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Marina Di Giacinto
Salvatore Federico
Fausto Gozzi
Elena Vigna

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Constrained portfolio choices in the decumulation phase of a pension plan

Marina Di Giacinto[*] Salvatore Federico[†] Fausto Gozzi [‡] Elena Vigna[§]

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Abstract

This paper deals with a constrained investment problem for a defined contribution (DC) pension fund where retirees are allowed to defer the purchase of the annuity at some future time after retirement.

This problem has already been treated in the unconstrained case in a number of papers. The aim of this work is to deal with the more realistic case when constraints on the investment strategies and on the state variable are present. Due to the difficulty of the task, we consider the basic model of [Gerrard, Haberman & Vigna, 2004], where interim consumption and annuitization time are fixed. The main goal is to find the optimal portfolio choice to be adopted by the retiree from retirement to annuitization time in a Black and Scholes financial market. We define and study the problem at two different complexity levels. In the first level (problem P1), we only require no short-selling. In the second level (problem P2), we add a constraint on the state variable, by imposing that the final fund cannot be lower than a certain guaranteed safety level. This implies, in particular, no ruin.

The mathematical problem is naturally formulated as a stochastic control problem with constraints on the control and the state variable, and is approached by the dynamic programming method. We give a general result of existence and uniqueness of regular solutions for the Hamilton-Jacobi-Bellman equation and, in a special case, we explicitly compute the value function for the problem and give the optimal strategy in feedback form.

A numerical application of the special case – when explicit solutions are available – ends the paper and shows the extent of applicability of the model to a DC pension fund in the decumulation phase.

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Keywords: pension fund, decumulation phase, constrained portfolio, stochastic optimal control, dynamic programming, Hamilton-Jacobi-Bellman equation.

[*]Dipartimento Istituzioni, Metodi Quantitativi e Territorio, Università degli studi di Cassino, Italy. Email: `digiacinto at unicas.it`

[†]Dipartimento di Scienze Economiche ed Aziendali, Luiss University, Roma, Italy. Email: `sfederico at luiss.it`

[‡]Dipartimento di Scienze Economiche ed Aziendali, Luiss University, Roma, Italy & Centro De Giorgi, Scuola Normale Superiore, Pisa, Italy. Email: `fgozzi at luiss.it`

[§]Dipartimento di Statistica e Matematica Applicata and Collegio Carlo Alberto, Università di Torino, Italy. Email: `elena.vigna at econ.unito.it`

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1 Introduction

In countries where immediate annuitization is the only option available to retiring members of defined contribution (DC) pension schemes, members who retire at a time of low bond yield rates have to accept a pension lower than the one available with higher bond yields (so-called annuity risk). In many countries, including Argentina, Australia, Brazil, Canada, Chile, Denmark, El Salvador, Japan, Peru, UK, US, the retiree is allowed to defer annuitization at some time after retirement, withdraw periodic income from the fund, and invest the rest of it in the period between retirement and annuitization. This allows the retiree to postpone the decision to purchase an annuity until a more propitious time. This flexibility is usually referred as "programmed withdrawal (option)".[1] For a detailed survey on the several forms of benefits provided by the programmed withdrawals option, we refer the interested reader to [Antolin, Pugh & Stewart, 2008]. There are often limits imposed on both the consumption and on how long the annuity purchase can be deferred. On

[1]Other equivalent expressions are: phased withdrawal, scheduled withdrawal, allocated annuities, allocated pensions, allocated income streams, and income drawdown.

the other hand, there is virtually unlimited freedom to invest the fund in a broad range of assets. While this option allows the retiree to aim to a final annuity higher than that purchasable at retirement, the evident drawback consists in the possibility of ruin, i.e. exhausting the fund while still alive. The three degrees of freedom of the retiree (amount of consumption, investment allocation, and time of annuitization), together with the important issue of ruin possibility, have been investigated in the actuarial and financial literature in many papers. Among others, [Albrecht & Maurer, 2002], [Blake, Cairns & Dowd, 2003], [Gerrard, Haberman & Vigna, 2004], [Gerrard, Haberman & Vigna, 2006], [Gerrard, Højgaard & Vigna, 2010], [Milevsky, 2001], [Milevsky, Moore & Young, 2006], and [Milevsky & Young, 2007].

In this paper, we consider the position of a representative participant of a defined contribution pension fund who retires and compulsorily has to purchase an annuity within a certain period of time after retirement. In the interim the accumulated capital is dynamically allocated while the pensioner withdraws periodic amounts of money to provide for daily life, in accordance with restrictions imposed by the scheme's rules or by legislation. In particular, we assume that an individual who retires acquires control of the fund at her disposal, which is invested in a market that consists of a risky and a riskless asset. The value of the risky asset is assumed to follow a geometric Brownian motion model. The retiree is given only one degree of freedom, namely the investment allocation. The income withdrawn from the fund in the unit time is assumed to be fixed and the retiree is obliged to annuitize at a fixed future time.

This kind of problem is naturally formulated as a stochastic optimal control problem, once the utility or loss function is selected. In the presence of a quadratic loss function, this problem has been considered in [Gerrard, Haberman & Vigna, 2004], [Gerrard, Haberman & Vigna, 2006], and in [Gerrard, Højgaard & Vigna, 2010]. In the first work, the control variable is the investment strategy, in the second one the control variables are the investment and the consumption policies, while in the last paper the retiree is allowed to choose the annuitization time, together with the investment-consumption policies. We notice that in all these papers the controls are unrestricted and, apart in [Gerrard, Højgaard & Vigna, 2010], the no-ruin constraint is absent. This can be explained by the considerable greater difficulty of the task, whenever constraints are added into the model. For this reason, in this paper we select the simplest of the above mentioned models, i.e. the model in [Gerrard, Haberman & Vigna, 2004], and add constraints on both the control and the state variable. In particular, we study what happens to the optimal investment strategy when a short-selling constraint and a final capital requirement are added to the basic model. Ongoing research aims at characterizing the optimal policy in the presence of the borrowing constraint. Furthermore, the extension to a model allowing for the three possible choices outlined, as well as restrictions on the controls, is in the agenda for future research. To the best of our knowledge, this is the first paper that deals with restrictions on the investment strategy together with the final capital requirement in the decumulation phase of a DC pension scheme.

From the methodological point of view, we tackle the problem by the dynamic programming approach, studying the associated Hamilton-Jacobi-Bellman (HJB) equation. The main issue with respect to the previous literature is the addition of the state constraints coming from the final capital requirement (see problem (P2) below). Indeed, while in [Gerrard, Haberman & Vigna, 2004] explicit solutions of the associated HJB equations are always found, the addition of a state constraint in our problem imposes suitable boundary conditions that make much more difficult (maybe impossible) to find explicit solutions of the associated HJB equation. Then, to use the machinery of the dynamic programming in its whole power, we need to prove existence and uniqueness of regular solutions of the HJB equation (as this is the departure point to prove existence of optimal feedback strategies and to perform a numerical application of them). In this case the regularity of solutions of the HJB equation is difficult to study since this is a fully nonlinear, degenerate parabolic equation. Similar equations have been studied in the autonomous case in [Di Giacinto, Federico & Gozzi, 2010] and [Zariphopoulou; 1994] finding regularity results, but we did not find results useful for this time-dependent case in the literature. Thus, we have used an *ad hoc* procedure to transform the equation into a "dual" one which turns out to be semilinear, and so more treatable. This allows us to prove a regularity result for the general problem and to find explicit solutions in a special but still realistic case.[2] We stress the fact that this regularity result may allow to prove the existence of optimal feedback strategies through a verification theorem. We did not do it for brevity in the general case, concentrating on the case when explicit solutions can be found (see Section 4.5 and 5).

The remainder of the paper is organized as follows. In Section 2, we introduce the model and list three problems to be solved, according to their difficulty. Namely, the problem of short-selling constraint (P1), the problem of short-selling constraint plus final capital requirement (P2), the problem of short-selling and borrowing constraints plus final capital requirement (P3). In Section 3, we tackle problem (P1) and solve it via the dynamic programming approach, proving various results about the value function, and providing both value function and optimal feedback policies in closed-form. Section 4 represents the theoretical core of the paper. Therein, we consider problem (P2), pass to the dual problem and use the viscosity approach to characterize the value function and to prove the regularity of the solution, which in general cannot be found in closed-form. In Section 5, we consider as a particular case the problem (P2) without the running cost, find the solution in closed-form, and show a numerical application that highlights the potential applicability to a DC pension plan. Section 6 concludes and outlines further research.

2 The model

In this section, we outline the model and describe the problem faced by the member of a pension scheme. In Subsection 2.1, we describe the financial market and define three different constrained optimization problems, ordered by complexity level. In Subsection 2.2, we specify the preferences of the individual, i.e. her loss function.

2.1 The financial market and three different constrained problems

In our model, we consider the position of an individual who chooses the income drawdown option at retirement. We assume that final annuitization is compulsory at a certain age. Thus, the individual withdraws a certain fixed income until she achieves the age when the purchase of the annuity is compulsory. Without loss of generality, we assume that the individual retires at time $s = 0$ and that compulsory annuitization occurs at time $s = T$. Bequests motives are absent and the only reason for taking programmed withdrawals is the hope of being better off than immediate annuitization when ultimate annuitization takes place. The fund is invested in two assets: a riskless asset with constant instantaneous rate of return $r \geq 0$, and a risky asset whose price follows a geometric Brownian motion with constant volatility $\sigma > 0$ and drift $\mu := r + \sigma\beta$, where $\beta > 0$ is the so-called Sharpe ratio or risk premium. The pensioner withdraws a fixed amount $b_0 > 0$ in the unit of time.

[2] This procedure has been used, e.g., in [Elie & Touzi, 2008], [Gao, 2008], [Gerrard, Højgaard & Vigna, 2010], [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007], [Schwartz & Tebaldi, 2006], and [Xiao , Zhai & Qin, 2007], but only to find explicit solutions and not to study the regularity of the solution of the HJB equation from a theoretical point of view.

Therefore, according to [Merton, 1969] the state equation that describes the dynamics of the fund wealth $X(\cdot)$ is the following

$$\begin{cases} dX(s) = [(r + (\mu - r)\theta(s))X(s) - b_0]\, ds + \sigma\theta(s)X(s)dB(s), \quad s \in [0, T], \\ X(0) = x_0, \end{cases} \tag{1}$$

where $x_0 > 0$ is the fund wealth at the retirement date $s = 0$, $B(\cdot)$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t^B)_{t\geq 0}, P)$ and $\theta(\cdot)$ is the investment strategy representing the share of portfolio invested in the risky asset.[3]

The dynamics of the wealth can be rewritten using a different control variable, namely the amount of money $\pi(\cdot)$ invested in the risky asset. In this case, the equation describing the dynamics of the fund wealth is

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)\pi(s) - b_0]\, ds + \sigma\pi(s)dB(s), \quad s \in [0, T], \\ X(0) = x_0. \end{cases} \tag{2}$$

The two control variables $\theta(\cdot)$ and $\pi(\cdot)$ are linked by the relationship

$$\pi(\cdot) = \theta(\cdot)X(\cdot).$$

Differently from the classical pure investment problem, we notice that the constraint $X(\cdot) \geq 0$ is not automatically satisfied due to the outflow term $b_0 > 0$ in the state equation (1) or (2). In such a situation, the no short-selling constraint in term of $\theta(\cdot)$ should be written as

$$\begin{cases} \theta(\cdot) \geq 0, & \text{if } X(\cdot) \geq 0, \\ \theta(\cdot) \leq 0, & \text{if } X(\cdot) < 0, \end{cases}$$

depending also on the current sign of the state variable. On the contrary, when the control variable is $\pi(\cdot)$, the no short-selling constraint can be written more easily as

$$\pi(\cdot) \geq 0.$$

For this reason, we choose to treat the problem using the representation with $\pi(\cdot)$ and (2), rather than $\theta(\cdot)$ and (1). We remark that the representation with $\theta(\cdot)$ and (1) would be more suitable in the case of no short-selling and no borrowing constraints (problem (P3) below): in this case the bilateral constraints on the investment strategy become simply $\theta(\cdot) \in [0, 1]$.

Almost all works present in the literature considering optimization problems in DC pension schemes have been solved without constraints on the control variables and the state variable. It is worth noticing that to the best of our knowledge only [Di Giacinto, Federico & Gozzi, 2010] solve a constrained portfolio selection problem in a DC pension scheme. This is due to the mathematical difficulty of the problem with constraints and justifies the choice of simplifying assumptions in this setup, such as the fixed consumption rate and the fixed annuitization time. We intend to relax these assumptions in future work. Given the hard mathematical tractability of this kind of problem, we here present three different constrained problems ordered by increasing complexity level.

(P1) No short-selling: the problem is written in terms of the amount of money invested in the risky asset, $\pi(\cdot)$, and the set of admissible strategies is

$$\{\pi(\cdot) \geq 0\}.$$

[3] $\mathbb{F}$ is the Brownian filtration augmented with the P-null sets, so it satisfies the usual conditions.

(P2) No short-selling and final capital requirement: the problem is written in terms of the amount of money invested in the risky asset, $\pi(\cdot)$, and the set of admissible strategies is

$$\{\pi(\cdot) \geq 0 \ : \ X(T) \geq S \text{ a.s.}\},$$

where $S \geq 0$.

(P3) No short-selling, no borrowing and final capital requirement: the problem is written in terms of the share of portfolio invested in the risky asset, $\theta(\cdot)$, and the set of admissible strategies is

$$\{0 \leq \theta(\cdot) \leq 1 \ : \ X(T) \geq S \text{ a.s.}\},$$

where $S \geq 0$.

Let us notice that in problems (P2) and (P3) the final capital requirement $X(T) \geq S \geq 0$ a. s. implies in particular no ruin, i.e. $X(\cdot) \geq 0$ (see Proposition 4.1-(5)). In this paper, we will address (P1) and (P2), leaving (P3) for further research.

2.2 The loss function

The preferences of the pensioner are described by the loss function

$$L(s,x) := (F(s) - x)^2, \tag{3}$$

where the target function $F(\cdot)$, i.e. the target that the agent wishes to track at any time $s \in [0,T]$, is given by

$$F(s) := \frac{b_0}{r} + \left(F - \frac{b_0}{r}\right) e^{-r(T-s)}.$$

The quantity $F \in (0, b_0/r)$ is the target fund desired at terminal time T and can be chosen arbitrarily. Typically, the final target F as well as the fixed consumption rate b_0 will depend on the initial wealth x_0 or on the replacement ratio achievable with it. The quantity F is also associated to the risk profile of the member: a high F is associated to a less risk averse retiree, and vice versa.

The choice of a quadratic loss function is motivated by the fact that, expectedly, it has been shown to produce an optimal portfolio that is mean-variance efficient (see [Højgaard & Vigna, 2007]). Indeed, there is no other portfolio that provides a (strictly) higher expected value with the same variance, and no other portfolio that provides a (strictly) lower variance with the same mean.

Furthermore, this choice of the target function has several advantages.

Firstly, its interpretation is pretty clear. Should the fund hit $F(s)$ at time $s \leq T$, the pensioner would be able to consume b_0 from s to T by investing the whole portfolio in the riskless asset, and achieve the desired target F at time T of compulsory annuitization. Clearly, in this case the loss function computed on the state trajectory corresponding to this riskless strategy would be 0 at any time $s \leq t \leq T$. As a matter of fact, it will be shown that if the fund is equal to the target, the optimal strategy is the null one.

Secondly, the possibility that deviations above the target can produce unreasonable positive loss is prevented. In fact, [Gerrard, Haberman & Vigna, 2004] show that the optimal fund never reaches the target, provided that at initial time $s = 0$ the fund x_0 is lower than the target $F(0)$.

Thirdly, it can be shown that, by choosing a loss function that does not penalize deviations above the target, such as

$$\tilde{L}(s,x) := \begin{cases} (F(s)-x)^2, & \text{if } x \le F(s), \\ 0, & \text{if } x > F(s), \end{cases}$$

the optimal policy in the region of interest (i.e. below the target level) is equal to that found with the loss function (3). This desirable feature occurs both in problem (P1) and in problem (P2) (see Remark 3.13 and Remark 4.5).

The general optimization problem consists in minimizing over the set of admissible strategies the functional

$$\mathbb{E}\left[\int_0^T \kappa e^{-\rho s} L(s,X(s))ds + e^{-\rho T} L(T,X(T))\right], \tag{4}$$

where ρ is the individual discount factor, $\kappa \ge 0$ is a weighting constant which measures the importance of the running cost in the period before annuitization relative to the final cost at time T.

3 Dynamic Programming for (P1)

In this section, we solve the problem (P1) using the dynamic programming approach and calculate explicitly the value function and the optimal feedback strategy. In order to do this, we define the problem for generic initial data $(t,x) \in [0,T] \times \mathbb{R}^+$. As specified above, the control strategy here is represented by the amount of money $\pi(\cdot)$ invested in the risky asset, so we consider the state equation

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)\pi(s) - b_0]\,ds + \sigma\pi(s)dB(s), \quad s \in [t,T], \\ X(t) = x, \end{cases} \tag{5}$$

where $x \in \mathbb{R}^+$. Let us define the filtration $\mathbb{F}^t := (\mathcal{F}_s^t)_{s\in[t,T]}$, where $\mathcal{F}_s^t$ is the σ-algebra generated by $(B(u) - B(t))_{u\in[t,s]}$ augmented with the P-null sets, and the set of admissible strategies

$$\Pi_{ad}(t) := \{\pi(\cdot) \in L^2(\Omega \times [t,T]; \mathbb{R}^+) \mid \pi(\cdot) \text{ is } \mathbb{F}^t\text{-prog. meas.}\}.$$

For any $\pi(\cdot) \in \Pi_{ad}(t)$ equation (5) admits a unique strong solution on $(\Omega, \mathcal{F}, P)$ (see [Karatzas & Shreve, 1998], Section 5.6.C), and we denote it by $X(\cdot; t, x, \pi(\cdot))$.

We are interested in solving the following optimization problem: for given $(t,x) \in [0,T] \times \mathbb{R}^+$,

$$\text{minimize} \quad J(t,x;\pi(\cdot)) := \mathbb{E}\left[\int_t^T \kappa e^{-\rho s} L(s, X(s;t,x,\pi(\cdot)))ds + e^{-\rho T} L(T, X(T;t,x,\pi(\cdot)))\right]$$

over the set of admissible strategies $\pi(\cdot) \in \Pi_{ad}(t)$.

3.1 Properties of the value function

The value function for the problem is defined as

$$V(t,x) := \inf_{\pi(\cdot)\in\Pi_{ad}(t)} J(t,x;\pi(\cdot)), \qquad (t,x) \in [0,T] \times \mathbb{R}.$$

Clearly, we have $V(t,x) \ge 0$ for every $(t,x) \in [0,T] \times \mathbb{R}$. We now prove some properties of the value function.

Lemma 3.1. *Let $t \in [0,T]$ and $x = F(t)$. Then $X(s;t,x,0) = F(s)$ for all $s \in [t,T]$. Moreover, $V(t,x) = 0$ and the optimal strategy is $\pi(\cdot) \equiv 0$.*

Proof. Let $t \in [0,T]$, $x \in \mathbb{R}$, and set $X(\cdot) := X(\cdot;t,x,0)$. The dynamics of $X(\cdot)$ are given by

$$\begin{cases} dX(s) = (rX(s) - b_0)\, ds, \quad s \in [t,T], \\ X(t) = x. \end{cases}$$

The dynamics of the target $F(\cdot)$ after t is given by

$$\begin{cases} dF(s) = (rF(s) - b_0)\, ds, \quad s \in [t,T], \\ F(t) = F(t). \end{cases}$$

Therefore $X(\cdot)$ and $F(\cdot)$ solve the same initial value problem (IVP), so they coincide.

Moreover, since we have $J(t,x;0) = 0$ and $V(\cdot,\cdot) \geq 0$, we get that $\pi(\cdot) \equiv 0$ is optimal for the initial (t,x) and $V(t,x) = 0$. □

Lemma 3.1 suggests that the graph of $F(\cdot)$ works as a barrier for the problem, so that we are led to separate the space $[0,T] \times \mathbb{R}$ in the two regions

$$U_1 := \{(t,x) \mid t \in [0,T],\ x \leq F(t)\}, \qquad U_2 := \{(t,x) \mid t \in [0,T],\ x \geq F(t)\}. \tag{6}$$

Notice that

$$U_1 \cup U_2 = [0,T] \times \mathbb{R}, \qquad U_1 \cap U_2 = \{(t, F(t)) \mid t \in [0,T]\}.$$

Let us define

$$\Pi^1_{ad}(t,x) = \{\pi(\cdot) \in \Pi_{ad}(t) \mid X(s;t,x,\pi(\cdot)) \in U_1\},\ \Pi^2_{ad}(t,x) = \{\pi(\cdot) \in \Pi_{ad}(t) \mid X(s;t,x,\pi(\cdot) \in U_2\}. \tag{7}$$

Lemma 3.2. *Let $(t,x) \in [0,T] \times \mathbb{R}$, $\pi(\cdot) \in \Pi_{ad}(t)$. Set $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$ and define the stopping time*

$$\tau := \inf\{s \geq t \mid X(s) = F(s)\}$$

with the convention $\inf \emptyset = T$.

Define the strategy

$$\pi^\tau(s) := \begin{cases} \pi(s), & \text{if } s < \tau, \\ 0, & \text{if } s \geq \tau. \end{cases}$$

Then $J(t,x;\pi^\tau(\cdot)) \leq J(t,x;\pi(\cdot))$. Furthermore, the value function admits the representation

$$V(t,x) = \begin{cases} \inf\limits_{\pi(\cdot) \in \Pi^1_{ad}(t,x)} J(t,x;\pi(\cdot)), & \text{on } U_1, \\ \inf\limits_{\pi(\cdot) \in \Pi^2_{ad}(t,x)} J(t,x;\pi(\cdot)), & \text{on } U_2. \end{cases}$$

Proof. It follows straightly from Lemma 3.1. □

Definition 3.3. *Let $(t,x) \in [0,T] \times \mathbb{R}$, $\delta > 0$; a strategy $\pi^\delta(\cdot) \in \Pi_{ad}(t)$ is called δ-optimal if*

$$J(t,x;\pi^\delta(\cdot)) \leq V(t,x) + \delta.$$

Proposition 3.4. *Let $t \in [0,T]$. The function $\mathbb{R} \to \mathbb{R}^+$, $x \mapsto V(t,x)$ is convex.*

Proof. *Step 1.* In this step we will prove that $x \mapsto V(t,x)$ is convex on $(-\infty, F(t)]$. Let us suppose $x, y \le F(t)$. Let $\delta > 0$ and let $\pi_x^\delta(\cdot), \pi_y^\delta(\cdot)$ two controls δ-optimal for x, y respectively, i.e.

$$J(t,x;\pi_x^\delta(\cdot)) \le V(t,x) + \delta, \qquad J(t,y;\pi_y^\delta(\cdot)) \le V(t,y) + \delta.$$

Set $X(s) := X(s;t,x,\pi_x^\delta(\cdot))$, $Y(s) := X(s;t,y,\pi_y^\delta(\cdot))$. Without loss of generality, thanks to Lemma 3.2, we can suppose $X(s), Y(s) \le F(s)$ for all $s \in [t,T]$. We want to prove that, for all $\gamma \in [0,1]$,

$$V(t, \gamma x + (1-\gamma)y) \le \gamma V(t,x) + (1-\gamma)V(t,y).$$

Fix $\gamma \in [0,1]$ and set $Z(s) := \gamma X(s) + (1-\gamma)Y(s)$; of course $Z(s) \le F(s)$, for all $s \in [t,T]$. We have

$$\begin{aligned}
\gamma V(t,x) + (1-\gamma)V(t,y) + \delta &\ge \gamma J(t,x;\pi_x^\delta(\cdot)) + (1-\gamma)J(t,y;\pi_y^\delta(\cdot)) \\
&= \gamma \mathbb{E}\left[\int_t^T \kappa e^{-\rho s}(F(s)-X(s))^2 ds + e^{-\rho T}(F(T)-X(T))^2\right] \\
&\quad + (1-\gamma)\mathbb{E}\left[\int_t^T \kappa e^{-\rho s}(F(s)-Y(s))^2 ds + e^{-\rho T}(F(T)-Y(T))^2\right] \\
&\ge \mathbb{E}\left[\int_t^T \kappa e^{-\rho s}(F(s)-Z(s))^2 ds + e^{-\rho T}(F(T)-Z(T))^2\right]
\end{aligned} \tag{8}$$

where the last inequality follows by convexity of $\xi \mapsto (F(s)-\xi)^2$. The dynamics of $Z(\cdot)$ is

$$\begin{aligned}
dZ(s) &= \gamma dX(s) + (1-\gamma)dY(s) \\
&= \gamma\left[rX(s) + (\mu - r)\pi_x^\delta(s) - b_0\right]ds + (1-\gamma)\left[rY(s) + (\mu-r)\pi_y^\delta(s) - b_0\right]ds \\
&\quad + \gamma\sigma\pi_x^\delta(s)dB(s) + (1-\gamma)\sigma\pi_y^\delta(s)dB(s) \\
&= \left[rZ(s) + (\mu-r)(\gamma\pi_x^\delta(s) + (1-\gamma)\pi_y^\delta(s)) - b_0\right]ds + \sigma\left(\gamma\pi_x^\delta(s) + (1-\gamma)\pi_y^\delta(s)\right)dB(s).
\end{aligned}$$

Thus, if we define $\pi_z(\cdot) := \gamma\pi_x^\delta(\cdot) + (1-\gamma)\pi_y^\delta(\cdot) \in \Pi_{ad}(t)$, we have $Z(s) = X(s;t,\gamma x + (1-\gamma)y, \pi_z(\cdot))$. Therefore we obtain

$$\mathbb{E}\left[\int_t^T \kappa e^{-\rho s}(F(s)-Z(s))^2 ds + e^{-\rho T}(F(T)-Z(T))^2\right] \ge V(t, \gamma x + (1-\gamma)y)). \tag{9}$$

Comparing (8) with (9) we get the claim in this case by the arbitrariness of δ.

Step 2. We can argue exactly as in Step 1 and conclude that $x \mapsto V(t,x)$ is convex on $[F(t), +\infty)$.

Step 3. We can notice that $V(t,\cdot)$ is non-negative and that, thanks to Lemma 3.1, $V(t,F(t)) = 0$, so that $F(t)$ is a minimum for $V(t,\cdot)$. Thus, the global convexity of $V(t,\cdot)$ follows from the convexity on the two half lines $(-\infty, F(t)]$, $[F(t), +\infty)$ and from the fact that it has a minimum in $F(t)$. □

Proposition 3.5. *The value function V is continuous on $[0,T] \times \mathbb{R}$.*

Proof. See [Yong & Zhou, 1999], Chapter 4, Proposition 3.1. □

Proposition 3.6. *Let $t \in [0,T]$; the function $x \mapsto V(t,x)$ is nonincreasing on $(-\infty, F(t)]$ and nondecreasing on $[F(t), +\infty)$.*

Proof. $V(t,\cdot)$ is convex and admits a minimum at $x = F(t)$, hence the claim. □

3.2 The HJB equation

As usual in the context of optimal control problems with finite horizon, the value function is associated to a nonlinear parabolic PDE with terminal boundary condition, which is the so-called Hamilton-Jacobi-Bellman (HJB) equation. We are going to define this equation. To this aim, let us define the Hamiltonian current-value

$$\begin{array}{rccc} \mathcal{H}_{cv}: & \mathbb{R}^2 \times [0, +\infty) & \longrightarrow & \mathbb{R}, \\ & (p, P; \pi) & \longmapsto & \frac{1}{2}\sigma^2 P \pi^2 + \sigma\beta p \pi. \end{array}$$

and the Hamiltonian

$$\begin{array}{rccc} \mathcal{H}: & \mathbb{R}^2 & \longrightarrow & \mathbb{R} \cup \{-\infty\}, \\ & (p, P) & \longmapsto & \inf_{\pi \geq 0} \mathcal{H}_{cv}(p, P; \pi). \end{array}$$

Given $(p, P) \in \mathbb{R} \times (0, +\infty)$, the function $\pi \mapsto \mathcal{H}_{cv}(p, P; \pi)$ has a unique minimum point on $[0, +\infty)$ given by

$$\pi^* = -\frac{\beta\, p}{\sigma\, P} \vee 0, \tag{10}$$

so in this case the Hamiltonian can be written as

$$\mathcal{H}(p, P) = \begin{cases} -\dfrac{\beta^2 p^2}{2P}, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases}$$

The HJB equation is

$$\begin{cases} v_t(t, x) + (rx - b_0) v_x(t, x) + \kappa e^{-\rho t}(F(t) - x)^2 + \mathcal{H}\left(v_x(t, x), v_{xx}(t, x)\right) = 0, \text{ on } [0, T] \times \mathbb{R}, \\ v(T, x) = e^{-\rho T}(F - T)^2, \ x \in \mathbb{R}. \end{cases} \tag{11}$$

Suppose that the value function is regular on the regions U_1 and U_2 defined by (6). Inspired by the previous section, which gives information on the signs of V_x, V_{xx} on the regions U_1 and U_2, we can split the HJB equation (11) in these two regions. Thus, V should satisfy the equation

$$\begin{cases} \kappa e^{-\rho t}(F(t) - x)^2 + v_t(t, x) + (rx - b_0) v_x(t, x) - \dfrac{\beta^2 v_x^2(t, x)}{2 v_{xx}(t, x)} = 0, & \text{on } U_1 \backslash \{(t, F(t)) \,|\, t \in [0, T]\}, \\ \kappa e^{-\rho t}(F(t) - x)^2 + v_t(t, x) + (rx - b_0) v_x(t, x) = 0, & \text{on } U_2 \backslash \{(t, F(t)) \,|\, t \in [0, T]\}, \end{cases} \tag{12}$$

with boundary conditions

$$\begin{cases} v(t, F(t)) = 0, & t \in [0, T], \\ v_x(t, F(t)) = 0, & t \in [0, T], \\ v(T, x) = e^{-\rho T}(F(T) - x)^2, & x \in \mathbb{R}. \end{cases} \tag{13}$$

Definition 3.7. *A function v is called a classical solution of* (12)-(13) *if*

- $v \in C^{1,1}([0, T] \times \mathbb{R}; \mathbb{R}) \cap C^{1,2}\big(([0, T] \times \mathbb{R}) \backslash \{(t, F(t)) \mid t \in [0, T]\}; \mathbb{R}\big)$,
- *v satisfies pointwise in classical sense* (12) *(the derivatives with respect to the time variable at $t = 0$ and $t = T$ have to be intended respectively as right and left derivative),*
- *v satisfies the boundary conditions* (13).

We look for an explicit classical solution of (12)-(13).

Lemma 3.8.

(1) Let $v_1(t,x) = e^{-\rho t} A_1(t)(F(t)-x)^2$, *where* $A_1(\cdot)$ *is the unique solution of*

$$\begin{cases} A_1'(t) = \left(\rho + \beta^2 - 2r\right) A_1(t) - \kappa, \\ A_1(T) = 1, \end{cases}$$

i.e., setting $a_1 := \rho + \beta^2 - 2r$,

$$A_1(t) = \begin{cases} \left(1 - \frac{\kappa}{a_1}\right) e^{-a_1(T-t)} + \frac{\kappa}{a_1}, & \text{if } a_1 \neq 0, \\ \kappa(T-t) + 1, & \text{if } a_1 = 0. \end{cases}$$

Then:

(a) $v_{1_x} \leq 0$ *on* U_1;

(b) $v_{1_{xx}} > 0$ *on* U_1;

(c) v_1 *solves*

$$\begin{cases} \kappa e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) - \dfrac{\beta^2 v_x^2(t,x)}{2v_{xx}(t,x)} = 0, \ \text{on } [0,T]\times\mathbb{R}, \\ v(T,x) = e^{-\rho T}(F(T)-x)^2, \quad x \in \mathbb{R}. \end{cases} \tag{14}$$

(2) Let $v_2(t,x) = e^{-\rho t} A_2(t)(F(t)-x)^2$, *where* $A_2(\cdot)$ *is the unique solution of*

$$\begin{cases} A_2'(t) = (\rho - 2r) A_2(t) - \kappa, \\ A_2(T) = 1, \end{cases}$$

i.e., setting $a_2 := \rho - 2r$,

$$A_2(t) = \begin{cases} \left(1 - \frac{\kappa}{a_2}\right) e^{-a_2(T-t)} + \frac{\kappa}{a_2}, & \text{if } a_2 \neq 0, \\ \kappa(T-t), & \text{if } a_2 = 0. \end{cases}$$

Then:

(a) $v_{2_x} \geq 0$ *on* U_2;

(b) $v_{2_{xx}} > 0$ *on* U_2;

(c) v_2 *solves*

$$\begin{cases} \kappa e^{-\rho t}(F(t)-x)^2 + v_t(t,x) + (rx - b_0)v_x(t,x) = 0, \quad \text{on } [0,T]\times\mathbb{R}, \\ v(T,x) = e^{-\rho T}(F(T)-x)^2, \quad x \in \mathbb{R}. \end{cases} \tag{15}$$

(3) For $t \in [0,T]$, *we have:*

(a) $v_1(t,F(t)) = v_2(t,F(t)) = 0$;

(b) $v_{1_t}(t,F(t)) = v_{2_t}(t,F(t)) = 0$;

(c) $v_{1_x}(t,F(t)) = v_{2_x}(t,F(t)) = 0$;

(d) $v_{1_{xx}}(t,F(t)) \neq v_{2_{xx}}(t,F(t))$.

Proof. First of all notice that, regardless of the signs of a_1, a_2, the functions $A_1(\cdot), A_2(\cdot)$ are strictly positive on $[0,T]$. Now let us consider, for $A(\cdot) \in C^1([0,T];\mathbb{R})$, the function

$$v(t,x) = e^{-\rho t} A(t)(F(t)-x)^2.$$

We have, using also that $F'(t) = rF(t) - b_0$,

$$\begin{aligned} v_t(t,x) &= -\rho e^{-\rho t} A(t)(F(t)-x)^2 + e^{-\rho t} A'(t)(F(t)-x)^2 + 2e^{-\rho t} A(t)(F(t)-x)(rF(t)-b_0), \\ v_x(t,x) &= -2e^{-\rho t} A(t)(F(t)-x), \\ v_{xx}(t,x) &= 2e^{-\rho t} A(t). \end{aligned}$$

Taking $A(\cdot) = A_1(\cdot)$ and $A(\cdot) = A_2(\cdot)$, the claims follow by simple computations. □

Proposition 3.9. *Define on* $[0,T] \times \mathbb{R}$ *the function*

$$v(t,x) := \begin{cases} v_1(t,x), & \text{if } (t,x) \in U_1, \\ v_2(t,x), & \text{if } (t,x) \in U_2, \end{cases} \tag{16}$$

where v_1, v_2 *are the functions defined in Lemma 3.8. Then* v *is a classical solution of* (12)-(13) *in the sense of Definition 3.7.*

Proof. It follows from Lemma 3.8. □

3.3 The Verification Theorem and the optimal feedback strategy

The aim of this subsection is to prove a Verification Theorem stating that the function v defined in (16) is actually the value function and moreover giving an optimal feedback strategy for the problem.

Lemma 3.10 (Fundamental identity)**.**

(1) Let $(t,x) \in U_1$*, let* v_1 *be the function defined in Lemma 3.8-(1), and let* $\pi(\cdot) \geq 0$ *be a strategy such that* $X(\cdot; t, x, \pi(\cdot)) \in U_1$*; then*

$$\begin{aligned} &v_1(t,x) = J(t,x;\pi(\cdot)) \\ &\quad + \mathbb{E}\left[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s)); \pi(s)\big)\Big)\, ds\right]. \end{aligned}$$

(2) Let $(t,x) \in U_2$*, let* v_2 *be the function defined in Lemma 3.8-(2), and let* $\pi(\cdot) \geq 0$ *be a strategy such that* $X(\cdot; t, x, \pi(\cdot)) \in U_2$*; then*

$$\begin{aligned} &v_2(t,x) = J(t,x;\pi(\cdot)) \\ &\quad + \mathbb{E}\left[\int_t^T \Big(\mathcal{H}\big(v_{2_x}(s,X(s)), v_{2_{xx}}(s,X(s))\big) - \mathcal{H}_{cv}\big(v_{2_x}(s,X(s)), v_{2_{xx}}(s,X(s)); \pi(s)\big)\Big)\, ds\right]. \end{aligned}$$

Proof. **(1)** Let v_1 be the function defined in Lemma 3.8-(1); by the same lemma v_1 solves (14) on U_1. Since $v_{1_x} \leq 0, v_{1_{xx}} > 0$ on U_1, we have

$$\mathcal{H}(v_{1_x}(t,x), v_{1_{xx}}(t,x)) = -\frac{\beta^2 v_x^2(t,x)}{2v_{xx}(t,x)}, \qquad (t,x) \in U_1,$$

so that v_1 solves the original HJB equation (11) on U_1.

Let us take $\pi(\cdot) \in \Pi_{ad}(t)$ such that the corresponding state trajectory $X(\cdot) := X(\cdot; t, x, \pi(\cdot))$ remains in U_1 and apply Dynkin's formula to $X(\cdot)$ with the function v_1. We obtain

$$
\begin{aligned}
v_1(T, X(T)) - v_1(t,x) = \mathbb{E}\Big[\int_t^T \Big(& v_{1_t}(s, X(s)) + (rX(s) - b_0)\, v_{1_x}(s, X(s)) \\
& + \mathcal{H}_{cv}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s)); \pi(s)\big)\Big)\, ds \Big],
\end{aligned}
$$

hence

$$
\begin{aligned}
v_1(t,x) = \mathbb{E}\Big[e^{-\rho T}(F - X(T))^2 - \int_t^T \Big(& v_{1_t}(s, X(s)) + (rX(s) - b_0)\, v_{1_x}(s, X(s)) \\
& + \mathcal{H}_{cv}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s)); \pi(s)\big)\Big)\, ds \Big].
\end{aligned}
$$

Taking into account the assumption on $\pi(\cdot)$ and the fact that, as we have shown, v_1 solves the original HJB equation (11) on U_1, we can write

$$
\begin{aligned}
v_1(t,x) = \ & \mathbb{E}\Big[e^{-\rho T}(F - X(T))^2 + \int_t^T \kappa e^{-\rho s}(F(s) - X(s))^2\, ds \\
& + \int_t^T \Big(\mathcal{H}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s)); \pi(s)\big)\Big) ds \Big] \\
= \ & J(t, x; \pi(\cdot)) \\
& + \mathbb{E}\left[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s))\big) - \mathcal{H}_{cv}\big(v_{1_x}(s, X(s)), v_{1_{xx}}(s, X(s)); \pi(s)\big)\Big) ds \right].
\end{aligned}
$$

(2) Let v_2 be the function defined in Lemma 3.8-(2); by the same lemma v_2 solves (15) on U_2. Since $v_{2_x} \geq 0, v_{2_{xx}} > 0$ on U_2, we get

$$
\mathcal{H}(v_{2_x}(t,x), v_{2_{xx}}(t,x)) = 0, \quad (t,x) \in U_2,
$$

so that v_2 solves the original HJB equation (11) on U_2. The proof follows the same line of the proof of the previous statement. □

Taking into account (10) and Lemma 3.8, we can write the feedback map as

$$
(t,x) \mapsto G(t,x) := \begin{cases} -\dfrac{\beta}{\sigma} \dfrac{v_{1_x}(t,x)}{v_{1_{xx}}(t,x)}, & \text{if } (t,x) \in U_1, \\ 0, & \text{if } (t,x) \in U_2. \end{cases} \tag{17}
$$

Since by Lemma 3.8 we have

$$
F(t) - x = -\frac{v_{1_x}(t,x)}{v_{1_{xx}}(t,x)}, \quad \text{on } U_1, \tag{18}
$$

the feedback map (17) becomes

$$
(t,x) \mapsto G(t,x) := \begin{cases} \dfrac{\beta}{\sigma}(F(t) - x), & \text{if } (t,x) \in U_1, \\ 0, & \text{if } (t,x) \in U_2. \end{cases}
$$

The corresponding closed loop equation is

$$\begin{cases} dX(s) = [rX(s) + \sigma\beta G(X(s)) - b_0]\, ds + \sigma G(X(s)) dB(s), \quad s \in [t,T], \\ X(t) = x. \end{cases} \tag{19}$$

The solution of the above equation has a desirable feature that is stated in the following lemma.

Lemma 3.11 (Closed loop equation)**.** *For every $(t,x) \in [0,T] \times \mathbb{R}$, there exists a unique $\mathbb{F}^t$-progressively measurable process $X_G(\cdot;t,x) \in L^2(\Omega \times [t,T];\mathbb{R})$ solution of* (19)*. Moreover*

(1) if $(t,x) \in U_1$, then $(s, X_G(s;t,x))$, $s \in [t,T]$, lives in U_1;

(2) if $(t,x) \in U_2$, then $X_G(\cdot;t,x)$ is deterministic and $(s, X_G(s;t,x))$, $s \in [t,T]$, lives in U_2.

Proof. The proof of the existence and uniqueness of $X_G(\cdot;t,x)$ is due to the Lipschitz continuity of the map G and it is standard (see, e.g., [Karatzas & Shreve, 1991], Chapter 5, Theorems 2.5 and 2.9). Let us prove the second part of the statement.

(1) Let $(t,x) \in U_1$. Consider the process $Q(\cdot)$ solution of

$$\begin{cases} dQ(s) = \left(r - \beta^2\right) Q(s)ds - \beta Q(s)dB(s), \quad s \in [t,T], \\ Q(t) = F(t) - x. \end{cases}$$

$Q(\cdot)$ is a geometric Brownian motion with non-negative starting point, so that it has to be non-negative for every $s \in [t,T]$. Consider now the process $\bar{X}(s) = F(s) - Q(s)$, $s \in [t,T]$. We have $\bar{X}(s) \leq F(s)$ for every $s \in [0,T]$ and it is straightforward to see that $\bar{X}$ solves (19). By uniqueness it must be $X_G(s;t,x) = \bar{X}(s) \leq F(s)$ for every $s \in [t,T]$ and the claim is proved.

(2) Let $(t,x) \in U_2$. The deterministic process $\bar{X}(s) = X(s;t,x,0)$ is a solution to (19) and $(s, \bar{X}(s)) \in U_2$ for every $s \in [t,T]$. By uniqueness it must be $X_G(s;t,x) = \bar{X}(s) \geq F(t)$ for every $s \in [t,T]$ and the claim is proved. □

By the previous lemma and by Lipschitz continuity of G, the feedback strategy $\pi_G^{t,x}(\cdot)$ defined by

$$\pi_G^{t,x}(s) := G(s, X_G(s;t,x)) \tag{20}$$

is admissible, that is $\pi_G^{t,x}(\cdot) \in \Pi_{ad}(t)$.

Theorem 3.12 (Verification Theorem and Optimal Feedback)**.** *Let $(t,x) \in [0,T] \times \mathbb{R}$ and let v be the function defined in* (16)*. Then $V(t,x) = v(t,x)$. Moreover, the control $\pi(\cdot) \in \Pi_{ad}(t)$ is optimal for the initial (t,x) if and only if*

$$\pi(s) = G(s, X(s;t,x,\pi(\cdot)), \quad P\text{-a.s. } \forall s \in [t,T]. \tag{21}$$

In particular, the feedback strategy $\pi_G^{t,x}(\cdot)$ defined in (20) *is the unique optimal strategy.*

Proof. Let $(t,x) \in U_1$ and $\pi(\cdot) \in \Pi^1_{ad}(t)$ (see (7)) and set $X(\cdot) := X(\cdot;t,x,\pi(\cdot))$. Thus, we can apply Lemma 3.10 to the process $X(\cdot)$ with v_1 obtaining

$$v_1(t,x) = J(t,x;\pi(\cdot)) + \mathbb{E}\Bigg[\int_t^T \Big(\mathcal{H}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s))\big) \\ - \mathcal{H}_{cv}\big(v_{1_x}(s,X(s)), v_{1_{xx}}(s,X(s)); \pi(s)\big)\Big) ds\Bigg] \leq J(t,x;\pi(\cdot)). \tag{22}$$

Taking into account Lemma 3.2, this shows that $v_1(t,x) \leq V(t,x)$.

Now consider $X(\cdot;t,x,\pi_G^{t,x}(\cdot)) = X_G(\cdot;t,x)$. From Lemma 3.11-(1) we have $(s, X_G(s;t,x)) \in U_1$ for every $s \in [t,T]$, so we can apply the fundamental identity to $X_G(\cdot)$ with v_1. Taking into account Lemma 3.8 and (18), we see that the feedback map $\pi_G^{t,x}(\cdot)$ minimizes, at any time $t \in [s,T]$, the current value Hamiltonian. Thus, we get in this case $v_1(t,x) = J(t,x;\pi_G^{t,x}(\cdot))$, which shows that

$$v_1(t,x) = V(t,x) = J(t,x;\pi_G^{s,x}(\cdot)).$$

The fact that an optimal strategy must satisfy (21) is consequence of $V = v_1$ and of (22).

Finally, the uniqueness of the optimal strategy is consequence of the characterization (21) and of the uniqueness of solutions to the closed loop equation stated in Lemma 3.11-(1).

If $(t,x) \in U_2$, we argue exactly in the same way with v_2, obtaining the claim also in this case. □

Remark 3.13. *If we replace the loss function* (3) *in the optimization problem with*

$$\tilde{L}(s,x) = \begin{cases} (F(s)-x)^2, & \text{if } x \leq F(s), \\ 0, & \text{if } x > F(s), \end{cases} \tag{23}$$

basically the answer to the problem is the same. Indeed, the argument used in Lemma 3.2 would show that in this case the value function $\tilde{V}$ would be the same on U_1 and the optimal feedback strategy would still be given by (20) *starting from $(t,x) \in U_1$. Instead, we would have $\tilde{V} \equiv 0$ on U_2: starting from $(t,x) \in U_2$, every strategy keeping the state in U_2 would be optimal (in particular the strategy $\pi(\cdot) \equiv 0$). In this case we would loose the uniqueness of the optimal feedback strategy. From the point of view of applications, the analysis of the problem in the region U_2 is not interesting for it is unrealistic, especially in the absence of bequest motives (as in the current paper). From the analysis performed above, it is obvious that with the quadratic loss function* (3) *the optimal wealth cannot exceed the target, and therefore the undesirable event of paying a loss due to excess of wealth cannot happen. However, this different formulation of the problem might be more appealing to financial advisors of pension funds. In fact, a model based on a loss function such as* (23) *can be immediately understood and accepted by any pensioner, without entering the mathematical technicalities of the model.*

Remark 3.14. *It turns out that starting from $(t,x) \in U_1$ the optimal strategy is equal to that found in the unconstrained case of [Gerrard, Haberman & Vigna, 2004]. Instead, if we start from $(t,x) \in U_2$, the optimal feedback strategy in our case is the null one, while in the unconstrained case is negative, so not admissible in our setting. Basically, this is due to the fact that, since the loss function is positive above the target, the optimal strategy without constraints pushes the pensioner to throw away wealth, meaning short-selling, which here is not allowed.*

4 Dynamic Programming for (P2)

In this section, we approach the problem (P2) rigorously. Here, we require the no short-selling constraint on the strategy and, moreover, the capital requirement $X(T) \geq S$ almost surely, where $0 \leq S < F$. It turns out that this constraint implies the "no-ruin" constraint, i.e. $X(t) \geq 0$ almost surely for every $t \in [0,T]$.

The contents of this section are divided in five subsections, as follows. In Subsection 4.1, the set of admissible strategies is studied. In Subsection 4.2, the results of Subsection 4.1 are used to reduce the problem (P2) on a bounded domain. In Subsection 4.3, the problem on the bounded domain is reduced to a problem in a rectangle, through a change of variable. In Subsection 4.4, the properties (monotonicity, convexity, continuity) of the value function H introduced in Subsection 4.3 are studied. Subsection 4.5 is the central one. Here the HJB equation is studied: using the theory of viscosity solutions and a suitable dual transformation, it is shown that the value function is the unique regular solution of it.

4.1 The set of admissible strategies

As before, given $t \in [0,T]$, let us define the filtration $\mathbb{F}^t := (\mathcal{F}_s^t)_{s\in[t,T]}$, where $\mathcal{F}_s^t$ is the σ-algebra generated by $(B(u)-B(t))_{u\in[t,s]}$ and completed by the P-null sets. Consider the equation

$$\begin{cases} dX(s) = [rX(s) + (\mu - r)\pi(s) - b_0]\, ds + \sigma\pi(s)dB(s), \quad s \in [t,T], \\ X(t) = x, \end{cases}$$

where $x \in \mathbb{R}$ and $\pi(\cdot) \in L^2(\Omega \times [t,T];\mathbb{R})$ is progressively measurable with respect to $\mathbb{F}^t$. This equation admits a unique strong solution on $(\Omega, \mathcal{F}, P)$ (see again [Karatzas & Shreve, 1998], Section 5.6.C) that we denote by $X(\cdot; s, x, \pi(\cdot))$. Let us define the set of the admissible strategies, depending on the initial (t,x), by

$$\Pi_{ad}^0(t,x) = \big\{\pi(\cdot) \in L^2(\Omega \times [t,T];\mathbb{R}) \mid \pi(\cdot) \text{ prog. meas. w.r.t. } \mathbb{F}^t,\ \pi(\cdot) \geq 0,\ X(T;t,x,\pi(\cdot)) \geq S\big\}.$$

Let us set

$$S(s) := \frac{b_0}{r} - \left(\frac{b_0}{r} - S\right) e^{-r(T-s)}, \tag{24}$$

for any $s \in [0,T]$. The function S represents a sort of safety level for the wealth. Should the fund hit this barrier at time s, the null strategy (i.e. $\pi(\cdot) \equiv 0$) from s to T would guarantee the fulfillment of the capital requirement. Moreover, it will be shown that the null strategy is indeed the only admissible one for $x = S(t)$, and therefore the optimal one.

Proposition 4.1. *Let $t \in [0,T]$, $x \in \mathbb{R}$. Then*

(1) $\Pi_{ad}^0(t,x) \neq \emptyset$ if and only if $0 \in \Pi_{ad}^0(t,x)$. This happens if and only if $x \geq S(t)$.

(2) If $x = S(t)$, then $\Pi_{ad}^0(t,x) = \{0\}$ and $X(s;t,x,0) = S(s)$ on $[t,T]$.

(3) Let $x \geq S(t)$. Then, $\pi(\cdot) \in \Pi_{ad}^0(t,x)$ if and only if

$$\pi(s) = \pi(s)\mathbf{1}_{\{t\leq s<\tau\}},$$

where

$$\tau := \inf\{s \in [t,T] \mid X(s;t,x,\pi(\cdot)) = S(s)\}$$

with the convention that $\inf\emptyset = T$.

(4) If $x > S(t)$, then $\Pi_{ad}^0(s,x) \supsetneq \{0\}$.

(5) The state constraint $X(T) \geq S$ is equivalent to

$$X(t) \geq S(t), \quad P\text{-a.s. } \forall t \in [0,T].$$

Proof. **(1)** Clearly, if $0 \in \Pi^0_{ad}(t,x)$, then $\Pi^0_{ad}(t,x) \neq \emptyset$. Conversely, suppose that $\Pi^0_{ad}(t,x) \neq \emptyset$ and let $\pi(\cdot) \in \Pi^0_{ad}(t,x)$. This means that $X(T;t,x,\pi(\cdot)) \geq S$ almost surely; therefore $\tilde{\mathbb{E}}[X(T;t,x,\pi(\cdot))] \geq S$, where $\tilde{\mathbb{E}}$ denotes the expectation under the probability $\tilde{P} = e^{-\beta B(T) - \frac{\beta^2}{2}T} \cdot P$ given by the Girsanov transformation. Writing the dynamics of $X(\cdot;t,x,\pi(\cdot))$ under $\tilde{P}$ and taking the expectations under $\tilde{\mathbb{E}}$, we see that

$$X(T;t,x,0) = \tilde{\mathbb{E}}[X(T;t,x,\pi(\cdot)] \geq S,$$

hence $0 \in \Pi^0_{ad}(t,x)$. This proves the first part of the claim.

For the second part, notice that the state equation yields

$$X(s;t,x,0) = \frac{b_0}{r} - \left(\frac{b_0}{r} - x\right) e^{r(s-t)},$$

so that from the expression of $S(\cdot)$ in (24) we obtain the claim.

(2) If $x = S(t)$, by the state equation and (24) we have $X(s;t,x,0) = S(s)$ on $[t,T]$; therefore $0 \in \Pi^0_{ad}(t,x)$. On the other hand, taking a strategy $\pi(\cdot) \in \Pi^0_{ad}(t,x)$ and arguing as before, one can see that in this case

$$S = X(T;t,x,0) = \tilde{\mathbb{E}}[X(T;t,x,\pi(\cdot)] \geq S.$$

Since $\pi(\cdot)$ is admissible, we have $X(T;t,x,\pi(\cdot)) \geq S$. Thus, it must be $X(T;t,x,\pi(\cdot)) \equiv S$. Therefore $\mathrm{Var}[X(T;t,x,\pi(\cdot))] = 0$ and this happens if and only if $\pi(\cdot) \equiv 0$.

(3) This is obvious, given the previous item.

(4) Let $x > S(t)$. Define the strategy

$$\pi^\tau(s) := \begin{cases} 1, & \text{if } s \in [t,\tau], \\ 0, & \text{if } s \in [\tau,T], \end{cases}$$

where

$$\tau := \inf\{s \in [t,T] \mid X(s;t,x,1) = S(s)\}.$$

Then, by the previous item, $\pi^\tau(\cdot) \in \Pi^0_{ad}(t,x)$. Moreover, since $x > S(t)$, we have $\tau > t$. Therefore $\pi(\cdot)$ is not identically null, so the claim.

(5) The claim reduces to show that, for every $\pi(\cdot) \in \Pi^0_{ad}(t,x)$, we have $X(s) \geq S(s)$ almost surely for any $s \in [t,T]$. This follows by items (2) and (3). □

4.2 The optimization problem and its reduction on a bounded domain

We are interested in the following optimization problem: for given $(t,x) \in [0,T] \times \mathbb{R}^+$,

$$\text{minimize } J(t,x;\pi(\cdot)) := \mathbb{E}\left[\int_t^T \kappa e^{-\rho s}(F(s) - X(s;t,x,\pi(\cdot)))^2\, ds + e^{-\rho T}(F(T) - X(T;t,x,\pi(\cdot)))^2\right] \quad (25)$$

over the set of admissible strategies $\pi(\cdot) \in \Pi^0_{ad}(x,t)$.

We denote the value function by W, i.e.

$$W(t,x) := \inf_{\pi(\cdot) \in \Pi^0_{ad}(t,x)} J(t,x;\pi(\cdot)), \quad t \in [0,T],\ x \in \mathbb{R},$$

with the agreement that $\inf \emptyset = +\infty$. Clearly we have $W \geq V$, where V is the value function of the state unconstrained problem (P1). Due to Proposition 4.1, W is finite (and non-negative) on the set

$$\mathcal{D} = \{(t,x) \in [0,T] \times \mathbb{R} \mid x \geq S(t)\}.$$

Moreover, we can restrict the problem to a bounded domain (that is more suitable for the viscosity solutions approach). Indeed, consider the set

$$\mathcal{C} := \{(t,x) \in [0,T] \times \mathbb{R} \mid S(t) \leq x \leq F(t)\} \subset \mathcal{D}.$$

Since the optimal strategy $\pi(\cdot) \equiv 0$ of the (state) unconstrained problem starting from a point of the set $(t,x) \in \mathcal{D} \setminus \mathcal{C}$ satisfies $X(s;t,x,0) \geq F(s) \geq S(s)$, we have $W = V$ on this set, where V is the value function of the (state) unconstrained problem studied in the previous section. In other words, the state constrained problem is already solved on the region $\mathcal{D} \setminus \mathcal{C}$ keeping the strategy $\pi(\cdot) \equiv 0$.

Remark 4.2. *From the practical point of view, differently from problem (P1), this problem is meaningless for initial data $(t,x) \in U_1 \backslash \mathcal{C}$, because in this region there are no admissible strategies. This has an immediate consequence in the application of the model. In particular, the subjective choice of the guaranteed final fund S cannot be too high. In fact, it must be*

$$S \leq x e^{r(T-t)} - \frac{b_0}{r}\left(e^{r(T-t)} - 1\right)$$

so that $(t,x) \in \mathcal{C}$ (see also Remark 5.6).

For the points belonging to $\mathcal{C}$ we have the following representation for the value function W.

Proposition 4.3. *Let $(t,x) \in \mathcal{C}$ and consider the set*

$$\Pi_{ad}(t,x) = \left\{\pi(\cdot) \in \Pi^0_{ad}(t,x) \mid S(s) \leq X(s;t,x,\pi(\cdot)) \leq F(s), \ \ s \in [t,T]\right\} \subset \Pi^0_{ad}(t,x).$$

Then we have

$$W(t,x) = \inf_{\pi(\cdot) \in \Pi_{ad}(t,x)} J(t,x;\pi(\cdot)).$$

Proof. It follows from Lemma 3.2 and Proposition 4.1-(5). □

Proposition 4.3 says that on the set $\mathcal{C}$ the original problem is equivalent to the problem with state constraint

$$S(s) \leq X(s) \leq F(s), \qquad s \in [t,T].$$

The analogue of Proposition 4.1 is the following.

Proposition 4.4. *Let $(t,x) \in \mathcal{C}$. Then*

(1) $0 \in \Pi_{ad}(t,x)$.

(2) If $x = S(t)$ (respectively, $x = F(t)$), then $\Pi_{ad}(t,x) = \{0\}$ and $X(s;t,x,0) = S(s)$ (respectively $X(s;t,x,0) = F(s)$) on $[t,T]$.

(3) $\pi(\cdot) \in \Pi_{ad}(t,x)$ *if and only if*

$$\pi(s) = \pi(s)\mathbf{1}_{\{t \le s < \tau\}},$$

where

$$\tau := \inf\{s \in [t,T] \mid X(s;t,x,\pi(\cdot)) \in \{S(s), F(s)\}\}$$

with the convention that $\inf \emptyset = T$.

(4) If $S(t) < x < F(t)$*, then* $\Pi_{ad}(s,x) \supsetneq \{0\}$.

Proof. The claims can be obtained exactly as in the proof of Proposition 4.1. □

Notice that, rephrasing the problem in these new terms, both the lateral boundaries

$$\partial_F^* \mathcal{C} := \{(t,x) \in [0,T] \times \mathbb{R} \mid x = F(t)\}, \quad \partial_S^* \mathcal{C} := \{(t,x) \in [0,T] \times \mathbb{R} \mid x = S(t)\}$$

are absorbing for the problem, in the sense that if $x = S(t)$ (respectively, $x = F(t)$), then the only admissible strategy is $\pi(\cdot) \equiv 0$ and $X(s;t,x,0) = S(s)$ for $s \in [t,T]$ (respectively, $X(s;t,x,0) = F(s)$ for $s \in [t,T]$).

Remark 4.5. *As in (P1), a relevant consequence of Proposition 4.3 is that if we replace the loss function* (3) *with*

$$\tilde{L}(s,x) = \begin{cases} (F(s)-x)^2, & \text{if } x \le F(s), \\ 0, & \text{if } x > F(s), \end{cases}$$

we have the same value function W *on* $\mathcal{C}$ *and, starting from* $(t,x) \in \mathcal{C}$*, we have the same optimal feedback strategy. Same comments as in Remark 3.13 apply.*

4.3 Reducing the problem to a rectangle

Here we introduce a transformation in order to work with a simpler stochastic control problem. The domain $\mathcal{C}$ will be transformed into a rectangle and our value function W will be related to the value function H of this new control problem.

Let us consider the diffeomorphism $\mathcal{L} : [0,T] \times [S,F] \to \mathcal{C}$,

$$(t,z) \longmapsto (t,x) = \mathcal{L}(t,z) = (t, \mathcal{L}_1(t,z)) := \left(t, ze^{-r(T-t)} + \frac{b_0}{r}\left(1 - e^{-r(T-t)}\right)\right).$$

Remark 4.6. *The relationship between* x *and* z *given by* $x = \mathcal{L}_1(t,z)$ *is clear:* z *is the fund that one would have at time* T *with the riskless strategy from* t *onwards. In other words,* $z = X(T;t,x,0)$*. In particular,* $F(t) = \mathcal{L}_1(t,F)$ *and* $S(t) = \mathcal{L}_1(t,S)$.

Let $(t,x) \in \mathcal{C}$ and $\pi(\cdot) \in \Pi_{ad}(t,x)$. By application of Ito's formula to the process

$$Z(s) = [\mathcal{L}_1(t,\cdot)]^{-1}(X(s;t,x,\pi(\cdot))), \quad s \in [t,T], \tag{26}$$

we see that Z solves

$$\begin{cases} dZ(s) = e^{r(T-s)}[(\mu - r)\pi(s)dt + \sigma\pi(s)dB(s)], & s \in [t,T], \\ Z(t) = z := [\mathcal{L}_1(t,\cdot)]^{-1}(x). \end{cases} \tag{27}$$

For $(t,z)\in[0,T]\times[S,F]$ define

$$\tilde\Pi_{ad}(t,z)=\left\{\pi(\cdot)\in L^2(\Omega\times[t,T];\mathbb{R})\,|\,\pi(\cdot)\,\text{is prog. meas. w.r.t.}\,\mathbb{F}^t,\ S\le Z(s;t,z,\pi(\cdot))\le F,\ s\in[t,T]\right\}.$$

Due to (26), we have $\tilde\Pi_{ad}(t,z)=\Pi_{ad}(t,\mathcal{L}_1(t,z))$.

Consider the objective functional

$$\tilde J(t,z;\pi(\cdot)):=\mathbb{E}\left[\int_t^T\kappa\,\eta(s)(F-Z(s))^2ds+\eta(T)(F-Z(T))^2\right],\tag{28}$$

where $\eta(s):=e^{-\rho s-2r(T-s)}$ and $Z(\cdot):=Z(\cdot;t,z,\pi(\cdot))$ follows the dynamics (27). Then consider the associated optimization problem: for given $(t,z)\in[0,T]\times[S,F]$,

$$\text{minimize}\quad\tilde J(t,z;\pi(\cdot))\quad\text{over}\quad\pi(\cdot)\in\tilde\Pi_{ad}(t,z).\tag{29}$$

As usual, define the value function for this problem as

$$H(t,z):=\inf_{\pi(\cdot)\in\tilde\Pi_{ad}(t,z)}\tilde J(t,z;\pi(\cdot)),\quad t\in[0,T],\ z\in[S,F].\tag{30}$$

We can easily see that

$$H(t,z)=W(t,\mathcal{L}_1(t,z)).\tag{31}$$

It follows that all the analysis that will be done for the problem (29) and for its associated value function H can be suitably rephrased for the problem (25) and for its associated value function W. Therefore, from now on within this section, we will study the problem (29) and the associated value function H, which is simpler. The advantage is that the lateral boundaries are now

$$[0,T)\times\{S\},\qquad[0,T)\times\{F\}.$$

They are absorbing for this new problem, in the sense that if $z=S$ (respectively, $z=F$), then the only admissible strategy is $\pi(\cdot)\equiv0$ and $Z(s;t,S,0)=S$ for all $s\in[t,T]$ (respectively, $Z(s;t,F,0)=F$ for all $s\in[t,T]$).

4.4 Properties of the value function

Here we prove some first qualitative properties of H.

Proposition 4.7. *Let $t\in[0,T]$. The function $[S,F]\to\mathbb{R}^+$, $z\mapsto H(t,z)$ is convex and nonincreasing. In particular it has minimum at F.*

Proof. It follows the line of the proofs of Proposition 3.4 and Proposition 3.6. □

Proposition 4.8. *The value function H is continuous on $[0,T]\times[S,F]$.*

Proof. The proof is in the Appendix. □

We are ready to present the Dynamic Programming Principle, which is the first step to study the HJB equation.

Theorem 4.9 (Dynamic Programming Principle)**.** *The value function H satisfies the dynamic programming equation, i.e. for every $t \in [0,T]$, $z \in [S,F]$, and $\tau \in [t,T]$ stopping time (possibly depending on $\pi(\cdot)$), the following functional equation holds true*

$$H(t,z) = \inf_{\pi(\cdot)\in\tilde{\Pi}_{ad}(t,z)} \mathbb{E}\left[\int_t^\tau \kappa\,\eta(s)(F - Z(s;t,z,\pi(\cdot)))^2 ds + H(\tau, Z(\tau;t,z,\pi(\cdot)))\right]. \tag{32}$$

Proof. We do not provide a precise proof here for brevity. [Yong & Zhou, 1999], Chapter 4, Theorem 3.3, provides a proof when the value function is known to be continuous. Such theorem is stated and proved under assumptions that are other than ours. In particular, there are no state constraints and τ is deterministic and independent of $\pi(\cdot)$. Nevertheless, the proof may be easily adapted to our case. For the proof in a more general case, when τ is a stopping time possibly depending on the control, see [Krylov, 1980], Chapter 3. □

Concerning the behaviour of H at the lateral boundaries we have the following.

Proposition 4.10.

(1) We have

$$H(t,F) = 0, \ \forall t \in [0,T). \tag{33}$$

(2) We have

$$H(t,S) = \psi(t) + \eta(T)(F-S)^2, \ \forall t \in [0,T), \tag{34}$$

where

$$\psi(t) = \int_t^T \kappa\,\eta(s)(F-S)^2 ds. \tag{35}$$

(3) For every $t \in [0,T)$ the function $z \mapsto H(t,z)$ is differentiable at F^- and

$$H_z(t,F) = 0, \ \forall t \in [0,T), \tag{36}$$

where the derivative is intended as left derivative.

Proof. The first two claims are simple consequences of the absorbing property of the boundaries $[0,T) \times \{S\}$, $[0,T) \times \{F\}$ for the control problem related to H.

We show the last claim for H. The existence of the left derivative of $z \mapsto H(t,z)$ at F is guaranteed by the convexity of such function. Moreover, since $H(t,F) = 0$ and $H(t,z) > 0$ for $z \in [S,F)$, it must be $H_z(t,F) \le 0$. On the other hand, taking the control $0 \in \tilde{\Pi}_{ad}(t,z)$, we see that

$$\frac{H(t,F) - H(t,z)}{F-z} \ge -(F-z)\int_t^T \kappa\,\eta(s)ds - \eta(T)(F-z) \overset{z\to F^-}{\longrightarrow} 0,$$

hence the claim. □

We will use the result above to provide the appropriate boundary conditions for the related HJB equation in the next subsection.

4.5 The HJB equation: existence, uniqueness and regularity

In this subsection we study the HJB equation for the value function H. Apart from some linear terms that have been discarded in the transformation of Subsection 4.3 above, this equation is the same as the one of Section 3.2 in the interior of $[0,T] \times [S,F]$. The difference, due to the presence of the state constraint, is that the domain is smaller in space and that suitable boundary conditions must be imposed. These facts, as usual, make the HJB equation much more difficult to treat. In particular, we are not able to find explicit solutions when $\kappa \neq 0$. Then we prove existence and uniqueness of regular solutions, as this provides a solid basis for the study of optimal strategies through the smooth verification theorem (see [Yong & Zhou, 1999], Chapter 5, Section 4.1). However, studying the regularity is difficult since the equation is degenerate, fully nonlinear, non-autonomous; so, to our knowledge, the approaches of the literature to similar problems (see, e.g., [Choulli, Taksar & Zhou, 2003], [Di Giacinto, Federico & Gozzi, 2010], [Duffie, Fleming, Soner & Zariphopoulou, 1997], [Zariphopoulou; 1994]) do not work.[4] For this reason, we provide existence and uniqueness of regular solutions using an *ad hoc* method described as follows.

(a) We prove existence and uniqueness of viscosity solutions with appropriate (mixed) boundary conditions (Subsection 4.5.1, Proposition 4.13).

(b) Through a dual approach, we associate to the boundary value problem (BVP) of point (a) another BVP for a semilinear PDE (beginning of Subsection 4.5.2).[5]

(c) We prove that existence of regular solutions to the BVP of point (a) is equivalent, under additional hypotheses on the solutions, to the existence of regular solutions of the BVP of point (b) (Proposition 4.18).

(d) We prove the existence and uniqueness of the regular solution (satisfying the additional hypotheses required at point (c)) of the BVP of point (b) (Theorem 4.19).

(e) We conclude by previous points (c)-(d) the regularity of the (unique) viscosity solution of point (a) (Corollary 4.20) plus the fact that it satisfies the additional hypotheses of point (c).

The current value Hamiltonian is

$$\begin{array}{rccc} \mathcal{H}_{cv}: & [0,T) \times \mathbb{R}^2 \times [0,+\infty) & \longrightarrow & \mathbb{R}, \\ & (t,p,P;\pi) & \longmapsto & \frac{1}{2}e^{2r(T-t)}\sigma^2 P\pi^2 + e^{r(T-t)}\sigma\beta p\pi \end{array}$$

and the Hamiltonian is

$$\begin{array}{rccc} \mathcal{H}: & [0,T) \times \mathbb{R}^2 & \longrightarrow & \mathbb{R} \cup \{-\infty\}, \\ & (t,p,P) & \longmapsto & \inf_{\pi \geq 0} \mathcal{H}_{cv}(t,p,P;\pi). \end{array}$$

Given $(t,p,P) \in [0,T) \times \mathbb{R} \times (0,+\infty)$, the function $\pi \mapsto \mathcal{H}_{cv}(t,p,P;\pi)$ has a unique minimum point on $[0,+\infty)$ given by

$$\pi^*(t,p,P) = \left(-\frac{\beta\, p}{\sigma P} e^{-r(T-t)}\right) \vee 0, \tag{37}$$

[4] In particular, the fact that the problem is not autonomous makes impossible to use the arguments of these papers.

[5] We stress the fact that the boundary conditions we use to state the BVPs at points (a) and (b) are the minimal ones to get existence and uniqueness of solutions to such BVPs. Indeed, such solutions will satisfy *ex post* also other boundary conditions.

hence in this case the Hamiltonian can be written as

$$\mathcal{H}(t,p,P) = \begin{cases} -\dfrac{\beta^2 p^2}{2P}, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases} \tag{38}$$

If $P \leq 0$ the Hamiltonian is

$$\mathcal{H}(t,p,P) = \begin{cases} -\infty, & \text{if } p < 0, \\ 0, & \text{if } p \geq 0. \end{cases}$$

We recall that in the Hamiltonian p is the formal argument where to insert H_z and P is the formal argument where to insert H_{zz} (if these derivatives exist). Therefore, due to Proposition 4.7, only negative values of p and positive values of P are consistent with H_z, H_{zz}. Nevertheless, we allow the formal arguments p, P of $\mathcal{H}$ to range in the whole $\mathbb{R}^2$, because a wider domain is needed to use the theory of viscosity solutions.

The Hamiltonian does not depend on t (so we suppress it as argument of $\mathcal{H}$) and the HJB equation reads as

$$h_t(t,z) + \kappa\eta(t)(F-z)^2 + \mathcal{H}(h_z(t,z), h_{zz}(t,z)) = 0, \quad (t,z) \in [0,T) \times (S,F). \tag{39}$$

We consider as boundary conditions, following the ones found in Proposition 4.10,

$$\begin{cases} h_z(t,F) = 0, \ t \in [0,T), \\ h(t,S) = \psi(t) + \eta(T)(F-S)^2, \ t \in [0,T), \\ h(T,z) = \eta(T)(F-z)^2, \ z \in [S,F]. \end{cases} \tag{40}$$

Remark 4.11. *In* (40) *we consider mixed (Dirichelet-Neumann) conditions as lateral boundary conditions. They correspond to* (34)-(36) *and will be suitable to characterize the function H as unique viscosity solution of the HJB equation. However, also the more natural couple of Dirichlet lateral boundary conditions corresponding to* (33)-(34) *would be suitable to this aim. The reason of our choice is that the proof of the equivalence explained at point (c) above (i.e. Proposition 4.18) becomes easier.*

4.5.1 Existence and uniqueness of viscosity solutions

In this subsection we study the HJB equation by the viscosity approach and we characterize the value function H as unique viscosity solution of (39) satisfying (40) in classical sense.

Definition 4.12.

- *A function $h \in C([0,T] \times [S,F]; \mathbb{R})$ is called a viscosity subsolution of* (39)*, if for every $(t_M, z_M) \in [0,T) \times (S,F)$ and $\varphi \in C^{1,2}([0,T) \times (S,F); \mathbb{R})$ such that $h - \varphi$ has a local maximum at (t_M, z_M), we have*

$$-\varphi_t(t_M, z_M) - \kappa\eta(t_M)(F-z_M)^2 - \mathcal{H}(\varphi_z(t_M, z_M), \varphi_{zz}(t_M, z_M)) \leq 0.$$

- *A function $h \in C([0,T] \times [S,F]; \mathbb{R})$ is called a viscosity supersolution of* (39)*, if for every $(t_m, z_m) \in [0,T) \times (S,F)$ and $\varphi \in C^{1,2}([0,T) \times (S,F); \mathbb{R})$ such that $h - \varphi$ has a local minimum at (t_m, z_m), we have*

$$-\varphi_t(t_m, z_m) - \kappa\eta(t_m)(F-z_m)^2 - \mathcal{H}(\varphi_z(t_m, z_m), \varphi_{zz}(t_m, z_m)) \geq 0.$$

- *A function $h \in C([0,T]\times[S,F];\mathbb{R})$ is called a viscosity solution of* (39) *if it is both a viscosity subsolution and a viscosity supersolution of* (39).

Proposition 4.13. *The value function H is the unique viscosity solution of* (39) *satisfying* (40) *in classical sense.*

Proof. (Uniqueness) The theory of comparison (providing uniqueness) for viscosity solutions of parabolic equations usually concerns either with Dirichlet-type boundary conditions (see, e.g., Theorem V.8.1 in [Fleming & Soner, 1993]) or with Neumann-type boundary conditions (see, e.g., Theorem 3.1 in [Barles, 1999]). Instead we have mixed-type boundary conditions (i.e. a Dirichlet-type boundary condition at $[0,T)\times\{S\}$ and a Neumann-type boundary condition at $[0,T)\times\{F\}$). Therefore we cannot apply directly the known theory. However, since we have two different boundary conditions on two unconnected parts of the boundary, the proof can be done as follows:

- where Dirichlet condition holds, we apply the argument of [Fleming & Soner, 1993];
- where Neumann condition holds, we apply the argument of [Barles, 1999].

(Existence) The boundary conditions follow from Proposition 4.10. The subsolution and supersolution properties are a straightforward modifications, e.g., of [Fleming & Soner, 1993], Section V.3, and are proved in the Appendix, for the reader's convenience. □

4.5.2 Regularity of the solution by means of the dual problem

In this subsection we prove the regularity of the solution H of (39)-(40) along the lines specified above. Let us start with the definition of classical solution.

Definition 4.14. *A function h is called a classical solution of* (39)-(40) *if*

- $h \in C([0,T]\times[S,F];\mathbb{R}) \cap C^{0,1}([0,T)\times(S,F];\mathbb{R}) \cap C^{1,2}([0,T)\times(S,F);\mathbb{R})$;
- *h satisfies pointwise in classical sense* (39) *on $[0,T)\times(S,F)$;*
- *h satisfies (in classical sense) the boundary Dirichlet-Neumann conditions* (40).

Remark 4.15. *Every classical solution of* (39)-(40) *is also a viscosity solution of* (39)-(40). *Conversely, every viscosity solution of* (39)-(40) *which is of class $C^{1,2}([0,T)\times(S,F);\mathbb{R})$ is also a classical solution. See, e.g., [Crandall, Ishii & Lions, 1992] for these statements.*

Now we associate to the fully nonlinear PDE (39) a semilinear PDE, by means of a dual transformation of the variables that has been already used in the case of HJB equations coming from optimal portfolio allocation problems (for which the nonlinearity in the equation takes the form v_x^2/v_{xx}). We refer, e.g., to [Elie & Touzi, 2008] and [Schwartz & Tebaldi, 2006] in a lifetime consumption and investment problem, to [Gao, 2008] and [Xiao , Zhai & Qin, 2007] in the accumulation phase of a pension fund, to [Milevsky, Moore & Young, 2006], [Milevsky & Young, 2007] and [Gerrard, Højgaard & Vigna, 2010] in the decumulation phase of a pension fund. We stress that, differently from our case, in all these papers the resulting PDE is linear (apart from [Schwartz & Tebaldi, 2006] where it is quasi-linear) and is treated finding explicit solutions. In our case, the resulting PDE is just semilinear and we are not able to find explicit solutions, so we are led to study the regularity of its solutions.[6]

Suppose that the unique viscosity solution H of (39)-(40) is such that $H \in C^{1,2}([0,T) \times (S,F);\mathbb{R})$. As value function of the stochastic control problem (29), we see that H satisfies for every $t \in [0,T)$

$$\text{(i) } H(t,z) > 0,\ \forall z \in (S,F); \qquad \text{(ii) } H(t,F) = 0; \qquad \text{(iii) } H_z(t,F) = 0. \tag{41}$$

In particular, property (i) above comes from (31) and from the fact that $W \geq V > 0$ on the set $\{(t,x) \in [0,T) \times \mathbb{R} \mid S(t) < x < F(t)\}$; properties (ii)-(iii) follow from Proposition 4.10. Due to (41) and to the convexity of $z \mapsto H(t,z)$, it must be also

$$H_z(t,z) < 0, \quad \forall z \in (S,F). \tag{42}$$

Moreover, suppose that for every $t \in [0,T)$

$$H_{zz}(t,z) > 0,\ \forall z \in (S,F) \tag{43}$$

and that

$$\lim_{z \downarrow S} H_z(t,z) = -\infty,\ \forall t \in [0,T). \tag{44}$$

Let us notice that assumption (43) above (that is related to the strict convexity of the value function) is classical in this kind of optimization problems, while assumption (44) is not standard. The intuition behind it is that the marginal loss when the fund approaches the safety level is huge. This is clear if one thinks to the main idea of this paper. The retiree takes the income drawdown option in order to be better off than immediate annuitization, and therefore she aims at reaching the target. Her worst scenario is (and has to be) falling into the safety level.

From (42) and taking into account the structure (38) for the Hamiltonian, we see that H satisfies in classical sense

$$h_t(t,z) + \kappa\eta(t)(F-z)^2 - \frac{\beta^2 h_z^2(t,z)}{2\, h_{zz}(t,z)} = 0,\ \forall (t,z) \in [0,T) \times (S,F). \tag{45}$$

Due to (41)-(iii), (43) and (44), for every $(t,y) \in [0,T) \times [0,+\infty)$, there exists a unique minimizer $g(t,y) \in (S,F]$ of the function $[S,F] \to \mathbb{R}^+$, $z \mapsto H(t,z) + zy$. It is characterized by the relationship

$$H_z(t, g(t,y)) = -y,\ \forall (t,y) \in [0,T) \times [0,+\infty). \tag{46}$$

Let us look at the behaviour of g for fixed $t \in [0,T)$. Since $H(t,\cdot)$ is twice differentiable on (S,F), from (46) we see that $g(t,\cdot)$ is differentiable on $(0,+\infty)$. By considering (41)-(iii), (43), and (44) we obtain

$$\text{(i) } g(t,y) \in (S,F),\ \forall y \in (0,+\infty); \qquad \text{(ii) } g_y(t,y) < 0,\ \forall y \in (0,+\infty). \tag{47}$$

At the boundaries, taking into account (41)-(iii) and (44), we have

$$\text{(i) } g(t,0) = F; \qquad \text{(ii) } \lim_{y \to +\infty} g(t,y) = S. \tag{48}$$

When $t = T$, the unique minimizer $g(T,y) \in [S,F]$ of

$$[S,F] \to \mathbb{R}, \qquad z \mapsto H(T,z) + zy$$

[6]The next section will provide the explicit solution in the special case when $\kappa = 0$.

is explicitly computable, since $H(T,\cdot)$ is known. Indeed,

$$g(T,y) = \left(F - \frac{y}{2\eta(T)}\right) \vee S.$$

We now write a BVP for the function g and then in Proposition 4.18 we show the equivalence between this problem and the original one for H. The BVP for g is

$$g_t(t,y) - 2\kappa\eta(t)(F - g(t,y))g_y(t,y) + \beta^2 y g_y(t,y) + \frac{\beta^2}{2} y^2 g_{yy}(t,y) = 0, \quad \text{on } [0,T) \times (0,+\infty), \tag{49}$$

with boundary conditions

$$\begin{cases} g(t,0) = F, \quad t \in [0,T); \\ g(T,y) = \left(F - \dfrac{y}{2\eta(T)}\right) \vee S, \quad y \in [0,+\infty). \end{cases} \tag{50}$$

Remark 4.16. *The two boundary conditions* (50) *refer directly to the first and the third of the boundary conditions* (40)*, while the second one of* (40) *has not a corresponding condition in* (50)*. This lack of symmetry in the representation of the problem is due to the fact that, while the three boundary conditions* (40) *are all needed to prove that H is the unique viscosity solution of the problem* (39)-(40)*, only the two boundary conditions* (50) *are necessary to prove that g is a classical solution of* (49)-(50)*. To be clear, in formulating the BVPs for H and g, we have taken the minimal boundary conditions to characterize the solution. Indeed, H and g both satisfy other properties that are needed to prove the equivalence of the two formulations, shown in Proposition 4.18. We put them out the BVPs to preserve the minimality of the boundary conditions.*

To go deeper, we may ask what is the equivalent of the second condition of (40) *in terms of g. To this aim we notice that the feedback map formally defining the optimal feedback strategy has the structure H_z/H_{zz} (despite some constants, see* (37)*). Due to* (46) *and* (57)*, it can be rewritten in terms of g assuming the structure yg_y. Supposing such map continuous, the boundary condition that should be considered the equivalent one of $h(t,S) = \psi(t) + \eta(T)(F-S)^2$, $t \in [0,T)$, is*

$$\lim_{y \to +\infty} y g_y(t,y) = 0. \tag{51}$$

Indeed, imposing that the value function H on the border $[0,T) \times \{S\}$ must be $\psi(t) + \eta(T)(F-S)^2$ is equivalent to imposing the null strategy on that border. Then the second condition of (40) *is equivalent to* (51) *by the assumed continuity of the feedback map. In fact, in order to prove the equivalence between these BVPs we need* (52)*, which implies* (51)*.*

Definition 4.17. *A function $g \in C([0,T] \times [0,+\infty);\mathbb{R}) \cap C^{1,2}([0,T) \times [0,+\infty);\mathbb{R})$ is called a classical solution of* (49)-(50) *if it satisfies pointwise in classical sense* (49) *on $[0,T) \times (0,+\infty)$ and the Dirichlet conditions* (50)*.*

Proposition 4.18. *Suppose that the unique viscosity solution H of* (39)-(40) *belongs to the class $C^{1,3}([0,T) \times (S,F);\mathbb{R})$ (so, by Remark 4.15, it is also the unique classical solution of* (39)-(40)*) and satisfies* (43)-(44)*. Let g be defined as above. Then g is a classical solution of* (49)-(50)*. Moreover, g satisfies* (47) *and* (48)*-(ii).*

Conversely, let $g \in C([0,T] \times [0,+\infty);\mathbb{R}) \cap C^{1,2}([0,T) \times (0,+\infty);\mathbb{R})$ be a classical solution of (49)-(50) *satisfying* (47)*,* (48)*-(ii). Suppose*

$$y^2 g_y(t,y) \overset{y \to +\infty}{\longrightarrow} 0 \text{ uniformly in } t \in [0,T), \tag{52}$$

$$[g(t,\cdot)]^{-1} \textit{ integrable at } S^+, \ \forall t \in [0,T), \tag{53}$$

and let

$$\begin{cases} h(t,z) := \psi(t) + \eta(T)(F-S)^2 - \displaystyle\int_S^z [g(t,\cdot)]^{-1}(\xi)d\xi, \quad (t,z) \in [0,T) \times [S,F], \\ h(T,z) = \eta(T)(F-z)^2, \quad z \in [S,F], \end{cases} \tag{54}$$

where $\psi(t)$ *is defined in* (35)*. Then* $h \in C([0,T] \times [S,F]) \cap C^{0,1}([0,T) \times (S,F];\mathbb{R}) \cap C^{1,3}([0,T) \times (S,F);\mathbb{R})$*, it is a classical solution of* (39)-(40) *and satisfies* (43)-(44)*. In particular, by Remark 4.15 and Proposition 4.13, h is the unique viscosity solution of* (39)-(40)*, therefore* $h = H$*.*

Proof. Let H be the unique viscosity solution of (39)-(40), suppose that H belongs to the class $C^{1,3}([0,T) \times (S,F);\mathbb{R})$, and that it satisfies (43)-(44). Due to (42), we know that H satisfies (45). Deriving this equation with respect to z we have

$$H_{tz}(t,z) = 2\kappa\,\eta(t)(F-z) + \frac{\beta^2}{2}\frac{2H_z(t,z)H_{zz}(t,z)^2 - H_z^2 H_{zzz}(t,z)}{H_{zz}(t,z)^2}, \quad (t,z) \in [0,T) \times (S,F). \tag{55}$$

Let g be defined as above. Since $H \in C^{1,3}([0,T) \times (S,F);\mathbb{R})$, we have that $g \in C^{1,2}([0,T) \times (0,+\infty);\mathbb{R})$. Deriving (46) with respect to t, with respect to y, and twice with respect to y we obtain

$$H_{tz}(t,g(t,y)) + H_{zz}(t,g(t,y))g_t(t,y) = 0, \tag{56}$$

$$H_{zz}(t,g(t,y))g_y(t,y) = -1, \tag{57}$$

$$H_{zzz}(t,g(t,y))g_y^2(t,y) + H_{zz}(t,g(t,y))g_{yy}(t,y) = 0. \tag{58}$$

Plugging (56), (57), and (58) into (55) we get (49). The boundary conditions (50) together with the required properties (47) and (48)-(ii) have already been proved in the construction of g above.

Conversely, let $g \in C([0,T] \times [0,+\infty);\mathbb{R}) \cap C^{1,2}([0,T) \times (0,+\infty);\mathbb{R})$ be a classical solution of (49)-(50) satisfying (47), (48)-(ii), (52), (53). First of all we note that, due to (47), (48)-(ii) and (50), the function $[g(t,\cdot)]^{-1}$ is well defined on $(S,F]$ for every $t \in [0,T)$. Let h be defined by (54). We clearly have

$$h \in C([0,T] \times [S,F]) \cap C^{0,1}([0,T) \times (S,F];\mathbb{R}) \cap C^{1,3}([0,T) \times (S,F);\mathbb{R}).$$

Moreover, from (50), (53) and (54) we obtain (40). Deriving (54) with respect to z and taking into account that, by (47) and (48)-(ii), if $z \to S^+$ then $[g(t,\cdot)]^{-1}(z) \to +\infty$, we obtain (44). Furthermore, (43) follows from (57) and from the fact that, due to (47)(i)-(48), we have $g(t,(0,+\infty)) = (S,F)$ for every $t \in [0,T)$, and that (47)-(ii) holds true.
Computing h_z by using the definition (54) of h, we see that h satisfies (46) with h in place of H. Therefore, arguing as in the first part of the present proof, we get that h satisfies (56)-(57)-(58) with h in place of H. Setting $z = g(t,y)$, using backward the argument of the first part of this proof, and taking into account that $g(t,(0,+\infty)) = (S,F)$ for every $t \in [0,T)$, we get that h solves (55) in $[0,T) \times (S,F)$. Integrating (55) with respect to z we see that

$$h_t(t,z) + \kappa\eta(t)(F-z)^2 - \frac{\beta^2 h_z^2(t,z)}{2\,h_{zz}(t,z)} = C(t). \quad (t,z) \in [0,T) \times (S,F).$$

It remains to show that $C(t) \equiv 0$. Since $C(t)$ does not depend on z, it can be expressed as

$$C(t) = h_t(t,z_0) + \kappa\eta(t)(F-z_0)^2 - \frac{\beta^2 h_z^2(t,z_0)}{2\,h_{zz}(t,z_0)}, \tag{59}$$

for every $z_0 \in (S, F)$. In particular $C(t)$ is continuous. Integrating (59) over the time variable in the interval $[t, T]$ we have

$$\int_t^T C(s)ds = h(T, z_0) - h(t, z_0) + \int_t^T \kappa\eta(s)(F - z_0)^2 ds - \frac{\beta^2}{2} \int_t^T \frac{h_z^2(s, z_0)}{h_{zz}(s, z_0)} ds. \tag{60}$$

Taking $z_0 \downarrow S$ in (60) and by using (35) we get

$$\int_t^T C(s)ds = - \lim_{z_0 \downarrow S} \frac{\beta^2}{2} \int_t^T \frac{h_z^2(s, z_0)}{h_{zz}(s, z_0)} ds. \tag{61}$$

Given $(s, y_0) \in [0, T) \times (0, +\infty)$, set $z_0(s) = g(s, y_0)$. By (46) and (57) we have

$$\frac{h_z^2(s, z_0(s))}{h_{zz}(s, z_0(s))} = -y_0^2 g_y(s, y_0).$$

Due to (47), (48)-(ii), and (52), if $y_0 \to +\infty$ then $z_0(s) \to S^+$. Moreover, this convergence is uniform with respect to $s \in [0, T)$. Indeed, by (52) we have for some $c_0 > 0$

$$g_y(s, y_0) \leq \frac{c_0}{y_0^2}, \ \forall (s, y_0) \in [0, T) \times [1, +\infty). \tag{62}$$

This implies that we can write

$$g(s, y_0) - S = \int_{y_0}^{+\infty} g_y(s, \xi) d\xi, \ \forall (s, y_0) \in [0, T) \times [1, +\infty).$$

Taking the supremum over $s \in [0, T)$ and using (62), we get for every $y_0 \in [1, +\infty)$

$$\sup_{s \in [0,T)} |g(s, y_0) - S| \leq \int_y^{+\infty} \sup_{s \in [0,T)} |g_y(s, \xi)| d\xi \leq \frac{c_0}{y_0} \overset{y \to +\infty}{\longrightarrow} 0. \tag{63}$$

Since $z_0(s) = g(s, y_0)$, we see that (63) shows that $z_0(s) \to S^+$ uniformly with respect to $s \in [0, T)$. So we have

$$\lim_{z_0 \downarrow S} \int_t^T \frac{h_z^2(s, z_0)}{h_{zz}(s, z_0)} ds = \lim_{y_0 \to +\infty} \int_t^T \frac{h_z^2(s, z_0(s))}{h_{zz}(s, z_0(s))} ds = - \lim_{y_0 \to +\infty} \int_t^T y_0^2 g_y(s, y_0) ds = 0.$$

Plugging the result above in (61), by arbitrariness of $t \in [0, T)$ and by continuity of $C(\cdot)$, we get $C(t) \equiv 0$, concluding the proof. □

By virtue of Proposition 4.18 above, in order to conclude the argument showing the $C^{1,2}$ interior regularity of H (hence of W, due to (31)), we must prove that there exists a function g satisfying the assumptions of the second part of such proposition. This is provided by the following result.

Theorem 4.19. *There exists a unique $g \in C([0, T] \times [0, +\infty); \mathbb{R}) \cap C^{1,2}([0, T) \times (0, +\infty); \mathbb{R})$ classical solution of* (49)-(50) *and it satisfies* (47), (48)-*(ii),* (52), *and* (53).

Proof. We show the claim for the equation

$$\begin{cases} -g_t + G(t, y, g, g_y, g_{yy}) = 0; \\ g(t, 0) = F, \quad t \in [0, T); \qquad g(T, y) = \left(F - \dfrac{y}{2\eta(T)} \right) \vee S, \quad y \in [0, +\infty); \end{cases} \tag{64}$$

$$G(t,y,g,p,P) = 2\kappa\eta(t)(F - ((g \wedge F) \vee S)) \cdot (p \wedge 0) - \beta^2 yp - \frac{\beta^2}{2} y^2 P, \quad g,p,P \in \mathbb{R}.$$

Then, due to (47), the claim will hold for our equation. We note that in this way G satisfies the standard assumptions of the theory of viscosity solutions (see [Crandall, Ishii & Lions, 1992]), i.e. it is degenerate and proper. We will use this theory.

We set

$$\mathcal{O} = [0,T] \times [0,+\infty), \qquad \partial_p \mathcal{O} = ([0,T) \times \{0\}) \cup (\{T\} \times [0,+\infty)).$$

By $BLSC(\mathcal{O})$, $BUSC(\mathcal{O})$ we denote respectively the space of lower semicontinuous functions and the space of upper semicontinuous functions on $\mathcal{O}$. The definition of viscosity sub and supersolution of (49) we use is the standard one and we do not give it ([Crandall, Ishii & Lions, 1992], Section 8).

Step 1: comparison for the equation. We observe that our equation satisfies a comparison principle in the class of bounded functions, i.e.

$$\begin{cases} u \in BUSC(\mathcal{O}) \text{ bounded viscosity subsolution of (64)}, \\ v \in BLSC(\mathcal{O}) \text{ bounded viscosity supersolution of (64)}, \\ u \leq v \text{ on } \partial_p \mathcal{O}, \end{cases} \quad \Longrightarrow \quad u \leq v \text{ on } \mathcal{O}.$$

We refer, e.g., to Theorem 8.2 of [Crandall, Ishii & Lions, 1992], Section 8: the proof can be adapted quite easily (due to the requirement of boundedness) to the case of our unbounded domain $(0,+\infty)$. In particular a viscosity solution, if it exists, must be unique.

Step 2: subsolution and supersolution. In order to get a continuous bounded viscosity solution of (64) by Perron's method, it is enough to exhibit a bounded viscosity subsolution and a bounded viscosity supersolution satisfying the Dirichlet boundary conditions of such equation (see the next item). So in this item we provide them. The key point is that we have an explicit solution g^0 of (64) when $\kappa = 0$, which satisfies all the statements of our theorem (see formula (76) in the next section). Then it is straightforward to check that $\underline{g} := g^0$ is a (classical, thus viscosity) bounded subsolution also for the case $\kappa > 0$.

From (78) we can see that $g^0(t,y) \geq F - \frac{1}{2}e^{\beta^2 T} y$ for every $[0,T) \times [0,+\infty)$. Notice that $g^0_y(t,y) < 0$ for every $[0,T) \times [0,+\infty)$. Setting $C_0 := \kappa e^{(\beta^2+\rho)T}$, we can see that the function

$$\bar{g} := \begin{cases} g^0\left(T - \dfrac{1 - e^{-C_0(T-t)}}{C_0}, e^{-C_0(T-t)} y\right), & \text{if } \kappa > 0, \\ g^0(t,y), & \text{if } \kappa = 0, \end{cases}$$

is a (classical, thus viscosity) bounded supersolution. Both $\underline{g}, \bar{g}$ satisfy the boundary conditions of (64), so we have concluded this step. We note that the comparison principle yields $\underline{g} \leq \bar{g}$ over $\mathcal{O}$.

Step 3: existence by Perron's method. Due to Step 1 and Step 2, we have existence of a (unique) bounded viscosity solution g of (64) by Perron's method (see [Crandall, Ishii & Lions, 1992], Section 4; the argument is adaptable to the parabolic case). Such a solution, g_{visc}, will be such that $S \leq \underline{g} \leq g_{\text{visc}} \leq \bar{g} \leq F$ over $\mathcal{O}$ and $S < \underline{g} \leq g_{\text{visc}}$ over $[0,T) \times [0+\infty)$.

Step 4: interior $C^{1,2}$ regularity. Our equation is a semilinear parabolic equation, which is uniformly parabolic when restricted to the compact sets contained in the interior part of the domain. Therefore, a localization procedure and the use of the classical theory of parabolic equations yield the interior $C^{1,2}$ regularity of the solution.

Let $0 < a < b < +\infty$, let g_{visc} the unique viscosity solution of (64) provided by the previous item, and consider the equation

$$\begin{cases} -g_t + G(t, y, g, g_y, g_{yy}) = 0; \\ g(t,a) = g_{\text{visc}}(t,a), \quad g(t,b) = g_{\text{visc}}(t,b), \quad t \in [0,T); \qquad g(T,y) = g_{\text{visc}}(T,y), \quad y \in [0,+\infty). \end{cases}$$

By comparison, also this equation admits a unique viscosity solution, which is still g_{visc}. On the other hand, Theorem 12.22 in [Lieberman, 1996] (using, e.g., the assumptions of Theorem 12.14 of the same book for the parabolic operator) yields the existence of a solution of the same equation, which is $C^{1,2}$ in $[0,T) \times (a,b)$. This solution is also a viscosity solution of the equation, hence must coincide with g_{visc}. This shows that g_{visc} belongs to the class $C^{1,2}([0,T) \times (a,b); \mathbb{R})$. By the arbitrariness of a, b, we get $g_{\text{visc}} \in C^{1,2}([0,T) \times (0,+\infty); \mathbb{R})$. Therefore $g := g_{\text{visc}}$ is the unique classical solution of (49)-(50).

Step 5: convexity. Let $g \in C^{1,2}([0,T) \times (0,+\infty); \mathbb{R})$ be the classical solution of (49)-(50) found in the previous step. We want to prove that $y \mapsto g(t,y)$ is convex for all $t \in [0,T]$. There is some literature on the convexity preserving property of second-order nonlinear equations (in particular, we refer to [Korevaar, 1983] and [Giga, Goto, Ishii & Sato, 1991], dealing with the parabolic case; the first one is working in a classical context, the second one in a viscosity context). However, our equation is not covered by these references, so we give the proof following mainly the arguments of Theorem 1.6 in [Korevaar, 1983].

The key is to prove a maximum principle for the so called concavity function, i.e.

$$C(t, y_0, y_1) := 2g\left(t, \frac{y_0 + y_1}{2}\right) - g(t, y_0) - g(t, y_1), \quad t \in [0,T], \; y_0, y_1 \in [0,+\infty).$$

It is clear that if

$$\sup_{[0,T]\times[0,+\infty)^2} C(t, y_0, y_1) \leq 0 \tag{65}$$

we would have the claim. Suppose by contradiction that (65) is false, i.e.

$$\sup_{[0,T]\times[0,+\infty)^2} C(t, y_0, y_1) > 0. \tag{66}$$

We note that, since $\underline{g} \leq g \leq \bar{g}$, we have $\limsup_{y_0^2+y_1^2 \to +\infty} C(t, y_0, y_1) \leq 0$ and this holds uniformly on $t \in [0,T]$ because it holds for g^0. Therefore, the supremum in (66) must be attained at some point $(\hat{t}, \hat{y}_0, \hat{y}_1) \in [0,T] \times [0,+\infty)^2$. However, since $g(T, \cdot)$ is convex, we must actually have $(\hat{t}, \hat{y}_0, \hat{y}_1) \in [0,T) \times [0,+\infty)^2$. Moreover, since $C(\hat{t}, 0, 0) = 0$, it must be either $\hat{y}_0 > 0$ or $\hat{y}_1 > 0$. First of all, suppose that one of them is 0, for example that $\hat{y}_0 = 0$. Then $\hat{y}_1 > 0$ and the first and second order conditions for the maximum yield (note that it can be $\hat{t} = 0$)

$$2g_t\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - g_t(\hat{t}, \hat{y}_1) \leq 0, \quad g_y\left(\hat{t}, \frac{\hat{y}_1}{2}\right) = g_y\left(\hat{t}, \hat{y}_1\right), \quad \hat{y}_1^2\left[\frac{1}{2}g_{yy}\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - g_{yy}\left(\hat{t}, \hat{y}_1\right)\right] \leq 0. \tag{67}$$

Using (67) and the fact that (64) is satisfied by g at $(\hat{t}, \hat{y}_1)$, $\left(\hat{t}, \frac{\hat{y}_1}{2}\right)$ we arrive to the inequality

$$\left(2g\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - F - g(\hat{t}, \hat{y}_1)\right) \cdot (a \wedge 0) \geq 0, \quad \text{where } a := g_y\left(\hat{t}, \frac{\hat{y}_1}{2}\right) = g_y(\hat{t}, \hat{y}_1). \tag{68}$$

If we were able to say that $a < 0$, then the above inequality would represent a contradiction of (66). Unfortunately, we do not have at this stage any information on g_y, so we are not able to get directly this contradiction. However, defining the function

$$g^\varepsilon(t,y) = e^{-\varepsilon(T-t)}g(t,y), \quad \varepsilon > 0,$$

we see that it solves

$$-g^\varepsilon_t + G\left(t, y, e^{\varepsilon(T-t)}g^\varepsilon, e^{\varepsilon(T-t)}g^\varepsilon_y, e^{\varepsilon(T-t)}g^\varepsilon_{yy}\right) + \varepsilon g^\varepsilon(t,y) = 0.$$

Arguing as above with this function and this equation we would get, in place of (68), the inequality

$$\left(2g^\varepsilon\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - F - g^\varepsilon(\hat{t},\hat{y}_1)\right) \cdot [(a \wedge 0) - \varepsilon)] \geq 0, \quad \text{where} \quad a := g^\varepsilon_y\left(\hat{t}, \frac{\hat{y}_1}{2}\right) = g^\varepsilon_y(\hat{t},\hat{y}_1).$$

Then

$$\left(2g^\varepsilon\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - F - g^\varepsilon(\hat{t},\hat{y}_1)\right) \leq 0, \ \forall \varepsilon > 0.$$

Taking the limit for $\varepsilon \to 0$, by uniform convergence,

$$\left(2g\left(\hat{t}, \frac{\hat{y}_1}{2}\right) - F - g(\hat{t},\hat{y}_1)\right) \leq 0,$$

contradicting (66). The same argument leads to a contradiction if we assume $\hat{y}_1 = 0$.

It remains to show that also the case $\hat{y}_0 > 0, \hat{y}_1 > 0$ leads to a contradiction. The argument is the same as the one before, it is just needed to work with the couple (y_0, y_1). Due to the fact that in this case $(\hat{t}, \hat{y}_0, \hat{y}_1)$ is an interior (with respect to the space variables y_0, y_1) maximum for C, we see that

$$a := g_y(\hat{t},\hat{y}_0) = g_y(\hat{t},\hat{y}_1) = g_y\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right). \tag{69}$$

and

$$D^2_{(y_0,y_1)}C(\hat{t},\hat{y}_0,\hat{y}_1) = \begin{pmatrix} \frac{1}{2}g_{yy}\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) - g_{yy}(\hat{t},\hat{y}_0) & \frac{1}{2}g_{yy}\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) \\ \frac{1}{2}g_{yy}\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) & \frac{1}{2}g_{yy}\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) - g_{yy}(\hat{t},\hat{y}_0) \end{pmatrix} \leq 0. \tag{70}$$

Applying (70) to the vector $(\hat{y}_0, \hat{y}_1)$ we get

$$-\hat{y}_0^2 g_{yy}(\hat{t},\hat{y}_0) - \hat{y}_1^2 g_{yy}(\hat{t},\hat{y}_1) + \left(\frac{\hat{y}_0+\hat{y}_1}{2}\right)^2 g_{yy}\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) \leq 0. \tag{71}$$

At $(\hat{t},\hat{y}_0), (\hat{t},\hat{y}_1), \left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right)$ equation (64) holds for g. This fact, together with (69), (71) and the fact that $C_t(\hat{t},\hat{y}_0,\hat{y}_1) \leq 0$, yields ($a$ is defined in (69))

$$(a \wedge 0)\left[2g\left(\hat{t}, \frac{\hat{y}_0+\hat{y}_1}{2}\right) - g(\hat{t},\hat{y}_0) - g(\hat{t},\hat{y}_1)\right] \geq 0.$$

Again, the problem to get the contradiction is that we do not have any information on g_y. But using again the approximation procedure shown above, we can conclude.

Step 6: monotonicity and other qualitative properties.

- *Properties* (47)-*(i) and* (48). We notice that (47)-(i) and (48) hold for $\underline{g}, \bar{g}$. Since $\underline{g} \le g \le \bar{g}$ they also hold true for g.
- *Property* (47)-*(ii).* The convexity property and the fact that $g \le \bar{g}$ force to have (47)-(ii). This can be proved by contradiction. Indeed, suppose that for some $t \in [0,T)$ the function $y \mapsto g(t,y)$ is not strictly decreasing. By convexity we would have the existence of some $\bar{y} \in [0,+\infty)$ such that $g(t,y) \ge g(t,\bar{y}) > S$ for all $y \ge \bar{y}$. This fact would contradict the fact that $\lim_{y\to+\infty} \bar{g}(t,y) = S$.
- *Property* (53). This property is a consequence of $g \le \bar{g}$. Indeed, due to the explicit expression of $\bar{g}(t,\cdot) - S$, we see that it is integrable over $[0,+\infty)$ for every $t \in [0,T)$. Therefore, also $g(t,\cdot) - S$ is integrable over $[0,+\infty)$. Passing to the inverse function we obtain (53).
- *Property* (52). Due to the convexity of g and to (47)-(ii), we have

  $$y(g(t,2y) - g(t,y)) \le y^2 g_y(t,y) < 0, \ \forall t \in [0,T), \ \forall y > 0.$$

  Now we observe that $0 < g(t,y) \le \bar{g}(t,y) \le c_0/y^2$ for some $c_0 > 0$ independent of $t \in [0,T)$. Then from the inequality above

  $$-\frac{c_0}{y} \le y^2 g_y(t,y) < 0, \ \forall y > 0, \ \forall t \in [0,T).$$

  Hence we get (52), concluding the proof. □

As corollary of Proposition 4.18 and Theorem 4.19 we get the following.

Corollary 4.20. *The value function H defined in* (30) *is of class $C^{1,3}$ in $[0,T) \times (S,F)$ and it is the unique classical solution of* (39)-(40). *Moreover it satisfies* (41)-(42)-(43)-(44).

Proof. The fact that H is of class $C^{1,3}$ in $[0,T)\times(S,F)$, is the unique classical solution of (39)-(40) and that it satisfies (43)-(44) follows from Proposition 4.18 and Theorem 4.19. It satisfies (41) and (42) by Proposition 4.10 and because of its regularity and convexity. □

At this point it would be natural to try to prove the existence of optimal feedbacks for our problem. For brevity we limit ourselves to perform this study only in the case $\kappa = 0$, when explicit solutions are available. This will be done in the next section.

5 A special case of (P2): solution of the problem without running cost

In this section, we focus on a special case of the problem (P2) and find a closed-form solution for it. Here we eliminate the running cost in the objective functional (4) by setting $\kappa = 0$. Clearly, our choice is mainly due to the mathematical tractability of the problem and the need of explicit solutions. However, we observe that the main assumption of this paper is that the retiree enters retirement at time $t = 0$ and takes the income drawdown option until compulsory annuitization at time $t = T$, with no action in the intervening period. Therefore, the desire of closeness to a target fund over time, although perfectly reasonable, does not seem to be strictly necessary and can be dropped without rendering the problem unrealistic or less interesting. In fact, we notice that finding optimal strategies that avoid ruin a priori is of great interest in itself, given that a substantial stream of literature addresses the relevant issue of avoiding ruin or minimizing its probability in income drawdown problems. See, among others, [Albrecht & Maurer, 2002], [Gerrard, Højgaard & Vigna, 2010], [Milevsky, Moore & Young, 2006], and [Milevsky & Robinson, 2000]. Notice that, similarly to this paper, also [Gerrard, Højgaard & Vigna, 2010] find optimal strategies that avoid short-selling and ruin. However, in their model the flexibility of choosing a guaranteed final fund $S > 0$ and the guarantee of a positive income b_0 from retirement to final annuitization are missing. Up to our knowledge, this is the first model in the literature on income drawdown option that allows the pensioner to choose a minimum guaranteed level of wealth at the time of ultimate annuitization.

Summarizing, we are interested in solving the following problem: for given $(t, x) \in \mathcal{C}$,

$$\text{minimize} \quad J(t, x; \pi(\cdot)) = \mathbb{E}\left[e^{-\rho T}(F - X(T))^2\right] \quad \text{over} \quad \pi(\cdot) \in \Pi_{ad}(t, x). \tag{72}$$

The problem rewritten in terms of Z as in Section 4 becomes: for given $(t, z) \in [0, T] \times [S, F]$,

$$\text{minimize } \tilde{J}(t, z, \pi(\cdot)) = \mathbb{E}\left[\eta(T)(F - Z(T))^2\right] = \eta(T)\mathbb{E}\left[(F - Z(T))^2\right] \text{ over } \pi(\cdot) \in \tilde{\Pi}_{ad}(t, z), \tag{73}$$

where $\eta(T) = e^{-\rho T}$. It is clear that we could assume without loss of generality $\rho = 0$. However, we note that in the proof of Theorem 4.19 we have used the explicit solution of the case $\kappa = 0$ also when $\rho > 0$ (more precisely, we have used the estimate (78) below). Therefore, to be consistent, here we cannot assume that $\rho = 0$. We will assume $\rho = 0$ only in the numerical application in Subsection 5.2.

In the case $\kappa = 0$, (49)-(50) read as

$$g_t(t, y) + \beta^2 y g_y(t, y) + \frac{\beta^2}{2} y^2 g_{yy}(t, y) = 0, \quad \text{on } [0, T) \times (0, +\infty), \tag{74}$$

and

$$\begin{cases} g(t, 0) = F, \quad t \in [0, T]; \\ g(T, y) = \left(F - \dfrac{y}{2\eta(T)}\right) \vee S, \quad y \in [0, +\infty). \end{cases} \tag{75}$$

As known, the classical solution of (74)-(75) is given by the Kolmogorov probabilistic representation

$$g(t, y) = \mathbb{E}\left[g(T, Y(T; t, y))\right], \quad (t, y) \in [0, T] \times [0, +\infty),$$

where $Y(\cdot; t, y)$ is the solution of

$$\begin{cases} dY(s) = \beta^2 Y(s)ds + \beta Y(s)dB(s), \quad s \in [t, T], \\ Y(t) = y. \end{cases}$$

Since the law of $Y(T; t, y)$ is known (it is the log-normal law), we can explicitly compute g.

Proposition 5.1. *The unique classical solution of* (74)-(75) *is the function*

$$\begin{cases} g(t, y) = (F - S)\Phi(k(t, y)) - \dfrac{y}{2\eta(T)} e^{\beta^2(T-t)}\Phi(k(t, y) - \beta\sqrt{T-t}) + S, \ (t, y) \in [0, T) \times [0, +\infty), \\ g(T, y) = \left(F - \dfrac{y}{2\eta(T)}\right) \vee S, \end{cases} \tag{76}$$

where

$$k(t, y) = \frac{-\log\left(\dfrac{y}{2\eta(T)(F - S)}\right) - \dfrac{\beta^2}{2}(T - t)}{\beta\sqrt{T - t}}$$

and where Φ is the cumulative distribution function of a standard normal random variable, i.e. $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{\xi^2}{2}} d\xi$.

Proof. It can be proved by direct computations. □

We notice that the function g defined in (76) corresponds to the function g^0 in the proof of Theorem 4.19.

The expression of g is related to the price function $p_{\text{put}}(t,y)$ of a European put option with strike price $2\eta(T)(F-S)$, in a Black-Scholes market where the spot rate of the riskless asset is β^2, the volatility of the risky asset is β, and the risk-premium is 0. Indeed,

$$g(t,y) = \frac{e^{\beta^2(T-t)}}{2\eta(T)}\, p_{\text{put}}(t,y) + S \in \begin{cases} (S,F], & \text{if } t \in [0,T), \\ [S,F], & \text{if } t = T. \end{cases} \tag{77}$$

As known, $p_{\text{put}}(t,\cdot)$ is convex, so we see that also $g(t,\cdot)$ is convex. Moreover, (77) or direct computations show that g_y is bounded. More precisely, $0 \geq g_y(t,\cdot) \geq \lim_{y\to 0^+} g_y(t,y)$ and

$$0 \geq \lim_{y\to 0^+} g_y(t,y) = -\frac{e^{\beta^2(T-t)}}{2\eta(T)} \geq -\frac{1}{2}e^{(\beta^2+\rho)T}. \tag{78}$$

5.1 The optimal feedback strategy

In this section, we study the optimal feedback strategy for the problem (72). Due to (37), Corollary 4.20, and Proposition 4.18, the optimal feedback map associated to the value function H is

$$G(t,z) := \begin{cases} -\dfrac{\beta}{\sigma}\dfrac{H_z(t,z)}{H_{zz}(t,z)}e^{-r(T-t)}, & (t,z) \in [0,T) \times (S,F), \\ 0, & (t,z) \in [0,T) \times \{S,F\}. \end{cases} \tag{79}$$

It is more suitable to rewrite it in terms of the solution g provided by (76). Using (56), (57), and (58), it reads as

$$G(t,z) = -\frac{\beta}{\sigma}\, e^{-r(T-t)}\ [g(t,\cdot)]^{-1}(z)\ g_y\left(t, [g(t,\cdot)]^{-1}(z)\right), \quad \text{on } [0,T) \times (S,F).$$

Then, taking into account (52) or the explicit expression (76), we see that G is continuous and bounded on $[0,T) \times [S,F]$.

Let $y^* = [g(t,\cdot)]^{-1}(z)$, let $Y^*(\cdot;t,y^*)$ be the solution of

$$\begin{cases} dY^*(s) = -\beta Y^*(s)dB(s), \quad s \in [t,T], \\ Y^*(t) = y^*, \end{cases} \tag{80}$$

and consider the process

$$Z^*(s;t,z) = g\left(s, Y^*(s;t,y^*)\right). \tag{81}$$

We notice that by definition of $Z^*(\cdot;t,z)$, by definition of $Y^*(\cdot;t,y^*)$ in (80), and since $g(t,\cdot) \in (S,F)$ on $(0,+\infty)$ for every $t \in [0,T)$ we have

$$Z^*(s;t,y^*) \in (S,F),\ \forall s \in [t,T]. \tag{82}$$

Ito's formula and the fact that g solves (74) yield that $Z^*(\cdot;t,z)$ solves the closed loop equation associated with the map G, i.e.

$$\begin{cases} dZ(s) = e^{r(T-t))}\left[(\mu-r)G(s,Z(s))ds + \sigma G(s,Z(s))dB(s)\right], \quad s \geq t \\ Z(t) = z \in (S,F), \end{cases} \tag{83}$$

on the interval $[t,T]$. Furthermore, equation (83) admits the solution $Z(\cdot) \equiv F$ (respectively, $Z(\cdot) \equiv S$) if $z = F$ (respectively, if $z = S$). So we also set $Z^*(\cdot;t,S) \equiv S$ and $Z^*(\cdot;t,F) \equiv F$. Then the feedback strategy

$$\pi^*_{t,z}(s) = \begin{cases} G(s, Z^*(s;t,z)), & s \in [t,T), \\ 0, & s = T, \end{cases} \tag{84}$$

is square integrable since G is bounded and so admissible. We now show that it is indeed the unique optimal strategy starting from (t,z). To prove the uniqueness, first we need to prove that the functional (28) is strictly convex on $\tilde{\Pi}_{ad}(t,z)$. This is the result of the following proposition.

Proposition 5.2. *Let $(t,z) \in [0,T) \times [S,F]$. Then the functional $\tilde{\Pi}_{ad}(t,z) \to \mathbb{R}$, $\pi(\cdot) \mapsto \tilde{J}(t,z;\pi(\cdot))$ is strictly convex.*

Proof. Let $(t,z) \in [0,T) \times [S,F]$ and take $\pi_1(\cdot), \pi_2(\cdot) \in \tilde{\Pi}_{ad}(t,z)$. Further, for $\lambda \in (0,1)$ let $\pi_\lambda(\cdot) := \lambda\pi_1(\cdot) + (1-\lambda)\pi_2(\cdot)$. Defining $Z_1(\cdot) := Z(\cdot;t,z,\pi_1(\cdot))$, $Z_2(\cdot) := Z(\cdot;t,z,\pi_2(\cdot))$, and $Z_\lambda(\cdot) = \lambda Z_1(\cdot) + (1-\lambda)Z_2(\cdot)$ we see that $Z_\lambda(\cdot) := Z(\cdot;t,z,\pi_\lambda(\cdot))$. Therefore, due to the convexity of $z \mapsto (F-z)^2$, we have

$$\begin{aligned} \tilde{J}(t,z;\pi_\lambda(\cdot)) &= \mathbb{E}\left[(F - Z_\lambda(T))^2\right] \le \lambda\mathbb{E}\left[(F - Z_1(T))^2\right] + (1-\lambda)\mathbb{E}\left[(F - Z_2(T))^2\right] \\ &= \lambda\tilde{J}(t,z;\pi_1(\cdot)) + (1-\lambda)\tilde{J}(t,z;\pi_2(\cdot)). \end{aligned}$$

Moreover, by linearity of the state equation we have $Z_1 \ne Z_2$ when $\pi_1 \ne \pi_2$ and by the strict convexity of $z \mapsto (F-z)^2$ the above inequality is strict; so the claim is proved. □

Theorem 5.3. *Let $(t,z) \in [0,T) \times [S,F]$. Then the strategy $\pi^*_{t,z}(\cdot)$ defined by (84) is the unique optimal strategy for the problem (73).*

Proof. Let $(t,z) \in [S,F]$. We notice that

$$Z^*(s) = Z(s;t,z,\pi^*_{s,z}(\cdot)),$$

where $Z^*(\cdot)$ is defined by (81) and $Z(\cdot;t,z,\pi^*_{t,z}(\cdot))$ is the solution of the state equation under the control $\pi^*_{t,z}(\cdot)$. Indeed, both of them solve the state equation under the control $\pi^*_{t,z}(\cdot)$.[7]

As we have observed, the admissibility of $\pi^*_{t,z}(\cdot)$ is consequence of the boundedness of G. Moreover,

(i) when $(t,z) \in [0,T) \times (S,F)$ then $Z^*(s) \in (S,F)$ for all $s \in [t,T)$, since $Y^*(s) \in (0,+\infty)$;

(ii) when $(t,z) \in [0,T) \times \{S\}$ (respectively $(t,z) \in [0,T) \times \{F\}$) then $Z^*(s) = S$ for all $s \in [t,T]$ (respectively $Z^*(s) = F$ for all $s \in [t,T]$) and $\pi^*_{t,z} \equiv 0$.

The fact that $\pi^*_{t,z}(\cdot)$ is optimal in the case (ii) is obvious, since $\pi^*_{t,z}(\cdot) \equiv 0$ is the only admissible strategy in this case; the fact that it is optimal in the case (i) follows (arguing as we have done for the problem (P1)) from the fact that H is a classical solution of the HJB equation (39) on $[0,T) \times (S,F)$ and that in this case the trajectory remains in the interior region.

The uniqueness of the optimal strategy straightly follows from the strict convexity of $\tilde{J}(t,z;\cdot)$ proved in Proposition 5.2. □

Remark 5.4. *The uniqueness of the optimal strategy yields the uniqueness of solutions for the closed loop equation (83). Indeed, suppose to have another solution $\tilde{Z}$ of the closed loop equation. Applying the Dynamic Programming Principle with the stopping time*

$$\tau := \inf\left\{s \in [t,T] \mid \tilde{Z}(s) \in \{S,F\}\right\},^{8}$$

[7] Note that this does not prove uniqueness of solutions for the closed loop equation (83); in the next remark, we argue to show it as consequence of the uniqueness of the optimal strategy.

[8] With the agreement that $\inf \emptyset = T$.

and using again the fact that H is a smooth solution of the HJB equation (39) *in* $[0,T) \times (S,F)$, *we can see that the strategy*

$$\tilde{\pi}_{t,z}(s) = \begin{cases} G(s, \tilde{Z}(s)), & s \in [t,T), \\ 0, & s = T, \end{cases}$$

is optimal for the problem. By uniqueness of optimal strategies it must be $\tilde{\pi}_{t,z} = \pi^*_{t,z}$, *hence also* $\tilde{Z} = Z^*$, *where* Z^* *is defined in* (81). *This shows what we have claimed.*

We finally provide a regularity result for the feedback map.

Proposition 5.5. *The map G defined in* (79) *is not Lipschitz continuous with respect to z. However, for every $t_0 \in [0,T)$ and $\alpha \in (0,1)$ G is α-Hölder continuous with respect to z uniformly in $t \in [0,t_0]$.*

Proof. The proof is in the Appendix. □

The previous result would be suitable to study directly the closed loop equation, proving existence and uniqueness of a strong solution, which we have proved in Subsection 4.6 by means of the process Y^* defined in (80). In this case, we would use the theory treated in [Yamada & Watanabe, 1971] to prove pathwise uniqueness and then existence of strong solutions. For a similar approach see, e.g., [Di Giacinto, Federico & Gozzi, 2010].

5.2 Numerical application

In this subsection we show a numerical application of the model presented so far. We consider the position of a male retiree aged 60 with initial wealth $x_0 = 100$. Consistently with [Gerrard, Haberman & Vigna, 2004], we set $T = 15$. The market parameters are $r = 0.03, \mu = 0.08, \sigma = 0.15$, implying a Sharpe ratio equal to $\beta = 0.33$. The amount withdrawn in the unit time, b_0, is set equal to the pension rate purchasable at retirement, using Italian projected mortality tables (RG48). Thus, we set $b_0 = 6.22$. This choice is consistent with previous literature on the topic.

The choice of the final target F and the final guarantee S are evidently subjective and depend on the member's risk aversion. High risk aversion will lead to a high guarantee and a low level of the target, while a high target and a low guarantee will be driven by low risk aversion. We have tested three levels of risk aversion. Thus, high risk aversion is associated to terminal safety level $S = \frac{2}{3} b_0 a_{75}$ and final target equal to $F = 1.5 b_0 a_{75}$, where a_{75} is the actuarial value of a unitary lifetime annuity issued to an individual aged 75; medium risk aversion is associated to terminal safety level $S = \frac{1}{2} b_0 a_{75}$ and final target equal to $F = 1.75 b_0 a_{75}$; low risk aversion is associated to terminal safety level equal to $S = 0$ and final target equal to $F = 2 b_0 a_{75}$. These values are reported in Table 1 below.

The interpretation of these choices is immediate. With high and medium risk aversion, the minimum pension rate guaranteed is, respectively, two third and half of the annuity rate that was possible to have on immediate annuitization at retirement, b_0; the targeted wealth is sufficient to fund a final pension that amounts to, respectively, 1.5 and 1.75 of b_0. With low risk aversion, ruin is always avoided but in the worst scenario no money is left for annuitization at age 75; on the other hand, the targeted pension pursued is twice b_0.

	S	F
High risk aversion	$\frac{2}{3}b_0a_{75}$	$1.5b_0a_{75}$
Medium risk aversion	$\frac{1}{2}b_0a_{75}$	$1.75b_0a_{75}$
Low risk aversion	0	$2b_0a_{75}$

Table 1: Terminal safety level S and final target F for different risk profiles.

Remark 5.6. *It is worth mentioning that even the most risk averse individual has some restrictions in choosing the minimum income guaranteed. Indeed, it is clear from the formulation of the problem*[9] *that the value of S has to satisfy $S \leq z_0 = x_0 e^{rT} - \frac{b_0}{r}(e^{rT} - 1)$. The most risk averse choice would be $S = z_0$, but in this case the only admissible strategy would be $\pi(\cdot) \equiv 0$, i.e. the whole fund wealth must be invested in the riskless asset (see Proposition (4.1)), and one would end up after 15 years with an annuity lower than that purchasable at retirement.*[10] *This choice makes little sense in a realistic framework, given that here the bequest motive is disregarded and the individual takes the income drawdown option only in the hope of being able to buy a better annuity than b_0. For this reason, we here consider only cases where $S < z_0$, which in this particular example translates into $S < 0.70\, b_0\, a_{75}$.*

We have carried out 1000 Monte Carlo simulations for the behaviour of the risky asset, with discretization step equal to one week. In order to do so, we have simulated the process Y^* given by equation (80) with starting point

$$Y^*(0) = y^* = [g(0, \cdot)]^{-1}(z_0)$$

and inserting the corresponding values of S, F and z_0 as above. With each risk aversion we have generated the same 1000 scenarios, by applying in each case the same stream of pseudo random numbers.

For each risk aversion choice, we report the following results:

- Evolution of the fund under optimal control during the 15 years time, by showing a graph with mean and standard deviation and a graph with some percentiles.
- Behaviour of the optimal investment strategy over the 15 years time, by showing a graph with some percentiles. Notice that we report the optimal share of portfolio invested in the risky asset $\theta^*(\cdot)$, rather than the optimal amount $\pi^*(\cdot)$. This is standard, and is done in order to facilitate comparisons between different situations.
- Distribution of the final annuity that can be bought with the final fund at age 75 and comparison with the annuity purchasable at retirement. The conversion of the final fund into annuity has been done with the same basis as above.

Figures 1–4 report results for high risk aversion, Figures 5–8 those for medium risk aversion, Figures 9–12 those for low risk aversion. In particular, Figures 1, 5, and 9 report, over 15 years time, the

[9] Recall that, due to Proposition 4.1, in the region $U_1 \backslash \mathcal{C}$ there are no admissible strategies (see Remark 4.2).

[10] This is clear, considering that the investment in an insurance product benefits from mortality credits, that enhance the riskless rate.

mean and dispersion of the fund trajectories, while Figures 2, 6, and 10 report their percentiles. Figures 3, 7, and 11 report some percentiles of the distribution of the optimal investment allocation $\theta^*(\cdot)$ over 15 years. Finally, Figures 4, 8, and 12 report the distribution of the final annuity upon annuitization at time T.

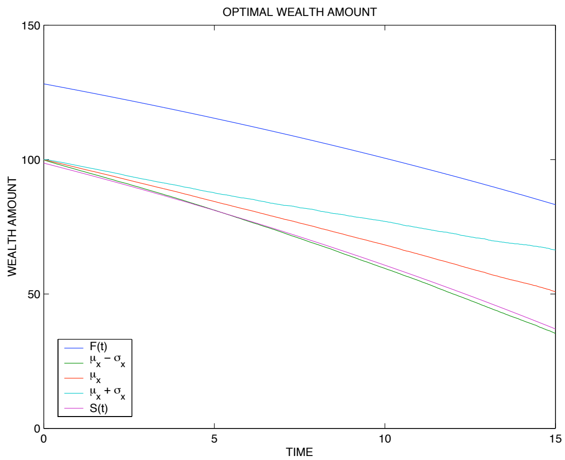

Figure 1: High risk aversion.

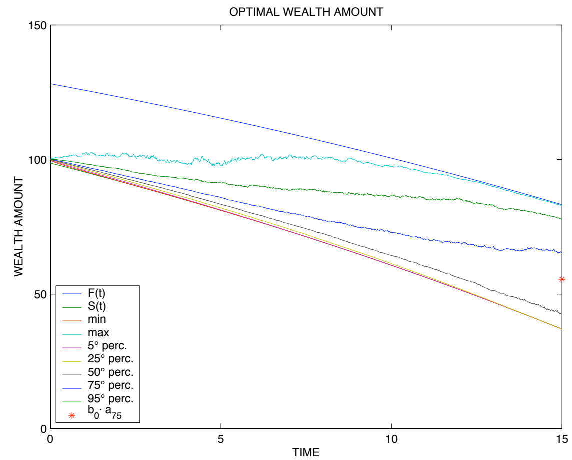

Figure 2: High risk aversion.

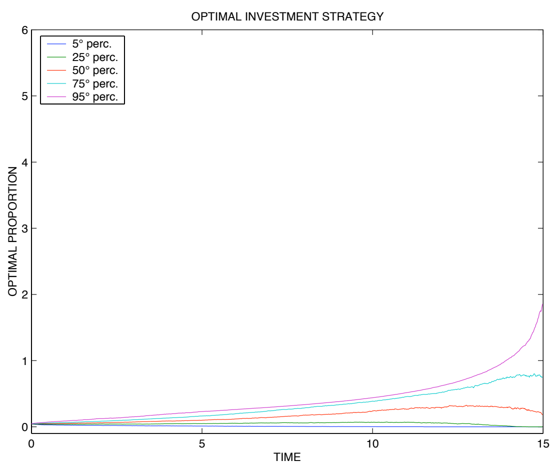

Figure 3: High risk aversion.

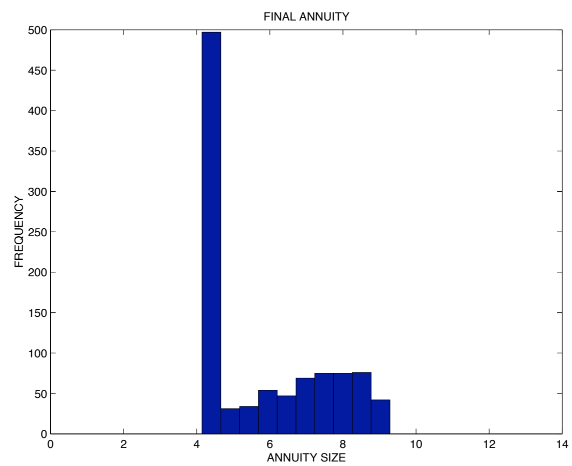

Figure 4: High risk aversion.

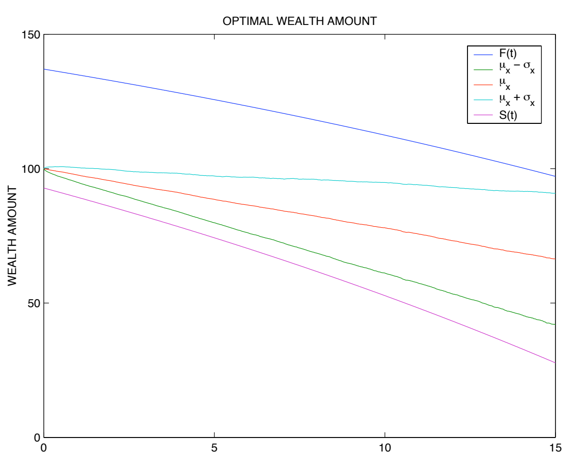

Figure 5: Medium risk aversion.

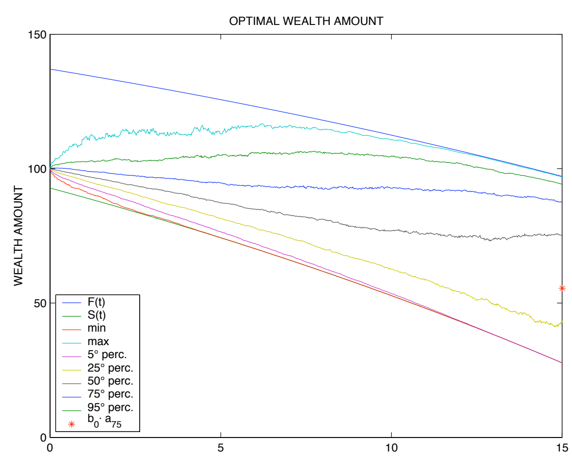

Figure 6: Medium risk aversion.

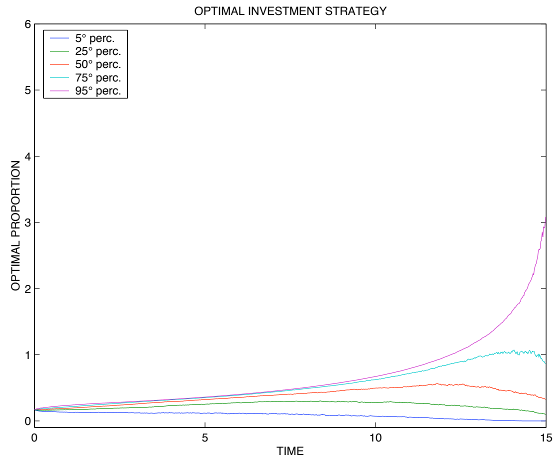

Figure 7: Medium risk aversion.

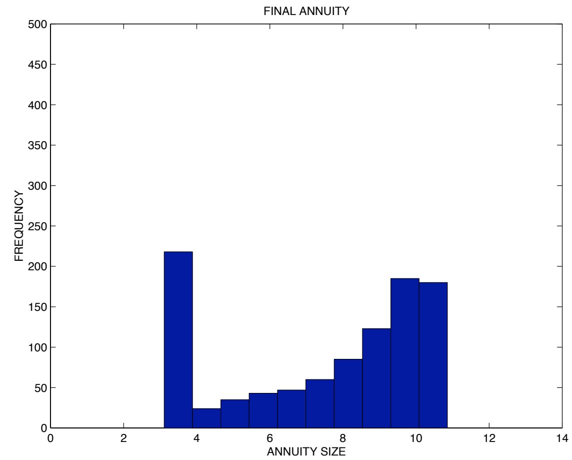

Figure 8: Medium risk aversion.

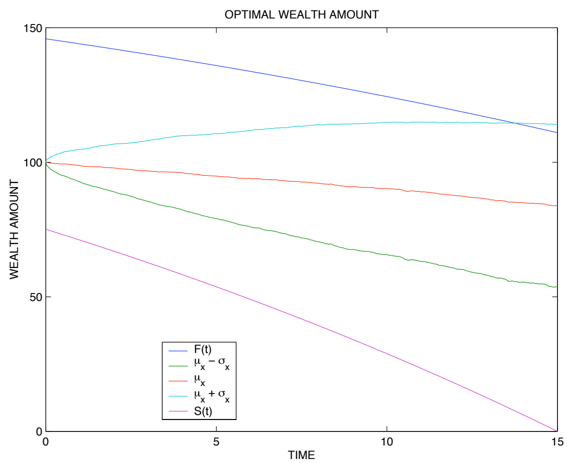

Figure 9: Low risk aversion.

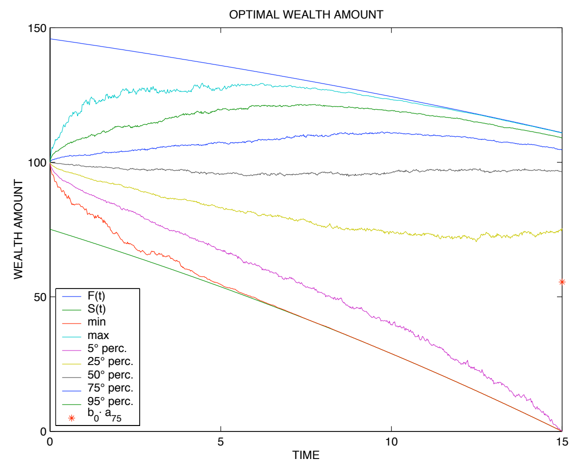

Figure 10: Low risk aversion.

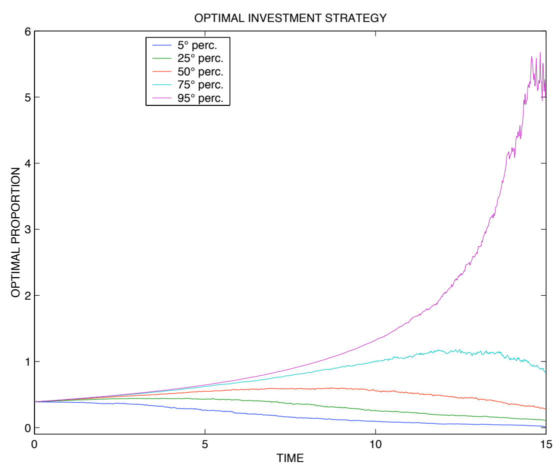

Figure 11: Low risk aversion.

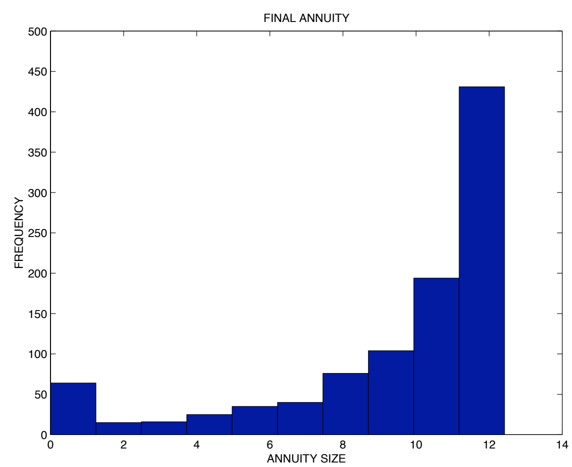

Figure 12: Low risk aversion.

From the graphs we can make the following comments:

- The wealth trajectories lie strictly[11] between the two barriers $S(t)$ and $F(t)$ for $t < T$. In fact, due to (81) and (82), the two bottom and upper absorbing barriers cannot be reached before time T.

- Due to our choice of S and F, when the risk aversion decreases, the boundaries for the wealth process become larger. Also the simulated wealth process results to be more spread out around the mean. This is due to the intuitive fact that the optimal strategies are more aggressive (see next item) and the range of final outcomes increases, both in the positive and in the negative direction.

- Inspection of Figures 3, 7, and 11 shows that when the risk aversion decreases, the optimal strategies become riskier. In fact, with high risk aversion the 95^{th} percentile of $\theta^*(\cdot)$ stays below 2 even immediately prior to time T, whereas with low risk aversion it lies between 5 and 6 close to T. On the other hand, clearly, all strategies are bounded away from 0.

- Comparing Figures 4, 8, and 12 it is immediate to see that the distribution of the final annuity becomes more and more spread when the risk aversion decreases. Moreover, with high risk aversion one can observe a considerable concentration around the guaranteed income $\frac{2}{3}b_0 = 4.15$. In fact, in almost 50% of the cases, the fund approaches $S(t)$ and stays close to it until T (this can be noticed also by thorough inspection of Figure 2). On the contrary, the distribution of final annuity looks very favourable in the case of low risk aversion, where in most of the cases the annuity lies between 9 and 12, and unfavourable scenarios leading to final income equal to 0 happen in ca 5% of the cases.

- We observe that the standard deviation of the investment allocation increases over time, especially towards time T. This can be observed in Figures 3, 7, and 11. The optimal share of portfolio becomes very variable in the 2-3 years before time T. This feature makes this case substantially different from the (state) unconstrained one, where the higher variability of the investment strategy is experienced in the first years after retirement. This interesting difference is evidently due to the inclusion of the absorbing lower barrier $S(t)$. The explanation can be the following. When time approaches T the risk of collapsing onto the safety level reduces remarkably, and many pensioners may be willing to take more risk than in previous years when the risk of locking their position into the safety level is more important.

One should not forget that the real goal of the pensioner who opts for phased withdrawals is to be better off than immediate annuitization when final annuitization takes place. Thus, it is of greatest interest to provide her with detailed information regarding the distribution of the final annuity achieved. To some extent, this has been already shown in Figures 4, 8, and 12. However, the histograms cannot report relevant information that are of immediate use for the member who has to choose a risk profile. In particular, for the member's decision making it is relevant the comparison between the final annuity achievable by taking income drawdown option and b_0, the pension rate purchasable at retirement. Table 2 reports useful statistics of the distribution of the final annuity achieved at age 75, for each risk aversion. The first nine line report mean, standard

[11] Looking at the graphs reporting the percentiles of the trajectories, however, it seems that in some cases the fund touches the bottom target $S(t)$. This is due to the approximation error made by the machine, that is unavoidable. In fact, for not too low values of x, $\Phi(x)$ is so close to 0 that it cannot be distinguished from it. The result is that in the practical applications for not too high values of y^* one has $\Phi\left(k(t,y^*)\right) = \Phi\left(k\left(t,y^*\right) - \beta\sqrt{T-t}\right) = 0$, which implies $g(t,y^*) = S$, meaning that the fund is on the safety level $S(t)$ – that is theoretically impossible.

deviation, min, max and some percentiles of the distribution of the final annuity. Lines 10 and 11 report, respectively, the guaranteed income S/a_{75} and the targeted income F/a_{75} (as chosen in Table 1), while the last line reports the probability (i.e. the frequency over 1000 scenarios) that the final annuity is higher than b_0.

	HIGH RISK AVERSION	MEDIUM RISK AVERSION	LOW RISK AVERSION
mean	5.70	7.44	9.40
st.dev.	1.74	2.73	3.38
min	4.15	3.11	0.00
5th perc.	4.15	3.11	0.00
25th perc.	4.15	4.87	8.45
50th perc.	4.75	8.41	10.80
75th perc.	7.33	9.80	11.72
95th perc.	8.71	10.55	12.21
max	9.29	10.86	12.42
guaranteed income S/a_{75}	4.15	3.11	0
targeted income F/a_{75}	9.33	10.89	12.44
prob(final annuity $> b_0$)	39.20%	68.80%	84.10%

Table 2: Distribution of final annuity at age 75 when the annuity on immediate annuitization is $b_0 = 6.22$.

The following comments can be made:[12]

- The mean of the final annuity is 5.70, 7.44, 9.40 with high, medium and low risk aversion, respectively. The probability of being able to afford a final annuity higher than $b_0 = 6.22$ is 39.20%, 68.80% and 84.10% with high, medium and low risk aversion, respectively.
- This shows that if the risk aversion is too high,[13] the price for having a high guarantee on the final income is that the chances of reaching the desired annuity reduce dramatically. In fact, in 60% of the cases the individual ends up with a final annuity lower than b_0 and, even worse, in almost 50% of the cases the individual receives exactly the guaranteed income, that is only two third of b_0. This is likely to be an undesirable result for the pensioner and it seems to indicate that if the member's risk aversion is too high, it is not convenient to take the income drawdown option. This feature was already observed by [Gerrard, Haberman & Vigna, 2006].
- On the other hand, with medium and low risk aversion the chances of being better off with annuitization at time T are almost 70% and 85%, respectively. This is an encouraging result,

[12]Notice that, due to the approximation error made by the machine (see previous footnote) the values indicated by the minimum and by the first low percentiles coincide with the guaranteed income.

[13]Observe, in fact, that the value of $S = 0.67\, b_0\, a_{75}$ is chosen to be very close to the upper boundary $z_0 = 0.70\, b_0\, a_{75}$.

given that from retirement to T the pensioner has withdrawn the prescribed rate of b_0 and that she was also guaranteed with a minimum lifetime income at retirement, or at worst against ruin.

- The low risk aversion profile could turn out to be particularly attractive to a member whose global post-retirement income was not heavily affected by the second pillar provision. In fact:
  - the chances of exceeding the immediate annuitization income b_0 are extremely high (84%);
  - in 750 cases out of 1000 the member ends up with an annuity higher than 8.45, that is well above $b_0 = 6.22$;
  - in about 50 cases out of 1000 the final annuity is null (see also Figure 12);
  - ruin never occurs.
- Clearly, the price to pay for having a favourable distribution of final income is to take more risk, which translates into more aggressive investment policies. This is highlighted by Fig.11, that reports the optimal investment strategies for low risk aversion. In more than 25% of the cases, the optimal strategy consists in borrowing considerable amounts of money to be invested in the risky asset. This kind of strategy is evidently not feasible in the presence of real world constraints. Hence, the importance and the need of approaching problem (P3) in future research.

6 Conclusions and further research

In this paper, we have considered the investment allocation problem for a member of a DC pension scheme, in the decumulation phase. The main novelty with respect to the previous literature on the topic is the addition of constraints on both the control and the state variable. Starting from the basic unconstrained model of [Gerrard, Haberman & Vigna, 2004], where interim consumption and annuitization time are fixed, we have defined and analyzed two kind of problems. In the first problem (P1), we have considered constraints on the control variable only. This problem can be solved in closed-form and turns out to be a generalization of the results in [Gerrard, Haberman & Vigna, 2004]. In the second problem (P2), which is completely new in this kind of literature and is the real core of the paper, we have added a constraint on the state variable. Namely, the wealth process must lie between two barriers: the bottom one representing a natural safety level for the fund, and the upper one representing a sort of target to be pursued. In particular, the presence of the bottom safety level implies that the undesirable event of ruin is avoided. The problem (P2) has been studied in its general formulation through the dynamic programming approach and the associated HJB equation. The value function has been shown to be the unique regular solution of the associated HJB equation, which is the departure point to find optimal strategies in feedback form. In a special, though not unrealistic, formulation of problem (P2) – namely, without the running cost – we have found both the value function and the optimal feedback strategy in closed-form. A numerical application of the special case, aimed at showing the impact of the model on retiree's choices, ends the paper.

From the methodological point of view the main novelty of the paper is the proof of the regularity of the value function of problem (P2) as a solution of the associated HJB equation. The known theory could not be applied to this equation due its features (it is degenerate parabolic and fully nonlinear). Therefore, we had to proceed with an *ad hoc* method using a suitable dual transformation of the original problem. Such a dual transformation has already been used in some papers (quoted in Subsection 4.5), but only when the resulting dual equation is linear, so it is approached finding explicit solutions. Here, the transformed equation is semilinear and degenerate at the boundary, so explicit solutions do not seem to be available. We have studied such equation through the viscosity approach, proving a suitable regularity result and providing a precise link between its solution and the value function of the original problem. To our knowledge, this is the first time that this dual transformation is used to prove regularity of solutions of the original HJB equation just through a theoretical argument and without looking for explicit solutions. Also for this reason we think that our method should work for more general classes of equations, so we expect that it could be extended to other cases, like e.g. problem (P3).

We would like to remark also the practical relevance of the analysis of the special case with no running cost. In fact, the model is quite flexible for it allows for subjective choices regarding both the safety and the target levels. These choices are typically driven by the risk profile of the pensioner. In particular, the less risk-averse pensioner can aim to a high target such as double the annuity, while still keeping the guarantee of avoiding ruin; the most risk-averse individual can aim to a lower target, while still guaranteeing a minimum income level upon final annuitization. This considerable flexibility allows the representation of the preferences and needs of most retirees. Moreover, the availability of closed-form expressions for the optimal policy makes this model quite useful for practical purposes. Indeed, supported by encouraging results of the numerical application performed, we believe that this model could be the starting point for a powerful decision-making tool in the decumulation phase of a DC pension plan. To the best of our knowledge, this is the first model in the literature on this topic that allows the pensioner to choose a minimum guaranteed level of wealth at the time of ultimate annuitization.

Due to the difficulty of the task, we have not analyzed the more important problem of short-selling and borrowing constraints plus final capital requirement (problem P3). This problem could be tackled at theoretical level again with the viscosity approach coupled with the dual transformation. The search for explicit solutions, at least in some special case, seems to be very challenging and is in the agenda for future research.

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Appendix

Proof of Proposition 4.8

We split the proof in several steps.

Step 1. Here we show that the function $[0,T] \to \mathbf{R}$, $t \mapsto H(t,z)+\int_0^t \kappa\,\eta(s)(F-z)^2 ds$ is nondecreasing for every $z \in [S,F]$. Let $t \in [0,T]$, $t' \in (t,T]$ and let $z \in [S,F]$. Let $\varepsilon > 0$ and let $\pi^{\varepsilon}_{t'}(\cdot) \in \tilde{\Pi}_{ad}(t',z)$ be an ε-optimal strategy for (t',z). Define $\pi^{\varepsilon}_{t}(\cdot) \in \tilde{\Pi}_{ad}(t,z)$ as

$$\pi^{\varepsilon}(s) := \begin{cases} 0, & \text{if } s \in [t,t'], \\ \pi^{\varepsilon}_{t'}(s), & \text{if } s \in [t',T]. \end{cases}$$

We have $Z(s;t,z,\pi_t^\varepsilon(\cdot)) \equiv z$ for $s \in [t,t']$, so

$$H(t,z) \leq J(t,z;\pi_t^\varepsilon(\cdot)) = \int_t^{t'} \kappa\,\eta(s)(F-z)^2 ds + J(t',z;\pi_{t'}^\varepsilon(\cdot)) \leq \int_t^{t'} \kappa\,\eta(s)(F-z)^2 ds + H(t',z) - \varepsilon.$$

Therefore, by the arbitrariness of ε we get

$$H(t,z) + \int_0^t \kappa\,\eta(s)(F-z)^2 ds \leq H(t',z) + \int_0^{t'} \kappa\,\eta(s)(F-z)^2 ds,$$

i.e. the claim of this step.

Step 2. Here we show that the function $t \mapsto H(t,z)$ is continuous for every $z \in [S,F]$. As usual in stochastic control problems with state constraints the continuity with respect to the time variable is the most difficult step. Fix $t \in [0,T)$, let $t' \in (t,T]$, set $\varepsilon = t'-t$ and take a generic control $\pi_t(\cdot) \in \tilde{\Pi}_{ad}(t,z)$. By Theorem 2.10, Chapter 1 of [Yong & Zhou, 1999], we can map in a natural way the strategy $\pi_t(\cdot) \in \tilde{\Pi}_{ad}(t,z)$ in a strategy belonging to $\tilde{\Pi}_{ad}(t',z)$. Indeed, there exists a process ψ on the space $(C[t,T], \mathcal{B}(C[t,T]))$, adapted with respect to the filtration $(\mathcal{B}_s(C[t,T]))_{s\in[t,T]}$, where $\mathcal{B}_s(C[t,T])$ is the σ-algebra on $C[t,T]$ induced by the projection

$$\pi : C[t,T] \longrightarrow (C[t,s], \mathcal{B}(C[t,s]))$$

$$\zeta(\cdot) \longmapsto \zeta(\cdot)|_{[t,s]}$$

such that

$$\pi_t(s) = \psi(s, B^t(\cdot)), \quad s \in [t,T],$$

where $B^t(\cdot) = B(\cdot) - B(t)$. Then we can consider the strategy

$$\pi_{t'}(s) = \psi(s - t' + t, B^{t'}(\cdot)), \quad s \in [t',T],$$

where $B^{t'}(\cdot) = B(\cdot) - B(t')$. We have $\pi_{t'}(\cdot) \in \tilde{\Pi}_{ad}(t',z)$ and

$$Z(s;t,z,\pi_t(\cdot)) \overset{\text{law}}{=} Z(s+\varepsilon;t',z,\pi_{t'}(\cdot)), \quad \forall s \in [t, T-\varepsilon).$$

Call $Z_t(\cdot) := Z(s;t,z,\pi_t(\cdot))$ and $Z_{t'}(\cdot) := Z(\cdot;t',z,\pi_{t'}(\cdot))$. We have

$$\left| \mathbb{E}\left[\int_t^T \kappa\,\eta(s)(F - Z_t(s))^2 ds \right] - \mathbb{E}\left[\int_{t'}^T \kappa\,\eta(s)(F - Z_{t'}(s))^2 ds \right] \right|$$
$$= \left| \mathbb{E}\left[\int_{T-\varepsilon}^T \kappa\,\eta(s)(F - Z_t(s))^2 ds \right] \right| \overset{\varepsilon\to 0}{\longrightarrow} 0, \quad (85)$$

and

$$\left|\mathbb{E}[(F - Z_t(T))^2] - \mathbb{E}[(F - Z_{t'}(T))^2]\right| = \left|\mathbb{E}[(F - Z_t(T))^2] - E[(F - Z_t(T-\varepsilon))^2]\right| \overset{\varepsilon\to 0}{\longrightarrow} 0. \quad (86)$$

Arguing with δ-optimal controls, the convergences (85)-(86) show that $H(\cdot,z)$ is upper semicontinuous on the right, hence by Step 1, right-continuous. A similar argument (mapping a strategy $\pi_{t'}(\cdot) \in \tilde{\Pi}_{ad}(t',z)$ into a strategy $\pi_t(\cdot) \in \tilde{\Pi}_{ad}(t,z)$) proves also the left-continuity, hence the claim of this step.

Step 3. Here we prove that H is continuous on the sets $[0,T] \times [S+\varepsilon, F]$ for any $\varepsilon > 0$. First of all notice that, since $H(t,\cdot)$ is convex and admits minimum at F, it is Lipschitz continuous on

$[S+\varepsilon, F]$ for every $t \in [0,T]$. Let us give an estimate of the Lipschitz constant uniform on $t \in [0,T]$. As said, $H(t,\cdot)$ is convex (so that the incremental ratios are increasing) and nonincreasing (so that the incremental ratios are negative); thus, if we set

$$M_{t,\varepsilon} := \left| \frac{H(t, S+\varepsilon) - H(t,S)}{\varepsilon} \right|,$$

we get that $M_{t,\varepsilon}$ is good as Lipschitz constant for $H(t,\cdot)$ in the interval $[S+\varepsilon, F]$. By Step 2 there exists

$$M_\varepsilon := \max_{t \in [0,T]} M_{t,\varepsilon},$$

so M_ε is the Lipschitz constant uniform on $t \in [0,T]$ we were looking for. Uniform (with respect to t) Lipschitz continuity with respect to z and continuity with respect to t (Step 2) yield the claim of this step.

Step 4. Here we prove that the function $[S,F] \to [0,+\infty)$, $z \mapsto H(t,z)$ is continuous at S^+ for every $t \in [0,T]$. Since $H(t,\cdot)$ is nonincreasing, it suffices to prove the claim on a sequence $z_n \to S^+$. Since the boundary is absorbing, we proceed with estimates on the state equation. Let D be the density of P with respect to the probability measure $\tilde{P}$ given by Girsanov's transformation, which belongs to $L^p(\Omega, \tilde{\mathbb{P}})$, for any $p \in [1,+\infty)$. For any $\pi(\cdot) \in \tilde{\Pi}_{ad}(t, S+\frac{1}{n^2})$, $s \in [t,T]$, by Hölder's and Markov's inequalities we have

$$\begin{aligned}
P\left\{Z\left(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\right) - S > \tfrac{1}{n}\right\} &= \mathbb{E}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}}\right] \\
&= \tilde{\mathbb{E}}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}} D\right] \\
&\le \left(\tilde{\mathbb{E}}\left[D^2\right]\right)^{\frac{1}{2}} \left(\tilde{\mathbb{E}}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}}\right]\right)^{\frac{1}{2}} \\
&= \left(\tilde{\mathbb{E}}\left[D^2\right]\right)^{\frac{1}{2}} \left(\tilde{P}\left\{Z(s;t,S+\tfrac{1}{n^2},\pi(\cdot)) - S > \tfrac{1}{n}\right\}\right)^{\frac{1}{2}} \\
&\le n^{\frac{1}{2}} \left(\tilde{\mathbb{E}}\left[D^2\right]\right)^{\frac{1}{2}} \left(\tilde{\mathbb{E}}\left[Z(s;t,S+\tfrac{1}{n^2},\pi(\cdot)) - S\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\tilde{\mathbb{E}}\left[Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)\right] \equiv S + \frac{1}{n^2}$, we have

$$P\left\{Z\left(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\right) - S > \tfrac{1}{n}\right\} \le \frac{\left(\tilde{\mathbb{E}}\left[D^2\right]\right)^{\frac{1}{2}}}{n^{\frac{1}{2}}}.$$

Moreover, we notice that $\mathbb{E}\left[\left(Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right) - S\right)^2\right] \le (F-S)^2$. Thus, again by Hölder's inequality, we can write

$$
\begin{aligned}
\mathbb{E}\big[\big(Z\big(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\big)-S\big)\big] &= \mathbb{E}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S\le\frac{1}{n}\right\}}\left(Z\left(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\right)-S\right)\right] \\
&\quad+\mathbb{E}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}}\left(Z\left(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\right)-S\right)\right] \\
&\le \frac{1}{n}+\mathbb{E}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}}\left(Z\left(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\right)-S\right)\right] \\
&\le \frac{1}{n}+\mathbb{E}\left[\big(Z\big(s;t,S+\tfrac{1}{n^2},\pi(\cdot)\big)-S\big)^2\right]^{\frac{1}{2}}\mathbb{E}\left[\mathbb{1}_{\left\{Z\left(s;t,S+\frac{1}{n^2},\pi(\cdot)\right)-S>\frac{1}{n}\right\}}\right]^{\frac{1}{2}} \\
&\le \frac{1}{n}+(F-S)\frac{\left(\tilde{\mathbb{E}}\left[D^2\right]\right)^{\frac{1}{4}}}{n^{\frac{1}{4}}}.
\end{aligned}
\tag{87}
$$

Observing that the previous inequality is uniform on $\pi(\cdot)\in\tilde{\Pi}_{ad}(t,S+\frac{1}{n^2})$ and taking into account the Lipschitz continuity of $z\mapsto(F-z)^2$, we have that (87) yields

$$
\left|H\left(t,S+\tfrac{1}{n^2}\right)-H(t,S)\right|\le C(n)\to 0 \text{ as } n\to\infty.
$$

Then the claim of this step follows.

Step 5. It remains only to prove the continuity at the boundary $[0,T]\times\{S\}$. By Step 4 we have

$$
H(t,S+\varepsilon)\uparrow H(t,S),\quad \forall t\in[0,T],\quad \text{when } \varepsilon\to 0.
$$

By Dini's Lemma we get that $H(\cdot,S+\varepsilon)\overset{\varepsilon\to 0}{\longrightarrow}H(\cdot,S)$ uniformly. This convergence is enough to obtain the claim. □

Proof of Proposition 4.13

Subsolution. This proof is standard. Let $\varphi\in C^{1,2}\big([0,T)\times(S,F);\mathbb{R}\big)$ and let $(t_M,z_M)\in[0,T)\times(S,F)$ be such that (t_M,z_M) is a local maximum point for $H-\varphi$. We can assume without loss of generality that

$$
H(t_M,z_M)=\varphi(t_M,z_M)\qquad\text{and}\qquad H(t,z)\le\varphi(t,z),\ \forall(t,z)\in[0,T)\times(S,F). \tag{88}
$$

Let $\pi\in[0,+\infty)$, set $Z(\cdot):=Z(\cdot;t_M,z_M,\pi)$, and let us define

$$
\tau^\pi=\inf\{t\ge t_M \mid (t,Z(t))\notin[0,T)\times(S,F)\}
$$

with the convention $\inf\emptyset=T$. Of course τ^π is a stopping time and by continuity of trajectories $\tau^\pi>t_M$ almost surely. By (88) we have, for any $t\in[t_M,\tau^\pi]$,

$$
H(t,Z(t))-H(t_M,z_M)\le\varphi(t,Z(t))-\varphi(t_M,z_M).
$$

Let $h\in(t_M,T]$ and set $\tau_h^\pi:=\tau^\pi\wedge h$; by Dynamic Programming Principle (32) we get, for any $\pi\in[0,+\infty)$,

$$
\begin{aligned}
0&\le\mathbb{E}\left[\int_{t_M}^{\tau_h^\pi}\kappa\,\eta(t)(F-Z(t))^2dt+H(\tau_h^\pi,Z(\tau_h^\pi))-H(t_M,z_M)\right] \\
&\le\mathbb{E}\left[\int_{t_M}^{\tau_h^\pi}\kappa\,\eta(t)(F-Z(t))^2dt+\varphi(\tau_h^\pi,Z(\tau_h^\pi))-\varphi(t_M,z_M)\right].
\end{aligned}
\tag{89}
$$

Applying Dynkin's formula to the function $\varphi(t,x)$ with the process $Z(\cdot)$, we obtain

$$\begin{aligned}&\mathbb{E}\left[\varphi(\tau_h^\pi, Z(\tau_h^\pi)) - \varphi(t_M, z_M)\right] \\ &\qquad = \mathbb{E}\left[\int_{t_M}^{\tau_h^\pi}\left[\varphi_t(t,Z(t)) + (\mu - r)\pi(t)Z(t)\varphi_z(t,Z(t)) + \frac{1}{2}\sigma^2\pi^2 Z(t)^2\varphi_{zz}(t,Z(t))\right]dt\right]\end{aligned}$$

and hence by (89) we have

$$\begin{aligned}0 \leq \mathbb{E}\Bigg[\int_{t_M}^{\tau_h^\pi}\Bigg[&\kappa e^{-\rho t}(F - Z(t))^2 dt + \varphi_t(t,X(t)) + (\mu - r)\pi(t)Z(t)\varphi_z(t,Z(t)) \\ &+ \frac{1}{2}\sigma^2\pi^2 Z(t)^2\varphi_{zz}(t,Z(t))\Bigg]dt\Bigg].\end{aligned}$$

Thus, for any $\pi \in [0,+\infty)$, we get

$$0 \leq \mathbb{E}\left[\int_{t_M}^{\tau_h^\pi}\left[\kappa\eta(t)(F - Z(t))^2 dt + \varphi_t(t,X(t)) + \mathcal{H}_{cv}(\varphi_x(t,X(t)), \varphi_{xx}(t,X(t));\pi)\right]dt\right],$$

and we can write for any $\pi \in [0,+\infty)$

$$0 \leq \mathbb{E}\left[\frac{1}{h - t_M}\int_{t_M}^{h}\mathbb{1}_{[t_M,\tau^\pi]}(t)\left[\kappa\eta(t)(F - Z(t))^2 dt + \varphi_t(t,X(t)) + \mathcal{H}_{cv}(\varphi_z(t,Z(t)), \varphi_{zz}(t,Z(t));\pi)\right]dt\right].$$

Now, passing to the limit for $h \to t_M$, by the continuity properties of φ and $\mathcal{H}_{cv}$, and by dominated convergence we have

$$-\varphi_t(t_M, z_M) - \kappa\eta(t)(F - z_M)^2 - \mathcal{H}_{cv}(\varphi_z(t_M,z_M), \varphi_{zz}(t_M,x_M);\pi) \leq 0.$$

From the arbitrariness of π we obtain that H is a subsolution on $[0,T) \times (S,F)$.

Supersolution. Let $\varphi \in C^{1,2}([0,T) \times (S,F);\mathbb{R})$ and $(t_m, z_m) \in [0,T) \times (S,F)$ be such that (t_m, z_m) is a local minimum point for $H - \varphi$. We can assume without loss of generality that

$$H(t_m, z_m) = \varphi(t_m, z_m) \qquad \text{and} \qquad H(s,x) \geq \varphi(t,z),\ \forall (t,z) \in [0,T) \times (S,F). \tag{90}$$

We must prove that

$$-\varphi_t(t_m, z_m) - \kappa\eta(t_m)(F - z_m)^2 - \mathcal{H}\left(\varphi_z(t_m,z_m), \varphi_{zz}(t_m,z_m)\right) \geq 0.$$

Let us suppose by contradiction that this relation is false. Then there exists $\nu > 0$ such that

$$-\varphi_t(t_m, z_{t_m}) - \kappa\eta(t_m)(F - z_m)^2 - \mathcal{H}\left(\varphi_z(t_m,z_m), \varphi_{zz}(t_m,z_m)\right) < -\nu < 0.$$

Setting

$$\tilde{\varphi}(t,z) = \varphi(t,z) - |z - z_m|^3,$$

we have

$$-\tilde{\varphi}_t(t_m, z_t) - \kappa\eta(t_m) - \mathcal{H}\left(\tilde{\varphi}_z(t_m,z_m), \tilde{\varphi}_{zz}(t_m,z_m)\right) < -\nu < 0,$$

and from (90) we get

$$H(t_m, z_m) = \tilde{\varphi}(t_m, z_m) \quad \text{and} \quad H(s,z) \geq \tilde{\varphi}(t,z) + |z - z_m|^3,\ \forall (t,z) \in [0,T) \times (S,F). \tag{91}$$

By continuity of $\tilde{\varphi}_t, \tilde{\varphi}_x, \tilde{\varphi}_{xx}$ and $\mathcal{H}$, there exists $\varepsilon > 0$ such that if

$$(t,z) \in B := [t_m, t_m + \varepsilon) \times (z_m - \varepsilon, z_m + \varepsilon) \subset [0,T) \times (S,F),$$

we have for every $\pi \in [0,+\infty)$

$$\begin{aligned} -\frac{\nu}{2} &\geq -\tilde{\varphi}_t(t,z) - \kappa\eta(t)(F-z)^2 - \mathcal{H}\left(\tilde{\varphi}_z(t,z), \tilde{\varphi}_{zz}(t,z)\right) \\ &\geq -\tilde{\varphi}_t(t,z) - \kappa\eta(t)(F-z)^2 - \mathcal{H}_{cv}\left(\tilde{\varphi}_z(s,z), \tilde{\varphi}_{zz}(t,z); \pi\right). \end{aligned} \tag{92}$$

Let us consider any admissible control strategy $\pi(\cdot) \in \Pi_{ad}(t_m, z_m)$ and let $Z(s) := Z(s; t_m, x_m, \pi(\cdot))$. Define the stopping time $\tau^\pi := \inf\{t \geq t_m \mid (t, Z(t)) \notin B\}$ with the convention $\inf \emptyset = T$; of course, by continuity of trajectories, $\tau^\pi > t_m$ almost surely. Now, we can apply (92) to $Z(t)$ for red any $t \in [t_m, \tau^\pi]$ getting

$$-\frac{\nu}{2} \geq -\tilde{\varphi}_t(t, Z(t)) - \kappa\eta(t)(F - Z(t))^2 - \mathcal{H}_{cv}\left(\tilde{\varphi}_z(t, Z(t)), \tilde{\varphi}_{zz}(t, Z(t)); \pi(t)\right). \tag{93}$$

Integrating (93) on $[t_m, \tau^\pi]$ and taking the expectations we obtain

$$-\frac{\nu}{2}\mathbb{E}\left[\tau^\pi - t_m\right] \geq -\mathbb{E}\left[\int_{t_m}^{\tau^\pi}\left[\tilde{\varphi}_t(t, X(t)) + \kappa\eta(t)(F - Z(t))^2 + \mathcal{H}_{cv}\left(\tilde{\varphi}_z(t, Z(t)), \tilde{\varphi}_{zz}(t, Z(t)); \pi(t)\right)\right] dt\right].$$

Applying Dynkin's formula to the function $\tilde{\varphi}(t,x)$ with the process $X(\cdot)$ on $[t_m, \tau^\pi]$ we have

$$\tilde{\varphi}(t_m, z_m) - \mathbb{E}\left[\tilde{\varphi}(\tau^\pi, Z(\tau^\pi))\right] \leq -\frac{\nu}{2}\mathbb{E}\left[\tau^\pi - t_m\right] + E\left[\int_{t_m}^{\tau^\pi} e^{-\rho t}(F - Z(t))^2\, dt\right],$$

and from (91) we get

$$\begin{aligned} &H(t_m, z_m) - \mathbb{E}\left[H(\tau^\pi, Z(\tau^\pi))\right] \\ &\qquad \leq -\frac{\nu}{2}\mathbb{E}\left[\tau^\pi - t_m\right] - \mathbb{E}\left[|Z(\tau^\pi) - z_m|^3\right] + \mathbb{E}\left[\int_{t_m}^{\tau^\pi} \kappa\eta(t)(F - Z(t))^2\, dt\right]. \end{aligned} \tag{94}$$

Now notice that

$$-\frac{\nu}{2}\mathbb{E}\left[\tau^\pi - t_m\right] - \mathbb{E}\left[|Z(\tau^\pi) - x_m|^3\right] \leq \left(-\frac{\nu\varepsilon}{2}\right) \vee (-\varepsilon^3) := -\delta < 0;$$

hence from (94) we obtain

$$H(t_m, z_m) \leq -\delta + \mathbb{E}\left[\int_{t_m}^{\tau^\pi} \eta(t)(F - Z(t))^2\, dt + H(\tau^\pi, Z(\tau^\pi))\right].$$

The arbitrariness of $\pi(\cdot) \in \Pi_{ad}(t_m, z_m)$ in the argument leads to contradict the dynamic programming principle (32), so we have proved that W is a supersolution on $[0,T) \times (S,F)$. □

Proof of Proposition 5.5

We prove the claim in several steps.

Step 1. First of all we notice that it is equivalent to prove the claim for the function

$$M(t,z) := -\frac{\sigma}{\beta}e^{r(T-t)}G(t,z) = \left[g(t,\cdot)\right]^{-1}(z)\, g_y\left(t, \left[g(t,\cdot)\right]^{-1}(z)\right), \quad (t,z) \in [0,T) \times [S,F].$$

Step 2. Since $M \in C^{1,2}([0,t_0] \times (S,F);\mathbb{R})$ $t_0 \in [0,T)$, it follows that M is Lipschitz continuous on every compact set contained in $[0,t_0] \times (S,F)$. This implies, in particular, that M is α- Hölder, $\alpha \in (0,1)$, on every compact set contained in $[0,t_0] \times (S,F)$.

Step 3. Here we prove that $M(t,\cdot)$ is continuous on $[S,F]$ (therefore on $\mathbb{R}$) for every $t \in [0,t_0]$. Thanks to Step 2 and to the definition of $M(t,\cdot)$ on $(-\infty,S] \cup [F,+\infty)$, we need to prove the claim only at the endpoint S from the right and at the endpoint F from the left. This is equivalent to prove that for every $t \in [0,t_0]$

$$\lim_{z \to S^+} M(t,z) = 0, \qquad \lim_{z \to F^-} M(t,z) = 0,$$

i.e.

$$\lim_{y \to +\infty} y g_y(t,y) = 0, \qquad \lim_{y \to 0^+} y g_y(t,y) = 0.$$

The limits above are true by straightforward computations.

Step 4. Here we prove that $M(t,\cdot)$ is Lipschitz continuous on $[F-\varepsilon,F]$, for some $\varepsilon > 0$, uniformly with respect to $t \in [0,t_0]$. To this aim, it suffices to show that $M_z(s,\cdot)$ is bounded on $[0,t_0] \times [F-\varepsilon,F]$ for some $\varepsilon > 0$. We have

$$M_z(t,z) = 1 + \frac{[g(t,\cdot)]^{-1}(z) \cdot g_{yy}\left(t,[g(t,\cdot)]^{-1}(z)\right)}{g_y\left(t,[g(t,\cdot)]^{-1}(z)\right)}.$$

Therefore we study the limit for $z \to F^-$ of $M_z(t,\cdot)$, or equivalently

$$\lim_{y \to 0^+} \frac{y g_{yy}(t,y)}{g_y(t,y)}.$$

Straightforward computations show that

$$\lim_{y \to 0^+} \frac{y g_{yy}(t,y)}{g_y(t,y)} = 0$$

uniformly in $t \in [0,t_0]$, which is enough to get the claim of this step.

Step 5. Here we prove that, for every $\alpha \in (0,1)$, the map $M(t,\cdot)$ is α-Hölder continuous on $[S,S+\varepsilon]$, for some $\varepsilon > 0$, uniformly with respect to $t \in [0,t_0]$, but not Lipschitz continuous (which corresponds to consider $\alpha = 1$ in the following computations). The argument below holds uniformly with respect to $t \in [0,t_0]$, so here we suppress without loss of generality t as argument. Also we assume without loss of generality $S = 0$.

We must show that $M_z(z) \cdot z^{1-\alpha}$ is bounded on $[0,\varepsilon]$ for some $\varepsilon > 0$. Therefore, we are led to study the limit for $z \to 0^+$ of $M_z(z) \cdot z^{1-\alpha}$, or equivalently

$$\lim_{y \to +\infty} \frac{y g_{yy}(y)}{g_y(y)} g(y)^{1-\alpha}.$$

We call

$$h(y) := h(t,y) := k(t,y) - \beta\sqrt{T-t}.$$

By computation of g_y and g_{yy} we have

$$\frac{yg_{yy}}{g_y}g^{1-\alpha}=$$
$$=\frac{\left[\frac{F}{y}e^{-\frac{k^2(\cdot)}{2}}-\frac{Fk(\cdot)}{y\beta\sqrt{T-t}}e^{-\frac{k^2(\cdot)}{2}}+\frac{h(\cdot)}{\beta\sqrt{T-t}}e^{\beta^2(T-t)-\frac{h^2(\cdot)}{2}}+\frac{1}{2}e^{\beta^2(T-t)+\rho T-\frac{h^2(\cdot)}{2}}\right]\left[F\Phi(k(\cdot))-\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha}}{-\frac{F}{y}e^{-\frac{k^2(\cdot)}{2}}+e^{\beta^2(T-t)-\frac{h^2(\cdot)}{2}}-\frac{1}{2}\beta\sqrt{2\pi(T-t)}e^{\beta^2(T-t)+\rho T}\phi(h(\cdot))} \tag{95}$$

On the other hand, after some algebra, it can be shown that

$$e^{-\frac{h^2(\cdot)}{2}}=K_0ye^{-\frac{k^2(\cdot)}{2}}, \tag{96}$$

where $K_0>0$ is some constant that does not depend on y. Therefore, after plugging (96) into (95) and simplifying by $e^{-\frac{h^2(\cdot)}{2}}$, we get

$$\frac{yg_{yy}}{g_y}g^{1-\alpha}=\frac{\left[\frac{FK_1}{y}-\frac{Fk(\cdot)K_1}{y^2\beta\sqrt{T-t}}+\frac{h(\cdot)}{\beta\sqrt{T-t}}e^{\beta^2(T-t)}+\frac{1}{2}e^{\beta^2(T-t)+\rho T}\right]\left[F\Phi(k(\cdot))-\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha}}{\left[-\frac{FK_1}{y^2}+e^{\beta^2(T-t)}-\frac{1}{2}\beta\sqrt{2\pi(T-t)}e^{\beta^2(T-t)+\rho T}\phi(h(\cdot))e^{\frac{h^2(\cdot)}{2}}\right]}, \tag{97}$$

where $K_1=\frac{1}{K_0}$. In order to show that $M(t,\cdot)$ is not Lipschitz, we need to compute the limit of (97) for $y\to+\infty$ in the case $\alpha=1$. To this end, notice that

$$\lim_{y\to+\infty}\frac{\phi(h(y))}{e^{-\frac{h^2(y)}{2}}}=\lim_{y\to+\infty}\frac{h'(y)e^{-\frac{h^2(y)}{2}}}{-\sqrt{2\pi}h'(y)h(y)e^{-\frac{h^2(y)}{2}}}=-\lim_{y\to+\infty}\frac{1}{\sqrt{2\pi}h(y)}=0,$$

where in the first equality we have used de L'Hôpital rule. Then

$$\lim_{y\to+\infty}\frac{yg_{yy}}{g_y}=\frac{1}{e^{\beta^2(T-t)}}\lim_{y\to+\infty}\left[\frac{FK_1}{y}-\frac{Fk(\cdot)K_1}{y^2\beta\sqrt{T-t}}+\frac{h(\cdot)}{\beta\sqrt{T-t}}e^{\beta^2(T-t)}+\frac{1}{2}e^{\beta^2(T-t)+\rho T}\right]=-\infty.$$

This shows that $M(t,\cdot)$ is not Lipschitz.

In order to show that $M(t,\cdot)$ is α-Hölder for $\alpha\in(0,1)$, we need to compute the limit of (97) for $y\to+\infty$. Observe that

$$\lim_{y\to+\infty}\left[F\Phi(k(\cdot))-\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha}=0$$

and

$$\lim_{y\to+\infty}\left[\frac{FK_1}{y}-\frac{Fk(\cdot)K_1}{y^2\beta\sqrt{T-t}}+\frac{h(\cdot)}{\beta\sqrt{T-t}}e^{\beta^2(T-t)}+\frac{1}{2}e^{\beta^2(T-t)+\rho T}\right]=\lim_{y\to+\infty}\frac{h(\cdot)}{\beta\sqrt{T-t}}e^{\beta^2(T-t)}=-\infty.$$

Then it suffices to show the boundness of the limit

$$\lim_{y\to+\infty}h(\cdot)\left[F\Phi(k(\cdot))-\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha} \tag{98}$$

to get the claim. The limit (98) is indeed 0. To show it, let us call $\lambda := \frac{1}{1-\alpha}$. Clearly, $\lambda \in (1, +\infty)$. Then

$$\begin{aligned}
&\lim_{y\to+\infty} h(\cdot)\left[F\Phi(k(\cdot)) - \frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha} = \\
&\quad = \lim_{y\to+\infty}\left(-\rho T + \ln 2F - \frac{3}{2}\beta^2(T-t) - \ln y\right)\left[F\Phi(k(\cdot)) - \frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha} = \\
&\quad = \lim_{y\to+\infty}(-\ln y)\left[F\Phi(k(\cdot)) - \frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha} = \qquad (99)\\
&\quad = -\lim_{y\to+\infty}\left[(\ln y)^\lambda\left(F\Phi(k(\cdot)) - \frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right)\right]^{1-\alpha} = \\
&\quad = -\left(\lim_{y\to+\infty}F\Phi(k(\cdot))(\ln y)^\lambda - \lim_{y\to+\infty}\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))(\ln y)^\lambda\right)^{1-\alpha} = -\left(L_1 - L_2\right)^{1-\alpha},
\end{aligned}$$

where

$$L_1 := \lim_{y\to+\infty} F\Phi(k(\cdot))(\ln y)^\lambda$$

and

$$L_2 := \lim_{y\to+\infty}\frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))(\ln y)^\lambda.$$

Since it can be easily shown that

$$\lim_{y\to+\infty}\frac{2F\Phi(k(\cdot))}{ye^{\beta^2(T-t)}\Phi(h(\cdot))} = 0,$$

one can see that the relationship between the two limits is

$$0 \le L_1 \le L_2. \qquad (100)$$

This allows us to compute only the limit L_2. Observing that

$$\ln y \le y \implies (\ln y)^\lambda \le y^\lambda,$$

we have

$$\begin{aligned}
&0 \le \lim_{y\to+\infty} y\Phi(h(\cdot))(\ln y)^\lambda \le \lim_{y\to+\infty} y^{\lambda+1}\Phi(h(\cdot)) = \lim_{y\to+\infty}\frac{\Phi(h(\cdot))}{y^{-(\lambda+1)}} = \lim_{y\to+\infty}\frac{h'(y)e^{-\frac{h^2(\cdot)}{2}}}{\sqrt{2\pi}(-\lambda-1)y^{-\lambda-2}} = \\
&= K_2\lim_{y\to+\infty}\frac{y^{\lambda+1}}{e^{\frac{h^2(\cdot)}{2}}} = K_2\lim_{y\to+\infty}\frac{y^{\lambda+1}}{K_3\left(e^{(\ln y)^2 - 2K_4\ln y}\right)^{\frac{1}{2\beta^2(T-t)}}} = \frac{K_2}{K_3}\left(\lim_{y\to+\infty}\frac{y^{2K_4+2\beta^2(T-t)(\lambda+1)}}{y^{\ln y}}\right)^{\frac{1}{2\beta^2(T-t)}},
\end{aligned}$$

where in the second equality we have used de L'Hôpital rule and where K_i, for $i = 2, 3, 4$, are constants depending at most on t, and $K_2 > 0, K_3 > 0$.

For $\alpha < 1$ we have $\lambda < +\infty$, implying that for $y \to +\infty$

$$2K_4 + 2\beta^2(T-t)(\lambda+1) - \ln y \to -\infty.$$

This is enough to show that

$$\lim_{y\to+\infty} y^{\lambda+1}\Phi(h(\cdot)) = 0.$$

Thus, due to (100) and (99), we have

$$\lim_{y\to+\infty} h(\cdot)\left[F\Phi(k(\cdot)) - \frac{y}{2}e^{\beta^2(T-t)}\Phi(h(\cdot))\right]^{1-\alpha} = 0,$$

hence the claim. □