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MINIMAX THEOREMS AND THE NASH EQUILIBRIA ON GENERALIZED CONVEX SPACES

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ABSTRACT. We obtain minimax theorems and the Nash equilibrium theorem for G-convex spaces. Our new results extend and unify a number of known results for particular types of G-convex spaces. Finally, we compare our new results with the celebrated minimax theorem of H. König.

1. INTRODUCTION

The numerous applications and generalizations of John von Neumann's classical minimax theorem [Ne] constitute an important chapter of modern convex analysis. One of the main purposes of these generalizations was to eliminate the underlying convexity structure from the original hypothesis.

On the other hand, the convexity of subsets of topological vector spaces was extended to convex spaces by Lassonde, to C-spaces (or H-spaces) by Horvath [H], and to G-convex spaces (or generalized convex spaces) by the author; for the literature, see [P1-6, PK1-6]. It is known that the KKM theory, fixed point theory, and other equilibrium results are now well-developed for these abstract convexities.

In this paper, we obtain minimax theorems and the Nash equilibrium theorem for G-convex spaces. Our new results extend and unify a number of known results for particular types of G-convex spaces.

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In Section 3, from a coincidence theorem, we deduce the von Neumann–Sion type minimax theorems for G-convex spaces. In Section 4, from a Fan–Browder type fixed point theorem, we deduce the Ky Fan intersection theorem, another minimax theorem, and the Nash equilibrium theorem for G-convex spaces. Finally, Section 5 deals with the comparison of our results with the celebrated minimax theorem due to H. König [Kö1,2].

2. Preliminaries

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty subset D of X, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. Note that $\phi_A|_{\Delta_J}$ can be regarded as ϕ_J.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$. We may write $\Gamma_A = \Gamma(A)$ for each $A \in \langle D \rangle$, and $(X, \Gamma) = (X, X; \Gamma)$. A subset C of X is said to be *Γ-convex* if for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma_A \subset C$. For details on G-convex spaces, see [P1-6, PK1-6], where basic theory was extensively developed.

Major examples of other G-convex spaces than convex spaces or C-spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joo's pseudoconvex spaces, and topological semilattices with path-connected intervals. For the literature, see [PK1-6].

Recently, we gave new examples of G-convex spaces and, simultaneously, showed that some abstract convexities of other authors are simple particular examples of our G-convexity; see [P6]. Such examples are L-spaces of Ben-El-Mechaiekh *et al.*, continuous images of C-spaces, Verma's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, generalized H-spaces of Stachó, and *mc*-spaces of Llinares. Moreover, Ben-El-Mechaiekh *et al.* [BC] gave examples of G-convex spaces (X, Γ) as follows: B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For topological spaces X and Y, a multimap $T : X \multimap Y$ is called an *acyclic map* if it is upper semicontinuous (u.s.c.) with compact acyclic values.

A polytope is a convex hull of a finite subset in a topological vector space.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X, Y)$ denotes the set of maps $F : X \multimap Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of finite compositions of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following properties:

(1) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;

(2) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and

(3) for any polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neill maps $\mathbb{N}$ (continuous with values consisting of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ in uniform spaces, admissible maps in the sense of Górniewicz, permissible maps of Dzedzej, and many others.

We introduce one more classes:

$F \in \mathfrak{A}_c^\kappa(X, Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K, Y)$ such that $\Gamma(x) \subset F(x)$ for each $x \in K$.

Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\kappa$. Any subclass of $\mathfrak{A}_c^\kappa$ will be called *admissible*. For details, see [PK1,2].

Recall that an extended real function $f : X \to \overline{\mathbf{R}}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. (u.s.c.)] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \overline{\mathbf{R}}$.

We begin with the following particular form of a coincidence theorem of Park and Kim [PK2,3, Theorem 1]:

Theorem 0. *Let (X,Γ) be a G-convex space, Y a Hausdorff space, and $F, G : X \multimap Y$ maps satisfying*

(0.1) *$F \in \mathfrak{A}_c^\kappa(X,Y)$ is compact;*

(0.2) *for each $y \in F(X)$, $G^-(y)$ is Γ-convex; and*

(0.3) *$\overline{F(X)} \subset \bigcup\{\operatorname{Int} G(x) : x \in X\}$.*

Then F and G have a coincidence point $x_0 \in X$; that is, $F(x_0) \cap G(x_0) \neq \emptyset$.

The following continuous selection theorem is due to the author [P4]:

Lemma 1. *Let Y be a Hausdorff space, (X,Γ) a G-convex space, and $T : Y \multimap X$ a map satisfying*

(1) *$T(y)$ is Γ-convex for each $y \in Y$; and*

(2) *$Y = \bigcup\{\operatorname{Int} T^-(x) : x \in X\}$.*

Then $T \in \mathbb{C}^\kappa(Y,X) \subset \mathfrak{A}_c^\kappa(Y,X)$. More precisely, for any nonempty compact subset K of Y, $T|_K$ has a continuous selection $f : K \to X$; that is, $f(y) \in T(y)$ for all $y \in K$, such that $f(K) \subset \Gamma_A$ for some $A \in \langle X \rangle$.

3. Minimax Theorems

In this section, we obtain the von Neumann–Sion type minimax theorems for G-convex spaces.

The following is basic:

Theorem 1. *Let (X,Γ) and (Y,Γ') be G-convex spaces with Y Hausdorff and $F, G : X \multimap Y$ maps such that*

(1.1) *F is compact, $F(x)$ is Γ'-convex for each $x \in X$, and $X = \bigcup\{\operatorname{Int} F^-(y) : y \in Y\}$; and*

(1.2) *$G^-(y)$ is Γ-convex for each $y \in F(X)$ and $\overline{F(X)} \subset \bigcup\{\operatorname{Int} G(x) : x \in X\}$.*

Then F and G have a coincidence point.

Proof. By Lemma 1, (1.1) implies that $F \in \mathfrak{A}_c^\kappa(X,Y)$ and F is compact. Moreover, (1.2) implies that conditions (0.2) and (0.3) are satisfied. Therefore, by Theorem 0, F and G have a coincidence point.

From Theorem 1, we deduce the following von Neumann–Sion type minimax theorem for G-convex spaces:

Theorem 2. *Let (X, Γ) and (Y, Γ') be G-convex spaces, Y Hausdorff compact, $f : X \times Y \to \overline{\mathbf{R}}$ an extended real function, and $\mu := \sup_{x \in X} \inf_{y \in Y} f(x, y)$. Suppose that*

(2.1) *$f(x, \cdot)$ is l.s.c. on Y and $\{y \in Y : f(x, y) < r\}$ is Γ'-convex for each $x \in X$ and $r > \mu$; and*

(2.2) *$f(\cdot, y)$ is u.s.c. on X and $\{x \in X : f(x, y) > r\}$ is Γ-convex for each $y \in Y$ and $r > \mu$.*

Then

$$\sup_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \sup_{x \in X} f(x, y).$$

Proof. Since $f(x, \cdot)$ is l.s.c. on the compact space Y, $p(x) := \min_{y \in Y} f(x, y)$ exists for each $x \in X$. Since $q(y) := \sup_{x \in X} f(x, y)$ is l.s.c. for each $y \in Y$, $q(y_0) = \min_{y \in Y} q(y)$ exists. Note that

$$p(x) = \min_{y \in Y} f(x, y) \leq f(x, y) \leq \sup_{x \in X} f(x, y) = q(y)$$

for all $x \in X$ and $y \in Y$. Therefore, we have

$$\sup_{x \in X} p(x) \leq \min_{y \in Y} q(y).$$

Suppose that the equality does not hold. Then there exists an $r > \mu$ such that

$$\mu = \sup_{x \in X} p(x) < r < \min_{y \in Y} q(y).$$

We define multimaps $F, G : X \multimap Y$ by

$$F(x) = \{y \in Y : f(x, y) < r\} \text{ and } G(x) = \{y \in Y : f(x, y) > r\}$$

for $x \in X$. Then $F(x)$ is nonempty and Γ'-convex by (2.1), and $G(x)$ is open since $f(x, \cdot)$ is l.s.c. Moreover,

$$F^{-}(y) = \{x \in X : f(x, y) < r\} \text{ and } G^{-}(y) = \{x \in X : f(x, y) > r\}$$

for $y \in Y$. Then $F^{-}(y)$ is open since $f(\cdot, y)$ is u.s.c. by (2.2), and $G^{-}(y)$ is nonempty and Γ-convex. Now, by applying Theorem 1, there exist an $x_0 \in X$ and a $y_0 \in Y$ such that $y_0 \in F(x_0) \cap G(x_0) \neq \emptyset$. This leads a contradiction

$$f(x_0, y_0) < r < f(x_0, y_0).$$

This completes our proof.

Corollary. *Under the hypothesis of Theorem 2, further if X is compact, then f has a saddle point $(x_0, y_0) \in X \times Y$ such that*

$$\max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y).$$

Proof. Since $f(x, \cdot)$ and $f(\cdot, y)$ are l.s.c. and u.s.c., resp., $p(x) := \min_{y \in Y} f(x, y)$ and $q(y) := \max_{x \in X} f(x, y)$ exist for each $x \in X$ and $y \in Y$. Since p is u.s.c. on X and q is l.s.c. on Y, we have $\max_{x \in X} p(x) = p(x_0)$ and $\min_{y \in Y} q(y) = q(y_0)$ for some $x_0 \in X$ and $y_0 \in Y$. Then (x_0, y_0) is a saddle point by Theorem 2. This completes our proof.

Particular Forms. We list historically well-known particular forms of Theorem 2 and Corollary in chronological order:

1. von Neumann [Ne], Kakutani [K]: X and Y are compact convex subsets of Euclidean spaces and f is continuous.

2. Nikaidô [Ni]: Euclidean spaces in the above are replaced by Hausdorff topological vector spaces, and f is continuous in each variable.

3. Sion [S]: X and Y are convex spaces in Theorem 2 and Corollary.

4. Komiya [K, Theorem 3]: X and Y are compact convex spaces in the sense of Komiya.

5. Bielawski [Bi, Theorem (4.13)]: X and Y are compact spaces having certain simplicial convexities.

6. Horvath [H, Prop. 5.2]: X and Y are C-spaces with Y compact.

Remark. In 4 and 6 above, Hausdorffness of Y is assumed since they used the partition of unity argument. However, 3 and 5 were based on the corresponding KKM theorems which need not the Hausdorffness of Y; see Theorem 5 below.

From Theorem 0 for the subclass $\mathbb{V}$ of $\mathfrak{A}_c^\kappa$, we have another minimax theorem:

Theorem 3. *Let (X,Γ) and (Y,Γ') be G-convex spaces, Y Hausdorff compact, $f : X \times Y \to \overline{\mathbf{R}}$ a l.s.c. function, and $\mu := \sup_{x\in X} \inf_{y\in X} f(x,y)$. Suppose that*

(3.1) *for each $r > \mu$ and $y \in Y$, $\{x \in X : f(x,y) > r\}$ is Γ-convex; and*

(3.2) *for each $r > \mu$ and $x \in X$, $\{y \in X : f(x,y) \le r\}$ is acyclic.*

Then

$$\sup_{x\in X} \min_{y\in Y} f(x,y) = \min_{y\in Y} \sup_{x\in X} f(x,y).$$

Proof. As in the proof of Theorem 2, we have

$$\sup_{x\in X} \min_{y\in Y} f(x,y) \le \min_{y\in Y} \sup_{x\in X} f(x,y).$$

Suppose that the equality does not hold. Then there exists an $r > \mu$ such that

$$\mu = \sup_{x\in X} \min_{y\in Y} f(x,y) < r < \min_{y\in Y} \sup_{x\in X} f(x,y).$$

We define multimaps $F, G : X \multimap Y$ by

$$F(x) = \{y \in Y : f(x,y) \le r\} \text{ and } G(x) = \{y \in Y : f(x,y) > r\}$$

for $x \in X$. Then $F(x)$ is nonempty and closed since $f(x,\cdot)$ is l.s.c. for each $x \in X$. On the other hand, $G(x)$ is open since $f(x,\cdot)$ is l.s.c. Moreover, for each $y \in Y$,

$$G^-(y) = \{x \in X : f(x,y) > r\}$$

is nonempty and Γ-convex by (3.1). Therefore, conditions (0.2) and (0.3) are satisfied.

Consider the graph of F

$$\mathrm{Gr}(F) = \{(x,y) \in X \times Y : f(x,y) \le r\}.$$

Since f is l.s.c., $\mathrm{Gr}(F)$ is closed in $X \times Y$. Since Y is compact, F is u.s.c. Note that each $F(x)$ is closed and acyclic by (3.2). Hence F is an acyclic map.

Therefore, by Theorem 0 for $\mathbb{V}$ instead of $\mathfrak{A}_c^\kappa$, there exists an $x_0 \in X$ such that $F(x_0) \cap G(x_0) \ne \emptyset$. This leads a contradiction as in the proof of Theorem 2.

Particular Forms. 1. von Neumann [Ne], Kakutani [K]: X and Y are compact convex subsets of Euclidean spaces, f is continuous, and Γ-convexity and acyclicity are replaced by convexity.

2. Nikaidô [Ni]: Euclidean spaces were replaced by Hausdorff topological vector spaces in the above.

4. The Nash Equilibrium Theorem

In this section, from a Fan-Browder type fixed point result for G-convex spaces, we deduce the Ky Fan intersection theorem, another minimax theorem, and the Nash equilibrium theorem for G-convex spaces.

The following is known:

Lemma 2. *Let (X, Γ) be a compact G-convex space and $T : X \multimap X$ a map such that*

(1) *$T(x)$ is nonempty and Γ-convex for each $x \in X$; and*

(2) *$T^{-}(y)$ is open for each $y \in X$.*

Then T has a fixed point.

It is known that if F is a single-valued map, then Theorem 0 holds without assuming the Hausdorffness of Y; see [PK2,3, P5]. Hence, Lemma 2 follows from Theorem 0 for the case $X = Y$ and $F = 1_X$, the identity map.

Lemma 2 is also equivalent to the corresponding KKM type theorem for which Hausdorffness of the space is known to be reduntant.

Given a cartesian product $X = \Pi_{i=1}^{n} X_i$ of sets, let $X^i = \Pi_{j \neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n).$$

From Lemma 2, we have the following Ky Fan type intersection theorem:

Theorem 4. *Let $X = \Pi_{i=1}^{n} X_i$, (X, Γ) be a compact G-convex space, and A_1, $A_2, \dots, A_n$ be n subsets of X such that*

(4.1) *for each $x \in X$ and each $i = 1, \dots, n$, the set $A_i(x) = \{y \in X : (y_i, x^i) \in A_i\}$ is Γ-convex and nonempty; and*

(4.2) *for each $y \in X$ and each $i = 1, \dots, n$, the set $A_i(y) = \{x \in X : (y_i, x^i) \in A_i\}$ is open.*

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define a map $T : X \multimap X$ by $T(x) = \bigcap_{i=1}^n A_i(x)$ for $x \in X$. Then each $T(x)$ is Γ-convex being an intersection of Γ-convex sets by (4.1). For each $x \in X$ and each i, there exists a $y^{(i)} \in A_i(x)$ by (4.1), or $(y_i^{(i)}, x^i) \in A_i$. Hence, we have $(y_1^{(1)}, \dots, y_n^{(n)}) \in \bigcap_{i=1}^n A_i(x)$. This shows $T(x) \neq \emptyset$. Moreover, $T^-(y) = \bigcap_{i=1}^n A_i(y)$ is open for each $y \in X$ by (4.2). Now, the conclusion follows from Lemma 2.

Remarks. 1. If each X_i is a compact G-convex space, so is X.

2. In view of Theorem 0, condition (4.2) can be replaced by the following:

(4.2)′ $X = \bigcup_{y \in X} \mathrm{Int}(\bigcap_{i=1}^n A_i(y))$.

3. For $n = 2$, Theorem 4 can be comparable to Theorem 1.

Particular Forms. 1. Ky Fan [F1, Theorem 2]: X_i are compact convex subsets of topological vector spaces in Theorem 4.

2. Bielawski [B, Proposition (4.12) and Theorem (4.15)]: Theorem 4 for X having a finitely local convexity, which is a particular type of his simplicial convexity.

3. Kirk, Sims, and Yuan [KSY, Theorem 5.2]: Theorem 4 for hyperconvex metric spaces, which are of extremely particular type of G-convex spaces.

From Theorem 4 for $n = 2$, we can deduce the following improved version of Corollary to Theorem 2:

Theorem 5. *Let* (X, Γ) *and* (Y, Γ') *be compact* G*-convex spaces and* $f : X \times Y \to \overline{\mathbf{R}}$ *a function satisfying conditions (2.1) and (2.2). Then*

(i) f *has a saddle point* $(x_0, y_0) \in X \times Y$*; and*

(ii) *we have*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. Just follow the proofs of Theorem 2 and its Corollary using Theorem 4 for the case $n = 2$ instead of Theorem 1.

Particular Forms. All of the examples given for Theorem 2 and Corollary follow from Theorem 5. Especially, Sion [S, Theorem 3.4] is a particular form of Theorem 5, and [S, Corollary 3.5] is a non-Hausdorff version of Theorem 2 and can be obtained from Theorem 5 by following his own method.

From Theorem 4, we also deduce the following Nash equilibrium theorem for G-convex spaces:

Theorem 6. *Let $X = \Pi_{i=1}^{n} X_i, (X, \Gamma)$ be a compact G-convex space, and $f_1, \ldots, f_n : X \to \mathbf{R}$ continuous functions such that*

(3) *for each $x \in X$, each $i = 1, \ldots, n$, and each $r \in \mathbf{R}$, the set $\{(y_i, x^i) \in X : f_i(y_i, x^i) > r\}$ is Γ-convex.*

Then there exists a point $x \in X$ such that

$$f_i(x) = \max_{y_i \in X_i} f_i(y_i, x^i) \quad \text{for} \quad i = 1, \ldots, n.$$

Proof. Let $\varepsilon > 0$ and, for each i, let

$$A_i^\varepsilon = \{x \in X : f_i(x) > \max_{y_i \in X_i} f_i(y_i, x^i) - \varepsilon\}.$$

Then the sets $A_1^\varepsilon, \ldots, A_n^\varepsilon$ satisfy conditions (4.1) and (4.2) of Theorem 4, and hence $\bigcap_{i=1}^{n} A_i^\varepsilon \neq \emptyset$. Then $H_\varepsilon = \bigcap_{i=1}^{n} \overline{A_i^\varepsilon}$ is a nonempty compact set. Since $H_{\varepsilon_1} \subset H_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$, we have $\bigcap_{\varepsilon > 0} H_\varepsilon \neq \emptyset$. Then $x \in \bigcap_{\varepsilon > 0} H_\varepsilon$ satisfies the conclusion.

Particular Forms. 1. Nash [N]: Each X_i is a compact convex subset of a Euclidean space in Theorem 6.

2. Fan [F2, Theorem 4]: X_i are compact convex subsets of real Hausdorff topological vector spaces in Theorem 6.

3. Bielawski [B, Theorem (4.16)]: Theorem 6 for X having a finitely local convexity.

4. Kirk, Sims, and Yuan [KSY, Theorem 5.3]: Theorem 6 for hyperconvex metric spaces.

5. Comparisons with the König minimax theorem

In 1992, H, König obtained a general minimax theorem based on connectedness, which is regarded the best result in this area. Therefore, it is quite natural to compare our results with his theorem which is obtained by quite different approach from ours. Now we follow H. König [Kö1,2]:

Minimax theorems consider functions $f : X \times Y \to \overline{\mathbf{R}}$ on the product of topological spaces X and Y, and the formations

$$f_* := \sup_{x \in X} \inf_{y \in Y} f(x,y) \leq \inf_{y \in Y} \sup_{x \in X} f(x,y) =: f^*.$$

The assertions are that the minimax relation $f_* = f^*$ holds true under the respective assumptions.

Let us define f to fulfill the [finite] condition $X(\geq)$ iff

$$X(H,\lambda,\geq) := \bigcap_{y \in H} [f(\cdot,y) \geq \lambda] := \{x \in X : f(x,y) \geq \lambda \ {}^{\forall} y \in H\} \subset X$$

is connected for all nonvoid [finite] $H \subset Y$ and for all real $\lambda > f_*$, and the obvious variant $X(>)$.

Likewise we define f to fulfill the [finite] condition $Y(\leq)$ iff

$$Y(H,\lambda,\leq) := \bigcap_{x \in H} [f(x),\cdot) \leq \lambda] := \{y \in Y : f(x,y) \leq \lambda \ {}^{\forall} x \in H\} \subset X$$

is connected for all nonvoid [finite] subsets $H \subset X$ and all real $\lambda > f_*$, and the obvious variant $Y(<)$.

The main result of H. König [Kö1] is as follows; see [Kö2]:

Theorem. *Let X and Y be topological spaces with Y compact and X connected, and let $f : X \times Y \to \overline{\mathbf{R}}$ fullfill the continuity condition*

(C) $f(\cdot,y) \in USC(X)$*(:= upper semicontinuous)* ${}^{\forall} y \in Y$ *and*
$f(x,\cdot) \in LSC(Y)$*(:= lower semicontinuous)* ${}^{\forall} x \in X$*;*

or the variant

(C′) $f \in LSC(X \times Y)$.

Then $f_* = f^*$ *whenever* f *fulfills some combinations of one of the conditions* $X(\geq)$ *and* $X(>)$ *with one of the finite conditions* $Y(\leq)$ *and* $Y(<)$, *provided that the combination of* $X(>)$ *with* $Y(<)$ *the space* Y *is Hausdorff.*

Comparing our results with the König theorem, we observe the following:

(i) If each Γ- and Γ'-convex subset is connected, Theorem 2 follows from the König theorem [for the case (C), $X(>)$ and $Y(<)$].

(ii) If each Γ- and Γ'-convex subset is connected, Theorem 3 without assuming the Hausdorffness of Y follows from the König theorem [for the case (C′), $X(>)$ and $Y(\leq)$].

(iii) Theorem 5 is not comparable to the König theorem [for the case (C), $X(>)$ and $Y(<)$]. Even when each Γ- and Γ'-convex subset is connected, Theorem 5 assumes the compactness of (X, Γ) instead of the Hausdorffness of Y in the König theorem.

Note that a Γ-convex set is not necessarily connected and we give the following important case for the connectivity:

Proposition. *In a G-convex space* (X, Γ), *if* $\Gamma_{\{x\}} = \{x\}$ *for each* $x \in X$, *then every* Γ*-convex set is connected.*

Proof. Let C be a Γ-convex subset of (X, Γ) and choose a point $x_0 \in C$. For any other point $x \in C$, we have $\Gamma_{\{x_0, x\}} \subset C$. Since there exists a continuous function $\phi_A : \Delta_1 = [0, 1] \to \Gamma_{\{x_0, x\}}$ for $A = \{x_0, x\} \in \langle X \rangle$ such that $\phi_A(0) \subset \Gamma_{\{x_0\}} = \{x_0\}$ and $\phi_A(1) \subset \Gamma_{\{x\}} = \{x\}$. Therefore, $\phi_A([0, 1])$ is a connected set in C containing x_0 and x. This implies the connectivity of C.

Finally, it is well-known that $\overline{\mathbf{R}}$ in our work can be replaced by an order complete, order dense, linearly ordered space.

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