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**Inaction and Long Term Properties in
the Housing Market**

Daniela Gamberini

Supervisor
Prof. Pierpaolo Benigno

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Chapter 1

Introduction

The role of housing prices is drawing an increasing interest in academic research due to major changes experienced in the recent financial crisis.

The integration of this asset in the financial world has been very important with a financial and economical impact at microeconomic and macroeconomic level.

Financial integration means that housing has gained importance in portfolio composition not only for households but also for banks and investors, mortgages and REIT are some examples.

Different models have been developed in the literature in the last few years, they span from DSGE models to Asset pricing models, from Search models to Finance models.

The aim of our entire work is focused on the behavior of housing purchases/sales, which constitutes the major driver for housing prices.

In the time considered for our investigation (from late '80 to present), changes related to the financial integration and to credit access may have affected substantially the path of housing prices. On the other hand, treating housing choice as a choice for durable consumption, it is still a viable approach which can capture its intrinsic and binding nature.

The main issue is linked to the fact that from the household's perspective because of financial and non-financial costs it is suboptimal to adjust durable consumption at any point in time, therefore the purchase is delayed until a certain critical level of the consumption over wealth ratio is reached.

Accumulation of wealth over the life cycle becomes crucial in the housing purchase process and to integrate a model for durable consumption with a realistic process for the household income is one of the aspect investigated in

this thesis.

Household's wealth (given investments in stocks and income from labor) and housing purchase have a long-run relationship which needs to be captured also by a quantitative framework when modeling agent's behavior.

This intuitive relationship between prices, purchases, wealth and extremely delayed adjustment of consumption can be easily translated in an intuition of long-memory property of housing prices time series.

Long memory property or Long-Range Dependence constitutes the statistical foundation for models in which choices are delayed in time. In literature, LRD has been detected in stock prices and volatility and it is explained by means of adjustment costs and therefore intervals of times in which the agent keeps its assets without making any action. This behavior is common and enhanced while considering durable consumption in general and especially considering housing purchase. Economics of Inaction, following the definition of Nancy Stokey, is a branch of economics looking and modeling this long-term behavior for durable assets.

The first and pioneer work, which linked literature for durable consumption and financial investment and considered the literature on asset prices and finance, was the Grossman and Laroque paper "Asset pricing and Optimal portfolio Choice in the presence of Illiquid durable consumption good". After their attempt, other works integrating Economics of Inaction and Finance literature hasn't been developed.

The second chapter of this thesis wants to fill the gap, constructing again a bridge between the two fields. The importance is given by the fact that only recently in financial literature income over life-cycle has been understood as a fundamental component which cannot be neglected for the motivation stated above.

Chapter 2

Economics of Inaction and Long-Range Dependence in Housing Prices

Abstract

Inaction regions are assumed when the agent faces transaction costs. This behavior should affect asset prices and their momentum.

Long-Range Dependence (LRD), i.e. high degree of persistence, has been detected in many Economic and Financial time Series, e.g. market indices and commodity prices, such that in the literature sometimes authors refer to LRD as a paradigm for Macroeconomics and Finance.

A statistical analysis for persistence is provided, this feature motivates and supports the theoretical literature on asset pricing considering habits and housing consumption, or literature on durable consumption. In particular, LRD refers to data displaying high-degree of persistence and long-lasting effect of unanticipated shocks. After investigating the whole series and the persistence properties, also a detection of shocks affecting the series is provided.

Keywords [Housing, Asset prices, Forecastability, Long Memory]

Introduction

In literature, several theoretical models have been developed for explaining asset prices behavior by means of habit persistence, housing/durable versus non-durable consumption. It is observed that sales and purchases do not adjust consumption instantaneously to the optimal level but they are delayed in time. One possible motivation consists in transaction costs, another in habits which can play a key role in pursuing an inaction region, the time in which no action is taken by the agent. Anyway, they are just different angles from which one can look at the same phenomenon.

Prices are affected by sales and purchases behavior and it seems that asset characterized by an inaction region, especially housing prices, they present momentum.

This work wants to exploit the degree of persistence in housing prices, giving a statistical foundation for the theoretical models.

The property of Long-Range Dependence is the natural statistical correspondent to inaction of agent, and inertia is crucial in explaining relatively slow variation in prices around the stochastic trend.

A large number of empirical studies on asset prices have investigated long range dependence properties of asset returns. The concepts of self-similarity, scaling, fractional processes and long range dependence have been repeatedly used to describe properties of financial time series such as stock prices, foreign exchange rates, market indices and commodity prices. It seems not only reasonable to search it also in the price dynamics of this asset but very likely to find, in fact all the motivations for finding LRD in asset prices are viable and enhanced when talking about the housing sector.

Some authors constructed a link between Investor's Inertia and Long Range Dependence, the former being the microeconomic foundation for this statistical models.

They especially refers to inertia in the stock exchange market, documented in the annual survey of individual shareownership developed by New York Stock Exchange (NISE) "Shareownership2011".

We can quote, as an example, "A limit theorem for Financial Markets with Inert Investors", Bayraktar, Horst, 2007. In their investigation they found that low level of trading activity, i.e long values for the duration of the inactivity periods, is responsible for high persistence in the log-returns in asset prices.

In the Housing Market, long-term investors have very long period of inac-

tion, in fact the durable purchase is commonly investigated in what is called "Economics of Inaction".

It can appear that Long Range dependence is good candidate for a suitable motivation of the "momentum" shown in the time series.

Another motivation for looking at this property in housing prices is given by the relationship between heterogeneity in agents' time scale and long memory investigated by Rama Cont [33], Agents can differ in the time horizon of the investment, for example, long-Term investors naturally focus on long term behavior of prices, whereas traders aim to exploit short-term fluctuations.

Also different bequest valuation functions (which of course change the the time horizon of the investor) can impact the behavior of price.

Certainly, applying also to Housing Prices the same arguments which constitute a foundation for LRD in the Stock Market , Long Memory should be found also in this particular Market.

An analysis of the properties of LRD in the time series of housing prices, has been pursued by Gil-Alana, [22]. Anyway, he doesn't explore models that can capture this behavior and the importance of a work with the extent of estimate model's parameter is interesting for forecastability purposes.

The importance of the investigation is straightforward. The role, which this particular asset plays in the economy and in the financial world, has been fully understood in the last few years. In this period the housing sector was no longer only "a durable good" but assumed different facets involving also financial role with aggregate mortgage default, mortgage backed securities and REITs.

A good forecastability in this sector would help not only household's decisions but also financial decisions.

This is why we will focus on the long memory property of the whole series without forgetting an analysis of structural changes, i.e. breaks, in the pattern of the stochastic trend.

Long-Range Dependence property is a feature of the series itself (univariate analysis) and if this feature is detected, it can provide a good description of the evolution of the process or after the series experiments a shock. Shock or breaks in the evolutionary pattern will be estimated to complement this work.

In section 1 a brief review of Long-Range dependence and Economics is provided, while in section 2 we will give instead a mathematical definition of Long-Range Dependence. Section 3 describes the dataset used for the in-

vestigation. In Section 4, analysis of long-range dependence is implemented, while in the following section an estimation of parameters for LRD models is provided. Instead, structural breaks are analyzed in the final section.

2.1 History of Long Range Dependence

Granger was the first who pointed out that non-parametrically estimated power spectra of many economic variables, such as industrial production and commodity price indexes, suggested the overwhelming importance of the low-frequency components. Mandelbrot observed a self-similar behaviour in the distribution of speculative prices, and proposed continuous and discrete time fractional models, such as fractional Brownian motion. However, initial sizeable empirical success of the long-range dependence concept in economics is certainly related to the autoregressive fractionally integrated moving average model (hereafter ARFIMA).

Empirical finance research had a tremendous impact in emphasizing the importance of the LRD paradigm, both empirically and theoretically. On the one hand, the availability of very large time series of high frequency data allowed easier and more convincing detection starting from 1990s. On the other hand, some empirical findings prompted the development of nonlinear time series models apt to fit the empirical distribution of asset returns, synthesized in a number of well known stylized facts, including slow decay of sample autocorrelations of squared returns. This has stimulated research aiming at establishing asymptotic theory such models and deriving their implications in terms of asset pricing and risk management.

Anyway, as far as we know, a part from few work of Gil-Alana which detected LRD in Housing Prices, without providing a model for the time series, other very few articles about Long memory property in the volatility of housing prices have been written.

In this articles the authors try to apply statistical models as ARCH and GARCH to fit housing Price data.

In another work, already quoted [Bayraktar and Horst, 2007] they present microfoundation argument for LRD in housing prices and develop a theoretical model in which it happens that the asset returns follow a Fractional Brownian Motion.

2.2 Long Range Dependence

In this section we want to provide a brief review of the definition of Long Range Dependence and of the methodology that applies in order to detect it.

Long-Range dependence in a stationary time series occurs when the covariances tend to zero like a power function and so slowly that their sums diverge. It is often observed in nature, for example in hydrology in economics and telecommunications, and it is closely related to self-similarity. Self-similarity refers to invariance in distribution under a suitable change of scale. To understand the relationship between self-similarity and long-range dependence, suppose that the self-similar process has stationary increments. These increments form a stationary time series which can display long-range dependence. Then a central limit-type theorem will yield a self-similar process with stationary increments. The intensity of long-range dependence is related to the scaling exponent of the self-similar process. We shall provide a summary of stationary processes with long-memory; we shall show in details fractional Brownian Motion, the Gaussian self-similar process with stationary increments; its increment process known as fractional Gaussian Noise which displays long-range dependence, and a large class of long-range dependent stationary sequences called FARIMA, which are commonly used in modeling such physical phenomena.

2.2.1 Definitions

Self-Similarity and Stationary Increments

A real valued process $Z = \{Z(t)\}_{t \in \mathbb{R}}$ is self-similar with index $H > 0$ (denoted $H - ss$) if for any $a > 0$,

$$\{Z(t)\} \stackrel{d}{=} \{Z(at)\}_{t \in \mathbb{R}} \stackrel{d}{=} \{a^H Z(t)\}_{t \in \mathbb{R}} \quad (2.1)$$

The parameter H is called **scaling exponent** of the process and it refers to the **Hurst Parameter**.

A real valued process $\{Z(t)\}_{t \in \mathbb{R}}$ has stationary increments if, for all $h \in \mathbb{R}$,

$$\{Z(t+h) - Z(h)\}_{t \in \mathbb{R}} \stackrel{d}{=} \{Z(t) - Z(0)\}_{t \in \mathbb{R}} \quad (2.2)$$

Examples

- Brownian motion is H -sssi with $H = 1/2$
- SaS Levy motion is H -sssi with $H = 1/\alpha$

Properties

1. $Z(0)=0$
2. If $H \neq 1$ then $E[Z(t)] = 0, \forall t \in \mathbb{R}$
3. By the property of stationary increments and self-similarity, the covariance function of H -sssi process Z is given by:

$$\Gamma_H(s, t) = E[Z(s)Z(t)] = \frac{1}{2}\{E[Z(s)^2]\} + \{E[Z(t)^2]\} - \{E[(Z(s) - Z(t))^2]\}$$
$$\Gamma_H(s, t) = \frac{\sigma^2}{2}\{|t|^{2H} + |s|^{2H} - |t - s|^{2H}\}$$

Series Exhibiting Long-Range Dependence

Let $B_H = \{B_H(t)\}_{t \geq 0}$ a fBm and $Z = \{Z_t\}_{t \in \mathbb{Z}}$ the fGn associated.

By the relation (3.16), the Z 's autocovariance function tends to 0 like a power function, as $k \rightarrow \infty$.

However, when $1/2 < H < 1$, it tends to 0 so slowly that $\sum_{k=-\infty}^{\infty} \gamma(k)$ diverges. In this case, one says that $Z = \{Z_k, k \in \mathbb{Z}\}$ exhibits *long-range dependence* or (*strong dependence* or has *long memory* or is a *1/f noise*).

It is also interesting to see how long-range dependence translates from the "time domain" to the "frequency domain".

$$\gamma(k) = \int_{-\pi}^{\pi} e^{i\nu k} f(\nu) d\nu, \quad k \in \mathbb{Z},$$

and

$$f(\nu) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{-i\nu k} \gamma(k), \quad \nu \in [-\pi, \pi].$$

Long-range dependence in the "time domain" translates into the behavior of the spectral density around the origin and it follows that for some constant " c "

$$f(\nu) = c \nu^{1-2H},$$

as $\omega \rightarrow 0$. Hence long-range dependence corresponds to the blow-up of the spectral density $f(\nu)$ at the origin.

2.3 Data Set

We will use in this research the SP/Case-Shiller Home Price Indices are designed to be a reliable and consistent benchmark of housing prices in the United States. Their purpose is to measure the average change in home prices in a particular geographic market. They are calculated monthly and cover 20 major metropolitan areas (Metropolitan Statistical Areas or MSAs), which are also aggregated to form two composites — one comprising 10 of the metro areas, the other comprising all 20.

The SP/Case-Shiller U.S. National Home Price Index (the U.S. national index) tracks the value of single-family housing within the United States. The index is a composite of single-family home price indices for the nine U.S. Census divisions and is calculated quarterly.

The indices measure changes in housing market prices given a constant level of quality. Changes in the types and sizes of houses or changes in the physical characteristics of houses are specifically excluded from the calculations to avoid incorrectly affecting the index value.

The SP/Case-Shiller indices are designed to measure, as accurately as possible, changes in the total value of all existing single-family housing stock. The methodology samples all available and relevant transaction data to create matched sale pairs for pre-existing homes.

The SP/Case-Shiller indices do not sample sale prices associated with new construction, condominiums, co-ops/apartments, multi-family dwellings, or other properties that cannot be identified as single-family.

The factors that determine the demand, supply, and value of housing are not the same across different property types. Consequently, the price dynamics of different property types within the same market often vary, especially during periods of increased market volatility. In addition, the relative sales volumes of different property types fluctuate, so indices that are segmented by property type will more accurately track housing values.

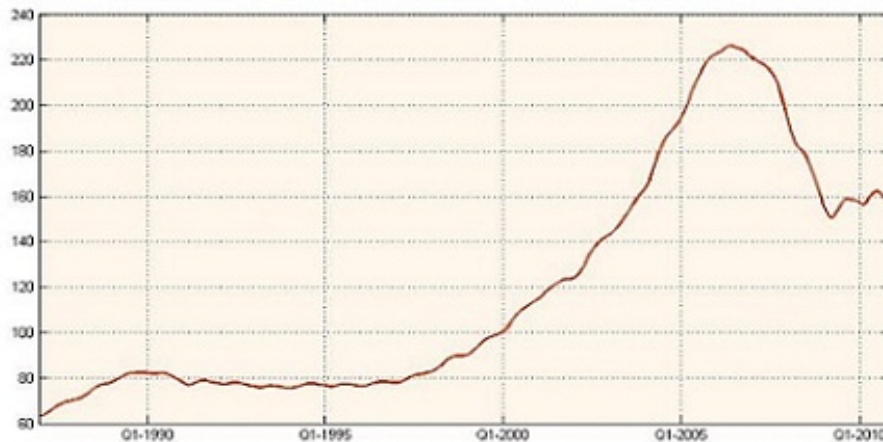


Figure 2.1: Housing price index from 1980 to 2011

2.4 The Analysis

Long Range Dependence is a property of stationary time series. Before implementing further investigations it is necessary to understand if the series is stationary or if there is the need to detrend data. Of course, housing prices exhibit a clear tendency to grow over time.

Anyway this could be due to inflation, then we will use real data.

The sample autocorrelation function can be a signal of long-Range Dependence or non stationarity, but The Dickey-Fuller test for non-stationarity accepts the unit-root null.

The analysis can be split into several sections. First, an examination of the difference process is pursued. Then, we will check again for stationarity. After that tools for detecting LRD will be implemented.

2.4.1 Detrending by Differencing

We tested using Dickey Fuller test for non stationarity the unit root null. The unit root test for non stationarity Autocorrelation, Partial Correlation and Stationarity tests indicates that the first difference of the series can be considered stationary



Figure 2.2: Housing Prices (the green line refers to real prices)

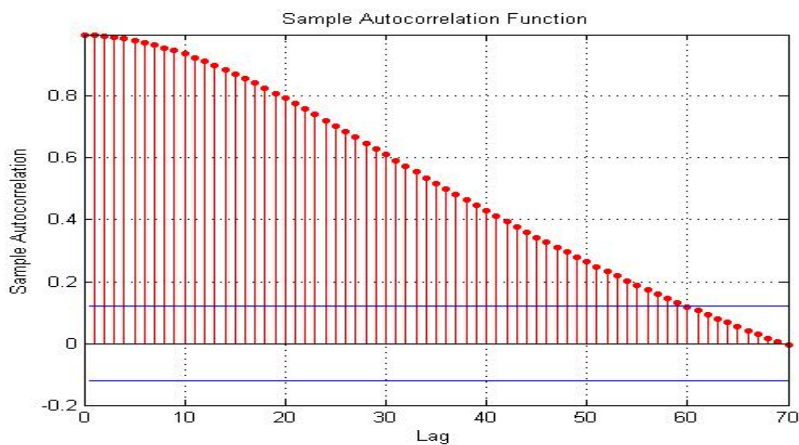


Figure 2.3: Sample autocorrelation function

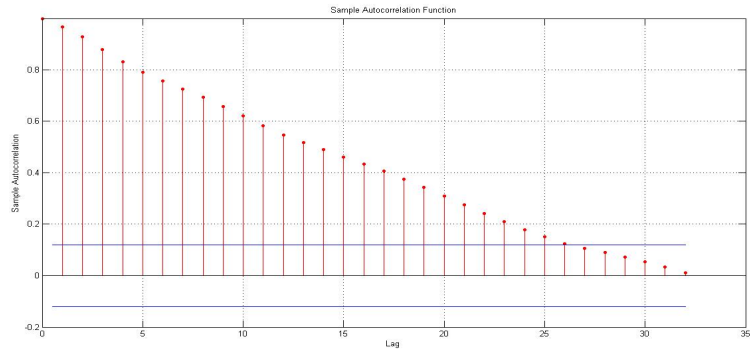


Figure 2.4: Autocorrelation-first difference

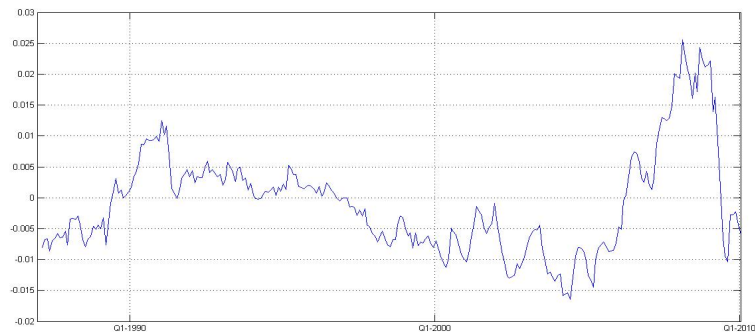


Figure 2.5: First Difference of the series

$$\Delta y_t = y_t - y_{t-1} \quad (2.3)$$

Moreover the stationary time series displays the following pattern for correlations which suggests the implementation of some more advanced techniques for long-range dependence detection.

2.4.2 Detecting Long Memory

We will start with an empirical approach and then we will develop more advanced techniques which will be based on spectral analysis.

Variance of the Sample Mean

The variance of the sample mean is equal to the variance of one sample of the observations divided by the sample size.

Let $\{X_1, X_2, \dots, X_n\}$ be a collection of observations randomly extracted from the same population at the time intervals: $t_i = 1, 2, \dots, n$

By Central Limit theorem it can be proved that

if $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$,

$$E(\bar{X}) = \mu; \quad \text{var}(\bar{X}) = \frac{\sigma^2}{n}. \quad (2.4)$$

Also the distribution of \bar{X} converges to the normal distribution $N(\mu, \sigma^2/n)$, with mean μ and variance σ^2/n , for $n \rightarrow \infty$. Therefore, if there is an indication of a slower decay of the variance of \bar{X} , it is useful to model this decay explicitly. The simplest approach one can think of is a decay proportional to $n^{-\alpha}$. In other words,

$$\text{var}(\bar{X}) \sim \sigma^2 c(\rho) n^{-\alpha}, \quad 0 < \alpha < 1. \quad (2.5)$$

where $c(\rho)$ is defined

$$c(\rho) = \lim_{n \rightarrow \infty} n^{\alpha-2} \sum_{i \neq j} \rho(i, j) \quad (2.6)$$

$$\sum_{k=-n+1}^{n-1} \rho(k) \sim n^{1-\alpha}, \quad n \rightarrow \infty. \quad (2.7)$$

As $\alpha < 1$ this implies:

$$\sum_{k=-\infty}^{\infty} \rho(k) = \infty. \quad (2.8)$$

Thus the correlations decay to zero so slowly that they are not summable. More specifically:

$$\rho(k) \sim c(\rho) |k|^{-\alpha}, \quad k \rightarrow \infty. \quad (2.9)$$

as $|k|$ tends to infinity and c_ρ is a finite positive constant.

The intuitive interpretation is that the process has long-memory.

The dependence between events that are far apart diminishes very slowly

with increasing distance. A stationary process with slowly decaying correlations is therefore called a stationary process with long memory or long range dependence.

Variance Plot As we have seen one of the striking property of long memory processes is that the variance of the sample mean converges slower to zero than n^{-1} . We have

$$\text{var}(\bar{X}_n) \approx cn^{2H-2}, \quad (2.10)$$

where $c > 0$. This suggest the following method for estimating H :

1. Let k be an integer. For different integers k in the range $2 \leq k \leq n/2$, and a sufficient number (say m_k) of subseries of length k calculate the sample means $\bar{X}_1(k), \bar{X}_2(k), \dots, \bar{X}_{m_k}(k)$ and the overall mean

$$\bar{X}(k) = m_k^{-1} \sum_{j=1}^{m_k} \bar{X}_j(k) \quad (2.11)$$

2. For each k , calculate the sample variance of the sample means $\bar{X}_j(k) (j = 1, \dots, m_k)$:

$$s^2(k) = (m_k - 1)^{-1} \sum_{j=1}^{m_k} (\bar{X}_j(k) - \bar{X}(k))^2.$$

3. Plot $\log s^2(k)$ against $\log k$.

For large values k , the points in this plot are expected to be scattered around a straight line with negative slope $2H - 2$. In the case of short range dependence or independence, the ultimate slope is $2\frac{1}{2} - 2 = -1$. Thus, the slope is steeper (more negative) for short memory processes. The problems in this method are in principle the same as for the R/S plot and the log-log correlogram.

It is convenient to draw a straight line with slope -1 as a reference line in the same picture. The variance plot gives us a rough idea about whether there is long memory in the data, provided that the long memory is strong enough. Slight departures from $H = \frac{1}{2}$ seem rather difficult to distinguish from $H = \frac{1}{2}$, even for large sample size.

The analysis of the variance of the sample mean shows that, from this first euristical approach, it is not possible to exclude long-range dependence but instead it a property which very likely could belong to the series.

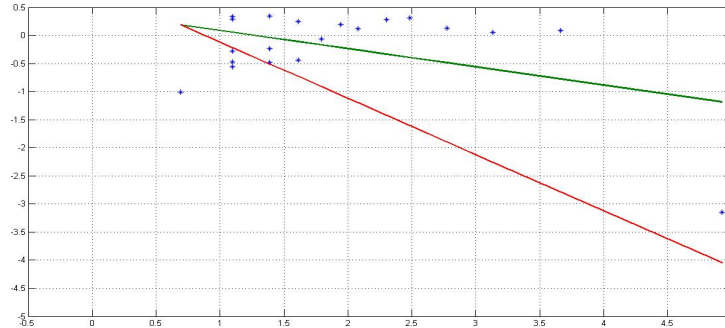


Figure 2.6: Variance of the Sample Mean

Spectral Analysis

The most powerful tool to detect long-memory is the spectral analysis. As we said several times the periodogram is the "sample analogous" of the spectral density. It is defined:

$$I(\nu_j) = \frac{1}{2\pi n} \left| \sum_{t=1}^n (x_t - \bar{x}_n) e^{it\nu_j} \right|^2 = \frac{1}{2\pi} \sum_{k=-(n-1)}^{n-1} \hat{\gamma}(k) e^{ik\nu_j}, \quad (2.12)$$

where: $\nu_j = 2\pi j/n$ con $j = 1, \dots, n^*$ e $n^* = \text{Int}[(n-1)/2]$, are the Fourier frequencies.

For processes displaying long-memory, in log scale, near the origin, the periodogram is like a straight line with negative slope.

It follows the graph of the periodogram for the series we want to examine

We can see that the periodogram is dominated by low-frequencies, which is a potential signal of long-memory property.

In the following section we are going to estimate the parameters related to the long-memory property in order to verify our hypothesis. Plotting in logarithmic scale:

This supports the property of long-range dependence.

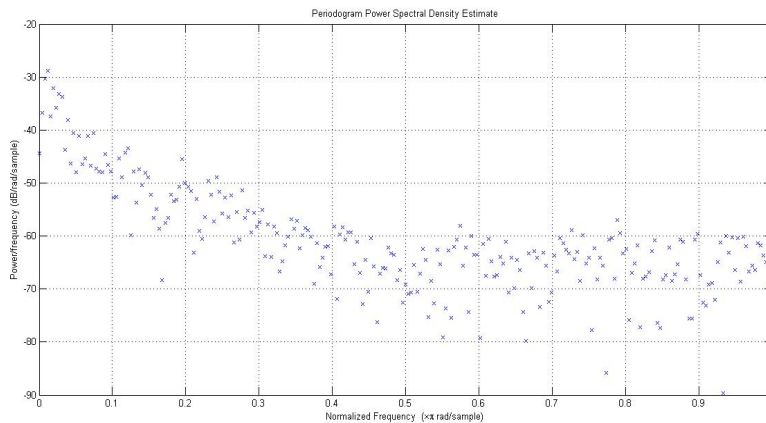


Figure 2.7: Periodogram

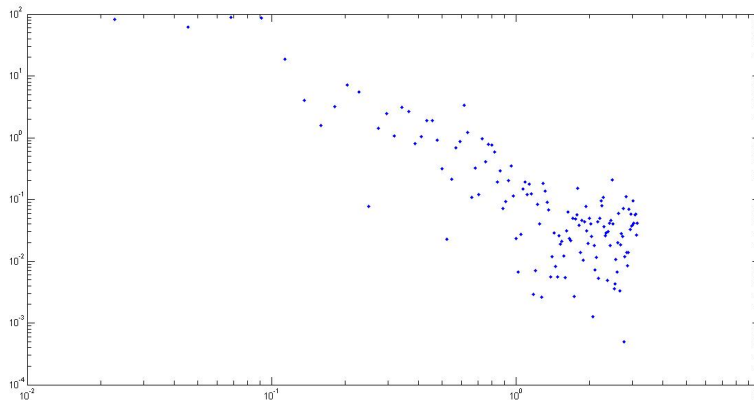


Figure 2.8: Log-Periodogram

2.5 Modelling housing prices

2.5.1 FARIMA model

Stationary models which belongs to the category of "long-memory" are FARIMA model and Fractional Gaussian Noise (fGn) Model.

In this section we will provide a brief review of FARIMA model and results found in the literature about its relationship with LRD are also presented. We will estimate the Hurst parameter, which is related to persistence and anti-persistence behavior.

Instead, the following section will focus on fractional Brownian Motion, given that fGn is the increment process of fBm, and in the estimates of the parameters for this model.

Fractional Gaussian noise can exhibit long-range dependence but its correlation is also specified at all lags. It may, therefore, not be suitable for modeling phenomena which display long-range dependence but where covariance at short lags differs from that of fGn. We shall introduce the so-called FARIMA class of time series models. They display long-range dependence but they also include additional parameters that allow fitting the covariance lags to the data at hand.

Essentially, FARIMA are an extension of the well-known ARMA model. The difference is that FARIMA models can also exhibit the long-range dependence property. The term ARMA stands for "autoregressive-moving average" and FARIMA (sometimes called ARFIMA) stands for "fractional autoregressive integrated moving average".

$\{X_t\}_{t \in \mathbb{Z}}$ is a Gaussian linear sequence if there are numbers c_j , $j \in \mathbb{Z}$, such that, for all $t \in \mathbb{Z}$,

$$X_t = \sum_{j=-\infty}^{\infty} c_{t-j} \epsilon_j = \sum_{j=-\infty}^{\infty} c_j \epsilon_{t-j} \quad (2.13)$$

where $\sum_{j=-\infty}^{\infty} c_j^2 < \infty$ and $\{\epsilon_j\}_{j \in \mathbb{Z}}$ are i.i.d. normal random variables, sometimes called *innovations*. If only past innovations matter, that is, $c_j = 0$, $j < 0$, then $\{X_t\}_{t \in \mathbb{Z}}$ is called *causal*. Observe that a Gaussian linear sequence $\{X_t\}_{t \in \mathbb{Z}}$ is stationary.

FARIMA models are particular cases of (Gaussian) linear sequence. They are usually denoted $FARIMA(p, d, q)$, where $p, q \in \mathbb{N} \cup \{0\}$ and $d \in \mathbb{R}$.

FARIMA(0,d,0) Let $d \in \mathbb{R}$. We say that a sequence $\{X_t\}_{t \in \mathbb{Z}}$ is a *FARIMA*(0, d , 0) if it satisfies the equation

$$\Delta^d X_t = \epsilon_t, \quad (2.14)$$

where $\{\epsilon_t\}_{t \in \mathbb{Z}}$ is i.i.d. $N(0, \sigma^2)$ sequence.

The operator Δ is called the difference operator. Its powers can be written in a more compact form as

$$\Delta^j = (I - B)^j, \quad j = 1, 2, \dots,$$

where B is the backward shift operator defined by $B^j X_t = X_{t-j}$, $j = 0, 1, 2, \dots$, and $I = B^0$. From eq.(14)

$$\Delta^d X_t = (I - B)^d X_t = \epsilon_t, \quad (2.15)$$

If the following series is convergent

$$X_t = (I - B)^{-d} \epsilon_t = \left(\sum_{j=0}^{\infty} b_j B^j \right) \epsilon_t = \sum_{j=0}^{\infty} b_j \epsilon_{t-j}$$

given

$$b_j = \prod_{k=1}^j \frac{k-1+d}{k} = \frac{\Gamma(j+d)}{\Gamma(j+1)\Gamma(d)}, \quad j \in \mathbb{N}, \quad (2.16)$$

the process is called *FARIMA*(0, d , 0) in the range $-\frac{1}{2} < d < \frac{1}{2}$.

Given the definition of a FARIMA process is easy to extend the differencing operator to the real line.

For instance $\Delta^{1.2} X_t = \epsilon_t$ meaning that $\Delta^{-0.2} X_t = \Delta \epsilon_t$, that is $\Delta^{-0.2} X_t = \epsilon_{t-t-1}$.

Recall that FARIMA(0,d,0) has LRD if and only if $0 < d < 1/2$.

Let's consider now the increments of an H -sssi process, $\gamma(k) = EX_i X_{i+k} \sim c|k|^{2H-2}$, for some constant c , as k tends to infinity. Comparing with the equation for a FARIMA process, we can investigate the correspondence between

the two exponents is $2H - 2 = 2d - 1$. This leading to

$$H = d + 1/2. \quad (2.17)$$

The range $0 < d < 1/2$ of long-range dependence corresponds to $1/2 < H < 1$.

2.5.2 FARIMA Model Estimation

For the estimation of the parameter we will use the Whittle approximation. It is a MLE estimator which allow to deal with large sample, avoiding computational problems.

$$\tilde{L}_W(\theta) = \frac{1}{\pi} \sum_{j=1}^{n^*} \log f(\nu_j; \theta) \frac{2\pi}{n} + \sum_{j=1}^{n^*} \frac{I(\nu_j)}{f(\nu_j; \theta)} \frac{2\pi}{n}$$

The method require a calibration of the expected spectral density $f(\nu; \theta)$ to the periodogram estimated by the available observations.

Considering a FARIMA(0,d,0) model, the spectral density will be

$$f(\nu; \theta) = \frac{\sigma_\epsilon^2}{2\pi} (2(1 - \cos\nu))^d = \frac{\sigma^2}{2\pi} \left(4 \sin^2 \frac{\nu}{2}\right)^d \quad (2.19)$$

Anyway, from the conclusions of our first examination, the best FARIMA model would be

$$FARIMA(a, d, 0) \quad (2.20)$$

Therefore we need to calculate the spectral density, which is associated to this stochastic process.

It is easy to show that if X_t , $t \in \mathbb{Z}$, is a FARIMA(p,d,q) sequence, then it has the spectral representation:

$$X_t = \int_{-\pi}^{\pi} e^{it\nu} (1 - e^{-i\nu})^{-d} \frac{\Theta_q(e^{-i\nu})}{\Phi_p(e^{-i\nu})} Z(d\nu), \quad (2.21)$$

Therefore it is straightforward to write the spectral density function:

$$f_X(\nu) = |1 - e^{-i\nu}|^{-2d} \left| \frac{\Theta_q(e^{-i\nu})}{\Phi_p(e^{-i\nu})} \right|^2 \frac{\sigma^2}{2\pi} = \quad (2.22)$$

$$= \left(4 \sin^2 \frac{\nu}{2}\right)^{-d} \left| \frac{\Theta_q(e^{-i\nu})}{\Phi_p(e^{-i\nu})} \right|^2 \frac{\sigma^2}{2\pi} \quad (2.23)$$

Notice that

$$C|\nu|^{-2d}, \quad (2.24)$$

for $\nu \rightarrow 0$, so that the range $0 < d < 1/2$ corresponds to the blow-up of the spectral density $f_X(\nu)$ at the origin, that is, to long-range dependence.

In the model that we have in mind FARIMA(p,d,0),

$$\Theta_q(e^{i\nu}) = 1 \quad (2.25)$$

Therefore

$$f_X(\nu) = |1 - e^{-i\nu}|^{-2d} \left| \frac{1}{\Phi_p(e^{-i\nu})} \right|^2 \frac{\sigma^2}{2\pi} \quad (2.26)$$

and recalling that

$$\Phi_p(e^{-i\nu}) = 1 - \Phi_1 e^{-i\nu} - \dots - \Phi_p e^{p(-i\nu)} \quad (2.27)$$

In the model we want to implement

$$\Phi_p(e^{-i\nu}) = 1 - \Phi_1 e^{-i\nu} \quad (2.28)$$

The spectral density will be

$$f_X(\nu) = |1 - e^{-i\nu}|^{-2d} \left| \frac{1}{1 - \Phi_1 e^{-i\nu}} \right|^2 \frac{\sigma^2}{2\pi} \quad (2.29)$$

being

$$e^{i\nu} = \cos\nu + i\sin\nu$$

we can write equation (40)

$$f_X(\nu) = |1 - e^{-i\nu}|^{-2d} \frac{1}{1 - 2\phi\cos\nu + \phi^2} \frac{\sigma^2}{2\pi} \quad (2.30)$$

and

$$f_X(\nu) = (2 - 2\cos\nu)^{-d} \left(\frac{1}{1 - 2\phi\cos\nu + \phi^2} \right)^{\frac{1}{2}} \frac{\sigma^2}{2\pi} \quad (2.31)$$

Estimation of the parameters

The estimation of the parameters lead to the following model

$$FARIMA(0.9, 0.3, 0) \quad (2.32)$$

In the following section we will present the fGN model and we will provide the estimate of the parameter

2.5.3 Fractional Gaussian Noise

The increments of an $H - sssi$ process $\{Z(t)\}_{t \in \mathbb{R}}$, $0 < H < 1$, i.e.,

$$X_k = Z(k+1) - Z(k), k \in \mathbb{Z}. \quad (2.33)$$

If $\{Z(t)\}_{t \in \mathbb{R}}$ is fBm, then $\{X(k)\}_{k \in \mathbb{Z}}$ is called fractional Gaussian Noise (fGn). Fractional Gaussian noise has the following properties:

- (a) $\{X_k\}_{k \in \mathbb{Z}}$ is stationary ,
- (b) $E[X_k] = 0$,
- (c) $E[X_k^2] = \sigma^2 = E[(Z(1))^2]$,
- (d) the autocovariance function of the process $\{X_k\}_{k \in \mathbb{Z}}$ is given by

$$\gamma(k) = E[X_i X_{i+k}] = \frac{\sigma^2}{2} (|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}) = \frac{\sigma^2}{2} \Delta^2 |k|^{2H}, \quad (2.34)$$

where Δ^2 denotes the second difference. ¹

- (e) Let $k \neq 0$. Then
 - $\gamma(k) = 0$, if $H = 1/2$,
 - $\gamma(k) < 0$, if $0 < H < 1/2$, and
 - $\gamma(k) > 0$, if $1/2 < H < 1$.

- (f) if $H \neq 1/2$, then

$$\gamma(k) \sim \sigma^2 H(2H-1) |k|^{2H-2}, \quad (2.35)$$

as $k \rightarrow \infty$.

¹ $\Delta^2 X_k = X_{k+1} - 2X_k + X_{k-1}$

Estimating the parameters

The estimated parameter by local whittle estimator is

$$H = 1 \tag{2.36}$$

2.6 Structural Breaks

Structural breaks are not inconsistent with the persistence property of a time series.

In fact, we can have persistence in time series before or after structural changes in the series.

We analyzed the structure of persistence and we experienced that one shock can affect the path of the housing prices for the following three years.

It seems clear from the series in level and clearer from its first difference that some major changes have been experienced.

Moreover, the parameter d for long memory is shown to be related [23] to the number of structural breaks which can be detected into the series. As d is approaching the range for non-stationarity, i.e. the value of d increases, the number of the breaks seems to be increasing.

A standard analysis is provided in which we try to investigate the changing points and the intensity of the breaks.

2.6.1 Estimation of the number and intensity of changing points

In this paragraph we will detect structural changes in housing price movements.

Series presenting long-memory can assume local trends and cyclicity at any frequency, then they adjust to the mean or to the stochastic mean and frequency in the long run are superimposed at any frequencies.

First we investigate the series in difference and then we will try to model a window of fifty observation by an AR model. We iterate the procedure with a rolling sample and see if and how the parameters is affected by major changes.

Given H_t the series of housing prices we will estimate

$$H_t = \xi H_{t-1} + \varepsilon \quad (2.37)$$

where $H_t \in [H_i, H_{i+50}]$ and for $i=[1, T-50]$.

We have individuated three main changes and they have been clustered to form three groups and calculated the mean within these groups.

$$H_t = -0.21H_{t-1} + \varepsilon \quad (2.38)$$

for $H_t \in [H_{t=0}, H_{t=150}]$

$$H_t = +0.34H_{t-1} + \varepsilon \quad (2.39)$$

for $H_t \in [H_{t=150}, H_{t=250}]$ q

$$H_t = +0.02H_{t-1} + \varepsilon \quad (2.40)$$

for $H_t \in [H_{t=250}, H_{t=299}]$

then we run a regression within the groups to estimate the standard deviation of the estimated parameters and plot the probability distribution associated to them.

For a better identification of the breaks we can see the flat graphics, in this the color scale goes from red to dark blue. Red is associated to the higher value of the probability density function.

Meaning that we can individuate the following breaks in the behavior of housing prices

Conclusions

The aim of the analysis is to test the hypothesis and to put a bridge between the so called "economics of inaction" and long-range dependence.

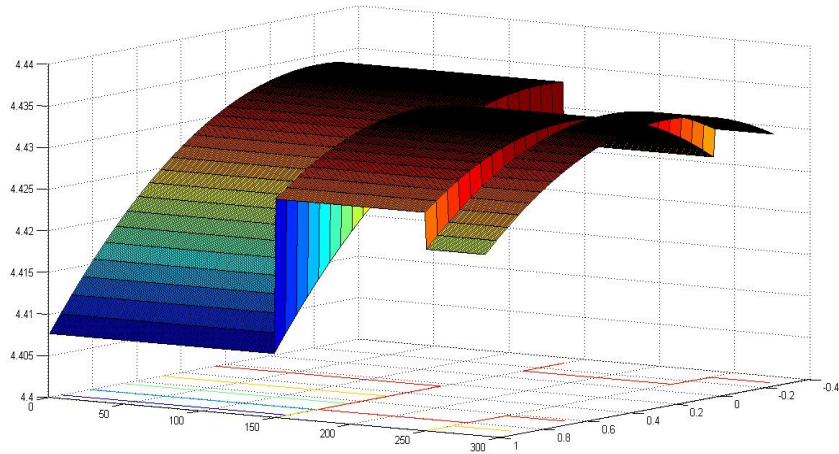


Figure 2.9: Structural breaks in the estimated parameters

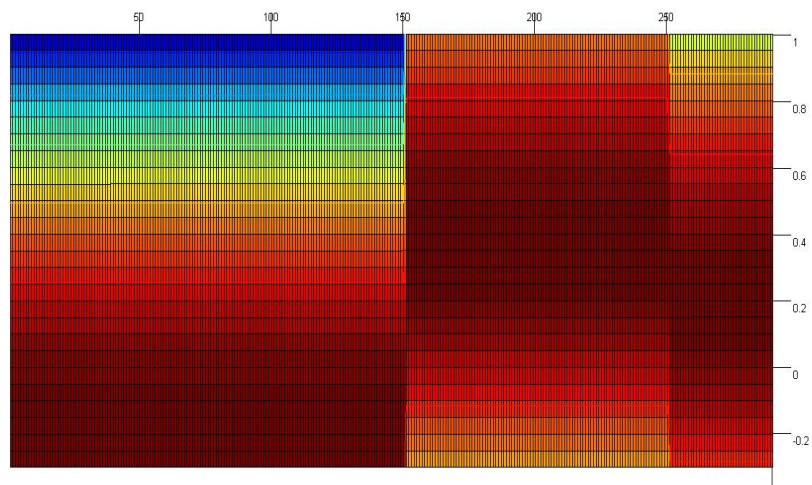


Figure 2.10: Structural breaks in the estimated parameters-2

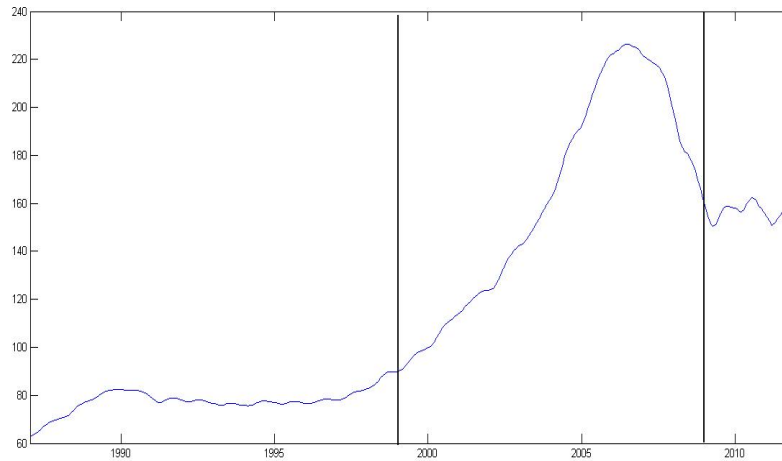


Figure 2.11: Housing prices time series and localization of breaks

It is the nature itself of the underlying asset suggesting the presence of long-memory property in the data. In fact, this property is detected in stock prices and for the markets in which, for financial and non-financial costs, the timing of purchased is delayed with respect to the optimal one in complete markets. Of course, the housing sector is the one which experiences an extremely wide inaction region.

This work consists in a detailed examination of the relationship between long-memory property and housing prices.

The first analysis is standard and stationarity had been tested. The Augmented Dickey fuller test cannot reject the null of unit root. Therefore, a stochastic trend in the data is revealed and it is one of the leading statistical factor in housing prices.

Second, the long-range dependence hypothesis is tested. After the euristical analysis, the graph of the variance of the sample mean, the presence of the property of interest cannot be rejected and further investigations are implemented.

The logarithm of the periodogram shows that a model for persistence should be appropriate for the series.

Two models, FARIMA and Fractional Gaussian Noise, are proposed and the relative parameters related to persistence had been estimated. As expected, the value of the Hurst exponent is very close to the unit, meaning that the persistence in the data is extremely high.

Finally, we wanted to test for structural breaks in the series. As we said several times persistence and structural breaks are not in contradiction. Persistence consists in a behavior of the series following or preceding shocks. These shock sometimes can lead to structural breaks in the time series and the bigger is the intensity of persistence, the bigger is the effects of these shocks.

For completeness we investigated for breaks and we found two main years in which the change in price affected substantially the path of the series for following years (1999 and 2009).

Anyway, The reason for a stochastic trend we found at the beginning of the analysis and the leading economical factor cannot be fully understood only by method applied for time-series.

Anyway, the LRD jointly with the structural break investigation can found a very reasonable counterpart in what experienced in the market.

Anyway, the motivations, which help in understanding why housing prices show "momentum", can help in understanding why instead we can observe a rapid drop in the year 2008.

In the year 2008, housing mortgages couldn't be afforded by an above than average number of households and they were forced to adjust consumption at their optimal level. The purchase of a new house couldn't be avoided or delayed as usual. The number of transactions increased in the same time and arguments of low transactions and delayed timing of purchase couldn't apply and at the same the reasons for long-range dependence. A structural break is evident.

After this break in which it seems that the series experienced therefore a jump, more stable prices and persistence followed, consistently with a FARIMA or fGN behavior.

Further developments imply a multivariate analysis that can explore the economical reason for the trend, which we observe statistically in the data and factors, which can force the price in rapid drops and to experience structural

breaks (i.e. jumps)

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Chapter 3

Stocks and Housing over the Life Cycle

Keywords [Housing, Durable Consumption, Stopping Time, Portfolio Choice]

Abstract

The aim of this research is to develop and implement a model for durable consumption and portfolio choice over the life cycle.

The purchase of a durable good involve financial and non financial costs that imply an inaction region, where the household decides to delay his choice even if his level of consumption over wealth is above or under the optimal one. The environment is an (S,s) model, well known in the literature for infrequent choice problems.

First, will be provided the optimal policy and the inaction region.

Labor income will play a key role in this dynamics together with returns on financial assets.

3.1 Introduction

The housing market dynamics in the recent years has drawn the attention not only of the economists but of politicians and households themselves. In fact, the housing markets enters in the economy system via different channels, as a primary good, as an investment, as a collateral, as the underlying of a derivative. The multiple facets that this asset presents is the reason that makes it unique and very complex to analyze.

Literature for asset pricing, durable consumption, DSGE models, search and matching studies tried to consider the influence of housing in the stock market, or tried to model its dynamics.

The recent financial crisis has highlighted once again the importance and the integration of housing in the financial economy.

The consumption-side of housing was studied in the past. The techniques used were the same as general durable goods, computers and cars.

The basic idea consists in modeling new purchases as infrequent choices. This is a natural assumption coming from behavioral observation, the size of the durable good is not adjusted continuously due to transactional costs, financial and non-financial ones, therefore the size of the durable good is optimal only when the new choice is made. The owner waits until the ratio between his wealth and the value of the asset reaches a critical level, which will be called threshold level, and a new purchases will substitute the old durable.

The natural framework for this research in an (S,s) model, which will be reviewed in the following section.

Instead financial implication have been introduced in order to improve asset pricing models. The importance of the housing sector from an asset pricing perspective is highlighted also by Piazzesi, Schneider and Tuzel (2007). Separating housing from non-housing consumption expenditure, they develop a model that overperform the standard CCAPM, implying higher equity premia and higher stock return volatility.

Financial literature considering housing is growing and works exploiting the relationship between investment in stocks and this asset are quite numerous. An important sector in this literature is exploiting the importance of labor income the life-cycle. Cocco, Gomes and Maenhout solved a realistically calibrated life-cycle model of consumption and portfolio choice with labor income and borrowing constraints. This is just an example but the importance of labor income while considering investment in stocks and the

purchase of durable goods is also reasonable and clearly experienced from household's perspective.

The present research is devoted to the development of a model for optimal consumption and portfolio choice over the life-cycle considering income from labor in an (S,s) framework.

The agent chooses the amount of durable (house) and the timing of purchase, τ . In fact, because of transactions costs and other financial and non-financial ones, the consumer prefers not to adjust his level of housing services continuously but rather to delay until the ratio of his consumption over wealth reaches a threshold level.

The state variable $z = \frac{c}{\bar{w}}$ moves stochastically in a range (S,s) which we call inaction range because, while the variable is in this set, no action is taken by the consumer. When z is in $] -\infty, s]$ and $[-S, +\infty[$ the old durable is replaced in order to adjust consumption immediately at his optimal level z^* . In the purchase a cost is involved which is proportional to the value of the new durable.

The agent can also invest in n risky assets and in the risk free asset. Income changes during his life according to the evolution of wealth.

To treat durable consumption as an infrequent choice is quite reasonable. The Grossman and Laroque (1990) model is well known in the literature for infrequent choice of housing stock. It nests portfolio choice and optimal policy for durable consumption in a (S,s) framework, introduced in order to model inventory by Arrow, Harris and Marschak (1951).

The (S,s) models are universally applied to solve problems where the core elements are uncertainty and fixed costs of adjustment.

"The (S,s) model opened the door to a quite startling range of important and challenging follow-up questions, many of great practical importance and analytic depth" ... "has become one of the touchstone models of economics" [Economic Theory and the World of practice: A celebration of (S,s) Model (2010); Caplin and Leahy].

Caballero (1991) analyzes the methodology of Ss Dynamics giving a tutorial of reference for every future work.

The Grossman and Laroque model is stylized and do not consider non-durable consumption choice. The only source of uncertainty is given by the the value of the assets (stocks and bonds) in the household's portfolio and income from labor is absent. They show that small costs of changing

consumption levels will lead consumption to be insensitive to wealth for very long periods of time. Moreover, they prove that costs in the transaction do not alter the optimal portfolio constituted by the mean-variance efficient one. The simple but powerful structure opened a new research line. Flavin and Nakagawa (2004) provided an extension of the original model introducing non-durable consumption creating a bridge between the model introduced in the '90 and the standard consumption beta-model. They find that the housing model generates many of the implications of the habit persistence, such as smooth non-durable consumption, state dependent risk-aversion and a small elasticity of intertemporal substitution despite moderate risk aversion.

The necessity of improving the model of Grossman and Laroque, considering also earning as an important and considerable part of it, is very reasonable. In the literature a lot of research relating consumption choice and earnings is developed. We considered the apport given by "Consumption Response to Income Changes" (Pistaferri, Japelli 2009),"Uncertainty and Consumer durables adjustment" (Pistaferri, Guiso, Bertola; 2002).

We presented above the fields that are going to interact: asset pricing and portfolio choice, durable consumption and earnings. The particular environment in which we allow them to interact, the (S,s) framework, force us to consider lifecycle patterns. Anyway there are several works that study how the dynamics of earnings and wages affect consumption choices over the life cycle. Labor economists and macroeconomists are the main contributor to this area of research. We will refer essentially to "Consumption and Portfolio Choice Over Life Cycle" (Cocco,2005) and "Earnings, Consumption and Lifecycle choices" (Meghir and Pistaferri, 2010).

Anyway, none of them consider transaction costs.

3.1.1 Discussion of "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Good"

The authors analyze a model of optimal consumption and portfolio selection in which consumption services are generated by holding a durable physical asset.

The financial assets are continuously chosen and determine the evolution of

wealth between adjustments and also enter in the return function directly. The only consumption is the service flow from the durable. Non-durable consumption is not considered in their investigation.

The reason lies in what is called "insulation effect": non-durable consumption is unaffected by wealth shocks, but the timing of the purchase and the size of the durable good is.

Evidences of this microeconomic behavior can be found also at aggregate level in which we can see that non-durable consumption is not very volatile, compared to the consumption of durables.

Non durables can be consider in first approximation just a constant over the life cycle and this cannot affect the household in this environment. If introduced, it would act just as a rescaling both in the state variable and in the barriers, and the results will result not affected.

Maintenance costs or, an alternative way to see the same time effect on the durable good, the offsetting depreciation are considered.

This effect is captured by the model jointly with the fact that the purchase is irreversible, meaning that it is an illiquid asset and the purchase and sale prices for capital are not equal.

The optimal policy is characterized by three different values, the two barriers and the return point.

This pioneer work is very important in the literature, because it is the first research trying to link financial investments, i.e. portfolio choice, and methodology coming from literature on durable consumption.

The current work wants to extend the research of Grossman and Laroque nesting the recent theories of portfolio choice over the life cycle, in which income from labor plays a crucial role.

The economical importance of updating the well-know "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid Durable Consumption Good" is motivated especially by the fact that stock market dynamics are not sufficient per se for explaining durable consumption choice (size and timing) or portfolio choice over the life-cycle.

3.1.2 (S,s) Models: An Introduction

(S,s) models are in the broad category of stochastic control models with fixed costs. The main use is to model infrequent choice, inventory problems (Merton), withdrawals (Lippi, Alvarez), car and computer purchases.

The appealing characteristic of (S,s) models is its conceptual simplicity.

Essentially, the agent is already endowed by the durable good and the expected time until a new adjustment occurs is modeled.

The one-side version of the model considers the fact that adjustments entail only an upgrade in the size/quantity of the good (car, cash,...).

In this framework two-sided (S,s) model is considered, the agent can choose if upgrading or down grading its asset depending on the evolution of wealth. The agent is endowed by a house, he observes the ratio of wealth and asset value changing over time. Financial and non financial costs prevent him to change the size of the durable good continuously and he waits, stays inactive, until a critical level of this ratio is reached, meaning that the house appreciated too much with respect to his total, liquid plus illiquid, wealth, or meaning that the value of the house is too small compared to his wealth and he is willing to adjust the size of the durable with a new purchase.

In spite of the simplicity of the underlying motivation for using these class of models, the difficulties which arise from a mathematical and numerical point of view are extremely high. Very often the solution to the Hamilton Jacobi Bellman equation has no close form and numerical methods, that must consider stability issues, need to be performed.

It follows a graphical representation of a generic state variable, for the sake of simplicity leaded by a Brownian Motion, in a (S,s) framework.

When the upper or lower barrier is reached, suddenly (i.e. instantaneously) the value is adjusted at the optimal one.

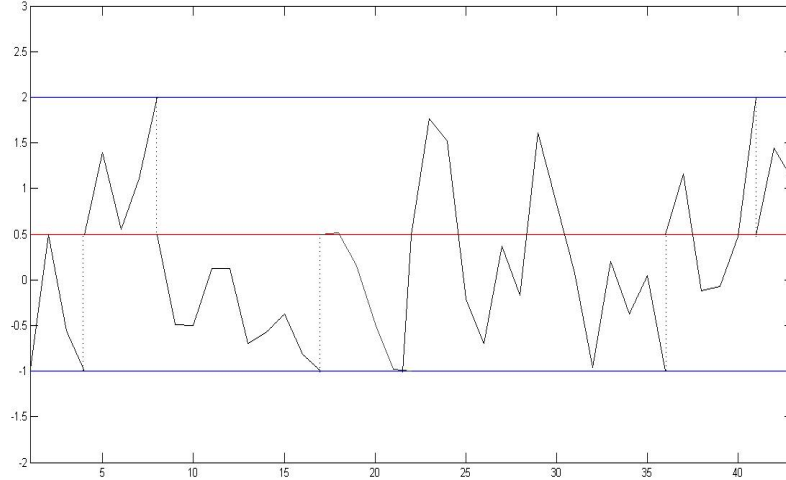


Figure 3.1: Example of a state-variable in a (s,S) range

3.2 The Model

3.2.1 Preferences

The economy is characterized on the real side by a unique consumption good, used as a numeraire. Then the flow of services is provided by the only tangible asset K (the house). This property depreciates at a rate β and $\beta > 0$, consistent with the utilization and the aging process.

Moreover, as we anticipated before, transaction fees exclude a perfect market and this is translated in an additional cost the household must pay. The price of the new asset is augmented by a fraction λ of its original value and $0 \leq \lambda \leq 1$

We assume that the utility function is an isoelastic utility function, i.e. exhibits constant relative risk aversion:

$$u(K) = \begin{cases} \frac{K^a}{a}, & \text{if } a < 1, a \neq 0 \\ \log(K), & \text{the limiting case for } a = 0 \end{cases} \quad (3.1)$$

The case in which $a = 0$ (log utility) must be treated in a separate section. We recall that

$$U(K) = \frac{K^a - 1}{a} \quad a < 1 \text{ and } a \neq 0$$

$$\lim_{a \rightarrow 0} \frac{K^a - 1}{a} = \lim_{a \rightarrow 0} \frac{e^{\log(K)a} - 1}{a} = \log(K)$$

Applying in the last equivalence H ˆopital's rule.

The parameter a governs the willingness to substitute consumption across both time period and states of the world. It implies an elasticity of substitution for consumption bundles across period and states is constant and equal to $\frac{1}{1-a}$.

The assumption of iso-elastic utility function is used for simplicity, following the broadly use of it in durable consumption models. These utility functions imply that if a given percentage asset allocation is optimal for some current level of wealth, that same percentage allocation is also optimal for all other levels of wealth. Satiation argument and the fact that housing is a primary good suggest that research on preferences about housing should be implemented in the future.

The agent enters in the economy at the age t_0 and live at most until the age T , being retired from the age of t_R . For the sake of simplicity, we take these values as exogenous and deterministic.

3.2.2 Labor Income

Before retirement, investor's age is t and his labor income L_t is exogenously given by:

$$\log(L_t) = f(t) + \nu_t + \varepsilon_t \quad \text{for } t \leq t_r \quad (3.2)$$

where $f(t)$ is a *deterministic* function of age and ν_t is a Wiener process given by

1. $\nu_0 = 0$

2. ν_t is continuous
3. ν_t has independent increment.

Thus before retirement, log income is the sum of a deterministic component that can be calibrated in order to capture the hump shape of earnings over the life cycle, and two random components, one transitory and one persistent.

The process for ν_t in Carrol(1997) and Gourinchas and Parker(2002) is taken to be a random walk. Since we are in a continuous framework we follow their study considering a Wiener process, the natural extension in the continuous case of a random walk.

Hubbard, Skinner and Zeldes (1995) estimate a general first-order autoregressive process and find the autoregressive coefficient to be very close to one.

This findings suggest instead that a unit root is present when we consider an autoregressive process of order one. We take this result and as before we use for the continuous case considering a "zero-drift" process.

Deterministic component of labor income

We assume that a deterministic component affects labor income over life-cycle. This factor is related to age in the way that $f(t)$ is a quadratic expansion of the age.

$$f(t) = e_1t + e_2t^2 + e_3t^3 + e_4t^4 \quad (3.3)$$

The coefficients of the polynomial are estimated in the appendix and a linear model is chosen for labor income as a function of the age.

Age is considered as the principal leading factor for human capital accumulation and labor income over life cycle. Therefore in the investigation it is considered in a first approximation as the only determinant.

3.2.3 Portfolio Composition

The choice, in which we are focused mainly, is the timing of Real Estate investment. Anyway there is another sector where opportunities for invest-

ment are important and to be considered: the Stock Market. On the financial side the economy is characterized by:

- the risk free asset

The amount of cash, whose accumulated value at time t , denoted $x(t)$, increases when trading does not take place, according to the following equation

$$B(t) = Be^{rt} \quad (3.4)$$

and

$$dB(t) = rB(t)dt \quad (3.5)$$

where r the rate of interest is assumed to be constant.

- multiple risky assets

There are n securities (equities and corporate bonds) indexed by j and having price $X_j(t)$, X denote the n -dimensional Price process. There are n Brownian motions $w_j(t)$ without drift. Instead diffusion terms and drift of the risky assets, $\mu'_X = (\mu_1, \mu_2, \dots, \mu_n)$ and Σ_X (the instantaneous positive definite covariance matrix), are constant over time.

Evolution of Prices

$$dX(t) = X(t) (\mu_X dt + \Sigma_X dw_t)$$

where

$$X(t) = \begin{pmatrix} X_1(t) & 0 & \dots & 0 \\ 0 & X_2(t) & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & \dots & 0 & X_n(t) \end{pmatrix}$$

In our model, we consider also income from labor which changes of course the amount invested in the financial assets.

3.2.4 Budget Equation

When the agent doesn't sell the house ($t < \tau$, where τ indicates the optimal stopping time), the evolution of the property value K , due to depreciation,

is given by

$$dK_t = -\alpha K_t dt \quad (3.6)$$

The total wealth of the consumer evolves according the appreciation/depreciation of the assets held in the portfolio (risk free asset B_t and risky assets X_t) income from labor, L_t and the durable asset K_t :

$$dQ_t = dK_t + dB_t + dX_t + L_t dt \quad (3.7)$$

Each time the consumer has to face an optimization problem in which he must decide whether to sell or not the current stock of housing.

$$dQ_t = -\alpha K_t dt + r_f B_t dt + X_t(\mu_X dt + dw_t) + L_t dt$$

The total wealth of the consumer, Q , each time is given by the house, the risk free asset, the value of the stocks augmented by the returns (either positive or negative) on these assets X_t , augmented by Labor income and diminished by the depreciation of the durable consumption good

$$Q_t = K_t + B_t + X_t + L_t \quad (3.8)$$

3.3 Bellman Equation

Each time the wealth of the agents evolves, according to the value of the durable asset, the stocks and income from labour:

$$dQ_t = -\alpha K_t dt + r_f B_t dt + X_t(\mu_X dt + \Sigma_X dw_t) + L_t dt$$

or alternatively

$$dQ_t = (Q_t - K_t) \left(\pi \frac{dX_t}{X_t} + (1 - \pi) \frac{dB_t}{B_t} \right) - \alpha K_t dt + L_t dt$$

reinvesting at each time L_t

$$dQ_t = (Q_t - K_t) \left(\pi \frac{dX_t}{X_t} + (1 - \pi) \frac{dB_t}{B_t} \right) - \alpha K_t dt + \pi \frac{L_t}{X_t} dt + (1 - \pi) \frac{L_t}{B_t} dt$$

$$dQ_t = (Q_t - K_t) (\pi \mu_X dt + \pi \Sigma_X dw_{X_t} + (1 - \pi) r_f dt) - \alpha K_t dt + \pi \frac{L_t}{X_t} dt + (1 - \pi) \frac{L_t}{B_t} dt$$

Of course at each point in time he has to decide whether to maintain the old durable or to purchase a new one. If he purchase a new one K_2 this means that $V(Q_t, K_t, L_t) = \max(Q_t - \lambda K_t, K_2, t)$. If instead $V(Q_t, K_t, t) \geq \max(Q_t - \lambda K_t, K_2, t)$ This means that not selling the house the agent will reach an higher level of utility and satisfies the following Hamilton-Jacoby-Bellman equation:

$$V(Q_t, K_t, t) = \max_{K_2, \tau, X_t} \mathbb{E} \left[\int_t^\tau e^{-\delta s} \frac{K_s^a}{a} ds + e^{-\delta \tau} V(Q_\tau - \lambda K_\tau, K_2, \tau) \right]$$

where τ is the optimal stopping time.

We can reduce the State-Space making use of a change of variable.

$$\begin{aligned} y_t &= \frac{Q_t}{K_t} - \lambda \\ x_t &= K_t^{-1} X_t \end{aligned}$$

Moreover, since $V(Q_t, K_t, t)$ we assume that

$$V(Q_t, K_t, t) = K_t^a V(y_t, 1, t) = K_t^a J(y_t, t) \quad (3.9)$$

$$H(y_t, t) = J(y_t + \lambda, t) \quad (3.10)$$

$$K_t^a H(y_t, t) = \max_{K_2, \tau, x_t} \mathbb{E} \left[\int_t^\tau e^{-\delta s} \frac{(K_s e^{-\alpha(s-t)})^a}{a} ds + e^{-\delta \tau} H(Q_\tau - \lambda K_\tau - \lambda K_2, \tau) \right]$$

working out the equation (see Mathematical Appendix), we obtain:

$$H(y, t = 0) = \max_{K_2, \tau, x_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta_2 \tau} (y_\tau)^a H \left(\left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-1} \right) \right] \quad (3.11)$$

We are solving the optimal stopping problem. Essentially, we model infrequent choice in the way that he takes no action until the best choice for him is to change the housing stock K in K_2 . Therefore, two different situations must be analyzed:

- It is optimal not to stop and delay the new purchase.
- It is optimal to re-adjust the consumption size.

When it is optimal to not change the housing stocks, we do not observe a discontinuity in the Wealth Function, i.e.

$$\lim_{t \rightarrow t^+} Q(t) = \lim_{t \rightarrow t^-} Q(t)$$

and, as we defined before,

$$dQ_t = -\alpha K_t dt + r_f B_t dt + X_t(\mu_X dt + dw_{X_t}) + L_t dt$$

When the agent faces a new purchase, he has to pay transaction fees, which of course introduces a jump in the the wealth process. The adjustment is instantaneous, meaning that at time τ he observes his consumption over wealth and given that it is optimal to adjust the housing stock, he buy a new house K_2 and pay the fees at the same time τ . This entails of course a discontinuity in the wealth process where

$$\lim_{t \rightarrow t^+} Q(t) \neq \lim_{t \rightarrow t^-} Q(t)$$

and

$$dQ_{(t^-)} = -\alpha K_{t^-} dt + r_f B_{t^-} dt + X_{t^-}(\mu_X dt + dw_{X_{t^-}}) + L_t dt$$

$$dQ_{(t^+)} = -\alpha K_{t^-} dt + r_f B_{t^-} dt + X_{t^-}(\mu_X dt + dw_{X_{t^-}}) + L_t dt - \lambda K_{t^-}$$

Combining the above equations

$$Q_{t^+} = Q_{t^-} - \lambda K_{t^-}$$

To work out the solution, we start from the value function at the time when it is optimal to stop $t = \tau$.

It is easy to see from the Bellman equation that

$$V(Q_{\tau+}, K_2, \tau) = V(Q_{\tau} - \lambda K_{\tau}, K_2, \tau)$$

and then

$$H(y_{\tau}, \tau) = (y_{\tau})^a H\left(\frac{Q_{\tau} - \lambda K_{\tau}}{K_2} - \lambda, \tau\right) \left(\frac{Q_{\tau} - \lambda K_{\tau}}{K_2}\right)^{-a}$$

While when it is not optimal to stop, i.e. we haven't discontinuity in the wealth process, the bellman equation is the following

$$H(y_t, t) = \max \mathbb{E}_t \left[\frac{e^{-\delta_2 t}}{a} ds + e^{-\delta_2 t} H(y_t + dy, t + dt) \right]$$

The previous equation must hold at each t , for $t \in (0, \tau)$
it can be considered also at time $t = 0$

$$H(y_t, t = 0) = \max \mathbb{E}_0 \left[\frac{e^{-\delta_2 t}}{a} dt + e^{-\delta_2 t} H(y_t + dy, t + dt) \right]$$

$$H(y, t = 0) = \max \mathbb{E}_0 \left[\int_0^t \frac{e^{-\delta_2 s}}{a} ds + e^{-\delta_2 t} H(y_t, t) \right]$$

By iteration it can be found that also the following must hold for each $t \leq \tau$

$$H(y_t, t) = \max \mathbb{E}_t \left[\int_t^{(t+\Delta t)} \frac{e^{-\delta_2 s}}{a} ds + e^{-\delta_2(t+\Delta t)} H(y_t + \Delta y, t + \Delta t) \right]$$

$$\max \mathbb{E}_t \left[e^{-\delta_2 \Delta t} H(y_t + \Delta y, t + \Delta t) - H(y_t, t) - \int_t^{(t+\Delta t)} \frac{e^{-\delta_2 s}}{a} ds \right] = 0$$

And multiplying everything for $\frac{1}{\Delta t}$

$$\max \mathbb{E}_t \left[\frac{1}{\Delta t} e^{-\delta_2 \Delta t} H(y_t + \Delta y, t + \Delta t) - \frac{1}{\Delta t} H(y_t, t) - \frac{1}{\Delta t} \int_t^{(t+\Delta t)} \frac{e^{-\delta_2 s}}{a} ds \right] = 0$$

$$\mathbb{E} \left[\frac{e^{-\delta_2 \Delta t} H(t + \Delta t, y_t + \Delta y) - H(t, y_t)}{\Delta t} \right] = \int_t^{t+\Delta t} \frac{e^{-\delta_2 s}}{a} ds = -\frac{e^{\delta_2 t}}{a}$$

and as Δy and Δt tend to 0

$$\mathbb{E} [dH(y_t, t)] = -\frac{e^{\delta_2 t}}{a} dt$$

3.4 The Dynamic Budget Constraint as a Diffusion Process

Starting from the framework in section (4) In the previous paragraph , we operated a change of variable reducing the State Space to a two dimension. There is the need now to make a restatement of the Budget Equation as a function of the new variable.

First, we want to calculate the differential of our new State-Variable y :

$$dy = \frac{dQ_t}{K_t} - \frac{Q_t dK_t}{K_t^2}$$

$$\frac{dQ_t}{K_t} = -\alpha dt + r_f \frac{B_t}{K_t} + \frac{X_t}{K_t} (\mu_X dt + \Sigma dw_{Xt}) + \frac{L_t}{K_t} dt$$

Renaming

$$x_t = \frac{X_t}{K_t}$$

$$l_t = \frac{L_t}{K_t}$$

We obtain:

$$\frac{dQ_t}{K_t} = -\alpha dt + r_f \frac{B_t}{K_t} + x_t (\mu_X dt + \Sigma dw_{Xt}) + l_t dt$$

The evolution over time of the asset value is $K_t = Ke^{-\alpha t}$. From here:

$$dK_t = -\alpha K_t dt$$

and

$$\frac{Q_t dK_t}{K_t^2} = -\alpha \frac{Q_t}{K_t} dt = -\alpha(y_t + \lambda) dt$$

$$dy_t = \alpha(y_t + \lambda - 1) dt + r_f \frac{B_t}{K_t} + x_t(\mu_X dt + \Sigma dw_{Xt}) + l_t dt$$

Now using the fact that

$$B_t = Q_t - K_t - X_t - L_t$$

we obtain:

$$dy_t = (r_f + \alpha)(y_t + \lambda - 1) dt + (1 - r_f)(l_t dt) + x_t(\mu_X dt + \Sigma dw_{Xt})$$

$$dy_t = [(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu_X] dt + x_t \Sigma dw_{Xt} \quad (3.12)$$

meaning that $y(t)$ is a Generalized Wiener Process

in fact can be written in the equivalent form:

$$dy_t = \beta_t dt + \xi_t dB \quad (3.13)$$

This enables us to go back to the Solution of the HJB Equation, since now we know that we can calculate the differential of H , dH by applying Ito' lemma.

3.5 The Dynamic Budget Constraint as a Diffusion Process II

In the previous paragraph, we operated a change of variable, considering the first framework in section(4). Now we consider the second framework

$$dy = \frac{dQ_t}{K_t} - \frac{Q_t dK_t}{K_t^2}$$

$$\frac{dQ_t}{K_t} = -\alpha dt + \left(\frac{Q_t}{K_t} - 1 \right) (\pi \mu_X dt + \pi \Sigma_X dw_{Xt} + (1-\pi)r_f dt) + \pi \frac{L_t}{K_t X_t} dt + (1-\pi) \frac{L_t}{B_t K_t} dt$$

Renaming

$$l_{xt} = \frac{L_t}{X_t K_t}$$

$$l_{bt} = \frac{L_t}{B_t K_t}$$

We obtain:

$$\frac{dQ_t}{K_t} = -\alpha dt + (y_t + \lambda - 1) (\pi \mu_X dt + \pi \Sigma_X dw_{Xt} + (1-\pi)r_f dt) + \pi l_{xt} dt + (1-\pi) l_{bt} dt$$

The evolution over time of the asset value is $K_t = K e^{-\alpha t}$. From here:

$$dK_t = -\alpha K_t dt$$

and

$$\frac{Q_t dK_t}{K_t^2} = -\alpha \frac{Q_t}{K_t} dt = -\alpha (y_t + \lambda) dt$$

$$dy_t = (y_t + \lambda - 1) (\alpha dt + \pi \mu_X dt + \pi \Sigma_X dw_{Xt} + (1-\pi)r_f dt) + \pi l_{xt} dt + (1-\pi) l_{bt} dt$$

meaning that $y(t)$ is a Generalized Wiener Process

in fact can be written in the equivalent form:

$$dy_t = \tilde{\beta}_t dt + \tilde{\xi}_t dB \tag{3.14}$$

This enables us to go back to the Solution of the HJB Equation, since now we know that we can calculate the differential of H , dH by applying Ito' lemma.

3.6 HJB Equation and Ito's Lemma for the Inaction Region

Starting from equation (14)

$$dy_t = \beta_t dt + \xi_t dB$$

and the function $H(y_t, t)$

We can apply Ito's Lemma

$$dH(t, y_t) = \left(\frac{\partial H}{\partial t} + \beta_t \frac{\partial H}{\partial y} + \frac{\xi_t^2}{2} \frac{\partial^2 H}{\partial y^2} \right) dt + \xi_t \frac{\partial H}{\partial y} dB_t \quad (3.15)$$

where

$$\beta_t \frac{\partial H}{\partial y} = \frac{\partial H(y_t, t)}{\partial y} [(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu]$$

$$\frac{\xi_t^2}{2} \frac{\partial^2 H}{\partial y^2} = \frac{1}{2} \frac{\partial^2 H(y_t, t)}{\partial y^2} \underline{x} \Sigma_X \underline{x}'$$

$$\xi_t \frac{\partial H}{\partial y} = \frac{\partial H(y_t, t)}{\partial y} \underline{x} \Sigma_X$$

Summing up

$$dH(t, y_t) = \partial_y H(y_t, t) [(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu] dt + \frac{1}{2} \partial_{yy} H(y_t, t) \underline{x} \Sigma \underline{x}' dt + \partial_y H(y_t, t) \underline{x} \Sigma dB_t + \partial_t H(y, t) dt \quad (3.16)$$

Moreover, considering (section 4)

$$\mathbb{E} [dH(t, y_t)] = \mathbb{E} [H(t + dt, y_t + dy) - H(t, y_t)] = -\frac{e^{-\delta_2 t}}{a} dt$$

for $\Delta t \rightarrow 0$ and $\Delta y \rightarrow 0$

$$\begin{aligned} \mathbb{E} [dH(t, y)] = & \partial_y H(y_t, t)[(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu] dt \\ & + \frac{1}{2} \partial_{yy} H(y_t) \underline{x} \Sigma_X \underline{x}' dt + \partial_t H(y, t) dt \end{aligned} \quad (3.17)$$

Finally,

$$\partial_y H(y_t, t)[(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu] + \frac{1}{2} \partial_{yy} H(y_t) \underline{x} \Sigma_X \underline{x}' + \partial_t H(y, t) + \frac{e^{-\delta t}}{a} = 0 \quad (3.18)$$

This second order differential equation describes the evolution of the value function over time.

It contains an explicit dependence on the subjective discount factor δ augmented by the quantity $a\alpha$, which allow to take into account the depreciation rate α of the asset and the risk aversion of the agent a .

The evolution is dependent on the risk-free rate r_f and the fraction of wealth over the value of the house y_t .

L_t represents the expected labor income at time t , this component will be estimated in one of the following sections.

The property of interest is the deterministic path of labor income over the life cycle, which will be captured from a regression that will take into account yearly effects (overall economic condition) and individual effects, in order to isolate the effect of age.

In the following section the evolution of the value function in the Grossman and Laroque formulation and a new approach in finding the numerical solution.

3.6.1 Optimal Portfolio and Consumption Rule

Recalling the equation (18)

$$\begin{aligned} \mathbb{E} [dH(t, y)] = & \partial_y H(y_t, t)[(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu] dt \\ & + \frac{1}{2} \partial_{yy} H(y_t) \underline{x} \Sigma_X \underline{x}' + \partial_t H(y, t) dt \end{aligned} \quad (3.19)$$

and recalling equation(13)

$$dy_t = [(r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu_X] dt + x_t \Sigma_X dw_{Xt}$$

we can derive the optimal portfolio choice.

$$Var[dy] \equiv \underline{x} \cdot \Sigma_X \cdot \underline{x}$$

$$E[dy] \equiv (r_f + \alpha)(y_t + \lambda - 1) + (1 - r_f)l_t + \underline{x}_t \cdot \mu_X$$

The optimal choice for \underline{x}_t is the one which maximizes eq.(18), or alternatively

$$\partial_y H(y_t, t) \underline{x}_t \cdot \mu_X + \frac{\partial_{yy} H(y_t, t)}{2} \underline{x}_t \cdot \Sigma_X \cdot \underline{x}_t \quad (3.20)$$

which leads to the following:

$$\underline{x}_t(y_t, t) = \frac{-\partial_y H(y_t, t)}{\partial_{yy} H(y_t, t)} \Sigma_X^{-1} \cdot \underline{\mu}_X \quad (3.21)$$

The form of the solution is not different from the one that Grossman and Laroque had found. Anyway, the policy will be affected since labor income changes significantly the structure of $H(y_t, t)$.

3.6.2 The Numerical Problem

The solution of the model purposed by this work entails to find out the numerical solution of the HJB equation with four boundaries conditions, these last ones related to the fact that the state variable can move freely only in a certain range between ξ_1 and ξ_2 .

Recalling equation (18) and considering one stock in the economy:

$$-\frac{1}{2} \left(\frac{\mu_X}{\Sigma_X} \right)^2 \frac{\partial_y H(y, t)^2}{\partial_{yy} H(y, t)} + H(y, t) + \partial_y H(y, t) [(r_f + \alpha)(y + \lambda - 1) + (1 - r_f)l_t] + \frac{e^{-\delta_2 t}}{a} + \partial_t H(y, t) = 0$$

The solution to this problem is characterized by a stopping rule for which

- if $\xi_{1t} < y_t < \xi_{2t}$ the household stays inactive [continue]
- if $y_t \leq \xi_{1t}$ or $y_t \geq \xi_{2t}$ he and adjust his consumption [stop] and $t = \tau$

meaning that:

$$y_t \in [\xi_{1t}, \xi_{2t}]$$

while

$$t = [t_1, t_2]$$

To solve the problem we need additional conditions at the boundary. They can be found applying backward induction techniques, which are derived from optimization and called **smooth pasting conditions**

Let from equation (13)

$$\Phi = \max_{K_2} H \left(\left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-1}, t \right) \quad (3.22)$$

We know that

$$H(y_t, t) \geq y_t^\alpha \Phi \quad (3.23)$$

for all $y_t \in [\xi_{1t}, \xi_{2t}]$

the value function reaches its minimum and at threshold level

$$H(y_t, t) = y_t^\alpha \Phi \quad (3.24)$$

When this value is reached, the household is not willing to accept his wealth over asset value increasing or decreasing anymore. Therefore he

changes the asset and this will be the value at the barrier.
This condition is called **value matching**

$$H(\xi_{1t}, \tau) = \xi_{1t}^a \Phi \quad (3.25)$$

and

$$H(\xi_{2t}, t) = \xi_{2t}^a \Phi \quad (3.26)$$

where also

$$H(\xi_{1t}, \tau) = \xi_{1t}^a \Phi$$

$$H(\xi_{2t}, \tau) = \xi_{2t}^a \Phi$$

Moreover, as usual, smooth pasting conditions apply, otherwise to stop at ξ_1 and ξ_2 is not optimal

$$\partial_y H(\xi_{1t}, t) = a \Phi \xi_{1t}^{(a-1)} \quad (3.27)$$

$$\partial_y H(\xi_{2t}, t) = a \Phi \xi_{2t}^{(a-1)} \quad (3.28)$$

Equation(19) and the set of boundaries conditions define the problem that cannot be solved in a closed form solution.

To solve the PDE, we need an additional boundary condition.

We assume that t_1 is the year in which the agent starts working but he will receive income from his labor only at the beginning of the following year. The meaning of this assumption is that at time t_1 his wealth is given by:

$$Q_{t_1} = K_{t_1} + B_{t_1} + X_{t_1}$$

At t_1 the problem is exactly the one solved by Grossman and Laroque and we will use this solution as initial condition. We will see the evolution of the value function as income from labor evolves starting from zero.

3.6.3 The Numerical Solution

To solve the problem numerically, there is the need to translate from a continuous time domain into a discrete domain.

The methods of finite difference seems the more appropriate for these frontier problems.

The solution at the boundaries is given by the boundaries conditions. The aim of the numerical method is to create a mapping of the solution in the domain with the use of a standard equally spaced grid.

It has been chosen the following method:

- **forward difference** to approximate the first derivative
- **central difference** to approximate the second derivative

In this framework the second set of boundaries conditions, provides the values for $H(\xi_{1+\Delta t}, t_0)$ and $H(\xi_{2+\Delta t}, t_0)$. The equation (19) will describe the pattern of the solution in the interval $(\xi_{1+\Delta t}, \xi_{2-\Delta t}) \times [t_1, t_2]$

$$-\frac{1}{2} \left(\frac{\mu_X}{\sigma_X} \right)^2 \frac{\partial_y H(y, t)^2}{\partial_{yy} H(y, t)} + H(y, t) + \partial_y H(y, t) [(r_f + \alpha)(y + \lambda - 1) + (1 - r_f)l_t] + \frac{e^{-\delta t}}{a} + \partial_t H(y, t) = 0$$

First, we will solve the equation without income from labor and therefore without evolution over time, i.e. the partial derivative of the value function with respect to time will be equal zero. This will constitute the additional boundary condition that we need in order to solve the PDE: $H(y, t_1)$

$$-\frac{1}{2} \left(\frac{\mu_X}{\sigma_X} \right)^2 \frac{\partial_y H(y, t_1)^2}{\partial_{yy} H(y, t_1)} + \delta_2 H(y, t_1) + \partial_y H(y, t_1) [(r_f + \alpha)(y + \lambda - 1)] + \frac{1}{a} = 0$$

From this equation we can work out the discrete version, considering only one stock in the economy.

$$H(y+2\Delta y, t_1) = -\frac{M \partial_y H(y, t_1)^2}{H(y, t_1) \delta + m_{t_1} \partial_y H(y, t_1) - 1/a} \Delta y^2 + \partial_y H(y, t_1) \Delta y + H(y + \Delta y, t_1)$$

where

$$m_{t_1} = (r_f + a)(y_{t_1} + \lambda - 1)$$

and

$$M = -\frac{1}{2} \left(\frac{\mu_X}{\sigma_X} \right)^2$$

Therefore $H(y_{t_1}, t_1)$ in the interval $(\xi_{1+\Delta t}, \xi_{2-\Delta t})$ can be expressed as a function of $H(y_{t_1} - \Delta y, t_1)$ and $H(y_{t_1} - 2\Delta y, t_1)$

$$H(y, t_1) = -\frac{M\partial_y H(y - \Delta y, t_1)^2}{H(y - 2\Delta y, t_1)\delta + m_{t_1}\partial_y H(y - \Delta y, t_1) - 1/a} \Delta y^2 + \partial_y H(y - \Delta y, t_1) \Delta y + H(y - \Delta y, t_1)$$

The procedure described above gives the boundary condition $H(y, t_1)$.

The complete solution will be given instead by a discretization of the equation in section (6.2).

We will use the following scheme

setting

$$H(y_k, t_j) \equiv H_k^j$$

$$M \frac{(H_{k+1}^j - H_k^j)^2}{(H_{k+1}^j - 2H_k^j + H_{k-1}^j)} + \frac{H_{k+1}^j - H_k^j}{\Delta y} [(r_f + a) + (y_k + \lambda - 1) + (1 - r_f)l_j] + \frac{e^{-\delta_2 t_j}}{a} + \frac{H_k^{j+1} - H_k^j}{\Delta t} = 0$$

To solve this equation we need first to set some parameters and to estimate others. Moreover, the model we have in mind for the labor income over the life cycle needs to be calibrated by an empirical analysis. One of the key values is K_t . As a first approximation we will consider this parameter constant and then we will offer a second model in which we allow housing prices to change.

First, we will calculate an average of housing prices index (provided by Standard and Poor) over the time period 1988/2011. Second, we will recover the

average value for deflated housing price (provided by the U.S. Census Bureau - Census of Housing) in the baseline year 2000. K_t will be given by these two quantities multiplied by each other and then divided by one-hundred. Instead the value for r_f is calculated considering the fact that house value is used as a numeraire. The average short term nominal interest rate is 5% between 1987 and 2005. In the same period the nominal inflation in housing prices was 7.1%. Therefore the risk free rate considered is -2.1% .

Table 3.1: Model parameters

Description	Parameter value
Retirement Age	65
Median Home values (2000)	119,600
Average price index (1985-2011)	120.57
K_t	144,200

Table 3.2: Model parameters

Description	Parameter value
α	0
μ	0.059
σ	0.22
λ	0.05
r_f	- 0.02

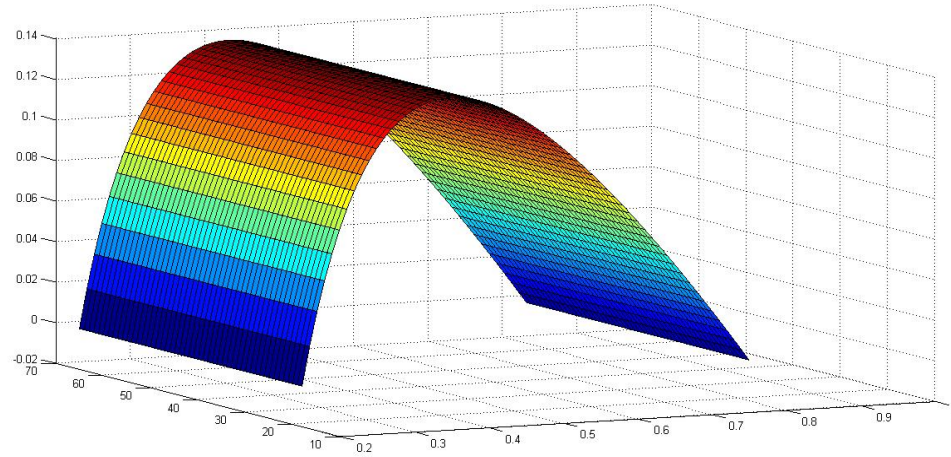


Figure 3.2: Evolution of the value function in the inaction region

$$H(y_t, t) - y_t^\alpha \Phi$$

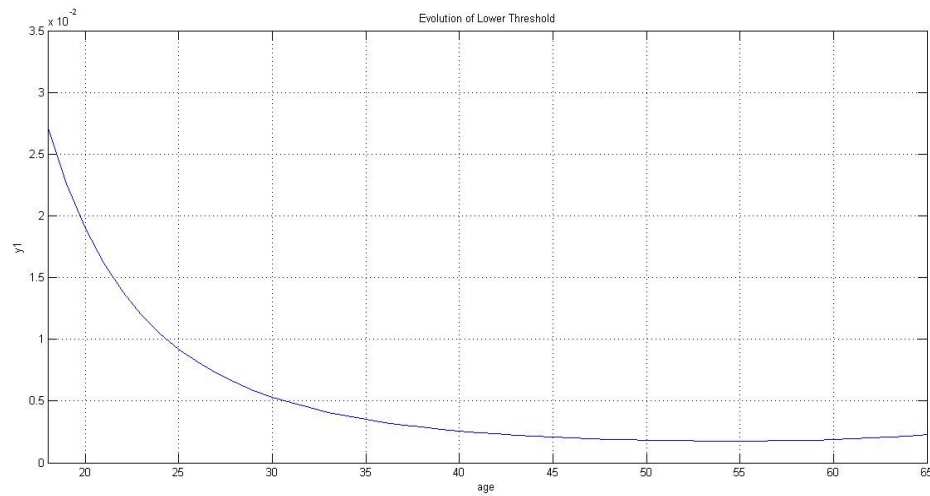


Figure 3.3: Evolution of the lower boundary over the time, i.e. age

3.7 Empirical Analysis

3.7.1 Estimation of Labor Income over the Life Cycle

The Data

This study requires the analysis of panel data where very large cross-sections, consisting of thousand of micro-units, followed through time. The most used are the National Longitudinal Survey of Labor Market Experience (NLS, <http://www.bls.gov/nls/nlsdoc.htm>) and the Michigan Panel Study of Income Dynamics (PSID, <http://psidonline.isr.umich.edu/>).

We study PSID Data for male head of family, there are roughly 23000 individual followed and asked repeatedly from 1968 to present (the panel is now in its 31st wave).

A graph of Labor Income (in real term) over the Life Cycle follows

We can observe the common hump-shaped curve. In order to understand the dynamic response of earnings to human capital accumulation, we run a regression with the aim to capture the effect of age, independently from the individual effect and from the effect of a particular year (generating 31 dummies, the number of periods the individuals are followed), i.e. the fixed effects, but leaving them implicit in the analysis. The regression has the functional form:

$$\ln(L_t, i) = a_0 + f(t) + \gamma_i + \nu_t + \varepsilon_{it} \quad (3.29)$$

where a_0 is the intercept and ε_{it} are i.i.d. $N(0, \sigma^2)$ random variables.

Moreover, $f(t)$ is the deterministic function which measures the effect of age on labor income.

$$f(t) = a_1t + a_2t^2 + a_3t^3 + a_4t^4 \quad (3.30)$$

3.7.2 Empirical Analysis - Labor Income

Some specifications are needed. We want to estimate the effect of human capital accumulation on earnings. Since in our model we assume that the agent will retire and will get a constant fraction of the last wage after the age of 65, we will try to model labor income from 18 to 65. Moreover, we assume that unemployment is considered as macroeconomic variable which depends strongly on global economic conditions and so on the year it is realized. For

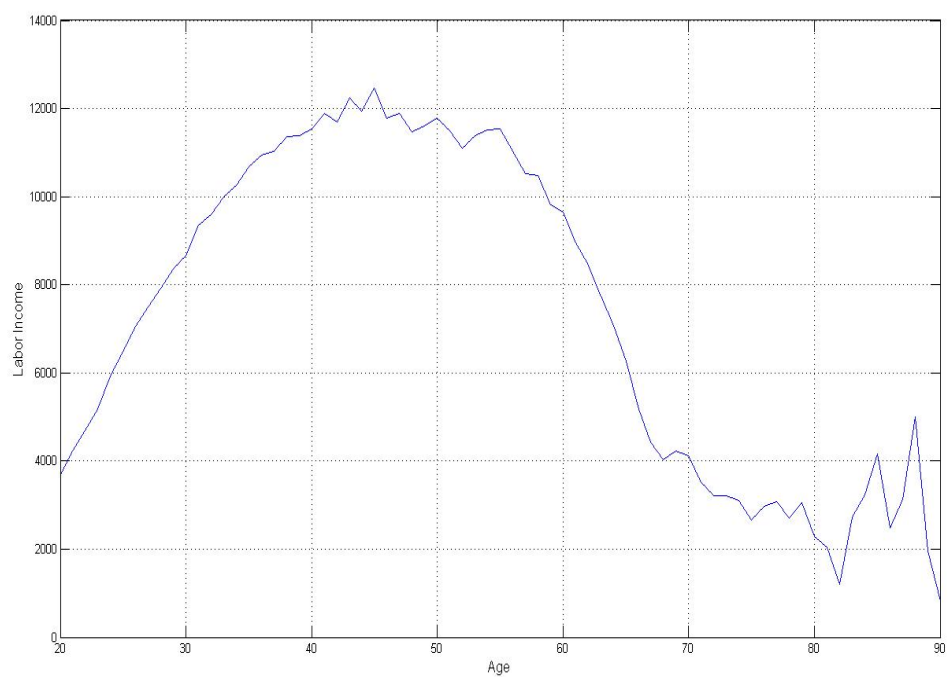


Figure 3.4: Labor income over the life cycle

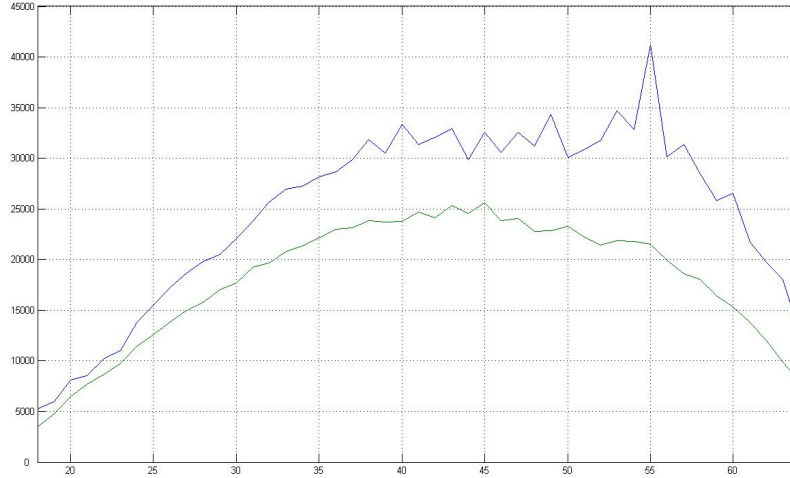


Figure 3.5: Labor income over the life cycle (green line includes unemployment and blue line exclude unemployment)

this reason we will exclude data for labor income equal to zero (equivalent to unemployment) from our investigations.

The graph presents the path of labor income over the life cycle where income can be also equal to zero, i.e. unemployment, at a given time for a certain individual - green line- and it presents also the path in the case period of unemployment are not included in the dataset - blue line-.

we will provide results for

- Quartic Expansion

$$L_{i,t} = a_0 + f(t) + \gamma_i + \nu_t + \varepsilon_{it}$$

$$f(t) = a_1t + a_2t^2 + a_3t^3 + a_4t^4$$

- Cubic Expansion

$$L_{i,t} = a_0 + f(t) + \gamma_i + \nu_t + \varepsilon_{it}$$

$$f(t) = a_1t + a_2t^2 + a_3t^3$$

- Quadratic Expansion

$$L_{i,t} = a_0 + f(t) + \gamma_i + \nu_t + \varepsilon_{it}$$

$$f(t) = a_1t + a_2t^2$$

- Linear Expansion

$$L_{i,t} = a_0 + f(t) + \gamma_i + \nu_t + \varepsilon_{it}$$

$$f(t) = a_1t$$

quartic expansion

```
. xtreg income eta eta2 eta3 eta4 dum1-dum31, r(id) fe
Fixed-effects (within) regression      Number of obs   =   20257
Group variable: id                    Number of groups =    2407

R-sq:  within = 0.1951                Obs per group:  min =     1
        between = 0.0000                avg           =    8.4
        overall = 0.0000                max           =    31

corr(u_i, Xb) = -1.0000                F(34,17816)    =   127.03
                                                Prob > F       =   0.0000
```

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
eta	-10233.81	1352.383	-7.57	0.000	-12884.61 -7583.01
eta2	373.7208	44.1587	8.46	0.000	287.1655 460.2762
eta3	-5.259715	.6340623	-8.30	0.000	-6.502539 -4.016891
eta4	.0243004	.0032563	7.46	0.000	.0179178 .030683
dum1	-55779.36	14481.48	-3.85	0.000	-84164.46 -27394.26
dum2	-54655.07	14076.17	-3.88	0.000	-82245.74 -27064.41
dum3	-54121.74	13746.58	-3.94	0.000	-81066.36 -27177.12
dum4	-53432.13	13402.9	-3.99	0.000	-79703.11 -27161.15
dum5	-52658.34	12992.39	-4.05	0.000	-78124.68 -27192
dum6	-52032.53	12624.68	-4.12	0.000	-76778.13 -27286.94
dum7	-51125.31	12244.76	-4.18	0.000	-75126.24 -27124.38
dum8	-50164.41	11868.9	-4.23	0.000	-73428.61 -26900.21
dum9	-49685.05	11486.83	-4.33	0.000	-72200.36 -27169.75
dum10	-48525.81	11143.24	-4.35	0.000	-70367.64 -26683.98
dum11	-47236.03	10815.5	-4.37	0.000	-68435.45 -26036.6
dum12	-46035.83	10433.62	-4.41	0.000	-66486.73 -25584.92
dum13	-44569.31	10060.06	-4.43	0.000	-64288.01 -24850.62
dum14	-43113.06	9719.81	-4.44	0.000	-62164.83 -24061.28
dum15	-41798.69	9345.623	-4.47	0.000	-60117.02 -23480.37
dum16	-40867.24	8982.81	-4.55	0.000	-58474.42 -23260.06
dum17	-39913.64	8622.033	-4.63	0.000	-56813.66 -23013.62
dum18	-37342.1	8235.978	-4.53	0.000	-53485.42 -21198.78
dum19	-36220.82	7892.962	-4.59	0.000	-51691.79 -20749.85
dum20	-34936.97	7528.121	-4.64	0.000	-49692.82 -20181.12
dum21	-33020.63	7152.727	-4.62	0.000	-47040.67 -19000.59
dum22	-31809.81	6788.699	-4.69	0.000	-45116.32 -18503.3
dum23	-29532.86	6441.669	-4.58	0.000	-42159.16 -16906.57
dum24	-28730.48	6061.588	-4.74	0.000	-40611.78 -16849.17
dum25	-28539.62	5714.508	-4.99	0.000	-39740.61 -17338.63
dum26	-25746.08	5289.732	-4.87	0.000	-36114.47 -15377.69
dum27	-20510.49	3972.624	-5.16	0.000	-28297.22 -12723.76
dum28	-15290.29	3263.454	-4.69	0.000	-21686.97 -8893.599
dum29	-8911.897	1945.98	-4.58	0.000	-12726.21 -5097.588
dum30	-6616.858	1463.37	-4.52	0.000	-9485.206 -3748.51
dum31	(dropped)				
_cons	-803970.5	121085.3	-6.64	0.000	-1041309 -566631.5
sigma_u	3.938e+08				
sigma_e	25080.462				
rho	1	(fraction of variance due to u_i)			

F test that all u_i=0: F(2406, 17816) = 6.78 Prob > F = 0.0000

(3.31)

cubic expansion

Fixed-effects (within) regression
 Group variable: **var1**

Number of obs = **20257**
 Number of groups = **2407**

R-sq: within = **0.1892**
 between = **0.0001**
 overall = **0.0017**

Obs per group: min = **1**
 avg = **8.4**
 max = **31**

corr(u_i, X_b) = **-0.9865**

F(32,17818) = **129.92**
 Prob > F = **0.0000**

var3	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
var4	2340.398	385.1369	6.08	0.000	1585.493	3095.304
var5	-25.53467	1.283886	-19.89	0.000	-28.05122	-23.01813
eta3	(dropped)					
dum1	-53509.85	14532.42	-3.68	0.000	-81994.81	-25024.89
dum2	-52514.85	14125.77	-3.72	0.000	-80202.74	-24826.96
dum3	-52031.89	13795.01	-3.77	0.000	-79071.46	-24992.32
dum4	-51451.3	13450.17	-3.83	0.000	-77814.94	-25087.67
dum5	-50836.33	13038.27	-3.90	0.000	-76392.62	-25280.05
dum6	-50204.37	12669.21	-3.96	0.000	-75037.25	-25371.49
dum7	-49469.47	12288.04	-4.03	0.000	-73555.22	-25383.71
dum8	-48662.69	11910.96	-4.09	0.000	-72009.34	-25316.05
dum9	-48400.3	11527.64	-4.20	0.000	-70995.6	-25805.01
dum10	-47405.58	11182.96	-4.24	0.000	-69325.27	-25485.89
dum11	-46257.3	10854.11	-4.26	0.000	-67532.41	-24982.18
dum12	-45077.15	10470.86	-4.31	0.000	-65601.06	-24553.25
dum13	-43687.99	10096.01	-4.33	0.000	-63477.15	-23898.83
dum14	-42396	9754.636	-4.35	0.000	-61516.03	-23275.96
dum15	-41204.29	9379.141	-4.39	0.000	-59588.32	-22820.26
dum16	-40330.56	9015.082	-4.47	0.000	-58001	-22660.13
dum17	-39420.75	8653.036	-4.56	0.000	-56381.54	-22459.96
dum18	-37000.05	8265.649	-4.48	0.000	-53201.53	-20798.58
dum19	-35943.01	7921.412	-4.54	0.000	-51469.75	-20416.28
dum20	-34743.3	7555.311	-4.60	0.000	-49552.44	-19934.16
dum21	-32896.99	7178.57	-4.58	0.000	-46967.68	-18826.29
dum22	-31680.16	6813.229	-4.65	0.000	-45034.75	-18325.57
dum23	-29327.38	6464.931	-4.54	0.000	-41999.27	-16655.48
dum24	-28580.35	6083.502	-4.70	0.000	-40504.61	-16656.1
dum25	-28428.39	5735.171	-4.96	0.000	-39669.88	-17186.9
dum26	-25494.52	5308.803	-4.80	0.000	-35900.29	-15088.75
dum27	-19977.38	3986.723	-5.01	0.000	-27791.74	-12163.02
dum28	-14661.8	3274.797	-4.48	0.000	-21080.73	-8242.884
dum29	-8422.898	1952.537	-4.31	0.000	-12250.06	-4595.736
dum30	-6328.434	1468.439	-4.31	0.000	-9206.716	-3450.152
dum31	(dropped)					
_cons	11979.93	23039.89	0.52	0.603	-33180.5	57140.35
sigma_u	473371.54					
sigma_e	25171.201					
rho	.99718047	(fraction of variance due to u _i)				

F test that all u_i=0: F(2406, 17818) = 7.23 Prob > F = 0.0000

quadratic expansion

```
. xtreg income eta eta2 dum1-dum31, i(id) fe
```

```
Fixed-effects (within) regression      Number of obs   =   20257
Group variable: id                    Number of groups =    2407

R-sq:  within = 0.1892                Obs per group:  min =     1
      between = 0.0001                avg           =    8.4
      overall  = 0.0017                max           =    31

corr(u_i, Xb) = -0.9865                F(32,17818)    =   129.92
                                          Prob > F       =    0.0000
```

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
eta	2340.398	385.1369	6.08	0.000	1585.493	3095.304
eta2	-25.53467	1.283886	-19.89	0.000	-28.05122	-23.01813
dum1	-53509.85	14532.42	-3.68	0.000	-81994.81	-25024.89
dum2	-52514.85	14125.77	-3.72	0.000	-80202.74	-24826.96
dum3	-52031.89	13795.01	-3.77	0.000	-79071.46	-24992.32
dum4	-51451.3	13450.17	-3.83	0.000	-77814.94	-25087.67
dum5	-50836.33	13038.27	-3.90	0.000	-76392.62	-25280.05
dum6	-50204.37	12669.21	-3.96	0.000	-75037.25	-25371.49
dum7	-49469.47	12288.04	-4.03	0.000	-73555.22	-25383.71
dum8	-48662.69	11910.96	-4.09	0.000	-72009.34	-25316.05
dum9	-48400.3	11527.64	-4.20	0.000	-70995.6	-25805.01
dum10	-47405.58	11182.96	-4.24	0.000	-69325.27	-25485.89
dum11	-46257.3	10854.11	-4.26	0.000	-67532.41	-24982.18
dum12	-45077.15	10470.86	-4.31	0.000	-65601.06	-24553.25
dum13	-43687.99	10096.01	-4.33	0.000	-63477.15	-23898.83
dum14	-42396	9754.636	-4.35	0.000	-61516.03	-23275.96
dum15	-41204.29	9379.141	-4.39	0.000	-59588.32	-22820.26
dum16	-40330.56	9015.082	-4.47	0.000	-58001	-22660.13
dum17	-39420.75	8653.036	-4.56	0.000	-56381.54	-22459.96
dum18	-37000.05	8265.649	-4.48	0.000	-53201.53	-20798.58
dum19	-35943.01	7921.412	-4.54	0.000	-51469.75	-20416.28
dum20	-34743.3	7555.311	-4.60	0.000	-49552.44	-19934.16
dum21	-32896.99	7178.57	-4.58	0.000	-46967.68	-18826.29
dum22	-31680.16	6813.229	-4.65	0.000	-45034.75	-18325.57
dum23	-29327.38	6464.931	-4.54	0.000	-41999.27	-16655.48
dum24	-28580.35	6083.502	-4.70	0.000	-40504.61	-16656.1
dum25	-28428.39	5735.171	-4.96	0.000	-39669.88	-17186.9
dum26	-25494.52	5308.803	-4.80	0.000	-35900.29	-15088.75
dum27	-19977.38	3986.723	-5.01	0.000	-27791.74	-12163.02
dum28	-14661.8	3274.797	-4.48	0.000	-21080.73	-8242.884
dum29	-8422.898	1952.537	-4.31	0.000	-12250.06	-4595.736
dum30	-6328.434	1468.439	-4.31	0.000	-9206.716	-3450.152
dum31	(dropped)					
_cons	11979.93	23039.89	0.52	0.603	-33180.5	57140.35
sigma_u	473371.54					
sigma_e	25171.201					
rho	.99718047	(fraction of variance due to u_i)				

```
F test that all u_i=0: F(2406, 17818) = 7.23 Prob > F = 0.0000
```

linear expansion

```
. xtreg income eta dum1-dum31, i(id) fe
```

```
Fixed-effects (within) regression      Number of obs   =   20257
Group variable: id                    Number of groups =    2407

R-sq:  within = 0.1712                Obs per group:  min =    1
        between = 0.1219                avg   =    8.4
        overall = 0.1351                max   =   31

corr(u_i, Xb) = -0.0779                F(31,17819)    =   118.73
                                          Prob > F       =    0.0000
```

income	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
eta	255.8508	374.6814	0.68	0.495	-478.5611	990.2628
dum1	-48877.47	14690.55	-3.33	0.001	-77672.36	-20082.57
dum2	-48548.08	14279.88	-3.40	0.001	-76538.04	-20558.12
dum3	-48658.56	13945.85	-3.49	0.000	-75993.78	-21323.34
dum4	-48550.92	13597.46	-3.57	0.000	-75203.27	-21898.57
dum5	-48437.72	13181.27	-3.67	0.000	-74274.29	-22601.15
dum6	-48284.34	12808.33	-3.77	0.000	-73389.91	-23178.77
dum7	-47822.18	12423.06	-3.85	0.000	-72172.58	-23471.78
dum8	-47307.08	12041.91	-3.93	0.000	-70910.4	-23703.76
dum9	-47082.38	11654.38	-4.04	0.000	-69926.09	-24238.67
dum10	-46206.46	11305.93	-4.09	0.000	-68367.18	-24045.74
dum11	-45156.78	10973.48	-4.12	0.000	-66665.86	-23647.7
dum12	-44082.53	10586.03	-4.16	0.000	-64832.18	-23332.88
dum13	-42904.29	10207.1	-4.20	0.000	-62911.19	-22897.39
dum14	-41629.49	9861.963	-4.22	0.000	-60959.9	-22299.09
dum15	-40455.72	9482.335	-4.27	0.000	-59042.01	-21869.42
dum16	-39598.97	9114.267	-4.34	0.000	-57463.81	-21734.12
dum17	-38636.41	8748.221	-4.42	0.000	-55783.77	-21489.05
dum18	-36173.68	8356.553	-4.33	0.000	-52553.34	-19794.03
dum19	-35150.06	8008.53	-4.39	0.000	-50847.56	-19452.56
dum20	-34014.15	7638.409	-4.45	0.000	-48986.17	-19042.13
dum21	-32120.69	7257.503	-4.43	0.000	-46346.1	-17895.28
dum22	-30969.46	6888.152	-4.50	0.000	-44470.91	-17468.02
dum23	-28597.33	6536.009	-4.38	0.000	-41408.54	-15786.12
dum24	-27845.6	6150.371	-4.53	0.000	-39900.93	-15790.28
dum25	-27627.98	5798.176	-4.76	0.000	-38992.97	-16263
dum26	-25043.06	5367.207	-4.67	0.000	-35563.31	-14522.81
dum27	-19811.14	4030.61	-4.92	0.000	-27711.53	-11910.75
dum28	-14540.08	3310.849	-4.39	0.000	-21029.67	-8050.499
dum29	-8477.039	1974.034	-4.29	0.000	-12346.34	-4607.741
dum30	-6455.576	1484.593	-4.35	0.000	-9365.522	-3545.63
dum31	(dropped)					
_cons	47342.46	23224.11	2.04	0.042	1820.956	92863.97
sigma_u	28762.504					
sigma_e	25448.351					
rho	.56090685	(fraction of variance due to u_i)				

```
F test that all u_i=0:      F(2406, 17819) =    6.93      Prob > F = 0.0000
```

3.8 Solution when Housing Prices changes over time

In the first section, we considered housing prices which doesn't vary with the time.

Now, we extend the work considering housing prices evolution and we model the evolution of prices with a geometric Brownian motion.

We are going to restate the problem in this new framework.

3.8.1 Budget Equation and State-Variable

The evolution of the property value is driven by depreciation and by changes in prices.

$$dQ_t = -\alpha K_t dt + r_f B_t dt + X_t(\mu_X dt + dw_X t) + \mu_K K_t dt + \sigma_K K_t dw_{K_t} \quad (3.32)$$

Recalling

$$dy = \frac{dQ_t}{K_t} - \frac{Q_t dK_t}{K_t^2} \quad (3.33)$$

and

$$y_t = \frac{Q_t}{K_t} - \lambda \quad (3.34)$$

$$\frac{dQ}{K_t} = -\alpha dt + r_f \frac{B_t}{K_t} dt + x_t(\mu_X dt + \sigma_X dw_{X_t}) + l_t dt + \mu_K dt + \sigma_K dw_{K_t}$$

$$\frac{Q_t dK_t}{K_t^2} = \frac{Q_t}{K_t} (-\alpha dt + \mu_K dt + \sigma_K dw_{K_t})$$

$$dy_t = [(r_f + \alpha + \mu_K)(y_t + \lambda - 1) + (1 - r_f)l_t + x_t \mu_X] dt + (y_t + \lambda - 1) \sigma_K dw_{K_t} \quad (3.35)$$

meaning that eq.17 is modified and the new equation we have to solve

$$-\frac{1}{2} \left(\frac{\mu_X}{\sigma_X} \right)^2 \frac{\partial_y H(y, t)^2}{\partial_{yy} H(y, t)} + H(y, t) + \partial_y H(y, t) [(r_f + \alpha + \mu_K)(y + \lambda - 1) + (1 - r_f)l_t] + \frac{e^{-\delta t}}{a} + \partial_t H(y, t) = 0 \quad (3.36)$$



Figure 3.6: Housing prices index

3.8.2 Empirical Analysis II - House Price index

The Data

Several measures of housing prices have been developed, which differs across country.

We investigate the U.S housing market and the best known indexes for residential real estate property are S&P/Case-Shiller Home Price Index

We want to treat housing prices in real term, as we did for labor income: The path suggests that it could be reasonable to model it with a Geometric Brownian Motion.

3.8.3 Estimation of the parameters for a Geometric Brownian Motion

Assuming that the process for prices can be well described by a GMB

$$dS_t = \mu S_t dt + \sigma S_t dW_t \quad (3.37)$$

We need to estimate the parameter μ and σ . It could be natural to think about implementing a MonteCarlo method in order to estimate the percentage drift and the percentage variance. Anyway, a problem arises when



Figure 3.7: Housing prices index (green line refers to real prices)

implementing the procedure.

After showing the results (in appendix also the codes), we provide an extensive proof about the fact that a MonteCarlo method is not appropriate for this estimates, when instead Maximum Likelihood performs sufficiently good.

Estimation via MonteCarlo method

The procedure we used at first is the following. Try first with an interval of values for μ and σ , for every μ and for every σ we run a Montecarlo simulation. For each combination we calculate the average mean square error. The best estimate would be the one which minimizes the average MSE.

We are going to show that this estimation procedure do not lead to a viable result.

Proof

Let $\{x(k)\}_{k=0,\dots,N-1}$ and $x(k) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma Z_1(t_k)$, $dt = \frac{t}{N-1}$, $Z_1(t_k)_{1k}$ are *i.i.d.* random variable with $Z_{1k} \sim N(0, 1)$, $\forall k \in K$

Let $Y(k) = \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} + \hat{\sigma} Z_2(t_k)\right)$ as above.

Then we want to estimate $(\hat{\mu}, \hat{\sigma})$ for the reference model that is going to describe the process of x_k using Mean Square Errors, as defined below:

$$(\hat{\mu}, \hat{\sigma}) = \text{Arg min}_{\mu, \sigma} \left[E \left(\frac{1}{N} \sum_k (x_k - y_k) \right)^2 \right] \quad (3.38)$$

Working out the equation

$$\begin{aligned} \mathbb{E} \left(\frac{1}{N} \sum_k (x_k - y_k) \right)^2 &= \mathbb{E} \left(\frac{1}{N} \sum_k (x_k^2 - Y(k)^2 - 2x_k Y(k)) \right) \\ &= \frac{1}{N} \sum_k x_k^2 + \frac{1}{N} \sum_k (Y(k)^2) - \frac{2}{N} \sum_k x_k \mathbb{E}(Y(k)) \\ &= \frac{1}{N} \sum_k x_k^2 + \frac{1}{N} \sum_k (\hat{\sigma}^2 (Y(t)) + \mathbb{E}(Y(k))^2 - 2x_k \mathbb{E}(Y(k))) \\ &= \sigma^2(x_k) + \mathbb{E}(x_k)^2 + \frac{1}{N} \sum_k \left(\hat{\sigma}^2 dt + \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right)^2 dt^2 - 2x_k \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right) dt \right) \\ &= \sigma^2 dt + \left(\mu - \frac{\sigma^2}{2} \right)^2 dt^2 + \hat{\sigma}^2 dt + \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right)^2 dt^2 - 2 \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right) \left(\mu - \frac{\sigma^2}{2} \right) dt^2 \\ &= F(\sigma, \mu; \hat{\sigma}, \hat{\mu}) \end{aligned}$$

Now we want to find $(\hat{\mu}, \hat{\sigma})$ which minimizes the function F

$$\begin{cases} \partial_\mu = 2 \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right) dt^2 - 2 \left(\mu - \frac{\sigma^2}{2} \right) dt^2 = 0 \\ \partial_\sigma = 2\sigma dt - \sigma \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right) dt^2 + 2\sigma \left(\hat{\mu} - \frac{\hat{\sigma}^2}{2} \right) dt^2 = 0 \end{cases} \quad (3.39)$$

This implies

$$\begin{cases} \hat{\mu} = \mu \\ \hat{\sigma} = 0 \end{cases} \quad (3.40)$$

This estimation procedure can be used for the estimate of μ but is completely unuseful for the estimation of σ .

We will see in the following chapter MLE outperform this methodology.

Estimation via MLE

Let the process $x_t = \left(\mu - \frac{\sigma^2}{2} dt\right) + \sigma dW_t$

$$Pr(\{x_k\}) \propto \prod_{k=1}^N \frac{e^{-\frac{(x_k - (\mu - \frac{\sigma^2}{2}) dt)^2}{2\sigma^2 dt}}}{\sqrt{2\pi\sigma^2 dt}} \quad (3.41)$$

Taking Logarithms

$$C(\mu, \sigma; \{x_t\}_k) = \sum_{k=1}^N \left(-\frac{(x_k - (\mu - \frac{\sigma^2}{2}) dt)^2}{2\sigma^2 dt} - \ln \sqrt{2\pi\sigma^2 dt} \right)$$

$$\sum_{k=1}^N -\frac{(x_k - (\mu - \frac{\sigma^2}{2}) dt)^2}{2\sigma^2 dt} - \frac{N}{2} \ln(2\pi\sigma^2 dt)$$

and maximizing the likelihood function

$$\partial_{\mu} C = \sum_{k=1}^N -\frac{2(x_k - (\mu - \frac{\sigma^2}{2} dt)) dt}{2\sigma^2(-dt)} = 0 \quad (3.42)$$

$$\partial_{\sigma} C = \sum_k \frac{-2 \left(x_k - \left(\mu - \frac{\sigma^2}{2} \right) dt \right) \sigma dt 2\sigma^2 dt + \left(x_k - \left(\mu - \frac{\sigma^2}{2} \right) dt \right)^2 4\sigma dt}{4\sigma^4 dt^2} - \frac{N}{2} \frac{4\pi\sigma dt}{2\pi\sigma^2 dt} = 0 \quad (3.43)$$

$$\begin{cases} \sigma = \sqrt{\frac{Var(x_t)}{dt}} \\ \mu = \frac{\mathbb{E}(x_t)}{dt} + \frac{\sigma^2}{2} \end{cases} \quad (3.44)$$

Alternative Approach

We derived MLE estimator of the parameters for a Geometric Brownian Motion.

An euristical approach leads to same result,

Given

$$x_t = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \sqrt{dt} Z_k \quad (3.45)$$

$$\mathbb{E}(x_t) = \left(\mu - \frac{\sigma^2}{2} \right) dt \quad (3.46)$$

$$Var(x_t) = \sigma^2 dt \quad (3.47)$$

Leading to the same result we found before (see eq. 30)

$$\begin{cases} \mu = \frac{\mathbb{E}(x_t)}{dt} + \frac{\sigma^2}{2} \\ \sigma = \sqrt{\frac{Var(x_t)}{dt}} \end{cases} \quad (3.48)$$

Description	Parameter value
μ_K	694
σ_K	33

3.9 Conclusions

The work provides a theoretical model for housing purchases treating the asset as durable and illiquid subject to infrequent adjustments.

This paper builds on literature about (S,s) models and durable consumption. Only a previous pioneer work considered financial choice in the sales/purchases process "Asset Pricing and Optimal Portfolio Choice in the Presence of Illiquid durable Consumption Goods"(1990).

Since 1990, literature on Finance and Asset Pricing improved the knowledge on households' portfolio choice, considering income from labor as a fundamental component of wealth and crucial in wealth accumulation.

The aim of this work is to construct again a bridge between literature on durable consumption and new results on Finance literature.

Therefore we defined a model for Labor income over life cycle, we considered that the agent can invest stocks and bonds.

The housing choice instead is given by an (S,s) rule, in which the household do not adjust his consumption at any point in time, given financial and non-financial costs in the transaction.

Anyway, at any point in time he invests his liquid wealth in the optimal portfolio.

The solution proposed originally by Grossman and Laroque, in which no income from labor is considered, results modified substantially.

There is no longer only one critical level associated to the threshold but it varies with the age, meaning that it is not the level of wealth which affect the choice but also the composition of wealth has an impact. In fact, Labor is modeled as a secure income with constant flow, while the stocks behavior is purely stochastic.

This model do not consider borrowing constraints and in a further work it would be interesting to examine the path of the solution in this case, which would strongly enhance our results.

In the second part of the paper we examine the evolution of the bellman

equation in case housing prices varies according to a geometric Brownian motion. After estimating the parameters, solution is provided and compared to the previous case it present a different pattern of the state variable in the inaction region, while the threshold are only slightly affected. Further work would include, as said before, borrowing constraints and an empirical estimation of the threshold.

3.10 Mathematical Appendix

$$K_t^a H(y_t, t) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_t^\tau e^{-\delta s} \frac{(K_s e^{-\alpha(s-t)})^a}{a} ds + e^{-\delta \tau} H(Q_\tau - \lambda K_\tau - \lambda K_2, \tau) \right]$$

factoring K_2 , reminding the Homogeneity of $H(y_t, t)$

$$K_t^a H(y_t, t) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_t^\tau e^{-\delta s} \frac{(K_s e^{-\alpha(s-t)})^a}{a} ds + e^{-\delta \tau} K_2^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \right]$$

where $K_\tau = K_t e^{-\alpha(\tau-t)}$, the evolution of the asset value due to depreciation

$$K_t^a H(y_t, t) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_t^\tau e^{-\delta s} \frac{(K_s e^{-\alpha(s-t)})^a}{a} ds + e^{-\delta \tau} K_2^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \right] \quad (3.49)$$

This holds at each t then also for $t = 0$. We will define $y_0 = y$

$$K^a H(y, t = 0) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_0^\tau e^{-\delta t} \frac{(K e^{-\alpha t})^a}{a} dt + e^{-\delta \tau} K_2^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \right]$$

dividing by K^a , and defining $\delta_2 = \delta + a\alpha$, we obtain:

$$H(y, t = 0) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta \tau} \left(\frac{K_2}{K} \right)^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \right]$$

$$H(y, t = 0) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta \tau} e^{-a\alpha \tau} \left(\frac{Q_\tau - \lambda K_\tau}{K e^{-\alpha \tau}} \right)^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-a} \right] \quad (3.50)$$

$$H(y, t = 0) = \max_{K_2, \tau, \underline{x}_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta_2 \tau} \left(\frac{Q_\tau - \lambda K_\tau}{K_\tau} \right)^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-a} \right] \quad (3.51)$$

$$H(y, t = 0) = \max_{K_2, \tau, x_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta_2 \tau} (y_\tau)^a H \left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-a} \right]$$

$$H(y, t = 0) = \max_{K_2, \tau, x_t} \mathbb{E} \left[\int_0^\tau \frac{e^{-\delta_2 t}}{a} dt + e^{-\delta_2 \tau} (y_\tau)^a H \left(\left(\frac{Q_\tau - \lambda K_\tau}{K_2} - \lambda, \tau \right) \left(\frac{Q_\tau - \lambda K_\tau}{K_2} \right)^{-1} \right) \right]$$

(3.52)

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