bigg\{\frac{1}{6}\Bigg[\pi^2-\Big[6\ln(1-\eta)\ln\big[1-(1-\eta)x\big]\Big]-3\ln^2\big[1-(1-\eta)x\big]\\
&-6\mathrm{Li}_2\Big(\frac{1-x}{1-(1-\eta)x}\Big)+6\mathrm{Li}_2(x-\eta x)\Bigg]-\mathrm{Li}_2(x)\Bigg\}\ln(x)\\
&+\Big[i\pi-\ln(1-\eta)\Big]\mathrm{Li}_2(\eta)+\Big[i\pi-\ln(1-\eta)\Big]\mathrm{Li}_2\Big(-\frac{\eta}{(1-\eta)(1-x)}\Big)\\
&-i\pi\mathrm{Li}_2\Big(-\frac{\eta x}{1-x}\Big)+\ln(1-\eta)\mathrm{Li}_2\Big(\frac{1-x}{1-(1-\eta)x}\Big)+\mathrm{Li}_3\Big(-\frac{\eta}{1-\eta}\Big)\\
&-\mathrm{Li}_3\Big(-\frac{x}{1-x}\Big)+\mathrm{Li}_3\Big(-\frac{\eta x}{1-x}\Big)-\mathrm{Li}_3(x-\eta x)\Bigg\}\\
&+\Bigg[\frac{\ln(x)\ln(1-(1-\eta)x)}{1-\eta}+\frac{\mathrm{Li}_2(x-\eta x)}{1-\eta}\Bigg]\ln(1-x)
\end{aligned}
\tag{B.41}
$$

$$
\begin{aligned}
g_{13}(x)&=\int_0^x dz\frac{-\big(\ln(1-z)\ln(z)+\mathrm{Li}_2(z)\big)}{1-(1-\eta)z}\\
&=\frac{\ln(1-x)\ln(x)\ln\big[1-(1-\eta)x\big]}{1-\eta}+\frac{\ln\big[1-(1-\eta)x\big]\mathrm{Li}_2(x)}{1-\eta}\\
&+\frac{1}{1-\eta}\Bigg\{\ln^2(x)\ln\big[1+(-1+\eta)x\big]-\ln(1-\eta)\ln^2\big[1+(-1+\eta)x\big]\\
&+\ln\big[1-(1-\eta)x\big]\mathrm{Li}_2\Big(\frac{\eta}{1-(1-\eta)x}\Big)-\ln\big[1-(1-\eta)x\big]\mathrm{Li}_2(x-\eta x)\\
&+\Bigg[\ln(1-\eta)\ln\big[1-(1-\eta)x\big]-\mathrm{Li}_2\Big(\frac{\eta}{1-(1-\eta)x}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\Big[-2\ln(x)+2\ln\big[1-(1-\eta)x\big]\Big]\ln(1-x)+2\ln(x)\ln\big[1-(1-\eta)x\big]\\
&-2\ln^2\big[1+(-1+\eta)x\big]+\mathrm{Li}_2(\eta)-\mathrm{Li}_2(x)-\mathrm{Li}_2\left(\frac{1-x}{1-(1-\eta)x}\right)\\
&+\mathrm{Li}_2(x-\eta x)+\zeta_2\Big]\ln(\eta)+\Big[-\ln(1-x)+\ln\big[1-(1-\eta)x\big]\Big]\ln^2(\eta)\\
&+\Big[-\ln^2(x)+\ln(x)\ln\big[1-(1-\eta)x\big]-\ln^2\big[1-(1-\eta)x\big]\Big]\ln(1-x)\\
&+\Big[-2\ln^2\big[1+(-1+\eta)x\big]-\mathrm{Li}_2(x)\\
&-\mathrm{Li}_2\left(\frac{1-x}{1-(1-\eta)x}\right)+\mathrm{Li}_2(x-\eta x)+\zeta_2\Big]\ln(x)+\ln^3\big[1-(1-\eta)x\big]\\
&+\ln\big[1-(1-\eta)x\big]\mathrm{Li}_2\left(\frac{1-x}{1-(1-\eta)x}\right)-\mathrm{Li}_3(\eta)+\mathrm{Li}_3(x)\\
&+\mathrm{Li}_3\left(\frac{\eta}{1-(1-\eta)x}\right)-\mathrm{Li}_3\left(\frac{\eta x}{1-(1-\eta)x}\right)\\
&-\mathrm{Li}_3\big[1-(1-\eta)x\big]-\mathrm{Li}_3(x-\eta x)+\zeta_3\Big\}.
\end{aligned}
\tag{B.42}
$$

Appendix C. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.nuclphysb.2020.115059.

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