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FUNCTIONAL BRK INEQUALITIES, AND THEIR DUALS, WITH APPLICATIONS

by

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Functional BRK Inequalities, and their Duals, with Applications

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Abstract

The inequality conjectured by van den Berg and Kesten in [9], and proved by Reimer in [6], states that for A and B events on S, a product of finitely many finite sets, and P any product measure on S,

$$P(A \Box B) \leq P(A)P(B),$$

where $A \Box B$ are the elementary events which lie in both A and B for 'disjoint reasons.' This inequality on events is the special case, for indicator functions, of the inequality having the following formulation. Let $\mathbf{X}$ be a random vector with n independent components, each in some space S_i (such as $\mathbf{R}^d$), and set $S = \prod_{i=1}^n S_i$. Say that the function $f : S \to \mathbf{R}$ depends on $K \subseteq \{1, \ldots, n\}$ if $f(\mathbf{x}) = f(\mathbf{y})$ whenever $x_i = y_i$ for all $i \in K$. Then for any given finite or countable collections of non-negative real valued functions $\{f_\alpha\}_{\alpha \in \mathcal{A}}$, $\{g_\beta\}_{\beta \in \mathcal{B}}$ on S which depend on K_α and L_β respectively,

$$E\left\{\sup_{K_\alpha \cap L_\beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X})\right\} \leq E\left\{\sup_\alpha f_\alpha(\mathbf{X})\right\} E\left\{\sup_\beta g_\beta(\mathbf{X})\right\}.$$

Related formulations, and functional versions of the dual inequality on events by Kahn, Saks, and Smyth [4], are also considered. Applications include order statistics, assignment problems, and paths in random graphs.

1 Introduction

For $\mathbf{x} = (x_1, \ldots, x_n) \in S$, where $S = \prod_{i=1}^n S_i$ any product space, and $K = \{k_1, \ldots, k_m\} \subseteq \mathbf{n} := \{1, \ldots, n\}$, define

$$\mathbf{x}_K = (x_{k_1}, \ldots, x_{k_m}) \quad \text{and} \quad [\mathbf{x}]_K = \{\mathbf{y} \in S : y_K = x_K\},$$

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the restriction of $\mathbf{x}$ to the indicated coordinates, and the collection of all elements in S which agree with $\mathbf{x}$ in those coordinates, respectively. For $A, B \subseteq S$ we say that $\mathbf{x} \in A, \mathbf{y} \in B$ disjointly if there exists

$$K, L \subseteq \mathbf{n}, K \cap L = \emptyset \text{ such that } \quad [\mathbf{x}]_K \subseteq A \quad \text{and} \quad [\mathbf{y}]_L \subseteq B, \tag{1}$$

and denote

$$A \Box B = \{\mathbf{x} : \mathbf{x} \in A, \mathbf{x} \in B \,\text{disjointly}\}. \tag{2}$$

The operation $A \Box B$ corresponds to elementary events which are in both A and B for disjoint 'reasons' in the sense that inclusion in A and B is determined on disjoint sets of coordinates.

Theorem 1.1 was conjectured in van den Berg and Kesten [9]. It was proved in [9] for A and B increasing sets and $S = \{0,1\}^n$, and it was also demonstrated there that Theorem 1.1 follows from its special case $S = \{0,1\}^n$. Using the latter fact, the conjecture was established in general by Reimer [6].

Theorem 1.1 *For $P = \prod_{i=1}^n P_i$ any product measure on $S = \prod_{i=1}^n S_i$, S_i finite,*

$$P(A \Box B) \le P(A)P(B). \tag{3}$$

Many useful formulations can be found in van den Berg and Fiebig [8], in addition to the following motivating example which appeared earlier in [9]. Independently assign a random direction to each edge $e = \{v_i, v_j\}$ of a finite graph, with $p_e(v_i, v_j) = 1 - p_e(v_j, v_i)$ the probability of the edge e being directed from vertex v_i to v_j. With V_1, V_2, W_1, W_2 sets of vertices, Theorem 1.1 yields that the product of the probabilities that there exist directed paths from V_1 to V_2 (event A) and from W_1 to W_2 (event B) is an upper bound to the probability that there exists two disjoint directed paths, one from V_1 to V_2 and another from W_1 to W_2 (event $A \Box B$).

The main thrust of this paper is to show how Theorem 1.1 implies inequalities in terms of functions, of which (3) is the special case of indicators, and similarly for the dual. These functional inequalities, and their dual, are stated in Theorems 1.2 and 1.5, and their proofs can be found in Section 3. Applications to order statistics, allocation problems, and random graphs are given in Section 2. Specializing to monotone functions, we derive related inequalities and stochastic orderings in Section 4; these latter results are connected to those of Alexander [1].

For each $i = 1, \ldots, n$, let $(S_i, \mathbb{S}_i)$ be measurable spaces, and set $S = \prod_{i=1}^n S_i$ and $\mathbb{S} = \bigotimes_{i=1}^n \mathbb{S}_i$, the product sigma algebra. Henceforth, all given real valued functions on S, such as $f_\alpha, g_\beta, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$ are assumed to be $(\mathbb{S}, \mathbb{B})$ measurable where $\mathbb{B}$ denotes the Borel sigma algebra of $\mathbf{R}$, and functions on S with values in $2^{\mathbf{n}}$, such as $K(\mathbf{x})$ in inequality (d) of Theorem 1.2 below, are assumed to be $(\mathbb{S}, 2^{2^{\mathbf{n}}})$ measurable. Measurability issues arise in definitions (11), (4), and (18), and are settled in Section 5. We also show in Section 5 that Theorem 1.2 applies to the completion of the measure space $(S, \mathbb{S})$ with respect to the measure P appearing in the theorem; similarly for Theorem 1.5.

For $K \subseteq \mathbf{n}$ we say that a function f defined on S depends on K if $\mathbf{x}_K = \mathbf{y}_K$ implies $f(\mathbf{x}) = f(\mathbf{y})$. The inequalities in Theorems 1.2 and 1.5 require one of two frameworks, the first of which is the following.

Framework 1 *$\{f_\alpha(\mathbf{x})\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta(\mathbf{y})\}_{\beta\in\mathcal{B}}$ are given collections of non-negative functions on S, such that f_α, g_β depend respectively on subsets of $\mathbf{n}$ K_α, L_β in $\mathcal{K} = \{K_\alpha\}_{\alpha\in\mathcal{A}}$ and $\mathcal{L} = \{L_\beta\}_{\beta\in\mathcal{B}}$, where $\mathcal{A}$ and $\mathcal{B}$ are finite or countable.*

The elements of $\mathcal{K}$ and $\mathcal{L}$ are not assumed to be distinct; we may have, say, $K_\alpha = K_\gamma$ for some $\alpha \neq \gamma$ and $f_\alpha \neq f_\gamma$. For notational brevity we may write α for K_α; for example, we may use $\alpha \cap \beta$ as an abbreviation for $K_\alpha \cap L_\beta$, and also $\mathbf{x}_\alpha$ for $\mathbf{x}_{K_\alpha}$.

The second framework needed is

Framework 2 *f and g are two given non-negative functions, and $\mathcal{K}$ and $\mathcal{L}$ any collections of subsets of $\mathbf{n}$. With P a probability measure on $(S, \mathbb{S})$ define for $K \in \mathcal{K}, L \in \mathcal{L}$,*

$$\underline{f}_K(\mathbf{x}) = \operatorname{ess}\inf_{\mathbf{y}\in[\mathbf{x}]_K} f(\mathbf{y}), \quad \text{and} \quad \underline{g}_L(\mathbf{x}) = \operatorname{ess}\inf_{\mathbf{y}\in[\mathbf{x}]_L} g(\mathbf{y}). \tag{4}$$

Our functional extension of the BKR inequality (3) is

Theorem 1.2 *Let $\mathbf{X} = (X_1, \ldots, X_n) \in S$ be a random vector and P a probability measure on $(S, \mathbb{S})$ such that $X_1, \ldots, X_n$ are independent. The following inequalities hold.*

1. *Under framework 1,*

$$E\left\{\sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} \leq E\left\{\sup_\alpha f_\alpha(\mathbf{X})\right\} E\left\{\sup_\beta g_\beta(\mathbf{X})\right\}. \tag{a}$$

2. *Under framework 2,*

$$E\left\{\max_{\substack{K\cap L=\emptyset\\ K\in\mathcal{K}, L\in\mathcal{L}}} \underline{f}_K(\mathbf{X})\underline{g}_L(\mathbf{X})\right\} \leq E\{f(\mathbf{X})\}\ E\{g(\mathbf{X})\}. \tag{b}$$

The special case of inequality (b) where $\mathcal{K} = \mathcal{L}$ are the collections of all subsets implies it in general.

In previous work the $\square$ operation was defined only for finite product spaces. Note, however, that (2) applies on any product space. Further, with $f(\mathbf{x})$ and $g(\mathbf{x})$ the indicator functions of A and B respectively, we can equivalently express $A\square B$ as the set whose indicator function is given by

$$\mathbf{1}_{A\square B}(\mathbf{x}) = \max_{K\cap L=\emptyset} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}), \tag{5}$$

where $\underline{f}_K, \underline{g}_L$ are given in (4). Therefore, inequality (b) of Theorem 1.2 specialized to the case that $\mathcal{K} = \mathcal{L}$ are all subsets of $\mathbf{n}$ and f and g are indicators, says that the original BKR inequality for events in finite spaces extends to more general spaces.

The following is a straightforward generalization of Theorem 1.2, stated here only for inequality (a) . Note that in (6) below, for large m the pairwise constraints $\alpha_i \cap \alpha_j = \emptyset$ are more restrictive, making the inequality less sharp.

Theorem 1.3 *Let $\mathbf{X} \in S$ be a random vector with independent coordinates. Then for given finite or countable collections of non-negative functions $\{f_{i,\alpha}\}_{\alpha\in\mathcal{A}_i}$ depending on $\{K_{i,\alpha}\}_{\alpha\in\mathcal{A}_i}$, $i = 1, \ldots, m$,*

$$E\left\{ \sup_{\substack{(\alpha_1,\ldots,\alpha_n)\in\mathcal{A}_1\times\cdots\times\mathcal{A}_n \\ \alpha_k\cap\alpha_l=\emptyset,\, k\neq l}} \prod_{i=1}^{m} f_{i,\alpha_i}(\mathbf{X}) \right\} \leq \prod_{i=1}^{m} E\left\{ \sup_{\alpha\in\mathcal{A}_i} f_{i,\alpha}(\mathbf{X}) \right\}. \tag{6}$$

Next we describe an inequality of Kahn, Saks, and Smyth [4], which may be considered dual to the BKR inequality (3), and provide a function version. We use simpler notation, more compatible with (3). With 'disjointly' defined in (1), denote

$$A \Diamond B = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x} \in A, \mathbf{y} \in B \,\text{disjointly}\}. \tag{7}$$

The following, which we call the KSS inequality, is dual to Theorem 1.1 and is given in [4].

Theorem 1.4 *If P denotes the uniform measure over $\{0,1\}^n \times \{0,1\}^n$, then for any $(A, B) \subseteq \{0,1\}^n \times \{0,1\}^n$,*

$$(P \times P)(A \Diamond B) \leq P(A \cap B). \tag{8}$$

Our functional extension of the KSS inequality is as follows.

Theorem 1.5 *Let $\mathbf{X} = (X_1, \ldots, X_n) \in S$ be a random vector, P any probability measure on $(S, \mathbb{S})$ such that $X_1, \ldots, X_n$ are independent, and $\mathbf{Y}$ an independent copy of $\mathbf{X}$. The following inequalities hold.*

1. *Under framework 1,*

$$E\left\{ \sup_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{Y}) \right\} \leq E\left\{ \sup_{\alpha,\beta} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X}) \right\}. \tag{a'}$$

2. *Under framework 2,*

$$E\left\{ \max_{\substack{K\cap L=\emptyset \\ K\in\mathcal{K}, L\in\mathcal{L}}} \underline{f}_K(\mathbf{X}) \underline{g}_L(\mathbf{Y}) \right\} \leq E\left\{ f(\mathbf{X}) g(\mathbf{X}) \right\}. \tag{b'}$$

In [4] the (implicit) $\Diamond$ operation was defined only on $\{0,1\}^n \times \{0,1\}^n$. Note, however, that (7) applies on any product space. Further, with $f(\mathbf{x})$ and $g(\mathbf{x})$ the indicator functions of A and B respectively, we can equivalently express $A \Diamond B$ as the set whose indicator function is given by

$$\mathbf{1}_{A\Diamond B}(\mathbf{x}, \mathbf{y}) = \max_{K\cap L=\emptyset} \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{y}), \tag{9}$$

where $\underline{f}_K, \underline{g}_L$ are given in (4). Therefore, inequality (a$'$) of Theorem 1.5 specialized to the case that $\mathcal{K} = \mathcal{L}$ are all subsets of $\mathbf{n}$ and f and g are indicators, says that the original KSS inequality for events in $\{0,1\}^n$ extends to more general spaces. Note by (5) and (9) that

$$\mathbf{1}_{A\Box B}(\mathbf{x}) = \mathbf{1}_{A\Diamond B}(\mathbf{x}, \mathbf{x}).$$

We next discuss further formulations of Theorems 1.2 and 1.5 which are of independent interest, and will be used in the proof. Under Framework 1, for any K and L subsets of $\mathbf{n}$, define

$$\tilde{f}_K(\mathbf{x}) = \sup_{\alpha: K_\alpha \subseteq K} f_\alpha(\mathbf{x}) \quad \text{and} \quad \tilde{g}_L(\mathbf{x}) = \sup_{\beta: L_\beta \subseteq L} g_\beta(\mathbf{x}). \tag{10}$$

For any given functions $K(\mathbf{x})$ and $L(\mathbf{x})$, under Framework 1, extend (10) to

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sup_{\alpha: K_\alpha \subseteq K(\mathbf{x})} f_\alpha(\mathbf{x}) \quad \text{and} \quad \tilde{g}_{L(\mathbf{x})}(\mathbf{x}) = \sup_{\beta: L_\beta \subseteq L(\mathbf{x})} g_\beta(\mathbf{x}), \tag{11}$$

and under Framework 2, extend (4) to

$$\underline{f}_{K(\mathbf{x})}(\mathbf{x}) = \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_{K(\mathbf{x})}} f(\mathbf{y}), \quad \text{and} \quad \underline{g}_{L(\mathbf{x})}(\mathbf{x}) = \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_{L(\mathbf{x})}} g(\mathbf{y}). \tag{12}$$

Proposition 1.1 *In Framework 1, inequality (a) of Theorem 1.2 and inequalities (c) and (d) below are equivalent.*

$$E\left\{\max_{K \cap L = \emptyset} \tilde{f}_K(\mathbf{X}) \tilde{g}_L(\mathbf{X})\right\} \leq E\left\{\sup_\alpha f_\alpha(\mathbf{X})\right\} E\left\{\sup_\beta g_\beta(\mathbf{X})\right\}. \tag{c}$$

$$E\left\{\tilde{f}_{K(\mathbf{X})}(\mathbf{X}) \tilde{g}_{L(\mathbf{X})}(\mathbf{X})\right\} \leq E\left\{\sup_\alpha f_\alpha(\mathbf{X})\right\} E\left\{\sup_\beta g_\beta(\mathbf{X})\right\}, \tag{d}$$

holding for any given $K(\mathbf{x})$ and $L(\mathbf{x})$ such that $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.

In Framework 2, inequality (b) of Theorem 1.2 and inequality (e) below are equivalent.

$$E\left\{\underline{f}_{K(\mathbf{X})}(\mathbf{X}) \underline{g}_{L(\mathbf{X})}(\mathbf{X})\right\} \leq E\left\{f(\mathbf{X})\right\} E\{g(\mathbf{X})\}, \tag{e}$$

holding for given $K(\mathbf{x}) \in \mathcal{K}$ and $L(\mathbf{x}) \in \mathcal{L}$ such that $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.

It is easy to see from (10) and (11) that a restriction of K or $K(\mathbf{x})$ to $\mathcal{K}$ and L or $L(\mathbf{x})$ to $\mathcal{L}$ would not change the quantities on the left hand sides of (c) and (d).

The special case of inequality (e) with $\mathcal{K} = \mathcal{L} = 2^{\mathbf{n}}$ and $L(\mathbf{x}) = K^c(\mathbf{x})$, where K^c denotes the complement of K, yields the inequality in general, that is, (e) is equivalent to

$$E\left\{\underline{f}_{K(\mathbf{X})}(\mathbf{X}) \underline{g}_{K^c(\mathbf{X})}(\mathbf{X})\right\} \leq E\left\{f(\mathbf{X})\right\} E\{g(\mathbf{X})\}$$

for any given $K(\mathbf{x})$. A similar comment holds for (c) and (d).

Proposition 1.2 *In Framework 1, inequality (a′) of Theorem 1.5 and inequalities (c′) and (d′) below are equivalent.*

$$E\left\{\max_{K,L} \tilde{f}_K(\mathbf{X}) \tilde{g}_L(\mathbf{Y})\right\} \leq E\left\{\sup_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X})\right\}. \tag{c′}$$

$$E\left\{\tilde{f}_{K(\mathbf{X},\mathbf{Y})}(\mathbf{X}) \tilde{g}_{L(\mathbf{X},\mathbf{Y})}(\mathbf{Y})\right\} \leq E\left\{\sup_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{X}) g_\beta(\mathbf{X})\right\}, \tag{d′}$$

holding for any given $K(\mathbf{x},\mathbf{y})$ and $L(\mathbf{x},\mathbf{y})$ replacing $K(\mathbf{x})$ and $L(\mathbf{x})$ in (11) respectively, and satisfying $K(\mathbf{x},\mathbf{y}) \cap L(\mathbf{x},\mathbf{y}) = \emptyset$.

In Framework 2 inequality (b′) of Theorem 1.5 and inequality (e′) below are equivalent.

$$E\left\{\underline{f}_{K(\mathbf{X},\mathbf{Y})}(\mathbf{X})\underline{g}_{L(\mathbf{X},\mathbf{Y})}(\mathbf{Y})\right\} \leq E\left\{f(\mathbf{X})g(\mathbf{X})\right\}, \tag{e'}$$

holding for given $K(\mathbf{x},\mathbf{y}) \in \mathcal{K}$ and $L(\mathbf{x},\mathbf{y}) \in \mathcal{L}$, replacing $K(\mathbf{x})$ and $L(\mathbf{x})$ in (12), respectively.

2 Applications

Example 2.1 ***Order Statistics Type Inequalities*** *Let $\mathbf{X} = (X_1,\ldots,X_n)$ be a vector of independent non-negative random variables with associated order statistics $X_{[n]} \leq \cdots \leq X_{[1]}$. Let $\mathcal{A} = \mathcal{B}$ be the collection of all the singleton subsets $\alpha \in \mathbf{n}$ and $f_\alpha(\mathbf{x}) = g_\alpha(\mathbf{x}) = x_\alpha$. Then*

$$\max_\alpha f_\alpha(\mathbf{X}) = X_{[1]}, \quad \max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X}) = X_{[1]}X_{[2]},$$

and inequality (a) of Theorem 1.2 provides the middle inequality in the string

$$EX_{[1]}EX_{[2]} \leq EX_{[1]}X_{[2]} \leq (EX_{[1]})^2 \leq EX_{[1]}^2.$$

The leftmost inequality is true since order statistics are always positively correlated (moreover they are associated as defined by Esary et al [2], and even MTP_2, see Karlin and Rinott [5]); the rightmost inequality follows from Jensen.

Theorem 1.2 allows a large variety of extensions of this basic order statistics inequality. For example, taking $\mathcal{A}$ and $\mathcal{B}$ to be all k and l subsets of $\mathbf{n}$ respectively, then with

$$f_\alpha(\mathbf{x}) = \prod_{j\in\alpha} x_j \quad \text{and} \quad g_\beta(\mathbf{x}) = \prod_{j\in\beta} x_j \tag{13}$$

we derive

$$E\left(\prod_{j=1}^{k+l} X_{[j]}\right) \leq E\left(\prod_{j=1}^{k} X_{[j]}\right) E\left(\prod_{j=1}^{l} X_{[j]}\right).$$

Dropping the non-negativity assumption on $X_1,\ldots,X_n$, we have for all $t > 0$,

$$Ee^{t(X_{[1]}+X_{[2]})} \leq [Ee^{tX_{[1]}}]^2 = Ee^{tX_{[1]}}Ee^{tY_{[1]}} = Ee^{t(X_{[1]}+Y_{[1]})},$$

with Y_i's being independent copies of the X_i's. Likewise, for all $t > 0$,

$$Ee^{-t(X_{[n]}+X_{[n-1]})} \leq [Ee^{-tX_{[n]}}]^2 = Ee^{-tX_{[n]}}Ee^{-tY_{[n]}} = Ee^{-t(X_{[n]}+Y_{[n]})}.$$

Moment generating function and Laplace orders are discussed in Shaked and Shanthikumar [7].

Returning to non-negative variables, a variation of (13) follows by replacing products with sums, that is,

$$f_\alpha(\mathbf{x}) = \sum_{j\in\alpha} x_j \quad \text{and} \quad g_\beta(\mathbf{x}) = \sum_{j\in\beta} x_j, \tag{14}$$

which for, $k=l=2$ say, yields

$$E \max_{\{i,j,k,l\}=\{1,2,3,4\}} (X_{[i]}+X_{[j]})(X_{[k]}+X_{[l]}) \le [E(X_{[1]}+X_{[2]})]^2.$$

Though the maximizing indices on the left hand side will be $\{1,2,3,4\}$ as indicated, the choice is not fixed and depends on the X's; however, by a simple majorization argument, $(X_{[1]}+X_{[2]})(X_{[3]}+X_{[4]})$ is not maximal apart from degenerate cases.

Definition (13) and (14) are special cases where f and g are increasing non-negative functions of k and l variables and

$$f_\alpha(\mathbf{x}) = f(\mathbf{x}_\alpha) \quad \textit{and} \quad g_\beta(\mathbf{x}) = g(\mathbf{x}_\beta); \tag{15}$$

when f and g are symmetric,

$$E \max_{\{i_1,\ldots,i_k,j_1,\ldots,j_l\}=\{1,\ldots,k+l\}} f(X_{[i_1]},\ldots,X_{[i_k]})g(X_{[j_1]},\ldots,X_{[j_l]}) \le Ef(X_{[1]},\ldots,X_{[k]})Eg(X_{[1]},\ldots,X_{[l]}).$$

We now give an example which demonstrates that these order statistics type inequalities can be considered in higher dimensions. Let $\mathbf{X}_1,\ldots,\mathbf{X}_n$ be independent vectors in $\mathbf{R}^m$, and for $\alpha,\beta \subseteq \mathbf{n}$ with $|\alpha|=|\beta|=3$ let f_α and g_β be given as in (15), where $f(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) = g(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ is, say, the area of the triangle formed by the given three vectors. Theorem 1.2 gives that the expected greatest product of the areas of two triangles with distinct vertices is bounded above by the square of the expectation of the largest triangular area.

To explore the dual inequality in these settings, let $\mathbf{X}$ be a vector of independent variables with support contained in $[0,1]$, and $\mathbf{Y}$ an independent copy. With $\mathcal{A}=\mathcal{B}$ the collections of all singletons α in $\mathbf{n}$, and $f_\alpha(\mathbf{x}) = x_\alpha, g_\beta(\mathbf{x}) = 1-x_\beta$, inequality (a′) of Theorem 1.5 gives

$$E\left\{\max_{\alpha\ne\beta} X_\alpha(1-Y_\beta)\right\} \le EX_{[1]}(1-X_{[n]}). \tag{16}$$

Note that $\max_{\alpha\ne\beta} X_\alpha(1-Y_\beta) \ne X_{[1]}(1-Y_{[n]})$; the right hand side might be larger because of the restriction $\alpha\ne\beta$. Removing the restriction $\alpha\ne\beta$ reverses (16), that is,

$$EX_{[1]}(1-X_{[n]}) \le EX_{[1]}E(1-X_{[n]}) = EX_{[1]}E(1-Y_{[n]}) = EX_{[1]}(1-Y_{[n]}) = E\left\{\max_{\alpha,\beta} X_\alpha(1-Y_\beta)\right\},$$

where the inequality follows by the negative association of $X_{[1]}$ and $1-X_{[n]}$.

Following our treatment of applications of Theorem 1.2 we can extend (16) as follows: with $\mathcal{A}$ and $\mathcal{B}$ the collection of all k and l subsets of $\mathbf{n}$ respectively, and

$$f_\alpha(\mathbf{x}) = \prod_{j\in\alpha} x_j \quad \textit{and} \quad g_\beta(\mathbf{x}) = \prod_{j\in\beta}(1-x_j),$$

we obtain

$$E\left\{\max_{\alpha\cap\beta=\emptyset} \prod_{i\in\alpha,j\in\beta} X_i(1-Y_j)\right\} \le E\left\{\prod_{1\le i\le k,1\le j\le l} X_{[i]}(1-X_{[n-j+1]})\right\}.$$

We now consider resource allocation problems of the following type. Suppose that two projects A and B have to be completed using n available resources represented by the components of a vector $\mathbf{x}$. Each resource can be used for at most one project, and an allocation is given by a specification of disjoint subsets of resources. For any given subsets $\alpha, \beta \subseteq \mathbf{n}$, let $f_\alpha(\mathbf{x})$ and $g_\beta(\mathbf{x})$ count the number of ways that projects A and B can be completed using the resources $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ respectively. The exact definitions of the projects and the counts are immaterial; in particular larger sets do not necessarily imply more ways to carry out a project. For an allocation α, β, $\alpha \cap \beta = \emptyset$, the total number of ways to carry out the two projects together is the product $f_\alpha(\mathbf{x})g_\beta(\mathbf{x})$. When the resources are independent variables, inequality (a) of Theorem 1.2 bounds the expected maximal number of ways of completing A and B together, by the product of the expectations of the maximal number of ways of completing each project alone. The bound is simple in the sense that it does not require understanding of the relation between the two projects. In particular, it can be computed without knowledge of the optimal allocation of resources.

Example 2.2 *With J a list of tasks, consider fulfilling task lists A and B from J, (not necessarily disjoint) in two distant cities using n workers. For worker $i \in \mathbf{n}$, let $\mathbf{x}_i \subseteq 2^J$ be the collection of possible assignments of tasks for worker i; a worker may be able to fill more than one task. For $\alpha, \beta \subseteq \mathbf{n}$, let $f_\alpha(\mathbf{x})$ equal the number of ways the collection of workers α can complete A, and $g_\beta(\mathbf{x})$ the number of ways the collection β can complete B. When the qualifications $\mathbf{X}_i, i \in \mathbf{n}$ are independent, Theorem 1.2 bounds the expectation of the maximal number of ways of fulfilling the task requirements in both cities, by the product of the expectations of the maximal numbers of ways that the requirements in each collection can be separately satisfied.*

Example 2.3 ***Paths on Graphs*** *Consider a graph $\mathcal{G}$ with n fixed vertices $v_j, j = 1, \ldots, n$ in $\mathcal{V}$, an arbitrary space, where for each pair of vertices the existence of the edge $\{v_i, v_j\}$ is determined independently using a probability rule based on v_i, v_j, perhaps depending only on $d(\{v_i, v_j\})$ for some function d. Let $\mathbf{X} = \{X_{\{i,j\}}\}$ where $X_{\{i,j\}}$ is the indicator that there exists an edge between v_i and v_j. For instance, with $\mathcal{V} = \mathbf{R}^m$ and $Z_{\{i,j\}}, 1 \le i, j \le n$ independent non-negative variables, we may take*

$$X_{\{i,j\}} = \mathbf{1}(d(\{\mathbf{v}_i, \mathbf{v}_j\}) < Z_{\{i,j\}}) \quad \textit{where} \quad d(\{\mathbf{v}_i, \mathbf{v}_j\}) = ||\mathbf{v}_i - \mathbf{v}_j||.$$

Note that since the variables $Z_{\{i,j\}}$ do not have to be identically distributed, we can set $Z_{i,i} = 0$ and avoid self loops should we wish to do so.

Let a path in the graph $\mathcal{G}$ from u to w be any ordered tuple of vertices $v_{i_1}, \ldots, v_{i_p}$ with $v_{i_1} = u, v_{i_p} = w$ and $X_{\{i_k, i_{k+1}\}} = 1$ for $k = 1, \ldots, p-1$, and having all edges $\{v_{i_k}, v_{i_{k+1}}\}$ distinct. For U, V and W subsets of $\{v_1, \ldots, v_n\}$ and $\alpha, \beta \subseteq \{\{i,j\} : 1 \le i, j \le n\}$, let $f_\alpha(\mathbf{X})$ be the number of paths in the graph from U to V which use only edges $\{v_i, v_j\}$ for $\{i,j\} \in \alpha$; in the same manner, let $g_\beta(\mathbf{X})$ be the number of paths in the graph from V to W which use only edges $\{v_i, v_j\}$ for $\{i,j\} \in \beta$.

The "projects" A and B in this framework are to create paths from U to V using α, and from V to W using β, respectively, which combine together, when $\alpha \cap \beta = \emptyset$, to give the overall project of creating a path from U to W passing through V. As the product $f_\alpha(\mathbf{X})g_\beta(\mathbf{X})$ for $\alpha \cap \beta = \emptyset$ is the number of paths from U to W via V for the given allocation, Theorem 1.2 provides a bound on the expected maximal number of such paths over all allocations in terms *of the product of the expectations of the maximal number of paths from U to W and from W to V when the paths are created separately. Though finding the optimal allocation may be tedious, the upper bound can be computed simply, for this case in particular by monotonicity of $f_\alpha(\mathbf{x}), g_\beta(\mathbf{x})$ in α and β for fixed $\mathbf{x}$, implying that the maximal number of paths created separately is attained when using all available edges, i.e. at $\alpha = \beta = \mathbf{n}$. However, the result holds even in situations where the existence of more edges does not lead to more paths, that is, in cases where the functions f_α, g_β are not monotone in α and β. One such example would be a situation where the existence of a particular edge mandates that all paths use it.*

This example easily generalizes to paths with multiple waypoints. We may also consider directed graphs where for $i < j$ the directed edge from v_i to v_j exists when $X_{ij} = 1$, from v_j to v_i when $X_{ij} = -1$ and $X_{ij} = 0$ when no edge exists, with X_{ij} independent. Returning to the graph example following the statement of Theorem 1.1, inequality (a) of Theorem 1.2 provides a bound on the expected maximal number of paths from V_1 to V_2 and W_1 to W_2 using disjoint edges.

For application of the dual inequality, consider for example two directed graphs on the same vertex set, determined by equally distributed and independent collections of edge indicators $\mathbf{X}$ and $\mathbf{Y}$, each having independent (but not necessarily identically distributed) components. Let $\alpha, \beta \subseteq \{(i,j) : 1 \le i < j \le n\}$, and $f_\alpha(\mathbf{X})$ be the number of directed paths in the first graph from U to V which use only $\mathbf{X}$ edges (v_i, v_j) with $(i,j) \in \alpha$; in the same manner, let $g_\beta(\mathbf{Y})$ be the number of directed paths in the second graph from V back to U which use only $\mathbf{Y}$ edges (v_i, v_j) with $(i,j) \in \beta$. Consider the expected maximal number of paths, over all α and β with $\alpha \cap \beta = \emptyset$, to go from U to V using the $\mathbf{X}$ edges α at most once and returning to U from V using the $\mathbf{Y}$ edges β at most once. Then Theorem 1.5 implies that this expectation is bounded by the expected maximal number of paths, over all α and β, to move from U to V using α, and then returning to U using β, all with $\mathbf{X}$ edges, but where vertices used on the forward trip may now be used for the return.

3 Proofs of Proposition 1.1 and Theorem 1.2

Proof of Proposition 1.1: We show $(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$ and $(b) \Leftrightarrow (e)$.

$(a) \Rightarrow (c)$: Apply inequality (a) to the finite collections $\{\tilde{f}_K\}_{K \in \mathcal{K}}, \{\tilde{g}_L\}_{L \in \mathcal{L}}$ and use

$$\sup_{\alpha \cap \beta = \emptyset} f_\alpha(\mathbf{x}) g_\beta(\mathbf{x}) = \max_{K \cap L = \emptyset} \left(\sup_{K_\alpha \subseteq K, L_\beta \subseteq L} f_\alpha(\mathbf{x}) g_\beta(\mathbf{x}) \right)$$

$$= \max_{K \cap L = \emptyset} \left(\sup_{K_\alpha \subseteq K} f_\alpha(\mathbf{x}) \sup_{L_\beta \subseteq L} g_\beta(\mathbf{x}) \right) = \max_{K \cap L = \emptyset} \tilde{f}_K(\mathbf{x}) \tilde{g}_L(\mathbf{x}). \tag{17}$$

$(c) \Rightarrow (d)$: Apply $\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) \tilde{g}_{L(\mathbf{x})}(\mathbf{x}) \le \max_{K \cap L = \emptyset} \tilde{f}_K(\mathbf{x}) \tilde{g}_L(\mathbf{x})$.

$(d) \Rightarrow (a)$: Note that the right hand side of (17) equals $\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) \tilde{g}_{L(\mathbf{x})}(\mathbf{x})$ for some $K(\mathbf{x})$ and $L(\mathbf{x})$ with $K(\mathbf{x}) \cap L(\mathbf{x}) = \emptyset$.

$(b) \Rightarrow (e)$: Apply $\underline{f}_{K(\mathbf{x})}(\mathbf{x}) \underline{g}_{L(\mathbf{x})}(\mathbf{x}) \le \max_{K \cap L = \emptyset} \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x})$.

$(e) \Rightarrow (b)$: Use the fact that there exist some disjoint $K(\mathbf{x}), L(\mathbf{x})$ taking values in the subsets of $\mathbf{n}$ such that

$$\max_{K \cap L = \emptyset} \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x}) \le \underline{f}_{K(\mathbf{x})}(\mathbf{x}) \underline{g}_{L(\mathbf{x})}(\mathbf{x}). \quad \blacksquare \tag{18}$$

Our next objective proving the inequalities of Framework 1, to be accomplished in Lemma 3.6. We start with a simple extension of (3), expressed in terms of indicator functions, from finite spaces to spaces endowed with finite sigma algebras.

Lemma 3.1 *Let Q be any probability product measure on $\mathcal{H}$, a finite product sigma algebra. Then, inequality (a) holds when expectations are taken with respect to Q, and $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$ are $\mathcal{H}$ measurable indicator functions.*

Proof: First note that (3) implies $Q(A\Box B) \le Q(A)Q(B)$, by a trivial identification between the atoms of finite sigma algebras. Let A and B be defined by their indicator functions

$$\mathbf{1}_A(\mathbf{x}) = \max_\alpha f_\alpha(\mathbf{x}), \quad \mathbf{1}_B(\mathbf{x}) = \max_\beta g_\beta(\mathbf{x}).$$

Then

$$\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) \le \max_{\alpha\cap\beta=\emptyset} \underline{\mathbf{1}_A}(\mathbf{x})_{K_\alpha}\underline{\mathbf{1}_B}(\mathbf{x})_{L_\beta} \le \max_{K\cap L=\emptyset} \underline{\mathbf{1}_A}(\mathbf{x})_K\underline{\mathbf{1}_B}(\mathbf{x})_L = \mathbf{1}_{A\Box B}(\mathbf{x}), \tag{19}$$

where the first inequality holds since $f_\alpha(\mathbf{x}) \le \mathbf{1}_A(\mathbf{x})_{K_\alpha}$ and therefore $f_\alpha(\mathbf{x}) = \underline{f_\alpha}(\mathbf{x})_{K_\alpha} \le \underline{\mathbf{1}_A}(\mathbf{x})_{K_\alpha}$, and the last equality is (5), and hence

$$E_Q\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} = Q(A\Box B) \le Q(A)Q(B) = E_Q\left\{\max_\alpha f_\alpha(\mathbf{X})\right\} E_Q\left\{\max_\beta g_\beta(\mathbf{X})\right\}. \blacksquare$$

For $\mathcal{C} = \prod_{i=1}^n \mathcal{C}_i$, where $\mathcal{C}_i$ are finite collections of elements of $\mathbb{S}_i$, let $\mathcal{F}_\mathcal{C}$ be the product sub-sigma algebra of $\mathbb{S}$ generated by $\mathcal{C}$. Then

$$\mathbb{S} = \mathcal{F}_\mathcal{J} \quad \text{with} \quad \mathcal{J} = \bigcup_\mathcal{C} \mathcal{F}_\mathcal{C}, \tag{20}$$

where the union is over all finite $\mathcal{C}$, and $\mathcal{F}_\mathcal{J}$ denotes the sigma algebra generated by the algebra $\mathcal{J}$ consisting of the rectangular sets.

We say a collection of functions is FP if it generates a finite product sub-sigma algebra of $\mathbb{S}$; note that a finite union of FP collections is FP.

Lemma 3.2 *Inequality (a) is true for P any probability product measure on $(S,\mathbb{S})$, and $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$, any finite collections of FP indicator functions.*

Proof: Let $\mathcal{H}$ be the sigma algebra generated by $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$, and $Q := P|_\mathcal{H}$, the restriction of P to the finite product sigma algebra $\mathcal{H}$. For h an $\mathcal{H}$ measurable indicator function, that is, for $h(\mathbf{x}) = \mathbf{1}_A(\mathbf{x})$ for some $A \in \mathcal{H}$, we have

$$E_Q h = Q(A) = P(A) = E_P h. \tag{21}$$

Since the product of $\mathcal{H}$ measurable indicators is an $\mathcal{H}$ measurable indicator, and the same is true for the maximum, we have by Lemma 3.1 and (21),

$$\begin{aligned} E_P\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} &= E_Q\left\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\right\} \\ &\le E_Q\left\{\max_\alpha f_\alpha(\mathbf{X})\right\} E_Q\left\{\max_\beta g_\beta(\mathbf{X})\right\} \\ &= E_P\left\{\max_\alpha f_\alpha(\mathbf{X})\right\} E_P\left\{\max_\beta g_\beta(\mathbf{X})\right\}. \quad \blacksquare \end{aligned}$$

Next we drop the restriction that the collection of indicator functions be FP.

Lemma 3.3 *Inequality (a) is true for any probability product measure P and finite collections $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ of $\mathbb{S}$ measurable indicator functions.*

Proof: For $\mathcal{R},\mathcal{S}$ subsets of $\mathcal{A}\cup\mathcal{B}$ satisfying $\mathcal{R}\cap\mathcal{S}=\emptyset$ and $\mathcal{R}\cup\mathcal{S}=\mathcal{A}\cup\mathcal{B}$, we proceed by induction on the cardinality of the set $\mathcal{S}$ in the statement $I(\mathcal{R},\mathcal{S})$: inequality (a) is true when $\{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}},\{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ are finite FP collections of indicator functions, and $\{f_\alpha\}_{\alpha\in\mathcal{S}\cap\mathcal{A}},\{g_\beta\}_{\beta\in\mathcal{S}\cap\mathcal{B}}$ are any finite collections of $\mathbb{S}$ measurable indicators. Lemma 3.2 shows that $I(\mathcal{A}\cup\mathcal{B},\emptyset)$ is true, and the conclusion of the present lemma is $I(\emptyset,\mathcal{A}\cup\mathcal{B})$. Assume for some such $\mathcal{R},\mathcal{S}$ with $\mathcal{S}\neq\mathcal{A}\cup\mathcal{B}$, that $I(\mathcal{R},\mathcal{S})$ is true. For $\gamma\in\mathcal{R}$ with, say $\gamma\in\mathcal{A}$, let $\mathcal{M}$ be the collection of all sets $A\subseteq S$ such that (a) holds for $f_\gamma=\mathbf{1}_A$, when $\{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}\setminus\{\gamma\}},\{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ is a finite FP collections of indicators, and $\{f_\alpha\}_{\alpha\in\mathcal{S}\cap\mathcal{A}},\{g_\beta\}_{\beta\in\mathcal{S}\cap\mathcal{B}}$ any collection of $\mathbb{S}$ measurable indicators. The singleton collection f_γ is FP for $A\in\mathcal{J}$ as in (20). Therefore, for $A\in\mathcal{J}$, the union $f_\gamma,\{f_\alpha\}_{\alpha\in\mathcal{R}\cap\mathcal{A}\setminus\{\gamma\}},\{g_\beta\}_{\beta\in\mathcal{R}\cap\mathcal{B}}$ is FP. By the induction hypothesis, $\mathcal{J}\subseteq\mathcal{M}$. Since $\mathcal{M}$ is a monotone class and $\mathcal{J}$ is an algebra which generates $\mathbb{S}$, the monotone class theorem implies $\mathbb{S}\subseteq\mathcal{M}$. This completes the induction. ∎

At this point we have proved the BKR inequality (3) on a general space $(S,\mathbb{S})$. To move to the functional inequalities we begin to relax the requirement that the functions be indicators.

Lemma 3.4 *Inequality (d) is true for any product measure P and finite collections of $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$ which assume finitely many non-negative values.*

Proof: We prove (d) by induction on m and l, the number of values taken on by the collections $\{f_\alpha\}_{\alpha\in\mathcal{A}},\{g_\beta\}_{\beta\in\mathcal{B}}$ respectively. By by Lemma 3.3 inequality (a) is true for finite collections of measurable indicators, and hence by Proposition 1.1, so is (d). Now the base case $m=2$, $l=2$ follows readily by extending from indicators to two valued functions by linear transformation.

Assume the result is true for some m and l at least 2, and consider a collection $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ assuming the values $0\le a_1<\ldots<a_{m+1}$; a similar argument applies to induct on l. For some k, $2\le k\le m$, define

$$A_{\alpha,k}=\{\mathbf{x}:f_\alpha(\mathbf{x})=a_k\},$$

and for $a_{k-1}\le a\le a_{k+1}$, let

$$h_\alpha^a(\mathbf{x})=f_\alpha(\mathbf{x})+(a-a_k)\mathbf{1}_{A_{\alpha,k}}(\mathbf{x}),$$

the function f_α with the value of a_k replaced by a. We shall prove that for all $a\in[a_{k-1},a_{k+1}]$ inequality (d) holds with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$. By the induction hypothesis we know it holds at the endpoints, that is, for $a\in\{a_{k-1},a_{k+1}\}$, since then the collection $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ takes on m values; the case $a=a_k$ suffices to prove the lemma, as that reduces to (d).

Given $\Gamma(\mathbf{x})$, a function with values in $2^{\mathcal{A}}$, with some abuse of notation denote

$$\tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x})=\sup_{\alpha:\alpha\in\Gamma(\mathbf{x})}f_\alpha(\mathbf{x}). \tag{22}$$

Note that $\tilde{f}_{K(\mathbf{x})}(\mathbf{x})$ in (11) corresponds to $\Gamma(\mathbf{x})=\{\alpha:K_\alpha\subseteq K(\mathbf{x})\}$, and similarly for $\tilde{g}_{L(\mathbf{x})}(\mathbf{x})$; for measurability issues see Section 5. For any given function $\Gamma(\mathbf{x})$ with values in $2^{\mathcal{A}}$, we have for all $a\in[a_{k-1},a_{k+1}]$,

$$C_\Gamma:=\{\mathbf{x}:\widetilde{h^a}_{\Gamma(\mathbf{x})}(\mathbf{x})=a,\ \tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x})\notin\{a_{k-1},a_{k+1}\}\}=\{\mathbf{x}:\tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x})=a_k\},$$

showing that C_Γ does not depend on a, and for $\mathbf{x} \in C_\Gamma^c$, the value $\sup_{\alpha\in\Gamma(\mathbf{x})} h_\alpha^a(\mathbf{x}) = a_j$, for some $j \neq k$, which again does not depend on a.

Letting $D = C_\Gamma$ for $\Gamma(\mathbf{x}) = \mathcal{A}$, the right hand side of (d), with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, equals $a\delta + \lambda$, where

$$\delta = P(D)\int \sup_\beta g_\beta(\mathbf{x})dP(\mathbf{x}) \quad \text{and} \quad \lambda = \int_{D^c} \sup_\alpha h_\alpha^a(\mathbf{x})dP(\mathbf{x}) \int \sup_\beta g_\beta(\mathbf{x})dP(\mathbf{x})$$

do not depend on a. Now, letting $E = C_\Gamma$ for $\Gamma(\mathbf{x}) = \{\alpha : K_\alpha = K(\mathbf{x})\}$, the left hand side of (d), with $\{h_\alpha^a\}_{\alpha\in\mathcal{A}}$ replacing $\{f_\alpha\}_{\alpha\in\mathcal{A}}$, equals $a\theta + \eta$, where

$$\theta = \int_E \tilde{g}_{L(\mathbf{x})}(\mathbf{x})dP(\mathbf{x}) \quad \text{and} \quad \eta = \int_{E^c} \widetilde{h^a}_{K(\mathbf{x})}(\mathbf{x})\tilde{g}_{L(\mathbf{x})}(\mathbf{x})dP(\mathbf{x})$$

do not depend on a. When $a \in \{a_{k-1}, a_{k+1}\}$ the collection $h_\alpha, \alpha \in \mathcal{A}$ takes on m values, so by the induction hypotheses,

$$a\theta + \eta \leq a\delta + \lambda, \quad \text{for } a \in \{a_{k-1}, a_{k+1}\}. \tag{23}$$

By taking a convex combination, we see that inequality (23) holds for all $a \in [a_{k-1}, a_{k+1}]$, so in particular for a_k, completing the induction. ∎

Lemma 3.5 *Inequality (d) is true for any probability product measure P and finite collections of non-negative $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$.*

Proof: Lemma 3.4 shows that the result is true for simple functions. By approximating the functions f_α, g_β below by simple functions, $f_{\alpha,k} \uparrow f_\alpha, g_{\beta,k} \uparrow g_\beta$ as $k \uparrow \infty$, and applying the monotone convergence theorem, we have the result for arbitrary non-negative functions. ∎

Lemma 3.6 *Inequality (d) holds for countable collections of non-negative $\mathbb{S}$ measurable functions $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{g_\beta\}_{\beta\in\mathcal{B}}$.*

Proof: Apply Lemma 3.5 to the finite collections $\{\tilde{f}_K\}_{K\in 2^{\mathbf{n}}}$ and $\{\tilde{g}_L\}_{L\in 2^{\mathbf{n}}}$ defined in (10); setting $\varphi_K = \tilde{f}_K$, for the left hand side of (d) use

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sup_{\alpha:K_\alpha\subseteq K(\mathbf{x})} f_\alpha(\mathbf{x}) = \sup_{K\subseteq K(\mathbf{x})} \sup_{\alpha:K_\alpha\subseteq K} f_\alpha(\mathbf{x}) = \sup_{K\subseteq K(\mathbf{x})} \tilde{f}_K(\mathbf{x}) = \tilde{\varphi}(\mathbf{x})_{K(\mathbf{x})},$$

and on the right hand side use

$$\sup_K \tilde{f}_K(\mathbf{x}) = \sup_\alpha f_\alpha(\mathbf{x}). \quad ∎$$

At this point we have completed proving all inequalities pertaining to Framework 1. The next proposition connects the two frameworks and completes the proofs of Theorem 1.2 and Proposition 1.1.

Proposition 3.1 *Inequality (a) holds in Framework 1 for all collections $\{f_\alpha\}_{\alpha\in\mathcal{A}}, \{g_\beta\}_{\beta\in\mathcal{B}}$ of given functions, if and only if inequality (b) holds in Framework 2 for all given functions f and g and collections $\mathcal{K}$ and $\mathcal{L}$.*

Proof:
$(a) \Rightarrow (b)$. Given f, g, $\mathcal{K}$ and $\mathcal{L}$ apply inequality (a) to the collections $\{\underline{f}_K(\mathbf{x})\}_{K\in\mathcal{K}}$ and $\{\underline{g}_L(\mathbf{x})\}_{L\in\mathcal{L}}$ as in (4), and use

$$\max_{K\in\mathcal{K}} \underline{f}_K(\mathbf{x}) \le f(\mathbf{x}) \quad \text{and} \quad \max_{L\in\mathcal{L}} \underline{g}_L(\mathbf{x}) \le g(\mathbf{x}) \quad a.e.$$

$(b) \Rightarrow (a)$: Given collections of functions f_α, g_β depending on K_α, L_β, define

$$f(\mathbf{x}) = \sup_\alpha f_\alpha(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = \sup_\beta g_\beta(\mathbf{x}). \tag{24}$$

Now letting $\underline{f}_K, \underline{g}_L$ be as in (4), we have

$$f_\alpha(\mathbf{x}) = \underline{f_\alpha}(\mathbf{x})_{K_\alpha} \le \underline{f}_{K_\alpha}(\mathbf{x}) \quad \text{and likewise} \quad g_\beta(\mathbf{x}) \le \underline{g}_{L_\beta}(\mathbf{x}).$$

Now, for α, β disjoint,

$$f_\alpha(\mathbf{x})g_\beta(\mathbf{x}) \le \underline{f}_{K_\alpha}(\mathbf{x})\underline{g}_{L_\beta}(\mathbf{x}) \le \max_{K\cap L=\emptyset} \underline{f}_K(\mathbf{x})\underline{g}_L(\mathbf{x}).$$

Taking supremum on the left hand side and then expectation, the result now follows using (24). ■

3.1 The Dual Inequality

The dual inequality (8) is stated only for uniform measure on $\{0,1\}^n \times \{0,1\}^n$. In order to parallel the proof of Theorem 1.2, we first need an analog of Theorem 1.1, showing that (8) is true for any product measure on a discrete finite product space S. This generalization was done for $A\Box B$ in [8]. For $A\Diamond B$, Lemma 3.4 of [8] carries over with minimal changes, e.g. one again considers a function $f : S' \to S$ between finite product spaces, but now

$$(f \times f)^{-1}(A\Diamond B) = \bigcup_{C_1,C_2} \left\{(f \times f)^{-1}(C_1 \times C_2)\right\},$$

where the union is over all C_1, C_2 such that C_1 is a maximal cylinder of A, C_2 is a maximal cylinder of B, and $C_1 \perp C_2$. Now Lemma 3.5 of [8] and the transformation f constructed there can be applied to prove the generalization of (8) to discrete spaces.

The proof of Theorem 1.5 and Proposition 1.2 follows in a nearly identical manner to that of Theorem 1.2 and Proposition 1.1. To prove (d$'$) , consider

$$C_\Gamma = \{(\mathbf{x},\mathbf{y}) : \widetilde{h^a}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) = a,\ \tilde{f}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) \notin \{a_{k-1}, a_{k+1}\}\} = \{(\mathbf{x},\mathbf{y}) : \tilde{f}_{\Gamma(\mathbf{x},\mathbf{y})}(\mathbf{x}) = a_k\}.$$

Setting $D = C_\Gamma$ for $\Gamma(\mathbf{x},\mathbf{y}) = \{\alpha : K_\alpha = K(\mathbf{x},\mathbf{y})\}$ we can write the left hand side of (d$'$) as $a\theta + \eta$, with

$$\theta = \int_D \tilde{g}_{L(\mathbf{x},\mathbf{y})}(\mathbf{y})dP(\mathbf{x})dP(\mathbf{y}) \quad \text{and} \quad \eta = \int_{D^c} \widetilde{h^a}_{K(\mathbf{x},\mathbf{y})}(\mathbf{x})\tilde{g}_{L(\mathbf{x},\mathbf{y})}(\mathbf{y})dP(\mathbf{x})dP(\mathbf{y}),$$

and using $E = C_\Gamma$ for $\Gamma = \mathcal{A}$, the right hand side becomes $a\delta + \lambda$, where

$$\delta = \int_D \sup_\beta g_\beta(\mathbf{x})dP(\mathbf{x}) \quad \text{and} \quad \lambda = \int_{D^c} \sup_{\alpha,\beta} h^a_\alpha(\mathbf{x})g_\beta(\mathbf{x})dP(\mathbf{x})$$

with θ, η, δ and λ not depending on a.

4 A PQD ordering inequality

Consider a collection $\{f_\alpha(\mathbf{x})\}_{\alpha=1}^m$ of functions which are all increasing or all decreasing in each component of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$. Let $\mathbf{X} = (X_1, \ldots, X_n) \in \mathbf{R}^n$ be a vector of independent random variables, $\mathbf{Y} = (Y_1, \ldots, Y_n)$ an independent copy of $\mathbf{X}$, and for each $\alpha = 1, \ldots, m$, let $K_\alpha \subseteq \mathbf{n}$, and

$$\mathbf{Z}_\alpha = (Z_{1,\alpha}, \ldots, Z_{n,\alpha}), \tag{25}$$

where $Z_{i,\alpha} = Y_i$ if $i \in K_\alpha$, and $Z_{i,\alpha} = X_i$, if $i \notin K_\alpha$. Now let

$$\mathbf{U} = (f_1(\mathbf{Z}_1), \ldots, f_m(\mathbf{Z}_m)) \quad \text{and} \quad \mathbf{V} = (f_1(\mathbf{X}_1), \ldots, f_m(\mathbf{X}_m)). \tag{26}$$

Inequalities between vectors below are coordinate-wise. When (27) below holds, we say that the components of $\mathbf{V}$ are more 'Positively Quadrant Dependent' than those of $\mathbf{U}$, and write $\mathbf{U} \le_{PQD} \mathbf{V}$.

Theorem 4.1 *For every* $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbf{R}^m$,

$$P(\mathbf{U} \ge \mathbf{c}) \le P(\mathbf{V} \ge \mathbf{c}) \quad \textit{and} \quad P(\mathbf{U} \le \mathbf{c}) \le P(\mathbf{V} \le \mathbf{c}). \tag{27}$$

Proof: For $k \in \{0, \ldots, n\}$ let $K_\alpha^k = K_\alpha \cap \{0, \ldots, k\}$, and with K_α replaced by K_α^k, let $\mathbf{Z}_\alpha^k$ be defined as in (25), and $\mathbf{U}^k$ be defined as in (26). We prove the first inequality in (27) by induction on k in $P(\mathbf{U}^k \ge \mathbf{c}) \le P(\mathbf{V} \ge \mathbf{c})$; the second follows in the same manner. The inequality is trivially true, with equality, when $k = 0$, since then $K_\alpha^k = \emptyset$ and $\mathbf{Z}_\alpha = \mathbf{X}$ for all $\alpha \in \mathbf{n}$. Now assume the inequality is true for $0 \le k < n$ and set

$$B = \{\alpha : k + 1 \in K_\alpha\}.$$

Then

$$\begin{aligned}
& P(\mathbf{U}^{k+1} \ge \mathbf{c}) \\
= \;& P(f_1(\mathbf{Z}_1^{k+1}) \ge c_1, \ldots, f_m(\mathbf{Z}_m^{k+1}) \ge c_m) \\
= \;& E[P(f_1(\mathbf{Z}_1^{k+1}) \ge c_1, \ldots, f_m(\mathbf{Z}_m^{k+1}) \ge c_m | X_l, Y_l, l \ne k+1)] \\
= \;& E[P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \notin B | X_l, Y_l, l \ne k+1) P(f_\alpha(\mathbf{Z}_\alpha^{k+1}) \ge c_\alpha, \alpha \in B | X_l, Y_l, l \ne k+1)] \\
= \;& E[P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \notin B | X_l, Y_l, l \ne k+1) P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \in B | X_l, Y_l, l \ne k+1)] \\
= \;& E[P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \notin B | X_l, l \ne k+1) P(f_\alpha(\mathbf{Z}_\alpha^k) \ge c_\alpha, \alpha \in B | X_l, l \ne k+1)] \\
\le \;& E[P(f_1(\mathbf{Z}_1^k) \ge c_1, \ldots, f_m(\mathbf{Z}_m^k) \ge c_m) | X_l, l \ne k+1] \\
= \;& P(\mathbf{U}^k \ge \mathbf{c}) \le P(\mathbf{V} \ge \mathbf{c}),
\end{aligned}$$

where the third equality follows from the independence of X_{k+1} and Y_{k+1} and the forth from the fact that $\{f_\alpha(\mathbf{Z}_\alpha^k)\}_{\alpha \in B}$ has the same distribution when either only X_{k+1} or only Y_{k+1} appears in the $k+1$st coordinate; the first inequality follows from the fact that conditioned on $X_l, l \ne k+1$, the variables $f_\alpha(\mathbf{Z}_\alpha^k)$ are all increasing functions of X_{k+1} and are therefore (conditionally) associated, and the second inequality is the induction hypothesis. ∎

Taking $\mathbf{c} = (c, \ldots, c)$ we immediately have

Corollary 4.1

$$P(\max_\alpha f_\alpha(\mathbf{Z}_\alpha) \le c) \le P(\max_\alpha f_\alpha(\mathbf{X}) \le c) \quad \text{or equivalently} \quad \max_\alpha f_\alpha(\mathbf{X}) \le_{ST} \max_\alpha f_\alpha(\mathbf{Z}_\alpha).$$

Application 1. Consider the framework of Theorem 1.2, with $f_\alpha(\mathbf{x}), g_\beta(\mathbf{x}), \alpha \in \mathcal{A}, \beta \in \mathcal{B}$ all increasing or all decreasing functions which depend on coordinates K_α, L_β. Define $\mathcal{D}$ to be a collection of functions

$$\mathcal{D} = \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{X}) : K_\alpha \cap L_\beta = \emptyset\},$$

and for $\mathbf{Y} = (Y_1, \ldots, Y_n)$ as above, set

$$\mathcal{D}^* = \{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{Y}) : K_\alpha \cap L_\beta = \emptyset\}.$$

By Theorem 4.1 we have

$$\mathcal{D}^* \le_{PQD} \mathcal{D}.$$

Applying Corollary 4.1,

$$\max_{\alpha\cap\beta=\emptyset}\{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{X})\} \le_{ST} \max_{\alpha\cap\beta=\emptyset}\{f_\alpha(\mathbf{X}) + g_\beta(\mathbf{Y})\}.$$

Exponentiating the last relation and replacing e^{f_α} by f_α, using obvious properties of the max, we obtain

$$\max_{\alpha\cap\beta=\emptyset}\{f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\} \le_{ST} \max_{\alpha\cap\beta=\emptyset}\{f_\alpha(\mathbf{X})g_\beta(\mathbf{Y})\}. \tag{28}$$

and therefore

$$E\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{X})\} \le E\{\max_{\alpha\cap\beta=\emptyset} f_\alpha(\mathbf{X})g_\beta(\mathbf{Y})\} \le E\{\max_\alpha f(\mathbf{X})\}\, E\{\max_\beta g(\mathbf{X})\},$$

for nonnegative monotone functions f_α and g_β. Thus the relation (28) is stronger than the BKR inequality for monotone sets, which was proved in [9]. Alexander [1] presents similar functional versions in this context.

As an example we return to order statistics as in Section 2.1. From (28) we get, for example, that

$$X_{[1]}X_{[2]} \le_{ST} X_{[1]}Y_{[2]} \vee Y_{[1]}X_{[2]}.$$

Generalizing by using the functions (13), we obtain for any $p + q = m$,

$$\prod_{j=1}^m X_{[j]} \le_{ST} \max_{\{i_1,\ldots,i_p\}\cup\{j_1,\ldots,j_q\}=\{1,\ldots,m\}} \prod X_{[i_q]}Y_{[j_q]}.$$

5 Appendix on Measurability

In this section we briefly deal with various measurability issues. The measurability of the function defined in (11) can be seen from

$$\tilde{f}_{K(\mathbf{x})}(\mathbf{x}) = \sum_K \tilde{f}_K(\mathbf{x})\mathbf{1}(K(\mathbf{x}) = K),$$

since the given function $K(\mathbf{x})$ is assumed measurable. Similarly for (22),

$$\tilde{f}_{\Gamma(\mathbf{x})}(\mathbf{x}) = \sum_{A \in 2^{\mathcal{A}}} \sup_{\alpha \in A} f_\alpha(\mathbf{x}) \mathbf{1}(\Gamma(\mathbf{x}) = A).$$

We next prove that given a non-negative, $(\mathbb{S}, \mathbb{B})$ measurable function $f : S \to \mathbf{R}$ and any $K \subseteq \mathbf{n}$, the function defined in (4) is $(\mathbb{S}, \mathbb{B})$ measurable. Letting

$$f_r(\mathbf{x}) = \min(f(\mathbf{x}), r)$$

and $dP(\mathbf{x}_L)$ be the marginal of P on the coordinates $\mathbf{x}_L$, we have

$$\lim_{p \to \infty} \left(\int (r - f_r(\mathbf{x}))^p dP(\mathbf{x}_{K^c}) \right)^{1/p} = \operatorname{ess} \sup_{\mathbf{y} \in [\mathbf{x}]_K} (r - f_r(\mathbf{y})) = r - \operatorname{ess} \inf_{\mathbf{y} \in [\mathbf{x}]_K} f_r(\mathbf{y}).$$

Tonelli's theorem (see e.g. [3]) now implies that $\operatorname{ess\,inf}_{\mathbf{y} \in [\mathbf{x}]_K} f_r(\mathbf{y})$ is measurable. Letting $r \uparrow \infty$ shows that (4) is measurable.

The only complication regarding measurability of the pair $(K(\mathbf{x}), L(\mathbf{x}))$ in (18) is that the maximum may not be uniquely attained, since otherwise we would simply have

$$\{\mathbf{x} : K(\mathbf{x}) = K, L(\mathbf{x}) = L\} = \bigcap_{K' \cap L' = \emptyset} \{\mathbf{x} : \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x}) \geq \underline{f}_{K'}(\mathbf{x}) \underline{g}_{L'}(\mathbf{x})\},$$

a finite intersection of measurable sets, so measurable. To handle the problem of non-uniqueness, let $\prec$ be an arbitrary total order on the finite collection of subsets of $\mathbf{n} \times \mathbf{n}$, so that when the max is not unique we can choose $(K(\mathbf{x}), L(\mathbf{x}))$ to be the first disjoint pair that attains the maximum. Then $\{\mathbf{x} : K(\mathbf{x}) = K, L(\mathbf{x}) = L\} = F \cap G$ where

$$F = \bigcap_{\substack{(K', L') \prec (K, L) \\ K' \cap L' = \emptyset}} \{\mathbf{x} : \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x}) > \underline{f}_{K'}(\mathbf{x}) \underline{g}_{L'}(\mathbf{x})\}$$

and

$$G = \bigcap_{\substack{(K', L') \succeq (K, L) \\ K' \cap L' = \emptyset}} \{\mathbf{x} : \underline{f}_K(\mathbf{x}) \underline{g}_L(\mathbf{x}) \geq \underline{f}_{K'}(\mathbf{x}) \underline{g}_{L'}(\mathbf{x})\}$$

and again measurability follows.

Finally, since all integrals in the inequalities are unchanged by modifications on P null sets, the results hold for $(S, \bar{\mathbb{S}})$, the completion of $(S, \mathbb{S})$ with respect to P.

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