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par

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Infinite dimensional stochastic calculus via regularization with financial motivations

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Titolo: Calcolo stocastico via regolarizzazione in dimensione infinita e motivazioni finanziarie.

Riassunto: Questa tesi di dottorato sviluppa certi aspetti del calcolo stocastico via regolarizzazione per dei processi a valori in uno spazio di Banach generale B. Viene introdotto un concetto orginale di $\chi-$variazione quadratica, dove χ è un sottospazio del duale de prodotto tensoriale $B \otimes B$, munito della topologia proiettiva. Una attenzione particolare é dedicata al caso in cui B é lo spazio della funzioni continue su l'intervallo $[-\tau, 0]$, $\tau > 0$. Viene dimostrata una classe di risultati di stabilità attraverso funzioni di classe C^1 di processi che ammettono una χ-variazione quadratica e viene dimostrata una formula di Itô per tali processi. I processi continui reali a variazione quadratica finita Y (ad esempio processi di Dirichelt o anche Dirichlet debole) giocano un ruolo significativo. Viene definito un processo associato chiamato processo *finestra* e indicato con $Y_t(\cdot)$ definito da $Y_t(y) = Y_{t+y}$ per $y \in [-\tau, 0]$. $Y(\cdot)$ è un processo a valori nello spazio di Banach $C[-\tau, 0]$. Se Y è un processo reale con varazione quadratica uguale a $[Y]_t = t$ e $h = F(Y_T(\cdot))$ dove F è una funzione di classe $C^2(H)$ Fréchet e $H = L^2([-T, 0])$, è possibile rappresentare h come somma di un numero reale H_0 più un integrale forward di tipo $\int_0^T \xi d^- Y$ dove ξ è un processo di cui diamo la forma esplicita. Questo generalizza la formula di Clark-Ocone valida quando Y è un moto Browniano standard W. Una delle motivazioni viene dalla teoria di copertura di opzioni che dipendono da tutta la traiettoria del sottostante o quando il prezzo dell'azione sottostante non è una semimartingala.

Titre: Calcul stochastique via régularisation en dimension infinie avec motivations financières.

Résumé: Ce document de thèse développe certains aspects du calcul stochastique via régularisation pour des processus à valeurs dans un espace de Banach général B. Il introduit un concept original de χ-variation quadratique, où χ est un sous-espace du dual d'un produit tensioriel $B \otimes B$, muni de la topologie projective. Une attention particulière est dévouée au cas où B est l'espace des fonctions continues sur $[-\tau, 0]$, $\tau > 0$. Une classe de résultats de stabilité de classe C^1 pour des processus ayant une χ-variation quadratique est établie ainsi que des formules d'Itô pour de tels processus. Un rôle significatif est joué par les processus réels à variation quadratique finie Y (par exemple un processus de Dirichlet, faible Dirichlet). Le processus naturel à valeurs dans $C[-\tau, 0]$ est le dénommé processus *fenêtre* $Y_t(\cdot)$ où $Y_t(y) = Y_{t+y}$. Si Y est un processus dont la variation quadratique vaut $[Y]_t = t$ et $h = F(Y_T(\cdot))$ où F est une fonction de classe $C^2(H)$ Fréchet where $H = L^2([-T, 0]$, il est possible de représenter h comme un nombre réel H_0 plus une intégrale progressive du type $\int_0^T \xi d^- Y$ o ξ est un processus donné explicitement. A certains égards, ceci généralise la formule de Clark-Ocone valide lorsque Y est un mouvement brownien standard W. Une des motivations vient de la théorie de la couverture d'options lorsque le prix de l'actif soujacent n'est pas une semimartingale.

Title: Infinite dimensional calculus via regularization with financial motivations.

Abstract: This paper develops some aspects of stochastic calculus via regularization to Banach valued processes. An original concept of χ-quadratic variation is introduced, where χ is a subspace of the dual of a tensor product $B \otimes B$ where B is the value space of the process. Particular interest is devoted to the case when B is the space of real continuous functions defined on $[-\tau, 0]$, $\tau > 0$. Itô formulae and stability of finite χ-quadratic variation processes are established. Attention is devoted to a finite real quadratic variation (for instance Dirichlet, weak Dirichlet) process X. The $C([-\tau, 0])$-valued process $Y(\cdot)$ defined by $Y_t(y) = Y_{t+y}$ where $y \in [-\tau, 0]$ is called *window* process. Let $T > 0$. If Y is a finite quadratic variation process such that $[Y]_t = t$ and $h = F(Y_T(\cdot))$ where F is a $C^2(H)$ Fréchet function with $H = L^2([-T, 0])$, it is possible to represent h as a sum of a real number H_0 plus a forward integral of type $\int_0^T \xi d^- Y$ where ξ will be explicitly given. This decomposition generalizes the Clark-Ocone formula which is true when Y is the standard Brownian motion W. The main motivation comes hedging theory of path dependent options without semimartingales in mathematical finance.

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Contents

Chapter 1

Introduction

Classical stochastic calculus and integration come back at least to Itô [27] and it has been developed successfully by a huge number of authors. The most classical Itô's integrator is Brownian motion but the theory naturally extends to martingales and semimartingales. Stochastic integration with respect to semimartingales is now quite established and performing. For that topic, there are also many monographs, among them [30], [37] for continuous integrators and [29] and [36] for jump processes. In order to describe models coming especially from physics and biology, useful tools are infinite dimensional stochastic differential equation for which the classical stochastic integrals needed to be generalized. Those integrals involve Banach valued stochastic processes. At our knowledge the seminal book is [33], which generalizes stochastic integrals and Itô formulae, in a general framework, to a class of integrators called π-processes. Let B be a Banach space and X a B-valued continuous process. Let Y be an elementary B^*-valued process i.e. it is a finite sum of functions of the type $c\mathbb{1}_{]a,b]}$, where $a < b$ and c is a non-anticipating B^*-valued random variable. The integral $\int_0^T \langle c\mathbb{1}_{]a,b]}, dX\rangle$ can be obviously defined by $\langle c, X_b - X_a\rangle$. The integral $\int_0^T \langle Y, dX\rangle$ can be deduced by linearity. If X is a so-called π-process and Y is an elementary process then the following inequality holds $\mathbb{E}\left[\int_0^T \langle Y, dX\rangle\right]^2 \leq \mathbb{E}\left[\int_0^T \|Y\|^2 d\alpha\right]$, where α is a suitable measure on predictable sets. In other words for a π-process X it is possible to write a generalization of the isometry property of real valued Itô integrals. If the Banach values space B is a Hilbert space then the concept of π-process generalizes the notion of square integrable martingale and bounded variation process. The infinite dimensional stochastic integration theory has known a big success in applications to different classes of (stochastic) partial differential equations. It concerned especially the case when B is a separable Hilbert space. The most frequent tools are the Da Prato-Zabczyk integral (see [11]) and Walsh integral ([52]). A recent book completing the Metivier-Pellaumail approach is the [16]. A significant theory of infinite dimensional stochastic integration was developed when B is an M-type 2 Banach spaces, see [13, 12] and continued by several authors as e.g. [5], [1]. Interesting issues in this direction concern the case when B

is a UMD space; one recent paper in this direction is [50]. A space which is neither a M-type 2 space nor a UMD space is $C([-\tau, 0])$ with $\tau > 0$, i.e. the Banach space of real continuous functions defined on $[-\tau, 0]$. This is the typical space in which stochastic integration is challenging. This context is natural when studying stochastic differential equations with functional dependence (as for instance delay equations). Due to the difficulty of stochastic integration and calculus in that space, most of the authors fit the problem in some ad hoc Hilbert space, see for instance [7]. A step in the investigation of stochastic integration for $C([-\tau, 0])$-valued and associated processes was done by [51].

The literature of stochastic integrals via regularizations and calculus concerns essentially real valued (and in some cases $\mathbb{R}^n$-valued) processes. This topic was studied first in [40] and [41, 42]. A recent survey on the subject is [44]. Important investigations in the case of jump integrators were performed by [14]. Given an integrand process $Y = (Y_t)_{t\in[0,T]}$ and an integrator $X = (X_t)_{t\in[0,T]}$, a significant notion is the forward integral of Y with respect to X, denoted by $\int_0^T Y d^- X$. When X is a (continuous) semimartingale and Y is a cadlag adapted process, that integral coincides with Itô's integral $\int_0^T Y dX$. Stochastic calculus via regularization is a theory which allows, in many specific cases to manipulate those integrals when Y is anticipating or X is not a semimartingale. If $X = W$ is a Brownian motion and Y is a (possibly anticipating) process with some Malliavin differentiability, then $\int_0^T Y d^- W$ equals Skorohod integral $\int_0^T Y \delta W$ plus of a trace term. A version of this calculus when B has infinite dimension was not yet developed. The aim of the present work is to set up the basis of such a calculus with values on Banach spaces in the (simplified) case when integrals are real valued. The central object is a forward integral of the type $\int_0^T \langle Y, d^- X\rangle$, when Y (resp. X) is a B^*-valued (resp. B-valued) process. We show that when $B = X$ is a Hilbert space, Y is a non-anticipating square integrable process and X is a Wiener process, $\int_0^T \langle Y, d^- X\rangle$ coincides with the Da Prato-Zabczyk integral.
One important object in calculus via regularization is the notion of the covariation $[X, Y]$ of two real processes X and Y. If $X = Y$, $[X, X]$ is called the so-called quadratic variation of X. If X is $\mathbb{R}^n$-valued process with components $X^1, \dots, X^n$, the generalization of the notion of quadratic variation $[X, X]$ is provided by the matrix $([X^i, X^j])_{i,j=1,\dots,n}$. If such a matrix indeed exists, one also says that X admits all its mutual covariations.
In this paper we introduce a sophisticated notion of quadratic variation which generalizes the former one. This is called χ-quadratic variation in reference to a subspace χ of the dual of $B\hat{\otimes}_\pi B$. When B is finite dimensional, if X admits all its mutual brackets, then X has a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$. A Banach valued *locally semi summable* process X in the sense of [16], has again a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$. We establish a general Itô's formula; we also show that if X has a χ-quadratic variation and $F : B \to \mathbb{R}$ is of class C^1 Fréchet with some supplementary properties on DF than $F(X)$ is a real finite quadratic variation process.
A specific attention is devoted to the case when $B = C([-\tau, 0])$ and X is a *window* process associated to a

real continuous process. Given $\tau > 0$ and a $(\mathcal{F}_t)$-classical real Brownian motion $W = (W_t)_{t\in[0,T]}$, we will call window Brownian motion the $C([-\tau,0])$-valued process $W(\cdot) = \big(W_t(\cdot)\big)_{t\in[0,T]} = \{W_t(u) := W_{t+u}; u \in [-\tau,0], t \in [0,T]\}$. $C([-\tau,0])$ is typical a non-reflexive Banach space. We obtain generalized Doob-Meyer decomposition for C^1-functionals of window Dirichlet processes. Motivated by financial applications, we finally establish a Clark-Ocone type decomposition for a class of random variables h depending on the paths of a finite quadratic variation process Y such that $[Y]_t = t$. This chapter is motivated by the hedging problem of path-dependent options in mathematical finance. If the noise is modeled by (the derivative of a) Brownian motion W, the classical martingale representation theorem and classical Clark-Ocone formula is a useful tool for finding a portfolio hedging strategy. One of our results consists in expressing a random variable $h = H(Y(\cdot))$, where H has some Fréchet regularity, as $h = H_0 + \int_0^T \xi_s d^- Y_s$ where H_0 is a real number and ξ is an non-anticipating process which are explicitly given. Previous formula extends Clark-Ocone formula to the case when Y is no longer a Brownian motion but it has the same quadratic variation. This generalizes some results included in [47, 3, 9] concerning the hedging of vanilla or Asiatic type options.

The paper is organised as follows. After this introduction, chapter 2 contains preliminary notations with some remarks on classical Dirichlet processes and Malliavin calculus and basic notions on tensor products analysis. In chapter 3, we define the integral via regularization for infinite dimension Banach valued processes and we establish link with notion of Da Prato-Zabczyk's stochastic integral. chapter 4 will be devoted to the definition of χ-quadratic variation and some related results and in chapter 5, we will evaluate the χ-quadratic variation for different classes of processes. In chapter 6, we give the definition of χ-covariation and we establish C^1 stability properties and some basic facts about weak Dirichlet processes and to the Fukushima-Dirichlet decomposition of process $F(t, D_t(\cdot))$ with a sufficient condition to guarantee that the resulting process is a true Dirichlet process. In chapter 7 we verify a C^2-Fréchet type Itô's formula. The final chapter 8 is devoted to the Clark-Ocone type formula.

Chapter 2

Preliminaries

2.1 General notations

In this chapter we recall some definitions and notations concerning the whole paper. Let A and B be two general sets such that $A \subset B$, $\mathbb{1}_A : B \to \{0,1\}$ will denote the indicator function of the set A, i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ if $x \notin A$. Let $\mathcal{R}$ be a sentence that may be true or not, in this case $\mathbb{1}_{\mathcal{R}} = 1$ if $\mathcal{R}$ is fulfilled and $\mathbb{1}_{\mathcal{R}} = 0$ if $\mathcal{R}$ is not fulfilled. It holds $\mathbb{1}_A(x) = \mathbb{1}_{\{x \in A\}}$. If m, n are positive natural numbers, we will denote by $\mathbb{M}_{m\times n}(\mathbb{R})$ the space of real valued matrix of dimension $m \times n$. When $m = n$, this is the space of squared real valued matrix $n \times n$, denoted simply by $\mathbb{M}_n(\mathbb{R})$. If $n = 1$, $\mathbb{M}_{m\times 1}(\mathbb{R})$ will be identified with $\mathbb{R}^m$, analogously if $n = 1$.
Throughout this paper we will denote by $(\Omega, \mathcal{F}, P)$ a fixed filtered probability space where $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ stands for a given filtration $(\mathcal{F}_t; 0 \leq t \leq T)$ fulfilling the usual conditions. Let $a < b$ be two real numbers, $C([a, b])$ will denote the Banach linear space of real continuous functions equipped with the uniform norm and $C_0([a, b])$ will denote the space of real continuous functions f on $[a, b]$ such that $f(a) = 0$. The letters B, E, F, G (respectively H) will denote Banach (respectively Hilbert) spaces over the scalar field $\mathbb{R}$. Given two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E, we say that $\|\cdot\|_1 \leq \|\cdot\|_2$ if for every $x \in E$ there is a positive constant c such that $\|x\|_1 \leq c\,\|x\|_2$. We say that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if they define the same topology, i.e. if there exist positive real numbers c and C such that $c\,\|x\|_2 \leq \|x\|_1 \leq C\,\|x\|_2$ for all $x \in E$.
The space of bounded linear mappings from E to F will be denoted by $L(E; F)$ and the topological dual space of B by B^*. If ϕ is a linear functional on B, we shall denote the value of ϕ at an element $b \in B$ either by $\phi(b)$ or $\langle \phi, b\rangle$. Throughout the paper the symbols $\langle \cdot, \cdot \rangle$ will denote always some type of duality that will change depending on the context. Let K be a compact space, $\mathcal{M}(K)$ will denote the dual space $C(K)^*$, i.e. the so-called set of finite signed measures on K. We will say that two positive (or signed or complex) measures μ and ν defined on a measurable space (Ω, Σ) are *singular* if there exist two disjoint

sets A and B in Σ whose union is Ω such that μ is zero on all measurable subsets of B while ν is zero on all measurable subsets of A. This will be denoted by $\mu \perp \nu$. Given a Banach space B and its topological bidual space B^{**} the application $J : B \to B^{**}$ will denote the natural injection between a Banach space and its bidual. J is an isometry with respect to the topology defined by the norm in B and $J(B)$ is weak star dense in B^{**}. For more informations about Banach spaces topologies, see [4, 54]. Let E, F, G be Banach spaces, we shall denote the space of G-valued bounded bilinear forms on the product $E \times F$ by $\mathcal{B}(E \times F; G)$ with the norm given by $\|\phi\| = \sup\{\|\phi(e, f)\|_G : \|e\|_E \leq 1; \|f\|_F \leq 1\}$. The letters X, Y, Z will denote Banach valued continuous processes indexed to time variable $t \in [0, T]$ with $T > 0$ (or $t \in \mathbb{R}$). A stochastic process X will be also denoted by $(X_t)_{t\in[0,T]}$ or $\{X_t; t \in [0, T]\}$. A B-valued stochastic process X is a map $X : \omega \times [0, T] \to B$ which will be always supposed to be measurable w.r.t. the product sigma-algebra. All the processes indexed by $[0, T]$ (respectively $\mathbb{R}^+$) will be naturally prolongated by continuity setting $X_t = X_0$ for $t \leq 0$ and $X_t = X_T$ for $t \geq T$ (respectively $X_t = X_0$ for $t \leq 0$). A sequence of continuous B-valued processes $(X^n)_{n\in\mathbb{N}}$ will be said to converge *ucp* (*uniformly convergence in probability*) to a process X if $\sup_{0\leq t\leq T} \|X^n - X\|_B$ converges to zero in probability when $n \to \infty$. The space $\mathcal{C}([0, T])$ will denote the linear space of continuous real processes equipped with the ucp topology and the metric $d(X, Y) = \mathbb{E}\left[\sup_{t\in[0,T]} |X_t - Y_t| \wedge 1\right]$. The space $\mathcal{C}([0, T])$ is not a Banach space but equipped with this metric is a Fréchet space (or F-space shortly) see Definition II.1.10 in [17]. For more details about F-spaces and their properties see chapter II.1 in [17].
We recall Lemma 3.1 from [43]. The mentioned lemma states that a sequence of continuous increasing processes converging at each time in probability to a continuous process, converges ucp.

Lemma 2.1. Let $(Z^\epsilon)_{\epsilon>0}$ be a family of continuous processes. We suppose
1) $\forall \epsilon > 0$, $t \to Z^\epsilon_t$ is increasing.
2) There is a continuous process $(Z_t)_{t>0}$ such that $Z^\epsilon_t \to Z_t$ in probability when ϵ goes to zero.
Then Z^ε converges to Z ucp.

We go on with our notations.
If X is a real continuous process indexed by $[0, T]$ and $0 < \tau \leq T$, we will call X *window process* the $C([-\tau, 0])$-valued process denoted by $(X_t(\cdot))_{t\in[0,T]}$ defined setting

$$(X_t(\cdot))_{t\in[0,T]} = (X_t(u) := X_{t+u}; u \in [-\tau, 0])_{t\in[0,T]} .$$

The symbols $X(\cdot)$ or $\{X_t(\cdot); t \in [0, T]\}$ will denote always the $(X_t(\cdot))_{t\in[0,T]}$ window process.

2.2 The forward integral for real valued processes

We will follow here a framework of calculus via regularizations started in [41]. At the moment many authors have contributed to this and we suggest the reader consult the recent fairly survey paper [44] on it. We first recall basic concepts and some one dimensional results about calculus via regularization. For simplicity, all the processes considered, except if stated otherwise, will be continuous processes. For two real valued processes X and Y we define the forward integral and the covariation as follows

$$\int_0^t X_r d^- Y_r = \lim_{\epsilon\to 0}\int_0^t X_r \frac{Y_{r+\epsilon}-Y_r}{\epsilon}dr \tag{2.1}$$

$$[X,Y]_t = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t (X_{r+\epsilon}-X_r)(Y_{r+\epsilon}-Y_r)dr \tag{2.2}$$

if those quantities exist in the sense of ucp with respect to t. This ensures that the forward integral defined in (2.1) and the covariation process defined in (2.2) are continuous processes. It can be seen that the covariation is a bilinear and symmetric operator. If $(X^1,\ldots,X^n)$ is a vector of continuous processes we say that it has all its mutual covariations (brackets) if $[X^i,X^j]$ exists for any $1\le i,j\le n$. If $X^1,\ldots,X^n$ have all their mutual covariations then by polarization (i.e. writing a bilinear form as a sum/difference of quadratic forms) we know that $[X^i,X^j]$ are locally bounded variation processes for $1\le i,j\le n$.

Lemma 2.2. Let $(X^1,\ldots,X^n)$ be a vector of continuous processes such that

$$\frac{1}{\epsilon}\int_0^t (X^i_{s+\epsilon}-X^i_s)(X^j_{s+\epsilon}-X^j_s)ds \tag{2.3}$$

converges in probability for every $1\le i,j\le n$ to some continuous process. Then $[X^i,X^j]$ exists for every $1\le i,j\le n$.

Proof. Let i, j be fixed. By bilinearity we can write

$$\frac{1}{\epsilon}\int_0^t \left[(X^i_{s+\epsilon}+X^j_{s+\epsilon})-(X^i_s+X^j_s)\right]^2 ds$$

which converges in probability for every t. By Lemma 2.1, it converges ucp. Again by bilinearity, it follows the result. □

Definition 2.3. If $[X,X]$ exists, even denoted by $[X]$, then X is said to be a **finite quadratic variation** process, $[X]$ is called the quadratic variation of X. We convene that

$$[X]_t = 0 \qquad \text{for } t<0. \tag{2.4}$$

If $[X]=0$, then X is said to be a zero quadratic variation process.

A bounded variation process is a zero quadratic variation process. If S^1, S^2 are $(\mathcal{F}_t)$-semimartingale then $[S^1,S^2]$ coincides with the classical bracket $\langle S^1,S^2\rangle$.

Remark 2.4.
1. Let S be an $(\mathcal{F}_t)$-continuous semimartingale (resp. Brownian motion), (Y_t) be an adapted cadlag (resp. such that $\int_0^T Y_r^2 dr<\infty$). Then $\int_0^{\cdot} Y_r d^- S_r$ exists and equals the classical Itô integral $\int_0^{\cdot} Y_r dS_r$, see chapter 3.5 in [44].

2. Let X (respectively Y) be a finite (respectively zero) quadratic variation process. Then (X, Y) has all its mutual covariations and $[X, Y] = 0$.

Definition 2.5. Let X and Y two real continuous processes. We call **covariation structure of** X the field $(u, v) \mapsto [X_{u+\cdot}, X_{v+\cdot}]$ whenever it exists for all $u, v \in \mathbb{R}$. We call **covariation structure of** X **and** Y the field $(u, v) \mapsto [X_{u+\cdot}, Y_{v+\cdot}]$ whenever it exists for all $u, v \in \mathbb{R}$.

An important fact about the covariation structure of semimartingale is the following.

Proposition 2.6. Let X and Y be two $(\mathcal{F}_t)$-continuous semimartingales. Then X admits a covariation structure such that $[X_{u+\cdot}, Y_{v+\cdot}] = 0$ for $u \neq v$.

2.3 Notations about processes

We introduce now some continuous processes that will appear in the paper.
W (respectively B^H and $B^{H,K}$) will denote a real $(\mathcal{F}_t)$ Brownian motion (resp. a fractional Brownian motion of Hurst parameter $H \in (0, 1]$ and a bifractional Brownian motion of parameters $H \in (0, 1)$ and $K \in (0, 1]$). The bifractional Brownian motion was introduced by Houdré and Villa in [25] and investigated by Russo and Tudor in [39]. In particular, [39] shows that the bifractional Brownian motion behaves similarly to a fractional Brownian motion with Hurst parameter HK and developed a related stochastic calculus. Other properties were established by [31] and [22].
X will be a real $(\mathcal{F}_t)$-semimartingale if X admits a decomposition $X = M + V$ where M is a $\mathcal{F}$-local square integrable integral, V is a locally bounded variation process and $V_0 = 0$.
D will be a real continuous $(\mathcal{F}_t)$-Dirichlet process if D admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$-local martingale and A is a zero quadratic variation process. The decomposition is unique if we require for instance $A_0 = 0$, see for instance [45]. A Dirichlet process is in particular a finite quadratic variation process. An $(\mathcal{F}_t)$-semimartingale is also an $(\mathcal{F}_t)$-Dirichlet process, a locally bounded variation process is in fact a zero quadratic variation process.
The concept of Dirichlet process can be weakened. We will make use of an extension of such processes, called weak Dirichlet processes, introduced in parallel in [20] and implicitly in [21]. Recent developments concerning the subject appear in [8, 10, 48]. Weak Dirichlet processes are not Dirichlet processes but they preserve a sort of orthogonal decomposition.
D will be a $(\mathcal{F}_t)$-weak Dirichlet process if D admits a decomposition $D = M + A$ where M is an $(\mathcal{F}_t)$ local martingale and A is a process such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$ local martingale N. For convenience, we will always suppose $A_0 = 0$. A will be said to be an $(\mathcal{F}_t)$**-martingale orthogonal process**. The decomposition is unique, see for instance Remark 3.5 in [24]. [10] made the following observation. If the underlying filtration $(\mathcal{F}_t)$ is the natural filtration associated with a Brownian motion W then the condition "A is a process such that $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$ local martingale N "

can be replaced with "A is a process with $[A, W] = 0$", see for instance [10]. An $(\mathcal{F}_t)$-Dirichlet process is also an $(\mathcal{F}_t)$-weak Dirichlet process, a zero quadratic variation process is in fact also an $(\mathcal{F}_t)$-martingale orthogonal process. An $(\mathcal{F}_t)$-weak Dirichlet process is not necessarily a finite quadratic variation process, but there are $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation that are not Dirichlet processes, see for instance [21]. In this paper we will see a general example of weak Dirichlet with finite quadratic variation which is not a Dirichlet process in Theorem 6.26.
If W (resp. B^H, $B^{H,K}, X, D, N$) are a Brownian motion (resp. a fractional Brownian motion of Hurst parameter $H \in (0,1]$, a bifractional Brownian motion of parameters $H \in (0,1)$ and $K \in (0,1]$, a semimartingale, a Dirichlet, a weak Dirichlet) real process, then $W(\cdot)$ (resp. $B_t^H(\cdot)$, $B_t^{H,K}(\cdot)$, $X(\cdot)$, $N(\cdot)$ and $D(\cdot)$ will be called *window Brownian motion* (resp. *window fractional Brownian motion* of Hurst parameter $H \in (0,1]$, *window bifractional Brownian motion* of parameters $H \in (0,1)$ and $K \in (0,1]$, *window semimartingale*, *window Dirichlet* or *window weak Dirichlet*). The window processes will constitute the main example of Banach valued process in the paper; in that case the state space is $C([-\tau, 0])$.

2.4 Direct sum of Banach spaces

We recall the definition of *direct sum* of Banach spaces given in [17]. The vector space E is said to be the direct sum of vector spaces E_1 and E_2, symbolically $E = E_1 \oplus E_2$, if E_i are subspaces of E with property that every $e \in E$ has a unique decomposition $e = e_1 + e_2$, $e_i \in E_i$. The map $P_i : E \to E_i$ given by $P_i(e) = e_i$ is the projection of E onto E_i. This map will be denoted by P_{E_i} if necessary. If E_i are topological linear spaces over the same field of scalars, E is a topological linear space, equipped with the product topology. If E_i are Banach spaces, E is a Banach space under either of the norms: (1) $\|e_1 + e_2\|_E := \max\{\|e_1\|_{E_1}, \|e_2\|_{E_2}\}$, (2) $\|e_1 + e_2\|_E = (\|e_1\|_{E_1}^p + \|e_2\|_{E_2}^p)^{1/p}$, with $1 \le p < +\infty$. These norms are equivalent to the product topology and there is a real positive constant C such that $\|e_i\|_{E_i} \le C\|e_1 + e_2\|_E$, for $i = 1, 2$ and all $e_1 \in E_1$ and $e_2 \in E_2$. If the norm is given by (1) or (2) with $p = 1$ the constant is 1. If the norm is given by (2) with $1 < p < \infty$ the constant will be $2^{1-1/p}$, it suffices to observe that the real function $f(x) = |x|^{1/p}$ is concave if $p > 1$.
Given $T \in (E_1 \oplus E_2)^*$, we have a unique decomposition of $T = T_1 + T_2$ with $T_1 \in E_1^*$ and $T_2 \in E_2^*$. We define T_1 by $T_1(e) = T(e)$ for all $e \in E_1$ and T_2 by $T_2^*(e) = T(e)$ for all $e \in E_2$. One may verify easily that (a) $T(e) = T(e_1 + e_2) = T_1(e_1) + T_2(e_2)$; (b) T_i are linear; (c) T_i are continuous. To prove (c) we use the fact that given a sequence $(e_i^n)_n$ in E_i it holds $\|e_i^n\|_{E_i} = \|e_i^n\|_E$ for any norm in E.
It may be seen that in the case of Banach spaces, if the norms are chosen appropriately, we have $E_1^* \oplus E_2^* = (E_1 \oplus E_2)^*$. Whenever the direct sum of normed linear spaces is used as a normed space, the norm will be explicitly mentioned. If, however, each of the spaces E_i is a Hilbert space then it will be always understood, sometimes without explicit mention, that E is the uniquely determined Hilbert space with scalar product $\langle e, f\rangle_E = \langle e_1 + e_2, f_1 + f_2\rangle_E = \sum_{i=1}^2 \langle e_i, f_i\rangle_i$, where $\langle \cdot, \cdot\rangle_i$ is the scalar product in E_i.

Thus the norm in a direct sum of Hilbert spaces is always given by (2) with $p = 2$ and, if necessary, will be called Hilbertian direct sum and will be denoted by $E_1 \oplus_h E_2$. We remark that in a direct sum of Hilbert spaces it holds $\langle e, f\rangle_E = 0$ for all $e \in E_1$ and $f \in E_2$. The extension to any finite number of summands is immediate. If E_1 and E_2 are closed normed subspace of E, it holds $\overline{Span\{E_1, E_2\}} = E_1 \oplus E_2$.

2.5 Tensor product of Banach spaces

In this chapter we recall basic concepts and results about tensor product of two Banach spaces E and F. For details and a more complete description of these arguments, the readers may refer to the appendix and [46, 15], the case with E and F Hilbert spaces is well developed in [34]. Let E and F be Banach spaces, the vector space $E \otimes F$ will denote the *algebraic tensor product*. The typical description of an element $u \in E \otimes F$ is $u = \sum_{i=1}^{n} \lambda_i\, e_i \otimes f_i$ where n is a natural number, $\lambda_i \in \mathbb{R}$, $e_i \in E$ and $f_i \in F$. We observe that we can consider the mapping $(e, f) \mapsto e \otimes f$ as a sort of multiplication on $E \times F$ with values in the vector space $E \otimes F$. This product is itself bilinear, so in particular the representation of u is not unique. The general element u can always be rewritten in the form $u = \sum_{i=1}^{n} x_i \otimes y_i$ where $x_i \in E$, $y_i \in F$. We say that a norm, α, on $E \otimes F$ is a *reasonable crossnorm* if $\alpha(e \otimes f) \leq \|e\|_E \, \|f\|_F$ for every $e \in E$ and $f \in F$ and if for every $\phi \in E^*$ and $\psi \in F^*$, the linear functional $\phi \otimes \psi$ on $E \otimes F$ is bounded and $\|\phi \otimes \psi\| := \{\sup |\phi \otimes \psi(u)|;\, u \in E \otimes F; \alpha(u) \leq 1\} \leq \|\phi\|_{E^*} \, \|\psi\|_{F^*}$. We can define two different norms in the vector space $E \otimes F$, the so-called called *projective norm*, denoted by π and defined by

$$\pi(u) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| \, : \, u = \sum_{i=1}^{n} x_i \otimes y_i \right\} \tag{2.5}$$

and the so-called *injective norm*, denoted by ϵ, defined by

$$\epsilon(u) = \sup \left\{ \left| \sum_{i=1}^{n} \phi(x_i)\psi(y_i) \right| \, : \, \phi \in E^*, \|\phi\| \leq 1; \psi \in F^*, \|\psi\| \leq 1 \right\} \tag{2.6}$$

Those norms are reasonable and it holds that α is a reasonable crossnorm if and only if

$$\epsilon(u) \leq \alpha(u) \leq \pi(u) \tag{2.7}$$

for every $u \in E \otimes F$, i.e. the projective one is the largest one and ϵ is the smallest one. Moreover for every reasonable crossnorm in $E \otimes F$ we have $\alpha(e \otimes f) = \|e\| \, \|f\|$ and $\|\phi \otimes \psi\| = \|\phi\| \, \|\psi\|$. We will work principally with the projective norm π, the injective norm ϵ and a particular reasonable norm denoted by h, so-called *Hilbert tensor norm*. The Hilbert norm is a reasonable crossnorm in the sense that , whenever E and F are Hilbert spaces then h derives from a scalar product $\langle \cdot, \cdot \rangle_h$ verifying $\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle_{E \otimes F} = \langle e_1, e_2\rangle_E \langle f_1, f_2\rangle_F$. Given a reasonable crossnorm α, we denote by $E \otimes_\alpha F$ the tensor product vector space $E \otimes F$ endowed with the norm α. Unless the spaces E and F are finite dimensional, this space is not complete. We denote its completion by $E\hat{\otimes}_\alpha F$. The Banach space $E\hat{\otimes}_\alpha F$ will be referred to as the α *tensor product* of the Banach spaces E, F. If E and F are Hilbert spaces the *Hilbertian tensor product* is a Hilbert space. We recall an important statement in the case of Hilbert spaces from chapter 6 in [34]. Let $(\Omega_1, \mathcal{F}_1, \mu_1)$ and $(\Omega_2, \mathcal{F}_2, \mu_2)$ be two measure spaces, then $L^2(\Omega_1, \mathcal{F}_1, \mu_1)\hat{\otimes}_h L^2(\Omega_2, \mathcal{F}_2, \mu_2) \cong L^2(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mu_1 \otimes \mu_2)$. The symbols $E\hat{\otimes}_\alpha^2$, $e\otimes^2$ and $e\otimes_\alpha^2$ will denote respectively the Banach space $E\hat{\otimes}_\alpha E$, the elementary element $e \otimes e$ of the algebraic tensor product $E \otimes F$ and $e \otimes e$ in the Banach space $E\hat{\otimes}_\alpha E$. An important role in the paper it will be played by topological duals of tensor product spaces denoted, as usual for a dual space of a Banach space, by $(E\hat{\otimes}_\alpha F)^*$ equipped with operator norm denoted by α^*. If $T \in (E\hat{\otimes}_\alpha F)^*$, $\alpha^*(T) = \sup_{\alpha(u)\leq 1} |T(u)|$. By (2.7) we deduce following relation between tensor dual norms

$$\epsilon^*(u) \geq \alpha^*(u) \geq \pi^*(u). \tag{2.8}$$

We spend some words on two special cases.
We have an isometric isomorphism between the Banach space of G-valued bounded bilinear forms on the product $E \times F$, denoted by $\mathcal{B}(E \times F; G)$, and the Banach space of G-valued bounded linear operators on $E\hat{\otimes}_\pi F$.

Proposition 2.7. Let $\tilde{B} : E \times F \to G$ be a continuous bilinear mapping, it exists a unique bounded linear operator $B : E\hat{\otimes}F \to G$ satisfying $B(e \otimes f) = \tilde{B}(e, f)$ for every $e \in E$, $f \in F$. We observe moreover that it exists a canonical identification between $\mathcal{B}(E \times F; G)$ and $L(E; L(F; G))$ which identifies $\tilde{B}$ with $\bar{B} : E \to L(F; G)$ by $\tilde{B}(e, f) = \bar{B}(e)(f)$. Thus we have a canonical identification $L(E\hat{\otimes}_\pi F; G) = \mathcal{B}(E \times F; G) = L(E; L(F; G))$. If we take G to be the scalar field $\mathbb{R}$, we obtain an isometric isomorphism between the dual space of the projective tensor product equipped with the norm π^* with the space of bounded bilinear forms equipped with the usual norm:

$$(E\hat{\otimes}_\pi F)^* = \mathcal{B}(E \times F) = L(E; F^*) \tag{2.9}$$

With this identification, the action of a bounded bilinear form B as a bounded linear functional on $E\hat{\otimes}_\pi F$ is given by

$$\langle \sum_{i=1}^n x_i \otimes y_i, B\rangle = \sum_{i=1}^n \tilde{B}(x_i, y_i) = \sum_{i=1}^n \bar{B}(x_i)(y_i) \tag{2.10}$$

It holds $\pi^*(B) = \|\tilde{B}\|$.

There is a chain relation of densely and continuous inclusions between the following Banach tensor products

$$E\hat{\otimes}_\pi F \subset E\hat{\otimes}_\alpha F \subset E\hat{\otimes}_\epsilon F \tag{2.11}$$

then for their dual spaces it follows that,

$$(E\hat{\otimes}_\epsilon F)^* \subset (E\hat{\otimes}_\alpha F)^* \subset (E\hat{\otimes}_\pi F)^* \tag{2.12}$$

continuously. This inclusion it fails to be a densely inclusion even if E and F are Hilbert spaces, in Proposition 5.35 we will give a probabilistic proof of this fact.

Remark 2.8. If E and F are Hilbert spaces, we can identify the Hilbert space $E\hat{\otimes}_h F$ with its dual $(E\hat{\otimes}_h F)^*$ via the Riesz-Fréchet's representation Theorem and we still have $E\hat{\otimes}_\pi F \subset E\hat{\otimes}_h F$ continuously and densely. The Banach space $E\hat{\otimes}_\pi F$ is not a reflexive space because it contains a copy of ℓ^1 (the proof can be found in [46], chapter 4.2), in particular it fails to be a Hilbert space even if E and F are Hilbert spaces. Then we can not use the Remark 1, after Theorem V.5 in [4] and the inclusion $(E\hat{\otimes}_h F)^* \subset (E\hat{\otimes}_\pi F)^*$ is still only continuous and not dense. We have the triple inclusion but only continuously

$$E\hat{\otimes}_\pi F \subset E\hat{\otimes}_h F = (E\hat{\otimes}_h F)^* \subset (E\hat{\otimes}_\pi F)^*$$

We recall another important identification that will be used throughout the paper in a significantly way, this identification can be applied to obtain a representation of a space of continuous functions of two variables as an injective tensor product of two spaces of continuous functions. Let K_1, K_2 be compact spaces, therefore we have

$$C(K_1)\hat{\otimes}_\epsilon C(K_2) = C(K_1; C(K_2)) = C(K_1 \times K_2) \tag{2.13}$$

In particular we have $\mathcal{M}(K_1 \times K_2) = (C(K_1)\hat{\otimes}_\epsilon C(K_2))^* \subset (C(K_1)\hat{\otimes}_\pi C(K_2))^*$. Let η_1, η_2 be two elements in $C([-\tau,0])$ (respectively $L^2([-\tau,0])$), the element $\eta_1 \otimes \eta_2$ in the algebraic tensor product $C([-\tau,0])\otimes^2$ (respectively $L^2([-\tau],0)\otimes^2$) will be identified with the element η in $C([-\tau,0]^2)$ (respectively $L^2([-\tau,0]^2)$) defined by $\eta(x,y) = \eta_1(x)\eta_2(y)$ for all x, y in $[-\tau,0]$. Then let μ be a measure on $\mathcal{M}([-\tau,0]^2)$, the pair duality $\langle \mu, \eta_1 \otimes \eta_2 \rangle$ has to be understood as the pair duality $\langle \mu, \eta \rangle = \int_{[-\tau,0]^2} \eta(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\mu(dx,dy)$.
We recall an interesting result involving Hilbertian tensor product and Hilbertian direct sum.

Remark 2.9. Let X and Y be Hilbert separables spaces such that $Y = Y_1 \oplus_h Y_2$ with the Hilbertian direct norm. Then $X\hat{\otimes}_h Y = (X\hat{\otimes}_h Y_1) \oplus_h (X\hat{\otimes}_h Y_2)$.

Proof. To prove the result we show that there is a isometric isomorphism between the two spaces. First of all we observe that if we consider the orthonormal basis for every Hilbert space, i.e. $(e_n)_{n\in N}$ for X, $(f_m)_{m\in M_1}$ for Y_1 and $(f_m)_{m\in M_2}$ for Y_2 then the Hilbert space $X\hat{\otimes}_h Y$ will have the basis $(e_n \otimes f_m)_{n\in N,\, m\in M_1\cup M_2}$, then the space is a tensor product of a direct sum. Moreover this is an isometry. Being a Hilbertian tensor product it suffices to verify the isometry for elementary tensor product $x \otimes y$ in $X\hat{\otimes}_h Y$, where $x \in X$, $y \in Y$ with unique decomposition $y = y_1 + y_2$, $y_i \in Y_i$ for $i = 1, 2$. The Hilbertian norm of the element $x \otimes y$ equals

$$\|x \otimes y\|^2 = \|x\|^2\|y\|^2 = \|x\|^2(\|y_1\|^2 + \|y_2\|^2) = \|x\|^2\|y_1\|^2 + \|x\|^2\|y_2\|^2 = \|x \oplus y_1\|^2 + \|x \oplus y_2\|^2$$

□

2.6 Subsets notation

Spaces $\mathcal{M}([-\tau,0])$ and $\mathcal{M}([-\tau,0]^2)$ and their subsets will play a central role. We will introduce some notations that will be used in the paper. Let $-\tau = a_N < a_{N-1} < \dots a_1 < a_0 = 0$ be $N+1$ fixed points in $[-\tau,0]$. Symbols a and A will refer respectively to the vector $(a_N, a_{N-1}, \dots, a_1, 0)$ and to the matrix $(A_{i,j})_{0\le i,j\le N} = (a_i, a_j)$. Vector a will identify $N+1$ points on $[-\tau,0]$ and matrix A will identify $(N+1)^2$ points on $[-\tau,0]^2$.

- Symbol $\mathcal{D}_i([-\tau,0])$, $\mathcal{D}_i$ shortly, will denote the one dimension set of Dirac's measure concentrated on $a_i \in [-\tau,0]$, i.e.

$$\mathcal{D}_i([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0]);\ s.t.\mu(dx) = \lambda\, \delta_{a_i}(dx) \text{ with } \lambda \in \mathbb{R}\} \tag{2.14}$$

  and we define the scalar product between $\mu^1 = \lambda^1 \delta_{a_i}$ and $\mu^2 = \lambda^2 \delta_{a_i}$ by $\langle \mu^1, \mu^2\rangle = \lambda^1\lambda^2$. $\mathcal{D}_i$ equipped with this scalar product is a Hilbert space. In particular for $a_0 = 0$, the space $\mathcal{D}_0$ will be the space of Dirac's measure concentrated on 0.

- Symbol $\mathcal{D}_{i,j}([-\tau,0]^2)$, $\mathcal{D}_{i,j}$ shortly, will denote the one dimensional set of Dirac's measure concentrated on $(a_i, a_j) \in [-\tau,0]^2$, i.e.

$$\mathcal{D}_{i,j}([-\tau,0]^2) := \{\mu \in \mathcal{M}([-\tau,0]^2);\ s.t.\mu(dx,dy) = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy) \text{ with } \lambda \in \mathbb{R}\} \cong \mathcal{D}_i \hat{\otimes}_h \mathcal{D}_j \tag{2.15}$$

  Let $\mu^1 = \lambda^1\, \delta_{a_i}(dx)\delta_{a_j}(dy)$ and $\mu^2 = \lambda^2\, \delta_{a_i}(dx)\delta_{a_j}(dy)$, $\mathcal{D}_{i\,j}$ is a Hilbert space equipped with the scalar product defined by $\langle \mu^1, \mu^2\rangle = \lambda^1\lambda^2$. The identification with $\mathcal{D}_i \hat{\otimes}_h \mathcal{D}_j$ is a trivial exercise. If $a_j = a_i = 0$, the space $\mathcal{D}_{0,0}$ will be the space of Dirac's measures concentrated on $(0,0)$.

- Symbol $\mathcal{D}_a([-\tau,0])$, $\mathcal{D}_a$ shortly, will denote the $N+1$ dimension set of weighted Dirac's measures concentrated on $(N+1)$ fixed points in $[-\tau,0]$ identified by a.

$$\mathcal{D}_a([-\tau,0]) := \{\mu \in \mathcal{M}([-\tau,0])\, s.t.\ \mu(dx) = \sum_{i=0}^{N} \lambda_i \delta_{a_i}(dx);\ \lambda_i \in \mathbb{R},\ i = 0, \dots, N\} \cong \bigoplus_{i=0}^{N} \mathcal{D}_i \tag{2.16}$$

  Let $\mu^1 = \sum_{i=0}^N \lambda_i^1 \delta_{a_i}(dx)$ and $\mu^2 = \sum_{i=0}^N \lambda_i^2 \delta_{a_i}(dx)$, $\mathcal{D}_a$ is a Hilbert space with respect to the scalar product $\langle \mu^1, \mu^2\rangle = \sum_{i=0}^N \lambda_i^1 \lambda_i^2$. It is obvious the isomorphism with $\bigoplus_{i=0}^N \mathcal{D}_i$, where the $(N+1)$ direct sum is equipped with the Hilbertian norm. $\mathcal{D}_a$ is a subspace of the Banach space $\mathcal{M}([-\tau,0])$.

- Symbol $\mathcal{D}_A([-\tau,0]^2)$, $\mathcal{D}_A$ shortly, will denote the $(N+1)^2$ dimensional set of measures concentrated on $(a_i,a_j)_{0\leq i,j\leq N}\in[-\tau,0]^2$, i.e.

$$\mathcal{D}_A([-\tau,0]^2):=\{\mu\in\mathcal{M}([-\tau,0]^2);\ s.t.\mu(dx,dy)=\lambda_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy)\text{ with }\lambda_{i,j}\in\mathbb{R},\,i,j=0,\dots,N\}\tag{2.17}$$

Let $\mu^1=\lambda^1_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy)$ and $\mu^2=\lambda^2_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy)$, $\mathcal{D}_A$ is a Hilbert space equipped with the scalar product defined by $\langle\mu^1,\mu^2\rangle=\sum_{0\leq i,j\leq N}\lambda^1_{i,j}\lambda^2_{i,j}$. Moreover we have the following useful identifications

$$\mathcal{D}_A\cong\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a=\mathcal{D}_a\hat{\otimes}^2_h=\left(\bigoplus_{i=0}^N\mathcal{D}_i\right)\hat{\otimes}^2_h\cong\bigoplus_{i,j=0}^N\mathcal{D}_i\hat{\otimes}_h\mathcal{D}_j=\bigoplus_{i,j=0}^N\mathcal{D}_{i,j}\tag{2.18}$$

In fact there is an isometric isomorphism between $\mathcal{D}^A$ and $\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$. Let $\mu=\lambda_{i,j}\,\delta_{a_i}(dx)\delta_{a_j}(dy)\in\mathcal{D}_A$ there is a unique element $\tilde{\mu}\in\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$ identified by $\tilde{\mu}=\sum_{i,j=0}^N\lambda_{i,j}\delta_{a_i}\otimes\delta_{a_j}$. The contrary follows in analogous way. The isometry is trivial by equality between scalar products. Let $\mu^{1,2,3,4}=\sum_{i=0}^N\lambda_i^{1,2,3,4}\delta_{a_i}(dx)$, four elements in $\mathcal{D}_a$, the Hilbertian tensor product $\mathcal{D}_a\hat{\otimes}_h\mathcal{D}_a$ is equipped with the scalar product $\langle\mu^1\otimes\mu^2,\mu^3\otimes\mu^4\rangle=\langle\mu^1,\mu^3\rangle\langle\mu^2,\mu^4\rangle=\left(\sum_{i=0}^N\lambda^1_i\lambda^3_i\right)\left(\sum_{i=0}^N\lambda^2_i\lambda^4_i\right)=\sum_{0\leq i,j\leq N}\lambda^1_i\lambda^3_i\lambda^2_j\lambda^4_j$. Other two identifications in (2.18) derives by (2.16), Remark 2.9 end (2.15).

Dirac's measures concentrated in vector a (in matrix A respectively) are mutually singular with respect to the Lebesgue measure on $[-\tau,0]$ (on $[-\tau,0]^2$ respectively), this shows the direct sum representation for $\mathcal{D}_a$ and $\mathcal{D}_A$. We will appreciate the importance of a direct sum representation with Proposition 4.19. As a proper subspace of $\mathcal{D}_A([-\tau,0]^2)$ we could consider the case with only the points on the diagonal of $[-\tau,0]^2$.

- Symbol $\mathcal{D}_d([-\tau,0]^2)$, $\mathcal{D}_d$ shortly, will denote the $N+1$ dimension set of weighted Dirac's measures concentrated on $(N+1)$ fixed points $(a_i,a_i)_{i=0,\dots,N}$ on the diagonal of $[-\tau,0]^2$, i.e.

$$\mathcal{D}_d([-\tau,0]^2):=\{\mu\in\mathcal{M}([-\tau,0]^2)\,s.t.\,\mu(dx)=\sum_{i=0}^N\lambda_i\delta_{a_i}(dx)\delta_{a_i}(dy);\,\lambda_i\in\mathbb{R},\,i=0,\dots,N\}\cong\bigoplus_{i=0}^N\mathcal{D}_{i,i}\tag{2.19}$$

This a Hilbert space.

Remark 2.10. There are naturally identification $\mathcal{D}_i\cong\mathcal{D}_{i,j}\cong\mathbb{R}$, $\mathcal{D}_a\cong\mathcal{D}_d\cong\mathbb{R}^{N+1}$ and $\mathcal{D}_A\cong\mathbb{M}_{(N+1)}(\mathbb{R})\cong\mathbb{R}^{N+1}\otimes\mathbb{R}^{N+1}$. All those spaces are finite dimensional separable Hilbert spaces.

We give others examples of infinite dimensional subsets of measure.

- $L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$, as well as $L^2([-\tau,0]^2) \cong L^2([-\tau,0])\hat{\otimes}_h^2$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$, both equipped with the norm derived from the usual scalar product.

- $\mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0])$. This is a direct sum in the space of measure $\mathcal{M}([-\tau,0])$. In fact for every measure $\mu \in \mathcal{M}([-\tau,0])$ the Lebesgue decomposition identify uniquely a measure μ_{ac} absolutely continuous w.r.t. Lebesgue measure and a measure μ_s singular w.r.t. Lebesgue measure such that $\mu = \mu_{ac} + \mu_s$. If moreover $\mu \in L^2([-\tau,0]) \cup \mathcal{D}_i([-\tau,0])$, the decomposition identifies uniquely the part in $\mu_{ac} \in L^2([-\tau,0])$ and $\mu_s \in \mathcal{D}_i([-\tau,0])$ and the sum is direct. As generalization of this case we have

- $\mathcal{D}_a([-\tau,0]) \oplus L^2([-\tau,0]) \cong \bigoplus_{i=0}^N \mathcal{D}_i([-\tau,0]) \oplus L^2([-\tau,0])$, this is a Hilbert separable subspace of $\mathcal{M}([-\tau,0])$.

- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$. The Hilbert structure of the tensor product derives as usual from the Hilbert structure in every Hilbert space.

- Symbol $Diag([-\tau,0]^2)$, $Diag$ shortly, will denote the subset of $\mathcal{M}([-\tau,0]^2)$ defined as follows

$$Diag([-\tau,0]^2) := \left\{\mu \in \mathcal{M}([-\tau,0]^2)\; s.t.\; \mu(dx,dy) = g(x)\delta_y(dx)dy;\; g \in L^\infty([-\tau,0])\right\} \tag{2.20}$$

  $Diag([-\tau,0]^2)$, equipped with the norm $\|\mu\|_{Diag([-\tau,0]^2)} = \|g\|_\infty$, is a Banach space. Let f be a function in $C([-\tau,0]^2)$, the pair duality between f and $\mu(dx,dy) = g(x)\delta_y(dx)dy \in Diag$ equals

$$\langle f,\mu\rangle = \int_{[-\tau,0]^2} f(x,y)\mu(dx,dy) = \int_{[-\tau,0]^2} f(x,y)g(x)\delta_y(dx)dy = \int_{-\tau}^0 f(x,x)g(x)dx \tag{2.21}$$

2.7 Fréchet derivative

The importance of tensor product and their duals comes first of all from Proposition 2.7. We recall some notions about differential calculus in Banach spaces, for more details reader can refer to [6].
Let B and G be Banach spaces and $U \subset B$ be an open subspace of B. A function $F : U \longrightarrow G$ is called Fréchet differentiable at $x \in U$ if it exists an linear bounded application $A_x : B \longrightarrow G$ such that

$$\lim_{h\to 0} \frac{\|F(x+h) - F(x) - A_x(h)\|_G}{\|h\|_B} = 0$$

If this limit exists we write $DF(x) = A_x$ the derivative of F at x. We define for a function F which is Fréchet differentiable for any $x \in U$ the application $DF : U \longrightarrow L(B;G)$ such that $x \mapsto DF(x)$. If DF is continuous F is said to be $C^1(B;G)$ or once Fréchet differentiable. Analogously this function DF may as well have a derivative, the second order derivative of F which will be a map $D^2F : U \longrightarrow L(B;L(B;G)) \cong \mathcal{B}(B\times B;G) \cong L(B\hat{\otimes}_\pi B;G)$. If D^2F is continuous F is said to be $C^2(B;G)$ or twice Fréchet differentiable. In particular if we consider a function $F : [0,T]\times B \longrightarrow \mathbb{R}$, F is $C^{1,2}([0,T]\times B)$, or $C^{1,2}$, means that F is once continuously differentiable with respect to time and it is twice continuously Fréchet differentiable with respect to the Banach space B. If $B = C([-\tau, 0])$, the different derivatives are such that

$$\begin{aligned}
&\partial_t F : [0,T] \times C([-\tau,0]) \longrightarrow \mathbb{R} \\
&DF : [0,T] \times C([-\tau,0]) \longrightarrow C([-\tau,0])^* \cong \mathcal{M}([-\tau,0]) \\
&D^2F : [0,T] \times C([-\tau,0]) \longrightarrow L\left(C([-\tau,0]); C([-\tau,0])^*\right) \cong \mathcal{B}(C([-\tau,0]) \times C([-\tau,0])) \cong \left(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0])\right)^*
\end{aligned}$$

For all η, h, h_1 and h_2 in $C([-\tau, 0])$ and $t \in [0, T]$ we will denote with $D_{dx}F(t, \eta)$ the measure such that

$$_{\mathcal{M}([-\tau,0])}\langle DF(t,\eta), h\rangle_{C([-\tau,0])} = DF(t,\eta)(h) = \int_{[-\tau,0]} h(x) D_{dx}F(t,\eta). \tag{2.22}$$

Moreover if $D^2(F)(t, \eta) \in \mathcal{M}([-\tau, 0]^2) \subset (C([-\tau, 0])\hat{\otimes}_\pi C([-\tau, 0]))^*$ for all $(t, \eta) \in [0, T] \times C([-\tau, 0])$ (which will happen in most of the treated cases) we will denote with $D^2_{dx\,dy}F(\eta)$, or $D_{dx}D_{dy}F(\eta)$, the measure on $[-\tau, 0]^2$ such that

$$\begin{aligned}
_{\mathcal{M}([-\tau,0]^2)}\langle D^2F(t,\eta), h_1 \cdot h_2\rangle_{C([-\tau,0]^2)} &= D^2F(t,\eta)(h_1,h_2) = D^2F(\eta)(h_1 \otimes h_2) = \\
&= \int_{[-\tau,0]^2} h_1(x)\, h_2(y)\, D^2_{dx\,dy}F(\eta)
\end{aligned} \tag{2.23}$$

Let $0 \le k \le +\infty$, we denote by $C^k(\mathbb{R}^n)$ the set of all function $g : \mathbb{R}^n \to \mathbb{R}$ which admits all partial derivatives of order $0 \le p \le k$. In particular let $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ be a function in $C^{1,2}([0, T] \times \mathbb{R}^n)$, $t \in [0, T]$, $x \in \mathbb{R}^n$, the symbols $\partial_t g(t, x)$, $\partial_i g(t, x)$ and $\partial^2_{ij} g(t, x)$ will denote respectively the partial derivative with respect to time, the partial derivative with respect to the i-th component and the second order mixed derivative with respect to j-th and i-th component evaluated in (t, x).
We denote by $C^\infty_p(\mathbb{R}^n)$ (resp. $C^\infty_b(\mathbb{R}^n)$ and $C^\infty_b(\mathbb{R}^n)$) the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \to \mathbb{R}$ such that g and all its partial derivatives have polynomial growth (resp. g and all its partial derivatives are bounded and g has compact support).

2.8 Malliavin calculus

We recall some notions of stochastic calculus of variations, i.e. Malliavin calculus, that we need in the sequel. We refer the reader to [35] for a presentation of the subject.. Let $W = \{W(h), h \in H\}$ be a stochastic process associated to a the Hilbert space H defined in a complete probability space $(\Omega, \mathcal{F}, P)$. W define a centered Gaussian family of random variables such that $\mathbb{E}\left[W(h)\, W(g)\right] = \langle h, g\rangle_H$. Let $\mathcal{S}$ denote the class of smooth random variables such that a random variable $F \in \mathcal{S}$ has the form

$$F = f(W(h_1), \ldots, W(h_n)) \tag{2.24}$$

where $f \in C_p^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n$ are in H and $n \geq 1$. We will denote by $\mathcal{S}_b$, $\mathcal{S}_0$ and $\mathcal{P}$ the classes of smooth random variables of the form (2.24) such that the function f belong to $C_b^\infty(\mathbb{R}^n)$, $C_b^\infty(\mathbb{R}^n)$ or f is polynomial. Note that $\mathcal{P} \subset \mathcal{S}$, $\mathcal{S}_0 \subset \mathcal{S}_b \subset \mathcal{S}$, and $\mathcal{P}$ and $\mathcal{S}_0$ are dense in $L^2(\Omega)$.
We define, as in Definition 1.2.1 in [35], the Malliavin's derivative of F, this operator will be denoted by D^m.

Definition 2.11. The derivative of a smooth random variable F of the form (2.24) is the H-valued random variable given by

$$D^m F = \sum_{i=1}^{n} \partial_i f(W(h_1), \ldots, W(h_n))\, h_i \tag{2.25}$$

The operator D^m is closable from $L^p(\Omega)$ to $L^p(\Omega; H)$ for any $p \geq 1$, then for any $p \geq 1$ we will denote the domain of D^m in $L^p(\Omega)$ by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth random variables $\mathcal{S}$ with respect to the norm $\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p]\right)^{1/p}$. For $p = 2$, the space $\mathbb{D}^{1,2}$ is a Hilbert space with the scalar product $\langle F, G\rangle = \mathbb{E}[FG] + [\langle DF, DG\rangle_H]$.
We recall Proposition 1.2.3 in [35] which will be useful for calculus.

Proposition 2.12. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function with bounded derivatives, and fix $p \geq 1$. Suppose that $F = (F^1, \ldots, F^m)$ is a random vector whose components belong to the space $\mathbb{D}^{1,p}$. Then $\varphi(F) \in \mathbb{D}^{1,p}$ and

$$D^m(\varphi(F)) = \sum_{i=1}^{m} \partial_i \varphi(F)\, D^m F^i \tag{2.26}$$

If we suppose that the Hilbert space H i an L^2 space of the form $L^2(T, \mathcal{B}, \mu)$, where μ is a sigma-finte atomless measure on a measurable space $(T, \mathcal{B})$, the derivative of a random variable $F \in \mathbb{D}^{1,2}$ will be a stochastic process denoted by $\{D_t^m F, t \in T\}$ due to the identification between the Hilbert space $L^2(\Omega; H)$ and $L^2(\Omega \times T)$.

Suppose that $W = \{W(t), t \in [0,1]\}$ is a one dimensional Brownian motion. In this case $W_t = \int_0^t dW_r = W(\mathbb{1}_{[0,t]})$ and $\mathbb{E}[W_s W_t] = \langle \mathbb{1}_{[0,s]}, \mathbb{1}_{[0,t]}\rangle_{L^2(T,\mathcal{B},\mu)} = t \wedge s$. As a lemma of martingale representation Theorem we know that any square integrable random variable F, measurable with respect to $\mathcal{F}_1$, can be written as

$$F = \mathbb{E}[F] + \int_0^1 H_s dW_s \tag{2.27}$$

where H_s is an adapted process such that $\mathbb{E}\left[\int_0^1 H_s^2 ds\right] < \infty$. When the variable F belong to the space $\mathbb{D}^{1,2}$, it turns out that the process H_s can be identified as the optional projection of the derivative of F. This is called the Clark-Ocone representation formula:

Proposition 2.13. Let $F \in \mathbb{D}^{1,2}$ and suppose that W is a one-dimensional Brownian motion. Then

$$F = \mathbb{E}[F] + \int_0^1 \mathbb{E}\left[D_t^m F | \mathcal{F}_t\right] dW_t \tag{2.28}$$

We recall some useful rules of stochastic calculus of variations. By Propositions 1.3.8 and 1.3.18 in [35] we obtain that if $(u_t)_{t\in[0,1]}$ is a square integrable adapted process

1. $D_s^m(W_t) = \mathbb{1}_{[0,t]}(s) = \mathbb{1}_{\{s\leq t\}}$
2. $D_s^m\left(\int_0^t u_r\, dW_r\right) = u_s \mathbb{1}_{\{s\leq t\}} + \int_s^t D_s^m(u_r)\, dW_r$
3. $D_s^m\left(\int_0^t u_r\, dr\right) = \int_s^t D_s^m(u_r)\, dr$

Our principal references about functional analysis are [17, 18, 19, 4, 54, 49].

Chapter 3

Calculus via regularization

In this chapter we will define a stochastic integral with respect to a Banach-valued stochastic process. We did not aim to have a full generalization: integral process will only be scalar. The difficulty in this construction is the fact that the stochastic integrator is infinite dimensional and is not necessarily a semimartingale. As a special case in fact it will be possible to consider the $C([-\tau, 0])$-valued window Brownian motion $W(\cdot)$ as stochastic integrator. Firstly we observe that although we can define the stochastic integral with respect to an infinite dimension martingale ([11, 33, 16]), we can not apply this definition to $W(\cdot)$ because, as we will see in the first paragraph, it is not any reasonable $C([-\tau, 0])$-valued martingale. Then we give a definition of a stochastic integral for Banach valued stochastic processes. The last part of this chapter is devoted to the Da Prato Zabczyck stochastic integral. We will be interested to show that whenever Da Prato-Zabczyck integral and forward integral both exist, they are equal.

3.1 Basic motivation: the window Brownian motion

Definition 3.1. Let B be a Banach space and X a B-valued stochastic process. We say that X is a *weakly semimartinglale* if, for every $\phi \in B^*$, $\langle \phi, X_t \rangle$ is a real semimartingale with respect to a filtration $(\mathcal{G}_t)$. It holds that if X is a B-valued martingale in the sense of [33], page 12, then it is also a weakly martingale.

We will show that the window Brownian motion is not even a weak semimartingale, then is not a martingale and we can not define a stochastic integral with the classical method for integration with respect to Banach valued semimartingale.

Proposition 3.2. The $C([-\tau, 0])$-valued window Brownian motion is not a weakly semimartingale.

Proof. Let $(\mathcal{F}_t)$ be the natural filtration generated by the real Brownian motion W_t. It suffices to show that it exists an element μ in $B^* = \mathcal{M}([-\tau, 0])$ such that $\langle \mu, W_t(\cdot) \rangle = \int_{[-\tau,0]} W_t(x)\mu(dx)$ is not a semimartingale with respect to any filtration. We will prove by contradiction: we suppose that $W(\cdot)$ is a weakly semimartingale, then in particular if we take $\mu = \delta_0 + \delta_{-\tau}$, the process $\langle \delta_0 + \delta_{-\tau}, W_t(\cdot)\rangle = W_t + W_{t-\tau} := X_t$ will be a semimartingale with respect to some filtration $(\mathcal{G}_t)$. At the same time $W_t + W_{t-\tau}$ is $\mathcal{F}_t$ adapted, then by Stricker's theorem X_t is a semimartingale with respect to filtration $(\mathcal{F}_t)$, for details about that theorem see Theorem 4, pag. 53 in [36]. Moreover we observe that $W_{t-\tau}$ is an $(\mathcal{F}_t)$ strongly predictable continuous process, where we recall that R is called strongly predictable with respect to a filtration $\mathbb{F}$, if it exists $\delta > 0$, such that $(R_{s+\epsilon})_{s\geq 0}$ is $\mathbb{F}$-adapted, for every $\epsilon \leq \delta$. This notion of *strongly predictable process* has been introduced in [9]. Then by Proposition 4.11 in [9], we have $[W_{\cdot-\tau}, N] = 0$ for every continuous $\mathcal{F}_t$-local martigale N, so $W_{t-\tau}$ is an $(\mathcal{F}_t)$-martingale orthogonal process. Since W_t an $(\mathcal{F}_t)$ martingale, process $X_t = W_t + W_{t-\tau}$ is an $(\mathcal{F}_t)$ weak Dirichlet process. By unicity of decomposition for an $(\mathcal{F}_t)$ weak Dirichlet process and for an $(\mathcal{F}_t)$ semimartingale $W_{t-\tau}$ is a bounded variation process. This generates a contradiction. In particular $W_{t-\tau}$ is a finite quadratic variation process, not a zero quadratic variation process, i.e. $W_t + W_{t-\tau}$ is an example of $\mathcal{F}_t$ weak Dirichlet with finite quadratic variation which is not a $\mathcal{F}_t$ Dirichlet process. □

3.2 Definition of the integral for Banach valued processes

In paragraph 2.2 we briefly recall the definition of forward integral for real valued processes. Now we define a forward stochastic integral for a Banach valued integrand processes and an integrator process with values in the dual of the Banach space. We did not aim to have a full generalization. Integral process will only be scalar.

Definition 3.3. Let $(X_t)_{t\in[0,T]}$ and $(Y_t)_{t\in[0,T]}$ respectively a B-valued and a B^*-valued continuous stochastic processes, i.e. $X : \Omega \times [0,T] \longrightarrow B$ and $Y : \Omega \times [0,T] \longrightarrow B^*$.
For every fixed $t \in [0,T]$ we define the **definite forward integral of Y with respect to X** denoted by $\int_0^t Y_s d^- X_s$ or by $\int_0^t \langle Y_s, d^- X_s\rangle$ as follows

$$\int_0^t Y_s d^- X_s = \int_0^t {}_{B^*}\langle Y_s, d^- X_s\rangle_B := \lim_{\epsilon\longrightarrow 0} \int_0^t {}_{B^*}\langle Y(s), \frac{X(s+\epsilon) - X(s)}{\epsilon}\rangle_B ds$$

The **forward stochastic integral of Y with respect to X** exists if the process

$$\left(\int_0^t Y_s, d^- X_s\right)_{t\in[0,T]} = \left(\int_0^t {}_{B^*}\langle Y_s, d^- X_s\rangle_B\right)_{t\in[0,T]}$$

admits a continuous version. In the sequel indexes B^* and B^* will be often omitted.

Remark 3.4. 1. We remember that even in the case $B = \mathbb{R}$, the notion of forward integral is a little bit relaxed with respect to the traditional one appearing for instance in [41] and recalled in paragraph 2.2 where the limit has to be ucp with respect to t and not only in probability for every fixed t.

2. We remark that if B is a Hilbert space H, then via the Riesz representation theorem, Definition 3.3 gives a definition also in the case X and Y both H-valued.

Remark 3.5. Let B and H be respectively Baanach and Hilbert spaces such that $B \subset H \cong H^* \subset B^*$. If X is a B-valued continuous process and Y is an H^*-valued process. Then

$$\int_0^t {}_{B^*}\langle Y_s, d^- X_s \rangle_B = \int_0^t {}_{H^*}\langle Y_s, d^- X_s \rangle_H \tag{3.1}$$

Remark 3.6. Those type of stochastic integral involves naturally anticipative stochastic integral even in elementary case, as we will see in example equation (7.16) of examples 7.3.

3.3 Link with Da Prato-Zabczyk's integral

Let F and H two separable Hilbert spaces. In the first part of this section we recall the definition stochastic Itô's type integral as it has been defined in [11] denoted by

$$\int_0^t Y_s \cdot dW_s^{dz} \ t \in [0, T]. \tag{3.2}$$

where W is a Wiener process on H and Y is a process with values being linear but not necessarily bounded operators from H to F. This integral will be also called **Da Prato-Zabczyk integral**. We will recall the definition of Hilbert space valued Wiener processes including cylindrical ones and some properties of the stochastic integral (3.2). In the second part we illustrate link with our integral. The central result will be Proposition 3.9. This is an equality result, in fact we will show that if Y is a cadlag H^*-valued process such that $\mathbb{E}\int_0^t \|Y_s\|^2_{H^*} ds < +\infty$ and W is a Q-Brownian motion W, Q being a nuclear operator on H, then the forward integral $\int_0^t \langle Y_s, d^- W_s \rangle$ exists as well as the Da Prato-Zabczyk integral $\int_0^t Y_s \cdot dW_s^{dz}$ and they are equals.

3.3.1 Notations

Let Q be a symmetric non negative operator in $L(H)$. We will consider first the case when Q is a trace class operator in H, i.e. $Q \in L^1(H)$. We assume that there exists a complete orthonormal system $\{e_i\}$ in H, and a bounded sequence of nonnegative real numbers λ_i such that $Qe_i = \lambda_i e_i$, for $i = 1, 2, \ldots$ An H-valued stochastic process $(W_t)_{t \geq 0}$ is called a Q-Wiener motion (or Q-Brownian process) if

(i) $W(0) = 0$.
(ii) W has continuous trajectories.
(iii) W has independent increments
(iv) We have

$$\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q), \ t \geq s \geq 0$$

Remark 3.7. Assume that W is a Q-Brownian motion, with $Q \in L^1(H)$ then the following statements hold

(1) W is a Gaussian process on H and $\mathbb{E}(W_t) = 0$ and $Var(W_t) = tQ$, $t \in [0, T]$.

(2) $\mathbb{E}(\langle W_t, h\rangle) = 0 \quad \forall h \in H$

(3) $\mathbb{E}(\langle W_t, h\rangle^2) = t\langle Qh, h\rangle \quad \forall h \in H$

(4) $\mathbb{E}(\langle W_t, h_1\rangle\langle W_t, h_2\rangle) = t\langle Qh_1, h_2\rangle \quad \forall h_1, h_2 \in H$

(5) $\mathbb{E}(\langle W_t, h_1\rangle\langle W_s, h_2\rangle) = t \wedge s\langle Qh_1, h_2\rangle \quad \forall h_1, h_2 \in H$

We anticipate that the Da Prato-Zabczyk quadratic variation of a Q-Wiener process in H with $Tr(Q) < +\infty$ is given by the formula $[W]^{dz}_t = t\,Q$. Firstly we summarize the definition of stochastic integral with respect to a Q-Brownian motion W with values in H, Q trace class operator.
Let F be a separable Hilbert space with complete orthonormal basis $\{f_j\}$ and let us fix a number $T > 0$. An $L(H; F)$-valued process $(\Phi_t)_{t\in[0,T]}$ taking only a finite number of values is said to be elementary if there exists a sequence $0 = t_0 < t_1 < \ldots < t_M = T$ and sequence $\Phi_0, \Phi_1, \ldots, \Phi_{M-1}$ of $L(H; F)$-valued random variables taking only a finite number of values such that Φ_m are $(\mathcal{F}_{t_m})$-measurable and $\Phi_t = \Phi_m$ for $t \in]t_m, t_{m+1}]$, $m = 0, \ldots, M-1$. For elementary processes Φ the Da Prato-Zabczyk stochastic integral is defined by the formula

$$\int_0^t \Phi_s \cdot dW^{dz}_s := \sum_{m=0}^{M-1} \Phi_m(W_{t_{m+1}\wedge t} - W_{t_m\wedge t})$$

We introduce the subspace $H_0 = Q^{1/2}(H)$ of H, which, endowed with the inner product

$$\langle u, v\rangle_0 = \sum_{i=1}^{\infty} \frac{1}{\lambda_i}\langle u, e_i\rangle\langle v, e_i\rangle = \langle Q^{-1/2}u, Q^{-1/2}v\rangle$$

is a Hilbert space. The space of Hilbert-Schmidt operators from H_0 to F, denoted by $L^2(H_0; F)$, is also a separable Hilbert space, equipped with the norm

$$\begin{aligned}\|\Phi\|^2_{L^2(H_0;F)} &= \sum_{i=1}^{\infty} \|\Phi g_i\|^2_F = \sum_{i,j=1}^{\infty} \lambda_i |\langle\Phi e_i, f_j\rangle|^2 = \|\Phi Q^{1/2}\|^2_{L^2(H;F)} = \\ &= \langle\Phi Q^{1/2}, \Phi Q^{1/2}\rangle_{L^2(H;F)} = Tr\left((\Phi Q^{1/2})(\Phi Q^{1/2})^*\right) = Tr(\Phi Q \Phi^*)\end{aligned}$$

where $g_i = \sqrt{\lambda_i}e_i$, $i = 1, 2, \ldots$, $\{e_i\}$ and $\{f_j\}$ are complete orthonormal bases in H_0, H and F. We remark here that the adjoint operator of $Q^{1/2}$ is $Q^{-1/2}$ from H_0 to H and that the operator $\Phi Q \Phi^*$ is of trace class being a composition of the Hilbert-Schmidt operator $(\Phi Q^{1/2})$ and its adjoint, which is also Hilbert-Schmidt by properties in [23]. Clearly $L(H; F) \subset L^2(H_0; F)$ but $L^2(H_0; F)$ contains also unbounded operators on H.
Let $(\Phi_t)_{t\in[0,T]}$ be a measurable $L^2(H_0; F)$-valued process; we define the norm by

$$|||\Phi|||^2_t = \mathbb{E}\int_0^t \|\Phi_s\|^2_{L^2(H_0;F)} ds = \mathbb{E}\int_0^t Tr(\Phi_s Q^{1/2})(\Phi_s Q^{1/2})^* ds \qquad t \in [0, T]$$

We denote with $\mathcal{N}_W^2(0,T;L^2(H_0;F))$ the Hilbert space of all $L^2(H_0;F)$ predictable processes with $|||\Phi|||_T < +\infty$.
Symbol $\mathcal{M}_T^2(H)$ will denote the space of all H-valued continuous square integrable martingales M. $\mathcal{M}_T^2(H)$ with the norm defined by $\|M\|^2_{\mathcal{M}_T^2(H)} = \mathbb{E}\left[\|M_T\|_H^2\right]$ is a Hilbert space.
If a process Φ is elementary and $|||\Phi|||_T < +\infty$, then the stochastic integral $\int_0^\cdot \Phi_s \cdot dW_s^{dz}$ is a continuous square integrable F-valued martingale on $[0,T]$ and it holds following isometry

$$\mathbb{E}\left\|\int_0^t \Phi_s \cdot dW_s^{dz}\right\|_F^2 = |||\Phi|||_t^2 \qquad\qquad 0 \le t \le T \tag{3.3}$$

The stochastic integral with respect to a Q-Brownian motion is an isometric transformation from the space of elementary processes equipped with the norm $|||\cdot|||$ into the space of F-valued square integrable martingale $\mathcal{M}_T^2(F)$. By the fact that elementary processes form a dense set in $\mathcal{N}_W^2(0,T;L^2(H_0;F))$ the definition of stochastic integral is extended to all elements in $\mathcal{N}_W^2(0,T;L^2(H_0;F))$ and (3.3) holds true.

Definition 3.8. For a general element $\Phi \in \mathcal{N}_W^2(0,T;L^2(H_0;F))$, we will denote **Brownian martingale** the martingale $M \in \mathcal{M}_T^2(F)$ given by the stochastic integral

$$M_\cdot = \int_0^\cdot \Phi_s \cdot dW_s^{dz} \tag{3.4}$$

By the so called localization procedure it is possible to extend the definition of the Da Prato-Zabczyk stochastic integral to $L^2(H_0;F)$-predictable processes satisfying even the weaker condition

$$\mathbb{P}\left[\int_0^T \|\Phi_s\|^2_{L^2(H_0;F)} ds < +\infty\right] = 1$$

In [11] the definition of stochastic integral with respect to a Q-Brownian motion is extended to a a cylindrical Brownian motion. Let Q be a general bounded, self-adjoint, non negative (to avoid complication we will assume strictly positive) operator on H, i.e. not necessarily such that $Tr(Q) < +\infty$. Let $H_0 = Q^{1/2}(H)$ with the induced norm and let H_1 be an arbitrary Hilbert space such that H is embedded continuously into H_1 and the embedding J of H_0 into H_1 is Hilbert-Schmidt. Let $\{g_j\}$ be an orthonormal and complete basis in H_0 and β_j a family of independent real valued standard Brownian motion then the the following series is convergent in $L^2(\Omega;H_1)$

$$W_t = \sum_{j=1}^{+\infty} g_j \beta j(t)$$

and we will call W_t a cylindrical Brownian motion on H. We recall that W_t is a Q_1 Brownian motion on H_1 with $Tr(Q_1) < +\infty$, $Q_1 = JJ^*$. We remark that a Q Brownian motion with $Tr(Q) < +\infty$ is H-valued and has the same expansion of a cylindrical Brownian motion in $L^2(\Omega;H)$. The definition of stochastic integral

is the same for a cylindrical Brownian motion because the class $\mathcal{N}^2_W(0;T;L^2(H_0;F))$ is independent of the space H_1 and the spaces $Q_1^{1/2}(H_1)$ are identical for all possible extension H_1.
We recall some properties of Brownian stochastic integral from chapter 4.4 in [11].
If $\Phi \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$, then the stochastic integral $M = \Phi \cdot W$

$$M_t = \int_0^t \Phi(s) \cdot dW_s^{dz} \tag{3.5}$$

is a continuous square integrable martingale in $\mathcal{M}^2_T(F)$ and its quadratic variation is of the form

$$[M]_t^{dz} = [\Phi \cdot W]_t^{dz} = \int_0^t \left(\Phi(s)Q^{1/2}\right)\left(\Phi(s)Q^{1/2}\right)^* ds$$

Moreover if $\Phi_1, \Phi_2 \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$ then

$$\mathbb{E}\left[\Phi_i \cdot W_t\right] = 0 \qquad \mathbb{E}\left[\|\Phi_i \cdot W_t\|^2\right] < +\infty \qquad s,t \in [0,T] \text{ and } i = 1,2$$

and the correlation operator is given by the formula

$$V(t,s) = Cor\left[\Phi_1 \cdot W_t, \Phi_2 \cdot W_t\right] = \mathbb{E}\int_0^{t\wedge s} \left(\Phi_1(r)Q^{1/2}\right)\left(\Phi_2(r)Q^{1/2}\right)^* dr$$

Moreover under the same hypotheses we have

$$\mathbb{E}\left[\langle \Phi_1 \cdot W_t, \Phi_2 \cdot W_s\rangle\right] = \mathbb{E}\int_0^{t\wedge s} Tr\left[\left(\Phi_1(r)Q^{1/2}\right)\left(\Phi_2(r)Q^{1/2}\right)^*\right] dr$$

We recall also that stochastic integration theory with respect to martingales $M \in \mathcal{M}^2_T(F)$, completely analogous to the one with respect to a Wiener process described in preceeding chapters, can be developped, see [33]. The role of the process tQ is played by the quadratic variation $[M]_t^{dz}$, $t \in [0,T]$. We will need this extension in the case when the martingale M is itself a stochastic integral, say $M = \Phi \cdot W$ with $\Phi \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$. Then the extension is straightforward, since we can define the stochastic integral $\Psi \cdot M$ for $\Psi \in \mathcal{N}^2_M(0,T;L^2(F_0;G))$ simply by

$$\Psi \cdot M_t^{dz} = \int_0^t \Psi(s) dM_s^{dz} := \int_0^t \Psi(s)\Phi(s) dW_s^{dz},\ t \in [0,T]. \tag{3.6}$$

Note that

$$[\Psi \cdot M]_t^{dz} = \int_0^t \left(\Psi(s)\Phi(s)Q^{1/2}\right)\left(\Psi(s)\Phi(s)Q^{1/2}\right)^* ds \tag{3.7}$$

We recall that every operator in $L(H;F)$ is also in $L^2(H_0;F)$. In fact if $T \in L(H;F)$ then is well defined $L^2(H_0;F)$ because $H_0 = Q^{1/2}(H)$ is a subspace of H. Moreover if we suppose $T \in L(H;F)$, then, using the fact that $g_j = \sqrt{\lambda_j}e_j$ and $\|Te_j\|_F \le \|T\|_{L(H;F)}$ being $\{e_j\}$ a complete orthonormal system for H, we have

$$\|T\|^2_{L^2(H_0;F)} = \sum_{j=1}^{+\infty} \|Tg_j\|_F^2 = \sum_{j=1}^{+\infty} \lambda_j \|Te_j\|_F^2 \le \sum_{j=1}^{+\infty} \lambda_j \|T\|^2_{L(H;F)} = Tr(Q) \cdot \|T\|^2_{L(H;F)} < +\infty$$

Then for $L(H;F)$ predictable process Y such that $\mathbb{E}\int_0^t \|Y_s\|^2_{L(H;F)} ds < \infty$ it holds

$$\mathbb{E}\int_0^t \|Y_s\|^2_{L^2(H_0;F)} ds \le Tr(Q)\mathbb{E}\int_0^t \|Y_s\|^2_{L(H;F)} ds < \infty$$

so $Y \in \mathcal{N}^2_W(0,T;L^2(H_0;F))$, then the stochastic integral integral $\int Y \cdot dW^{dz}$ in the sense of [11] is a well defined F-valued process.

3.3.2 Main result

We consider $F = \mathbb{R}$.

Proposition 3.9. Let W a H-valued Q-Brownian motion with $Q \in L^1(H)$, i.e. $Tr(Q) = \sum_{j=1}^{+\infty} \lambda_j < +\infty$, and Y be a $L(H;\mathbb{R}) = H^*$ cadlag process such that $\mathbb{E}\int_0^t \|Y_s\|^2_{H^*} ds < \infty$. Then, for every $t \in [0,T]$,

$$\int_0^t \langle Y_s, d^- W_s\rangle = \int_0^t Y_s \cdot dW_s^{dz}$$

Proof. By the hypothesis we obtain that Y in $\mathcal{N}^2_W(0,T;L^2(H_0;\mathbb{R})))$. On the right hand we have a $\mathcal{M}^2_T(\mathbb{R})$ process because it is a stochastic integral for a process $Y \in \mathcal{N}^2_W(0,T;L^2(H_0;\mathbb{R}))$. We want to show that

$$\int_0^t \langle Y_s, \frac{W_{s+\epsilon} - W_s}{\epsilon}\rangle ds \xrightarrow{\mathbb{P}} \int_0^t Y_u \cdot dW_u^{dz} \tag{3.8}$$

We can represent $(W_{s+\epsilon} - W_s)$ as a H-valued stochastic integral in the sense of [11] with respect to the $L(H;H)$ elementary process identity on H. This integral, that we will denote with dW^{dz^*}, is with values in $\mathcal{M}^2_T(H)$ because the identity process belong to $\mathcal{N}^2_W(0,t;L^2(H_0;H))$.

$$W_{s+\epsilon} - W_s = \int_s^{s+\epsilon} dW_u^{dz^*}$$

Then the left hand in (3.8) gives

$$\frac{1}{\epsilon}\int_0^t \langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle ds = \frac{1}{\epsilon}\int_0^t \int_s^{s+\epsilon} Y_s \cdot dW_u^{dz}\, ds = \frac{1}{\epsilon}\int_0^t \int_{u-\epsilon}^u Y_s\, ds \cdot dW_u^{dz} \tag{3.9}$$

The first equality in (3.9) is true because, for a fixed $\epsilon > 0$ and $s \in [0,t]$, it holds

$$\langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle = \int_s^{s+\epsilon} Y_s \cdot dW_u^{dz}$$

Y_s is an elementary process so the definition for the right hand stochastic integral gives

$$\int_s^{s+\epsilon} Y_s \cdot dW_u^{dz} = \langle Y_s, W_{s+\epsilon}\rangle - \langle Y_s, W_s\rangle = \langle Y_s, W_{s+\epsilon} - W_s\rangle = \langle Y_s, \int_s^{s+\epsilon} dW_u^{dz^*}\rangle$$

The second equality in (3.9) is true by the Fubini's stochastic theorem in [11]. The term $\int_{u-\epsilon}^{u} Y_s\, ds$ has to be understood as a random Bochner type integral with values in H^*. We remark that $\int_{u-\epsilon}^{u} Y_s\, ds \in \mathcal{N}_W^2(0,t;L^2(H_0;\mathbb{R}))$ because it is bounded, in fact $\| \int_{u-\epsilon}^{u} Y_s\, ds\|_{H^*} \leq \frac{1}{\epsilon}\int_{u-\epsilon}^{u} \|Y_s\|_{H^*} ds \leq \sup_s \|Y_s\|_{H^*}$. It follows that the last integral in (3.9) is well defined. We will prove the following convergence in probability showing an $L^2(\Omega;\mathbb{R})$ convergence via the isometry property for the $\mathcal{M}_T^2(\mathbb{R})$ stochastic integral.

$$\int_0^t \int_{u-\epsilon}^{u} \frac{Y_s}{\epsilon}\, ds \cdot dW_u^{dz} - \int_0^t Y_u \cdot dW_u^{dz} \xrightarrow{\mathbb{P}} 0$$

It holds

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \int_{u-\epsilon}^{u} \frac{Y_s}{\epsilon}\, ds \cdot dW_u^{dz} - \int_0^t Y_u \cdot dW_u^{dz}\right]^2 &= \mathbb{E}\left[\int_0^t \left(\int_{u-\epsilon}^{u} \frac{Y_s}{\epsilon}\, ds - Y_u\right) \cdot dW_u^{dz}\right]^2 = \\
&= \mathbb{E}\left[\int_0^t \left\|\int_{u-\epsilon}^{u} \frac{Y_s}{\epsilon} ds - Y_u\right\|^2_{L^2(H_0;\mathbb{R})} du\right] \leq \\
&\leq \mathbb{E}\left[\int_0^t \left\|\int_{u-\epsilon}^{u} \frac{Y_s - Y_u}{\epsilon} ds\right\|^2_{H^*} du\right] \leq \\
&\leq \mathbb{E}\left[\int_0^t \left(\int_{u-\epsilon}^{u} \frac{\|Y_s - Y_u\|_{H^*}}{\epsilon} ds\right)^2 du\right] \to 0
\end{aligned}$$

We know that $\|Y_s - Y_u\|_{H^*} \to 0$ for for all $s \to u$, u continuity point for Y, moreover Y is cadlag, then it has a countable numbers of jumps so the result integrating is then $\int_{u-\epsilon}^{u} \frac{\|Y_s - Y_u\|_{H^*}}{\epsilon} ds \to 0$ □

In the special case $G = \mathbb{R}$, we obtain a similar result with respect to Brownian martingale

Proposition 3.10. Let M be a F-valued Brownian martingale $M \in \mathcal{M}_T^2(F)$ defined as a stochastic integral $M = \Phi \cdot W$, where $\Phi \in \mathcal{N}_W^2(0,T;L^2(H_0;\mathbb{R}))$. Let Y be a $L(F;\mathbb{R}) = F^*$-valued cadlag process such that $\mathbb{E}\int_0^T \|Y(s)\|^2_{F^*} ds < +\infty$.

Then for every $t \in [0,T]$

$$\int_0^t \langle Y_s, d^- M_s\rangle = \int_0^t Y_s \cdot dM_s^{dz}$$

We introduce now a new concept of quadratic variation.

Chapter 4

Chi-quadratic variation

4.1 Comments

In this chapter we will define a concept of quadratic variation which is suitable for Banach spaces. Let B be a Banach space.

Definition 4.1. A closed linear subspace χ of $(B\hat{\otimes}_\pi B)^*$, endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B\hat{\otimes}_\pi B)^*}. \tag{4.1}$$

will be called a **Chi-subspace of** $(B\hat{\otimes}_\pi B)^*$.

The result below follows immediately by the definition

Proposition 4.2. Any closed subspace of a Chi-subspace is a Chi-subspace.

We first try to explain why our concept is more general than other notions in the literature. The classical notions appear for instance in [33] (resp. [16]) for some classes of B-valued processes where B is a Hilbert (resp. Banach) space. One typical class is the family of π-processes which are not so far to Banach valued semimartingales, since their notion is constantly related to Itô type stochastic integrals. We remark that [11] introduces slight different notion of quadratic variation for B-valued martingales with B Hilbert separable space.
In that framework of infinite dimension valued stochastic process appear two concepts of quadratic variation: the *real quadratic variation* and the *tensor quadratic variation*. Let X be a B-valued stochastic process; in the language of regularizations, the first concept can be caracterised as the real-valued increasing continuous process which is ucp limit of $1/\epsilon \int_0^\cdot \|X_{s+\epsilon} - X_s\|_B^2 \, ds$ which equals

$$\frac{1}{\epsilon}\int_0^\cdot \left\|(X_{s+\epsilon} - X_s)\otimes^2\right\|_{B\otimes_\pi B} \, ds$$

according to 2.5. The second one, in [33, 16], appears to be related to expression of the type

$$X_t \otimes^2 - X_0 \otimes^2 - \int_{]0,t]} (X_{s^-} \otimes dX_s + dX_s \otimes X_{s^-});$$

in our framework, it corresponds to a $B\hat{\otimes}_\pi B$-valued process which is the ucp limit with respect to the projective topology π of

$$\frac{1}{\epsilon}\int_0^t (X_{s+\epsilon} - X_s) \otimes^2 \, ds, \tag{4.2}$$

where those integrals have to be considered in the Bochner sense.
In fact, the tensor quadratic variation is the natural object intervening in Itô's formula which expands $F(X)$ for some C^2-Fréchet B-valued function. To ensure that it has bounded variation, the classical procedure consists in showing that the real quadratic variation exists. In fact the variation of tensor quadratic variation is dominated by the variation of real quadratic variation, which is clearly of bounded variation being an increasing process.
Unfortunately, the existence of the real quadratic variation is a very requiring and rarely verified condition. For instance, the window Brownian motion $W(\cdot)$, which is our fundamental example, does not have, in principle, the real quadratic variation. In fact, even if for fixed ϵ the quantity

$$\int_0^t \frac{\|W_{s+\epsilon}(\cdot) - W_s(\cdot)\|^2_{C([-\tau,0])}}{\epsilon} ds$$

exists, it is not possible to control its limit for ϵ going to zero. The projective norm π is too strong for the convergence of the approximate tensor quadratic variation

$$\frac{1}{\epsilon}\int_0^t (W(\cdot)_{s+\epsilon} - W(\cdot)_s) \otimes^2 \, ds. \tag{4.3}$$

One possible relaxation could be to require a (strong) convergence with respect to a weaker tensor topology as the Hilbertian or the injective ϵ-topology, however this route was not easily practicable for us. As announced, our notion of convergence makes use of a subspace χ of $(B\hat{\otimes}_\pi B)^*$; when χ coincides with the whole space $(B\hat{\otimes}_\pi B)^*$ our convergence coincides the classical weak topology in $(B\hat{\otimes}_\pi B)$.
Our χ-quadratic variation generalizes the concept of tensor quadratic variation at two levels. Let X be a B-valued stochastic process.

- Firstly replacing the (strong) convergence in (4.2) with a weak type convergence.
- Secondly the choice of a suitable subspace χ of $(B\hat{\otimes}_\pi B)^*$ gives a degree of freedom.

As we will see in **??**, whenever X admits one of the classical quadratic variation (in the sense of [21, 11, 33, 16]), it admits a χ-quadratic variation with χ equal to the whole space. This corresponds to the *elementary* situation for us.

A window Brownian motion $X = W(\cdot)$ admits a χ- quadratic variation a priori only for strict subspaces χ. This will be particularly helpful in applications, in particular for obtaining some generalized Clark-Ocone formulae.

4.2 Examples of Chi-subspaces

Before providing the definition of the so-called χ-quadratic variation for a B-valued stochastic process, we will give some examples of Chi-subspaces that we will use frequently in the paper. We recall that a Chi-subspace has to be a topological subspace of $(B\hat{\otimes}_\pi B)^*$ such that (4.1) is verified. As a preliminary result we show that a finite direct sum of Chi-subspaces is still a Chi-subspace.

Proposition 4.3. Let $\chi_1, \cdots, \chi_n$ be Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$ such that $\chi_i \bigcap \chi_j = \{0\}$ for any $1 \leq i \neq j \leq n$. Then the normed space $\chi = \chi_1 \oplus \cdots \chi_n$ is a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$.

Proof. It is enough to prove the result for the case $n = 2$. Let $\mu \in \chi$, then it admits decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \chi_1, \quad \mu_2 \in \chi_2$. It holds $\|\mu\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_1\|_{(B\hat{\otimes}_\pi B)^*} + \|\mu_2\|_{(B\hat{\otimes}_\pi B)^*}$. By assumption, (4.1) for χ_1 and χ_2 implies that $\|\mu_i\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_i\|_{\chi_i}$ for $i = 1, 2$. It follows $\|\mu\|_{(B\hat{\otimes}_\pi B)^*} \leq \|\mu_1\|_{\chi_1} + \|\mu_2\|_{\chi_2}$, i.e. the norm (2) with $p = 1$ in the Banach space χ. Any norms defined in a direct sum of Banach spaces is equivalent to the product topology, then (4.1) is also verified for any norm. $\square$

Example 4.4. Let B be a general Banach space.

- $\chi = (B\hat{\otimes}_\pi B)^*$. This corresponds to our elementary case. We will show in chapter **??** that whenever a process admits a quadratic variation in the sense of [11, 33, 21] then it admits a $(B\hat{\otimes}_\pi B)^*$-quadratic variation.

Example 4.5. Let $B = C([-\tau, 0])$.
This is the natural value space of all window (continuous) processes. We list some examples of Chi-subspaces which are valuable for quadratic variations of window processes. Our basic reference subspace of $(C([-\tau, 0]\hat{\otimes}_\pi C([-\tau, 0]))^*$ will be $\mathcal{M}([-\tau, 0]^2)$ equipped with the usual total variation norm, denoted by $\|\cdot\|_{Var}$. This is in fact a proper subspace as it will be illustrated in the following lines. Condition (4.1) may be trivially verified using properties of projective tensor products, see section 2.5. All other Chi-subspaces will be included in $\mathcal{M}([-\tau, 0]^2)$. Moreover we will show those χ are Chi-subspaces of $\mathcal{M}([-\tau, 0]^2)$. In particular, they fulfill the (4.1) type relation $\|\cdot\|_\chi \geq \|\cdot\|_{Var}$. As a consequence, they will also be Chi-subspaces of $(B\hat{\otimes}_\pi B)^*$.

- $\mathcal{M}([-\tau, 0]^2)$. This space, equipped with the total variation norm, is a Banach space. We can identify this space with the dual of the injective tensor product; in fact

$$\begin{aligned}\mathcal{M}([-\tau,0]^2) &= \big(C([-\tau,0]^2)\big)^* = \big(C([-\tau,0])\hat{\otimes}_\epsilon \mathcal{C}([-\tau,0])\big)^* = \mathcal{B}_I(C([-\tau,0]),C([-\tau,0]))\\ &\subset \mathcal{B}(C([-\tau,0]),C([-\tau,0])) = \big(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0])\big)^*.\end{aligned}$$

In particular by properties of tensor product, (4.1) is verified because $\|\mu\|_{\epsilon^*} = \|\mu\|_{Var} \geq \|\mu\|_{(B\hat{\otimes}_\pi B)^*}$ for every $\mu \in \mathcal{M}([-\tau,0]^2)$.

- $L^2([-\tau,0]^2)$. This is a Hilbert subspace of $\mathcal{M}([-\tau,0]^2)$ and for $\mu \in L^2$ it holds obviously that $\|\mu\|_{Var} \leq \|\mu\|_{L^2([-\tau,0]^2)}$.
- $\mathcal{D}_{ij}([-\tau,0]^2)$ for every $i,j = 0,\ldots,N$. If $\mu = \lambda\, \delta_{a_i}(dx)\delta_{a_j}(dy)$, $\|\mu\|_{Var} = |\lambda| = \|\mu\|_{\mathcal{D}_{i,j}}$.
- $\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$. For a general element in this space $\mu = \lambda\delta_{a_i}(dx)\phi(y)dy$, $\phi \in L^2([-\tau,0])$, we have $\|\mu\|_{Var} \leq \|\mu\|_{L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_i([-\tau,0])} = |\lambda| \cdot \|\phi\|_{L^2}$.
- $\chi^2([-\tau,0]^2) := (L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0]))\hat{\otimes}_h^2$. This space will be denoted frequently shortly by χ^2. This is a well defined Hilbert space with the scalar product which derives from the scalar products in every Hilbert space and it is a subset of $\mathcal{M}([-\tau,0]^2)$ and consequently also of $(B\hat{\otimes}_\pi B)^*$.

Remark 4.6. 1. We could have shown that $\chi^2([-\tau,0]^2) \subset \mathcal{M}[-\tau,0]^2$ through an argument of tensor product theory. In fact if H is a Hilbert space such that $H \subset \mathcal{M}([-\tau,0])$ it holds $H\hat{\otimes}_h^2 \subset H\hat{\otimes}_\epsilon^2 \subset \mathcal{M}([-\tau,0])\hat{\otimes}_\epsilon^2 = C^*([-\tau,0])\hat{\otimes}_\epsilon^2 \subset (C([-\tau,0])\hat{\otimes}_\epsilon)^* = (C([-\tau,0]^2)^* = \mathcal{M}([-\tau,0]^2)$ because the ϵ-topology respects subspaces, see pag. 47 on [46]. Our $H = L^2 \oplus \mathcal{D}_a$ which is a Hilbert subset of $\mathcal{M}([-\tau,0])$ as required.

2. It will be useful have a direct sum representation, whenever it is possible, of the Chi-subspaces involved. In this case using once Remark 2.9, we obtain:

$$\chi^2([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2 \tag{4.4}$$

Using again Remark 2.9 with (2.16) and (2.17) we can expand every addend in the right-hand side of (4.4), into a sum of elementary addends. For instance we have $L^2\hat{\otimes}_h\mathcal{D}_a = \bigoplus_{i=0}^N \big(L^2\hat{\otimes}_h\mathcal{D}_i\big)$ and $\mathcal{D}_a\hat{\otimes}_h^2 = \mathcal{D}_A = \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}$ so that (4.4) equals

$$L^2([-\tau,0]^2) \oplus \bigoplus_{i=0}^N \big(L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_i([-\tau,0])\big) \oplus \bigoplus_{i=0}^N \big(\mathcal{D}_i([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])\big) \oplus \bigoplus_{i,j=0}^N \mathcal{D}_{i,j}([-\tau,0]^2) \tag{4.5}$$

Being χ^2 a finite direct sum of Chi-subspaces, Proposition 4.3 confirms that it is Chi-subspace.

- As a particular case of $\chi^2([-\tau,0]^2)$ we will denote $\chi^0([-\tau,0]^2)$, χ^0 shortly, the subspace of measures defined as

$$\chi^0([-\tau,0]^2) := (\mathcal{D}_0([-\tau,0]) \oplus L^2([-\tau,0]))\hat{\otimes}_h^2.$$

Again using Remark 2.9, we obtain:

$$\chi^0([-\tau,0]^2) = L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_0([-\tau,0]) \oplus \mathcal{D}_0([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \oplus \mathcal{D}_{0,0}([-\tau,0]^2) \tag{4.6}$$

Remark 4.7. For every μ in $\chi^2([-\tau,0]^2)$ there exist $\mu_1 \in L^2([-\tau,0]^2)$, $\mu_2 \in L^2([-\tau,0])\hat{\otimes}_h\mathcal{D}_a([-\tau,0])$, $\mu_3 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$ and $\mu_4 \in \mathcal{D}_a([-\tau,0])\hat{\otimes}_h^2$ such that

$$\mu = \mu_1 + \mu_2 + \mu_3 + \mu_4, \tag{4.7}$$

with $\mu_1 = \phi_1$, $\mu_2 = \sum_{i=0,\dots,N} \phi_2 \otimes \alpha_i\delta_{a_i}$, $\mu_3 = \sum_{i=0,\dots,N} \beta_i\delta_{a_i} \otimes \phi_3$ and $\mu_4 = \sum_{i,j=0,\dots,N} \lambda_{i,j}\delta_{a_i} \otimes \delta_{a_j}$, where $\phi_1 \in L^2([-\tau,0]^2)$, ϕ_1, $\phi_2 \in L^2([-\tau,0])$ and $\lambda_{i,j}$, α_i, β_i are real numbers for every $i,j = 0,\dots,N$. Components μ_1, μ_2 and μ_3 are singular with respect to the Dirac's measure on $\{a_i,a_j\}_{0\le i,j\le N}$, then $\mu_k(\{a_i,a_j\}) = 0$ for $k = 1,2,3$. For a general μ it follows

$$\mu(\{a_i,a_j\}) = \mu_4(\{a_i,a_j\}) = \lambda_{i,j} \tag{4.8}$$

If in particular $\mu \in \chi^0([-\tau,0]^2)$ then it can be uniquely decomposed into

$$\mu = \phi_1 + \phi_2 \otimes \alpha\delta_0 + \beta\delta_0 \otimes \phi_3 + \lambda\delta_0 \otimes \delta_0, \tag{4.9}$$

where $\phi_1 \in L^2([-\tau,0]^2)$, ϕ_2, ϕ_3 are functions in $L^2([-\tau,0])$ and λ, α, β are real numbers and

$$\mu(\{0,0\}) = \mu_4(\{0,0\}) = \lambda. \tag{4.10}$$

- $Diag([-\tau,0]^2)$. Let $\mu \in Diag$, we have $\|\mu\|_{Var} \le \tau\,\|\mu\|_{Diag}$, and (4.1) follows.

- $\chi^3([-\tau,0]^2) := \chi^2([-\tau,0]^2) \oplus Diag([-\tau,0]^2)$. The sum is direct and obviously it is a subset of $\mathcal{M}([-\tau,0]^2)$. As a consequence of Proposition 4.3, χ^3 is a Chi-subspace. This is Banach space with any norm in the direct sum, it fails to be a Hilbert space because $Diag$ is not Hilbert. We select here the norm (2), with $p = 2$. Let μ be an element in $\chi^3([-\tau,0]^2)$ with decomposition $\mu = \mu_1 + \mu_2$, $\mu_1 \in \chi^2([-\tau,0]^2)$ and $\mu_2 \in Diag([-\tau,0]^2)$, we define

$$\|\mu\|^2_{\chi^3([-\tau,0]^2)} = \|\mu_1\|^2_{\chi^2([-\tau,0]^2)} + \|\mu_2\|^2_{Diag([-\tau,0]^2)} \tag{4.11}$$

- $\chi^6([-\tau,0]^2)$ where

$$\chi^6([-\tau,0]^2) := \mathcal{D}_d([-\tau,0]^2) \oplus L^2([-\tau,0]^2) \oplus L^2([-\tau,0])\hat{\otimes}_h \mathcal{D}_a([-\tau,0]) \oplus \mathcal{D}_a([-\tau,0])\hat{\otimes}_h L^2([-\tau,0]) \tag{4.12}$$

This is a subspace of $\mathcal{M}([-\tau,0]^2)$ and it is a Chi-subspace because of Proposition 4.3.

The following examples are academic and they will not be used in the sequel in a relevant way. Some of them involves discrete infinite measures.

- $\chi^4([-\tau,0]^2) = \mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)$ with

$$\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2) := \left\{ \mu \in \mathcal{M}([-\tau,0]^2) : \mu = \sum_{i,j\in\mathbb{N}} \lambda_{i,j}\delta_{(\alpha_i,\alpha_j)}; \lambda_{i,j} \in \mathbb{R}, \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\} < +\infty \right\} \tag{4.13}$$

where $(\alpha_i)_{i\in\mathbb{N}}$ and $(\alpha_j)_{j\in\mathbb{N}}$ are two sequences of given points in $[-\tau,0]$, then an element of χ^4 is a discrete measure concentrated on a countable sequence of fixed points $(\alpha_i,\alpha_j)_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ on the square $[-\tau,0]^2$. The space $\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)$ equipped with the norm $\|\mu\|_{\mathcal{D}^{\mathbb{N}\times\mathbb{N}}([-\tau,0]^2)} = \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\}$, is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$.
To be a Chi-subspace it remains to show $\|\mu\|_{Var} \le \|\mu\|_{\chi^4}$. For an element $\mu\in\chi^4$ the total variation norm is $\|\mu\|_{Var([-\tau,0]^2)} = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}|$ and it is finite. In particular $\|\mu\|_{Var([-\tau,0]^2)} = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}| = \sum_{i,j\in\mathbb{N}} |\lambda_{i,j}|i^2j^2 \frac{1}{i^2j^2} \le \sup_{i,j}\{|\lambda_{i,j}|i^2j^2\} \sum_{i,j\in\mathbb{N}} \frac{1}{i^2j^2} = \|\mu\|_{\chi^4}\frac{\pi^4}{36}$.

- Let $\{\mu_i\}_{i=1,\dots,N}$ be N fixed mutually singular measures in $\mathcal{M}([-\tau,0]^2)$ with $\|\mu_i\|_{Var}=1$. We define the space $\chi^5([-\tau,0]^2)$ as the space

$$\chi^5([-\tau,0]^2) := Span(\{\mu_i\}_{i=1,\dots,N}) = \left\{ \mu = \sum_{i=1,\dots,N} \lambda_i\mu_i;\ \mu_i \in \mathcal{M}([-\tau,0]^2),\ \lambda_i\in\mathbb{R} \right\}. \tag{4.14}$$

The space χ^5 equipped with the norm $\|\mu\|_{\chi^5} = \sqrt{\sum_{i=1}^N \lambda_i^2}$, is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ of finite dimension N. The norm $\|\cdot\|_{\chi^5}$ is compatible with the induced topology defined by $\mathcal{M}([-\tau,0]^2)$. By Proposition 4.2, χ^5 is a Chi-subspace. We observe that $\|\mu\|_{Var} = \sum_{i=1}^N |\lambda_i| \le \|\mu\|_{\chi^5} = \sqrt{\sum_{i=1}^N \lambda_i^2}$.

- Let μ be a fixed finite measure on $[-\tau,0]^2$ singular with respect to the Lebesgue measure.

$$\chi^\mu([-\tau,0]^2) = \{\nu\in\mathcal{M}([-\tau,0]^2); d\nu = g\,d\mu, g\in L^\infty(d\mu)\} \tag{4.15}$$

Without restriction of generality we can consider μ being a positive measure. χ^μ is the space of measure absolutely continuous with respect μ with density in $L^\infty(d\mu)$. The space χ^μ equipped with the norm $\|\nu\|_{\chi^\mu} := \|g\|_{L^\infty}$ is a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ and it is therefore isomorphic to $L^\infty(d\mu)$. For a general measure $\nu\in\chi^\mu$ it holds $\|\nu\|_{Var} \le \|g\|_{L^\infty}\|\mu\|_{Var} = C\,\|\nu\|_{\chi^\mu}$ with C a constant, so it is a Chi-subspace. We illustrate two significant cases:

1. Let I be a countable set. Let $\{\mu_i\}_{i\in I}$, singular non-negative finite measure. The set constituted by measures ν of the type $\nu = \sum_{i\in I} g_i\mu_i$, g_i Borel bounded functions coincides with χ^μ with $\mu = \sum_{i\in I}\mu_i$. In fact $d\nu = g\,d\mu$, with $g = \sum_{i\in I} g_i \frac{d\mu_i}{d\mu}$. We observe that $\|\nu\|_{\chi^\mu} = \sup_i \left\{\|g_i\|_{L^\infty(\mu_i)}\right\} = \|g\|_{L^\infty(\mu)}$. By definition of μ we have $\mu_i << \mu$ and $d\mu_i\ d\mu$ belong to $L^\infty(d\mu)$, in fact for avery set A we have $\mu(A) = \int_A \mathbb{1}_A d\mu$ and for every $i \in I$ we have $\mu_i(A) = \int_A f_i d\mu$ by the Radon-Nikodym theorem because $\mu_i << \mu$. On the other hand $\mu_i(A) = \int_A d\mu_i$ then $f_i = d\mu_i/d\mu$. To prove $f_i \in L^\infty(d\mu)$ we take a general set A. The following integral $\mu(A) - \mu_i(A) = \int_A \mathbb{1}_A - f_i d\mu$ is always greater or equal to 0, μ being a sum of positive measures. So we conclude that $f_i \le \mathbb{1}_A$ μ-a.e.
2. As a special case of previous example we can take $\mu_i = \delta_{(a_i,b_i)}$, where $(a_i, b_i) \in [-\tau, 0]^2$ for $i \in I = \{= 1, \ldots, N\}$, then $\nu = \sum_{i=1}^N \lambda_i\delta_{(a_i,b_i)}$ and easily $\|\nu\| = \max_{1\le i\le N}\{|\lambda_i|\}$.

- Another example of Chi-subspace is $L^2([-\tau, 0]^2) \oplus \chi^\mu([-\tau, 0]^2)$, where μ is a given measure in $\mathcal{M}([-\tau, 0]^2)$. This is a Chi-subspace again because of Proposition 4.3.

Example 4.8. Let $B = H = L^2([-\tau, 0])$.
For processes with values in the Hilbert space $H = L^2([-\tau, 0])$, χ has to be a subset of $(L^2([-\tau, 0]\hat{\otimes}_\pi L^2([-\tau, 0]))^*$. We recall that $(L^2([-\tau, 0]\hat{\otimes}_\pi L^2([-\tau, 0]))^* = \mathcal{B}(L^2([-\tau, 0], L^2([-\tau, 0]))$. This Banach space contains two significant Chi-subspaces; the first one is naturally associated with $L^2([-\tau, 0]^2$, the second one with $L^\infty([-\tau, 0])$.

We observe that $L^2([-\tau, 0]^2) = L^2([-\tau, 0])\hat{\otimes}_h^2 \equiv (L^2([-\tau, 0])\hat{\otimes}_h^2)^* \subset (L^2([-\tau, 0])\hat{\otimes}_\pi^2)^*$, where $\equiv$ is the usual Riesz identification and the last inclusion is continuous as we have seen in Remark 2.8. The space $L^2([-\tau, 0]^2)$ identifies a subspace of bilinear bounded (continuous) forms on $(L^2([-\tau, 0]) \times L^2([-\tau, 0]))$. In fact for every $f \in L^2([-\tau, 0]^2)$ we can associate a bilinear operator

$$T^f : L^2([-\tau, 0]) \times L^2([-\tau, 0]) \longrightarrow \mathbb{R} \qquad (g, h) \mapsto T^f(g, h) = \int_{[-\tau,0]^2} g(x)h(y)f(x, y)\,dx\,dy. \quad (4.16)$$

Definition 4.9. We will denote by $L^2_{\mathcal{B}}([-\tau, 0]^2)$ the set of all bilinear maps T^f. This space equipped with the norm $\|T^f\|_{L^2_{\mathcal{B}}([-\tau,0]^2)} := \|f\|_{L^2([-\tau,0]^2)}$, is a Hilbert space which indeed coincides with $L^2([-\tau, 0]^2)^*$.

Proposition 4.10. $L^2_{\mathcal{B}}([-\tau, 0]^2)$ is included properly in $\mathcal{B}(L^2([-\tau, 0], L^2([-\tau, 0]))$.

Remark 4.11. We anticipate that $L^2_{\mathcal{B}}([-\tau, 0]^2)$ is not densely included in $\mathcal{B}(L^2([-\tau, 0], L^2([-\tau, 0]))$. This will be shown in Proposition 5.35 using a probabilistic argument.

Proof of Proposition 4.10. We can prove that the bilinear bounded form defined by

$$T : L^2([-\tau, 0]) \times L^2([-\tau, 0]) \longrightarrow \mathbb{R} \qquad (g, h) \mapsto T(g, h) = \int_{[-\tau,0]} g(x)h(x)\,dx = \langle g, h\rangle_{L^2([-\tau,0])}$$

does not belong to $L^2_{\mathcal{B}}([-\tau, 0]^2)$.
We denote the Hilbert space $L^2([-\tau, 0])$ by H. To show that $T \notin L^2([-\tau, 0]^2)$ we will use the identification

between $L^2([-\tau,0]^2) = H\hat{\otimes}_h^2$ and the space $L^2\big(H;H^*\big)$ of Hilbert-Schmidt operators from H to H^*. We show that the operator $\bar{T}$ in $L(H;H^*)$ associated canonically to T in $\mathcal{B}(H,H)$ is not Hilbert-Schmidt. Canonical identification between $\mathcal{B}(E,F)$ and $L(E;F^*)$ give us

$$_{F^*}\langle \bar{T}(e), f\rangle_F = T(e,f) \qquad\qquad e \in E,\ f\in F$$

In our case $E = F =: H$ and $T(e,f) = {_H}\langle f, g\rangle_H$.
Let $(e_i)_{i\in\mathbb{N}}$ and $(e_j^*)_{j\in\mathbb{N}}$ be respectively the usual basis of H and its dual H^*. Using Parseval's identity and the Riesz isomorphism $e_i \mapsto e_i^*$ we obtain

$$\sum_{i=1}^{+\infty} \|\bar{T}(e_i)\|_{H^*}^2 = \sum_{i=1}^{+\infty}\sum_{j=1}^{\infty} {_{H^*}}\langle \bar{T}(e_i), e_j^*\rangle_{H^*}^2 = \sum_{i=1}^{+\infty}\sum_{j=1}^{\infty} {_{H^*}}\langle \bar{T}(e_i), e_j\rangle_{H}^2 = \sum_{i=1}^{+\infty}\sum_{j=1}^{\infty} T(e_i,e_j) =$$
$$= \sum_{i=1}^{+\infty}\sum_{j=1}^{\infty} {_H}\langle e_i, e_j\rangle_H^2 = \sum_{i=1}^{+\infty} 1 = +\infty$$

Then $\bar{T}$ is not Hilbert-Schmidt and consequently $T \notin L^2([-\tau,0]^2) = L^2([-\tau,0])\hat{\otimes}_h L^2([-\tau,0])$. □

Below we describe the announced Chi-subspaces.

- $\chi = L^2_{\mathcal{B}}([-\tau,0]^2)$ equipped with its norm. We verify directly condition (4.1). We recall the isometry between $(L^2([-\tau,0]\hat{\otimes}_\pi L^2([-\tau,0]))^*$ and $\mathcal{B}(L^2([-\tau,0], L^2([-\tau,0]))$, i.e. the usual norm of the bilinear operator T^f, denoted by $\|\cdot\|$, is equal to the norm of the corresponding element in $(L^2([-\tau,0])\hat{\otimes}_\pi^2)^*$. So it is enough to remark that

$$\|T^f\| = \sup_{\|g\|\le 1, \|f\|\le 1} |T(g,h)| \le \|f\|_{L^2([-\tau,0]^2)} = \|T^f\|_{L^2_{\mathcal{B}}([-\tau,0]^2)}$$

  Condition (4.1) should have been verified also using relations (2.11) and (2.12).

- $\chi = Diag_{\mathcal{B}}([-\tau,0]^2)$ where $Diag_{\mathcal{B}}([-\tau,0]^2)$ is the following set

$$\left\{ T^f \in \mathcal{B}\big(L^2([-\tau,0]), L^2([-\tau,0])\big),\ \text{s.t. } T^f(g,h) = \int_{[-\tau,0]} g(x)h(x)f(x)\,dx\ ; f\in L^\infty([-\tau,0]) \right\} \quad (4.17)$$

  By definition it is a subspace of $\mathcal{B}\big(L^2([-\tau,0]), L^2([-\tau,0])\big)$ and every operator T^f is determined by a function in $f \in L^\infty([-\tau,0])$. This space equipped with the norm $\|T^f\|_{Diag_{\mathcal{B}}([-\tau,0]^2)} := \|f\|_{L^\infty([-\tau,0])}$ is a Banach space. We verify condition (4.1). Let $T \in Diag_{\mathcal{B}}([-\tau,0]^2)$, we have

$$\|T\| = \sup_{\|g\|\le 1,\ \|h\|\le 1} |T(g,h)| = \sup_{\|g\|\le 1,\ \|h\|\le 1} \left| \int_{[-\tau,0]} g(x)h(x)f(x)dx \right| \le \|f\|_{L^\infty([-\tau,0])} = \|T\|_{Diag_{\mathcal{B}}([-\tau,0]^2)}$$

Remark 4.12. This space has been denoted with $Diag_{\mathcal{B}}$ because it has a strong relation with the space of measures $Diag$ defined in (2.20). In fact let φ be a function in $L^\infty([-\tau,0])$, we can associate a measure $\mu^\varphi \in Diag([-\tau,0]^2)$ and an operator $T^\varphi \in Diag_{\mathcal{B}}([-\tau,0]^2)$. The measure is identified by $\mu^\varphi(dx,dy) = \varphi(x)\delta_y(dx)dy$. The bilinear operator is identified by $T^\varphi(g,h) = \int_{[-\tau,0]} g(x)h(x)\varphi(x)\,dx$. Let η_1, η_2 be two elements in $C([\tau,0]) \subset L^2([-\tau,0])$,

$$
{}_{\mathcal{M}([-\tau,0]^2)}\langle \mu^\varphi, \eta_1 \otimes \eta_2 \rangle_{C([-\tau,0]^2)} = \langle \mu^\varphi(dx,dy), \eta_1(x)\cdot\eta_2(y)\rangle = \int_{[-\tau,0]^2} \eta_1(x)\eta_2(y)\varphi(x)\delta_y(dx)dy = \\
= \int_{[-\tau,0]} \eta_1(x)\eta_2(x)\varphi(x)dx
$$

and

$$
\langle T^\varphi, \eta_1 \otimes \eta_2 \rangle = T(\eta_1,\eta_2) = \int_{[-\tau,0]} \eta_1(x)\eta_2(x)\varphi(x)dx
$$

For instance if φ is the constant function equal to 1, then diagonal measure μ^1 corresponds to the inner product in $L^2([-\tau,0])$ in the sense that

$$
\langle \mu^1, \eta_1 \otimes \eta_2 \rangle = T^1(\eta_1,\eta_2) = \langle \eta_1,\eta_2\rangle_{L^2([-\tau,0])}.
$$

Remark 4.13. We recall that the bilinear functions in $L^2_{\mathcal{B}}([-\tau,0]^2)$ identified with $L^2([-\tau,0]^2)$, can be also observed as a subspace of $\mathcal{M}([-\tau,0]^2)$.

4.3 Definition of χ-quadratic variation and some related results

In this section, we introduce the definition of the χ-quadratic variation of a B-valued stochastic process X.

Let χ be a Chi-subspace, X be a B-valued stochastic process and $\epsilon > 0$. We denote by $[X,X]^\epsilon$, or simply by $[X]^\epsilon$, the following application

$$
[X]^\epsilon : \chi \longrightarrow \mathcal{C}([0,T])
$$

defined by

$$
\phi \mapsto \left(\int_0^t \langle \phi, \frac{J\left((X_{s+\epsilon} - X_s)\otimes^2\right)}{\epsilon} \rangle \, ds \right)_{t\in[0,T]}
$$

where the canonical injection J between a space and its bidual was introduced in section 2.1. In the sequel J will be often omitted. With this application it is possible to associate another one, denoted by $\widetilde{[X,X]}^\epsilon$, or simply by $\widetilde{[X]}^\epsilon$, defined by

$$
\widetilde{[X]}^\epsilon(\omega,\cdot) : [0,T] \longrightarrow \chi^*
$$

such that

$$t \mapsto \left(\phi \mapsto \frac{1}{\epsilon}\int_0^t \langle \phi, J\left((X_{s+\epsilon}-X_s)\otimes^2\right)\rangle\, ds\right).$$

We observe that it is of bounded variation.

Remark 4.14. We recall that $\chi \subset (B\hat{\otimes}_\pi B)^*$ then $(B\hat{\otimes}_\pi B) \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$. In this context $\langle \cdot,\cdot\rangle$ indicates the duality between the space χ and its dual χ^*. ϕ is in fact an element of χ and $(X_{s+\epsilon}-X_s)\otimes^2 \in B\hat{\otimes}_\pi B$, then $J\left((X_{s+\epsilon}-X_s)\otimes^2\right) \in (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$. In the case $B = C([-\tau,0])$, we will identify $\eta_1\otimes\eta_2$ in $(B\hat{\otimes}_\pi B) \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$ with the element η in $C([-\tau,0]^2)$ defined by $\eta(x,y) = \eta_1(x)\cdot\eta_2(y)$. In this context, all the considered Chi-subspaces will be subspaces of $\mathcal{M}([-\tau,0]^2)$ and the pair duality between χ and χ^* will be compatible with the pair duality between a measure μ and the continuous function η.

Definition 4.15. Let χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued stochastic process. We say that X **admits a χ-quadratic variation** if the following assumptions are fulfilled.

H1 For all $(\epsilon_n)\downarrow 0$ it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi\le 1}\left|\langle\phi, \frac{(X_{s+\epsilon_{n_k}}-X_s)\otimes_\pi^2}{\epsilon_{n_k}}\rangle\right| ds = \sup_k \frac{1}{\epsilon_{n_k}}\int_0^T \left\|J\left((X_{s+\epsilon_{n_k}}-X_s)\otimes_\pi^2\right)\right\|_{\chi^*} ds \quad < \infty \tag{4.18}$$

H2 (i) It exists an application $\chi \longrightarrow \mathcal{C}([0,T])$, denoted by $[X,X]$ or simply by $[X]$, such that $[X,X]^\epsilon(\phi) \xrightarrow{ucp} [X,X](\phi)$ when $\epsilon\to 0_+$ for every $\phi\in\chi\subset(B\hat{\otimes}_\pi B)^*$.

(ii) There is a bounded variation process $\widetilde{[X,X]}(\omega,\cdot):[0,T]\longrightarrow\chi^*$ such that $\widetilde{[X,X]}(\omega,t)(\phi) = [X,X](\phi)(\omega,t)$. This application will be denoted also by $\widetilde{[X]}$.

Remark 4.16.
1. Under Assumption **H2**(i), for fixed (ω,t) the application $\phi\mapsto[X,X](\phi)(\omega,t)$ is linear, however for fixed (ω,t) it could be not continuous.
2. The **H2**(ii) condition can be omitted in most cases using Corollary 4.30.

When X admits a χ-quadratic variation, we will call **χ-quadratic variation** of X the χ^*-valued process $(\widetilde{[X]})_{0\le t\le T}$ defined for every $\omega\in\Omega$ and $t\in[0,T]$ by $\phi\mapsto\widetilde{[X]}(\omega,t)(\phi) = [X](\phi)(\omega,t)$. Sometimes, with a slight abuse of notation, even $[X]$ will be called χ-quadratic variation and it will be confused with $\widetilde{[X]}$.

Remark 4.17.
1. A practical criterion to verify Condition **H1** is

$$\frac{1}{\epsilon}\int_0^T \left\|J\left((X_{s+\epsilon}-X_s)\otimes_\pi^2\right)\right\|_{\chi^*} ds \le B(\epsilon) \tag{4.19}$$

where $B(\epsilon)$ converges in probability. In fact convergence in probability is equivalent with convergence a.s. of a subsequence, and the convergence implies the boundness.

2. A consequence of Condition **H1** is that for all $(\epsilon_n) \downarrow 0$ it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \|\widetilde{[X]}^{\epsilon_{n_k}}\|_{Var[0,T]} < \infty \quad a.s. \tag{4.20}$$

In fact $\|\widetilde{[X]}^{\epsilon}\|_{Var[0,T]} \leq \frac{1}{\epsilon}\int_0^T \|J\left((X_{s+\epsilon}-X_s)\otimes_\pi^2\right)\|_{\chi^*} ds$. This implies that for a χ-valued continuous stochastic process Y the integral $\int_0^t \langle Y_s, d\widetilde{[X]}_s^{\epsilon_{n_k}}\rangle$ is a well-defined Lebesgue-Stieltjes type integral for almost all $\omega \in \Omega$.

Definition 4.18. We say that a continuous B-valued process X admits **global quadratic variation** if it admits a χ-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$. We will also say that X is a finite quadratic variation process.

Proposition 4.19. Let X be a B-valued process and χ_1, χ_2 be two Chi-subspaces. Let $\chi = \chi_1 \oplus \chi_2$. If X admits χ_i-quadratic variation $[X]_i$ for $i = 1, 2$ then it admits a χ-quadratic variation $[X]$ and it holds $[X](\phi) = [X]_1(\phi_1) + [X]_2(\phi_2)$ for all $\phi \in \chi$ with unique decomposition $\phi = \phi_1 + \phi_2$.

Proof. χ is a Chi-subspace because of Proposition 4.3. We remark that for all possible norm in $\chi_1 \oplus \chi_2$ we have $\|\phi\|_\chi \geq \|\phi_i\|_{\chi_i}$. Then condition **H1** follows immediately by inequality

$$\begin{aligned}\int_0^T \sup_{\|\phi\|_{\chi_1\oplus\chi_2}\leq 1} \left|\langle \phi, (X_{s+\epsilon}-X_s)\otimes^2\right| ds \leq & \int_0^T \sup_{\|\phi_1\|_{\chi_1}\leq 1} \left|\langle \phi_1, (X_{s+\epsilon}-X_s)\otimes^2\right| ds + \\ & + \int_0^T \sup_{\|\phi_2\|_{\chi_2}\leq 1} \left|\langle \phi_2, (X_{s+\epsilon}-X_s)\otimes^2\right| ds\end{aligned}$$

Condition **H2**(i) follows by linearity; in fact

$$\begin{aligned}[X]^\epsilon(\phi) &= \int_0^t \langle \phi_1+\phi_2, (X_{s+\epsilon}-X_s)\otimes^2\rangle ds = \\ &= \int_0^t \langle \phi_1, (X_{s+\epsilon}-X_s)\otimes^2\rangle ds + \int_0^t \langle \phi_2, (X_{s+\epsilon}-X_s)\otimes^2\rangle ds \xrightarrow[\epsilon\longrightarrow 0]{ucp} [X]_1(\phi_1) + [X]_2(\phi_2)\end{aligned}$$

We also have $\widetilde{[X]}(t)(\phi) = [X](\phi)(t) = \widetilde{[X]}_1(t)(\phi_1) + \widetilde{[X]}_2(t)(\phi_2)$. $\widetilde{[X]}$ has bounded variation because $\|\widetilde{[X]}\|_{Var[0,T]} \leq \|\widetilde{[X]}_1\|_{Var[0,T]} + \|\widetilde{[X]}_2\|_{Var[0,T]}$ a.s., then **H2**(ii) follows. Finally X admits χ-quadratic variation $[X]$ and it is equal to $[X](\phi) = [X]_1(\phi_1) + [X]_2(\phi_2)$. □

Proposition 4.20. Let X be a B-valued stochastic process and χ_1 χ_2 two Chi-subspaces such that $\chi_1 \subset \chi_2 \subset (B\hat{\otimes}_\pi B)^*$ densely embedded, i.e. $\|\phi\|_{\chi_1} \geq \|\phi\|_{\chi_2} \geq \|\phi\|_{(B\hat{\otimes}_\pi B)^*}$ for all $\phi \in \chi_1$.
If X admits a χ_2-quadratic variation $[X]_2$, then it admits also a χ_1-quadratic variation $[X]_1$ and it holds $[X]_1(\phi) = [X]_2(\phi)$ for all $\phi \in \chi_1$.

Proof. We remark that $(X_{s+\epsilon}-X_s)\otimes^2$ is an element in $(B\hat{\otimes}_\pi B)\subset(B\hat{\otimes}_\pi B)^{**}\subset\chi_2^*\subset\chi_1^*$. Assumption **H1** follows immediately using the inequality $\left\|(X_{s+\epsilon}-X_s)\otimes^2\right\|_{\chi_1^*}\leq\left\|(X_{s+\epsilon}-X_s)\otimes^2\right\|_{\chi_2^*}$. Assumption **H2**(i) is trivially verified because for all $\phi\in\chi_1$ by hypothesis we have $[X]^\epsilon(\phi)\xrightarrow[\epsilon\to 0]{ucp}[X]_2(\phi)$. Moreover $\widetilde{[X]}_1(t)(\phi)=[X]_1(\phi)(t)=\widetilde{[X]}_2(t)(\phi)$ and $\|\widetilde{[X]}_1\|_{Var[0,T]}\leq\|\widetilde{[X]}_2\|_{Var[0,T]}$. So that also point (ii) of condition **H2** is established. We conclude that X admits χ_1-quadratic variation and it holds $[X]_1(\phi)=[X]_2(\phi)$ for all $\phi\in\chi_1$. □

Remark 4.21.

1. On the contrary, let χ_1 χ_2 be two Chi-subspaces such that $\chi_1\subset\chi_2\subset(B\hat{\otimes}_\pi B)^*$ as for Proposition 4.20, it may happens that a B-valued process X does not admit a $(B\hat{\otimes}_\pi B)^*$-quadratic variation or not even a χ_2-quadratic variation but it admits a χ_1-quadratic variation. For this reason the fact to introduce a subspace of $(B\hat{\otimes}_\pi B)^*$ give much more possibilities of calculus.

2. It is obvious that if Condition **H1** is verified for χ_2 than Condition **H1** is verified for χ_1. In fact if $A:=\{\phi\in\chi_1\,;\|\phi\|_{\chi_1\leq 1}\}$ and $B:=\{\phi\in\chi_2\,;\|\phi\|_{\chi_2\leq 1}\}$, then $A\subset B$ and $\int_0^t\sup_A|\langle\phi,(X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes^2\rangle|ds\leq\int_0^t\sup_B|\langle\phi,(X_{s+\epsilon}(\cdot)-X_s(\cdot))\otimes^2\rangle|ds$.

3. We anticipate that the $C([-\tau,0])$-valued window Brownian motion admits a χ^2-quadratic variation but it does not have $\mathcal{M}([-\tau,0]^2)$-quadratic variation. This will be seen in details in chapter 5.

We continue with some general properties of χ-quadratic variation.

Lemma 4.22. If the sequence of random variables defined, for every ϵ, by $\frac{1}{\epsilon}\int_0^T\|J\left((X_{s+\epsilon}-X_s)\otimes^2\right)\|_{\chi^*}\,ds$ converge to 0 in probability then X admits a zero χ-quadratic variation.

Proof. Condition **H1** is verified because of Remark 4.17(1). We verify **H2**(i) directly. For every fixed $\phi\in\chi$ we have

$$|[X,X]^\epsilon(\phi)(t)|=\left|\int_0^t\langle\phi,\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\,ds\right|\leq\int_0^t\left|\langle\phi,\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right|\,ds\leq\int_0^T\left|\langle\phi,\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right|\,ds$$

then we obtain

$$\sup_{t\in[0,T]}|[X,X]^\epsilon(\phi)(t)|\leq\int_0^T\left|\langle\phi,\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right|\,ds\leq\|\phi\|_\chi\frac{1}{\epsilon}\int_0^T\|J\left((X_{s+\epsilon}-X_s)\otimes^2\right)\|_{\chi^*}\,ds\xrightarrow{\mathbb{P}}0$$

in probability by the hypothesis of the lemma. This allows to conclude. □

An important proposition used later to prove Itô's formula is the following.

Proposition 4.23. Let $F^n:\chi\longrightarrow\mathcal{C}([0,T])$ be a sequence of linear continuous maps and $\widetilde{F}^n(\omega,\cdot):[0,T]\longrightarrow\chi^* a.s.$ such that $\widetilde{F}^n(\omega,t)(\phi):=F^n(\phi)(\omega,t)$. We suppose the following:

i) it exists a linear continuous map $F:\chi\longrightarrow\mathcal{C}([0,T])$ such that for all $t\in[0,T]$ and for every $\phi\in\chi$ $F^n(\phi)(\cdot,t)\longrightarrow F(\phi)(\cdot,t)$ in probability.
Moreover it exists $\widetilde{F}(\omega,\cdot):[0,T]\longrightarrow\chi^*$ of bounded variation defined by $\widetilde{F}(\omega,t)(\phi):=F(\phi)(\omega,t)$.

ii) for all (n_k) it exists (n_{k_j}) such that $\sup_j \|\widetilde{F}^{n_{k_j}}\|_{Var[0,T]} < \infty$.

Then for every $t \in [0,T]$ and every continuous process $H : \Omega \times [0,T] \longrightarrow \chi$

$$\int_0^t \langle H(\cdot, s), d\widetilde{F}^n(\cdot, s)\rangle \longrightarrow \int_0^t \langle H(\cdot, s), d\widetilde{F}(\cdot, s)\rangle \qquad \text{in probability.} \tag{4.21}$$

Proof. Let $t \in [0,T]$ fixed. Let $\delta > 0$ and a subdivision of $[0,t]$ $0 = t_0 < t_1 < \cdots < t_m = t$ with mesh smaller than δ. Let (n_k) be a sequence diverging to infinity and (n_{k_j}) a subsequence according to **ii**. We denote

$$I(n)(\omega) := \int_0^t \langle H(\omega, s), d\widetilde{F}^n(\omega, s)\rangle - \int_0^t \langle H(\omega, s), d\widetilde{F}(\omega, s)\rangle$$

Up to a further subsequence, that in general will always be denoted by (n_{kj}), we need to prove that

$$I(n_{k_j})(\omega) \to 0 \text{ a. s.} \tag{4.22}$$

In fact, for $\omega \in \Omega$ we have

$$\begin{aligned}
\left|I(n_{k_j})(\omega)\right| &= \left|\sum_{i=1}^m \left(\int_{t_{i-1}}^{t_i} \langle H(\omega, s), d\widetilde{F}^{n_{k_j}}(\omega, s)\rangle - \langle H(\omega, s), d\widetilde{F}(\omega, s)\rangle\right)\right| \leq \\
&\leq \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega, s) - H(\omega, t_{i-1}) + H(\omega, t_{i-1}), d\widetilde{F}^{n_{k_j}}(\omega, s)\rangle + \right. \\
&\qquad \left. - \int_{t_{i-1}}^{t_i} \langle H(\omega, s) - H(\omega, t_{i-1}) + H(\omega, t_{i-1}), d\widetilde{F}(\omega, s)\rangle\right| \leq \\
&\leq I_1(n_{k_j})(\omega) + I_2(n_{k_j})(\omega) + I_3(n_{k_j})(\omega)
\end{aligned}$$

where

$$\begin{aligned}
I_1(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega, s) - H(t\omega_{,i-1}), d\widetilde{F}^{n_{k_j}}(\omega, s)\rangle\right| \leq \varpi(H(\omega, \cdot), \delta) \sup_j \|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]} \\
I_2(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega, s) - H(\omega, t_{i-1}), d\widetilde{F}(\omega, s)\rangle\right| \leq \varpi(H(\omega, \cdot), \delta)\, \|\widetilde{F}(\omega)\|_{Var[0,T]} \\
I_3(n_{k_j})(\omega) &= \sum_{i=1}^m \left|\int_{t_{i-1}}^{t_i} \langle H(\omega, t_{i-1}), d(\widetilde{F}^{n_{k_j}}(\omega, s) - \widetilde{F}(\omega, s))\rangle\right| = \\
&= \sum_{i=1}^m \left|\langle H(\omega, t_{i-1}), \widetilde{F}^{n_{k_j}}(\omega, t_i) - \widetilde{F}(\omega, t_i) - \widetilde{F}^{n_{k_j}}(\omega, t_{i-1}) + \widetilde{F}(\omega, t_{i-1})\rangle\right| \leq \\
&\leq \sum_{i=1}^m |F^{n_{k_j}}(H(\omega, t_{i-1}))(\omega, t_i) - F(H(\omega, t_{i-1}))(\omega, t_i)| + \\
&\qquad \sum_{i=1}^m |F^{n_{k_j}}(H(\omega, t_{i-1}))(\omega, t_{i-1}) - F(H(\omega, t_{i-1}))(\omega, t_{i-1})|
\end{aligned}$$

The notation $\varpi(H(\omega,\cdot),\delta)$ indicates the modulus of continuity for H and it is a random variable, in fact it depends on ω in the sense that

$$\varpi(H(\omega,\cdot),\delta)=\sup_{|s-t|\le\delta}\|H(\omega,s)-H(\omega,t)\|_{\chi}$$

Assumption **i** implies that $I_3(n_{k_j})(\cdot)\to 0$ in probability because $F^n(\phi)(\cdot,t)\to F(\phi)(\cdot,t)$ for all $\phi=H(\omega,t_i)$, $i=0,\dots,m$. After extracting a further subsequence we can suppose that $I_3(n_{k_j})(\cdot)\to 0$ a.s. for $j\to+\infty$. Therefore a.s.

$$\limsup_{j\to\infty}|I(n_{k_j})(\omega)|\le\left(\sup_j\|\widetilde{F}^{n_{k_j}}(\omega)\|_{Var[0,T]}+\|\widetilde{F}(\omega)\|_{Var[0,T]}\right)\varpi(H(\omega,\cdot),\delta) \tag{4.23}$$

Since $\delta>0$ is arbitrary and H is uniformly continuous on $[0,t]$ so that $\varpi(H(\cdot,\cdot),\delta)\to 0$ a.s. for $\delta\to 0$, then $\limsup_{j\to\infty}|I(n_{k_j})(\cdot)|=0$ a.s..
This concludes (4.22) and the proof of the Propositon. $\square$

Corollary 4.24. Let X be a B-valued stochastic process with χ-quadratic variation and H a continuous measurable process $H:\Omega\times[0,T]\longrightarrow\chi$. Then for every $t\in[0,T]$

$$\int_0^t\langle H(\cdot,s),d\widetilde{[X]}^{\epsilon}(\cdot,s)\rangle\longrightarrow\int_0^t\langle H(\cdot,s),d\widetilde{[X]}(\cdot,s)\rangle \tag{4.24}$$

in probability.

Proof. The proof follows from Proposition 4.23 and definition of χ-quadratic variation. Of course the ucp convergence implies in fact the convergence in probability for every $t\in[0,T]$. $\square$

An important theorem for Banach valued stochastic integration is given below. It will be a consequence of Banach-Steinhaus type result for Fréchet spaces, see Theorem II.1.18, pag. 55 in [17]. We start with a remark.

Remark 4.25.

1. In the mentioned Banach-Steinhaus theorem intervenes the following notion. Let E be a Fréchet spaces, F-space shortly. A subset B of E is called **bounded** if for all $\epsilon>0$ it exists δ_ϵ such that for all $0<\alpha\le\delta_\epsilon$, αB is included in the open ball $\mathcal{B}(0,\epsilon)$.

2. Let (Y^n) be a sequence of random elements with values in a Banach space $(B,\|\cdot\|_B)$ such that $\sup_n\|Y^n\|_B\le Z$ a.s. for some positive random variable Z. Then (Y^n) is bounded in the F-space of random elements equipped with the convergence in probability equipped with the metric

$$d(X,Y)=\mathbb{E}\left[\|X-Y\|\wedge 1\right].$$

In fact by Lebesgue dominated convergence theorem we have $\lim_{\gamma\to 0}\mathbb{E}[\gamma Z\wedge 1]=0$.

3. In particular taking $B = C([0,T])$ a sequence of continuous processes (Y^n) such that $\sup_{t\le T} |Y_t^n| \le Z$ a.s. is bounded for the usual metric in $\mathcal{C}([0,T])$ equipped with the topology related to the ucp convergence.

Theorem 4.26. Let $F^n : \chi \longrightarrow \mathcal{C}([0,T])$ be a sequence of linear continuous maps such that $F^n(\phi)(0) = 0$ a.s. and such $F^n(\phi)(\omega,t) = \tilde{F}^n(\omega,t)(\phi)$ where $\tilde{F}^n(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ a.s.
We suppose the following:

i) $\sup_n \|\tilde{F}^n\|_{Var} < \infty$ a.s.

ii) There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathcal{C}([0,T])$ such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$.

Then there is a linear and continuous extension $F : \chi \longrightarrow \mathcal{C}([0,T])$ and there is $\tilde{F} : \Omega \times [0,T] \longrightarrow \chi^*$ such that $\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t)$. Moreover the following properties hold:

a) For every $\phi \in \chi$, $F^n(\phi) \xrightarrow{ucp} F(\phi)$. In particular for every $t \in [0,T]$, $\phi \in \chi$, $F^n(\phi)(\cdot,t) \xrightarrow{\mathbb{P}} F(\phi)(\omega,t)$.

b) $\|\tilde{F}\|_{Var} < \infty$ a.s.

Remark 4.27. 1) Given $G : \chi \longrightarrow C([0,T])$ we can associate $\tilde{G} : [0,T] \longrightarrow \chi^*$ setting $\tilde{G}(t)(\phi) = G(\phi)(t)$. $\tilde{G} : [0,T] \longrightarrow \chi^*$ has bounded variation if

$$\|\tilde{G}\|_{Var[0,T]} = \sup_{\sigma\in\Sigma_{[0,T]}} \sum_{(t_i)_i\in\sigma} \left\|\tilde{G}(t_{i+1}) - \tilde{G}(t_i)\right\|_{\chi^*} = \sup_{\sigma\in\Sigma_{[0,T]}} \sum_{(t_i)_i\in\sigma} \sup_{\|\phi\|_\chi\le 1} |G(\phi)(t_{i+1}) - G(\phi)(t_i)| < \infty$$

This quantity is called **total variation** of $\tilde{G}$.
For example if $G(\phi) = \int_0^t \dot{G}_s(\phi)\, ds$ then $\|G\|_{Var} = \int_0^T \sup_{\|\phi\|_\chi\le 1} |\dot{G}_s(\phi)|\, ds$.

2) Unfortunately the situation is more complicated for stochastic processes. Let $F : \chi \longrightarrow \mathcal{C}([0,T])$. For every $\phi \in \chi$, $F(\phi) \in C([0,T])$ a.s. and we associate $\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t)$. It may happen that for fixed $\omega \in \Omega$, $t \in [0,T]$ the linear form $\tilde{F}(\omega,t)$ is not continuous. In fact given $\phi_n \longrightarrow \phi$ in χ, $F(\phi_n) \longrightarrow F(\phi)$ in $\mathcal{C}([0,T])$ with the ucp convergence, then there is only a subsequence such that $F(\phi_{n_k}) \longrightarrow F(\phi)$ a.s. in $C([0,T])$.

Proof of the Theorem 4.26.

a) We recall that $\mathcal{C}([0,T])$ is an F-space. It holds $\sup_{t\in[0,T]} |F^n(\phi)(t)| \le \sup_n \|\tilde{F}^n\|_{Var}\|\phi\| < \infty$ a.s. by the hypothesis. By Remark 4.25(2,3) it follows that the set $\{F^n(\phi)\}$ is a bounded subset of the F-space $\mathcal{C}([0,T])$ for every fixed $\phi \in \mathcal{S}$.
We can apply the mentioned Banach-Steinhaus type theorem, which implies the existence of $F : \chi \longrightarrow \mathcal{C}([0,T])$ linear and continuous such that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \chi$. So **a)** is established.

b.1) Let (n_k) be a sequence, for $\omega \in \Omega$, $t \in [0,T]$ and $\phi \in \chi$ we set

$$\tilde{F}(\omega,t)(\phi) = F(\phi)(\omega,t) \qquad \text{and} \qquad \tilde{F}^{n_k}(\omega,t)(\phi) = F^{n_k}(\phi)(\omega,t).$$

We first prove the existence of a suitable version $\tilde{F}$ of F such that $\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ is weakly star continuous ω a.s.
Since χ is separable, we consider a dense countable subset $\mathcal{D} \subset \chi$. Point **a)** implies that for a fixed $\phi \in \mathcal{D}$ there is a further subsequence (n_k) such that $F^{n_k}(\phi)(\omega,t) \longrightarrow F(\phi)(\omega,t)$ for all $t \in [0,T]$ a.s. Since $\mathcal{D}$ is countable there is a null set N_0 and subsequence (n_k) such that

$$F^{n_k}(\phi)(\omega,t) \longrightarrow F(\phi)(\omega,t) \qquad \forall\, t \in [0,T],\ \forall\, \phi \in \mathcal{D},\ \forall \omega \notin N_0 \tag{4.25}$$

Let $t \in [0,T]$, $\omega \notin N_0$ the sequence

$$\tilde{F}^{n_k}(\omega,t) : \chi \longrightarrow \mathbb{R}$$

are linear continuous forms such that

- $\tilde{F}^{n_k}(\omega,t)(\phi) \longrightarrow \tilde{F}(\omega,t)(\phi)$ for all $\phi \in \mathcal{D}$, because of (4.25).
- $\sup_k |\tilde{F}^{n_k}(\omega,t)(\phi)| \le \sup_k \sup_{\|\phi\|\le 1} |\tilde{F}^{n_k}(\omega,t)(\phi)|\, \|\phi\| \le \sup_k \|\tilde{F}^{n_k}(\omega,t)\|_{Var} \|\phi\| < \infty$ a.s.

Banach Steinhaus thereom for Banach spaces implies the existence of

$$\tilde{F}(\omega,t) : \chi \longrightarrow \mathbb{R}$$

linear continuous form extending previous map $\tilde{F}(\omega,t)$ from $\mathcal{D}$ to χ with

$$\tilde{F}^{n_k}(\omega,t)(\phi) \longrightarrow \tilde{F}(\omega,t)(\phi) \qquad \forall\, t \in [0,T],\ \forall \phi \in \chi,\ \forall \omega \notin N_0$$

Moreover, for every $\omega \notin N_0$ the application

$$\tilde{F}(\omega,\cdot) : [0,T] \longrightarrow \chi^* \qquad t \mapsto \tilde{F}(\omega,t)$$

is weakly star continuous, i.e. for $t_n \to t$, $\tilde{F}(\omega,t_n)(\phi) \to \tilde{F}(\omega,t)(\phi)$ for all $\phi \in \chi$. $\tilde{F}(\omega,\cdot)$ is measurable from $[0,T]$ to χ^* being limit of measurable functions.

b.2) We prove now that the χ^*-valued process $\tilde{F}$ has bounded variation.
Let $\omega \notin N_0$ fixed again. Let $(t_i)_{i=0}^N$ be a subdivision of $[0,T]$ and let $\phi \in \chi$. Since the functions

$$F^{t_i,t_{i+1}} : \phi \longrightarrow \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\right)(\phi) \qquad F^{n_k,t_i,t_{i+1}} : \phi \longrightarrow \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i)\right)(\phi)$$

belong to χ^*, Banach-Steinhaus theorem says

$$\sup_{\|\phi\|\le 1} \left| \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i)\right)(\phi) \right| = \|F^{t_i,t_{i+1}}\|_{\chi^*} \le \lim\inf_{k\to\infty} \|F^{n_k,t_i,t_{i+1}}\|_{\chi^*} =$$

$$= \liminf_{k\to\infty} \sup_{\|\phi\|\leq 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i) \right)(\phi) \right|$$

Taking the sum over $i = 0, \dots, (N-1)$ we get

$$\sum_{i=0}^{N-1} \sup_{\|\phi\|\leq 1} \left| \left(\tilde{F}(t_{i+1}) - \tilde{F}(t_i) \right)(\phi) \right| \leq \sum_{i=0}^{N-1} \liminf_{k\to\infty} \sup_{\|\phi\|\leq 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i) \right)(\phi) \right| \leq$$

$$\leq \sup_k \sum_{i=0}^{N-1} \sup_{\|\phi\|\leq 1} \left| \left(\tilde{F}^{n_k}(t_{i+1}) - \tilde{F}^{n_k}(t_i) \right)(\phi) \right| \leq \sup_k \|\tilde{F}^{n_k}\|_{Var}$$

where the second inequality is justified by the relation $\liminf a_i^n + \liminf b_i^n \leq \sup(a_i^n + b_i^n)$. Taking the sup over all subdivision $(t_i)_{i=0}^N$ we obtain

$$\|\tilde{F}\|_{Var} \leq \sup_k \|\tilde{F}^{n_k}\|_{Var} < \infty$$

This shows finally the fact that $\tilde{F}(\omega, \cdot) : [0, T] \longrightarrow \chi^*$ has bounded variation.

□

Corollary 4.28. We can replace condition **ii)** in Therem 4.26 with

ii') There is a subset $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$ and a linear application $F : \mathcal{S} \longrightarrow \mathcal{C}([0,T])$ such that for every $\phi \in \mathcal{S}$.

- $F^n(\phi)(t) \longrightarrow F(\phi)(t)$ for every $t \in [0,T]$ in probability
- $F^n(\phi)$ is an increasing process.

Proof. Since for every $\phi \in \mathcal{S}$, $F(\phi)$ is an increasing process, Lemma 2.1 implies that $F^n(\phi) \longrightarrow F(\phi)$ ucp for every $\phi \in \mathcal{S}$, so **ii)** is established. □

Important implications of Theorem 4.26 and its Corollary 4.28 are Corollaries 4.29 and 4.30, which gives us easier conditions for existence of χ-quadratic variation. We will replace **H2** with convergence in probability in a generator system $\mathcal{S}$ of χ.

Corollary 4.29. Let B be a Banach space, $\chi \subset (B\hat{\otimes}_\pi B)^*$ be a Chi-subspace and X a B-valued stochastic process. We suppose the following

H0' There is $\mathcal{S} \subset \chi$ such that $\overline{Span(\mathcal{S})} = \chi$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s)\otimes_\pi^2}{\epsilon_{n_k}} \rangle \right| ds \quad < +\infty$$

and

H2' There is $\mathcal{T} : \chi \longrightarrow \mathcal{C}([0,T])$ such that $[X,X]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ ucp for all $\phi \in \mathcal{S}$.

Then X admits a χ-quadratic variation and it is equal to $\mathcal{T}$.

Proof. Condition **H1** is verified by assumption. Conditions **H2**(i) and (ii) follow by Theorem 4.26. □

Corollary 4.30. Let B be a Banach space, $\chi \subset (B\hat{\otimes}_\pi B)^*$ be a Chi-subspace and X a B-valued stochastic process. We suppose the following

H0" There are subsets $\mathcal{S}$, $\mathcal{S}^p$ of χ such that $\overline{Span(\mathcal{S})} = \chi$, $Span(\mathcal{S}) = Span(\mathcal{S}^p)$ and $\mathcal{S}^p$ is constituted by **positive definite** elements ϕ in the sense that $\langle \phi, b \otimes b \rangle \geq 0$ for all $b \in B$.

H1 For every sequence $(\epsilon_n) \downarrow 0$ there is a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi \leq 1} \left| \langle \phi, \frac{(X_{s+\epsilon_{n_k}} - X_s)\otimes_\pi^2}{\epsilon_{n_k}} \rangle \right| ds \quad < +\infty$$

and

H2" There is $\mathcal{T} : \chi \longrightarrow \mathcal{C}([0,T])$ such that $[X,X]^\epsilon(\phi)(t) \to \mathcal{T}(\phi)(t)$ in probability for every $\phi \in \mathcal{S}$ and for every $t \in [0,T]$.

Then X admits a χ-quadratic variation and it is equal to $\mathcal{T}$.

Proof. We verify the conditions of Corollary 4.29. Conditions **H0'** and **H1** are verified by assumption. We observe that for every $\phi \in \mathcal{S}^p$, $[X,X]^\epsilon(\phi)$ is an increasing process. By linearity, it follows that for any $\phi \in \mathcal{S}^p$, $[X,X]^\epsilon(\phi)(t)$ converges in probability to $\mathcal{T}(\phi)(t)$ for any $t \in [0,T]$. Lemma 2.1 implies that $[X,X]^\epsilon(\phi)$ converges ucp for every $\phi \in \mathcal{S}^p$ and therefore in $\mathcal{S}$. Conditions **H2'** of Corollary 4.29 is now verified. □

Chapter 5

Evaluations of χ-quadratic variations

In this chapter, for simplicity of exposition, we will consider in most of the cases $\tau = T$. Only when it is really necessary in view of furthers applications we develop computations also for $\tau < T$. We start with some examples of χ-quadratic variation calculated directly through the definition.

Proposition 5.1. Let X be a real valued process with Hölder continuous paths of parameter $\gamma > 1/2$. Then the $C([-T,0])$-valued process $X(\cdot)$ admits a zero global quadratic variation.

Proof. Since the injection $J : B \to B^{**}$ is isometric, quantity

$$\frac{1}{\epsilon}\int_0^T \left\|(X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\right\|_{(B\hat{\otimes}_\pi B)^{**}} ds \tag{5.1}$$

equals

$$\begin{aligned}\frac{1}{\epsilon}\int_0^T \left\|(X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\right\|_{B\hat{\otimes}_\pi B} ds &= \frac{1}{\epsilon}\int_0^T \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|^2_{C[-T,0]}\, ds \\ &= \frac{1}{\epsilon}\int_0^T \sup_{u\in[-T,0]} \left|X_{s+u+\epsilon} - X_{s+u}\right|^2 ds\end{aligned}$$

Since X is a.s. γ-Hölder continuous this is bounded by $Z(\epsilon) := \epsilon^{2\gamma-1}\, Z\, T$ where Z is a non-negative finite random variable. This implies that (5.1) converges to zero a.s. for $\gamma > \frac{1}{2}$. Lemma 4.22 implies that the process admits zero global quadratic variation. $\square$

Remark 5.2. By Proposition 4.20 every window process associated to a continuous process with Hölder continuous paths of parameter $\gamma > 1/2$ admits zero χ-quadratic variation for every Chi-subspace χ, for instance $\chi = \mathcal{M}([-T,0]^2)$.

Definition 5.3. The **fractional Brownian motion** B^H of Hurst parameter $H \in (0,1]$ is a centered Gaussian process with covariance

$$R^H(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

If $H = 1/2$ it corresponds to a classical Brownian motion. The process is Hölder continuous of order γ for any $\gamma \in (0, H)$. This follows from the Kolmogorov criterion, see [30].

Definition 5.4. The **bifractional Brownian motion** $B^{H,K}$ is a centered Gaussian process with covariance

$$R^{H,K}(t,s) = \frac{1}{2^K}\left((t^{2H} + s^{2H})^K - |t-s|^{2HK}\right)$$

with $H \in (0,1)$ and $K \in (0,1]$. Notice that if $K = 1$, then $B^{H,1}$ coincides with a fractional Brownian motion with Hurst parameter $H \in (0,1)$.
We recall some properties about quadratic variation in the particular case $HK = 1/2$ from Proposition 1 in [39]. If $K = 1$, then $H = 1/2$ and it is a Brownian motion. If $K \neq 1$, it provides an example of a Gaussian process, having non-zero finite quadratic variation which in particular equals $2^{1-K}t$, so, modulo a constant, the same as Brownian motion. The process is Hölder continuous of order γ for any $\gamma \in (0, HK)$. This follows again from Kolmogorov criterion.

Remark 5.5. As consequences of Proposition 5.1 we have the following properties.

1. The fractional window Brownian motion $B^H(\cdot)$ with $H > 1/2$ admits a zero global quadratic variation.

2. The bifractional window Brownian motion $B^{H,K}(\cdot)$ with $KH > 1/2$ admits a zero global quadratic variation.

Remark 5.6. We recall that a Brownian motion W has Hölder continuous paths of parameter $\gamma < 1/2$ so that we can not use Proposition 5.1.

Remark 5.7. In principle the window Brownian motion $W(\cdot)$ does not admit even a $\mathcal{M}([-T,0]^2)$-quadratic variation because the first condition is not verified. However we do not have a formal proof of this. Presumably the window Brownian motion $W(\cdot)$ does not admit a global quadratic variation. In fact setting $B = C([-\tau, 0])$, it is possible to show that the expectation

$$\lim_{\epsilon \to 0} \mathbb{E}\left[\int_0^T \frac{1}{\epsilon} \|W_{u+\epsilon}(\cdot) - W_u(\cdot)\|_B^2 \, du\right] = +\infty. \tag{5.2}$$

This is a consequence of the following result.

Proposition 5.8. Let W be a classical Brownian motion. Let $0 < \tau_1 < \tau_2$, then there are positive constants C_1, C_2 such that

$$C_1 \leq \mathbb{E}\left[\sup_{u \in [\tau_1, \tau_2]} \frac{|W_{u+\epsilon} - W_u|^2}{\epsilon \ln(1/\epsilon)} du\right] \leq C_2$$

Proof. See [26]. $\square$

The following proposition constitutes an existence result of a χ-quadratic variations calculated with the help of Corollaries 4.29 and 4.30.

Proposition 5.9. Let X be a real continuous process with finite quadratic variation $[X]$. Then

1) $X(\cdot)$ admits zero χ-quadratic variation, where $\chi = L^2([-T,0]^2)$.

2) $X(\cdot)$ admits zero χ-quadratic variation for every $i \in \{0,\dots,N\}$, where $\chi = L^2([-T,0])\hat{\otimes}_h \mathcal{D}_i([-T,0])$.

If moreover the covariation $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\dots,N\}$, then

3) $X(\cdot)$ admits χ-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t. \tag{5.3}$$

where $\chi = \mathcal{D}_{i,j}([-T,0]^2)$.

Proof. Example 4.5 says that the three involved sets χ are Chi-subspaces. Let $\{e_j\}_{j\in\mathbb{N}}$ be a basis for $L^2([-T,0])$; $\{f_i = \delta_{a_i}\}$ is clearly a basis for $\mathcal{D}_i$. Then $\{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ is a basis of $L^2([-T,0]^2)$, $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is a basis of $L^2([-T,0])\hat{\otimes}_h\mathcal{D}_i([-T,0])$ and $\{f_i \otimes f_j\}$ is a basis of $\mathcal{D}_{i,j}([-T,0]^2)$. We will show the result using Corollary 4.30. To verify Condition **H1** we consider

$$A(\epsilon) := \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_\chi \le 1} \left|\langle \phi, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds$$

for the three Chi-subspaces. In all the three situations we will show the existence of a family of random variables $\{B(\epsilon)\}$ converging in probability to some random variable B, such that $A(\epsilon) \le B(\epsilon)$ a.s. In particular this clearly implies Assumption **H1**.

1) Suppose $\chi = L^2([-T,0]^2)$. By Cauchy Schwarz inequality we have

$$\begin{aligned} A(\epsilon) &\le \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|_{L^2([-T,0]^2)}\le 1} \|\phi\|_{L^2([-T,0]^2)} \cdot \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|^2_{L^2([-T,0])} \le \\ &\le \frac{1}{\epsilon}\int_0^T \int_0^s (X_{u+\epsilon} - X_u)^2 \, du\, ds \le B(\epsilon) \end{aligned}$$

where

$$B(\epsilon) = T\int_0^T \frac{(X_{u+\epsilon} - X_u)^2}{\epsilon} du$$

which converges in probability to $T\,[X]_T$.

2) We proceed now similarly for $\chi = L^2([-T,0])\hat{\otimes}_h\mathcal{D}_i([-T,0])$. We consider ϕ of the form $\phi = \tilde{\phi}\otimes\lambda_i\delta_{\{a_i\}}$, where $\tilde{\phi}$ is an element of $L^2([-T,0])$, $\lambda_i\delta_{\{a_i\}}$ is an element of $\mathcal{D}_i$. We first recall

$$\|\phi\|_{L^2([-T,0])\hat{\otimes}_h\mathcal{D}_i} = \left\|\tilde{\phi}\right\|_{L^2([-T,0])} \cdot \left\|\lambda_i\delta_{\{a_i\}}\right\|_{\mathcal{D}_i} = \sqrt{\int_{[-T,0]} \tilde{\phi}(s)^2\, ds}\sqrt{\lambda_i^2}$$

Then

$$
\begin{aligned}
A(\epsilon) = \frac{1}{\epsilon}\int_0^T & \sup_{\|\phi\|_{L^2([-T,0])\hat{\otimes}_h \mathcal{D}_i}\leq 1} \left| \lambda_i \left(X_{s+\epsilon}(a_i) - X_s(a_i)\right) \int_{[-T,0]} \left(X_{s+\epsilon}(x) - X_s(x)\right) \tilde{\phi}(x)\, dx \right| ds \leq \\
& \leq \frac{1}{\epsilon}\int_0^T \sup_{\|\phi\|\leq 1} \left(\sqrt{\lambda_i^2} \sqrt{\left(X_{s+\epsilon}(a_i) - X_s(a_i)\right)^2} \right) \cdot \\
& \qquad \cdot \left(\left\|\tilde{\phi}\right\|_{L^2([-T,0])} \sqrt{\int_{[-T,0]} \left(X_{s+\epsilon}(x) - X_s(x)\right)^2\, dx} \right) ds \leq \\
& \leq \int_0^T \frac{1}{\epsilon} \sqrt{\left(X_{s+\epsilon}(a_i) - X_s(a_i)\right)^2} \sqrt{\int_{[-T,0]} \left(X_{s+\epsilon}(x) - X_s(x)\right)^2\, dx}\, ds \leq B(\epsilon)
\end{aligned}
$$

where

$$
B(\epsilon) = \int_0^T \frac{(X_{y+\epsilon} - X_y)^2}{\epsilon}\, dy
$$

sequence that converges in probability to $[X]_T$.

3) For the last case $\chi = \mathcal{D}_{i,j}([-T,0]^2)$ we consider a general element $\phi = \lambda_i \delta_{\{a_i\}} \otimes \mu_j \delta_{\{a_j\}} = \lambda_i \mu_j \delta_{\{(a_i,a_j)\}}$, λ_i, μ_j in $\mathbb{R}$. Its norm is defined by $\|\phi\|_{\mathcal{D}_{i,j}} = \sqrt{\lambda_i^2 \mu_i^2}$. Then, again using Cauchy Schwarz inequality,

$$
\begin{aligned}
A(\epsilon) = \frac{1}{\epsilon}\int_0^T & \sup_{\|\phi\|_{\mathcal{D}_{i,j}}\leq 1} \left| \lambda_i \mu_j \left(X_{s+a_i+\epsilon} - X_{s+a_i}\right) \left(X_{s+a_j+\epsilon} - X_{s+a_j}\right) \right| ds \leq \\
& \leq \frac{1}{\epsilon}\int_0^T \left| \left(X_{s+a_i+\epsilon} - X_{s+a_i}\right) \left(X_{s+a_j+\epsilon} - X_{s+a_j}\right) \right| ds \leq \\
& \leq \sqrt{\int_0^T \frac{\left(X_{s+a_i+\epsilon} - X_{s+a_i}\right)^2}{\epsilon} ds} \sqrt{\int_0^T \frac{\left(X_{s+a_j+\epsilon} - X_{s+a_j}\right)^2}{\epsilon} ds} \leq B(\epsilon)
\end{aligned}
\tag{5.4}
$$

where

$$
B(\epsilon) = \int_0^T \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds
$$

We verify now Conditions **H0”** and **H2”**.

1) A general element in $\{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ is different of two positive definite elements in $\{e_i \otimes^2, (e_i + e_j)\otimes^2\}_{i,j\in\mathbb{N}}$. Therefore we set $\mathcal{S} = \{e_i \otimes e_j\}_{i,j\in\mathbb{N}}$ and $\mathcal{S}^p = \{e_i\otimes^2, (e_i + e_j)\otimes^2\}_{i,j\in\mathbb{N}}$. This implies **H0”**. It remains to verify

$$
[X(\cdot)]^\epsilon (e_i \otimes e_j)(t) \longrightarrow 0 \tag{5.5}
$$

in probability for any $i, j \in \mathbb{N}$ in order to conclude the validity of Condition **H2”**. Without restriction of generality we can suppose $\{e_i\}_{i\in\mathbb{N}} \in C^1([-T,0])$. We fix $\omega \in \Omega$, fixed but omitted. We have

$$
[X(\cdot)]^\epsilon (e_i \otimes e_j)(t) = \int_0^t \gamma(s,\epsilon) \int_{-s}^0 e_i(x) \left(X_{s+x+\epsilon} - X_{s+x}\right) dx\, ds \tag{5.6}
$$

where

$$\gamma(s,\epsilon) = \frac{1}{\epsilon}\int_{-s}^{0} e_j(y)\,(X_{s+y+\epsilon} - X_{s+y})\,dy$$

Now for every $s \in [0,T]$, a.s.

$$\gamma(s,\epsilon) \longrightarrow X_s e_j(0) - \int_{-s}^{0} X_{s+y} de_j(y)$$

and for every $s \in [0,T]$, $\epsilon > 0$,

$$|\gamma(s,\epsilon)| \leq \sup_{t\in[0,T]} |X_t|\ \|e_j\|_\infty + \sup_{t\in[0,T]} |X_t| \int_0^T \dot{e_j}(y)dy$$

Consequently Lebesgue dominated convergence theorem implies that

$$[X(\cdot)]^\epsilon(e_i \otimes e_j)(t) \longrightarrow 0$$

a.s. and therefore (8.1).

2) A general element in $\{e_j \otimes f_i\}_{j\in\mathbb{N}}$ is different of two positive definite elements of type $\{e_j\otimes^2, f_i\otimes^2, (e_j + f_i)\otimes^2\}_{j\in\mathbb{N}}$. This shows **H0"**. It remains to show that

$$[X(\cdot)]^\epsilon\,(e_j \otimes f_i)\,(t) \longrightarrow 0 \tag{5.7}$$

in probability. In fact the left-hand side equals

$$\int_0^t \gamma(s,\epsilon)\,(X_{s+a_i+\epsilon} - X_{s+a_i})\ ds$$

A similar argument as for point 1) shows the result.

3) A general element $f_i \otimes f_j$ is different of two positive definite elements $(f_i + f_j)\otimes^2$ and $f_i \otimes^2 + f_j\otimes^2$. So that Condition **H0"** is fulfilled. Concerning Condition **H2"** we have, for $0 \leq i,j \leq N$,

$$[X(\cdot)]^\epsilon\,(f_i \otimes f_j)\,(t) = \frac{1}{\epsilon}\int_0^t (X_{s+a_i+\epsilon} - X_{s+a_i})\left(X_{s+a_j+\epsilon} - X_{s+a_j}\right) ds$$

This converges to $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ which exists by hypothesis.

This finally concludes the proof of Proposition 5.9. □

We recall that $\mathcal{D}_d$, $\mathcal{D}_A$, χ^2, χ^0 and χ^6 were defined respectively at (2.19), (2.17), (4.4), (4.6) and (4.12).

Corollary 5.10. Let X be a real continuous process with finite quadratic variation $[X]$. Then for every $i \in \{0,\ldots,N\}$

4) $X(\cdot)$ admits a $\mathcal{D}_{i,i}([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \mu(\{a_i, a_i\})[X_{\cdot+a_i}, X_{\cdot+a_i}]_t. \tag{5.8}$$

5) $X(\cdot)$ admits a $\mathcal{D}_d([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X]_{t+a_i}. \tag{5.9}$$

6) $X(\cdot)$ admits a $\chi^0([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \mu(\{0,0\})[X]_t. \tag{5.10}$$

7) $X(\cdot)$ admits a $\chi^6([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X]_{t+a_i} \tag{5.11}$$

Corollary 5.11. Let X be a real continuous process with covariation structure $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ for every $i, j = 0, \ldots, N$, in particular it is has finite quadratic variation process $[X]$. Then

8) $X(\cdot)$ admits a $\mathcal{D}_A([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t \tag{5.12}$$

9) $X(\cdot)$ admits a $\chi^2([-T,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t \tag{5.13}$$

Proof of Corollaries 5.10 and 5.11. Space χ^2 admits a finite direct sum decomposition given by (4.5). Also χ^6, χ^0, $\mathcal{D}_d$ and $\mathcal{D}_A$ admit a finite direct sum decomposition by definition. Results follow using Propositions 4.19 and 5.9 □

We mention a particular case of Corollary 5.11 that we will frequently meet in the paper.

Remark 5.12. Let X be a real continuous process with covariation structure such that $[X_{\cdot+a_i}, X_{\cdot+a_j}] = 0$ for $i \neq j$. In this case the $\chi^2([-T,0]^2)$-quadratic variation of $X(\cdot)$ simplifies in

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X_{\cdot+a_i}]_t = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X_\cdot]_{t+a_i} \tag{5.14}$$

Remark 5.13. We remark that in Corollary 5.10 (respectively in Corollary 5.11) the quadratic variation $[X]$ of the real finite quadratic variation process X (respectively the covariation structure of X) not only insure the existence of χ-quadratic variation but complete determines the χ-quadratic variation. For example if X is a real finite quadratic variation process such that $[X]_t = t$, then $X(\cdot)$ has the same χ^0-quadratic variation as the window Brownian motion. On the contrary in Remark 5.12 all the covariation structure of X is necessary to insure the existence of χ-quadratic variation even if it is completely determined only by the quadratic variation $[X]$.

Now we list two corollaries of Propositions 5.9 and 4.19 that will be useful in the application to Dirichlet processes in chapter 6.3.

Corollary 5.14. Let V be a real continuous zero quadratic variation process. Then the associated window process $V(\cdot)$ has zero $\mathcal{D}_A([-\tau,0]^2)$-quadratic variation. In particular the associated window process of a real bounded variation process has zero $\mathcal{D}_A([-\tau,0]^2)$-quadratic variation.

Corollary 5.15. Let V be a real continuous zero quadratic variation process. Then $V(\cdot)$ has zero $\chi^2([-\tau,0]^2)$-quadratic variation.

Proposition 5.16. Let V a real absolutely continuous process such that $V' \in L^2([0,T])$ ω-a.s. Then the associated window process $V(\cdot)$ has zero $\mathcal{M}([-\tau,0]^2)$-quadratic variation.

Proof. By hypothesis $V_t - V_0 = \int_0^t g(s)\, ds$ ω-a.s., with $g \in L^2([0,T])$ ω a.s..
Using Lemma 4.22, it will be enough to show the convergence to zero in probability of the quantity

$$\int_0^t \frac{1}{\epsilon} \sup_{\|\mu\|_{Var}\leq 1} \left| \int_{[-\tau,0]^2} \left(V_{s+\epsilon}(x_1) - V_s(x_1)\right)\left(V_{s+\epsilon}(x_2) - V_s(x_2)\right)\, \mu(dx_1,dx_2) \right| ds \tag{5.15}$$

for which we will even show the a.s. convergence. (5.15) equals

$$\int_0^{t\wedge\tau} \frac{1}{\epsilon} \sup_{\|\mu\|_{Var}\leq 1} \left| \int_{[-s,0]^2} \left(V_{s+\epsilon}(x_1) - V_s(x_1)\right)\left(V_{s+\epsilon}(x_2) - V_s(x_2)\right)\, \mu(dx_1,dx_2) \right| ds +$$
$$\int_{t\wedge\tau}^{t} \frac{1}{\epsilon} \sup_{\|\mu\|_{Var}\leq 1} \left| \int_{[-\tau,0]^2} \left(V_{s+\epsilon}(x_1) - V_s(x_1)\right)\left(V_{s+\epsilon}(x_2) - V_s(x_2)\right)\, \mu(dx_1,dx_2) \right| ds$$

The proof for both integrals is similar, so we concentrate on the first one. It gives

$$\int_0^{t\wedge\tau} \frac{1}{\epsilon} \sup_{\|\mu\|_{Var}\le 1} \left| \int_{[-s,0]^2} \int_{s+x_1}^{s+x_1+\epsilon} g(u)du \int_{s+x_2}^{s+x_2+\epsilon} g(u)du\, \mu(dx_1,dx_2)\right| ds \le$$
$$\le \int_0^{t\wedge\tau} \frac{1}{\epsilon} \sup_{\|\mu\|_{Var}\le 1} \int_{[-s,0]^2} \sqrt{\epsilon \int_{s+x_1}^{s+x_1+\epsilon} g^2(u)du} \sqrt{\epsilon \int_{s+x_2}^{s+x_2+\epsilon} g^2(u)du}\, |\mu|(dx_1,dx_2)ds \le$$
$$\le \int_0^{t\wedge\tau} \sup_{x\in[-s,0]} \left[\int_{s+x}^{s+x+\epsilon} g^2(u)du\right] \sup_{\|\mu\|_{Var}\le 1} \int_{[-s,0]^2} |\mu|(dx_1,dx_2)ds \le$$
$$\le \int_0^{t\wedge\tau} \sup_{x\in[-\tau,0]} \left[\int_{s+x}^{s+x+\epsilon} g^2(u)du\right] ds$$

Previous term converges to zero a.s. by dominated convergence theorem. □

Example 5.17. We list some examples of processes fulfilling the assumptions of Corollary 5.11 or only those of Corollary 5.10. If we only know the quadratic variation but we do not know the whole covariation structure of the process we use Corollary 5.10. We insist on the fact that to study the χ^2-quadratic variation it is necessary to know the whole covariation structure of X.

1) All continuous real semimartingale X. In fact X is a finite quadratic variation process and it holds $[X_{\cdot+a_i}, X_{\cdot+a_j}] = 0$ for $i \neq j$, see Proposition 2.6.

2) In particular if X is the Brownian motion W. In fact $[W]_t = t$ and $[W_{\cdot+a_i}, W_{\cdot+a_j}] = 0$ for $i \neq j$ because W is a semimartingale.

3)

Proposition 5.18. Let $B^{H,K}$ be a bifractional Brownian motion with $HK = 1/2$. Then $[B^{H,K}] = 2^{1-K}t$ and $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$.

Remark 5.19.
- If $K = 1$, then $H = 1/2$ and $B^{H,K}$ is a Brownian motion, case already treated.
- In the case $K \neq 1$ we recall that the bifractional Brownian motion $B^{H,K}$ is not a semimartingale, see Proposition 6 from [39].

Proof. Proposition 1 in [39] says that $B^{H,K}$ has finite quadratic variation equals to $[B^{H,K}] = 2^{1-K}t$. By Proposition 1 and Theorem 2 in [32] there are two constants α and β depending on K, a centered Gaussian process $X^{H,K}$ with absolutely continuous trajectories on $[0,+\infty[$ and a standard Brownian motion W such that $\alpha X^{H,K} + B^{H,K} = \beta W$. Then

$$[\alpha X^{H,K}_{\cdot+a_i} + B^{H,K}_{\cdot+a_i}, \alpha X^{H,K}_{\cdot+a_j} + B^{H,K}_{\cdot+a_j}] \quad = \quad \beta^2 [W_{\cdot+a_i}, W_{\cdot+a_j}] \tag{5.16}$$

Using bilinearity of the covariation we expand the left-hand side in (5.16) into a sum of four terms

$$\alpha^2[X^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha[B^{H,K}_{\cdot+a_i}, X^{H,K}_{\cdot+a_j}] + \alpha[X^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] + [B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] \tag{5.17}$$

Since $X^{H,K}$ has bounded variation then first three terms on (5.17) vanish using Proposition 1 in [44]. On the other hand term the right-hand side in (5.16) is equal to zero for $i \neq j$ since W is a semimartingale, see point 1. We conclude that $[B^{H,K}_{\cdot+a_i}, B^{H,K}_{\cdot+a_j}] = 0$ for $i \neq j$. □

4) Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, M local martingale and A zero quadratic variation process. Then X satisfies the hypotheses of the Corollary 5.11, in particular of Remark 5.12. In fact $[X]_t = [M]_t$ and $[X_{\cdot+a_i}, X_{\cdot+a_j}] = 0$ for $i \neq j$. Consequently the associated window Dirichlet process admits a χ^2-quadratic variation.
   More details about Dirichlet processes and their properties will be given in chapter 6.3.

5) Let D be a $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $D = W + A$, W being a $(\mathcal{F}_t)$-Brownian motion and the process A which is adapted and $[A, N] = 0$ for any continuous $(\mathcal{F}_t)$-local martingale N. Moreover we suppose that A is a finite quadratic variation process. Then D is an example of finite quadratic variation process in fact $[D] = [W] + [A]$. However the covariation structure is not determined by D. This is an example where we only can use Corollary 5.10 but not Corollary 5.11.

We will show now that, under the same assumptions of Corollary 5.11, a finite quadratic variation process X admits a $Diag([-\tau, 0]^2)$-quadratic variation, with $0 < \tau \leq T$. We will see that in the case $\tau < T$ there will be an additive term. This will be used in Example 7.3 of application of the Itô's formula to the window Brownian motion.

Proposition 5.20. Let $0 < \tau \leq T$. Let X be a real continuous process with finite quadratic variation $[X]$. Then $X(\cdot)$ admits a $Diag([-\tau, 0]^2)$-quadratic variation, where $Diag([-\tau, 0]^2)$ was defined in (2.20). We have

$$[X(\cdot)] : Diag([-\tau, 0]^2) \longrightarrow \mathcal{C}([0, T])$$

such that

$$\mu \mapsto [X(\cdot)]_t(\mu) = \int_{-\tau}^{0} g(y)[X]_{t+y} dy \qquad t \in [0, T], \tag{5.18}$$

where μ is a generic element in $Diag([-\tau, 0]^2)$ of type $\mu(dx, dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau, 0])$.

Proof. We recall that for a generic element μ we have $\|\mu\|_{Diag} = \|g\|_\infty$.

Firstly we show the result for $\tau = T$. We verify Condition **H1**. We can write

$$\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{Diag}\le 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds = \frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{Diag}\le 1} \left|\int_{[-T,0]^2} (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2 \;\mu(dx,dy)\right| ds =$$
$$= \frac{1}{\epsilon}\int_0^T \sup_{\|g\|_\infty\le 1} \left|\int_{-T}^0 g(x)\left(X_{s+\epsilon}(x) - X_s(x)\right)^2 dx\right| ds =$$
$$= \int_0^T \sup_{\|g\|_\infty\le 1} \left|\int_0^s \frac{\left(X_{x+\epsilon} - X_x\right)^2}{\epsilon} g(x-s)\,dx\right| ds \le T[X]_T$$

So in particular Condition **H1** is verified by Remark 4.17.
Concerning the remaining conditions, we will use the setting of Corollary 4.29. $\mathcal{S}$ will be the set of non-negative bounded functions, so that **H0'** is obviously verified. It remains to prove Condition **H2'**. Using Fubini's theorem, we obtain

$$\begin{aligned}
[X(\cdot)]^\epsilon_t(\mu) &= \frac{1}{\epsilon}\int_0^t \langle \mu(dx,dy), (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\, ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-T,0]^2} \left(X_{s+\epsilon}(x) - X_s(x)\right)\left(X_{s+\epsilon}(y) - X_s(y)\right) g(x)\delta_x(dy)dx\; ds = \\
&= \frac{1}{\epsilon}\int_0^t \int_{[-T,0]} \left(X_{s+\epsilon}(x) - X_s(x)\right)^2 g(x)dx\; ds = \\
&= \int_{-t}^0 g(x)\int_{-x}^t \frac{\left(X_{s+x+\epsilon} - X_{s+x}\right)^2}{\epsilon}\, ds\, dx = \\
&= \int_{-t}^0 g(x)\int_0^{t+x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds\; dx
\end{aligned} \tag{5.19}$$

Every function g can be written as $g^+ - g^-$, i.e. a difference of two non-negative functions: its positive part g^+ and its negative part g^-. So without loss of generality we can consider a non-negative function g, so process (5.19) will be an increasing process. It can be shown that

$$\left(\int_{-t}^0 g(x)\int_0^{t+x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds\; dx\right)_{t\in[0,T]} \xrightarrow{ucp} \left(\int_{-t}^0 g(x)[X]_{t+x}\, dx\right)_{t\in[0,T]} \tag{5.20}$$

In fact we have

$$\begin{aligned}
\left|[X(\cdot)]^\epsilon_t(\mu) - \int_{-t}^0 g(x)[X]_{t+x}\,dx\right| &= \left|\int_{-t}^0 g(x)\left(\int_0^{t+x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds - [X]_{t+x}\right) dx\right| \le \\
&\le \int_{-T}^0 |g(x)| \sup_{t\in[0,T]} \left|\int_0^{t+x} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds - [X]_{t+x}\right| dx \le \\
&\le \int_{-T}^0 |g(x)| \sup_{t\in[0,T]} \left|\int_0^{t} \frac{\left(X_{s+\epsilon} - X_s\right)^2}{\epsilon}\, ds - [X]_{t}\right| dx
\end{aligned} \tag{5.21}$$

By the dominated convergence theorem and the ucp convergence for the real quadratic variation of the process X, the last term in (6.25) converges to zero. Consequently ucp convergence in (5.20) follows. We conclude the proof for $\tau = T$ by applying Corollary 4.29.
For the case $\tau < T$ Condition **H1** is already verified. Developing $[X(\cdot)]^{\epsilon}_t(\mu)$ we obtain

$$\int_0^t \int_{-\tau}^0 \frac{(X_{s+x+\epsilon} - X_{s+x})^2}{\epsilon} g(x) dx ds = \int_0^{t\wedge\tau} \int_{-s}^0 \frac{(X_{s+x+\epsilon} - X_{s+x})^2}{\epsilon} g(x) dx ds + \\ + \int_{t\wedge\tau}^t \int_{-\tau}^0 \frac{(X_{s+x+\epsilon} - X_{s+x})^2}{\epsilon} g(x) dx ds = I_1^{\epsilon}(t) + I_2^{\epsilon}(t)$$

By the same argument of the case $\tau = T$ we obtain the convergence ucp of the first addend $I_1^{\epsilon}(t)$ to

$$\int_0^{t\wedge\tau} g(-x)[X]_{(t\wedge\tau)-x} dx$$

The second term $I_2^{\epsilon}(t)$ requires some more computations. It converges ucp to

$$\int_{-\tau}^0 g(x)\left([X]_{t+x} - [X]_{\tau+x}\right) dx$$

Then we conclude the result. We remark that when $t \leq \tau$ (in particular this is always the case when $\tau = T$ because $0 \leq t \leq T$) we only have the first term because the second term is zero. When $\tau \leq t \leq T$, we have to sum the two terms. More explicitly we obtain

$$[X(\cdot)]_t(\mu) = \begin{cases} \displaystyle\int_0^t g(-x)[X]_{t-x} dx & 0 \leq t \leq \tau \\ \displaystyle\int_0^{\tau} g(-x)[X]_{\tau-x} dx + \int_0^{\tau} g(-x)\left([X]_{t-x} - [X]_{\tau-x}\right) dx = \int_0^{\tau} g(-x)[X]_{t-x} dx & \tau < t \leq T \end{cases}$$

With the usual property $[X]_t = 0$ for $t < 0$ we rewrite the result in the compact form (5.18).

□

Direct consequences of Propositions 5.11, 5.10, 5.20 and 4.19 are the two corollaries below.

Corollary 5.21. Let $0 < \tau \leq T$ and X be a real continuous process with finite quadratic variation $[X]$ and covariation structure $[X_{\cdot+a_i}, X_{\cdot+a_j}]$ for $i, j \in \{0, \ldots, N\}$. Then $X(\cdot)$ admits a $\chi^3([-\tau, 0]^2)$-quadratic variation where $\chi^3([-\tau, 0]^2) = \chi^2([-\tau, 0]^2) \oplus Diag([-\tau, 0]^2)$. The $\chi^3([-\tau, 0]^2)$-quadratic variation is

$$[X(\cdot)]_t(\mu) = \sum_{i,j=0}^{N} \mu(\{a_i, a_j\})[X_{\cdot+a_i}, X_{\cdot+a_j}]_t + \int_{-t}^0 g(x)[X]_{t+x} dx$$

where μ is a generic element in $Diag([-\tau, 0]^2)$ of type $\mu(dx, dy) = g(x)\delta_y(dx)dy$, with associated g in $L^{\infty}([-\tau, 0])$.

Corollary 5.22. Let $0 < \tau \leq T$ and X be a real continuous process with finite quadratic variation $[X]$. Then $X(\cdot)$ admits a $\mathcal{D}_d([-\tau,0]^2) \oplus Diag([-\tau,0]^2)$-quadratic variation which equals

$$[X(\cdot)]_t(\mu) = \sum_{i=0}^{N} \mu(\{a_i, a_i\})[X_{\cdot+a_i}]_t + \int_{-t}^{0} g(x)[X]_{t+x} dx$$

where μ is a generic element in $Diag([-\tau,0]^2)$ of type $\mu(dx,dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$.

Example 5.23. For further calculations we indicate a direct application of Proposition 5.20.

1. Suppose that $[X]$ is absolutely continuous with $A_t = \frac{d[X]_t}{dt}$. For $\mu \in Diag([-\tau,0]^2)$, $\mu(dx,dy) = g(x)\delta_y(dx)dy$, with associated g in $L^\infty([-\tau,0])$, we have

$$\int_0^T \langle \mu, d\widetilde{[X(\cdot)]} \rangle_t = \int_0^T \int_{-t}^{0} g(x) A_{t+x} dx\, dt$$

2. In particular if $A \equiv 1$, as for standard Brownian motion,

$$\int_0^T \langle \mu, d\widetilde{[X(\cdot)]} \rangle_t = \int_0^T \int_{-t}^{0} g(x) dx\, dt \tag{5.22}$$

We go on with evaluation of χ-quadratic variation. We first recall that $\chi^4([-T,0]^2)$ and $\chi^5([-\tau,0]^2)$ were defined respectively at (4.13) and (4.14). In the next examples, the knowledge of the whole covariation structure of the process is needed.

Proposition 5.24. Let X be a real continuous process with finite quadratic variation $[X]$ and covariation structure $[X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]$ for every $i,j \in \mathbb{N}$. Then $X(\cdot)$ admits a $\chi^4([-T,0]^2)$-quadratic variation equals to

$$[X(\cdot)]_t(\mu) = \sum_{i,j \in \mathbb{N}} \mu(\{\alpha_i, \alpha_j\})[X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]_t \tag{5.23}$$

where μ is a general element in χ^4 which can be written $\mu = \sum_{i,j\in\mathbb{N}} \lambda_{i,j}\delta_{(\alpha_i,\alpha_j)}$.

Proof. We make use of Corollary 4.30. We recall that for a general $\mu \in \chi^4$ the norm is $\|\mu\|_{\chi^4} =$

$\sup_{i,j}\{|\lambda_{i,j}|i^2j^2\}$, then

$$\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^4}\leq 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds =$$

$$= \frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^4}\leq 1} \left|\sum_{i,j\in\mathbb{N}} \lambda_{i,j}(X_{s+\alpha_i+\epsilon} - X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon} - X_{s+\alpha_j})\right| ds =$$

$$= \int_0^T \sup_{\|\mu\|\leq 1} \left|\sum_{i,j\in\mathbb{N}} \lambda_{i,j}i^2j^2 \frac{(X_{s+\alpha_i+\epsilon} - X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon} - X_{s+\alpha_j})}{\epsilon i^2 j^2}\right| ds \leq$$

$$\leq \sum_{i,j\in\mathbb{N}} \frac{1}{i^2j^2}\sqrt{\int_0^T \frac{(X_{s+\alpha_i+\epsilon} - X_{s+\alpha_i})^2}{\epsilon}ds}\sqrt{\int_0^T \frac{(X_{s+\alpha_j+\epsilon} - X_{s+\alpha_j})^2}{\epsilon}ds} \leq$$

$$\leq \sum_{i,j\in\mathbb{N}} \frac{1}{i^2j^2}\int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon}ds =$$

$$= \left(\frac{\pi^2}{6}\right)^2 \int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon}ds \xrightarrow{\mathbb{P}} \frac{\pi^4}{36}[X]_T$$

Condition **H1** follows by using Remark 4.17 .
We set $\mathcal{S} = \{\delta_{(\alpha_i,\alpha_i)},\}_{i,j\in\mathbb{N}}$ and $\mathcal{S}^p = \{\delta_{\alpha_i}\otimes^2, \left(\delta_{\alpha_i} + \delta_{\alpha_j}\right)\otimes^2\}_{i,j\in\mathbb{N}}$ and **H0"** is verified. Also Condition **H2"** is proved, in fact for every element in $\mathcal{S}$ we have

$$\int_0^t \frac{(X_{s+\alpha_i+\epsilon} - X_{s+\alpha_i})(X_{s+\alpha_j+\epsilon} - X_{s+\alpha_j})}{\epsilon}ds \xrightarrow{\mathbb{P}} [X_{\cdot+\alpha_i}, X_{\cdot+\alpha_j}]_t$$

The result is established by Corollary 4.30. □

Proposition 5.25. Let X be a real continuous process with given covariation structure $[X_{\cdot+x}, X_{\cdot+y}]$ for every $x, y \in [-\tau, 0]$, in particular X is a finite quadratic variation process. Then $X(\cdot)$ admits a $\chi^5([-\tau,0]^2)$-quadratic variation

$$[X(\cdot)]_t(\mu) = \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \mu(dx,dy) = \sum_{i=1}^N \lambda_i \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \mu_i(dx,dy) \tag{5.24}$$

where μ is a general element in $\chi^5([-\tau,0]^2)$ which can be written as $\mu = \sum_{i=1}^N \lambda_i\mu_i$, μ is a linear composition of N fixed measures $(\mu_i)_{i=1,\dots,N}$ with total variation 1.

Proof. We verify **H1**. We recall that $\|\mu\|^2_{\chi^5} = \sum_{i=1}^N \lambda_i^2$, then

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^T \sup_{\|\mu\|_{\chi^5}\leq 1} \left|\langle \mu, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds = \\
&= \int_0^T \sup_{\|\mu\|_{\chi^5}\leq 1} \left| \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} \mu(dx,dy) \right| ds = \\
&= \int_0^T \sup_{\|\mu\|^2_{\chi^5}\leq 1} \left| \sum_{i=1}^N \lambda_i \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} \mu_i(dx,dy) \right| ds \leq \\
&\leq \int_0^T \sum_{i=1}^N \sup_{\sum \lambda_i^2 \leq 1} \left\{ |\lambda_i| \left| \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} \mu_i(dx,dy) \right| \right\} ds
\end{aligned}$$

In particular $|\lambda_i| \leq 1$ for every i, then using Fubini's theorem and Cauchy Schwarz inequality previous quantity is less or equal to

$$\begin{aligned}
&\sum_{i=1}^N \int_0^T \int_{[-\tau,0]^2} \frac{|(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})|}{\epsilon} |\mu_i|(dx,dy) ds = \\
&= \sum_{i=1}^N \int_{[-\tau,0]^2} \int_0^T \frac{|(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})|}{\epsilon} ds\, |\mu_i|(dx,dy) \leq \\
&\leq \sum_{i=1}^N \int_{[-\tau,0]^2} \sqrt{\int_0^T \frac{(X_{s+x+\epsilon} - X_{s+x})^2}{\epsilon} ds} \sqrt{\int_0^T \frac{(X_{s+y+\epsilon} - X_{s+y})^2}{\epsilon} ds}\, |\mu_i|(dx,dy) \leq \\
&\leq \sum_{i=1}^N \int_{[-\tau,0]^2} \int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds\, |\mu_i|(dx,dy) \xrightarrow{\mathbb{P}} [X]_T \sum_{i=1}^N \int_{[-\tau,0]^2} |\mu_i|(dx,dy) = \\
&= [X]_T \sum_{i=1}^N |\mu_i|([-\tau,0]^2) < +\infty
\end{aligned}$$

By Remark 4.17, Condition **H1** is verified. Since the signed measure μ_i can be decomposed in differences of positive and negative parts μ_i^+ and μ_i^-, setting $\mathcal{S} = \{\mu_i\}_{i\in\{1,\ldots,N\}}$ and $\mathcal{S}^p = \{\mu_i^+, \mu_i^-\}_{i\in\{1,\ldots,N\}}$ **H0"** is verified. To verify Condition **H2"** we consider a fixed positive measure μ with unitary total variation. It holds

$$\int_0^t \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} \mu(dx,dy)\, ds \xrightarrow{\mathbb{P}} \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \mu(dx,dy) \tag{5.25}$$

By Fubini's theorem and passing to subsequence we have to show the deterministic convergence for every $\omega \notin N$ of the quantity

$$\int_{[-\tau,0]^2} \int_0^t \frac{(X_{s+x+\epsilon_{n_k}} - X_{s+x})(X_{s+y+\epsilon_{n_k}} - X_{s+y})}{\epsilon_{n_k}} ds\, \mu(dx,dy) = \int_{[-\tau,0]^2} \gamma^{\epsilon_{n_k}}(x,y) \mu(dx,dy)$$

where

$$\gamma^{\epsilon_{n_k}}(x,y) = \int_0^t \frac{(X_{s+x+\epsilon_{n_k}} - X_{s+x})(X_{s+y+\epsilon_{n_k}} - X_{s+y})}{\epsilon_{n_k}} ds$$

It holds that $\gamma^{\epsilon}(x,y) \xrightarrow{\mathbb{P}} [X_{\cdot+x}, X_{\cdot+y}]_t$ and $|\gamma^{\epsilon_{n_k}}| \le \sup_k \frac{1}{\epsilon_{n_k}} \int_0^t (X_{s+\epsilon_{n_k}} - X_s)^2 ds \in L^1([-\tau,0]^2,\mu)$. Then, by the dominated convergence theorem applied for every $\omega \notin N$, we conclude with the convergence in probability. The fact that the term $\int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \mu(dx,dy)$ is measurable, i.e. $[X_{\cdot+x}, X_{\cdot+y}]_t \in L^1([-\tau,0]^2,\mu)$ for every $x,y \in [-\tau,0]$, is insured by the existence of the quadratic variation since by the Kunita-Watanabe inequality we have $|[X_{\cdot+x}, X_{\cdot+y}]| \le \sqrt{[X_{\cdot+x}][X_{\cdot+y}]} \le [X]$. Again by applying Corollary 4.30 the result is established. $\square$

Remark 5.26. As a particular case of Proposition 5.25 we observe that if X is a real continuous process with finite quadratic variation $[X]$ and covariation structure such that $[X_{\cdot+x}, X_{\cdot+y}] = 0$ for $x \neq y$. Then the $\chi^5([-\tau,0]^2)$-quadratic variation of $X(\cdot)$ equals

$$[X(\cdot)]_t(\mu) = \int_{[-\tau,0]^2} [X]_{t+x} \mathbb{1}_D(x,y)\mu(dx,dy) \tag{5.26}$$

where μ is a general element $\chi^5([-\tau,0]^2)$ which can be written as $\mu = \sum_{i=1}^N \lambda_i \mu_i$, and $D = \{(x,y) \in [-\tau,0]^2; x = y\}$ is the diagonal of the square $[-\tau,0]^2$.

Another significant example is the following. Let μ be a fixed measure on $[-\tau,0]^2$ finite and singular with respect to Lebesgue measure, we recall definition of χ^μ in (4.15).

Proposition 5.27. Let μ be a given positive, finite and singular measure on $[-\tau,0]^2$ and X be a real process with finite quadratic variation $[X]$ admitting a covariation process $[X_{\cdot+x}, X_{\cdot+y}]$ for every $x,y \in [-\tau,0]$. Then $X(\cdot)$ admits a χ^μ-quadratic variation which equals, for a measure $d\nu = g\, d\mu$ with $g \in L^\infty(d\mu)$,

$$[X(\cdot)]_t(\nu) = \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \nu(dx,dy) \tag{5.27}$$

Proof. We have

$$\frac{1}{\epsilon}\int_0^T \sup_{\|\nu\|_{\chi^\mu}\le 1} \left|\langle \nu, (X_{s+\epsilon}(\cdot) - X_s(\cdot))\otimes^2\rangle\right| ds =$$

$$= \int_0^T \sup_{\|g\|_{L^\infty(d\mu)}\le 1} \left| \int_{[-\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} g(x,y)\mu(dx,dy)\right| ds \le$$

$$= \int_0^T \int_{[-\tau,0]^2} \left| \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon}\right| |\mu|(dx,dy) ds \le$$

$$\le \int_{[-\tau,0]^2} \int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds\, |\mu|(dx,dy) \xrightarrow{\mathbb{P}} [X]_T \int_{[-\tau,0]^2} |\mu|(dx,dy) = [X]_T\, |\mu|([-\tau,0]^2) < +\infty$$

So **H1** is established using Remark 4.17. It is always possible to write g as difference of its positive and negative part such that $g = g^+ - g^-$, with g^+ and g^- positive functions. Setting $\mathcal{S} = \{g\}_{g\in L^\infty(d\mu)}$ and $\mathcal{S}^p = \{g^+, g^-\}_{g\in L^\infty(d\mu)}$ Condition **H0"** is verified. Moreover for every $d\nu = g\, d\mu$ it holds

$$\int_0^t \int_{[\tau,0]^2} \frac{(X_{s+x+\epsilon} - X_{s+x})(X_{s+y+\epsilon} - X_{s+y})}{\epsilon} g(x,y)\mu(dx,dy)\, ds \xrightarrow{\mathbb{P}} \int_{[-\tau,0]^2} [X_{\cdot+x}, X_{\cdot+y}]_t \nu(dx,dy)$$

Also Condition **H2"** is verified and result follows again by Corollary 4.30. $\square$

For further applications now we consider window processes as processes with values in the Hilbert space $L^2([-\tau,0])$ and we will compute some χ-quadratic variations. We start with a preliminary result.

Proposition 5.28. *Let X be a real continuous process with finite quadratic variation $[X]_t = t$. Then*

$$\frac{1}{\epsilon}\int_0^t \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|^2_{L^2([-\tau,0])}\, ds \xrightarrow[\epsilon\to 0]{ucp} \begin{cases} \dfrac{t^2}{2} & 0 \le t \le \tau \\ \tau\left(t - \dfrac{\tau}{2}\right) & \tau < t \le T \end{cases}$$

Proof. The random variables are increasing with respect to time, so by using Lemma 2.1 it will be enough to show convergence in probability for every fixed $t \in [0,T]$. We have

$$\begin{aligned}
&\frac{1}{\epsilon}\int_0^t \|X_{s+\epsilon}(\cdot) - X_s(\cdot)\|^2_{L^2([-\tau,0])}\, ds = \frac{1}{\epsilon}\int_0^t \int_{-\tau}^0 (X_{s+r+\epsilon} - X_{s+r})^2\, dr\, ds = \\
&= \int_0^{t\wedge\tau}\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\, ds + \int_{t\wedge\tau}^t \int_{-\tau}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\, ds = \\
&= \begin{cases} \displaystyle\int_0^t\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\, ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{t^2}{2} & 0 \le t \le \tau \\ \displaystyle\int_0^\tau\int_{-s}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\, ds + \int_\tau^t\int_{-\tau}^0 \frac{(X_{s+r+\epsilon} - X_{s+r})^2}{\epsilon} dr\, ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{\tau^2}{2} + \tau(t-\tau) & \tau < t \le T \end{cases}
\end{aligned}$$

$\square$

Remark 5.29. Let X be a real continuous process with finite quadratic variation $[X]_t = t$. We consider $X(\cdot)$ process with values in $L^2([-\tau,0])$. As consequences of Proposition 5.28 we have

1. Condition **H1** for existence of $(L^2([-\tau,0])\hat{\otimes}^2_\pi)^*$-quadratic variation of $X(\cdot)$ is verified. Let χ be a Chi-subspace of $(L^2([-\tau,0])\hat{\otimes}^2_\pi)^*$, using Remark 4.21.2, it follows that Condition **H1** for existence of χ-quadratic variation of $X(\cdot)$ is verified.
2. We are not able to prove the existence of a $(L^2([-\tau,0])\hat{\otimes}^2_\pi)^*$-quadratic variation because we can not prove Condition **H2**, i.e. that it exists an application $[X(\cdot)]$, such that $[X(\cdot)]^\epsilon(\phi) \xrightarrow[\epsilon\to 0]{ucp} [X(\cdot)](\phi)$ for every $\phi \in (L^2([-\tau,0])\hat{\otimes}^2_\pi)^* = \mathcal{B}(L^2([-\tau,0]), L^2([-\tau,0]))$. Nevertheless we have an expression of this limit for some particular $\phi \in (L^2([-\tau,0])\hat{\otimes}^2_\pi)^*$. For instance if we fix the bilinear bounded

operator $\phi : L^2([-\tau,0]) \times L^2([-\tau,0]) \to \mathbb{R}$, defined by $(h,g) \mapsto \phi(h,g) = \langle h,g\rangle_{L^2([-\tau,0])}$ we can show that $[X(\cdot)]^\epsilon(\phi) \xrightarrow[\epsilon\to 0]{ucp} [X(\cdot)](\phi)$. This ucp convergence is exactly the one that has been shown in Proposition 5.28.

Corollary 5.30. Let X be a real continuous process with zero quadratic variation $[X] = 0$. Then the $L^2([-\tau,0])$-valued window process $X(\cdot)$ admits zero global quadratic variation.

Proof. The result follows immediately using lemma 4.22. □

We recall the definitions of Chi-subspaces $L^2_{\mathcal{B}}([-\tau,0]^2)$ and $Diag_{\mathcal{B}}([-\tau,0]^2)$ given respectively in Definition 4.9 and in (4.17).

Proposition 5.31. Let X a real continuous process with finite quadratic variation $[X]$. We consider $X(\cdot)$ process with values in $L^2([-\tau,0])$.

1. $X(\cdot)$ admits zero $L^2_{\mathcal{B}}([-\tau,0]^2)$-quadratic variation.
2. $X(\cdot)$ admits a $Diag_{\mathcal{B}}([-\tau,0]^2)$-quadratic variation which equals, for every $T^f \in Diag_{\mathcal{B}}([-\tau,0]^2)$,

$$[X(\cdot)]_t(T^f) = \int_{-\tau}^{0} f(y)[X]_{t+y}dy \qquad t \in [0,T] \tag{5.28}$$

remembering that $[X]_u = 0$ for $u < 0$. In particular that quadratic variation is non zero.

Proof.

1. The proof is exactly the same that we have seen in Proposition 5.9 for the $L^2([-\tau,0]^2)$-quadratic variation of the window process with values in $C([-\tau,0])$.
2. The proof is practically the same in Proposition 5.20. More explicitly, we have

$$[X(\cdot)]_t(T^f) = \begin{cases} \displaystyle\int_0^t f(-y)[X]_{t-y}dy & 0 \le t \le \tau \\ \displaystyle\int_0^\tau f(-y)[X]_{t-y}dy & \tau < t \le T \end{cases}$$

□

Remark 5.32. Let consider $H = L^2([-\tau,0]^2)$ and $E = L^2([-\tau,0])\hat{\otimes}^2_\pi$. We recall that H is the Hilbert tensor product of $L^2([-\tau,0])$ with itself. Consequently E is densely embedded into H because of (2.11).

1. The norms $\|\cdot\|_{H^*} \ge \|\cdot\|_{E^*}$ by (2.7). Moreover they are not equivalent. In fact, consider $(e_i)_{i\mathbb{N}}$ an orthonormal basis of $L^2([-\tau,0])$ and set $g_n = \sum_{i=1}^n e_i \otimes e_i$. It holds $\|g_n\|^2_{H^*} = n$. On the other hand, let h and f in $L^2([-\tau,0])$, $g_n(h,f) = \sum_{i=1}^n \langle h,e_i\rangle\langle f,e_i\rangle = \sum_{i=0}^n \langle h\otimes f, e_i\otimes e_i\rangle$. So $|g_n(h,f)| \le \sqrt{\sum_{i=1}^n \langle h,e_i\rangle^2 \sum_{j=1}^n \langle f,e_j\rangle^2} = \|h\|\,\|f\|$, where the last equality comes by Parseval's identity. Then $\|g_n\|_{E^*} = \sup_{\|h\|,\|f\|\le 1} |g_n(h,f)| \le 1$.

2. Let $g \in E^*$ of the form $g(h,f) = \langle h, f\rangle_{L^2([-\tau,0])}$. We have

$$g(h,f) = \langle h, f\rangle_{L^2([-\tau,0])} = \sum_{i=0}^{\infty} \langle h, e_i\rangle\langle f, e_i\rangle = \lim_{n\to+\infty} \sum_{i=0}^{n} \langle h\otimes f, e_i \otimes e_i\rangle = \lim_{n\to+\infty} g_n(h,f)$$

   This means that for a h and f in $L^2([-\tau,0])$, $g_n(h,f) \to g(h,f)$. However g_n does not converge to g according to E^*. In fact the sequence (g_n) is not Cauchy. For $m,n \in \mathbb{N}$, $m > n$, for h, f in $L^2([-\tau,0])$ we have $(g_n - g_m)(h,f) = \sum_{i=m+1}^{n} \langle e_i \otimes e_i, h\otimes f\rangle_H$. Taking $h = f = e_i$, previous quantity equals 1 so that $\|g_n - g_m\|_{E^*} = 1$

We would like to comment on a well-known functional analytical result, see for instance pag. 55 in [38], which is the following.

Theorem 5.33. Let E be a reflexive Banach subspace densely embedded in a Hilbert space H. Then H^* is densely embedded in E^*.

Remark 5.34.

1. In fact by an unfortunate miss print of [38] the reflexivity assumption on E does not appear.

2. Using arguments related to covariation maps we can provide a probabilistic proof that the theorem statement is wrong if E is not reflexive.

Proposition 5.35. With previous conventions H^* is not densely embedded in E^*.

Proof. Let $W(\cdot)$ be a window Brownian motion considered with values in $L^2([-\tau,0])$. Point 1 of Proposition 5.31 says that $W(\cdot)$ has zero H^*-quadratic variation. We suppose ab absurdo that H^* is densely embedded in E^*. We recall by Remark 5.29.1 that Condition **H1** related to definition of E^*-quadratic variation is always verified. Setting $\mathcal{S} = H^*$, then Conditions **H0'** and **H2'** of Corollary 4.29 are verified. Consequently $W(\cdot)$ has a E^*-quadratic variation. Since the quadratic variation $[W(\cdot)] : E^* \longrightarrow \mathcal{C}([0,T])$ is continuous, it must be identically zero. This contradicts Point 2 of the same Proposition 5.31. □

Chapter 6

Stability of χ-quadratic variation and of χ-covariation

In this chapter we will firstly introduce the definition of a so-called $\chi-$covariation between two Banach valued processes X and Y and secondly we will concentrate about some stability results of the χ-covariation through a real function C^1 in the Fréchet sense.

6.1 The notion of χ-covariation

Let B_1, B_2 be two Banach spaces.

Definition 6.1. A closed linear subspace χ of $(B_1\hat{\otimes}_\pi B_2)^*$, endowed with its own norm, such that

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B_1\hat{\otimes}_\pi B_2)^*}. \tag{6.1}$$

will be called a **Chi-subspace of** $(B_1\hat{\otimes}_\pi B_2)^*$.

Definition 6.2. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let X (respectively Y) be B_1 (respectively B_2) valued stochastic processes. We say that X **and** Y **admit a** χ**-covariation** if

H1 For all (ϵ_n) it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \int_0^T \sup_{\|\phi\|_\chi\leq 1} \left|\langle \phi, \frac{(X_{s+\epsilon_{n_k}}-X_s)\otimes(Y_{s+\epsilon_{n_k}}-Y_s)}{\epsilon_{n_k}}\rangle\right| ds \quad < \infty \tag{6.2}$$

H2 (i) It exists an application denoted by $[X,Y]:\chi\longrightarrow \mathcal{C}([0,T])$ such that $[X,Y]^\epsilon(\phi)\longrightarrow [X,Y](\phi)$ in the ucp topology when $\epsilon\to 0_+$ for every $\phi\in\chi\subset(B_1\hat{\otimes}_\pi B_2)^*$.

(ii) Moreover we can associate a bounded variation function $\widetilde{[X,Y]}(\omega,\cdot) : [0,T] \longrightarrow \chi^*$ defined by $\widetilde{[X,Y]}(\omega,t)(\phi) := [X,Y](\phi)(\omega,t)$

If X, Y admit a χ-covariation we will denote with $(\widetilde{[X,Y]})_{0\leq t\leq T}$ defined for every $\omega \in \Omega$ and $t \in [0,T]$ the χ^*-valued process $\phi \mapsto \widetilde{[X,Y]}(\omega,t)(\phi) := [X,Y](\phi)(\omega,t) = \lim^{ucp} \frac{1}{\epsilon}\int_0^t \langle \phi, (X_{s+\epsilon} - X_s) \otimes (Y_{s+\epsilon} - Y_s)\rangle ds =: [X,Y]_t(\phi)$.

If the covariation exists for $\chi = (B_1\hat{\otimes}_\pi B_2)^*$, we say that X and Y admit a **global covariation**.

We enunciate now an important proposition used later to prove Itô's formula and other results of stability. It is a corollary of Proposition 4.23.

Corollary 6.3. Let B_1, B_2 be two Banach spaces and χ be a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$. Let X and Y be two stochastic processes with values in B_1 and B_2 admitting a χ-covariation and H a continuous measurable process $H : \Omega \times [0,T] \longrightarrow \chi$. Then for every $t \in [0,T]$

$$\int_0^t \langle H(\cdot,s), d\widetilde{[X,Y]}^{\epsilon}(\cdot,s)\rangle \longrightarrow \int_0^t \langle H(\cdot,s), d\widetilde{[X,Y]}(\cdot,s)\rangle \tag{6.3}$$

in probability.

Proof. The proof follows from Proposition 4.23 and definition of χ-covariation. □

Remark 6.4. 1) The statement of Propositions 4.19 and 4.20 related to the χ-quadratic variation of Banach valued process X can be immediately extended to the case of $\chi-$covariation of two Banach valued processes X and Y. We obtain sufficient condition for the existence of χ-covariation, if χ is a finite direct sum of Chi-subspaces, for instance the space $\chi^2([-\tau,0]^2)$.

2) Analogously the statement of Corollaries 4.29 and 4.30 related to the χ-quadratic variation of a Banach valued process X can be extended to the case of $\chi-$covariation of two Banach valued processes X and Y. Their proofs make use of Theorem 4.26. In most of the cases, the bounded variation property of the χ^*-valued process $\widetilde{[X,Y]}$ will be automatically satisfied. As an interesting consequence of this last property we obtain a sufficient condition in the real case to have the bounded variation of the covariation process. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X,Y]$ exist in the sense of 2.2, then $[X,Y]$ has bounded variation in fact via polarity can be written as difference of positive functions. But this assumption is strong, in fact there exist processes not necessarily with finite quadratic variation process that admit covariation. For instance an $(\mathcal{F}_t)$-weak Dirichlet process D with decomposition $M + A$ and an $(\mathcal{F}_t)$-local martingale N admit covariation $[D,N] = [M,N]$ even if D may not have a finite quadratic variation. Using Theorem 4.26 we obtain a sufficient condition for the bounded variation of real covariation which does not involve the existence of $[X]$ and $[Y]$. We have the following proposition.

Proposition 6.5. Let X and Y be two real continuous processes such that

i) $[X,Y]$ exists and

ii) for all (ϵ_n) it exists a subsequence (ϵ_{n_k}) such that

$$\sup_k \frac{1}{\epsilon_{n_k}} \int_0^T \left| X_{s+\epsilon_{n_k}} - X_s \right| \cdot \left| Y_{s+\epsilon_{n_k}} - Y_s \right| ds \quad < \infty \tag{6.4}$$

Then the real covariation process $[X,Y]$ has bounded variation.

Proof. We just observe that if the processes are real valued, space $(\mathbb{R}\hat{\otimes}_\pi\mathbb{R})^*$ coincides with $\mathbb{R}$. Processes X and Y admit global covariation which coincides with the classical covariation $[X,Y]$ defined in 2.2. Taking into account Remark 6.4 we obtain that $[X,Y]$ has bounded variation. □

As an immediate corollary we show that all finite quadratic variation processes X and Y admitting a covariation $[X,Y]$ satisfy hypothesis of Proposition 6.5.

Corollary 6.6. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X,Y]$ exist. Then Condition ii) of Proposition 6.5 is verified, in particular as it is well known the real covariation process $[X,Y]$ has bounded variation.

Proof. Using Cauchy Schwarz inequality we have that

$$\frac{1}{\epsilon}\int_0^T |X_{s+\epsilon} - X_s| \cdot |Y_{s+\epsilon} - Y_s|\, ds \le \sqrt{\int_0^T \frac{(X_{s+\epsilon} - X_s)^2}{\epsilon} ds}\sqrt{\int_0^T \frac{(Y_{s+\epsilon} - Y_s)^2}{\epsilon} ds} =: A(\epsilon)$$

where the sequence $A(\epsilon)$ converges in probability to $\sqrt{[X]_T[Y]_T}$. The convergence implies the boundness and the result follows. □

The proof of the propositions below can be provided taking into account Remark 6.4.

Proposition 6.7. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X,Y]$ exist. Then

1) $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation, where $\chi = L^2([-T,0]^2)$.

2) $X(\cdot)$ and $Y(\cdot)$ admit zero χ-covariation for every $i \in \{0,\dots,N\}$, where $\chi = L^2([-T,0])\hat{\otimes}_h\mathcal{D}_i([-T,0])$.

If moreover the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\dots,N\}$, then

3) $X(\cdot)$ and $Y(\cdot)$ admits χ-covariation which equals

$$[X(\cdot),Y(\cdot)]_t(\mu) = \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]_t. \tag{6.5}$$

where $\chi = \mathcal{D}_{i,j}([-T,0]^2)$.

Proof. The proof is practically the same of Proposition 5.9. □

Concerning the χ-covariation with $\chi = \mathcal{D}_{i,j}([-T,0]^2)$ we can even relax the hypothesis. It holds the following proposition.

Proposition 6.8. Under the same assumptions as in Proposition 6.5 and if the covariation $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$ exists for a given $i,j \in \{0,\dots,N\}$, then $X(\cdot)$ and $Y(\cdot)$ admits χ-quadratic variation which equals (5.3) where $\chi = \mathcal{D}_{i,j}([-\tau,0]^2)$.

Proof. The proof is practically the same of Proposition 6.7, but to show Assumption **H1** in 5.4 we will use (6.4) instead of the finite quadratic variation property of processes X and Y. □

Remark 6.9. In particular assumptions of Proposition 6.5 give sufficient conditions for a χ-covariation with $\chi = \mathcal{D}_{0,0}([-\tau,0]^2)$ of $X(\cdot)$ and $Y(\cdot)$

Theorem 6.10. Let X and Y be two real continuous processes with finite quadratic variation $[X]$ and $[Y]$ and with given covariation structure $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$, for every $i,j = 0,\dots,N$. Then $X(\cdot)$ and $Y(\cdot)$ admit the following $\chi^2([-\tau,0]^2)$-covariation

$$[X(\cdot),Y(\cdot)] : \chi^2([-\tau,0]^2)([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T]) \qquad \mu \mapsto \sum_{i,j=0}^{N} \mu(\{a_i,a_j\})[X_{\cdot+a_i}, Y_{\cdot+a_j}]$$

Theorem 6.11. Let X and Y be two real continuous processes such that $[X]$, $[Y]$ and $[X,Y]$ exist. Then $X(\cdot)$ and $Y(\cdot)$ admit the following $\chi^0([-\tau,0]^2)$-covariation

$$[X(\cdot),Y(\cdot)] : \chi^0([-\tau,0]^2) \longrightarrow \mathcal{C}([0,T]) \qquad \mu \mapsto \mu(\{0,0\})[X,Y]$$

Remark 6.12.

1. We remark that the existence of $\chi^0([-\tau,0]^2)$-covariation only requires the existence of the real covariations $[X,Y]$, $[X]$ and $[Y]$. We do not need the existence of $[X_{\cdot+a_i}, Y_{\cdot+a_j}]$, for every $i,j = 0,\dots,N$.

2. Let D be a real $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation and decomposition $M+A$, M being an $(\mathcal{F}_t)$-local martingale and let N be a real $(\mathcal{F}_t)$-martingale. Then $D(\cdot)$ and $N(\cdot)$ admit χ^0-covariation given by $[D(\cdot),N(\cdot)](\mu) = \mu(\{0,0\})[M,N]$ for every $\mu \in \chi^0$. This follows from Theorem 6.11, because D and N are with finite quadratic variation processes and $[D,N] = [M,N]$.

3. Let D be a real $(\mathcal{F}_t)$-Dirichlet process with decomposition $M+A$, M being the $(\mathcal{F}_t)$-local martingale part and let N be a real $(\mathcal{F}_t)$-local martingale. Then $D(\cdot)$ and $N(\cdot)$ admit a χ^2-covariation given by $[D(\cdot),N(\cdot)](\mu) = \sum_{i,j=0}^N \mu(\{a_i,a_j\})[D_{\cdot+a_i}, N_{\cdot+a_j}]. = \sum_{i,j=0}^N \mu(\{a_i,a_j\})[M_{\cdot+a_i}, N_{\cdot+a_j}] = \sum_{i=0}^N \mu(\{a_i,a_i\})[M_{\cdot+a_i}, N_{\cdot+a_i}]$. This follows again from Theorem 6.11 and Proposition 2.6

6.2 The stability of χ-covariation in the Banach space framework

In this section, we analyze the stability of χ-covariation for Banach valued processes transformed through C^1 Fréchet differentiable functions.
We recall first what happens in multidimensional case. We have a stability result concerning the covariation for vector valued processes through a C^1 function.

Proposition 6.13. Let $X = (X^1, \ldots, X^n)$ be a vector of real processes having all its mutual covariation, F, $G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)]_\cdot = \sum_{i,j=1}^{n} \int_0^\cdot \partial_i F(X) \partial_j G(X) d[X^i, X^j]$$

This include the case of Propostion 2.1 in [42], setting $n = 2$, $F(x, y) = f(x)$, $G(x, y) = g(y)$, $f, g \in C^1(\mathbb{R})$. Now we develop those type of stability results in the Banach framework.
Let X, Y B-vaued stochastic processes and F , $G : B \longrightarrow \mathbb{R}$ of class C^1 in the Fréchet sense. We are mainly interested in the three following situations.

1. The case when X admits a χ-quadratic variation and $(x, y) \to DF(x) \otimes DF(y)$ is a continuous application from $B \times B$ to χ. Then the quadratic variation of the real process $F(X)$ exists and equals
$$[F(X)]_\cdot = \int_0^\cdot \langle DF(X_s) \otimes DF(X_s), d\widetilde{[X(\cdot)]}_s$$

2. The case when X admits a χ-quadratic variation and $(x, y) \to DF(x) \otimes DG(y)$ is continuous application from $B \times B$ to χ. Then the covariation of $F(X)$ and $G(X)$ exists and equals
$$[F(X), G(X)]_\cdot = \int_0^\cdot \langle DF(X_s) \otimes DG(X_s), d\widetilde{[X(\cdot)]}_s$$

3. The case when X and Y admit χ-covariation and $(x, y) \to DF(x) \otimes DG(y)$ is a continuous application from $B \times B$ to χ. Then the covariation of $F(X)$ and $G(Y)$ exists and equals
$$[F(X), G(Y)]_\cdot = \int_0^\cdot \langle DF(X_s) \otimes DG(Y_s), d\widetilde{[X(\cdot), Y(\cdot)]}_s$$

Remark 6.14.

1. Let $S, T : B_0 \longrightarrow \mathbb{R}$ be linear continuous forms. $S \otimes T$ is the unique linear continuous form from $B_0 \hat{\otimes}_\pi B_0$ to $\mathbb{R} \hat{\otimes}_\pi \mathbb{R} \equiv \mathbb{R}$ such that $S \otimes T(b_1 \otimes b_2) = S(b_1) \cdot T(b_2)$ and $\|S \otimes T\| = \|S\| \, \|T\|$, see Proposition 2.3 in [46].

2. Suppose that B_0 is a Hilbert space. Then S (respectively T) can be identified via Riesz with $\mathcal{S}$ (respectively $\mathcal{T}$) element of B_0. In this case $S \otimes T \in (B_0 \hat{\otimes}_h B_0)^*$ and it will be identified via Riesz with $\mathcal{S} \otimes \mathcal{T}$, tensor product in $B_0 \hat{\otimes}_h B_0$. That Riesz identification will be omitted in the sequel.

3. With previous conventions, let x and y be fixed, $DF(x)$ (respectively $DF(y)$) are linear continuous forms from B to $\mathbb{R}$. Then above term $DF(x) \otimes DF(y)$ denotes the unique linear continuous form from $B\hat{\otimes}_\pi B$ to $\mathbb{R}$ as explained in point 1. We insist on the fact that "a priori" $DF(x) \otimes DF(y)$ does not denote an element of some tensor product $B^* \otimes B^*$. With a little abuse of notation we will denote the application $f \otimes f$ by $f\otimes^2$.

As corollary of Remark 6.14.2 we have the following result. This corollary will be used in chapter 6.3.

Corollary 6.15. Let F^1 and F^2 be two functions from $C([-\tau,0])$ to $\mathcal{D}_a \oplus L^2([-\tau,0])$ such that $\eta \mapsto F^j(\eta) = \sum_{i=0,\dots N} \lambda_i^j(\eta)\delta_{a_i} + g^j(\eta)$ with $\eta \in C([-T,0])$, $\lambda_i^j : C([-\tau,0]) \longrightarrow \mathbb{R}$ and $g^j : C([-\tau,0]) \longrightarrow L^2([-T,0])$ continuous for $j = 1,2$. Then for any (η_1,η_2), $(F^1 \otimes F^2)(\eta_1,\eta_2)$ will be identified with the true tensor product $F^1(\eta_1) \otimes F^2(\eta_2)$ which belongs to $\chi^2([-\tau,0]^2)$. In fact we have

$$F^1(\eta_1) \otimes F^2(\eta_2) = \sum_{i,j=0,\dots,N} \lambda_i^1(\eta_1)\lambda_j^2(\eta_2)\delta_{a_i} \otimes \delta_{a_j} + g^1(\eta_1) \otimes \sum_{i=0,\dots,N} \lambda_i^2(\eta_2)\delta_{a_i} +$$

$$+ \sum_{i=0,\dots,N} \lambda_i^1(\eta_1)\delta_{a_i} \otimes g^2(\eta_2) + g^1(\eta_1) \otimes g^2(\eta_2) \tag{6.6}$$

Theorem 6.16. Let B be a Banach space, χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X^1, X^2 two B-valued continuous stochastic process admitting a χ-covariation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be two functions of class C^1 in the Fréchet sense. We suppose moreover that the following applications

$$\begin{aligned} DF^i(\cdot) \otimes DF^j(\cdot) : B \times B &\longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \\ (x,y) &\mapsto DF(x) \otimes DF(y) \end{aligned}$$

are continuous for $i,j = 1,2$.
Then the covariation between $F^i(X^i)$ and $F^j(X^j)$ exists and is given by

$$[F^i(X^i), F^j(X^j)]_\cdot = \int_0^\cdot \langle DF^i(X_s^i) \otimes DF^j(X_s^j), d[\widetilde{X^i,X^j}]_s\rangle \tag{6.7}$$

Remark 6.17. If there exists a χ^*-valued stochastic process H Bochner integrable such that $[\widetilde{X^i,X^j}]_s = \int_0^s H_u \, du$. Then

$$[F^i(X^i), F^j(X^j)]_\cdot = \int_0^\cdot \langle DF^i(X_s^i) \otimes DF^j(X_s^j), H_s\rangle \, ds$$

Proof. This proof makes use in an essential manner of Corollary 6.3. Without restriction of generality we only consider the case $F^1 = F^2 = F$ and $X^1 = X^2 = X$. In this case previous result reduces to Corollary 4.24.

By definition of quadratic variation between real processes in [42] we know that $[F(X)].$ is the limit in the ucp sense of the quantity

$$\int_0^{\cdot} \frac{(F(X_{s+\epsilon}) - F(X_s))^2}{\epsilon} ds$$

Since Lemma 2.1, it will be enough to show the convergence in probability for a fixed $t \in [0, T]$.
Using a Taylor's expansion we have

$$\begin{aligned}\frac{1}{\epsilon}\int_0^t (F(X_{s+\epsilon}) - F(X_s))^2 ds = \frac{1}{\epsilon}\int_0^t \Big(\langle DF(X_s), X_{s+\epsilon} - X_s\rangle + \\ + \int_0^1 \langle DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s), X_{s+\epsilon} - X_s\rangle \, d\alpha\Big)^2 ds = \\ = A_1(\epsilon) + A_2(\epsilon) + A_3(\epsilon)\end{aligned}$$

where

$$\begin{aligned}
A_1(\epsilon) &= \frac{1}{\epsilon}\int_0^t \langle DF(X_s), X_{s+\epsilon} - X_s\rangle^2 ds = \\
&= \int_0^t \langle DF(X_s) \otimes DF(X_s), \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle ds \\
A_2(\epsilon) &= \frac{2}{\epsilon}\int_0^t \langle DF(X_s), X_{s+\epsilon} - X_s\rangle \cdot \\
&\qquad \cdot \int_0^1 \langle DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s), X_{s+\epsilon} - X_s\rangle d\alpha \, ds = \\
&= 2\int_0^t \int_0^1 \langle DF(X_s) \otimes (DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s)), \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle d\alpha \, ds \\
A_3(\epsilon) &= \frac{1}{\epsilon}\int_0^t \Big(\int_0^1 \langle DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s), X_{s+\epsilon} - X_s\rangle d\alpha\Big)^2 ds \leq \\
&\leq \frac{1}{\epsilon}\int_0^t \int_0^1 \langle DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s), X_{s+\epsilon} - X_s\rangle^2 d\alpha \, ds = \\
&= \int_0^t \int_0^1 \langle (DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s))\otimes^2, \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon}\rangle d\alpha \, ds
\end{aligned}$$

Using Corollary 4.24 it follows

$$A_1(\epsilon) \xrightarrow{\mathbb{P}} \int_0^t \langle DF(X_s) \otimes DF(X_s), d\widetilde{[X]}_s\rangle$$

It remains to show the convergence in probability of $A_2(\epsilon)$ and $A_3(\epsilon)$ to zero.
Concerning $A_2(\epsilon)$ we observe that we can decompose as follows

$$DF(X_s)\otimes(DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s)) = DF(X_s)\otimes DF((1-\alpha)X_s + \alpha X_{s+\epsilon}) - DF(X_s)\otimes DF(X_s) \tag{6.8}$$

and concerning $A_3(\epsilon)$ we have

$$\begin{aligned}(DF((1-\alpha)X_s+\alpha X_{s+\epsilon})-DF(X_s))\otimes^2 = {} & DF((1-\alpha)X_s+\alpha X_{s+\epsilon})\otimes^2 + \\ & + DF(X_s)\otimes DF(X_s) + \\ & - DF((1-\alpha)X_s+\alpha X_{s+\epsilon})\otimes DF(X_s) + \\ & - DF(X_s)\otimes DF((1-\alpha)X_s+\alpha X_{s+\epsilon})\,.\end{aligned} \tag{6.9}$$

Using (6.8), we obtain

$$\begin{aligned}|A_2(\epsilon)| & \leq 2\int_0^t\int_0^1 \left|\langle DF(X_s)\otimes(DF((1-\alpha)X_s+\alpha X_{s+\epsilon})-DF(X_s))\,,\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right| d\alpha\, ds \leq \\ & \leq \int_0^t\int_0^1 \|DF(X_s)\otimes DF((1-\alpha)X_s+\alpha X_{s+\epsilon})-DF(X_s)\otimes DF(X_s)\|_\chi \left\|\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\right\|_{\chi^*} d\alpha\, ds\end{aligned} \tag{6.10}$$

For fixed $\omega\in\Omega$ we denote by $\mathcal{V}(\omega) := \{X_t(\omega);\, t\in[0,T]\}$ and

$$\mathcal{U}=\mathcal{U}(\omega)=\overline{conv(\mathcal{V}(\omega))}, \tag{6.11}$$

i.e. the set $\mathcal{U}$ is the closed convex hull of the compact subset $\mathcal{V}(\omega)$ of B. From (6.10) we deduce

$$|A_2(\epsilon)| \leq \varpi_{DF\otimes DF}^{\mathcal{U}\times\mathcal{U}}(\epsilon)\int_0^t \left\|\frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\right\|_{\chi^*} ds,$$

where $\varpi_{DF\otimes DF}^{\mathcal{U}\times\mathcal{U}}$ is the continuity modulus of the application $DF(\cdot)\otimes DF(\cdot): B\times B\longrightarrow\chi$ restricted to $\mathcal{U}\times\mathcal{U}$. We recall that

$$\varpi_{DF\otimes DF}^{\mathcal{U}\times\mathcal{U}}(\delta) = \sup_{\|(x_1,y_1)-(x_2,y_2)\|_{B\times B}\leq\delta} \|DF(x_1)\otimes DF(y_1)-DF(x_2)\otimes DF(y_2)\|_\chi$$

where the space $B\times B$ is equipped with the norm equal to the norm obtained summing the norms of the two components.
According to Theorem 5.35 from [2], $\mathcal{U}(\omega)$ is compact, then the function $DF(\cdot)\otimes DF(\cdot)$ on $\mathcal{U}(\omega)\times\mathcal{U}(\omega)$ is uniformly continuous and $\varpi_{DF\otimes DF}^{\mathcal{U}\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero.
Let (ϵ_n) converging to zero; Condition **H1** in the definition of χ-quadratic variation, implies the existence of a subsequence (ϵ_{n_k}) such that $A_2(\epsilon_{n_k})$ converges to zero a.s. We can now conclude that $A_2(\epsilon)\to 0$ in probability.

With similar arguments, using (6.9), we can show that $A_3(\epsilon) \to 0$ in probability. We observe in fact

$$
\begin{aligned}
|A_3(\epsilon)| \le & \int_0^t \int_0^1 \left\| DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) \otimes^2 - DF(X_s) \otimes DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right)\right\|_\chi \cdot \\
& \qquad \cdot \left\| \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha\, ds + \\
& + \int_0^t \int_0^1 \left\| DF\left((1-\alpha)X_s + \alpha X_{s+\epsilon}\right) \otimes DF(X_s) - DF(X_s)\otimes^2 \right\|_\chi \left\| \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon} \right\|_{\chi^*} d\alpha\, ds \le \\
& \le 2\varpi^{\mathcal{U}\times\mathcal{U}}_{DF\otimes DF}(\epsilon) \int_0^t \left\| \frac{(X_{s+\epsilon} - X_s)\otimes^2}{\epsilon} \right\|_{\chi^*} ds
\end{aligned}
$$

□

Corollary 6.18. Let B, B_0 be Banach spaces such that $B_0 \supset B$, χ a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ such that $\chi = (B_0\hat{\otimes}_\pi B_0)^*$ and X a B-valued stochastic process admitting χ-quadratic variation. Let $F^1, F^2 : B \longrightarrow \mathbb{R}$ be functions of class C^1 in the sense of Fréchet differentiable such that applications

$$DF^i : B \longrightarrow B_0^* \subset B^*$$

are continuous, $i = 1, 2$.
Then the covariation of $F^i(X)$ and $F^j(X)$ exists and it is given by

$$[F^i(X), F^j(X)]_\cdot = \int_0^\cdot \langle DF^i(X_s) \otimes DF^j(X_s), d\widetilde{[X]}_s \rangle \tag{6.12}$$

Proof. For any given $x, y \in B$, $i, j = 1, 2$, by characterization of $DF^i(x) \otimes DF^j(y)$ given in Remark 6.14, it follows that the following applications

$$DF^i(x) \otimes DF^j(y) : B_0\hat{\otimes}_\pi B_0 \longrightarrow \mathbb{R}$$

are continuous for $i, j = 1, 2$. The result follows from Theorem 6.16. □

Remark 6.19. Under the same assumptions as Corollary 6.18 we suppose moreover that B_0 is a Hilbert space. In this case for any $x, y \in B$, $DF(x) \otimes DG(y)$ belongs to $\left(B_0\hat{\otimes}_h B_0\right)^*$ and it will we associated to the true tensor product in the sense of Remark 6.14.2.
In view of further applications we will see an important application of this corollary in Proposition 6.15, which set $B = C([-\tau, 0])$. This is the case when the Fréchet derivative of F and G are in $D_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ as in Corollary 6.15.

Example 6.20. Let X be a $\mathbb{R}^n$-valued stochastic process with finite quadratic variation $[X] : (\mathbb{R}^n\hat{\otimes}_\pi\mathbb{R}^n)^* \longrightarrow \mathcal{C}([0,T])$ and the associated $\widetilde{[X]}(\omega, \cdot) : [0,T] \longrightarrow (\mathbb{R}^n\hat{\otimes}_\pi\mathbb{R}^n)^{**} \cong (\mathbb{R}^n\hat{\otimes}_\pi\mathbb{R}^n)$. This space can be identified with the space of matrices $\mathbb{M}_{n\,n}(\mathbb{R})$. In fact the tensor product between a vector $x = (x_1, \ldots, x_n)^*$ and a

vector $y = (y_1, \ldots, y_n)$ can be linked to the matrix $(x_i y_j)_{i,j=1,\ldots,n} \in \mathbb{M}_{n\,n}(\mathbb{R})$, see chapter **??**.
Let $F, G : \mathbb{R}^n \longrightarrow \mathbb{R} \in C^1(\mathbb{R}^n)$. Then

$$[F(X), G(X)]_\cdot = \int_0^\cdot \langle DF(X_s) \otimes DG(X_s), d\widetilde{[X]}_s \rangle$$

which coincides with result recalled in Proposition 6.13.

6.3 Dirichlet processes

We formulate now some results about the stability of a window Dirichlet process and some related Fukushima type decomposition. If X is a Dirichlet process and $F : C([-\tau, 0]) \longrightarrow \mathbb{R}$ of class C^1 such that the first derivative belongs to $\mathcal{D}_0([-\tau, 0]^2) \oplus L^2([-\tau, 0])$, then Theorem 6.25 says that $F(X(\cdot))$ is a Dirichlet process.
First we need a preliminary result on measure theory.

Lemma 6.21. Let B be a topological direct sum $B_1 \oplus B_2$ where B_1, B_2 are Banach spaces equipped with some norm $\| \cdot \|_{B_i}$. We denote by P_i the projectors $P_i : \chi \to B_i$, $i \in 1, 2$. For $\tilde{g} : [0, T] \to B^*$, we define $\tilde{g}_i : [0, T] \to B_i^*$ setting $\tilde{g}_i(t)(\eta) := \tilde{g}(t)(\eta)$ for all $\eta \in B_i$, i.e. the restriction of $\tilde{g}(t)$ to B_i^*. Let $\tilde{g}_i$ continuous with bounded variation and $f : [0, T] \to B$ with projections $f_i := P_i(f)$ defined from $[0, T]$ to B_i by $P_i(f(s))$. Then

1) f in $L^1_B(\tilde{g})$ (for instance cadlag) iff f_i in $L^1_{B_i}(\tilde{g}_i)$ and it holds

$$\int_0^t {}_B\langle f(s), d\tilde{g}(s) \rangle_{B^*} = \int_0^t {}_{B_1}\langle f_1(s), d\tilde{g}_1(s) \rangle_{B_1^*} + \int_0^t {}_{B_2}\langle f_2(s), d\tilde{g}_2(s) \rangle_{B_2^*} \tag{6.13}$$

2) In particular if $\tilde{g}_2(t) \equiv 0$ then

$$\int_0^t {}_B\langle f(s), d\tilde{g}(s) \rangle_{B^*} = \int_0^t {}_{B_1}\langle f_1(s), d\tilde{g}_1(s) \rangle_{B_1^*} \tag{6.14}$$

3) Suppose that $B_1 = \mathcal{D}_{i,j}([-\tau, 0]^2)$, B_2 be a Chi-subspace of $\mathcal{M}([-\tau, 0]^2)$ and $\tilde{g}_2(t) \equiv 0$. Moreover there exists $g_1 : [0, T] \to \mathbb{R}$ continuous with bounded variation such that $\tilde{g}_1(t)(\mu) = \langle \mu, \delta_{a_i} \otimes \delta_{a_j} \rangle g_1(t) = \lambda\, g_1(t)$ for every element $\mu = \lambda\, \delta_{a_i} \otimes \delta_{a_j}$ in $\mathcal{D}_{ij}$, $\lambda = \mu(\{a_i, a_j\})$ real number. Then (6.14) equals

$$\int_0^t {}_B\langle f(s), d\tilde{g}(s) \rangle_{B^*} = \int_0^t {}_{B_1}\langle f_1(s), \delta_{a_i} \otimes \delta_{a_j} \rangle_{B_1^*} dg_1(s) = \int_0^t f(s)(\{a_i, a_j\}) dg_1(s) \tag{6.15}$$

Proof.

1) By hypothesis on $\tilde{g}_i$ we deduce that $\tilde{g} : [0, T] \to B_i^*$ has bounded variation. If $f : [0, T] \to \chi$ in L^1_B, then $f_i = P_i(f) : [0, T] \to B_i$, $i = 1, 2$ belongs to $L^1_{B_i}$ by the property $\|P_i f\|_{B_i} \leq \|f\|_B$. As we

can see in the appendix vector valued integrations on $L^1_B(\tilde{g})$, as well as on $L^1_{B_i}(\tilde{g}_i)$, is defined by density on step functions. First we show the result for a step function $f:[0,T]\to B^*$ defined by $f(s)=\sum_{j=1}^N \phi_{A_j}(s)f_j$ with ϕ_{A_j} indicator functions of the subsets A_j of $[0,T]$ and $f_j\in B$. We have $f_j=f_{1j}+f_{2j}$ by projections, so

$$\begin{aligned}\int_0^T \langle f(s), d\tilde{g}(s)\rangle &= \sum_{j=1}^N \int_{A_j}\langle f_j, d\tilde{g}(s)\rangle = \sum_{j=1}^N \langle f_j, \int_{A_j} d\tilde{g}(s)\rangle = \sum_{j=1}^N \langle f_j, \tilde{g}(A_j)\rangle = \\ &= \sum_{j=1}^N \langle f_{1j}, \tilde{g}_1(A_j)\rangle + \sum_{j=1}^N \langle f_{2j}, \tilde{g}_2(A_j)\rangle = \\ &= \int_0^T \langle f_1(s), d\tilde{g}_1(s)\rangle + \int_0^T \langle f_2(s) d\tilde{g}_2(s)\rangle\end{aligned}$$

A general function f in $L^1_B(\tilde{g})$ is a sum of f_1+f_2, $f_i\in L^1_{B_i}(\tilde{g}_i)$ for $i=1,2$. Both f_1 and f_2 can be approximate by step functions. The result follows by an approximation argument.

2) Follows easily by 1).

3) It is a consequence of 2) and Theorem 30 in Chapter 1, paragraph 2 of [16]. Obviously g_1 has bounded variation and $\tilde{g}_1(t)=g_1(t)\delta_{(a_i,a_j)}=g_1(t)\delta_{a_i}\otimes\delta_{a_j}$. Consequently, by inspection

$$\int_0^t {}_{B_1}\langle f_1(s), d\tilde{g}_1(s)\rangle_{B_1^*} = \int_0^t f_1(s)(\{a_i,a_j\})dg_1(s) = \int_0^t f(s)(\{a_i,a_j\})dg_1(s)$$

and the result follows.

□

Remark 6.22. Let χ be a Banach subspace of $\mathcal{M}([-\tau,0]^2)$ containing $\mathcal{D}_{i,j}([-\tau,0]^2)$. A typical example of application of Lemma 6.21 is given by $\chi_1=\mathcal{D}_{i,j}([-\tau,0]^2)$ and $\chi_2=\{\mu\in\chi \mid \mu(\{a_i,a_j\})=0\}$. Any $\mu\in\chi$ can be decomposed into $\mu_1+\mu_2$, where $\mu_1=\mu(\{a_i,a_j\})\delta_{(a_i,a_j)}$ and $\mu_2\in\chi_2$. This framework will be the one of proposition below.

Lemma 6.21 will be applied considering $\tilde{g}$ as the χ-covariation of two processes X and Y

Proposition 6.23. Let χ_2 be a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$ such that $\mu(\{a_i,a_j\})=0$ for a given $i,j\in\{0,\dots,N\}$ and $\mu\in\chi_2$. We set $\chi=\mathcal{D}_{i,j}([-\tau,0]^2)\oplus\chi_2$, Let X, Y be two real continuous processes such that $X(\cdot)$ and $Y(\cdot)$ admit a zero χ_2-covariation and a $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation. Then

1) $[X_{\cdot+a_i},Y_{\cdot+a_j}]$ exists and the $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation is given by

$$[X(\cdot),Y(\cdot)]:\mathcal{D}_{i,j}([-\tau,0]^2)\longrightarrow\mathcal{C}([0,T]) \qquad \mu\mapsto\mu(\{a_i,a_j\})[X_{\cdot+a_i},Y_{\cdot+a_j}].$$

2) χ is a Chi-subspace of $\mathcal{M}([-\tau,0]^2)$.

3) $X(\cdot)$ and $Y(\cdot)$ admit a χ-covariation of the type

$$[X(\cdot),Y(\cdot)]:\chi\longrightarrow\mathcal{C}([0,T]) \qquad [X(\cdot),Y(\cdot)](\mu)=\mu(\{a_i,a_j\})[X_{\cdot+a_i},Y_{\cdot+a_j}]$$

4) for every χ-valued process Z with locally bounded paths (for instance cadlag) we have

$$\int_0^{\cdot}\langle Z_s, d\widetilde{[X(\cdot),Y(\cdot)]}_s\rangle=\int_0^{\cdot} Z_s(\{a_i,a_j\})d[X_{\cdot+a_i},Y_{\cdot+a_j}]_s \tag{6.16}$$

Proof.

1) is a consequence of the fact that $X(\cdot)$ and $Y(\cdot)$ admit a $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation.

2) follows by Proposition 4.3.

3) Since $\mathcal{D}_{i,j}([-\tau,0]^2)$ and χ_2 are Chi-subspaces, $X(\cdot)$ and $Y(\cdot)$ admit $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation and χ_2-covariation, then Remark 6.4 point 1) implies that $X(\cdot)$ and $Y(\cdot)$ admit χ-covariation which equals the sum of $\mathcal{D}_{i,j}([-\tau,0]^2)$-covariation and χ_2-covariation. For μ in χ with decomposition $\mu_1+\mu_2$, $\mu_1\in\mathcal{D}_{i,j}([-\tau,0]^2)$ and $\mu_2\in\chi_2$ we have

$$\begin{aligned}[X(\cdot),Y(\cdot)](\mu)&=[X(\cdot),Y(\cdot)](\mu_1)+[X(\cdot),Y(\cdot)](\mu_2)=\\&=[X(\cdot),Y(\cdot)](\mu_1)=\\&=\mu_1(\{a_i,a_j\})[X_{\cdot+a_i},Y_{\cdot+a_j}]=\\&=\mu(\{a_i,a_j\})[X_{\cdot+a_i},Y_{\cdot+a_j}].\end{aligned}$$

4) Since both side of (6.16) are continuous processes, it is enough to show that they are equals a.s. for every fixed $t\in[0,T]$. This follows for almost all $\omega\in\Omega$ using Lemma 6.21 where $f=Z(\omega)$ and $\tilde{g}=\widetilde{[X(\cdot),Y(\cdot)]}(\omega)$.

□

Remark 6.24. Proposition 6.23 will be used in the sequel especially in the case $a_i=a_j=0$.

Theorem 6.25. Let X be a real continuous $(\mathcal{F}_t)-$Dirichlet process with decomposition $X=M+A$, M local martingale and A is a zero quadratic variation process with $A_0=0$. Let $F:C([-\tau,0])\longrightarrow\mathbb{R}$ be a Fréchet differentiable function such that the range of DF is $\mathcal{D}_0([-\tau,0])\oplus L^2([-\tau,0])$. Moreover we suppose that $DF:C([-\tau,0])\longrightarrow\mathcal{D}_0([-\tau,0])\oplus L^2([-\tau,0])$ is continuous.
Then $F(X(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\bar{M}_{\cdot}=\int_0^{\cdot} DF\big(X_s(\cdot)\big)(\{0\})dM_s+F\big(X_0(\cdot)\big).$$

Proof. We need to show that $[\bar{A}]=0$ where $\bar{A}:=F(X(\cdot))-\bar{M}$. From now on in this proof we denote $\alpha_0(\eta)=DF(\eta)(\{0\})$. By the linearity of the real covariation we have $[\bar{A}]=A_1+A_2-2A_3$ where

$$\begin{aligned}
A_1 &= [F(X_\cdot(\cdot))]\\
A_2 &= \left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right]\\
A_3 &= \left[F(X(\cdot)),\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right]
\end{aligned}$$

Since X is a finite quadratic variation process, then by Corollary 5.10 its window process $X(\cdot)$ admits $\chi^0([-\tau,0]^2)$-quadratic variation $[X(\cdot)]$. Moreover by Corollary 6.15 and Remark 6.19 the map $DF\otimes DF : C([-\tau,0])\times C([-\tau,0])\longrightarrow \chi^0([-\tau,0]^2)$ is a continuous application. Applying Theorem 6.16 and (6.16) of Proposition 6.23 we obtain

$$\begin{aligned}
A_1 &= \int_0^\cdot \langle DF(X_s(\cdot))\otimes DF(X_s(\cdot)), d\widetilde{[X_\cdot(\cdot)]}_s\rangle =\\
&= \int_0^\cdot \alpha_0^2(X_s(\cdot))d[X]_s = \int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s
\end{aligned}$$

Term A_2 is the quadratic variation of an Itô's integral because the stochastic process $\alpha_0\big(X_s(\cdot)\big)$ is $(\mathcal{F}_s)$-adapted, so that

$$A_2=\int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s$$

It remains to prove that $A_3=\int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s$.We define $G: C([-\tau,0])\longrightarrow\mathbb{R}$ by $G(\eta)=\eta(0)$. We observe that $\bar{M}=G(\bar{M}(\cdot))$ where $\bar{M}(\cdot)$ denotes as usual the window process associated to $\bar{M}$. G is Fréchet differentiable and $DG(\eta)=\delta_0$, therefore DG is continuous from $C([-\tau,0])$ to $\mathcal{D}_0([-\tau,0])\oplus L^2([-\tau,0])$. Moreover by Corollary 6.15 we know that $DF\otimes DG : C([-\tau,0])\times C([-\tau,0])\longrightarrow\chi^0([-\tau,0]^2)$ is a continuous application. Remark 6.12 point 2. says that the $\chi^0([-\tau,0]^2)$-covariation between $X(\cdot)$ and $\bar{M}(\cdot)$ exists and it is given by

$$[X(\cdot),\bar{M}(\cdot)](\mu)=\mu(\{0,0\})[X,\bar{M}]. \tag{6.17}$$

By usual properties of stochastic calculus we have $[X,\bar{M}]=[M,\bar{M}]+[A,\bar{M}]=\big[M,\int_0^\cdot a_0\big(X_s(\cdot)\big)dM_s\big]=\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)d[M]_s$. Finally again applying Theorem 6.16, equation (6.16) in Proposition 6.23 and result (6.17) we obtain

$$\begin{aligned}
A_3 &= [F(X(\cdot)),G(\tilde{M}(\cdot))] =\\
&= \int_0^\cdot \langle DF(X_s(\cdot))\otimes DG(\bar{M}_s(\cdot)), d\widetilde{[X(\cdot),\bar{M}(\cdot)]}_s\rangle =\\
&= \int_0^\cdot \alpha_0(X_s(\cdot))d[X,\bar{M}]_s = \int_0^\cdot \alpha_0^2(X_s(\cdot))d[M]_s
\end{aligned}$$

□

Theorem 6.26. Let X be a real continuous $(\mathcal{F}_t)$-Dirichlet process with decomposition $X = M + A$, M local martingale and A zero quadratic variation process with $A_0 = 0$. Let $F : C([-\tau, 0]) \longrightarrow \mathbb{R}$ be a Fréchet differentiable function such that $DF : C([-\tau, 0]) \longrightarrow \mathcal{D}_a([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous. Denoting $\alpha_i(\eta) = DF(\eta)(\{a_i\})$, we have the following.

1) $F(X(\cdot))$ is a finite quadratic variation process and

$$[F(X(\cdot))] = \sum_{i=0,\ldots,N} \int_0^t \alpha_i^2(X_s(\cdot))d[M_{\cdot+a_i}]_s \tag{6.18}$$

2) $F(X(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with decomposition $F(X(\cdot)) = \bar{M} + \bar{A}$, where $\bar{M}$ is a local martingale defined by $\bar{M}_\cdot := \int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s + F\left(X_0(\cdot)\right)$ and $\bar{A}$ is the $(\mathcal{F}_t)$-martingale orthogonal process.

3) Moreover $\bar{A}$ is a finite quadratic variation process and

$$[\bar{A}]_t = \sum_{i=1,\ldots,N} \int_0^t \alpha_i^2(X_s(\cdot))d[M_{\cdot+a_i}]_s \tag{6.19}$$

4) In particular $\{F(X_t(\cdot)); t \in [0, -a_1]\}$ is a Dirichlet process with local martingale component $\bar{M}$.

Proof.

1) By Corollary 6.15 we know that $DF \otimes DF : C([-\tau, 0]) \times C([-\tau, 0]) \longrightarrow \chi^2([-\tau, 0]^2)$ and it is a continuous map. Applying Theorem 6.16, equation (6.16) in Proposition 6.23 and Example 5.17 point 4) we obtain

$$\begin{aligned}
[F(X(\cdot))]_t &= \int_0^t \langle DF(X_s(\cdot)) \otimes DF(X_s(\cdot)), d\widetilde{[X_s(\cdot)]}\rangle = \\
&= \int_0^t \sum_{i,j=0\ldots,N} \alpha_i(X_s(\cdot))\alpha_j(X_s(\cdot))d[X_{\cdot+a_i}, X_{\cdot+a_j}]_s = \\
&= \sum_{i=0,\ldots,N} \int_0^t \alpha_i^2(X_s(\cdot))d[M_{\cdot+a_i}]_s
\end{aligned}$$

and (6.18) is proved.

2) To show that it $F(X(\cdot))$ is a weak Dirichlet process we need to show that $[F(X(\cdot)) - \int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s, N]$ is zero for every $(\mathcal{F}_t)$-continuous local martingale N. Again setting $G : C([-\tau, 0]) \longrightarrow \mathbb{R}$ setting $G(\eta) = \eta(0)$. It holds $N_t = G(N_t(\cdot))$. Function G is Fréchet differentiable with $DG : C([-\tau, 0]) \longrightarrow \mathcal{D}_0([-\tau, 0])$, $DG(\eta) = \delta_0$. Corollary 6.15 says that $DF \otimes DG : C([-\tau, 0]) \times C([-\tau, 0]) \longrightarrow \chi^2([-\tau, 0]^2)$

and it is a continuous map. Theorem 6.10 implies that $X(\cdot)$ and $N(\cdot)$ admit a $\chi^2([-\tau,0]^2)$-covariation which equals

$$[X(\cdot),N(\cdot)](\mu)=\mu(\{0,0\})[M,N]. \tag{6.20}$$

By Theorem 6.16 and (6.20) we have

$$\begin{aligned}[F(X(\cdot)),N]_t=[F(X(\cdot)),G(N(\cdot))]_t &= \int_0^t \langle DF(X_s(\cdot))\otimes DG(N_s(\cdot)), d[\widetilde{X(\cdot),N(\cdot)}]_s\rangle = \\ &= \int_0^t \alpha_0(X_s(\cdot))d[M,N]_s \end{aligned} \tag{6.21}$$

On the other hand $\alpha_0\big(X_s(\cdot)\big)$ is $(\mathcal{F}_s)$-adapted so $\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s$ is an Itô's integral then by usual properties of stochastic calculus

$$\left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s, N\right]_t = \int_0^t \alpha_0\big(X_s(\cdot)\big)d[M,N]_s$$

and the result follows.

3) Moreover $[\bar{A}]=[F(X(\cdot))]+[\bar{M}]-2[F(X(\cdot)),\bar{M}]$. The first bracket is equal to (6.18). The second term is

$$\left[\int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s$$

Setting $N_t=\int_0^t \alpha_0(X_s(\cdot))dM_s$, (6.21) gives

$$\left[F(X(\cdot)), \int_0^\cdot \alpha_0\big(X_s(\cdot)\big)dM_s\right] = \int_0^t \alpha_0^2(X_s(\cdot))d[M]_s$$

and (6.19) follows.

4) It is an easy consequence of (6.19) since $(\bar{A}_t)_{t\in[0,-a_1[}$ is a zero quadratic variation process.

$\square$

Remark 6.27. 1. Theorem 6.26 gives a class of examples of $(\mathcal{F}_t)$-weak Dirichlet processes with finite quadratic variation which are not $(\mathcal{F}_t)$-Dirichlet processes.

2. An example of $F: C([-\tau,0])\longrightarrow\mathbb{R}$ Fréchet differentiable such that $DF: C([-\tau,0])\longrightarrow \mathcal{D}_a([-\tau,0])\oplus L^2([-\tau,0])$ continuously is, for instance, $F(\eta)=\sum_{i=0}^N f_i\big(\eta(a_i)\big)$, with $f_i\in C^1(\mathbb{R})$. We have $DF(\eta)=\sum_{i=0}^N f_i'\big(\eta(a_i)\big)\delta_{a_i}$.

3. Let $a \in [-\tau, 0[$ and W be a classical $(\mathcal{F}_t)$-Brownian motion, process X defined as $X_t := W_{t+a}$ is an $(\mathcal{F}_t)$-weak Dirichlet process that is not $(\mathcal{F}_t)$-Dirichlet. This result can be proved of course also directly, see Proposition 4.11 in [9].

We now go on with a result concerning weak Dirichlet processes. Let D be a real continuous $(\mathcal{F}_t)$-Dirichlet process. In [24], Proposition 3.10, it was proved that given $u \in C^{0,1}(\mathbb{R}_+ \times \mathbb{R})$, $X_t = u(t, D_t)$ is an $(\mathcal{F}_t)$-weak Dirichlet process. Let $F \in C^{0,1}(\mathbb{R}_+ \times C([-\tau, 0]))$ in the Fréchet sense. Similarly to [24] we cannot expect $X = F(\cdot, D_\cdot(\cdot))$ to be a Dirichlet process. In general it will not even be a finite quadratic variation process if the dependence on t is very irregular. However we will show that X is a weak Dirichlet process, even if D is weak Dirichlet process with finite quadratic variation.

Theorem 6.28. Let D be an $(\mathcal{F}_t)$-weak Dirichlet process with finite quadratic variation where M is the local martingale part. Let $F : [0, T] \times C([-\tau, 0]) \longrightarrow \mathbb{R}$ continuous. We suppose moreover that $(t, \eta) \mapsto DF(t, \eta)$ exists with values in $\mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$ and $DF : [0, T] \times C([-\tau, 0]) \longrightarrow \mathcal{D}_0([-\tau, 0]) \oplus L^2([-\tau, 0])$ is continuous.
Then $F(\cdot, D_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\bar{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s \tag{6.22}$$

where $D^{\delta_0} F(s, D_s(\cdot))$ denotes $DF(s, D_s(\cdot))(\{0\})$, i.e. the Dirac zero component of the first derivative of F calculated in $(s, D_s(\cdot))$.

Proof. In the sequel we will denote real process $\bar{M}^F$ simply by $\bar{M}$. We need to show that for any $(\mathcal{F}_t)$-continuous local martingale N

$$\left[F(\cdot, D(\cdot)) - \bar{M}, N_\cdot\right]_t = 0 \quad a.s. \tag{6.23}$$

Since the covariation of semimartingales coincides with the classical covariation

$$\left[\bar{M}, N\right]_t = \int_0^t D^{\delta_0} F(s, D_s(\cdot)) d[M, N]_s \tag{6.24}$$

It remains to check that, for every $t \in [0, T]$,

$$[F(\cdot, D(\cdot)), N]_t = \int_0^t D^{\delta_0} F(s, D_s(\cdot)) d[M, N]_s$$

For this we have to evaluate the ucp limit of

$$\int_0^t \Big(F(s+\epsilon, D_{s+\epsilon}(\cdot)) - F(s, D_s(\cdot))\Big) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds \tag{6.25}$$

if it exists. (6.25) can be written as the sum of the two terms

$$I_1(t, \epsilon) = \int_0^t \Big(F(s+\epsilon, D_{s+\epsilon}(\cdot)) - F(s+\epsilon, D_s(\cdot))\Big) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds\,,$$

$$I_2(t, \epsilon) = \int_0^t \Big(F(s+\epsilon, D_s(\cdot)) - F(s, D_s(\cdot))\Big) \frac{N_{s+\epsilon} - N_s}{\epsilon} ds$$

First we prove that $I_1(t,\epsilon)$ converges to $\int_0^t D^{\delta_0}F(s,D_s(\cdot))d[M,N]_s$. If $G : C([-\tau,0]) \to \mathbb{R}$ is again the function $G(\eta) = \eta(0)$, then G is of class C^1 and $DG(\eta) = \delta_0$ for all $\eta \in C([-\tau,0])$ so that $DG : C([-\tau,0]) \longrightarrow \mathcal{D}_0([-\tau,0])$ is continuous. In particular it holds the equality $\eta(0) = G(\eta(\cdot)) = \langle \delta_0, \eta\rangle$. We express

$$\begin{aligned} I_1(t,\epsilon) &= \int_0^t \langle DF(s+\epsilon, D_s(\cdot)), (D_{s+\epsilon}(\cdot) - D_s(\cdot))\rangle \frac{N_{s+\epsilon} - N_s}{\epsilon} ds + R_1(t,\epsilon) \\ &= \int_0^t \langle DF(s+\epsilon, D_s(\cdot)), (D_{s+\epsilon}(\cdot) - D_s(\cdot))\rangle \frac{\langle \delta_0, N_{s+\epsilon}(\cdot) N_s(\cdot)\rangle}{\epsilon} ds + R_1(t,\epsilon) \end{aligned} \tag{6.26}$$

Now we have

$$\begin{aligned} R_1(t,\epsilon) = \int_0^t \left[\int_0^1 \langle DF\big(s+\epsilon, (1-\alpha)D_s(\cdot) + \alpha D_s(\cdot)\big) - DF\big(s+\epsilon, D_s(\cdot)\big), \big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big)\rangle d\alpha\right] \times \\ \times \frac{\langle \delta_0, N_{s+\epsilon}(\cdot) - N_s(\cdot)\rangle}{\epsilon} ds = \\ = \int_0^t \int_0^1 \langle DF\big(s+\epsilon, (1-\alpha)D_s(\cdot) + \alpha D_s(\cdot)\big) \otimes \delta_0 - DF\big(s+\epsilon, D_s(\cdot)\big) \otimes \delta_0, \\ \frac{\big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big) \otimes \big(N_{s+\epsilon}(\cdot) - N_s(\cdot)\big)}{\epsilon}\rangle d\alpha\, ds \end{aligned}$$

We recall that for $t \in [0,T]$, $\eta_1, \eta_2 \in C([-\tau,0])$ the map $DF(t,\eta_1) \otimes DG(\eta_2)$ equals the tensor product $DF(t,\eta_1) \otimes \delta_0$ by hypothesis and by Riesz identification of $DF(t,\eta_1)$. By Corollary 6.15 we know that map $DF \otimes \delta_0 : [0,T] \times C([-\tau,0]) \longrightarrow \chi^0([-\tau,0]^2)$ and it is a continuous map.
We denote by $\mathcal{U} = \mathcal{U}(\omega)$ the closed convex hull of the compact subset $\mathcal{V}$ of $C([-\tau,0])$ defined, for every ω, by

$$\mathcal{V} = \mathcal{V}(\omega) := \{D_t(\omega);\, t \in [0,T]\}$$

According to Theorem 5.35 from [2], $\mathcal{U}(\omega) = \overline{conv(\mathcal{V})(\omega)}$ is compact, then the function $DF(\cdot,\cdot) \otimes \delta_0$ on $[0,T] \times \mathcal{U}$ is uniformly continuous and we denote by $\varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}$ the continuity modulus of the application $DF(\cdot,\cdot)\otimes\delta_0$ restricted to $[0,T]\times\mathcal{U}$. $\varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. So we have

$$\sup_{t\in[0,T]} |R_1(t,\epsilon)| \le \int_0^T \varpi^{[0,T]\times\mathcal{U}}_{DF(\cdot,\cdot)\otimes\delta_0}(\epsilon) \left\| \frac{\big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big) \otimes \big(N_{s+\epsilon}(\cdot) - N_s(\cdot)\big)}{\epsilon} \right\|_{\chi^0([-\tau,0]^2)} ds \tag{6.27}$$

We recall by Remark 6.12, point 2. that $D(\cdot)$ and $N(\cdot)$ admit $\chi^2([-\tau,0]^2)$-covariation. In particular using condition **H1** and (6.27) claim $R_1(t,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ follows.
On the other hand the first addend in (6.26) can be rewritten as

$$\int_0^t \langle DF\big(s, D_s(\cdot)\big) \otimes \delta_0, \frac{\big(D_{s+\epsilon}(\cdot) - D_s(\cdot)\big) \otimes \big(N_{s+\epsilon}(\cdot) - N_s(\cdot)\big)}{\epsilon}\rangle ds + R_2(t,\epsilon) \tag{6.28}$$

where $R_2(t,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ arguing similarly as for $R_1(t,\epsilon)$. Using bilinearity and Corollary 4.24 the integral in (6.28) converges then ucp to

$$\int_0^t \langle DF\big(s, D_s(\cdot)\big) \otimes \delta_0, d[\widetilde{D(\cdot), N(\cdot)}]_s\rangle \tag{6.29}$$

By (6.16) in Proposition 6.23 in the case $a_i = a_j = 0$, (6.29) equals

$$\int_0^t D^{\delta_0} F(s, D_s(\cdot))d[D,N]_s = \int_0^t D^{\delta_0} F(s, D_s(\cdot))d[M,N]_s \tag{6.30}$$

It remains to show that $I_2(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$.
By stochastic Fubini's theorem we obtain

$$I_2(t,\epsilon) = \int_0^t \xi(\epsilon, r) dN_r$$

where

$$\xi(\epsilon, r) = \frac{1}{\epsilon}\int_{0\vee(r-\epsilon)}^r F(s+\epsilon, D_s(\cdot)) - F(s, D_s(\cdot))ds$$

Proposition 2.26, chapter 3 of [30] says that $I_2(\cdot,\epsilon) \xrightarrow[\epsilon\to 0]{ucp} 0$ if

$$\int_0^T \xi^2(\epsilon, r)d[N]_r \xrightarrow[\epsilon\to 0]{} 0 \tag{6.31}$$

We fix $\omega \in \Omega$ and we can even show that the convergence in (6.31) holds pointwise. We denote by $\varpi_F^{[0,T]\times\mathcal{U}}$ the continuity modulus of the application F restricted to the compact set $[0,T]\times\mathcal{U}$. For every $r \in [0,T]$ we have

$$|\xi(\epsilon, r)| \le \sup_{r\in[0,T]} |F(r+\epsilon, D_r(\cdot)) - F(r, D_r(\cdot))| \le \varpi_F^{[0,T]\times\mathcal{U}}(\epsilon)$$

which converges to zero for ϵ going to zero since function F on $[0,T]\times\mathcal{U}$ is uniformly continuous on the compact set and $\varpi_F^{[0,T]\times\mathcal{U}}$ is, as usual, a positive, increasing function on $\mathbb{R}^+$ converging to zero when the argument converges to zero. By Lebesgue's dominated convergence theorem we finally obtain (6.31). $\square$

Chapter 7

Itô's formula

We are now able to state an Itô's formula for stochastic processes with values in a general Banach space.

Theorem 7.1. Let B be a Banach space, χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued continuous process admitting a χ-quadratic variation. Let $F : [0,T] \times B \longrightarrow \mathbb{R}$ be a function once continuously differentiable with respect to the first variable t and of class C^2 in the Fréchet sense with respect to the second variable such that

$$D^2 F : [0,T] \times B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ continuous with respect to } \chi \tag{7.1}$$

Then for every $t \in [0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DF(s, X_s), d^- X_s\rangle_B$$

exists and following formula holds.

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \langle DF(s, X_s), d^- X_s\rangle + \frac{1}{2}\int_0^t \langle D^2 F(s, X_s), d\widetilde{[X]}_s\rangle \tag{7.2}$$

Proof. We fix a $t \in [0,T]$ and we observe that the quantity

$$I_0(\epsilon, t) = \int_0^t \frac{F(s+\epsilon, X_{s+\epsilon}) - F(s, X_s)}{\epsilon} ds \tag{7.3}$$

converges ucp for $\epsilon \to 0$ to $F(t, X_t) - F(0, X_0)$ since $F(X_s)$ is continuous. At the same time, using the Taylor's expansion, (7.3) can be written as the sum of two terms:

$$I_1(\epsilon, t) = \int_0^t \frac{F(s+\epsilon, X_{s+\epsilon}) - F(s, X_{s+\epsilon})}{\epsilon} ds \tag{7.4}$$

and

$$I_2(\epsilon, t) = \int_0^t \frac{F(s, X_{s+\epsilon}) - F(s, X_s)}{\epsilon} ds \tag{7.5}$$

First we prove that

$$I_1(\epsilon,t) \longrightarrow \int_0^t \partial_t F(s,X_s)ds \tag{7.6}$$

in probability for every fixed $t \in [0,T]$. In fact

$$I_1(\epsilon,t) = \int_0^t \partial_t F(s,X_{s+\epsilon})ds + R_1(\epsilon,t) \tag{7.7}$$

where

$$R_1(\epsilon,t) = \int_0^t \int_0^1 \partial_t F\big(s+(1-\alpha)\epsilon, X_{s+\epsilon}\big) - \partial_t F(s,X_{s+\epsilon})d\alpha ds$$

We have

$$\sup_{t\in[0,T]} |R_1(\epsilon,t)| \le T\,\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$$

where $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus in ϵ of the application $\partial_t F : [0,T]\times B \longrightarrow \mathbb{R}$ restricted to $[0,T]\times\mathcal{U}$. We recall that set $\mathcal{U}$ is been defined in (6.11) and there we have proved also that it is a compact set, as well as $[0,T]\times\mathcal{U}$. By the continuity of the $\partial_t F$ as function from $[0,T]\times B$ to $\mathbb{R}$ follows that the restriction on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{\partial_t F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. We deduce that $R_1(\epsilon,t)\to 0$ ucp as $\epsilon\to 0$.
On the other hand the first term in (7.7) can be rewritten as

$$\int_0^t \partial_t F(s,X_s)ds + R_2(\epsilon,t)$$

where $R_2(\epsilon,t)\to 0$ ucp arguing as for $R_1(\epsilon,t)$. Convergence (7.6) is now established
The second term $I_2(\epsilon,t)$ in (7.5), may also be approximated by using the Taylor's expansion and it can be written as the sum of three terms:

$$\begin{aligned}
I_{21}(\epsilon,t) &= \int_0^t \langle DF(s,X_s), \frac{X_{s+\epsilon}-X_s}{\epsilon}\rangle ds\\
I_{22}(\epsilon,t) &= \frac{1}{2}\int_0^t \langle D^2F(s,X_s), \frac{(X_{s+\epsilon}-X_s)^{\otimes 2}}{\epsilon}\rangle ds\\
I_{23}(\epsilon,t) &= \int_0^t \left[\int_0^1 \alpha\langle D^2F\left(s,(1-\alpha)X_{s+\epsilon}+\alpha X_s\right) - D^2F(s,X_s), \frac{(X_{s+\epsilon}-X_s)^{\otimes 2}}{\epsilon}\rangle\, d\alpha\right] ds
\end{aligned}$$

By Corollary 4.24,

$$I_{22}(\epsilon,t) \xrightarrow[\epsilon\to 0]{\mathbb{P}} \frac{1}{2}\int_0^t \langle D^2F(s,X_s), d\widetilde{[X]}_s\rangle$$

for every $t \in [0,T]$.
We study now $I_{23}(\epsilon,t)$ and we will show that $I_{23}(\epsilon,t) \xrightarrow{\mathbb{P}} 0$. We have

$$\begin{aligned}
|I_{23}(\epsilon,t)| &\leq \frac{1}{\epsilon}\int_0^t \int_0^1 \alpha \left|\langle D^2F\left(s,(1-\alpha)X_{s+\epsilon}+\alpha X_s\right) - D^2F(s,X_s), (X_{s+\epsilon}-X_s)\otimes^2\rangle\right| \, d\alpha \, ds \leq \\
&\leq \frac{1}{\epsilon}\int_0^t \int_0^1 \alpha \left\|D^2F\left(s,(1-\alpha)X_{s+\epsilon}+\alpha X_s\right) - D^2F(s,X_s)\right\|_{\chi} \left\|(X_{s+\epsilon}-X_s)\otimes^2\right\|_{\chi^*} \, d\alpha \, ds \leq \\
&\leq \varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)\int_0^t \sup_{\|\phi\|_\chi\leq 1}\left|\langle\phi, \frac{(X_{s+\epsilon}-X_s)\otimes^2}{\epsilon}\rangle\right| \, ds
\end{aligned}$$

where $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}(\epsilon)$ is the continuity modulus of the application $D^2F : [0,T]\times B \longrightarrow \chi$ restricted to the compact set $[0,T]\times\mathcal{U}$. We recall that $\mathcal{U}$ was defined in (6.11) and there we also proved that it is a compact set. So again D^2F on $[0,T]\times\mathcal{U}$ is uniformly continuous and $\varpi_{D^2F}^{[0,T]\times\mathcal{U}}$ is a positive, increasing function on $\mathbb{R}^+$ converging to 0 when the argument converges to zero. Taking into account condition **H1** in the definition of χ-quadratic variation, $I_{23}(\epsilon,t)\to 0$ in probability when ϵ goes to zero.
Since I_0, I_1, I_{22} and I_{23} converge in probability for every fixed $t\in[0,T]$, it follows

$$I_{21}(\epsilon,t) \longrightarrow \int_0^t \langle DF(s,X_s), d^-X_s\rangle$$

in probability. This insure by definition that the forward integral exists.
This in particular also implies the so-called Itô's formula (7.2). □

As corollary of Theorem 7.1 we have the so-colled Itô's formula in the homogeneous case, i.e. without the dependence on the time variable t.

Corollary 7.2. Let B be a Banach space, χ be a Chi-subspace of $(B\hat{\otimes}_\pi B)^*$ and X a B-valued continuous process admitting a χ-quadratic variation. Let $G : B \longrightarrow \mathbb{R}$ a function of class C^2 Fréchet such that

$$D^2G : B \longrightarrow \chi \subset (B\hat{\otimes}_\pi B)^* \text{ continuous with respect to } \chi \tag{7.8}$$

Then for every $t\in[0,T]$ the forward integral

$$\int_0^t {}_{B^*}\langle DG(X_s), d^-X_s\rangle_B$$

exists and following formula holds:

$$G(X_t) = G(X_0) + \int_0^t {}_{B^*}\langle DG(X_s), d^-X_s\rangle_B + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2G(X_s), d\widetilde{[X]}_s\rangle_{\chi^*} \tag{7.9}$$

Proof. The proof is just an application of Theorem 7.1 without the dependence on time variable t □

The Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ constitutes a degree of freedom of validity of Itô's formula. In order to find the suitable expansion for $F(t,X_t)$ we may proceed as follows

- Let $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 we compute the second order derivative D^2F if it exists.
- We check the existence of a Chi-subspace χ of $(B\hat{\otimes}_\pi B)^*$ for which the range of $D^2F : [0,T] \times B \longrightarrow (B\hat{\otimes}_\pi B)^*$ is included in χ and it is continuous with respect to the topology of χ.
- We verify that X admits a χ-quadratic variation

We observe that whenever X admits a global quadratic variation, i.e. $\chi = (B\hat{\otimes}_\pi B)^*$, previous points reduce to check that $F \in C^{1,2}$. When X is a semimartingale we rediscover the classical Itô's formula.

We illustrate now an application of Corollary 7.2 for window processes $X(\cdot)$, where X is a real continuous finite quadratic variation process. $X(\cdot)$ can be reasonably observed in the two following perspectives:

a) $X(\cdot)$ is $C([-\tau,0])$-valued and χ has to be a Chi-subspace of $(C([-\tau,0])\hat{\otimes}_\pi C([-\tau,0]))^*$. Related examples of such χ are listed in Example 4.5.

b) $X(\cdot)$ is $L^2([-\tau,0])$-valued and χ has to be a Chi-subspace of $(L^2([-\tau,0])\hat{\otimes}_\pi L^2([-\tau,0]))^*$. Related examples of such χ are listed in Examples 4.8.

Let $G : L^2([-\tau,0]) \longrightarrow \mathbb{R}$ be defined by

$$G(\eta) = \int_{-\tau}^{0} \eta^2(s)ds = \|\eta\|^2_{L^2([-\eta,0])} \tag{7.10}$$

G is a continuous function as well as its restriction F to $C([-\tau,0])$.
We have

$$D^2G : L^2([-\tau,0]) \longrightarrow Diag_{\mathcal{B}}([-\tau,0]^2) \tag{7.11}$$

In fact it is constant and equal to the double inner product in $L^2([-\tau,0])$, i.e. the bilinear map such that

$$(f,g) \mapsto 2\langle f,g\rangle_{L^2([-\tau,0])}.$$

Also the restriction F is C^2 Fréchet in fact

$$D^2F : C([-\tau,0]) \longrightarrow Diag([-\tau,0]^2) \tag{7.12}$$

and it is the constant Radon measure on $[-\tau,0]^2$, defined by

$$\mu(dx,dy) = 2\mathbb{1}_{[-\tau,0]}(x)\delta_y(dx)dy \tag{7.13}$$

Being constant previous maps are both continuous with respect to the corresponding χ-topology. We illustrate an application of Corollary 7.2 to function $F(X_t(\cdot))$ and $G(X_t(\cdot))$. The proposition below gives in particular a representation of a forward type integral.

Proposition 7.3. Let $0 < \tau \le T$ and X a continuous real process such that $[X]_t = t$. We set $B = C([-\tau, 0])$. Then for the B-valued window process $X(\cdot)$ it holds

$$2\int_0^t {}_{B^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_B = \|X_t(\cdot)\|^2_{L^2([-\tau,0])} - \int_0^{t\wedge\tau} (t-y)dy \tag{7.14}$$

Proof. We apply Itô's formula of Corollary 7.2 to $F(X_t(\cdot))$. In this case we have for η, h, h_1 and h_2 in $C([-\tau, 0])$

$$DF(\eta)(h) = 2\int_{-\tau}^0 \eta(s)h(s)ds$$
$$D^2F(\eta)(h_1, h_2) = 2\int_{-\tau}^0 h_1(s)h_2(s)ds = 2\langle h_1, h_2\rangle_{L^2([-\eta,0])}$$

wwhere D^2F was given in (7.13). In terms of measures, it gives

$$\begin{aligned} D_{dx}F(\eta) &= 2\mathbb{1}_{[-\tau,0]}(x)\eta(x)dx \\ D^2_{dx\,dy}F(\eta) &= 2\mathbb{1}_{[-\tau,0]}(x)\delta_y(dx)dy \end{aligned} \tag{7.15}$$

We set $\chi = Diag([-\tau, 0]^2)$. By Proposition 5.20 $X(\cdot)$ admits χ-quadratic variation given by (5.18). For every $t \in [0, T]$, by Corollary 7.2, we obtain

$$\int_0^t \langle DF(X_s(\cdot)), \frac{X_{s+\epsilon}(\cdot) - X_s(\cdot)}{\epsilon}\rangle ds \xrightarrow[\epsilon\to 0]{\mathbb{P}} 2\int_0^t \langle X_s(\cdot), d^- X_s(\cdot)\rangle \tag{7.16}$$

The χ-quadratic variation (5.18) in the case $[X]_t = t$ and for a general diagonal measure $\mu(dx, dy) = g(x, y)\delta_y(dx)dy$ is given by

$$[X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} g(-y)(t-y)dy = \begin{cases} \int_0^t g(-y)(t-y)dy & 0 \le t \le \tau \\ \int_0^\tau g(-y)(t-y)dy & \tau < t \le T \end{cases}$$

We denote $D^2F(X_s(\cdot))$ by μ for every $s \in [0, T]$, where $\mu = 2\mathbb{1}_{[-\tau,0]}(x)\delta_y(dx)dy$. So the second order derivative term in Itô's formula became a trivial case of Lebesgue-Stieltjes integral:

$$\frac{1}{2}\int_0^t \langle D^2F(X_s(\cdot)), d\widetilde{[X(\cdot)]}_s\rangle = \frac{1}{2}\langle \mu, \widetilde{[X(\cdot)]}_t\rangle - \frac{1}{2}\langle \mu, \widetilde{[X(\cdot)]}_0\rangle = \frac{1}{2}[X(\cdot)]_t(\mu) = \int_0^{t\wedge\tau} (t-y)dy =$$
$$= \begin{cases} \frac{t^2}{2} & 0 \le t \le \tau \\ \tau(t - \frac{\tau}{2}) & \tau < t \le T \end{cases}$$

This concludes the proof. □

Remark 7.4. Even in the case when X is a classical Brownian motion W, the forward integral $\int_0^t \langle W_s(\cdot), d^- W_s\rangle$ is of anticipating type, see for example [53] for similar considerations.

Remark 7.5. 1) We set now $H = L^2([-\tau, 0])$. Expressing $G(X_t(\cdot))$ where $X(\cdot)$ is seen as a H-valued process and G is defined as in (7.10), we can obtain

$$2\int_0^t {}_{H^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_H = \|X_t(\cdot)\|^2_{L^2([-\tau,0])} - \int_0^{t\wedge\tau} (t-y)dy \tag{7.17}$$

This follows again by Corollary 7.2 and the fact that $X(\cdot)$ admits a $Diag_{\mathcal{B}}([-\tau,0]^2)$-quadratic variation given by (5.28), see Proposition 5.31.

2) Remark 3.5 implies that

$$\int_0^t {}_{B^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_B = \int_0^t {}_{H^*}\langle X_s(\cdot), d^- X_s(\cdot)\rangle_H$$

so that point 1) provides another proof of Proposition 7.3.

Remark 7.6. In the case $X = W$ a Brownian motion, formula (7.17) was established in Example 8.7 of [53]. Their techniques use Skorohod anticipating calculus and they only could be applied because X is a Gaussian. Our considerations did not make any assumption on the law of X.

Chapter 8

A generalized Clark-Ocone formula

8.1 Introduction

In this chapter we will consider $\tau = T$. Let $Y = (Y_t)_{t\in[0,T]}$ be a stochastic process such that $[Y]_t = t$. The main aim of this chapter consists in looking for classes of functionals $H : C([-T,0]) \longrightarrow \mathbb{R}$ for which there is $H_0 \in \mathbb{R}$, ξ an adapted process with respect to the canonical filtration of Y such that

$$H = H_0 + \int_0^T \xi_s d^- Y_s \tag{8.1}$$

Moreover we look for an explicit expression for H_0 and ξ.
We do not aim to find out the full generality under which (8.1) is possible. We start with the following toy model.

$$H = f(Y_T) \qquad \text{so that} \qquad H(\eta) = f(\eta(0)).$$

We recall, see [47, 3, 9], that whenever $u \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ with $u(T,x) = f(x)$, then we can choose

$$H_0 = u(0,Y_0) \qquad \xi_t = \partial_x u(t,Y_t).$$

We recall that in that case

$$\int_0^T H_s d^- Y_s$$

is the improper integral

$$\lim_{t\to T} \int_0^t H_s d^- Y_s$$

We will investigate the following cases

1. $H \in C^2(L^2([-T,0]))$ such that $D^2H(\eta) \in L^2([-T,0]^2)$ with polynomial growth such that $DH(\eta) \in H^1([-T,0])$ for every $\eta \in C([-T,0])$. This can be generalized to the case $H = C^2(L^2([-T,0]) \oplus \mathcal{D}_0([-T,0]))$ which includes the previous toy model. However we have chosen not to formulate this full generality.

2. $H(\eta) = \|\eta\|^2_{L^2([-T,0])}$.

3. $H(\eta) = f\left(\int_{-T}^0 \eta(s)ds\right)$ and $f : \mathbb{R} \to \mathbb{R}$ such that $f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$.

4. $H(\eta) = f\left(\int_{-T}^0 \varphi_1(u+T)d\eta(u), \dots, \int_{-T}^0 \varphi_n(u+T)d\eta(u)\right)$ where $\varphi_i : [0,T] \to \mathbb{R}$ in $C^2([0,T];\mathbb{R})$ and $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable and bounded function.

We need now to develop some technical preliminaries.

8.2 Some technical preliminaries

Let us consider a process $(Y_t)_{t\geq 0}$ such that $[Y]_t = t$. We make the usual convention of prolongation by continuity for $t \leq 0$. In this chapter we aim at representing

$$h = H(Y_T(\cdot)) \qquad \text{where} \qquad H : L^2([-\tau,0]) \longrightarrow \mathbb{R}$$

is of class C^2, in the form

$$H = H_0 + \int_0^T \xi_s d^- Y_s \tag{8.2}$$

Let now consider a standard Brownian motion W and its canonical filtration $(\mathcal{F}_t)$.

Notation 8.1. For $0 < s < t < T$, $\eta \in C([-T,0])$ we define the "flow"

$$X_t^{s,\eta}(x) = \begin{cases} \eta(x+t-s) & x \in [-T, s-t] \\ \eta(0) + W_t(x) - W_s & x \in [s-t, 0] \end{cases} \tag{8.3}$$

$(X_t^{s,\eta})_{0\leq s\leq t\leq T,\, \eta\in C([-T,0])}$ is a $C([-T,0])$-valued random field. We observe that $(X_t^{s,\eta})_{0\leq s\leq t\leq T,\, \eta\in C([-T,0])}$ is continuous with respect to the three variables.

Lemma 8.2. The following flow property holds, for $0 < s < t < r < T$,

$$X_r^{s,\eta} = X_r^{t,X_t^{s,\eta}} \tag{8.4}$$

Proof. We have, for $\tilde{\eta} \in C([-T,0])$,

$$X_r^{t,\tilde{\eta}}(x) = \begin{cases} \tilde{\eta}(x+r-t) & x \in [-T, t-r] \\ \tilde{\eta}(0) + W_r(x) - W_t & x \in [t-r, 0] \end{cases}$$

We substitute $\tilde{\eta} = X_s^{t,\eta}$ to get

$$X_r^{t,X_s^{t,\eta}}(x) = \left\{ \begin{array}{ll} \eta(x+r-s) & x \in [-T, s-r] \\ \eta(0) + W_t(x+r-t) - W_s & x \in [s-r, t-r] \\ \eta(0) + (W_t - W_s) + W_r(x) - W_t & x \in [t-r, 0] \end{array} \right\} = X_r^{s,\eta}(x)$$

This concludes the proof of the Lemma. □

Remark 8.3. We have

$$X_T^{t,\eta}(x) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T] \\ \eta(0) + \overline{W}_{T-t}(x) & x \in [t-T, 0] \end{cases}$$

where $\overline{W}$ is a standard Brownian motion.

Given $H : L^2([-T,0]) \longrightarrow \mathbb{R}$, we express

$$\mathbb{E}\left[H(W_T(\cdot))|\mathcal{F}_t\right] = u(t, W_t(\cdot)) \tag{8.5}$$

where $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$. Clearly Lemma 8.2 implies $W_T(\cdot) = X_T^{t,W_t(\cdot)}$, so

$$V_t = \mathbb{E}\left[H\big(X_T^{t,X_t^{0,0}}\big)|\mathcal{F}_t\right] = \mathbb{E}\left[H\big(X_T^{t,W_t(\cdot)}\big)|\mathcal{F}_t\right] = u(t, W_t(\cdot))$$

with

$$u(t,\eta) = \mathbb{E}\left[H\big(X_T^{t,\eta}\big)\right] \tag{8.6}$$

In the sequel η will always be a generic function in $C([-T,0])$.
That function u will play a crucial role in this chapter. In particular, given Y_t a real continuous process, we will evaluate an Itô's type expansion of $u(t, Y_t(\cdot))$.
By definition 8.1 it follows the following homogeneity property.

Remark 8.4. We have

$$u(t,\eta) = \mathbb{E}\left[H\big(X_{T-t}^{0,\eta}\big)\right] \tag{8.7}$$

We need an ulterior preliminary tool.

Lemma 8.5. Let $(t_n)_n$ a sequence in $[0,T]$ such that $t_n \to t_0$. Then

$$X_{T-t_n}^{0,\eta} \xrightarrow[n\to\infty]{a.s. \text{ in } C([-T,0])} X_{T-t_0}^{0,\eta} \tag{8.8}$$

Proof. For every ω, $\sup_{x\in[-T,0]}|X^{0,\eta}_{T-t_n} - X^{0,\eta}_{T-t_0}|$ equals

$$\begin{aligned}
&\sup_{x\in[-T,0]} |\,(\eta(x+T-t_n)-\eta(x+T-t_0))\,\mathbb{1}_{[-T,t-T]}(x) + (W_{T-t_n}(x)-W_{T-t_0}(x))\,\mathbb{1}_{[t-T,0]}(x)| \leq \\
&\leq \sup_{x\in[-T,0]} |\eta(x+T-t_n)-\eta(x+T-t_0)| + \sup_{x\in[-T,0]} |W_{T-t_n}(x)-W_{T-t_0}(x)| \leq \\
&\leq \varpi_\eta(\epsilon) + \varpi_{W(\omega)}(\epsilon) \xrightarrow[\epsilon\to 0]{} 0
\end{aligned}$$

where ϖ_η and $\varpi_{W(\omega)}$ are respectively the modulus of continuity of η and the Brownian motion for every fixed ω. Since η and $W(\omega)$ are uniformly continuous on the compact set $[0,T]$ both modulus of continuity converge to zero when $\epsilon \to 0$. In particular $X^{0,\eta}_{T-t_n} \xrightarrow[n\to\infty]{a.s.\text{ in } C([-T,0])} X^{0,\eta}_{T-t_0}$. $\square$

We continue stating two Fréchet regularity results about u defined in (8.6).

Theorem 8.6. Let u defined by (8.6) and $H \in C^2(L^2([-T,0]))$ such that the second order Fréchet derivative D^2H belong to $L^2([-T,0]^2)$ and has polynomial growth (for instance bounded).

1) Then $u \in C^{0,2}([0,T]\times C([-T,0]))$ and $Du(t,\eta)$ and $D^2(t,\eta)$ are given by (8.12) and (8.13).

2) If moreover $DH(\eta) \in H^1([-T,0])$, i.e. function $x \mapsto D_xH(\eta)$ is in $H^1([-T,0])$, every fixed η. Then $u \in C^{1,2}([0,T]\times C([-T,0]))$ and $\partial_t u(t,\eta)$ is given by (8.24).
Moreover u satisfies

$$\partial_t u(t,\eta) + \langle D^{ac}u(t,\eta), d\eta\rangle + \frac{1}{2}D^2_{0,0}u(t,\eta) = 0. \tag{8.9}$$

Proof of 1) of Theorem 8.6. • **Continuity of function u with respect to time t.**
We consider a sequence $(t_n)_n$ in $[0,T]$ such that $t_n \xrightarrow[n\to\infty]{} t_0$. By Assumption $H \in C^0(L^2([-T,0]))$ and therefore also $H \in C^0(C([-T,0]))$. Consequently, by Lemma 8.5, it follows

$$H\big(X^{0,\eta}_{T-t_n}\big) \xrightarrow[n\to\infty]{a.s.} H\big(X^{0,\eta}_{T-t_0}\big) \tag{8.10}$$

By a Taylor's expansion, given for instance by Theorem 5.6.1 in [6], the fact that D^2H has polynomial growth implies that H also has also polynomial growth. Therefore there is $p \geq 1$ such that

$$|H(\zeta)| \leq const\left(1 + \sup_{t\in[0,T]}|\zeta(t)|^p\right)$$

We observe that

$$\begin{aligned}
|H(X^{0,\eta}_{T-t})| &\leq const\left(1+\left\|X^{0,\eta}_{T-t}\right\|^p\right) \leq \\
&\leq const\left(1+\sup_{t\leq T}|\eta_t|^p + \sup_{t\leq T}|W_t|^p\right)
\end{aligned}$$

By Lebesgue dominated convergence theorem, the fact that $\sup_{t\leq T}|W_t|^p$ is integrable and (8.10), it follows that

$$u(t_n,\eta)=\mathbb{E}\left[H\big(X^{0,\eta}_{T-t_n}\big)\right]\xrightarrow[n\to\infty]{}\mathbb{E}\left[H\big(X^{0,\eta}_{T-t_0}\big)\right]=u(t_0,\eta) \tag{8.11}$$

The continuity is now established by Remark 8.4.

• **First Fréchet derivative**

We express now the derivatives of u with respect to derivatives of H. We start with $Du : [0,T]\times C([-T,0])\longrightarrow\mathcal{M}([-T,0])$. We have

$$D_{dx}u(t,\eta)=D^{\delta_0}_{dx}u(t,\eta)+D^{ac}_x u(t,\eta)dx \tag{8.12}$$

where

$$D^{\delta_0}_{dx}u(t,\eta)=\mathbb{E}\left[\int_{t-T}^0 D_sH\big(X^{t,\eta}_T\big)ds\right]\delta_0(dx)$$

$$D^{ac}_x u(t,\eta)dx=\mathbb{E}\left[D_{x-T+t}H\big(X^{t,\eta}_T\big)\right]\mathbb{1}_{[-t,0]}(x)dx=\begin{cases}0 & x\in[-T,-t]\\ \mathbb{E}\left[D_{x-T+t}H\big(X^{t,\eta}_T\big)\right] & x\in\,]-t,0]\end{cases}$$

• **Second Fréchet derivative**

We discuss the second derivative $D^2u : [0,T]\times C([-T,0])\longrightarrow(C([-T,0])\hat{\otimes}_\pi C([-T,0]))^*\cong\mathcal{B}(C([-T,0]),C([-T,0]))$. We obtain

$$\begin{aligned}D^2_{dx,dy}u(t,\eta)=&\,\mathbb{E}\Big[D_{y-T+t}D_{x-T+t}H\big(X^{t,\eta}_T\big)\Big]\mathbb{1}_{[-t,0]}(x)\,\mathbb{1}_{[-t,0]}(y)dx\,dy+\\&+\mathbb{E}\left[\int_{t-T}^0 D_sD_{x-T+t}H\big(X^{t,\eta}_T\big)ds\right]\mathbb{1}_{[-t,0]}(x)dx\,\delta_0(dy)+\\&+\mathbb{E}\left[\int_{t-T}^0 D_{y-T+t}D_sH\big(X^{t,\eta}_T\big)ds\right]\mathbb{1}_{[-t,0]}(y)dy\,\delta_0(dx)+\\&+\mathbb{E}\left[\int_{[t-T,0]^2} D_{s_1}D_{s_2}H\big(X^{t,\eta}_T\big)ds_1\,ds_2\right]\delta_0(dx)\,\delta_0(dy)\end{aligned} \tag{8.13}$$

Clearly $D^2u(t,\eta)\in(\mathcal{D}_0\oplus L^2([-T,0]))\hat{\otimes}^2_h$.

It is possible to show that all the term in first and second derivative are well defined using similar technique used in the first part of the proof. We omit this technicality for simplicity. □

Proof of 2) of Theorem 8.6. We will denote $D'H(\eta)$ the first Fréchet derivative in $L^2([-T,0])$ of H, every fixed η.

• **Derivability with respect to time t.**

In order to express $\partial_t u$, let $\epsilon>0$ and try to express the quantity

$$\frac{u(t+\epsilon,\eta)-u(t,\eta)}{\epsilon} \tag{8.14}$$

The flow property (8.4) gives $X_T^{t,\eta} = X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}$, then

$$u(t,\eta) = \mathbb{E}\left[H\big(X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\big)\right] \tag{8.15}$$

By derivability of H in $L^2([-T,0])$ we have

$$\begin{aligned} H\big(X_T^{t+\epsilon,\eta}\big) - H\big(X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\big) &= \langle DH\left(X_T^{t,\eta}\right), X_T^{t+\epsilon,\eta} - X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\rangle + \\ &+ \int_0^1 \langle DH\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - DH\left(X_T^{t,\eta}\right), X_T^{t+\epsilon,\eta} - X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\rangle d\alpha = \\ &= \int_{-T}^0 D_x H\left(X_T^{t,\eta}\right)\left(X_T^{t+\epsilon,\eta}(x) - X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}(x)\right) dx + R(\epsilon,t,\eta) \end{aligned} \tag{8.16}$$

where

$$R(\epsilon,t,\eta) = \int_0^1 \langle DH\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - DH\left(X_T^{t,\eta}\right), X_T^{t+\epsilon,\eta} - X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\rangle d\alpha$$

We need to evaluate

$$X_T^{t+\epsilon,\eta}(x) - X_T^{t+\epsilon,\gamma}(x) \qquad x \in [-T,0] \qquad\qquad \text{in view of} \qquad \gamma = X_{t+\epsilon}^{t,\eta} \tag{8.17}$$

(8.17) gives

$$X_T^{t+\epsilon,\eta}(x) - X_T^{t+\epsilon,\gamma}(x) = \begin{cases} \eta(x+T-t-\epsilon) - \gamma(x+T-t-\epsilon) & x \in [-T, t-T+\epsilon] \\ \eta(0) - \gamma(0) = -W_{t+\epsilon}(0) + W_t & x \in [t-T+\epsilon, 0] \end{cases} \tag{8.18}$$

where $\gamma(0) = X_{t+\epsilon}^{t,\eta}(0) = \eta(0) + W_{t+\epsilon}(0) - W_t$. Moreover we have, by (8.3),

$$\gamma(x+T-t-\epsilon) = X_{t+\epsilon}^{t,\eta}(x+T-t-\epsilon) = \begin{cases} \eta(x+T-t) & x \in [-T, t-T] \\ \eta(0) + W_T(x) - W_t & x \in [t-T, t-T+\epsilon] \end{cases}$$

Finally we obtain an expression for (8.17), (8.18) gives in fact

$$X_T^{t+\epsilon,\eta}(x) - X_T^{t+\epsilon,\gamma}(x) \begin{cases} \eta(x+T-t-\epsilon) - \eta(x+T-t) & x \in [-T, t-T] \\ \eta(x+T-t-\epsilon) - \eta(0) - W_T(x) + W_t & x \in [t-T, t-T+\epsilon] \\ W_t - W_{t+\epsilon} & x \in [t-T+\epsilon, 0] \end{cases} \tag{8.19}$$

Consequently, using (8.15), (8.16) and (8.19), (8.14) can be written as sum of the following terms

$$\frac{u(t+\epsilon,\eta) - u(t,\eta)}{\epsilon} = \frac{1}{\epsilon}\mathbb{E}\left[H\big(X_T^{t+\epsilon,\eta}\big) - H\big(X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\big)\right] = I_1(\epsilon,t,\eta) + I_2(\epsilon,t,\eta) + I_3(\epsilon,t,\eta) + \frac{1}{\epsilon}\mathbb{E}\left[R(\epsilon,t,\eta)\right] \tag{8.20}$$

where

$$
\begin{aligned}
I_1(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-T}^{t-T} D_xH\left(X_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(x+T-t)}{\epsilon}dx\right] = \\
&= -\mathbb{E}\left[\int_{-t}^{0} D_{x-T+t}H\left(X_T^{t,\eta}\right)\frac{\eta(x)-\eta(x-\epsilon)}{\epsilon}dx\right] \\
I_2(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{t-T+\epsilon} D_xH\left(X_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(0)-W_T(x)+W_t}{\epsilon}dx\right] + \\
&- \mathbb{E}\left[\int_{t-T}^{t-T-\epsilon} D_xH\left(X_T^{t,\eta}\right)\frac{W_t-W_{t+\epsilon}}{\epsilon}dx\right] \\
I_3(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{t-T}^{0} D_xH\left(X_T^{t,\eta}\right)\frac{W_t-W_{t+\epsilon}}{\epsilon}dx\right]
\end{aligned}
$$

and $\frac{1}{\epsilon}\mathbb{E}\left[R(\epsilon,t,\eta)\right]$ is equal to

$$
\frac{1}{\epsilon}\int_0^1 \mathbb{E}\left[\int_{-T}^0 \left(D_xH\left(\alpha X_T^{t+\epsilon,\eta}+(1-\alpha)X_T^{t+\epsilon,X_{t+\epsilon}^{t,\eta}}\right)-D_xH\left(X_T^{t,\eta}\right)\right)\left(X_T^{t+\epsilon,\eta}(x)-X_T^{t+\epsilon,X_{t+\epsilon}^{t,\eta}}(x)\right)dx\right]d\alpha \tag{8.21}
$$

• First we prove that $I_1(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_1(t,\eta) := I_{11}(t,\eta)+I_{12}(t,\eta)+I_{13}(t,\eta)$ where

$$
\begin{aligned}
I_{11}(t,\eta) &= \mathbb{E}\left[D_TH\left(X_T^{t,\eta}\right)\eta(-t)\right] \\
I_{12}(t,\eta) &= -\mathbb{E}\left[\int_{-t}^0 D'_{x-T+t}\left(X_T^{t,\eta}\right)\eta(x)dx\right] \\
I_{13}(t,\eta) &= -\mathbb{E}\left[D_{t-T}H\left(X_T^{t,\eta}\right)\eta(0)\right]
\end{aligned}
$$

In fact $I_1(\epsilon,t,\eta)$ can be rewritten as sum of three terms

$$
\begin{aligned}
I_{11}(\epsilon,t,\eta) &= \mathbb{E}\left[\int_{-t}^{-t+\epsilon} D_{x-T+t}H\left(X_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right] \\
I_{12}(\epsilon,t,\eta) &= -\mathbb{E}\left[\int_{-t}^0 \frac{D_{x+\epsilon-T+t}H\left(X_T^{t,\eta}\right)-D_{x-T+t}H\left(X_T^{t,\eta}\right)}{\epsilon}\eta(x)dx\right] \\
I_{13}(\epsilon,t,\eta) &= -\mathbb{E}\left[\int_0^{\epsilon} D_{x-T+t}H\left(X_T^{t,\eta}\right)\frac{\eta(x-\epsilon)}{\epsilon}dx\right]
\end{aligned}
$$

By hypothesis function $x \mapsto D_xH(\eta)$ in H^1, for every fixed η and where we have denoted its derivative by $D'H(\eta)$. Then in particular $x \mapsto D_xH(\eta)$ is a continuous function.By application of finite increments theorem and dominated convergence theorem the following limit holds $I_{1i}(\epsilon,t,\eta) \xrightarrow[\epsilon\to 0]{} I_{1i}(t,\eta)$ for $i=1,2,3$.

• Secondly we prove that $I_2(\epsilon,t,\eta) \underset{\epsilon\to 0}{\longrightarrow} 0$. In fact, by using mean theorem, there exists $\bar{x}=\bar{x}(\epsilon)\in [t-T,t-T+\epsilon]$ such that

$$\begin{aligned} I_2(\epsilon,t,\eta) &= \int_{t-T}^{t-T+\epsilon} \mathbb{E}\left[D_x H\left(X_T^{t,\eta}\right)\frac{\eta(x+T-t-\epsilon)-\eta(0)-W_T(x)+W_{t+\epsilon}}{\epsilon}\right]dx \\ &= \mathbb{E}\left[D_{\bar{x}} H\left(X_T^{t,\eta}\right)\left(\eta(\bar{x}+T-t-\epsilon)-\eta(0)-W_T(\bar{x})+W_{t+\epsilon}\right)\right] \end{aligned}$$

We have that when ϵ goes to zero then $\bar{x}$ goes to $(t-T)$ and by continuity of flow (8.3), in particular in $x=(t-T)$, it follows that

$$D_{\bar{x}}H\left(X_T^{t,\eta}\right)\eta(\bar{x}+T-t-\epsilon)-\eta(0)-W_T(\bar{x})+W_{t+\epsilon} \underset{\epsilon\to 0}{\longrightarrow} D_{t-T}H\left(X_T^{t,\eta}\right)\left(\eta(0)-\eta(0)-W_T(t-T)+W_t\right)=0 \ a.s.$$

Again by using dominated convergence theorem we conclude that $I_2(\epsilon,t,\eta)$ converges to zero.

• Finally we prove that

$$I_3(\epsilon,t,\eta) \underset{\epsilon\to 0}{\longrightarrow} -\mathbb{E}\left[\int_{[t-T,0]^2} D_{s_1}D_{s_2}H\big(X_T^{t,\eta}\big)ds_1\, ds_2\right] =: I_3(t,\eta)$$

Using Skorohod integral term $I_3(\epsilon,t,\eta)$ can be rewritten as follows

$$-\frac{1}{\epsilon}\int_{t-T}^{0}\mathbb{E}\left[D_x H\left(X_T^{t,\eta}\right)\int_t^{t+\epsilon}\delta W_s\right]dx \tag{8.22}$$

We observe that $D_xH\left(X_T^{t,\eta}\right)\in\mathbb{D}^{1,2}$ then using integrations by parts in Malliavin's calculus, following equality holds

$$\mathbb{E}\left[D_x H\left(X_T^{t,\eta}\right)\int_t^{t+\epsilon}\delta W_s\right] = \mathbb{E}\left[\int_t^{t+\epsilon} D_r^m\left[D_x H\left(X_T^{t,\eta}\right)\right]dr\right]$$

We calculate, using results of [35], chapter 1, pag. 32.

$$D_r^m\left[D_x H\left(X_T^{t,\eta}\right)\right] = \int_{r-T}^{0} D_s D_x H\left(X_T^{t,\eta}\right)ds \tag{8.23}$$

Using Fubini's Theorem and then integrating with respect to variable r we obtain that (8.22) equals

$$\begin{aligned} I_3(\epsilon,t,\eta) &= -\frac{1}{\epsilon}\int_{t-T}^{0}\mathbb{E}\left[\int_t^{t+\epsilon}\int_{r-T}^{0} D_s D_x H\left(X_T^{t,\eta}\right)ds\, dr\right]dx = \\ &= -\frac{1}{\epsilon}\int_{t-T}^{0}\mathbb{E}\left[\int_{t-T}^{0}\int_t^{t+\epsilon} D_s D_x H\left(X_T^{t,\eta}\right)dr\, ds\right]dx = \\ &= -\int_{t-T}^{0}\mathbb{E}\left[\int_{t-T}^{0} D_s D_x H\left(X_T^{t,\eta}\right)ds\right]dx \end{aligned}$$

Then the result follows, in fact $I_3(\epsilon,t,\eta)=I_3(t,\eta)$.

• We study now the term $1/\epsilon\mathbb{E}\left[R(\epsilon,t,\eta)\right]$. By using (8.19) and the fact that $H\in C^2(L^2([-T,0]))$, (8.21) can be rewritten as the sum of the following terms

$$A_1(\epsilon,t,\eta) = \int_0^1 \mathbb{E}\left[\int_{-T}^{t-T} \left(D_x H\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - D_x H\left(X_T^{t,\eta}\right)\right) \times \right.$$
$$\left. \times \frac{\eta(x+T-t-\epsilon) - \eta(x+T-t)}{\epsilon} dx\right] d\alpha$$

$$A_2(\epsilon,t,\eta) = \int_0^1 \mathbb{E}\left[\int_{t-T}^{t-T-\epsilon} \left(D_x H\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - D_x H\left(X_T^{t,\eta}\right)\right) \times \right.$$
$$\left. \times \frac{\eta(x+T-t-\epsilon) - \eta(0) - W_T(x) + W_{t+\epsilon}}{\epsilon} dx\right] d\alpha$$

$$A_3(\epsilon,t,\eta) = A_{31}(\epsilon,t,\eta) + A_{32}(\epsilon,t,\eta)$$

where

$$A_{31}(\epsilon,t,\eta) = \frac{1}{2}\mathbb{E}\left[\langle D^2 H\left(X_T^{t,\eta}\right), \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\rangle|_{[t-T,0]^2}\right] =$$
$$= \frac{1}{2}\mathbb{E}\left[\int_{[t-T,0]^2} D_x D_y H\left(X_T^{t,\eta}\right) \frac{(W_t - W_{t+\epsilon})^2}{\epsilon} dx\, dy\right]$$

$$A_{32}(\epsilon,t,\eta) = \int_0^1 \mathbb{E}\left[\langle\left(D^2 H\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - D^2 H\left(X_T^{t,\eta}\right)\right), \frac{(W_t - W_{t+\epsilon})^2}{\epsilon}\rangle|_{[t-T,0]^2}\right] =$$
$$= \int_0^1 \mathbb{E}\left[\int_{[t-T,0]^2} \left(D_x D_y H\left(\alpha X_T^{t+\epsilon,\eta} + (1-\alpha) X_T^{t+\epsilon, X_{t+\epsilon}^{t,\eta}}\right) - D_x D_y H\left(X_T^{t,\eta}\right)\right) \times \right.$$
$$\left. \times \frac{(W_t - W_{t+\epsilon})^2}{\epsilon} dx\, dy\right] d\alpha$$

With the usual technique we obtain that $A_1(\epsilon,t,\eta)$, $A_2(\epsilon,t,\eta)$ and $A_{32}(\epsilon,t,\eta)$ go to zero as ϵ goes to zero. On the contrary we observe that $X_t^{t,\eta}|_{[t-T,0]}$ is independent from $(W_{t+\epsilon} - W_t)$, as well as its second derivative. By the factorization of the expectation we obtain then

$$A_{31}(\epsilon,t,\eta) = \frac{1}{2}\int_{[t-T,0]^2} \mathbb{E}\left[D_x D_y H\left(X_T^{t,\eta}\right)\right] dx\, dy =: A_{31}(t,\eta)$$

• Concerning the derivative with respect to t, we obtain that $\partial_t u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$ and we get $\partial_t u(t,\eta) = I_1(t,\eta) + I_3(t,\eta) + A_{31}(t,\eta)$ which gives

$$\partial_t u(t,\eta) = \mathbb{E}\left[D_T H\left(X_T^{t,\eta}\right)\eta(-t)\right] - \mathbb{E}\left[\int_{-t}^0 D'_{x-T+t}\left(X_T^{t,\eta}\right)\eta(x) dx\right] - \mathbb{E}\left[D_{t-T} H\left(X_T^{t,\eta}\right)\eta(0)\right] +$$
$$- \frac{1}{2}\int_{[t-T,0]^2} \mathbb{E}\left[D_x D_y H\left(X_T^{t,\eta}\right)\right] dx\, dy \tag{8.24}$$

□

Remark 8.7. It remains clear that dependence with respect to time t is a *difficult* problem. In general we will make this assumption, but in most of the examples that we will see u will be $C^{1,2}$ even with weaker assumption on functional H.

Corollary 8.8. Let u defined by (8.6) and Y a real continuous process with $[Y]_t = t$. If $H \in C^2(L^2([-T,0]))$ and D^2H with polynomial growth and such that D^2H belongs to $L^2([-T,0]^2)$, then $u \in C^{1,2}([0,T] \times C([-T,0]))$ and following representation formula holds

$$u(T, Y_T(\cdot)) = H_0 + \int_0^T \xi_i d^- W_t \tag{8.25}$$

with $H_0 = u(0, Y_0(\cdot))$ and $\xi_t = D^{\delta_0} u(t, Y_t(\cdot))$.

Proof. We recall that $D^2 u(t,\eta)$ belong to $(\mathcal{D}_0 \oplus L^2([-T,0]))\otimes_h^2$ continuously, then we apply Itô's formula (7.2) to equation u in $(t, Y_t(\cdot))$. We obtain, with the previous notations,

$$\begin{aligned}
u(T, Y_T(\cdot)) &= u(0, Y_0(\cdot)) + \int_0^T I_1(t, Y_t(\cdot))dt + \int_0^T I_{31}(t, Y_t(\cdot))dt + \int_0^T A_{31}(t, Y_t(\cdot)) \\
&+ \int_0^T D^{\delta_0} u(t, Y_t(\cdot)) d^- Y_t + \int_0^T \langle D^{ac} u(t, Y_t(\cdot)), d^- Y_t(\cdot)\rangle + \\
&+ \frac{1}{2}\int_0^T \langle D^2 u(t, Y_t(\cdot)), d\widetilde{[Y(\cdot)]}_t\rangle = \\
&= u(0, Y_0(\cdot)) + \int_0^T D^{\delta_0} u(t, Y_t(\cdot)) d^- Y_t
\end{aligned}$$

In fact we recall by (8.24) that

$$\int_0^T I_1(t, Y_t(\cdot))dt = -\int_0^T \langle D^{ac} u(t, Y_t(\cdot)), d^- Y_t(\cdot)\rangle$$

and that

$$\int_0^T I_{31}(t, Y_t(\cdot))dt + \int_0^T A_{31}(t, Y_t(\cdot)) = -\frac{1}{2}\int_0^T \langle D^2 u(t, Y_t(\cdot)), d\widetilde{[Y(\cdot)]}_t$$

□

A more signicant achievement concerns the comparison with Clark-Ocone formula. It is illustrated in the following Lemma.

Lemma 8.9. Let u defined by (8.6) fulfilling assumption of Theorem 8.6 and Y aqual to the Brownian motion W. Then

$$\int_0^T \mathbb{E}\left[D_t^m H(W_T(\cdot))|\mathcal{F}_t\right] dW_t = \int_0^T \langle D^{\delta_0} u(t, W_t(\cdot)), d^- W_t(\cdot)\rangle \tag{8.26}$$

Proof. We try to compare with the classical Clark-Ocone formula. We calculate, using results of [35], chapter 1, pag. 32.

$$D_t^m H(W_T(\cdot)) = \int_{t-T}^{0} D_s H(W_T(\cdot))ds$$

Taking the expectation with respect to $(\mathcal{F}_t)$ we obtain

$$\mathbb{E}\left[D_t^m H(W_T(\cdot))|\mathcal{F}_t\right] = \mathbb{E}\left[\int_{t-T}^{0} D_s H\big(X_T^{t,W_t(\cdot)}\big)|\mathcal{F}_t\right] = \Gamma(W_t(\cdot))$$

where

$$\Gamma(\eta) = \mathbb{E}\left[\int_{t-T}^{0} D_s H\big(X_T^{t,\eta}\big)\right]$$

Now we observe that

$$\int_0^T \mathbb{E}\left[D_t^m H(W_T(\cdot))|\mathcal{F}_t\right] dW_t = \int_0^T \Gamma(W_t(\cdot))dW_t = \int_0^T \langle D^{\delta_0} u(t, W_t(\cdot)), d^- W_t(\cdot)\rangle$$

□

Remark 8.10. Using previous Lemma 8.9 with (8.25) we have a generalization of Clark-Ocone formula. We will see in the sequel that a Clark-Ocone formula it is found also in weaker assumption, as in application 8.5.

Let us consider an $(\mathcal{G}_t)$-martingale M square integrable and $h = H(M_T(\cdot))$ with $H : L^2([-T, 0]) \longrightarrow \mathbb{R}$ with linear growth. We are interested in sufficient condition so that

$$h = \mathbb{E}[h] + \int_0^T \xi_s dM_s \tag{8.27}$$

where (ξ_s) is explicit. We start with a Corollary for Theorem 6.28 that give us sufficient conditions for such representation.

Corollary 8.11. Suppose there is $u : [0, T] \times C([-T, 0]) \longrightarrow \mathbb{R}$ with property of Theorem 6.28 statement such that $\mathbb{E}\left[h|\mathcal{G}_t\right] = u(t, M_t(\cdot))$. Then

$$h = \mathbb{E}[h] + \int_0^T D^{\delta_0} u(s, M_s(\cdot))dM_s$$

where as usual $D^{\delta_0} u(s, M_s(\cdot))$ denotes $Du(s, M_s(\cdot))(\{0\})$.

Proof. According to Theorem 6.28, $u(\cdot, D_\cdot(\cdot))$ is a $(\mathcal{G}_t)$-weak Dirichlet process with martingale part given in (6.22). Since $u(\cdot, M_\cdot(\cdot))$ is obviously a $(\mathcal{G}_t)$-martingale being a conditional expectation with respect to filtration $(\mathcal{G}_t)$, then uniqueness of the decomposition of weak Dirichlet processes allows to conclude. In particular the $(\mathcal{F}_t)$-martingale orthogonal process is zero. It holds

$$H = u(0, M_0(\cdot)) + \int_0^T D^{\delta_0} u(s, M_s(\cdot))dM_s$$

□

8.3 Some particular representations

In this chapter we will develop explicitly some calculus with Itô's formula in some path dependent options. In particular we will retrieve the terms appearing in the Clark-Ocone's formula. As first example we will consider $S = W$ the Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$.

8.3.1 First example

We consider the contingent claim defined with the function $H : L^2([-T,0]) \longrightarrow \mathbb{R}$ by $H(\eta) = \|\eta\|^2_{L^2}$, i.e.

$$H(W_T(\cdot)) = \int_{-T}^{0} W_T(s)^2 ds = \int_0^T W_s^2 ds$$

$H(W_T(\cdot))$ is $\mathcal{F}_T-$measurable and is in $\mathbb{D}^{1,2}$, then, by Clark-Ocone's formula (2.28), we have

$$H = \mathbb{E}\left[H\right] + \int_0^T \mathbb{E}\left[D_t^m H|\mathcal{F}_t\right] dW_t \tag{8.28}$$

where the Malliavin's derivative $D_t^m H$ can be easily calculated as follows.

$$D_t^m H = D_t^m \left(\int_0^T W_s^2 ds\right) = \int_t^T D_t^m (W_s^2) ds = \int_t^T 2W_s D_t^m (W_s) ds = \int_t^T 2W_s ds$$

Consequently, using usual properties of the conditional expectation,

$$\mathbb{E}\left[D_t^m H|\mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T 2W_s ds|\mathcal{F}_t\right] = 2\int_t^T \mathbb{E}\left[W_s|\mathcal{F}_t\right] ds = 2W_t(T-t)$$

Then (8.28) gives

$$H = \mathbb{E}\left[H\right] + 2\int_0^T W_t(T-t) dW_t \tag{8.29}$$

We will retrieve this representation using Itô's formula (7.2). For all $t \in [0,T]$ we define the real stochastic process V by

$$V_t = \mathbb{E}\left[H|\mathcal{F}_t\right] \tag{8.30}$$

So (8.30) gives

$$V_t = \int_0^t W_s^2 ds + W_t^2(T-t) + \frac{(T-t)^2}{2} = \int_{-t}^0 W_t^2(u) du + W_t^2(0)(T-t) + \frac{(T-t)^2}{2} = \phi(t, W_t(\cdot))$$

with $\phi : [0,T] \times C([-T,0]) \to \mathbb{R}$

$$\phi(t,\eta) = \int_{-T}^0 \eta^2(s) ds + \eta(0)^2(T-t) + \frac{(T-t)^2}{2} \tag{8.31}$$

In particular $H = V_T = \phi(T, W_T(\cdot))$ and trivially $\mathbb{E}[H] = \mathbb{E}\left[\int_0^T W_s^2 ds\right] = \frac{T^2}{2}$.
We want to apply Itô's formula (7.2) to the function $\phi \in C^{1,2}$ applied to $W_t(\cdot)$. First of all we evaluate the different derivatives of ϕ.

$$\begin{aligned}
&\partial_t \phi(t,\eta) = -\eta^2(0) - (T-t) \\
&D_{dx}\phi(t,\eta) = 2\eta(x)dx + 2\eta(0)(T-t)\delta_0(dx) \\
&D^2_{dx\,dy}\phi(t,\eta) = 2\delta_y(dx)\,dy + 2(T-t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy)\,dx + 2(T-t)\delta_0(dx)\delta_0(dy)
\end{aligned}$$

It happens that $D^2\phi(t,\eta)$ belongs to the subspace $Diag \oplus \mathcal{D}_{0,0}$ of $\mathcal{M}([-T,0]^2)$ and $(t,\eta) \to D^2\phi(t,\eta)$ is continuous from $[0,T] \times C([-T,0]$ into $Diag \oplus \mathcal{D}_{0,0}$ We recall that the window Brownian motion admits a $Diag \oplus \mathcal{D}_{0,0}$-quadratic variation given by

$$\begin{aligned}
&[W.(\cdot)]_t : Diag \oplus \mathcal{D}_{00} \longrightarrow \mathcal{C}([0,T]) \\
&\qquad\qquad \mu_1 + \mu_2 \longrightarrow \int_{-t}^0 g(y)(t+y)dy + \alpha[W]_t = \int_{-t}^0 g(y)(t+y)dy + \alpha t
\end{aligned} \tag{8.32}$$

where $\mu_1(dx,dy) = g(y)\delta_y(dx)\,dy$, with $g \in L^\infty([-T,0])$ is a general diagonal measure and $\mu_2(dx,dy) = \alpha\delta_0(dx)\delta_0(dy)$, $\alpha \in \mathbb{R}$, is a general Dirac's measure on $\{0,0\}$ and $d\widetilde{[W(\cdot)]}(t)(\mu_1+\mu_2) = d_t\left(\int_{-t}^0 g(y)(t+y)dy\right)+$ $\alpha dt = \int_{-t}^0 g(y)dydt + \alpha dt$ or $[W(\cdot)](\mu_1+\mu_2)(t) = \int_0^t \left(\int_{-s}^0 g(y)dy\right) ds + \alpha\, t$. We apply Itô's formula for ϕ:

$$\begin{aligned}
\phi(T, W_T(\cdot)) &= \phi(0, W_0(\cdot)) + \int_0^T \partial_t\phi(t, W_t(\cdot))dt + \int_0^T \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot)\rangle + \\
&+ \frac{1}{2}\int_0^T \langle D^2\phi(t, W_t(\cdot)), d\widetilde{[W(\cdot)]}_t\rangle = \frac{T^2}{2} + I_1 + I_2 + I_3 = \\
&= \mathbb{E}[H] + I_1 + I_2 + I_3
\end{aligned} \tag{8.33}$$

For the other terms we have

$$\begin{aligned}
I_1 &= \int_0^T (t - T - W_t^2)dt = \int_0^T (t - W_t^2)\,dt - T^2 \\
I_2 &= \int_0^T \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot)\rangle = \lim_{\epsilon\to 0}\int_0^T \langle D\phi(t, W_t(\cdot)), \frac{W_{t+\epsilon}(\cdot) - W_t(\cdot)}{\epsilon}\rangle = \lim_{\epsilon\to 0}(I_{21}(\epsilon) + I_{22}(\epsilon)) \\
I_{21}(\epsilon) &= 2\int_0^T \int_{-t}^0 W_t(r)\frac{W_{t+\epsilon}(r) - W_t(r)}{\epsilon}dr\,dt = \\
&= 2\int_0^T \int_0^t W_u \frac{W_{u+\epsilon} - W_u}{\epsilon} du\,dt \xrightarrow{\mathbb{P}} 2\int_0^T \int_0^t W_u dW_u dt = \int_0^T (W_t^2 - t)\,dt
\end{aligned}$$

and

$$I_{22}(\epsilon) = 2\int_0^T W_t(0)(T-t)\frac{W_{t+\epsilon}(0) - W_t(0)}{\epsilon}dt \xrightarrow{\mathbb{P}} 2\int_0^T W_t(T-t)dW_t$$

Previous convergence of $I_{22}(\epsilon)$ holds because Itô's integral coincides with forward integral, see Remark 2.4 2. This limiting term is the same as in (8.29). Coming back to (8.32) and (8.33) it follows

$$I_3 = \int_0^T d_t \left(\int_{-t}^0 (t+y)dy \right) + \int_0^T (T-t)dt = \int_0^T tdt + \int_0^T (T-t)dt = T^2$$

so $d_t \left(\int_{-t}^0 (t+y)dy \right) = d_t(\frac{t^2}{2}) = tdt$. Finally (8.33) gives

$$H = \phi(T, W_T(\cdot)) = \mathbb{E}[H] - T^2 + 2\int_0^T W_t(T-t)dW_t + T^2 = \mathbb{E}[H] + 2\int_0^T W_t(T-t)dW_t \qquad (8.34)$$

that is exactly (8.29).

8.3.2 Second example

We consider the contingent claim defined with the function $H : C([-T,0]) \longrightarrow \mathbb{R}$

$$H(\eta) = \left(\int_{-T}^0 \eta(s)ds \right)^2. \qquad (8.35)$$

$H(W_T(\cdot))$ is $\mathcal{F}_T$-measurable and it belongs $\mathbb{D}^{1,2}$. We compute first the Malliavin's derivative of H denoted by $D_t^m H$; it gives

$$D_t^m H = D_t^m \left(\int_0^T W_s ds \right)^2 = 2\left(\int_0^T W_s ds \right) D_t^m \left(\int_0^T W_s ds \right) = 2(T-t)\int_0^T W_s ds$$

Consequently, using usual properties of the conditional expectation,

$$\mathbb{E}\left[D_t^m H | \mathcal{F}_t\right] = \mathbb{E}\left[2(T-t)\int_0^T W_s ds | \mathcal{F}_t \right] = 2(T-t)\int_0^t W_s ds + 2(T-t)^2 W_t$$

Clark-Ocone formula (8.28) gives

$$H = \mathbb{E}\left[H\right] + 2\int_0^T (T-t)\left(\int_0^t W_s\, ds \right) dW_t + 2\int_0^T (T-t)^2 W_t dW_t \qquad (8.36)$$

and

$$\mathbb{E}\left[H\right] = \mathbb{E}\left[\left(\int_0^T W_s ds \right)^2 \right] = \mathbb{E}\left[\left(TW_T - \int_0^T s dW_s \right)^2 \right] = T^3 + \int_0^T s^2 ds - 2\frac{T^3}{2} = \frac{T^3}{3}$$

because it is a difference of mean-zero Gaussian random variables with covariance $\int_0^T sTds = T^3/2$. We will retrieve this representation using Itô's formula (7.2). For all $t \in [0,T]$ we define the martingale V via the conditional expectation of r.v. H:

$$V_t = \mathbb{E}\left[H|\mathcal{F}_t\right] = \left(\int_{-t}^0 W_t(s)ds + W_t(0)(T-t) \right)^2 + \frac{(T-t)^3}{3} = \phi(t, W_t(\cdot))$$

with $\phi : [0,T] \times C([-T,0]) \to \mathbb{R}$

$$\phi(t,\eta) = \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right)^2 + \frac{(T-t)^3}{3} \tag{8.37}$$

In particular $H = V_T = \phi(T, W_T(\cdot))$ and trivially $\phi(0, W_0(\cdot)) = T^3/3 = \mathbb{E}[H]$.
Since we will apply Itô's formula to the function $\phi \in C^{1,2}$ applied to the window Brownian motion, we need to evaluate the corresponding derivatives.

$$\begin{aligned}
\partial_t \phi(t,\eta) &= -2\eta(0) \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right) - (T-t)^2 \\
D_{dx}\phi(t,\eta) &= 2 \left(\int_{-T}^{0} \eta(s)ds + \eta(0)(T-t) \right) \left(\mathbb{1}_{[-T,0]}(x)dx + (T-t)\delta_0(dx) \right) \\
D^2_{dx\,dy}\phi(t,\eta) &= 2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy + \\
&\quad + 2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy) + \\
&\quad + 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy + \\
&\quad + 2(T-t)^2\delta_0(dx)\,\delta_0(dy)
\end{aligned}$$

We observe that for any (t,η), $D^2\phi(t,\eta)$ belongs to $(L^2([-T,0]) \oplus \mathcal{D}_0)\hat{\otimes}_h^2$ and $D^2\phi : [0,T] \times C([-T,0]) \to (L^2([-T,0]) \oplus \mathcal{D}_0)\hat{\otimes}_h^2$ is continuous. We recall that the window Brownian motion admits a $(L^2([-T,0]) \oplus \mathcal{D}_0)\hat{\otimes}_h^2$-quadratic variation given by Corollary 5.11. Itô's formula gives

$$\begin{aligned}
\phi(T, W_T(\cdot)) &= \phi(0, W_0(\cdot)) + \int_0^T \partial_t \phi(t, W_t(\cdot))dt + \int_0^T \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot) \rangle + \\
&\quad + \frac{1}{2} \int_0^T \langle D^2\phi(t, W_t(\cdot)), d\widetilde{[W(\cdot)]}_t \rangle = \\
&= \frac{T^3}{3} + I_1 + I_2 + I_3 = \mathbb{E}[H] + I_1 + I_2 + I_3
\end{aligned} \tag{8.38}$$

Moreover we have

$$\begin{aligned}
I_1 &= -2 \int_0^T W_t \int_{-T}^{0} W_t(s)ds\,dt - 2\int_0^T W_t^2 (T-t)dt - \int_0^T (T-t)^2 dt \\
&= -2 \int_0^T W_t \int_0^t W_u du\,dt - 2\int_0^T W_t^2 (T-t)dt - \int_0^T (T-t)^2 dt \\
I_2 &= \int_0^T \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot) \rangle = \lim_{\epsilon \to 0} \int_0^T \langle D\phi(t, W_t(\cdot)), \frac{W_{t+\epsilon}(\cdot) - W_t(\cdot)}{\epsilon} \rangle = \\
&= \lim_{\epsilon \to 0} (I_{21}(\epsilon) + I_{22}(\epsilon))
\end{aligned}$$

where

$$I_{21}(\epsilon) = 2\int_0^T \left(\int_0^t W_s\,ds\right)\left(\int_0^t \frac{W_{s+\epsilon}-W_s}{\epsilon}ds\right)dt + 2\int_0^T (T-t)W_t\left(\int_0^t \frac{W_{s+\epsilon}-W_s}{\epsilon}ds\right)dt$$
$$I_{22}(\epsilon) = 2\int_0^T (T-t)\left(\int_0^t W_u du\right)\frac{W_{t+\epsilon}-W_t}{\epsilon}dt + 2\int_0^T (T-t)^2 W_t \frac{W_{t+\epsilon}-W_t}{\epsilon}dt$$

We observe that for any (t,η) in $[0,T]\times C([-T,0])$ the first Fréchet derivative $D\phi(t,\eta)$ belongs to $L^2([-T,0])\oplus\mathcal{D}_0$. With our notation in $I_{21}(\epsilon)$ appears the $L^2([-T,0])$ contribution and in $I_{22}(\epsilon)$ appears the $\mathcal{D}_0$ contribution of $D\phi(t,W_t(\cdot))$. Since $\int_0^t \frac{W_{s+\epsilon}-W_s}{\epsilon}ds \to W_t$ a.s. when $\epsilon\to 0$ and by Lebesgue dominated convergence theorem we have

$$I_{21}(\epsilon) \xrightarrow{\mathbb{P}} 2\int_0^T W_t \int_0^t W_s\,ds\,dt + 2\int_0^T W_t^2(T-t)dt =: I_{21}$$

and again because Remark 2.4 2 we obtain convergence in probability of $I_{22}(\epsilon)$

$$I_{22}(\epsilon) \xrightarrow{\mathbb{P}} 2\int_0^T (T-t)\int_0^t W_s\,ds\,dW_t + 2\int_0^T (T-t)^2 W_t dW_t =: I_{22}$$

Expression I_{22} coincides with the stochastic integral appearing in Clark-Ocone's formula and we observe that it is given by the Dirac contribution of the first derivative. Recalling the χ^2-quadratic variation for the window Brownian motion in Corollary 5.11 we obtain

$$I_3 = \frac{1}{2}\int_0^T 2(T-t)^2 dt = \int_0^T (T-t)^2 dt$$

Finally (8.38) gives

$$H = \phi(T, W_T(\cdot)) = \mathbb{E}[H] + 2\int_0^T (T-t)\int_0^t W_s\,ds\,dW_t + 2\int_0^T (T-t)^2 W_t dW_t$$

which corresponds exactly to (8.36).

Before the next section we introduce a particular function that will be useful.

Notation 8.12. Let the function $p:[0,T]\times\mathbb{R}\to\mathbb{R}$ be the density function of a gaussian random variable with expected value 0 and variance, which depends on $t\in[0,T]$, denoted by σ_t^2. Equivalently the standard deviation, i.e. the square root of variance, will be denoted by σ_t. The function $t\to\sigma(t):=\sigma_t$ from $[0,T]$ to $\mathbb{R}^+$ has to be differentiable and its derivative will be denoted by σ_t'.

$$p(t,x) = \frac{1}{\sigma_t\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma_t^2}}$$

When the variance has null value, $\sigma_T = 0$, we have to consider the case of a random variable with mean zero and null variance, i.e. a dirac function concentrated in 0. To consider the limit case $p(T,dx)=\delta_0(dx)$

we introduce, with a little abuse of notation, the function $p : [0,T] \times \mathcal{B}(\mathbb{R}) \longrightarrow \mathcal{M}(\mathbb{R})$, which is a measure on $\mathbb{R}$ for every $t \in [0,T]$, defined as follow

$$p(t,dx) = \begin{cases} = p(t,x)dx = \frac{1}{\sigma_t\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_t^2}} dx & \text{if } t \in [0,T] \\ = \delta_0(dx) & \text{if } t = T \end{cases} \tag{8.39}$$

For the real function p it holds

$$\partial_t p(t,x) = \sigma_t \sigma_t' \partial_{xx} p(t,x) \tag{8.40}$$

As particular case we have

1. $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$. For instance σ_t is the standard deviation of the random variable $\int_t^T (W_r - W_t)dr$. It holds in particular

$$\partial_t p(t,x) = \left[-\frac{(T-t)^2}{2}\right] \partial_{xx} p(t,x) \tag{8.41}$$

   We remark that $\sigma_T = 0$.

2. $\sigma_t = \sqrt{\int_t^T \varphi^2(s)ds}$, whith ϕ as in (8.54) then we have

$$\partial_t p(t,x) = \left[-\frac{1}{2}\varphi_t^2\right] \partial_{xx} p(t,x) \tag{8.42}$$

   Also in this example we remark that $\sigma_T = 0$.

8.4 Toy model with $H(W_T(\cdot)) = f\left(\int_{-T}^{0} W_T(s)ds\right)$

We consider a general case where the value of the option at time T is given by $H(W_T(\cdot))$, H is a function $H : C([-T,0]) \longrightarrow \mathbb{R}$ defined by

$$H(\eta) = f\left(\int_{-T}^{0} \eta(s)ds\right) \tag{8.43}$$

and $f : \mathbb{R} \to \mathbb{R}$ is a function such that

$$f\left(\int_0^T W_s ds\right) \in L^1(\Omega) \qquad \text{i.e.} \qquad \mathbb{E}\left[\left|f\left(\int_0^T W_s ds\right)\right|\right] = \int_{\mathbb{R}} |f(y)|\, p(0,y)dy < +\infty \tag{8.44}$$

This hypothesis allows to compute the conditional expectation of H.

Theorem 8.13. Let H and f be defined such that (8.43) and (8.44) hold. Then we have

$$H = H_0 + \int_0^T A_t d^- W_t$$

where $H_0 = \mathbb{E}[H]$ and $(A_t)_{t\in[0,u]}$ is the process defined by

$$A_t := (T-t)\int_{\mathbb{R}} f\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t) + x\right)\partial_x p(t,x)dx \tag{8.45}$$

Proof. The value of the option at time T is

$$H(W_T(\cdot)) = f\left(\int_{-T}^{0} W_T(s)ds\right)$$

The option that we have considered in (8.35) can be see in this class of options in the special case $f(x) = x^2$. Let $(\mathcal{F}_t)$ be the Brownian filtration, we define, for every $t \in [0,T]$, the real stochastic process V as the conditional expectation of the option H with respect to $(\mathcal{F}_t)$:

$$V_t = \mathbb{E}\left[f\left(\int_0^T W_s ds\right)|\mathcal{F}_t\right] = \mathbb{E}\left[f\left(\int_0^t W_s\,ds + W_t(T-t) + \int_t^T (W_s - W_t)ds\right)|\mathcal{F}_t\right] = \phi(t, W_t(\cdot))$$

where ϕ is the function from $[0,T]\times C([-T,0])$ to $\mathbb{R}$ defined by

$$\begin{aligned}\phi(t,\eta) &= \mathbb{E}\left[f\left(\int_{-T}^{0}\eta(r)dr + \eta(0)(T-t) + \int_t^T (W_r - W_t)dr\right)\right] = \\ &= \int_{\mathbb{R}} f\left(\int_{-T}^{0}\eta(r)dr + \eta(0)(T-t) + x\right)p(t,dx)\end{aligned} \tag{8.46}$$

Where $p(t,dx)$ is the function defined in (8.39), with $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$ because $\int_t^T (W_r - W_t)dr$ is a centered gaussian random variable with standard deviation σ_t.
In $t = T$, $p(T,dx) = \delta_0(dx)$, then $\phi(T, W_T(\cdot)) = f\left(\int_0^T W_s ds\right) = H = V_T$.
From now on we will consider $t < T$, then $p(t,dx) = p(t,x)dx$.
We have also $\phi(0, W_0(\cdot)) = \mathbb{E}\left[f\left(\int_{-T}^0 W_0(s)ds + W_0 T + \int_0^T (W_s - W_0)ds\right)\right] = \mathbb{E}\left[f\left(\int_0^T W_s ds\right)\right] = \mathbb{E}[H]$.
We need the linear change of variables $z = \left(\int_{-T}^0 \eta(r)dr + \eta(0)(T-t) + x\right)$ to obtain another expression of ϕ which is in $C^{1,2}([0,T[\times B)\cap C^0([0,T]\times B)$, $B = C([-T,0])$.

$$\begin{aligned}\phi(t,\eta) &= \int_{\mathbb{R}} f(z)p\left(t, dz - \int_{-T}^{0}\eta(r)dr - \eta(0)(T-t)\right) = \\ &= \int_{\mathbb{R}} f(z)p\left(t, z - \int_{-T}^{0}\eta(r)dr - \eta(0)(T-t)\right)dz\end{aligned}$$

In order to apply the Itô's formula to ϕ from 0 to $u < T$ we evaluate the different derivatives.

$$\begin{aligned}
\partial_t \phi(t,\eta) &= \int_{\mathbb{R}} f(z)\eta(0)\partial_x p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz + \\
&\quad + \int_{\mathbb{R}} f(y)\partial_t p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \\
D_{dx}\phi(t,\eta) &= -\int_{\mathbb{R}} f(z)\partial_x p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \Big(\mathbb{1}_{[-T,0]}(x)dx + (T-t)\delta_0(dx)\Big) \\
D^2_{dx\,dy}\phi(t,\eta) &= \int_{\mathbb{R}} f(z)\partial_{xx} p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz \cdot \Big(A_1 + A_2 + A_3 + A_4\Big)
\end{aligned}$$

where

$$\begin{aligned}
A_1 &= \mathbb{1}_{[-T,0]^2}(x,y)dx\,dy \\
A_2 &= (T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy) \\
A_3 &= (T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy \\
A_4 &= (T-t)^2\delta_0(dx)\,\delta_0(dy)
\end{aligned}$$

and $\partial_x p(t,x) = \frac{1}{\sigma_t\sqrt{2\pi}}\left(-\frac{x^2}{\sigma_t^2}\right)e^{-\frac{x^2}{2\sigma_t^2}}$ and $\partial_t p(t,x) = \left[-\frac{(T-t)^2}{2}\right]\partial_{xx}p(t,x)$ by (8.41).

We have $D^2\phi : [0,T]\times C([-T,0]) \to (L^2([-T,0])\oplus\mathcal{D}_0)\hat{\otimes}_h^2$ and we recall that the window of a real process having finite quadratic admits a $(L^2([-T,0])\oplus\mathcal{D}_0)\hat{\otimes}_h^2$-quadratic variation which is determined only by the $\mathcal{D}_{00}$ component, which is in this case

$$A_4 \cdot \int_{\mathbb{R}} f(z)\partial_{xx}p\left(t, z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right) dz.$$

Now we apply the Itô's formula for ϕ from 0 to $u < T$. We obtain

$$\begin{aligned}
\phi(u, W_u(\cdot)) = H &= \phi(0, W_0(\cdot)) + \int_0^u \partial_t\phi(t, W_t(\cdot))dt + \int_0^u \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot)\rangle + \\
&\quad + \frac{1}{2}\int_0^u \langle D^2\phi(t, W_t(\cdot)), d\widetilde{[W(\cdot)]}_t\rangle = \\
&= \mathbb{E}[H] + I_1 + I_2 + I_3
\end{aligned} \tag{8.47}$$

For simplicity we make another change of variable $x = \left(z - \int_{-T}^0 \eta(r)dr - \eta(0)(T-t)\right)$. Concerning the first term I_1 and using (8.41) we obtain

$$\begin{aligned}
I_1 &= \int_0^u \int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) W_t\, \partial_x p(t,x) dx\, dt + \\
&+ \int_0^u \int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_t p(t,x) dx\, dt = \\
&= \int_0^u \int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) W_t\, \partial_x p(t,x) dx\, dt + \\
&- \frac{1}{2} \int_0^u \int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) (T-t)^2 \partial_{xx} p(t,x) dx\, dt
\end{aligned}$$

We continue with the second term I_2 obtaining

$$\begin{aligned}
I_2 &= \int_0^u \langle D\phi(t, W_t(\cdot)), d^- W_t(\cdot)\rangle = \lim_{\epsilon\to 0} \int_0^u \langle D\phi(t, W_t(\cdot)), \frac{W_{t+\epsilon}(\cdot) - W_t(\cdot)}{\epsilon}\rangle = \lim_{\epsilon\to 0} (I_{21}(\epsilon) + I_{22}(\epsilon)) \\
I_{21}(\epsilon) &= -\int_0^u \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_x p(t,x) dx\right] (T-t) \frac{W_{t+\epsilon} - W_t}{\epsilon} dt \\
I_{22}(\epsilon) &= -\int_0^u \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_x p(t,x) dx\right] \left[\int_0^t \frac{W_{u+\epsilon} - W_u}{\epsilon} du\right] dt
\end{aligned}$$

We have if it is the Brownian motion the term $I_{21}(\epsilon)$ converges and it is the Itô's integral being the process $(\mathcal{F}_t)$-adapted:

$$\begin{aligned}
I_{21}(\epsilon) &\xrightarrow{\mathbb{P}} -\int_0^u \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_x p(t,x) dx\right] (T-t) dW_t \\
I_{22}(\epsilon) &\xrightarrow{\mathbb{P}} -\int_0^u \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_x p(t,x) dx\right] W_t\, dt
\end{aligned}$$

Finally concerning the term I_3 we have, after the study of the appropriate χ-quadratic variation for the window of a process with finite quadratic variation

$$\begin{aligned}
I_3 &= \frac{1}{2} \int_0^u (T-t)^2 \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_{xx} p(t,x) dx\right] d[W]_t = \\
&= \frac{1}{2} \int_0^u (T-t)^2 \left[\int_{\mathbb{R}} f\left(\int_{-T}^0 W_t(r)dr + W_t(T-t) + x\right) \partial_{xx} p(t,x) dx\right] dt
\end{aligned}$$

So (8.47) gives explicitly

$$\phi(u, W_u(\cdot)) = \mathbb{E}[H] - \int_0^u A_t\, dW_t \tag{8.48}$$

where $(A_t)_{t\in[0,u]}$ is the process defined by (8.45) If $u \to T$ in probability we have exactly

$$H = \phi(T, W_T(\cdot)) = \mathbb{E}[H] - \int_0^T A_t d^- W_t \tag{8.49}$$

with $(A_t)_{t\in[0,T]}$ the process defined as in (8.45). □

Next proposition gives sufficient condition such that, let X be a Gaussian random variable, $f(X)$ belong to $L^1(\Omega)$.

Proposition 8.14. Given $T > 0$ and X a Gaussian random variable with mean 0 and variance $T^3/3$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

i $f \in L^1_{loc}(\mathbb{R})$

ii f *subexponential*, i.e. there exist $M > 0$ and $\gamma > 0$ such that $|f(y)| \leq e^{\gamma|y|}$ for $|y| > M$

then $f(X) \in L^1(\Omega)$, i.e. $\mathbb{E}\left[|f(X)|\right] = \int_{\mathbb{R}} |f(y)|p(0,y)dy < +\infty$

Remark 8.15. Proposition 8.14 gives sufficient conditions, for instance, such that $f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$, fulfill the assumption in the previous toy model (we show sufficient condition to apply Itô's formula).

Proposition 8.16. Let H be an option with values in time T equal to $H(W_T(\cdot)) = f\left(\int_0^T W_s ds\right)$ and $f : \mathbb{R} \to \mathbb{R}$ is a real function once differentiable and with polynomial growth, denoted by $f \in C^1_{pol}(\mathbb{R})$. Then the integral in (8.49) is an Itô's integral and

$$A_t = \mathbb{E}\left[D_t^m H|\mathcal{F}_t\right]$$

Remark 8.17. If $f \in C^1_{pol}(\mathbb{R})$ then $f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$, via the remark in fact we can show that $f \in L^1_{loc}(\mathbb{R})$ and it is subexponential.

Proof of the Proposition. Let $(\mathcal{F}_t)$ be the Brownian filtration, $H(W_T(\cdot)) = f\left(\int_0^T W_s ds\right)$ is $(\mathcal{F}_T)-$measurable and it is in $\mathbb{D}^{1,2}$ because $\int_0^T W_s ds \in \mathbb{D}^{1,2}$ and $f \in C^1_{pol}(\mathbb{R})$ by hypothesis. Then we compute the Clark-Ocone's formula as in (8.28) (that is a representation theorem).
We compute tha Malliavin's derivative of H denoted by $D_t^m H$:

$$\begin{aligned}
D_t^m H = D_t^m f\left(\int_0^T W_s ds\right) &= f'\left(\int_0^T W_s ds\right) D_t^m\left(\int_0^T W_s ds\right) = \\
&= f'\left(\int_0^T W_s ds\right)\int_t^T D_t^m(W_s)ds = \\
&= f'\left(\int_0^T W_s ds\right)\int_t^T \mathbb{1}_{[0,s]}(t)ds = \\
&= f'\left(\int_0^T W_s ds\right)(T-t)
\end{aligned}$$

Again we compute the conditional expectation

$$\begin{aligned}\mathbb{E}\left[D_t^m H|\mathcal{F}_t\right] &= \mathbb{E}\left[f'\left(\int_0^T W_s ds\right)(T-t)|\mathcal{F}_t\right] = \\ &= \mathbb{E}\left[f'\left(\int_0^t W_s ds + W_t(T-t) + \int_t^T (W_s - W_t)ds\right)(T-t)|\mathcal{F}_t\right] = \\ &= F(t, W_t(\cdot))\end{aligned}$$

where $F : [0,T] \times C([-T,0]) \to \mathbb{R}$ is defined by

$$\begin{aligned}F(t,\eta) &= \mathbb{E}\left[f'\left(\int_{-T}^0 \eta(s)ds + \eta(0)(T-t) + \int_t^T (W_s - W_t)ds\right)(T-t)\right] = \\ &= \int_{\mathbb{R}} f'\left(\int_{-T}^0 \eta(s)ds + \eta(0)(T-t) + x\right)(T-t)p(t,x)dx = \\ &= -(T-t)\int_{\mathbb{R}} f\left(\int_{-T}^0 \eta(s)ds + \eta(0)(T-t) + x\right)\partial_x p(t,x)dx =\end{aligned}$$

The last equality is obtained via integration by parts. The function p is the one with variation $\sigma_t^2 = \frac{(T-t)^3}{3}$, because $\int_t^T (W_s - W_t)ds$ is a gaussian random variable with mean zero and variance $\frac{(T-t)^3}{3}$.
Then (8.28) gives

$$H = \mathbb{E}\left[H\right] - \int_0^T (T-t)\int_{\mathbb{R}} f\left(\int_{-T}^0 \eta(s)ds + \eta(0)(T-t) + x\right)\partial_x p(t,x)dx\, dW_t \tag{8.50}$$

Moreover the limit for $u \to T$ is with the Itô's integral because f continuous.

□

Now we want to illustrate an example where we have a 'representation theorem' via the improper forward integral of a random variable that is not in $L^2(\Omega)$. We recall that a random variable in $L^2(\Omega)$ admits a representation via the representation martingale theorem or the Clark-Ocone's formula.

Proposition 8.18. Let $f : \mathbb{R} \to \mathbb{R}$ be a positive function such that $f(W_T) \in L^1(\Omega)$. Let $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a function in $C^{1,2}([0,T[\times\mathbb{R})$ such that

$$\begin{cases} u(t,x) = \int_{\mathbb{R}} p_{T-t}(x-y)f(y)dy \\ u(T,x) = f(x)\end{cases}$$

Then

$$f(W_T) = u(0,W_0) + \int_0^T \partial_x u(s,W_s)d^- W_s \tag{8.51}$$

The last integral is the *improper forward integral*, i.e. is the limit in probability for $t \to T$ of the forward integral whenever it exists.

Proof. Let N be a fixed number. We pose $f^N = f \wedge N$.
By construction $u(t, W_t) = \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right]$. We know the martingale process $\mathbb{E}\left[f(W_T)|\mathcal{F}_t\right]$ admits a cadlag version because the Brownian filtration $(\mathcal{F}_t)_t$ fulfill the usual assumption. Then by Doob's second martingale convergence theorem (controllare!) it exist a process denoted by N_{T^-} such that

$$\mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] \longrightarrow N_{T^-} \quad \text{a.s.}$$

We want to compare $f(W_T)$ and N_{T^-} and show that they are equals a.s.

$$\begin{aligned} f(W_T) - N_{T^-} = & \left(f(W_T) - f^N(W_T)\right) + \left(f^N(W_T) - \mathbb{E}\left[f^N(W_T)|\mathcal{F}_t\right]\right) + \\ & + \left(\mathbb{E}\left[f^N(W_T)|\mathcal{F}_t\right] - \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right]\right) + \left(\mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] - N_{T^-}\right) = I_1 + I_2 + I_3 + I_4 \end{aligned}$$

We observe that $f^N(W_T) \in L^2(\Omega)$, in fact it is also in L^∞ because it is bounded.
We fix N large enough and for each N we choose a suitable cadlag version for $\mathbb{E}\left[f^N(W_T)|\mathcal{F}_t\right]$ that will be prolonged.
Now we consider

$$\begin{aligned} \liminf_{t\to T} |f(W_T) - N_{T^-}| \le & \liminf_{t\to T} |I_1| + \liminf_{t\to T} |I_2| + \liminf_{t\to T} |I_3| + \liminf_{t\to T} |I_4| \le \\ \le & \left|f(W_T) - f^N(W_T)\right| + \liminf_{t\to T} \mathbb{E}\left[f^N(W_T) - f(W_T)|\mathcal{F}_t\right] \end{aligned}$$

because $\liminf_{t\to T} |I_4|$ for the suitable versione by the convergence a.s. zero by dominated Lebesgue's theorem We take the expectation and with Fatou's lemma we have

$$\mathbb{E}\left[|f(W_T) - N_{T^-}|\right] \le \mathbb{E}\left[\left|f^N(W_T) - f(W_T)\right|\right]$$

Now we choose just $N > N_0$ such that $\mathbb{E}\left[f^N(W_T) - f(W_T)\right] < \epsilon$ since $f^N(W_T) \xrightarrow{L^1(\Omega)} f(W_T)$, i.e. $\mathbb{E}\left[\left|f(W_T) - f^N(W_T)\right|\right] \to 0$. This allows to conclude that $N_{T^-} = f(W_T)$ a.s.
Now we apply the Itô's formula to $u(t, W_t)$ for $t < T$ and we have

$$u(t, W_t) = u(0, W_0) + \int_0^t \partial_x u(s, W_s) dW_s$$

On the left side, taking the limit in probability we obtain

$$\lim_{t\to T} u(t, W_t) = \lim_{t\to T} \mathbb{E}\left[f(W_T)|\mathcal{F}_t\right] = N_{T^-} = f(W_T) \quad \text{a.s.}$$

On the right side we obtain the improper integral $u(0, W_0) + \int_0^t \partial_x u(s, W_s) dW_s \xrightarrow{t\to T} u(0, W_0) + \int_0^T \partial_x u(s, W_s) d^- W_s$.
The result is now established. □

8.5 Toy model

We will consider the window Brownian process $W_t(\cdot)$. Let $\mathcal{F}_t$ be the Brownian filtration, $\mathcal{F}_t = \sigma\left(W_s; s \le t\right) = \sigma\left(W_t(\cdot)\right)$. For all $i = 1, \ldots, n$, let $\varphi_i : [0, T] \to \mathbb{R}$ be $C^2([0, T]; \mathbb{R})$, then they are in

particular $H^1([0,T];\mathbb{R})$-valued functions, i.e. it exists $\dot{\varphi}_i \in L^2([0,T])$ such that $\varphi_i(T)-\varphi_i(0)=\int_0^T \dot{\varphi}_i(s)ds$. Moreover we have that $\dot{\varphi}_i$ are bounded variation functions for all $i=1\dots,n$. We consider, without restriction of generality, $\varphi_1,\dots,\varphi_n$ orthogonal in the space $L^2([0,T])$ and we pose, for every i, $\varphi_i(t)=0$ for $t\notin[0,T]$. Obviously we have $\varphi_i(0^-)=0$ and $\varphi_i(T^+)=0$.
Let $f:\mathbb{R}^n\to\mathbb{R}$ be a measurable and bounded function (or linear increasing). We consider the function $H:C([-T,0])\to\mathbb{R}$ defined by

$$H(\eta)=f\left(\int_{-T}^0 \varphi_1(u+T)d\eta(u),\dots,\int_{-T}^0 \varphi_n(u+T)d\eta(u)\right)$$

This is well defined because by integration by parts we have $\int_{-T}^0 \varphi_i(u+T)d\eta(u)=\varphi_i(t)\eta(t-T)-\int_0^t \eta(s-T)d\varphi_i(s)$. So for the stochastic process $W_T(\cdot)$ we have

$$\begin{aligned}H(W_T(\cdot))&=f\left(\int_0^T \varphi_1(s)dW_s,\dots,\int_0^T \varphi_n(s)dW_s\right)=\\&=f\left(\int_{-T}^0 \varphi_1(u+T)dW_T(u),\dots,\int_{-T}^0 \varphi_n(u+T)dW_T(u)\right)\end{aligned}\tag{8.52}$$

We remark that by integration by parts for stochastic processes

$$\begin{aligned}\int_0^t \varphi_i(s)dW_s&=\int_{-t}^0 \varphi_i(u+t)dW_t(u)=\varphi_i(t)W_t(0)-\varphi_i(0)W_t(-t)-\int_{-t}^0 W_t(u)d\varphi_i(u+t)=\\&=\varphi_i(t)W_t-\int_0^t W_s d\varphi_i(s)\end{aligned}\tag{8.53}$$

i.e. it is just a pathwise integral. We calculate the conditional expectation and we have

$$\begin{aligned}&\mathbb{E}[H|\mathcal{F}_t]=\mathbb{E}\left[f\left(\int_0^T \varphi_i(s)dW_s,\dots,\int_0^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right]=\\&=\mathbb{E}\left[f\left(\int_0^t \varphi_1(s)dW_s+\int_t^T \varphi_1(s)dW_s,\dots,\int_0^t \varphi_n(s)dW_s+\int_t^T \varphi_n(s)dW_s\right)|\mathcal{F}_t\right]=\\&=\Psi\left(t,\int_0^t \varphi_1(s)dW_s,\dots,\int_0^t \varphi_n(s)dW_s\right)=\\&=\Psi\left(t,\int_{-t}^0 \varphi_1(u+t)dW_t(u),\dots,\int_{-t}^0 \varphi_n(u+t)dW_t(u)\right)=\\&=\Psi\left(t,\int_{-T}^0 \varphi_1(u+t)dW_t(u),\dots,\int_{-T}^0 \varphi_n(u+t)dW_t(u)\right)\end{aligned}$$

where the function $\Psi:[0,T]\times\mathbb{R}^n\longrightarrow\mathbb{R}$ is defined by

$$\begin{aligned}&\Psi(t,y_1,\dots,y_n)=\mathbb{E}\left[f\left(y_1+\int_t^T \varphi_1(s)dW_s,\dots\dots,y_n+\int_t^T \varphi_n(s)dW_s\right)\right]\\&\Psi(T,y_1,\dots,y_n)=f\left(y_1,\dots\dots,y_n\right)\end{aligned}\tag{8.54}$$

If we fix the hypothesis that the gaussian vector $\left(\int_t^T \varphi_1(s)dW_s, \ldots, \int_t^T \varphi_n(s)dW_s\right)$ has a variance covariance matrix Σ_t defined by $(\Sigma_t)_{i,j} = \int_t^T \varphi_i(s)\varphi_j(s)ds$ that is invertible for every $t > 0$ then $\Psi \in C^{1,2}([0,T[\times\mathbb{R}^n)$. In fact the function $p : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$p(t, z_1, \ldots, z_n) = \sqrt{\frac{1}{(2\pi)^n det\,(\Sigma_t)}} \exp\left\{-\frac{(z_1, \ldots, z_n)\Sigma_t^{-1}(z_1, \ldots, z_n)^*}{2}\right\}$$

is the density function of the gaussian vector $\left(\int_t^T \varphi_1(s)dW_s, \ldots, \int_t^T \varphi_n(s)dW_s\right)$. The function Ψ becomes

$$\begin{aligned}\Psi(t, y_1, \ldots, y_n) &= \int_{\mathbb{R}^n} f\,(y_1 + z_1, \ldots, y_n + z_n)\, p(t, z_1, \ldots, z_n) dz_1 \cdots dz_n = \\ &= \int_{\mathbb{R}^n} f\,(\tilde{z}_1, \ldots, \tilde{z}_n)\, p(t, \tilde{z}_1 - y_1, \ldots, \tilde{z}_n - y_n) d\tilde{z}_1 \cdots d\tilde{z}_n\end{aligned}$$

The function p is a solution $C^{1,2}([0,T] \times \mathbb{R}^n)$ of

$$\partial_t p(t, z_1, \ldots, z_n) = -\frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial^2_{ij} p(t, z_1, \ldots, z_n)$$

We remark that when $(\varphi_i)_{i=1,\ldots,n}$ is an orthogonal system in $L^2([0,T])$ the variance covariance matrix Σ_0 is a diagonal matrix in $M^{n,n}(\mathbb{R})$.
From the relation for p we deduce that the function Ψ is a solution $C^{1,2}([0,T[\times\mathbb{R}^n)$ of

$$\partial_t \Psi(t, z_1, \ldots, z_n) = -\frac{1}{2}\sum_{i,j=1}^n \varphi_i(t)\varphi_j(t)\partial^2_{ij}\Psi(t, z_1, \ldots, z_n) \tag{8.55}$$

Finally we define a function

$$u : [0,T[\times C([-T,0]) \longrightarrow \mathbb{R} \qquad u \in C^{1,2}([0,T] \times C([-T,0]))$$

by

$$u(t, \eta) = \Psi\left(t, \varphi_1(t)\eta(0) - \int_0^t \eta(s-t)d\varphi_1(s), \ldots, \varphi_n(t)\eta(0) - \int_0^t \eta(s-t)d\varphi_n(s)\right)$$

We note that φ_i is a bounded variation function for every $i = 1, \ldots, n$, then the term

$$\varphi_i(t)\eta(0) - \int_0^t \eta(s-t)d\varphi_i(s) = \varphi_i(t)\eta(0) - \int_0^t \eta(s-t)\dot{\varphi}_i(s)ds$$

is well defined in the Riemann-Stietjies sense and will be denoted for simplicity by

$$\begin{aligned}\int_0^t \varphi_i(s)d\eta(s-t) = \int_{-t}^0 \varphi_i(s+t)d\eta(s) &:= \varphi_i(t)\eta(0) - \int_0^t \eta(s-t)d\varphi_i(s) = \\ &= \eta(0)\varphi_i(t) - \eta(-t^-)\varphi_i(0^-) - \int_0^t \eta(s-t)\dot{\varphi}_i(s)ds = \\ &= \eta(0)\varphi_i(t) - \int_0^t \eta(s-t)\dot{\varphi}_i(s)ds\end{aligned} \tag{8.56}$$

The last equality comes from $\varphi_i(0^-) = 0$. We recall that in general let $g, f : [a, b] \to \mathbb{R}$ be càdlàg (continue à droite and limité à gauche) and g of bounded variation we have

$$\int_{[a,b]} g\, df = g(b)\, f(b) - g(a^-)\, f(a^-) - \int_{[a,b]} f\, dg \tag{8.57}$$

With this notation we have another expression for the function u

$$\begin{aligned} u(t,\eta) =& \Psi\left(t, \int_{-t}^{0} \varphi_1(s+t)d\eta(s), \dots, \int_{-t}^{0} \varphi_n(s+t)d\eta(s)\right) = \\ &= \mathbb{E}\left[f\left(\int_{-t}^{0} \varphi_1(s+t)d\eta(s) + \int_{t}^{T} \varphi_1(s)dW_s, \dots\dots, \int_{-t}^{0} \varphi_n(s+t)d\eta(s) + \int_{t}^{T} \varphi_n(s)dW_s\right)\right] \end{aligned} \tag{8.58}$$

We can verify, as a property, that for every window process $X_t(\cdot)$ such that $X_t(0)$ is a real process with finite quadratic variation $[X.(0)] = t$

$$u(T, X_T(\cdot)) = \mathbb{E}\left[f\left(\int_{-T}^{0} \varphi_1(s+t)dX_T(s), \dots\dots, \int_{-T}^{0} \varphi_n(s+t)dX_T(s)\right)\right] \tag{8.59}$$

oppure u evaluated in $(t, W_t(\cdot))$ verifies the equation for the conditional expectation, in fact by (8.53) we have

$$u\left(t, W_t(\cdot)\right) = \Psi\left(t, \varphi_1(t)W_t - \int_0^t W_s d\varphi_1(s), \dots, \varphi_n(t)W_t - \int_0^t W_s d\varphi_n(s)\right) = \mathbb{E}[H|\mathcal{F}_t] \tag{8.60}$$

In the following proposition we will use that function u to apply Itô's formula.

Proposition 8.19. Let $u : [0, T] \times C([-T, 0]) \to \mathbb{R}$ be the function in $C^{1,2}$ defined in (8.58) via a function Ψ such that is a solution $C^{1,2}([0, T[\times\mathbb{R}^n)$ of (8.55). Let $X(\cdot)$ be a window process such that $X(0)$ is a real finite quadratic variation process with $[X(0)]_t = t$. Then

1.
$$u(T, X_T(\cdot)) = u(0, X_0(\cdot)) + \int_0^T A_s d^- X_s \tag{8.61}$$
with the process A_s defined by
$$A_s = \sum_{i=1}^{n} \partial_i \Psi\left(s, \int_{-s}^{0} \varphi_1(r+s)d^- X_s(r), \dots, \int_{-s}^{0} \varphi_n(r+s)d^- X_s(r)\right)\varphi_i(s) \tag{8.62}$$

2. If $X_t(\cdot)$ is the window Brownian motion $W_t(\cdot)$ and if $u(T, X_T(\cdot))$ is square integrable then A_s is exactly the adapted process given by martingale representation theorem
$$A_s = \sum_{i=1}^{n} \partial_i \Psi\left(s, \int_{-s}^{0} \varphi_1(r+s)dW_s(r), \dots, \int_{-s}^{0} \varphi_n(r+s)dW_s(r)\right)\varphi_i(s)$$

3. If $X_t(\cdot)$ is the window Brownian motion $W_t(\cdot)$ and if $f \in C_b^1(\mathbb{R}^n)$ then

$$A_t = \mathbb{E}\left[D_t^m H | \mathcal{F}_t\right] \tag{8.63}$$

Proof.

1. The stochastic process $X(\cdot)$ admits a $(\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$. We have to verify that the second order derivative with respect to the space is in $(\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$ to apply the Itô's formula for the function u.
The first derivative $D_t u : [0,T] \times C([-T,0]) \to \mathbb{R}$ with respect to the space $[0,T]$ is

$$\begin{aligned}
\partial_t u(t,\eta) =& \partial_t \Psi\left(t, \int_{-t}^0 \varphi(s+t) d\eta(s)\right) + \\
&+ \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) d\eta(s), \ldots, \int_{-t}^0 \varphi_n(s+t) d\eta(s)\right)\right) \cdot \left(\partial_t \int_{-t}^0 \varphi_i(s+t) d\eta(s)\right) = \\
&= \partial_t \Psi\left(t, \int_{-t}^0 \varphi(s+t) d\eta(s)\right) + \\
&+ \sum_{i=1}^n \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) d\eta(s), \ldots, \int_{-t}^0 \varphi_n(s+t) d\eta(s)\right)\right) \cdot \left(\int_{-t}^0 \dot{\varphi}_i(s+t) d\eta(s)\right)
\end{aligned} \tag{8.64}$$

The last equality because by integration by parts

$$\begin{aligned}
\partial_t \left(\int_{-t}^0 \varphi_i(s+t) d\eta(s)\right) &= \partial_t \left(\eta(0)\varphi_i(t) - \int_{-t}^0 \eta(s)\dot{\varphi}_i(s+t) ds\right) = \\
&= \eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0) - \int_{-t}^0 \eta(s)\ddot{\varphi}_i(s+t) ds = \\
&= \eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0) - \left[\eta(0)\dot{\varphi}_i(t) - \eta(-t)\dot{\varphi}_i(0) - \int_{-t}^0 \dot{\eta}(s)\dot{\varphi}_i(s+t) ds\right] = \\
&= \int_{-t}^0 \dot{\varphi}_i(s+t) d\eta(s)
\end{aligned} \tag{8.65}$$

We remark that the term $\int_{-t}^0 \dot{\varphi}_i(s+t) d\eta(s)$ is an integral that exists for every continuous function η. In fact $\dot{\varphi}$, being C^1, is also of bounded variation and then the integral has a sense simply via integration by part. On the other hand the integral

$$\int_{-t}^0 \dot{\varphi}_i(t+s) dW_t(s) = \int_{-t}^0 \dot{\varphi}_i(t+s) dW_{t+s} = \int_0^t \dot{\varphi}_i(u) dW_u$$

is a well defined Itô's integral for (8.53) and it is exactly equal to the one in (8.65) if we fix $\eta = W_t(\cdot)$.
We go on with the evaluation of the derivative: the first derivative $D_x u : [0,T] \times C([-T,0]) \to$

$\mathcal{M}([-T,0])$ with respect the space $C([-T,0])$ is a measure

$$
\begin{aligned}
D_{ds}u(t,\eta) = &\sum_{i=1}^{n}\left(\partial_i\Psi\left(t,\int_{-t}^{0}\varphi_1(s+t)d\eta(s),\ldots,\int_{-t}^{0}\varphi_n(s+t)d\eta(s)\right)\right)\cdot\\
&\cdot\left(\varphi_i(t)\delta_0(ds)-\mathbb{1}_{[-t,0]}(s)d\varphi_i(s+t)\right)=\\
=&\sum_{i=1}^{n}\left(\partial_i\Psi\left(t,\int_{-t}^{0}\varphi_1(s+t)d\eta(s),\ldots,\int_{-t}^{0}\varphi_n(s+t)d\eta(s)\right)\right)\cdot\\
&\cdot\left(\varphi_i(t)\delta_0(ds)-\mathbb{1}_{[-t,0]}(s)\dot{\varphi}_i(ds+t)\right)
\end{aligned}\tag{8.66}
$$

We recall that, for avery i, $d\varphi_i(s+t)=\dot{\varphi}_i(ds+t)$ and $\dot{\varphi}_i$ is of bounded variation because it is supposed in C^1. The measure $\mathbb{1}_{[-t,0]}(\cdot)d\varphi_i(\cdot+t)$ on $[-T,0]$ is a measure such that on a continuous function $h\in C([-T,0])$ we have $\langle h,\mathbb{1}_{[-t,0]}(\cdot)d\varphi_i(\cdot+t)\rangle=\int_{-T}^{0}h(s)\mathbb{1}_{[-t,0]}(s)d\varphi_i(s+t)=\int_{-t}^{0}h(s)d\varphi_i(s+t)=\int_{-t}^{0}h(s)\dot{\varphi}_i(ds+t)$. Moreover the measure $d\varphi_i$ is singular with respect to the Dirac measure δ_0.
For the second derivative with respect to the space $D^2_{xx}u:[0,T]\times C([-T,0])\to(C([-T,0])\hat{\otimes}^2_\pi)^*$ with respect the space $C([-T,0])$ we have

$$
\begin{aligned}
D^2_{dv,dz}u(t,\eta) = &\sum_{i,j=1}^{n}\left(\partial^2_{i,j}\Psi\left(t,\int_{-t}^{0}\varphi_1(s+t)d\eta(s),\ldots,\int_{-t}^{0}\varphi_n(s+t)d\eta(s)\right)\right)\cdot\\
&\cdot\Bigg(\varphi_i(t)\varphi_j(t)\delta_0(dv)\delta_0(dz)-\varphi_i(t)\mathbb{1}_{[-t,0]}(v)\dot{\varphi}_j(dv+t)\delta_0(dz)+\\
&-\varphi_j(t)\mathbb{1}_{[-t,0]}(z)\dot{\varphi}_i(dz+t)\delta_0(dv)+\mathbb{1}_{[-t,0]}(v)\mathbb{1}_{[-t,0]}(z)\dot{\varphi}_i(dv+t)\dot{\varphi}_j(dz+t)\Bigg)
\end{aligned}\tag{8.67}
$$

We recall that, for all $i=1,\ldots,n$, φ_i are C^2 functions, then in particular $\dot{\varphi}_i$ are in L^2. Then $D^2u:[0,T]\times C([-T,0])\to(\mathcal{D}_0\oplus L^2)\hat{\otimes}^2_h$ continuously.
We can apply the Itô's formula for the function u and the process X. We have

$$
\begin{aligned}
u(T,X_T(\cdot)) = &u(0,X_0(\cdot))+\int_0^T\partial_t u(t,X_t(\cdot))dt+\int_0^T\langle Du(t,X_t()\cdot),d^-X_t(\cdot)\rangle+\\
&+\frac{1}{2}\int_0^T\langle D^2u(t,X_t(\cdot)),d\widetilde{[X(\cdot)]}_t\rangle = u(0,X_0(\cdot))+I_1+I_2+I_3
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int_0^T \partial_t \Psi\left(t, \int_{-t}^0 \varphi(s+t) dX_t(s)\right) dt + \\
&\quad + \sum_{i=1}^n \int_0^T \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \times \\
&\qquad\qquad \times \left(\int_{-t}^0 \dot{\varphi}_i(s+t) d^- X_t(s)\right) dt = \\
&= I_{11} + I_{12} \\
I_2 &= \lim_{\epsilon \to 0} \left(I_{21}(\epsilon) + I_{22}(\epsilon)\right) \qquad \text{in probability with}
\end{aligned}
$$

$$
\begin{aligned}
I_{21}(\epsilon) &= \sum_{i=1}^n \int_0^T \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \varphi_i(t) \frac{X_{t+\epsilon}(0) - X_t(0)}{\epsilon} dt \\
I_{22}(\epsilon) &= -\sum_{i=1}^n \int_0^T \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \cdot \\
&\quad \cdot \left(\int_{-t}^0 \dot{\varphi}_i(ds+t) \frac{X_{t+\epsilon}(s) - X_t(s)}{\epsilon}\right) dt
\end{aligned}
$$

and finally

$$
\begin{aligned}
I_3 &= \frac{1}{2} \sum_{i,j=1}^n \int_0^T \left(\partial_{i,j}^2 \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \varphi_i(t) \varphi_j(t) d[X.(0)]_t = \\
&= \frac{1}{2} \sum_{i,j=1}^n \int_0^T \left(\partial_{i,j}^2 \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \varphi_i(t) \varphi_j(t) dt
\end{aligned}
$$

The Itô's formula in particular tell us the convergence in probability for the sum $I_{21}(\epsilon) + I_{22}(\epsilon)$

$$
I_{21}(\epsilon) + I_2(\epsilon) = \int_0^T \langle D_x(t, X_t(\cdot)), \frac{X_{t+\epsilon}(\cdot) - X_t(\cdot)}{\epsilon} dt\rangle \xrightarrow{\mathbb{P}} \int_0^T \langle D_x(t, X_t(\cdot)), d^- X_t(\cdot)\rangle \tag{8.68}
$$

Moreover we will show

$$
\begin{aligned}
I_{22}(\epsilon) &\xrightarrow{a.s.} = I_{22} = -I_{12} \\
I_{22} &= -\sum_{i=1}^n \int_0^T \left(\partial_i \Psi\left(t, \int_{-t}^0 \varphi_1(s+t) dX_t(s), \ldots, \int_{-t}^0 \varphi_n(s+t) dX_t(s)\right)\right) \left(\int_0^t \dot{\varphi}_i(u) d^- X_u(0)\right) dt
\end{aligned}
\tag{8.69}
$$

In fact $I_{22}(\epsilon)$ is always convergent pathwise, i.e. for almost all ω. We show firstly the a.s. convergence of $\int_{-t}^0 \dot{\varphi}_i(ds+t) \frac{X_{t+\epsilon}(s) - X_t(s)}{\epsilon}$ to $\int_0^t \dot{\varphi}_i(s) d^- X_s$. Even if a priori it is an anticipating integral, $\dot{\varphi}$ is of bounded variation, so the integral has a sense via an integration by parts. We have in fact

$$\begin{aligned}\int_{-t}^{0} \dot{\varphi}_i(ds+t)\frac{X_{t+\epsilon}(s)-X_t(s)}{\epsilon} &= \int_0^t \dot{\varphi}_i(s+t)\frac{X_{s+\epsilon}-X_s}{\epsilon}ds = \\ &= \int_t^{t+\epsilon} \frac{\dot{\varphi}_i(u-\epsilon)}{\epsilon}X_u du + \int_\epsilon^t \frac{\dot{\varphi}_i(u-\epsilon)-\dot{\varphi}_i(u)}{\epsilon}X_u du - \int_0^\epsilon \frac{\dot{\varphi}_i(u)}{\epsilon}X_u du \\ &\xrightarrow{a.s.} \dot{\varphi}_i(t)X_t - \int_0^t X_u d\dot{\varphi}_i(u) - \dot{\varphi}_i(0)X_0 = \int_0^t \dot{\varphi}_i(u)d^- X_u\end{aligned}$$

By the dominated convergence theorem we have the a.s. convergence for $I_{22}(\epsilon)$. Consequently by (8.68) and (8.69) we deduce the convergence in probability for $I_{21}(\epsilon)$ to I_{21}

$$I_{21}(\epsilon) \xrightarrow{\mathbb{P}} \sum_{i=1}^{n}\int_0^T \left(\partial_i\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dX_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dX_t(s)\right)\right)\varphi_i(t)d^- X_t(0) = I_{21} \tag{8.70}$$

The first result follows from the Itô's formula with (8.55):

$$\begin{aligned}u(T, X_T(\cdot)) &= u(0, X_0(\cdot)) + I_{11} + I_{21} + I_3 = \\ &= u(0,X_0) + \sum_{i=1}^{n}\int_0^T \left(\partial_i\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dX_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dX_t(s)\right)\right)\varphi_i(t)d^- X_t \\ &+ \int_0^T \partial_t\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dX_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dX_t(s)\right)dt + \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^T \left(\partial^2_{i,j}\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dX_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dX_t(s)\right)\right)\varphi_i(t)\varphi_j(t)dt = \\ &= u(0,X_0) + \sum_{i=1}^{n}\int_0^T \left(\partial_i\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dX_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dX_t(s)\right)\right)\varphi_i(t)d^- X_t\end{aligned} \tag{8.71}$$

2. For the second result we pose as window process the window Brownian motion $W_t(\cdot)$. Clearly the process $\partial_i\Psi\left(t, \int_{-t}^0 \varphi_1(s+t)dW_t(s), \dots, \int_{-t}^0 \varphi_n(s+t)dW_t(s)\right)\varphi_i(t)$ is $\mathcal{F}_t$-adapted, then the forward integral coincides with the classical Itô's integral:

$$H = u(T, W_T(\cdot)) = u(0, W_0(\cdot)) + \int_0^T \sum_{i=1}^{n} \partial_i\Psi\left(t, \int_0^t \varphi_1(s)dW_s, \dots, \int_0^t \varphi_n(s)dW_s\right)\varphi_i(t)dW_t$$

At the same time we know, by the martingale representation theorem, that it exists a $\mathcal{F}_t$-adapted process A_t such that the square integrable martingale $H = u(T, X_T(\cdot))$ hs the following decomposition

$$u(T, X_T(\cdot)) = \mathbb{E}\left[u(T, X_T(\cdot))\right] + \int_0^T A_s dW_s$$

The second result follows by uniqueness of decomposition.

$$A_t = \sum_{i=1}^{n} \partial_i \Psi \left(t, \int_0^t \varphi_1(s) dW_s, \dots, \int_0^t \varphi_n(s) dW_s \right) \varphi_i(t)$$

3. For the last result we observe that if $f \in C_b^1(\mathbb{R}^n)$ then, using Proposition 2.12, we know the decomposition given by the Clark-Ocone's formula (8.28). By unicity of decomposition we have an expression for the conditional expectation of the Malliavin's derivative of H. In particular the expression will be independent from the derivatives of function f, in fact

$$\begin{aligned}
D_t^m H = D_t^m &\left[f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) \right] = \\
&= \sum_{i=1}^{n} \partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) D_t^m \left[\int_0^T \varphi_i(s) dW_s \right] = \\
&= \sum_{i=1}^{n} \partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) \left(\varphi_i(t) + \int_t^T D_t^m \left[\varphi_i(s) \right] dW_s \right) = \\
&= \sum_{i=1}^{n} \partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) \varphi_i(t)
\end{aligned}$$

$$\begin{aligned}
\mathbb{E} \left[D_t^m H | \mathcal{F}_t \right] &= \mathbb{E} \left[\sum_{i=1}^{n} \partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) \varphi_i(t) | \mathcal{F}_t \right] = \\
&= \sum_{i=1}^{n} \mathbb{E} \left[\partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) | \mathcal{F}_t \right] \varphi_i(t)
\end{aligned}$$

By the definition of Ψ in (8.54) we have just to verify that for every $i = 1, \dots, n$ we have

$$\mathbb{E} \left[\partial_i f \left(\int_0^T \varphi_i(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) | \mathcal{F}_t \right] = \partial_i \Psi \left(t, \int_0^t \varphi_1(s) dW_s, \dots, \int_0^t \varphi_n(s) dW_s \right)$$

This is trivial in fact it will be n functions such that

$$\begin{aligned}
&\mathbb{E} \left[\partial_i f \left(\int_0^T \varphi_1(s) dW_s, \dots, \int_0^T \varphi_n(s) dW_s \right) | \mathcal{F}_t \right] = \\
&= \mathbb{E} \left[\partial_i f \left(\int_0^t \varphi_1(s) dW_s + \int_t^T \varphi_1(s) dW_s, \dots, \int_0^t \varphi_n(s) dW_s + \int_t^T \varphi_n(s) dW_s \right) | \mathcal{F}_t \right] = \\
&= \Psi^i \left(t, \int_0^t \varphi_1(s) dW_s, \dots, \int_0^t \varphi_n(s) dW_s \right)
\end{aligned}$$

where $\Psi^i : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Psi^i(t, y_1, \dots, y_n) = \mathbb{E} \left[\partial_i f \left(y_1 + \int_t^T \varphi_1(s) dW_s, \dots, y_n + \int_t^T \varphi_n(s) dW_s \right) \right]$$

Now it is easy to know that Ψ^i is exactly the function $\partial_i \Psi$, then the result.

$\square$

Remark 8.20. In the previous proposition we have an expression for the conditional expectation of the Malliavin's derivative dependent only from the derivative of Ψ even if the hypothesis $f \in C^1(\mathbb{R}^n)$.

Lemma 8.21. Ψ is a $C^{1,2}([0,T[\times\mathbb{R}^n)$ solution of (8.55), as in the hypotheses of the proposition, under the following assumptions:

1. $f \in C^2(\mathbb{R}^n)$ and $\varphi_i \in C^2([0,T])$.

2. For all $t > 0$ the matrix Σ_t defined by $(\Sigma_t)_{i,j} = \int_t^T \varphi_i(s)\varphi_j(s)ds$ is invertible.
   This is the example that we have seen at the beginning of this chapter.

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Titolo: Calcolo stocastico via regolarizzazione in dimensione infinita e motivazioni finanziarie

Riassunto: Questa tesi di dottorato sviluppa certi aspetti del calcolo stocastico via regolarizzazione per dei processi a valori in uno spazio di Banach generale B. Viene introdotto un concetto orginale di $\chi-$variazione quadratica, dove χ è un sottospazio del duale de prodotto tensoriale $B \otimes B$, munito della topologia proiettiva. Una attenzione particolare é dedicata al caso in cui B é lo spazio della funzioni continue su l'intervallo $[-\tau, 0]$, $\tau > 0$. Viene dimostrata una classe di risultati di stabilità attraverso funzioni di classe C^1 di processi che ammettono una χ-variazione quadratica e viene dimostrata una formula di Itô per tali processi. I processi continui reali a variazione quadratica finita Y (ad esempio processi di Dirichelt o anche Dirichlet debole) giocano un ruolo significativo. Viene definito un processo associato chiamato processo *finestra* e indicato con $Y_t(\cdot)$ definito da $Y_t(y) = Y_{t+y}$ per $y \in [-\tau, 0]$. $Y(\cdot)$ è un processo a valori nello spazio di Banach $C[-\tau, 0]$. Se Y è un processo reale con varazione quadratica uguale a $[Y]_t = t$ e $h = F(Y_T(\cdot))$ dove F è una funzione di classe $C^2(H)$ Fréchet e $H = L^2([-T, 0])$, è possibile rappresentare h come somma di un numero reale H_0 più un integrale forward di tipo $\int_0^T \xi d^- Y$ dove ξ è un processo di cui diamo la forma esplicita. Questo generalizza la formula di Clark-Ocone valida quando Y è un moto Browniano standard W. Una delle motivazioni viene dalla teoria di copertura di opzioni che dipendono da tutta la traiettoria del sottostante o quando il prezzo dell'azione sottostante non è una semimartingala.

Title: Infinite dimensional calculus via regularization with financial motivations

Abstract: This paper develops some aspects of stochastic calculus via regularization to Banach valued processes. An original concept of χ-quadratic variation is introduced, where χ is a subspace of the dual of a tensor product $B \otimes B$ where B is the value space of the process. Particular interest is devoted to the case when B is the space of real continuous functions defined on $[-\tau, 0]$, $\tau > 0$. Itô formulae and stability of finite χ-quadratic variation processes are established. Attention is devoted to a finite real quadratic variation (for instance Dirichlet, weak Dirichlet) process X. The $C([-\tau, 0])$-valued process $Y(\cdot)$ defined by $Y_t(y) = Y_{t+y}$ where $y \in [-\tau, 0]$ is called *window* process. Let $T > 0$. If Y is a finite quadratic variation process such that $[Y]_t = t$ and $h = F(Y_T(\cdot))$ where F is a $C^2(H)$ Fréchet function with $H = L^2([-T, 0])$, it is possible to represent h as a sum of a real number H_0 plus a forward integral of type $\int_0^T \xi d^- Y$ where ξ will be explicitly given. This decomposition generalizes the Clark-Ocone formula which is true when Y is the standard Brownian motion W. The main motivation comes hedging theory of path dependent options without semimartingales in mathematical finance.

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