GLOBAL SOLUTIONS TO THE CAUCHY PROBLEM OF THE STOKES APPROXIMATION EQUATIONS FOR TWO-DIMENSIONAL COMPRESSIBLE FLOW

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Abstract

We prove the existence of global weak and classical solutions of the Stokes Approximation equations for two-dimensional compressible viscous flows. The initial data can be arbitrarily large. This is achieved by new a priori estimates for the pressure and the corresponding effective pressure which is defined to be the pressure minus the divergence of the velocity.

1 Introduction

The Navier-Stokes system for a compressible viscous fluid for the barotropic motion has the form

$$(1.1)\qquad \begin{aligned} &\rho(u_t + u\cdot\nabla u) = \mu\triangle u + (\mu+\lambda)\nabla(\operatorname{div} u) - \nabla P,\\ &\rho_t + \operatorname{div}(\rho u) = 0, \quad P = P(\rho), \quad x\in R^n, t\geq 0. \end{aligned}$$

Here, the unknowns are the density $\rho = \rho(x,t) \geq 0$ and velocity $u = u(x,t) \in R^n$, $x \in R^n$, $t \geq 0$, while P is the pressure governed by the equation of state $P = \rho^\gamma$, $\gamma > 1$; μ and λ are viscosity constant, $\mu > 0$, $3\lambda + 2\mu \geq 0$. div and ∇ are the usual spatial divergence and gradient and $\triangle$ is the Laplace operator.

The system (1.1) has been investigated most thoroughly for one-dimensional flows with plane waves [4]–[6]. For multi-dimensional flows, local existence theorems are known for solutions with arbitrary norm and the global existence is established for solutions close to equilibrium states [2], [3], [11]. Recently in [12], V.A.Vaigant and A.V.Kazhikhov proved the global existence to the periodic initial-boundary value problem with arbitrary initial data for two-dimensional flows in the case of $\lambda = \rho^\beta$, $\beta \geq 3$. With a similar growth condition on $P(\rho)$, P.L. Lions also obtained global weak solutions in [9], but the detail of the proof is not given there.

On the other hand, approaches to the Navier-Stokes problem have been intensively sought with simple hydrodynamic models. One of the best-known simplifications of the Navier-Stokes System is the Stokes approximation

$$(1.2)\qquad \begin{aligned} &\bar\rho u_t = \mu\triangle u + (\mu+\lambda)\nabla(\operatorname{div} u) - \nabla P,\\ &\rho_t + \operatorname{div}(\rho u) = 0, \quad P = P(\rho), \end{aligned} \qquad x\in R^n, t\geq 0,$$

where $\bar\rho = const. > 0$ is the mean density. This is a good approximations for strongly viscous fluids. For the system (1.2) the global existence is known only for the periodic boundary condition [12], and its proof does not apply to the Cauchy problem because the natural solutions are then such that $\rho(x) \to \bar\rho$ as $|x| \to \infty$ so that ρ and P cannot be in L^1. In this paper we prove the global existence of such weak and classical solutions to the Cauchy problem in R^2 with large initial data. The estimate which replaces the L^1 estimate of the pressure available in the periodic case is a bound of the pressure by the corresponding effective pressure *.

For simplicity, we take $\bar{\rho} = 1, \lambda = 0$, and study the system,

$$u_t = \triangle u - \bigtriangledown P, \quad P = P(\rho) = \rho^\gamma, \qquad (1.3)$$
$$\rho_t + \operatorname{div}(\rho u) = 0, \quad x \in R^2, t > 0,$$

under the condition

$$\rho \to 1, \quad v \to 0 \quad \text{as } |x| \to \infty, \qquad (1.4)$$

and with the initial data

$$u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), \quad x \in R^2. \qquad (1.5)$$

To state our result more precisely, we need the definitions of weak and classical solutions which are similar to those given in [12].

Definition 1 *A weak solution to problem (1.3),(1.5) is a pair of functions* (u,ρ) *such that* $\rho, P(\rho), u, \bigtriangledown u \in L^1_{loc}(R^2 \times (0,\infty))$ *and for all test functions* $\phi \in C_0^\infty(R^2 \times (-\infty,\infty))$,

$$\int_{R^2} \rho_0 \phi(\cdot,0)dx + \int_0^\infty \int_{R^2} (\rho\phi_t + \rho u \cdot \bigtriangledown \phi)dxdt = 0, \qquad (1.6)$$

and

$$\int_{R^2} u_0\phi(\cdot,0)dx + \int_0^\infty \int_{R^2} [u\phi_t + P(\rho)\bigtriangledown \phi + \bigtriangledown u \cdot \bigtriangledown \phi]dxdt = 0. \qquad (1.7)$$

hold good.

Definition 2 *A classical solution to problem (1.3)-(1.5) is a pair of functions* (u,ρ), $u \in C^2(R^2 \times (0,\infty))$, $\rho \in C^1(R^2 \times (0,\infty))$, *such that (1.3),(1.5) is satisfied everywhere in* $(R^2 \times (0,\infty))$.

As stated above,we shall look for a solution such that $P(\rho) \to P(\bar{\rho}) = 1$ as $|x| \to \infty$. To this end we need

$$\psi_\alpha(\rho) = \rho \int_1^\rho \frac{1}{s^2} \frac{|s^\gamma - 1|^\alpha}{s^\gamma - 1} ds. \qquad (1.8)$$

*After the submission of the present paper, P.L. Lions kindly informed us about his book [10] which was just in press. There,the Cauchy problem for (1.2) is also solved, but under the conditioin $\rho \to 0$ ($|x| \to \infty$). Apparently, this case can be solved also by our method replacing the function $\psi_\alpha(\rho)$ defined in (1.8) below by $P^{\alpha-1}(\rho)$, and actually, the remaining part of the proof is very similar to each other.

Note that $\psi_\alpha(\rho) > 0$ for all $\rho > 0$, $\psi_\alpha(\rho) \sim 1$ for small ρ and $\psi_\alpha(\rho) \sim \frac{1}{\rho}P(\rho)^\alpha$ for large ρ. Moreover, it holds that

$$\rho\psi'_\alpha(\rho) - \psi_\alpha(\rho) = \frac{|\rho^\gamma - 1|^\alpha}{\rho^\gamma - 1}. \tag{1.9}$$

Our main results are

Theorem 1 *Suppose the initial data $u_0 \in W^{1,q}(R^2)$ for any $q \in [2,4]$ and $\psi_\alpha(\rho_0) \in L^1(R^2)$ for any $\alpha \in [2,4]$, and suppose $P(\rho_0) - 1 \in L^2(R^2)$, $0 < \rho_1 \le \rho_0(x) \le \rho_2 < \infty$ where ρ_1, ρ_2 are constants. Then the Cauchy problem for (1.3), (1.5) with the initial data (ρ_0, u_0) has a global weak solution (ρ, u) and for any fixed $T > 0$*

$$u \in L^\infty(0,T;W^{1,2}(R^2)) \cap L^q(0,T;W^{1,q}(R^2)), \quad \textit{for any } q \in [2,4] \tag{1.10}$$

$$\psi_\alpha(\rho) \in L^\infty(0,T;L^1(R^2)), \quad \textit{for any } \alpha \in [2,4] \tag{1.11}$$

$$P(\rho) - 1 \in L^q(0,T;L^q(R^2)), \quad \textit{for any } q \in [2,4], \tag{1.12}$$

and there are positive constants $\underline{M}, \overline{M}$ such that $\underline{M} \le \rho(x,t) \le \overline{M}$.

Theorem 2 *Suppose the initial data (u_0, ρ_0) satisfy the conditions of Theorem 1 and suppose further $u_0 \in W^{l+1,q}(R^2)$, $P(\rho_0) - 1 \in W^{l,q}(R^2)$, for some $q \ge 2, l \ge 2$. Then (ρ, u) of Theorem 1 is a classical solution with*

$$\frac{\partial^k P(\rho)}{\partial t^k} \in L^\infty(0,T;W^{l-k,q}(R^2)), \tag{1.13}$$

$$\frac{\partial^k u}{\partial t^k} \in L^q(0,T;W^{l-k+1,q}(R^2)), \tag{1.14}$$

for any $0 \le k \le l$.

2 A priori estimates

In this section we derive L^p estimates for smooth solutions to the problem (1.3) and (1.5). These estimates will be used to construct both classical and weak solutions. The first energy estimate can be obtained by multiplying the first equation in system (1.3) by u and the second equation by $\psi'_2(\rho)$, followed by integrating over R^2:

$$\frac{d}{dt}\int_{R^2}[\frac{1}{2}|u(x,t)|^2 + \psi_2(\rho(x,t))]dx + \int_{R^2}|\nabla u(x,t)|^2 dx = 0. \tag{2.1}$$

In the following lemma, which is a key ingredient in obtaining a priori estimates, we derive a bound for the pressure $P-1$ in term of the bound of the effective pressure B in $L^\alpha(R^2\times[0,T])$. where B is defined in the form ([2],[13])

$$B = P - 1 - \operatorname{div} u. \tag{2.2}$$

Lemma 1 *There exists a constant $C>0$ and*

$$\begin{aligned}&\int_{R^2}\psi_\alpha(\rho(x,t))dx + \int_0^t \|P(\tau)-1\|^\alpha_{L^\alpha(R^2)}d\tau\\ &\le C[\int_{R^2}\psi_\alpha(\rho_0)dx + \int_0^t \|B\|^\alpha_{L^\alpha(R^2)}d\tau]\end{aligned} \tag{2.3}$$

holds for any exponent $\alpha>1$ and any $t\in[0,T]$.

Proof. For $\alpha>1$, we multiply the second equation in system (1.3) by $\psi'_\alpha(\rho)$ to obtain

$$\frac{\partial}{\partial t}\psi_\alpha + \operatorname{div}(u\psi_\alpha) + \frac{|P-1|^\alpha}{P-1}\operatorname{div} u = 0. \tag{2.4}$$

Since $\operatorname{div} u = P-1-B$, integrating over $[0,t]\times R^2$, we obtain

$$\begin{aligned}&\int_{R^2}\psi_\alpha(\rho(x,t))dx + \int_0^t\int_{R^2}|P-1|^\alpha dx\\ &\le \int_{R^2}\psi_\alpha(\rho(x,0))dx + \int_0^t\int_{R^2}|P-1|^{\alpha-1}|B|dxd\tau.\end{aligned} \tag{2.5}$$

According to the Young inequality $a^{\alpha-1}b\le \varepsilon a^\alpha + C_\varepsilon b^\alpha$ we obtain (2.3).

The main result of this section is the following.

Lemma 2 *If $u_0\in W^{1,2}(R^2)$, $\psi_\alpha(\rho_0)\in L^1(R^2)$ for $\forall\alpha\in[2,4]$ and $B(\cdot,0)\in L^2(R^2)$, then there is a constant $C(T)>0$ depending only on these norms and $T>0$ such that*

$$\begin{aligned}&\sup_{0\le t\le T}(\|B\|^2_{L^2(R^2)} + \|\operatorname{rot} u\|^2_{L^2(R^2)} + \|u\|^4_{L^4(R^2)})\\ &+\int_0^T(\|\operatorname{rot}\operatorname{rot} u\|^2_{L^2(R^2)} + \||u|\nabla u\|^2_{L^2(R^2)} + \|\nabla B\|^2_{L^2(R^2)})dt\\ &\le C(T),\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\int_{R^2}\psi_\alpha(\rho)dx + \int_0^T(\|B\|^\alpha_{L^\alpha(R^2)} + \|P(\rho)-1\|^\alpha_{L^\alpha(R^2)} + \|\nabla u\|^\alpha_{L^\alpha(R^2)})dt\\ &\le C(T).\end{aligned} \tag{2.7}$$

$$\int_0^T \|u\|^4_{L^\infty(R^2)} \le C(T). \tag{2.8}$$

Here (2.7) holds for any $\alpha\in[2,4]$.

Proof. We shall make repeated use of various standard Sobolev inequalities and embedding theorems. The most basic of those is the bound

$$\int_{R^n} |w|^{\frac{n}{n-1}} dx \leq C(n)\Big(\int_{R^n} |\bigtriangledown w| dx\Big)^{\frac{n}{n-1}} \tag{2.9}$$

for $w \in W^{1,1}(R^n)$ (see [14]). Applying Hölder's inequality in an elementary way, we then easily derive from (2.9) the estimates

$$\Big(\int_{R^2} |w|^q dx\Big)^{\frac{1}{2}} \leq C\Big(\int_{R^2} |w|^{q-2}\Big)^{\frac{1}{2}}\Big(\int_{R^2} |\bigtriangledown w|^2 dx\Big)^{\frac{1}{2}} \tag{2.10}$$

for $q \geq 2$ when $n = 2$. (2.10) shows that $u_0 \in L^4(R^2)$ if $u_0 \in W^{1,2}(R^2)$.

Since $\triangle u = \bigtriangledown \operatorname{div} u - \operatorname{rot}\operatorname{rot} u$, the first equation of (1.3) can be rewritten as

$$u_t + \operatorname{rot}\operatorname{rot} u = -\bigtriangledown B. \tag{2.11}$$

Multiplied by u_t and integrated by parts over R^2, this gives

$$\frac{1}{2}\frac{d}{dt}\int_{R^2} |\operatorname{rot} u|^2 dx + \int_{R^2} |u_t|^2 dx = \int_{R^2} B \operatorname{div} u_t dx. \tag{2.12}$$

On the other hand, since rot and $\bigtriangledown$ are orthogonal, (2.11) also gives

$$\int_{R^2} |u_t|^2 dx = \int_{R^2} |\bigtriangledown B|^2 dx + \int_{R^2} |\operatorname{rot}\operatorname{rot} u|^2 dx.$$

Finally, since $\operatorname{div} u_t = P_t - B_t$, we get

$$\int_{R^2} B \operatorname{div} u_t dx = -\frac{1}{2}\frac{d}{dt}\int_{R^2} |B|^2 + \int_{R^2} B P_t dx.$$

Now we can write (2.12) as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{R^2}(|B|^2 + |\operatorname{rot} u|^2)dx + \int_{R^2}(|\bigtriangledown B|^2 + |\operatorname{rot}\operatorname{rot} u|^2)dx \\ &= \int_{R^2}[Pu \bigtriangledown B - (\gamma - 1)PB \operatorname{div} u]dx. \end{aligned} \tag{2.13}$$

Here we have used the equality

$$P_t + \operatorname{div}(Pu) + (\gamma - 1)P \operatorname{div} u = 0, \tag{2.14}$$

which comes from the second equation of (1.3) with $P(\rho) = \rho^\gamma$.

Next we multiply the first equation in (1.3) by $u|u|^{q-2}$ for an arbitrary finite $q \geq 2$ and integrate over R^2 to obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^2}|u|^q dx + \int_{R^2}(q|u|^{q-2}|\nabla u|^2 + q(q-2)|u|^{q-2}|\nabla|u||^2)dx \\
&= q\int_{R^2}(P-1)\operatorname{div}(u|u|^{q-2})dx.
\end{aligned} \tag{2.15}
$$

Consequently, taking $q = 4$ in (2.15) and adding (2.13) to (2.15) we arrive at the inequality

$$
\begin{aligned}
&\frac{d}{dt}\int_{R^2}(|B|^2 + |\operatorname{rot} u|^2 + |u|^4)dx \\
&+\int_{R^2}(|\nabla B|^2 + |\operatorname{rot}\operatorname{rot} u|^2 + |u|^2|\nabla u|^2)dx \\
&\leq C_1\int_{R^2}(|P-1||u|^2|\nabla u| + P|u||\nabla B| + P|\nabla u||B|)dx.
\end{aligned} \tag{2.16}
$$

We write the right-hand side of (2.16) as $C_1(I_1+I_2+I_3)$. Applying the Schwartz inequality, I_1 can be estimated as

$$
\begin{aligned}
I_1 &\equiv \int_{R^2}|P-1||u|^2|\nabla u|dx \\
&\leq \int_{R^2}(|B| + |\operatorname{div} u|)|u|^2|\nabla u|dx \\
&\leq C\int_{R^2}(|B|^2|u|^2 + |\nabla u|^2|u|^2)dx.
\end{aligned} \tag{2.17}
$$

It follows from the embedding inequality (2.10) with $q = 4$ that

$$
\begin{aligned}
\int_{R^2}|B|^2|u|^2dx &\leq \|B\|_{L^4}^2\|u\|_{L^4}^2 \leq C\|B\|_{L^2}\|\nabla B\|_{L^2}\|u\|_{L^2}\|\nabla u\|_{L^2}, \\
\int_{R^2}|u|^2|\nabla u|^2dx &\leq \||u|^2\|_{L^4}\|\nabla u\|_{L^2}\|\nabla u\|_{L^4} \leq C\||u|^2\|_{L^2}^{\frac{1}{2}}\|\nabla|u|^2\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}.
\end{aligned}
$$

By virtue of (2.1), we have $\|u\|_{L^2(R^2)} \leq C$. Then we can bound the right side of (2.17) by

$$
\begin{aligned}
I_1 &\leq \varepsilon_1\|\nabla B\|_{L^2}^2 + \varepsilon_2\||u||\nabla u|\|_{L^2}^2 + \varepsilon_3\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \\
&+C(\|\nabla u\|_{L^2}^2\|B\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|u\|_{L^4}^4),
\end{aligned} \tag{2.18}
$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are arbitrary positive numbers and $C > 0$ is a constant depending only on them. The second term can be estimated similarly as

$$
\begin{aligned}
I_2 &\leq \textstyle\int_{R^2}(|B| + |\nabla u| + 1)|u||\nabla B|dx \\
&\leq \varepsilon_1\|\nabla B\|_{L^2}^2 + C(\||B||u|\|_{L^2}^2 + \||u||\nabla u|\|_{L^2}^2 + \|u\|_{L^2}^2) \\
&\leq \varepsilon_1\|\nabla B\|_{L^2}^2 + \varepsilon_2\||u||\nabla u|\|_{L^2}^2 + \varepsilon_3\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \\
&+C(\|\nabla u\|_{L^2}^2\|B\|_{L^2}^2 + \|\nabla u\|_{L^2}^2\|u\|_{L^4}^4 + 1).
\end{aligned} \tag{2.19}
$$

For the third term we have

$$
(2.20)\qquad \begin{aligned} I_3 &\le \int_{R^2}(|\nabla u|+|B|+1)|\nabla u||B|dx \\ &\le \|B\|_{L^4}\|\nabla u\|_{L^4}\|\nabla u\|_{L^2}+\|B\|_{L^4}^2\|\nabla u\|_{L^2} \\ &\quad +\|\nabla u\|_{L^2}^2+\|B\|_{L^2}^2. \end{aligned}
$$

Note that

$$
(2.21)\qquad \|B\|_{L^4}\|\nabla u\|_{L^4}\|\nabla u\|_{L^2}\le \varepsilon_3\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2+C\|B\|_{L^4}^2\|\nabla u\|_{L^2},
$$

and $\|B\|_{L^4}^2\|\nabla u\|_{L^2}\le C\|\nabla B\|_{L^2}\|B\|_{L^2}\|\nabla u\|_{L^2}$, we then easily derive the estimates of I_3 as follows:

$$
(2.22)\qquad \begin{aligned} I_3 &\le \varepsilon_3\|\nabla u\|_{L^4}^2\|\nabla u\|_{L^2}+\varepsilon_1\|\nabla B\|_{L^2}^2 \\ &\quad +C(\|\nabla u\|_{L^2}^2+1)\|B\|_{L^2}^2+C\|\nabla u\|_{L^2}^2. \end{aligned}
$$

Let us take $\varepsilon_1\le\frac{1}{6C_1},\varepsilon_2\le\frac{1}{4C_1}$, and integrate (2.16) over $[0,t]$ with the right-hand side estimated by (2.18), (2.19) and (2.22). Then we obtain

$$
(2.23)\qquad \begin{aligned} &\textstyle\int_{R^2}(|B|^2+|\operatorname{rot}u|^2+|u|^4)dx \\ &\textstyle+\int_0^t\int_{R_2}(|\nabla B|^2+|\operatorname{rot}\operatorname{rot}u|^2+|u|^2|\nabla u|^2)dx \\ &\textstyle\le\varepsilon\int_0^t\|\nabla u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\textstyle+C\int_0^t(1+\|\nabla u\|_{L^2}^2)(\|B\|_{L^2}^2+\|u\|_{L^4}^4)+C \end{aligned}
$$

for any $\varepsilon=\frac{1}{6C_1}\varepsilon_3>0$. Set $\Lambda(t)=\|\nabla u\|_{L^2}^2$. From the energy equality (2.1), we have

$$
(2.24)\qquad \int_0^t\Lambda(\tau)d\tau\le C.
$$

The function $y(t)$ and $z(t)$ are defined as

$$
(2.25)\qquad y(t)=\sup_{0<\tau<t}(\|B(\cdot,\tau)\|_{L^2(R^2)}^2+\|\operatorname{rot}u(\cdot,\tau)\|_{L^2(R^2)}^2+\|u(\cdot,\tau)\|_{L^4(R^2)}^4),
$$

$$
(2.26)\qquad \begin{aligned} z(t) &=\|\nabla B(\cdot,t)\|_{L(R^2)}^2+\|\operatorname{rot}\operatorname{rot}u(\cdot,t)\|_{L^2(R^2)}^2 \\ &\quad+\||u(\cdot,t)||\nabla u(\cdot,t)|\|_{L^2(R^2)}^2. \end{aligned}
$$

Then inequality (2.23) can be strengthened as follows:

$$
(2.27)\qquad \begin{aligned} y(t)+\textstyle\int_0^t z(\tau)d\tau &\le\varepsilon\textstyle\int_0^t\Lambda(\tau)^{\frac12}\|\nabla u\|_{L^4}^2d\tau \\ &\quad+C\textstyle\int_0^t(\Lambda(\tau)+1)y(\tau)d\tau+C. \end{aligned}
$$

Applying the Schwartz inequality to the first term on the right-hand side we have

$$
\begin{aligned}
&\int_0^t \Lambda^{\frac{1}{2}}(\tau)\| \nabla u\|_{L^4}^2 d\tau \\
&\le (\int_0^t \Lambda(\tau)d\tau)^{\frac{1}{2}}(\int_0^t \| \nabla u\|_{L^4}^4 d\tau)^{\frac{1}{2}} \\
&\le C\| \nabla u\|^2_{L^4((0,t)\times R^2)}.
\end{aligned} \tag{2.28}
$$

Let us show that

$$
\| \nabla u\|_{L^4(R^2\times[0,T])} \le C(\|P-1\|_{L^4(R^2\times[0,T])}+1). \tag{2.29}
$$

Define φ and $\tilde{u}$ by

$$
\varphi_t - \triangle \varphi = P-1, \quad \varphi(x,0)=0, \tag{2.30}
$$

$$
\tilde{u}_t - \triangle \tilde{u} = 0, \quad \tilde{u}(x,0)=u_0(x). \tag{2.31}
$$

Then,

$$
u = -\nabla \varphi + \tilde{u}, \tag{2.32}
$$

and by the well-known parabolic estimate, [8],

$$
\begin{aligned}
&\|\varphi_t\|_{L^\beta(R^2\times[0,T])} + \|\nabla^2\varphi\|_{L^\beta(R^2\times[0,T])} \\
&\le C\|P-1\|_{L^\beta(R^2\times[0,T])}, \quad 1<\beta<\infty,
\end{aligned} \tag{2.33}
$$

while it is easy to see that

$$
\| \nabla \tilde{u}\|_{L^4(R^2\times[0,T])} \le T^{\frac{1}{4}}\| \nabla u_0\|_{L^4(R^2])}. \tag{2.34}
$$

Thus (2.29) follows with $\beta = 4$ in (2.33) and (2.34). Taking into account Lemma 1, we conclude that

$$
\begin{aligned}
\int_0^t \Lambda^{\frac{1}{2}}(\tau)\| \nabla u\|_{L^4}^2 d\tau &\le C(\int_0^t \| \nabla u\|_{L^4}^4 d\tau)^{\frac{1}{2}} \le C(\int_0^t \|P-1\|_{L^4}^4)^{\frac{1}{2}} + C \\
&\le \; C(\int_0^t \|B\|_{L^4}^4 d\tau)^{\frac{1}{2}} + C \le C(\int_0^t \|B\|_{L^2}^2\| \nabla B\|_{L^2}^2 d\tau)^{\frac{1}{2}} + C.
\end{aligned}
$$

Finally, we find that

$$
(\int_0^t \|B\|_{L^2}^2\| \nabla B\|_{L^2}^2 d\tau)^{\frac{1}{2}} \le y(t)^{\frac{1}{2}}(\int_0^t z(\tau)d\tau)^{\frac{1}{2}} \le y(t) + \int_0^t z(\tau)d\tau \tag{2.35}
$$

Here we used the definitions (2.25) (2.26) and took into account the fact that $y(t)$ is a monotone nondecreasing function. Then, by choosing ε sufficiently small, (2.27) obviously implies the inequality

$$y(t) + \int_0^t z(\tau)d\tau \leq C\int_0^t (\Lambda(\tau)+1)y(\tau)d\tau + C. \tag{2.36}$$

Using the usual Gronwall inequality, we obtain that for any fixed time T, $y(t)$ and $\int_0^t z(\tau)d\tau$ are bounded in [0,T]. Now we have

$$\begin{aligned}&\sup_{0<\tau<T}(\|B\|_{L^2}^2 + \|\operatorname{rot} u\|_{L^2}^2 + \|u\|_{L^4}^4)\\ &\quad + \int_0^T (\|\operatorname{rot}\operatorname{rot} u\|_{L^2} + \||u|\bigtriangledown u\|_{L^2}^2 + \|\bigtriangledown B\|_{L^2}^2)d\tau\\ &\leq C(T),\end{aligned} \tag{2.37}$$

and

$$\int_0^t\int_{R^2} |B|^q dx d\tau \leq \int_0^t \|B\|_{L^2}^{q-2}\|\bigtriangledown B\|_{L^2}^2 d\tau \leq C(T), \quad \forall q \in [2,4]. \tag{2.38}$$

Hence, it follows from Lemma 1 that

$$\int_0^t\int_{R^2} (|P(\rho)-1)|^q + |\bigtriangledown u|^q)dxdt \leq C(T), \quad \forall q\in[2,4]. \tag{2.39}$$

The inequality (2.8) can be obtained by using the embedding inequality

$$\|w\|_{L^\infty(R^n)} \leq C\|w\|_{W^{1,q}(R^n)} \quad q>2. \tag{2.40}$$

Now we complete the proof of Lemma 2.

3 Upper and lower bounds for the density

The estimates obtained in the preceding section permit us to establish that the density ρ is bounded from below and above.

Lemma 3 *If $u_0 \in W_q^1(R^2)$ for any $q \in [2,4]$, $\psi_\alpha(\rho_0) \in L^1(R^2)$ for any $\alpha \in [2,4]$, $B(\cdot,0) \in L^2(R^2)$, and $0<\rho_1 \leq \rho_0(x) \leq \rho_2 < \infty$ with some constants ρ_1, ρ_2, then for any fixed T, there exist positive constants $\underline{M}(T)$ and $\overline{M}(T)$ such that*

$$0 < \underline{M}(T) \leq \rho(x,t) \leq \overline{M}(T) < \infty, \quad (x,t)\in R^2\times[0;T]. \tag{3.1}$$

Proof. Introduce the Eulerian coordinates (x,t) which are related to the Lagrangean ones (y,t) through the relations

$$x(y,\tau)=y+\int_0^\tau u(x(y,s),s)ds. \tag{3.2}$$

Then we can rewrite (2.14) into the form

$$P_t+u\cdot\nabla P+\gamma P^2-\gamma P(B+1)=0, \tag{3.3}$$

$$\frac{d}{dt}\frac{1}{P}+\gamma\frac{1}{P}(B+1)=\gamma, \tag{3.4}$$

where $\frac{d}{dt}=\frac{\partial}{\partial t}+u\cdot\nabla$ is the material derivative. Solving this differential equation gives

$$P(x(y,t),t)=\frac{P(\rho_0(y))\exp\{\gamma\int_0^t(B(x(y,\tau),\tau)+1)d\tau\}}{1+\gamma P(\rho_0(y))\int_0^t(\exp\{\gamma\int_0^\tau(B(x(y,s),s)+1)ds\})d\tau}. \tag{3.5}$$

Clearly, the lemma follows if it holds that

$$\left|\int_0^t B(x(y,\tau),\tau)d\tau\right|\le\bar{B}(T), \tag{3.6}$$

for any $t\in[0,T]$ and $y\in R^2$ with some constant $\bar{B}(T)>0$. To prove this, let φ and $\tilde{u}$ be as in (2.30) and (2.31), and notice that

$$\begin{aligned} B &= P-1-\operatorname{div}u=P-1+\Delta\varphi-\operatorname{div}\tilde{u}=\varphi_t-\operatorname{div}\tilde{u}\\ &= \varphi_t+u\cdot\nabla\varphi-u\cdot\nabla\varphi-\operatorname{div}\tilde{u}\\ &= \frac{d}{dt}\varphi+|\nabla\varphi|^2-\tilde{u}\cdot\nabla\varphi-\operatorname{div}\tilde{u}. \end{aligned} \tag{3.7}$$

Integrate this along the Lagrangean coordinate, and since $\varphi(x,0)=0$, we have

$$\begin{aligned} \int_0^t B(x(y,\tau),\tau)d\tau &= \varphi(x(y,t),t)+\int_0^t(|\nabla\varphi(x(y,\tau),\tau)|^2\\ &- \tilde{u}(x(y,\tau),\tau)\cdot\nabla\varphi(x(y,\tau),\tau)\\ &- \operatorname{div}\tilde{u}(x(y,\tau),\tau))dt. \end{aligned} \tag{3.8}$$

Hence,

$$\begin{aligned} &\left|\int_0^t B(x(y,\tau),\tau)d\tau\right|\le\|\varphi\|_{L^\infty(R^2\times(0,t))}\\ &\quad+C\int_0^t(\|\nabla\varphi\|^2_{L^\infty}+\|\tilde{u}\|^2_{L^\infty}+\|\nabla\tilde{u}\|_{L^\infty})d\tau. \end{aligned} \tag{3.9}$$

We shall estimate the right hand side. First, by the Sobolev theorem

$$
\begin{aligned}
\|\varphi\|_{L^\infty(R^2\times(0,t))} &\leq C \sup_{\tau\in(0,t)} (\|\varphi(\cdot,\tau)\|_{L^4} + \|\nabla\varphi(\cdot,\tau)\|_{L^4}) \\
&\leq C \sup_{\tau\in(0,t)} (\|\varphi(\cdot,\tau)\|_{L^4} + \|u(\cdot,\tau)\|_{L^4} \\
&\quad +\|\tilde{u}(\cdot,\tau)\|_{L^4}),
\end{aligned}
\tag{3.10}
$$

because $\nabla\varphi = -u + \tilde{u}$. Due to (2.33) and Lemmas 1 and 2, we get

$$
\|\varphi(\cdot,t)\|_{L^4} \leq \int_0^t \|\varphi_t(\cdot,\tau)\|_{L^4} d\tau \leq t^{\frac{3}{4}} \|\varphi_t\|_{L^4(R^2\times(0,t))} \leq Ct^{\frac{3}{4}},
\tag{3.11}
$$

$$
\|u(\cdot,t)\|_{L^4(R^2)} \leq C,
\tag{3.12}
$$

and clearly,

$$
\|\tilde{u}\|_{L^4} \leq \|u_0\|_{L^4}.
\tag{3.13}
$$

Therefore,

$$
\|\varphi\|_{L^\infty(R^2\times(0,t))} \leq C(t).
\tag{3.14}
$$

On the other hand, again by the Sobolev theorem,

$$
\int_0^t \|\nabla\varphi(\cdot,\tau)\|^2_{L^\infty} d\tau \leq C \int_0^t (\|\nabla\varphi(\cdot,\tau)\|^2_{L^4} + \|\nabla^2\varphi(\cdot,\tau)\|^2_{L^4}) d\tau.
\tag{3.15}
$$

The last term can be estimated by (2.33) combined with Lemma 1 and 2, while

$$
\int_0^t \|\nabla\varphi(\cdot,\tau)\|^2_{L^4} \leq C \int_0^t (\|u(\cdot,\tau)\|^2_{L^4} + \|\tilde{u}(\cdot,\tau)\|^2_{L^4}) d\tau \leq C(T)
\tag{3.16}
$$

by (3.12) and (3.13). Finally,

$$
\int_0^t \|\tilde{u}(\cdot,\tau)\|^2_{L^\infty} d\tau \leq \int_0^t \|u_0\|^2_{L^\infty} d\tau \leq Ct\|u_0\|^2_{W^{1,4}},
\tag{3.17}
$$

and using the well-known $L^p - L^q$ decay estimate of the solution of the heat equation [8],

$$
\|\tilde{u}(\cdot,\tau)\|_{L^p(R^n)} \leq Ct^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|u_0\|_{L^q(R^n)}, \quad 1 \leq p \leq q \leq \infty,
\tag{3.18}
$$

we have

$$
\int_0^t \|\nabla\tilde{u}(\cdot,\tau)\|_{L^\infty(R^2)} d\tau \leq C \int_0^t \tau^{-\frac{1}{4}} \|\nabla u_0\|_{L^4(R^2)} d\tau.
\tag{3.19}
$$

Combining these estimates proves (3.6), and hence substantiates the lemma.

4 Existence of classical solutions

In this section we apply the a priori estimates of Sections 2-3 to complete the proof of Theorem 2 stated in Section 1. To estimate higher derivatives of u and P, we need the following estimate. A Similar estimate was stated in [12], but the proof is not found there. We gives a slightly different version of it.

Lemma 4 *For any $q > 2$ we have*

$$\tag{4.1} \begin{aligned} &\| \nabla u(x,t)\|_{L^\infty(R^2\times[0,T])} \\ &\leq C\|P\|_{L^\infty(R^2\times[0,T])}\ln(1+\| \nabla P\|_{L^\infty(0,T;L^q(R^2))}) + C, \end{aligned}$$

where $C > 0$ is a constant depending only on q.

Proof. From the first equation in (1.3) we have

$$\tag{4.2} \begin{aligned} u(x,t) = & \int_0^t \int_{R^2} K(x-\xi,t-\tau)\nabla_\xi P(\xi,\tau)d\xi d\tau \\ & +\int_{R^2} K(x-\xi,t)u_0(\xi)d\xi, \end{aligned}$$

where $K(x,t)$ is the fundamental solution of $\frac{\partial}{\partial t} - \Delta$ in $R^2 \times [0,\infty)$ given by

$$\tag{4.3} K(x,t) = \frac{1}{2\pi t}\exp\{-\frac{|x|^2}{t}\}.$$

Let $\Sigma_d = \{\xi \in R^2; \frac{1}{d} \leq |x-\xi| \leq d, d \geq 1\}$ and from (4.2) we have

$$\tag{4.4} \begin{aligned} |\nabla u(x,t)| \leq & \int_0^t \int_{R^2\setminus\Sigma_d} |\nabla_x^2 K(x-\xi,t-\tau)(P(\xi,\tau)-P(x,\tau))|d\xi d\tau \\ + & \int_0^t \int_{\Sigma_d} |\nabla_x^2 K(x-\xi,t-\tau)(P(\xi,\tau)-P(x,\tau))|d\xi d\tau \\ + & \int_{R^2} |K(x-\xi,t)\nabla u_0(\xi)|d\xi. \end{aligned}$$

We employ the standard notation for Hölder norms,

$$\tag{4.5} \langle w\rangle_\alpha = \sup_{\substack{x,y\in R^2\\ x\neq y}} \frac{|w(x)-w(y)|}{|x-y|}$$

for functions $w : R^2 \to R$. Noting that

$$\tag{4.6} |\nabla^2 K(x,t)| \leq Ct^{-2}\exp\{-C\frac{|x|^2}{t}\},$$

we get the following estimates.

$$
\begin{aligned}
\int_0^t \int_{R^2\setminus\Sigma_d} &\le C\int_0^t \int_{R^2\setminus\Sigma_d} \frac{|x-\xi|^\alpha}{(t-\tau)^2} \exp\{-C\frac{|x-\xi|^2}{t-\tau}\}\langle P(\cdot,\tau)\rangle_\alpha d\xi d\tau \\
&\le C\{\int_0^t\int_0^{\frac{1}{d}} + \int_0^t\int_d^\infty\}\frac{r^{\alpha+1}}{\tau^2}\exp\{-C\frac{r^2}{\tau}\}drd\tau \sup_{0\le\tau\le t}\langle P(\cdot,\tau)\rangle_\alpha \\
&\le C(1+t^\alpha)\sup_{0\le\tau\le t}\langle P(\cdot,\tau\rangle_\alpha d^{-\alpha}, \quad \forall 0<\alpha<1,
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
\int_0^t \int_{\Sigma_d} &\le C\int_0^t \int_{\Sigma_d} \frac{1}{(t-\tau)^2}\exp\{-C\frac{|x-\xi|^2}{t-\tau}\}\|P\|_{L^\infty(R^2\times[0,t])}d\xi d\tau \\
&\le C\int_0^t\int_{\frac{1}{d}}^d \frac{r}{\tau^2}\exp\{-C\frac{r^2}{\tau}\}drd\tau\|P\|_{L^\infty(R^2\times[0,t])} \\
&\le C\|P\|_{L^\infty(R^2\times[0,t])}\ln d.
\end{aligned} \tag{4.8}
$$

The conclusion of the lemma will then follow by taking $d = \max\{1, \sup_{L^\infty(R^2\times[0,T])}\langle P(\cdot,\tau)\rangle_\alpha\}$, and using the Soblev inequality

$$
\langle P\rangle_\alpha \le C\|\nabla P\|_{L^q(R^2)}, \quad \alpha = 1-\frac{2}{p}, \quad p>2. \tag{4.9}
$$

∎

This lemma allows us to derive estimates of higher derivatives of u and ρ.

Lemma 5 *If u_0 and ρ_0 satisfy the assumption of Theorem 2 in section 1, then there is a constant $C(T)>0$ depending on these norms in the space indicated there and on $T>0$ such that*

$$
\sup_{0\le t\le T}\|\frac{\partial^k P(\rho)}{\partial t^k}\|_{W^{l-k,q}(R^2)} \le C(T) \tag{4.10}
$$

and

$$
\|\frac{\partial^k u}{\partial t^k}\|_{W^{l+1-k,q}((0,T)\times R^2)} \le C(T). \tag{4.11}
$$

Proof. Differentiation of the second equation of (1.3) with respect to the space variables x yields the equation

$$
\frac{\partial\nabla P}{\partial t} + (u\cdot\nabla)\nabla P + (\nabla P\cdot\nabla)u + \gamma\,\mathrm{div}\, u\nabla P + \gamma P\nabla\mathrm{div}\, u = 0. \tag{4.12}
$$

Multiplying this by $r|\nabla P|^{r-2}\nabla P$ for $r>2$, and integrating over R^2 give the inequality

$$
\begin{aligned}
&\tfrac{d}{dt}\|\nabla P\|^r_{L^r(R^2)} \\
&\le C(\|\nabla^2 u\|^r_{L^r(R^2)} + (\|\nabla u\|_{L^\infty(R^2\times[0,T])}+1)\|\nabla P\|^r_{L^r(R^2)}),
\end{aligned} \tag{4.13}
$$

with the help of Lemma 3. Then for the non-negative function

$$y_r(t) = \sup_{0<\tau<t} \| \nabla P\|^r_{L^r(R^2)}, \tag{4.14}$$

we obtain by integrating (4.13) over $(0, t)$,

$$y_r(t) \leq y_r(0) + C\int_0^t y_r(\tau)[1 + \ln(1 + y_r(\tau))^{\frac{1}{r}}]d\tau + C. \tag{4.15}$$

Here we used Lemma 4 and the estimate (cf, (2.33)).

$$\|u_t\|_{L^r(R^2\times(0,T))} + \|\nabla^2 u\|_{L^r(R^2\times(0,T))} \leq C + \| \nabla P\|_{L^r(R^2\times(0,T))}. \tag{4.16}$$

(4.15) gives immediately a bound for $y_r(t)$, and consequently (4.16) yields

$$\|u_t\|_{L^r(R^2\times(0,T))} + \|\nabla^2 u\|_{L^r(R^2\times(0,T))} \leq C, \tag{4.17}$$

while the second equation of (1.3) leads to

$$\sup_{0\leq\tau\leq T} \|\frac{\partial P}{\partial t}\|_{L^r(R^2)} \leq C. \tag{4.18}$$

Proceeding with further differentiations, we obtain the series of estimates (4.10) and (4.11). ∎

Lemma 6 is now sufficient to construct smooth global solutions to (1.3) for arbitrary large initial data. See [11].

5 Existence of weak solutions

For proving the existence of weak solutions, let (ρ_0, u_0) be initial data as described in Theorem 1 and $j_\delta = j_\delta(x)$ be the standard mollifier, and define

$$\rho_0^\delta = j_\delta * \rho_0, \quad u_0^\delta = j_\delta * u_0. \tag{5.1}$$

Then there is a smooth global solution (ρ^δ, u^δ) of (1.3) with initial data $(\rho_0^\delta, u_0^\delta)$, thanks to Theorem 2. The a priori estimates of lemmas 2 and 4 then apply to show the

Lemma 6 *There exist a subsequence $\delta = \delta_k \to 0$ such that:*

$$\rho^{\delta_k} \to \rho \quad \textit{weakly in } L^q((0,T)\times R^2) \quad \forall q, \quad 1\leq q<\infty$$

$$P(\rho^{\delta_k}) \to \bar{P} \quad \textit{weakly in } L^q((0,T)\times R^2) \quad \forall q, \quad 1\leq q<\infty$$

$$u^{\delta_k} \to u \quad \textit{weakly in } L^2(0,T; W^{1,2}(R^2))$$

$$B^{\delta_k} \to B \quad \textit{weakly in } L^2(0,T; W^{1,2}(R^2))$$

with some limits $\rho, \bar{P}, u, B$.

Now we show that the limit pair (ρ, u) is indeed a weak solution of (1.3) with initial data (ρ_0, u_0). First, we show that $\{u^\delta\}$ and $\{B^\delta\}$ are compact sets in the space $L^2(0,T;L^2(\Omega))$ for any bounded open set $\Omega \subset R^2$. To this end, we need

Lemma 7 *$\{u_t^\delta\}$ and $\{B_t^\delta\}$ are bounded in $L^2(0,T;W^{-1,2}(\Omega))$ for any open set Ω in R^2.*

Proof. From (1.3) we have that

$$
\begin{aligned}
|\langle u_t^\delta, \phi\rangle| &= |\langle \triangle u^\delta - \nabla P(\rho^\delta), \phi\rangle| \\
&= |\int_{R^2} (\nabla u^\delta \nabla \phi - (P(\rho^\delta) - 1)\operatorname{div}\phi) dx| \\
&\le (\| u^\delta\|_{W^{1,2}(R^2)} + \|P(\rho^\delta) - 1\|_{L^2(R^2)})\|\phi\|_{W_0^{1,2}(\Omega)}, \quad \forall \phi \in W_0^{1,2}(\Omega).
\end{aligned} \tag{5.2}
$$

In the same way, note that B satisfies

$$
B_t - \triangle B = P_t = -\operatorname{div}(Pu) - (\gamma - 1)P \operatorname{div} u, \tag{5.3}
$$

we have,

$$
\begin{aligned}
|\langle B_t^\delta, \phi\rangle| &= |\langle \triangle B^\delta - \operatorname{div}(P(\rho^\delta)u^\delta) - (\gamma-1)P(\rho^\delta)\operatorname{div} u^\delta, \phi\rangle| \\
&\le (\|B^\delta\|_{W^{1,2}(R^2)} + \gamma\|P(\rho^\delta)\|_{L^\infty(R^2)}\|u^\delta\|_{W^{1,2}(R^2)})\|\phi\|_{W_0^{1,2}(\Omega)}
\end{aligned} \tag{5.4}
$$

for any $\phi \in W_0^{1,2}(\Omega)$. Thus we have proved the lemma. ∎

On the other hand, $\{u^\delta\}$ and $\{B^\delta\}$ are bounded in $L^2(0,T;W^{1,2}(\Omega))$ by Lemma 6, and note that $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow W^{-1,2}(\Omega)$, where the symbol $\hookrightarrow$ is the continuous embedding while $\hookrightarrow\hookrightarrow$ is the continuous and compact one, so that we can apply one of celebrated compensated compactness theorems (see [7] Aubin-Simon theorem) to conclude that $\{u^\delta\}$ and $\{B^\delta\}$ are compact sets in the space $L^2(0,T;L^2(\Omega))$ for any bounded open set Ω in R^2. We can now choose a subsequence $\delta = \delta_k \to 0$ such that for any Ω,

$$
u^{\delta_k} \to u \quad \text{strongly in } L^2((0,T)\times\Omega), \tag{5.5}
$$

$$
B^{\delta_k} \to B \quad \text{strongly in } L^2((0,T)\times\Omega), \tag{5.6}
$$

$$
\rho^{\delta_k} u^{\delta_k} \to \rho u \quad \text{weakly in } L^2((0,T)\times\Omega), \tag{5.7}
$$

This means that we can go to the limit in the equation (1.6) for (ρ^δ, u^δ). Thus the limit (ρ, u) also satisfies (1.6).

Next we shall show that $P(\rho) = \bar{P}$. From the continuity equation (1.6) we have the following equality for the limit (ρ, u) (see [1] Lemma 2.1),

$$\frac{\partial P}{\partial t} + \operatorname{div}(Pu) + (\gamma - 1)P \operatorname{div} u = 0, \tag{5.8}$$

in the distribution sense. Since $B = \bar{P} - 1 - \operatorname{div} u$ holds for the limit $u, \bar{P}, B$, we have

$$\frac{\partial P}{\partial t} + \operatorname{div}(Pu) + (\gamma - 1)P(\bar{P} - 1 - B) = 0. \tag{5.9}$$

On the other hand, we have from (1.3) for any $\delta > 0$

$$\frac{\partial P(\rho^\delta)}{\partial t} + \operatorname{div}(P(\rho^\delta)u^\delta) + (\gamma - 1)P(\rho^\delta)(P(\rho^\delta) - 1 - B^\delta) = 0. \tag{5.10}$$

Passing to the limit gives the equation

$$\frac{\partial \bar{P}}{\partial t} + \operatorname{div}(\bar{P}u) - (\gamma - 1)\bar{P}B + (\gamma - 1)\overline{P^2} - (\gamma - 1)\bar{P} = 0, \tag{5.11}$$

which holds in the distribution sense. Then the difference $\Psi = \bar{P} - P$ satisfies

$$\frac{\partial \Psi}{\partial t} + \operatorname{div}(\Psi u) + (\gamma - 1)(\overline{P^2} - P^2) = (\gamma - 1)\Psi(B + 1). \tag{5.12}$$

Since for any smooth convex function $F(\rho)$, the difference $\bar{F} - F(\rho)$ is nonnegative, where ρ and $\bar{F}$ are weak limits of $\{\rho^\delta\}$ and $\{F(\rho^\delta)\}$ respectively, we can integrate (5.12) over $R^2 \times (0, t)$ to obtain the inequality

$$0 \le \int_{R^2} \Psi dx \le \int_0^t \int_{R^2} (B + 1)\Psi dx. \tag{5.13}$$

For the limit B we have

Lemma 8

$$\int_0^T \|B\|_{L^\infty(R^2)} \le C. \tag{5.14}$$

Proof We decompose (5.3) for

$$B_{1t}^\delta - \triangle B_1^\delta = 0, \quad B_1^\delta(x, 0) = P(\rho_0^\delta) - 1 - \operatorname{div} u_0^\delta, \tag{5.15}$$

$$\begin{aligned} &B_{2t}^\delta - \triangle B_2^\delta = -\operatorname{div}(P(\rho^\delta)u^\delta) - (\gamma - 1)P(\rho^\delta) \operatorname{div} u^\delta, \\ &B_2^\delta(x, 0) = 0. \end{aligned} \tag{5.16}$$

Owing to well-known estimates of solutions to the heat equation (see [8]), we have

$$\int_0^t \|B_1^\delta\|_{L^\infty(R^2)} \leq C, \tag{5.17}$$

$$\int_0^t \|B_2^\delta\|^q_{W_1^q(R^2)} \leq C \quad 2 \leq q \leq 4 \tag{5.18}$$

Then use the embedding inequality and pass to the limit to conclude (5.14). ∎

Now we arrive at the integrated inequality

$$\int_{R^2} \Psi dx \leq \int_0^t \|B(\cdot,\tau\|_{L^\infty(R^2)} \int_{R^2} \Psi(x,\tau) dx d\tau, \quad \Psi(x,0) = 0. \tag{5.19}$$

By the Gronwall inequality, we see that $\Psi = 0$ which means $\bar{P} = P(\rho)$. The existence of weak solutions is therefore established.

References

[1] R.J. DiPerna and P.-L. Lions, *"Ordinary differential equations, transport theory and Sobolev spaces."* Invent. Math. 98(1989), 511-547.

[2] D. Hoff, *"Global Solutions of the Navier-Stokes equations for Multidimensional Compressible Flow with Discontinuous Initial Data,"* J. Diff. Eq., 120(1995), 215-254.

[3] D. Hoff, *"Strong Convergence to Global Solutions for Multimensional Flows of Compressible, Viscous Fluids with Polytropic Equations of State and Discontinuous Initial Data,"* Arch. Rat. Mech. Anal. 132(1995) 1-14.

[4] N. Itaya, *"On the initial value problem of the motion of compressible viscous fluid, especially on the problem of uniqueness"* J. Math., Kyoto Univ.,16(1976), 413-427.

[5] N. Itaya, *"A survey on the generalized Burgers' equation with a pressure model term,* J. Math. Kyoto Univ., 16(1976), 1-18.

[6] A.V. Kazhikhov and V.V. Shelukhin, *"Unique global solution in time of initial boundary value problems for one-dimensional equations of a viscous gas",* Prikl. Mat. Mech., 41(1977), 282-291.

[7] A.V. Kazhikhov and V.V. Shelukhin, *"The verification compactness method,"* Actual Questions of the modern Mathematics, V.2, Novosibirsk Univ., (1996), 51-60.

[8] O.A. Ladyzenskaja, V.A. Solonnikov and N.N. Uralceva, *"Linear and Quasi-linear Equations of Parabolic Type,"* American Mathematical Society Providence, Rhode Island, 1968.

[9] P.-L. Lions, *"Existence global de solutions pour les équations de Navier-Stokes compressible isentropiques,"* C.R.Acad. Sci. Paris 316(1993), 1335-1340.

[10] P.-L. Lions, *"Mathematical Topics in Fluid Mechanics, Compressible Models"* Clarendon Press, Oxford, in press

[11] A. Matsumura and T. Nishida, *"The Initial Value Problem for the Equations of Motion of Viscous and Heat-conductive gases,"* J. Math. Kyoto Univ. 20-1 (1980), 67-104.

[12] V.A. Vaigant and A.V. Kazhikhov, *"Global Solutions to the Potential Flow Equations for a Compressible Viscous Fluid at Small Reynolds Numbers,"* Diff. Eq., 30(1994), 6, 935-946.

[13] V.A. Vaigant and A.V. Kazhikhov, *"On Existence of Global Solutions to the Two-dimensional Navier-Stokes Equations for a Compressible Viscous Fluid,"* Siberian Math. J., 36(1995), N6, 1108-1141.

[14] W.P. Ziemer, *"Weakly Differentiable Functions"* Springer-Verlag (1989)