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Mapping properties of weakly singular periodic volume potentials in Roumieu classes

M. Dalla Riva^{*,†}, M. Lanza de Cristoforis[‡], and P. Musolino[†]

Abstract

The analysis of the dependence of integral operators on perturbations plays an important role in the study of inverse problems and of perturbed boundary value problems. In this paper we focus on the mapping properties of the volume potentials with weakly singular periodic kernels. Our main result is to prove that the map which takes a density function and a periodic kernel to a (suitable restriction of the) volume potential is bilinear and continuous with values in a Roumieu class of analytic functions. Such result extends to the periodic case some previous results obtained by the authors for non periodic potentials and it is motivated by the study of perturbation problems for the solutions of boundary value problems for elliptic differential equations in periodic domains.

Keywords: periodic volume potentials; integral operators; Roumieu classes; periodic kernels; special nonlinear operators.

2010 Mathematics Subject Classification: 31B10, 47H30.

1 Introduction

This paper deals with the mapping properties of certain integral operators which arise in periodic potential theory. More precisely, the authors continue the work of [10, 22] where it has been shown that volume potentials associated to a parameter dependent analytic family of weakly singular kernels depend real-analytically upon the density function and on the parameter. The results of [10] have found application in the analysis of volume potentials corresponding to an analytic family of fundamental solutions of second order differential operators with constant coefficients. Here, instead, the authors extend the results of [10] to the case where the kernels are (spatially) periodic functions and thus provide a useful tool to study the integral operators which appear in the analysis of periodic boundary value problems.

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Several authors have studied the dependence of integral operators upon perturbations, with particular attention to layer and volume potentials associated to elliptic differential equations. For example, Potthast has proved in [36, 37, 38] Fréchet differentiability results for the dependence of layer potentials for the Helmholtz equation upon the support of integration in the framework of Schauder spaces. In the frame of inverse problems, we mention the works by Charalambopoulos [6], Haddar and Kress [14], Hettlich [15], Kirsch [20], and Kress and Päivärinta [21]. Moreover, Costabel and Le Louër [7, 8, 27] obtained analogous results in the framework of Sobolev spaces on Lipschitz domains.

The authors of the present paper and collaborators have developed a method based on functional analysis and on potential theory to prove analyticity results for the solution of boundary value problems upon perturbations of the domain and of the data. Indeed, classical potential theory enables to represent the solution of a boundary value problem in terms of integral operators. Therefore, in order to study the behavior of the solution upon perturbation, one needs to investigate perturbation results for layer and volume potentials. Thus [25, 26] have analyzed the layer potentials associated to the Laplace and Helmholtz equations, whereas [9] has investigated the case of layer potentials corresponding to second order complex constant coefficient elliptic differential operators, and [23] has considered a periodic analog.

The aim of this paper is to analyze the behavior of volume potentials corresponding to periodic kernels. More precisely, here we prove that the bilinear map which takes a weakly singular periodic kernel and a density function to the corresponding periodic volume potential is continuous with values in a Roumieu class of analytic functions (see Theorems 3.35 and 3.40). As shown by Preciso [39, 40], such a space of analytic functions is the convenient choice in order to ensure analyticity results for the composition operators in Schauder spaces (see also [10, §1, 6]). We note that, by replacing the weakly singular periodic kernels by periodic analogs of the fundamental solution of elliptic differential operators with constant coefficients, the results of the present paper allow to analyze perturbation problems for periodic boundary value problems for non-homogeneous differential equations by a potential theoretic method (cf. *e.g.*, [10, §5]). The reduction of periodic problems to integral equations is a powerful tool to investigate singular perturbation problems in periodic domains. As an example, we mention the asymptotic extension method of Ammari and Kang [2], Ammari, Kang, and Lee [3], and the functional analytic approach of the authors [11, 24]. As it is well known, periodic boundary value problems have a large variety of applications. Here we refer for example to Ammari and Kang [2, Chs. 2, 8], Kapanadze, Mishuris, and Pesetskaya [18, 19], Milton [30, Ch. 1], Mishuris and Slepyan [32], Mityushev, Pesetskaya, and Rogosin [33], and Movchan, Movchan, and Poulton [34]. In particular, such problems are relevant in the computation of effective properties of composite materials, which in turn can be justified by the homogenization theory (cf. *e.g.*, Allaire [1, Ch. 1], Bakhvalov and Panasenko [4], Bensoussan, Lions, and Papanicolaou [5, Ch. 1]).

The paper is organized as follows. In section 2, we introduce some basic notation. In section 3 we prove our main Theorems 3.35 and 3.40, where we consider volume potentials with periodic kernel taken in a suitable weighted space of analytic functions and density function belonging to a Roumieu class. We show that the map which takes the pair consisting of the kernel and of the density to a suitable restriction of the corresponding volume potential is bilinear and continuous with values in a Roumieu class. In particular, in Theorem 3.35

we consider the restriction of the volume potential to a periodic infinite union of bounded sets, while in Theorem 3.40 we study the restriction to the complement of a periodic infinite union of bounded sets. In the last Section 4, we present a class of kernels which are of the type considered in our main Theorems 3.35 and 3.40 and which play an important role in the treatment of periodic boundary value problems. Such a class consists of the q -periodic analogs of the fundamental solution of elliptic partial differential operators of second order.

2 Notation

We denote the norm on a normed space \mathcal{X} by $\|\cdot\|_{\mathcal{X}}$. Let \mathcal{X} and \mathcal{Y} be normed spaces. We endow the space $\mathcal{X} \times \mathcal{Y}$ with the norm defined by $\|(x, y)\|_{\mathcal{X} \times \mathcal{Y}} \equiv \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{Y}}$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$, while we use the Euclidean norm for \mathbb{R}^n . For standard definitions of Calculus in normed spaces, we refer to Deimling [12]. The symbol \mathbb{N} denotes the set of natural numbers including 0. Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\text{cl}\mathbb{D}$ denotes the closure of \mathbb{D} , and $\partial\mathbb{D}$ denotes the boundary of \mathbb{D} . The symbol $|\cdot|$ denotes the Euclidean modulus in \mathbb{R}^n or in \mathbb{C} . For all $R \in]0, +\infty[$, $x \in \mathbb{R}^n$, x_j denotes the j -th coordinate of x , and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let Ω be an open subset of \mathbb{R}^n . The space of m times continuously differentiable complex-valued functions on Ω is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. Let $f \in (C^m(\Omega))$. Then Df denotes the gradient of f . Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions f whose derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\text{cl}\Omega$ is denoted $C^m(\text{cl}\Omega)$. The subspace of $C^m(\text{cl}\Omega)$ whose functions have m -th order derivatives that are Hölder continuous with exponent $\alpha \in]0, 1[$ is denoted $C^{m, \alpha}(\text{cl}\Omega)$ (cf. *e.g.*, Gilbarg and Trudinger [13]). Let $\mathbb{D} \subseteq \mathbb{C}^n$. Then $C^{m, \alpha}(\text{cl}\Omega, \mathbb{D})$ denotes $\{f \in (C^{m, \alpha}(\text{cl}\Omega))^n : f(\text{cl}\Omega) \subseteq \mathbb{D}\}$. The subspace of $C^m(\text{cl}\Omega)$ of those functions f such that $f|_{\text{cl}(\Omega \cap \mathbb{B}_n(0, R))} \in C^{m, \alpha}(\text{cl}(\Omega \cap \mathbb{B}_n(0, R)))$ for all $R \in]0, +\infty[$ is denoted $C_{\text{loc}}^{m, \alpha}(\text{cl}\Omega)$.

Now let Ω be a bounded open subset of \mathbb{R}^n . Then $C^m(\text{cl}\Omega)$ and $C^{m, \alpha}(\text{cl}\Omega)$ are endowed with their usual norm and are well known to be Banach spaces (cf. *e.g.*, Troianiello [41, §1.2.1]). For the definition of a bounded open Lipschitz subset of \mathbb{R}^n , we refer for example to Nečas [35, §1.3]. We say that a bounded open subset Ω of \mathbb{R}^n is of class C^m or of class $C^{m, \alpha}$, if it is a manifold with boundary imbedded in \mathbb{R}^n of class C^m or $C^{m, \alpha}$, respectively (cf. *e.g.*, Gilbarg and Trudinger [13, §6.2]). We denote by ν_Ω the outward unit normal to $\partial\Omega$. For standard properties of functions in Schauder spaces, we refer the reader to Gilbarg and Trudinger [13] and to Troianiello [41] (see also [25, §2]).

We denote by $d\sigma$ the area element of a manifold imbedded in \mathbb{R}^n . We retain the standard notation for the Lebesgue spaces.

We note that throughout the paper ‘analytic’ means always ‘real analytic’. For the definition and properties of analytic operators, we refer to Deimling [12, §15].

If Ω is an open subset of \mathbb{R}^n , $k \in \mathbb{N}$, $\beta \in]0, 1[$, we set

$$C_b^k(\text{cl}\Omega) \equiv \{u \in C^k(\text{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^k(\text{cl}\Omega)$ with its usual norm

$$\|u\|_{C_b^k(\text{cl}\Omega)} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \text{cl}\Omega} |D^\gamma u(x)| \quad \forall u \in C_b^k(\text{cl}\Omega).$$

Then we set

$$C_b^{k,\beta}(\text{cl}\Omega) \equiv \{u \in C^{k,\beta}(\text{cl}\Omega) : D^\gamma u \text{ is bounded } \forall \gamma \in \mathbb{N}^n \text{ such that } |\gamma| \leq k\},$$

and we endow $C_b^{k,\beta}(\text{cl}\Omega)$ with its usual norm

$$\|u\|_{C_b^{k,\beta}(\text{cl}\Omega)} \equiv \sum_{|\gamma| \leq k} \sup_{x \in \text{cl}\Omega} |D^\gamma u(x)| + \sum_{|\gamma|=k} |D^\gamma u : \text{cl}\Omega|_\beta \quad \forall u \in C_b^{k,\beta}(\text{cl}\Omega),$$

where $|D^\gamma u : \text{cl}\Omega|_\beta$ denotes the β -Hölder constant of $D^\gamma u$.

We now fix once for all a natural number

$$n \in \mathbb{N} \setminus \{0, 1\}$$

and a periodicity cell

$$Q \equiv \Pi_{j=1}^n]0, q_{jj}[,$$

with

$$(q_{11}, \dots, q_{nn}) \in]0, +\infty[^n.$$

Then we denote by q the diagonal matrix

$$q \equiv \begin{pmatrix} q_{11} & 0 & \dots & 0 \\ 0 & q_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & q_{nn} \end{pmatrix} \quad (2.1)$$

and by $m_n(Q)$ the n -dimensional measure of the fundamental cell Q . Clearly,

$$q\mathbb{Z}^n \equiv \{qz : z \in \mathbb{Z}^n\}$$

is the set of vertices of a periodic subdivision of \mathbb{R}^n corresponding to the fundamental cell Q . Let $\text{cl}\Omega \subseteq Q$. Then we introduce the periodic domains

$$\mathbb{S}[\Omega] \equiv \bigcup_{z \in \mathbb{Z}^n} (qz + \Omega), \quad \mathbb{S}[\Omega]^- \equiv \mathbb{R}^n \setminus \text{cl}\mathbb{S}[\Omega].$$

A function u^i from $\text{cl}\mathbb{S}[\Omega]$ to \mathbb{R} is q -periodic if

$$u^i(x + q_{hh}e_h) = u^i(x) \quad \forall x \in \text{cl}\mathbb{S}[\Omega],$$

for all $h \in \{1, \dots, n\}$, and a function u^o from $\text{cl}\mathbb{S}[\Omega]^-$ to \mathbb{R} is q -periodic if

$$u^o(x + q_{hh}e_h) = u^o(x) \quad \forall x \in \text{cl}\mathbb{S}[\Omega]^-,$$

for all $h \in \{1, \dots, n\}$. Here $\{e_1, \dots, e_n\}$ denotes the canonical basis of \mathbb{R}^n .

We now introduce some functional spaces of q -periodic functions. If $k \in \mathbb{N} \cup \{\infty\}$ and $\beta \in]0, 1]$, then we set

$$C_q^k(\text{cl}\mathbb{S}[\Omega]) \equiv \{u \in C_b^k(\text{cl}\mathbb{S}[\Omega]) : u \text{ is } q\text{-periodic}\},$$

which we regard as a Banach subspace of $C_b^k(\text{clS}[\Omega])$, and

$$C_q^{k,\beta}(\text{clS}[\Omega]) \equiv \left\{ u \in C_b^{k,\beta}(\text{clS}[\Omega]) : u \text{ is } q\text{-periodic} \right\},$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\text{clS}[\Omega])$, and

$$C_q^k(\text{clS}[\Omega]^-) \equiv \left\{ u \in C_b^k(\text{clS}[\Omega]^-) : u \text{ is } q\text{-periodic} \right\},$$

which we regard as a Banach subspace of $C_b^k(\text{clS}[\Omega]^-)$, and

$$C_q^{k,\beta}(\text{clS}[\Omega]^-) \equiv \left\{ u \in C_b^{k,\beta}(\text{clS}[\Omega]^-) : u \text{ is } q\text{-periodic} \right\},$$

which we regard as a Banach subspace of $C_b^{k,\beta}(\text{clS}[\Omega]^-)$. We also set

$$C_q^k(\mathbb{R}^n \setminus q\mathbb{Z}^n) \equiv \left\{ u \in C^k(\mathbb{R}^n \setminus q\mathbb{Z}^n) : u(x + qz) = u(x) \forall x \in \mathbb{R}^n \setminus q\mathbb{Z}^n, \forall z \in \mathbb{Z}^n \right\}.$$

Next, we turn to introduce the Roumieu classes. For all bounded open subsets Ω of \mathbb{R}^n and $\rho \in]0, +\infty[$, we set

$$C_{\omega,\rho}^0(\text{cl}\Omega) \equiv \left\{ u \in C^\infty(\text{cl}\Omega) : \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C_{\omega,\rho}^0(\text{cl}\Omega)} \equiv \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}\Omega)} \quad \forall u \in C_{\omega,\rho}^0(\text{cl}\Omega).$$

As it is well known, the Roumieu class $(C_{\omega,\rho}^0(\text{cl}\Omega), \|\cdot\|_{C_{\omega,\rho}^0(\text{cl}\Omega)})$ is a Banach space.

If $\text{cl}\Omega \subseteq Q$ and $\rho > 0$, then we set

$$C_{q,\omega,\rho}^0(\text{clS}[\Omega]) \equiv \left\{ u \in C_q^\infty(\text{clS}[\Omega]) : \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}\Omega)} < +\infty \right\},$$

and

$$\|u\|_{C_{q,\omega,\rho}^0(\text{clS}[\Omega])} \equiv \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}\Omega)} \quad \forall u \in C_{q,\omega,\rho}^0(\text{clS}[\Omega]).$$

Similarly, we set

$$C_{q,\omega,\rho}^0(\text{clS}[\Omega]^-) \equiv \left\{ u \in C_q^\infty(\text{clS}[\Omega]^-) : \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}Q \setminus \Omega)} < +\infty \right\},$$

and

$$\|u\|_{C_{q,\omega,\rho}^0(\text{clS}[\Omega]^-)} \equiv \sup_{\beta \in \mathbb{N}^n} \frac{\rho^{|\beta|}}{|\beta|!} \|D^\beta u\|_{C^0(\text{cl}Q \setminus \Omega)} \quad \forall u \in C_{q,\omega,\rho}^0(\text{clS}[\Omega]^-).$$

The periodic Roumieu class $(C_{q,\omega,\rho}^0(\text{clS}[\Omega]), \|\cdot\|_{C_{q,\omega,\rho}^0(\text{clS}[\Omega])})$ and the periodic Roumieu class $(C_{q,\omega,\rho}^0(\text{clS}[\Omega]^-), \|\cdot\|_{C_{q,\omega,\rho}^0(\text{clS}[\Omega]^-)})$ are Banach spaces.

3 Periodic volume potentials corresponding to general kernels in Roumieu classes

We set

$$\tilde{Q} \equiv \Pi_{j=1}^n] - q_{jj}/2, q_{jj}/2[,$$

and

$$\mathbb{Z}_Q \equiv \{z \in \mathbb{Z}^n : qz \in \partial Q\} .$$

Obviously, \mathbb{Z}_Q has 2^n elements, and

$$Q \subseteq \bigcup_{z \in \mathbb{Z}_Q} (qz + \text{cl}\tilde{Q}) ,$$

and

$$m_n \left(Q \setminus \bigcup_{z \in \mathbb{Z}_Q} (qz + \tilde{Q}) \right) = 0 .$$

By exploiting the absolute continuity of a measure associated to an integrable function, we can prove the following elementary lemma (see also [10, §3]).

Lemma 3.1. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be q -periodic. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_{x+E} |f| dx \leq \epsilon ,$$

for all measurable subsets E of Q such that $m_n(E) \leq \delta$ and for all $x \in \mathbb{R}^n$.

Then we have the following.

Lemma 3.2. *Let $\lambda \in]0, n[$. Let $h \in C^0_q(\mathbb{R}^n \setminus q\mathbb{Z}^n)$. Let*

$$\sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)| |x|^\lambda < +\infty . \quad (3.3)$$

Then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\int_E |h(x-y)| dy \leq \epsilon ,$$

for all measurable subsets E of Q such that $m_n(E) \leq \delta$ and for all $x \in \mathbb{R}^n$.

Proof. We first prove that $h|_Q \in L^1(Q)$. Condition (3.3) implies the summability of h in \tilde{Q} . Moreover, by the periodicity of h we have

$$\int_Q |h(\eta)| d\eta \leq \sum_{z \in \mathbb{Z}_Q} \int_{qz + \tilde{Q}} |h(\eta)| d\eta = 2^n \int_{\tilde{Q}} |h(\eta)| d\eta . \quad (3.4)$$

Hence, the summability of h in Q follows by (3.4). Since $h|_Q \in L^1(Q)$ and h is q -periodic, we have $h \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $h(-\cdot)$ is also locally integrable and q -periodic. Therefore, there exists $\delta > 0$ such that

$$\int_E |h(x-y)| dy = \int_{-x+E} |h(-\eta)| d\eta \leq \epsilon \quad \forall x \in \mathbb{R}^n ,$$

for all measurable subsets E of Q such that $m_n(E) \leq \delta$. □

Next we introduce the following class of q -periodic functions which are singular in the punctured periodicity cell $\tilde{Q} \setminus \{0\}$.

Definition 3.5. Let $\lambda \in]0, +\infty[$. Then we denote by $A_{q,\lambda}^0$ the set of the functions $h \in C_q^0(\mathbb{R}^n \setminus q\mathbb{Z}^n)$ such that

$$\sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)| |x|^\lambda < +\infty,$$

and we set

$$\|h\|_{A_{q,\lambda}^0} \equiv \sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)| |x|^\lambda \quad \forall h \in A_{q,\lambda}^0.$$

One can readily verify that $(A_{q,\lambda}^0, \|\cdot\|_{A_{q,\lambda}^0})$ is a Banach space.

Proposition 3.6. Let $\lambda \in]0, n[$. Let Ω be a bounded open subset of \mathbb{R}^n such that $\text{cl}\Omega \subseteq Q$. Then the following statements hold.

(i) If $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(\Omega)$ and if $x \in \mathbb{R}^n$, then the function from Ω to \mathbb{R} which takes $y \in \Omega$ to $h(x-y)\varphi(y)$ is integrable.

(ii) If $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(\Omega)$, then the function $\mathcal{P}^+[h, \varphi]$ from \mathbb{R}^n to \mathbb{R} which takes $x \in \mathbb{R}^n$ to

$$\mathcal{P}^+[h, \varphi](x) \equiv \int_{\Omega} h(x-y)\varphi(y) dy \quad \forall x \in \mathbb{R}^n,$$

is continuous and q -periodic.

(iii) $\mathcal{P}^+[h, \varphi]$ is bounded and

$$\|\mathcal{P}^+[h, \varphi]\|_{L^\infty(\mathbb{R}^n)} \leq 2^n \int_{\tilde{Q}} |y|^{-\lambda} dy \|h\|_{A_{q,\lambda}^0} \|\varphi\|_{L^\infty(\Omega)},$$

for all $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(\Omega)$.

(iv) If $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(Q \setminus \text{cl}\Omega)$ and if $x \in \mathbb{R}^n$, then the function from $Q \setminus \text{cl}\Omega$ to \mathbb{R} which takes $y \in Q \setminus \text{cl}\Omega$ to $h(x-y)\varphi(y)$ is integrable.

(v) If $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(Q \setminus \text{cl}\Omega)$, then the function $\mathcal{P}^-[h, \varphi]$ from \mathbb{R}^n to \mathbb{R} which takes $x \in \mathbb{R}^n$ to

$$\mathcal{P}^-[h, \varphi](x) \equiv \int_{Q \setminus \text{cl}\Omega} h(x-y)\varphi(y) dy \quad \forall x \in \mathbb{R}^n,$$

is continuous and q -periodic.

(vi) $\mathcal{P}^-[h, \varphi]$ is bounded and

$$\|\mathcal{P}^-[h, \varphi]\|_{L^\infty(\mathbb{R}^n)} \leq 2^n \int_{\tilde{Q}} |y|^{-\lambda} dy \|h\|_{A_{q,\lambda}^0} \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)},$$

for all $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(Q \setminus \text{cl}\Omega)$.

Proof. If $(h, \varphi) \in A_{q,\lambda}^0 \times L^\infty(\Omega)$, then we have

$$|h(x-y)\varphi(y)| \leq \sum_{z \in \mathbb{Z}_Q} \chi_{qz+\tilde{Q}}(y) |h(x-y)| \|\varphi\|_{L^\infty(\Omega)} \quad \text{for a.a. } y \in \Omega, \quad (3.7)$$

for all $x \in \mathbb{R}^n$. Since h is q -periodic and \tilde{Q} is a q -periodicity cell, we have

$$\begin{aligned} \int_{qz+\tilde{Q}} |h(x-y)| dy &= \int_{x+qz+\tilde{Q}} |h(x-y)| dy \\ &= \int_{qz+\tilde{Q}} |h(-y)| dy = \int_{\tilde{Q}} |h(-y)| dy \\ &= \int_{\tilde{Q}} |h(-y)| | -y |^\lambda \frac{1}{|y|^\lambda} dy \leq \int_{\tilde{Q}} \frac{1}{|y|^\lambda} dy \|h\|_{A_{q,\lambda}^0} < +\infty \quad \forall x \in \mathbb{R}^n, \end{aligned} \quad (3.8)$$

for all $z \in \mathbb{Z}_Q$. Since \mathbb{Z}_Q has 2^n elements, statement (i) follows. Next we prove the continuity of $\mathcal{P}^+[h, \varphi]$. Let $x_0 \in \mathbb{R}^n$. Let $\epsilon > 0$. By Lemma 3.2, there exists $\delta \in]0, \frac{1}{2}[$ such that

$$\int_{(\xi+\delta\tilde{Q}) \cap Q} |h(x-y)| dy \leq \epsilon/2^{n+1} \quad \forall (\xi, x) \in \mathbb{R}^n \times \mathbb{R}^n. \quad (3.9)$$

Since $\delta \in]0, 1/2[$, the set

$$\mathbb{Z}(x_0, \delta) \equiv \left\{ z \in \mathbb{Z}^n : (x_0 + qz + \delta \text{cl}\tilde{Q}) \cap \text{cl}Q \neq \emptyset \right\} \quad (3.10)$$

has at most 2^n elements. Since the function $|h(x-y)|$ is continuous on the compact set

$$\left(x_0 + \frac{\delta}{2} \text{cl}\tilde{Q} \right) \times \left(\text{cl}Q \setminus \left(\bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta\tilde{Q}) \right) \right), \quad (3.11)$$

the function $|h(x - y)|$ has a maximum $\gamma(x_0, \delta)$ on such a set. Then by (3.9), we have that

$$\begin{aligned}
& \left| \mathcal{P}_q^+[h, \varphi](x) - \mathcal{P}_q^+[h, \varphi](x_0) \right| \\
& \leq \left| \int_{\Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})} h(x - y) \varphi(y) dy \right. \\
& \quad \left. - \int_{\Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})} h(x_0 - y) \varphi(y) dy \right| \\
& \quad + \int_{\bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q}) \cap Q} |h(x - y)| dy \|\varphi\|_{L^\infty(\Omega)} \\
& \quad + \int_{\bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q}) \cap Q} |h(x_0 - y)| dy \|\varphi\|_{L^\infty(\Omega)} \\
& \leq \left| \int_{\Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})} h(x - y) \varphi(y) dy \right. \\
& \quad \left. - \int_{\Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})} h(x_0 - y) \varphi(y) dy \right| \\
& \quad + (2^n(\epsilon/2^{n+1}) + 2^n(\epsilon/2^{n+1})) \|\varphi\|_{L^\infty(\Omega)} \\
& \leq \left\{ \int_{\Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})} |h(x - y) - h(x_0 - y)| dy \right. \\
& \quad \left. + 2^{n+1}(\epsilon/2^{n+1}) \right\} \|\varphi\|_{L^\infty(\Omega)} \quad \forall x \in \left(x_0 + \frac{\delta}{2} \tilde{Q} \right),
\end{aligned}$$

and that

$$|h(x - y)| \leq \gamma(x_0, \delta), \quad |h(x - y) - h(x_0 - y)| \leq 2\gamma(x_0, \delta), \quad (3.12)$$

for all $x \in x_0 + \frac{\delta}{2} \tilde{Q}$ and $y \in \Omega \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl} \tilde{Q})$. Then the above mentioned continuity of h and the Dominated Convergence Theorem imply that

$$\limsup_{x \rightarrow x_0} |\mathcal{P}_q^+[h, \varphi](x) - \mathcal{P}_q^+[h, \varphi](x_0)| \leq \|\varphi\|_{L^\infty(\Omega)} \epsilon,$$

and the continuity of $\mathcal{P}_q^+[h, \varphi]$ at x_0 follows. Since \mathbb{Z}_Q has 2^n elements, inequalities (3.7) and (3.8) imply the validity of statement (iii).

Next we consider statement (iv). If $(h, \varphi) \in A_{q, \lambda}^0 \times L^\infty(Q \setminus \text{cl} \Omega)$, then we have

$$|h(x - y) \varphi(y)| \leq \sum_{z \in \mathbb{Z}_Q} \chi_{qz + \tilde{Q}}(y) |h(x - y)| \|\varphi\|_{L^\infty(Q \setminus \text{cl} \Omega)} \quad \text{for a.a. } y \in Q \setminus \text{cl} \Omega, \quad (3.13)$$

for all $x \in \mathbb{R}^n$. By (3.8), we have

$$\int_{qz + \tilde{Q}} |h(x - y)| dy \leq \int_{\tilde{Q}} \frac{1}{|y|^\lambda} dy \|h\|_{A_{q, \lambda}^0} \quad \forall x \in \mathbb{R}^n. \quad (3.14)$$

Since \mathbb{Z}_Q has 2^n elements, statement (iv) follows. We now consider statement (v). Let $x_0 \in \mathbb{R}^n$. Let $\epsilon > 0$. Let $\delta \in]0, 1/2[$ be as in (3.9). Since $\delta < 1/2$, the set $\mathbb{Z}(x_0, \delta)$ has at

most 2^n elements (see (3.10)). Since the function $|h(x - y)|$ is continuous on the compact set in (3.11), the function $|h(x - y)|$ has a maximum $\gamma(x_0, \delta)$ on the set in (3.11). Then (3.9) implies that

$$\begin{aligned}
& \left| \mathcal{P}_q^- [h, \varphi](x) - \mathcal{P}_q^- [h, \varphi](x_0) \right| \\
& \leq \left| \int_{(Q \setminus \text{cl}\Omega) \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q})} h(x - y) \varphi(y) dy \right. \\
& \quad \left. - \int_{(Q \setminus \text{cl}\Omega) \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q})} h(x_0 - y) \varphi(y) dy \right| \\
& \quad + \int_{\bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q}) \cap Q} |h(x - y)| dy \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)} \\
& \quad + \int_{\bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q}) \cap Q} |h(x_0 - y)| dy \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)} \\
& \leq \left| \int_{(Q \setminus \text{cl}\Omega) \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q})} h(x - y) \varphi(y) dy \right. \\
& \quad \left. - \int_{(Q \setminus \text{cl}\Omega) \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q})} h(x_0 - y) \varphi(y) dy \right| \\
& \quad + (2^n (\epsilon/2^{n+1}) + 2^n (\epsilon/2^{n+1})) \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)} \\
& \leq \left\{ \int_{(Q \setminus \text{cl}\Omega) \setminus \bigcup_{z \in \mathbb{Z}(x_0, \delta)} (x_0 + qz + \delta \text{cl}\tilde{Q}) \cap Q} |h(x - y) - h(x_0 - y)| dy \right. \\
& \quad \left. + 2^{n+1} (\epsilon/2^{n+1}) \right\} \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)} \quad \forall x \in \left(x_0 + \frac{\delta}{2} \tilde{Q} \right).
\end{aligned}$$

Then inequality (3.12), and the continuity of h and the Dominated Convergence Theorem imply that

$$\limsup_{x \rightarrow x_0} \left| \mathcal{P}_q^- [h, \varphi](x) - \mathcal{P}_q^- [h, \varphi](x_0) \right| \leq \|\varphi\|_{L^\infty(Q \setminus \text{cl}\Omega)} \epsilon,$$

and the continuity of $\mathcal{P}_q^+ [h, \varphi]$ at x_0 follows. Since \mathbb{Z}_Q has 2^n elements, inequalities (3.13) and (3.14) imply the validity of statement (vi). \square

Next we introduce the following.

Definition 3.15. Let $\lambda \in]0, +\infty[$. Then we denote by $A_{q,\lambda}^1$ the set of the functions $h \in C_q^1(\mathbb{R}^n \setminus q\mathbb{Z}^n)$ such that

$$h \in A_{q,\lambda}^0, \quad \frac{\partial h}{\partial x_j} \in A_{q,\lambda+1}^0 \quad \forall j \in \{1, \dots, n\},$$

and we set

$$\|h\|_{A_{q,\lambda}^1} \equiv \|h\|_{A_{q,\lambda}^0} + \sum_{j=1}^n \left\| \frac{\partial h}{\partial x_j} \right\|_{A_{q,\lambda+1}^0} \quad \forall h \in A_{q,\lambda}^1.$$

One can readily verify that $(A_{q,\lambda}^1, \|\cdot\|_{A_{q,\lambda}^1})$ is a Banach space.

Proposition 3.16. *Let $\lambda \in]0, n-1[$. Let Ω be a bounded open subset of \mathbb{R}^n such that $\text{cl}\Omega \subseteq Q$. Then the following statements hold.*

(i) *If $(h, \varphi) \in A_{q,\lambda}^1 \times L^\infty(\Omega)$ and if $x \in \mathbb{R}^n$, then the functions from Ω to \mathbb{R} which take $y \in \Omega$ to $h(x-y)\varphi(y)$ and to $\frac{\partial h}{\partial x_j}(x-y)\varphi(y)$ for $j \in \{1, \dots, n\}$ are integrable.*

(ii) *If $(h, \varphi) \in A_{q,\lambda}^1 \times L^\infty(\Omega)$, then $\mathcal{P}^+[h, \varphi] \in C_q^1(\mathbb{R}^n)$ and*

$$\frac{\partial}{\partial x_j} \mathcal{P}^+[h, \varphi] = \mathcal{P}^+\left[\frac{\partial h}{\partial x_j}, \varphi\right] \quad \text{in } \mathbb{R}^n, \quad (3.17)$$

for all $j \in \{1, \dots, n\}$.

(iii) *If $(h, \varphi) \in A_{q,\lambda}^1 \times L^\infty(Q \setminus \text{cl}\Omega)$ and if $x \in \mathbb{R}^n$, then the functions from $Q \setminus \text{cl}\Omega$ to \mathbb{R} which take $y \in Q \setminus \text{cl}\Omega$ to $h(x-y)\varphi(y)$ and to $\frac{\partial h}{\partial x_j}(x-y)\varphi(y)$ for $j \in \{1, \dots, n\}$ are integrable.*

(iv) *If $(h, \varphi) \in A_{q,\lambda}^1 \times L^\infty(Q \setminus \text{cl}\Omega)$, then $\mathcal{P}^-[h, \varphi] \in C_q^1(\mathbb{R}^n)$ and*

$$\frac{\partial}{\partial x_j} \mathcal{P}^-[h, \varphi] = \mathcal{P}^-\left[\frac{\partial h}{\partial x_j}, \varphi\right] \quad \text{in } \mathbb{R}^n,$$

for all $j \in \{1, \dots, n\}$.

Proof. Statements (i), (iii) follow by Proposition 3.6 (i), (iv) applied to the functions $h, \frac{\partial h}{\partial x_j}$.

We now prove statement (ii). By Proposition 3.6 (ii), the two functions $\mathcal{P}^+[h, \varphi]$ and $\mathcal{P}^+\left[\frac{\partial h}{\partial x_j}, \varphi\right]$ are continuous and q -periodic in \mathbb{R}^n for all $j \in \{1, \dots, n\}$. Then it suffices to prove that $\frac{\partial}{\partial x_j} \mathcal{P}^+[h, \varphi]$ exists and that (3.17) holds. In order to prove (3.17), we proceed by a standard argument. Let $g \in C^\infty(\mathbb{R})$ be such that

$$g(t) = 0 \quad \forall t \in]-\infty, 1/8], \quad g(t) = 1 \quad \forall t \in [1/4, +\infty[.$$

Then we set

$$g_\delta(y) \equiv \begin{cases} g\left(\frac{|y-qb|}{\delta}\right) & \forall y \in qb + \mathbb{B}_n(0, \delta/4), \quad \forall b \in \mathbb{Z}^n, \\ 1 & \forall y \in \mathbb{R}^n \setminus \bigcup_{b \in \mathbb{Z}^n} (qb + \mathbb{B}_n(0, \delta/4)), \end{cases} \quad (3.18)$$

for all $\delta \in]0, 1[$. By definition, $g_\delta \in C_q^\infty(\mathbb{R}^n)$ and

$$\begin{aligned} g_\delta(x) &= 0 & \forall x \in \bigcup_{b \in \mathbb{Z}^n} (qb + \mathbb{B}_n(0, \delta/8)), \\ g_\delta(x) &= 1 & \forall x \in \mathbb{R}^n \setminus \bigcup_{b \in \mathbb{Z}^n} (qb + \mathbb{B}_n(0, \delta/4)), \end{aligned}$$

for all $\delta \in]0, 1[$. Then we set

$$u_\delta^+(x) \equiv \int_{\Omega} g_\delta(x-y)h(x-y)\varphi(y) dy \quad \forall x \in \mathbb{R}^n,$$

for all $\delta \in]0, 1[$. Clearly, $g_\delta(x - y)h(x - y)$ is of class C^1 in the variable $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. We now show that $u_\delta^+ \in C_q^1(\mathbb{R}^n)$ by applying the classical theorem of differentiability for integrals depending on a parameter. Let $x_0 \in \mathbb{R}^n$. Then we have

$$\begin{aligned} & \left| \frac{\partial}{\partial x_j} \{g_\delta(x - y)h(x - y)\varphi(y)\} \right| \\ & \leq \left| \frac{\partial}{\partial x_j} \{g_\delta(x - y)\} h(x - y)\varphi(y) \right| + \left| g_\delta(x - y) \frac{\partial}{\partial x_j} \{h(x - y)\} \varphi(y) \right| \end{aligned}$$

for all $x \in \mathbb{R}^n$ and for almost all $y \in \Omega$ and for all $\delta \in]0, 1[$. By (3.18), if $z \in \mathbb{Z}^n$ we have

$$\left| \frac{\partial}{\partial \xi_j} g_\delta(\xi) \right| = \left| \frac{\partial}{\partial \xi_j} \left(g \left(\frac{|\xi - qz|}{\delta} \right) \right) \right| = \left| g' \left(\frac{|\xi - qz|}{\delta} \right) \frac{\xi_j - q_{jj}z_j}{|\xi - qz|} \frac{1}{\delta} \right| \leq \sup |g'| \frac{1}{\delta},$$

for all $\xi \in qz + \mathbb{B}_n(0, \delta/4)$ and for all $\delta \in]0, 1[$, and

$$\frac{\partial}{\partial \xi_j} g_\delta(\xi) = 0 \quad \forall \xi \in \mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/4)),$$

and accordingly

$$\left| \frac{\partial}{\partial \xi_j} g_\delta(\xi) \right| \leq \frac{1}{\delta} \sup |g'| \quad \forall \xi \in \mathbb{R}^n,$$

for all $\delta \in]0, 1[$. Since the functions h and $\frac{\partial h}{\partial x_j}$ are q -periodic and continuous in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, the same functions are bounded on $\mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/16))$. Since g_δ vanishes on $\bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/8))$, we have

$$\begin{aligned} & \left| \frac{\partial}{\partial x_j} \{g_\delta(x - y)h(x - y)\varphi(y)\} \right| \\ & \leq \frac{1}{\delta} \sup |g'| \left(\sup_{\mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/16))} |h| \right) |\varphi(y)| \\ & \quad + \max_{j \in \{1, \dots, n\}} \left(\sup_{\mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/16))} \left| \frac{\partial h}{\partial x_j} \right| \right) \|g\|_{L^\infty(\mathbb{R})} |\varphi(y)|, \end{aligned}$$

for all $y \in \Omega$ and $x \in \mathbb{R}^n$. Since $\varphi \in L^1(\Omega)$, the differentiability theorem for integrals depending on a parameter implies that

$$\frac{\partial u_\delta^+}{\partial x_j}(x) = \int_\Omega \frac{\partial}{\partial x_j} [g_\delta(x - y)h(x - y)] \varphi(y) dy \quad \forall x \in \mathbb{R}^n.$$

Hence, $u_\delta^+ \in C_q^1(\mathbb{R}^n)$. In order to prove that $\mathcal{P}^+[h, \varphi] \in C_q^1(\mathbb{R}^n)$, it suffices to show that

$$\lim_{\delta \rightarrow 0} u_\delta^+ = \mathcal{P}^+[h, \varphi] \quad \text{uniformly in } \mathbb{R}^n, \quad (3.19)$$

$$\lim_{\delta \rightarrow 0} \frac{\partial u_\delta^+}{\partial x_j} = \mathcal{P}^+\left[\frac{\partial h}{\partial x_j}, \varphi\right] \quad \text{uniformly in } \mathbb{R}^n, \quad (3.20)$$

for all $j \in \{1, \dots, n\}$. We first prove (3.19). Since $1 - g_\delta(x - y) = 0$ when $x - y \in \mathbb{R}^n \setminus \bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(0, \delta/4))$ and $\mathbb{B}_n(0, \delta/4) \subseteq \tilde{Q}$ for all $\delta \in]0, \min_{j=1, \dots, n} q_{jj}[$ and \mathbb{Z}_Q has 2^n elements, we have

$$\begin{aligned} & |\mathcal{P}^+[h, \varphi](x) - u_\delta^+(x)| \\ &= \left| \int_{(\bigcup_{z \in \mathbb{Z}^n} (qz + \mathbb{B}_n(x, \delta/4))) \cap \Omega} (1 - g_\delta(|x - y|)) h(x - y) \varphi(y) dy \right| \\ &\leq (1 + \|g\|_{L^\infty(\mathbb{R})}) \|\varphi\|_{L^\infty(\Omega)} \int_{\bigcup_{z \in \mathbb{Z}_Q} (qz + \mathbb{B}_n(0, \delta/4))} |h(\eta)| d\eta \\ &\leq (1 + \|g\|_{L^\infty(\mathbb{R})}) \|\varphi\|_{L^\infty(\Omega)} 2^n \|h\|_{A_{q, \lambda}^0} \int_{\delta \tilde{Q}} |y|^{-\lambda} dy \quad \forall x \in \mathbb{R}^n, \end{aligned}$$

for all $\delta \in]0, \min\{1, \min_{j=1, \dots, n} q_{jj}\}[$. Since $|y|^{-\lambda}$ is integrable in \tilde{Q} , $\lim_{\delta \rightarrow 0} \int_{\delta \tilde{Q}} |y|^{-\lambda} dy = 0$ and the limiting relation (3.19) follows.

We now consider the limiting relation (3.20). Since g' vanishes outside of the interval $[1/8, 1/4]$, the same argument we have exploited above shows that

$$\begin{aligned} & \left| \mathcal{P}^+ \left[\frac{\partial h}{\partial x_j}, \varphi \right](x) - \frac{\partial u_\delta^+}{\partial x_j}(x) \right| \\ &\leq \left| \int_{\Omega} (1 - g_\delta(x - y)) \frac{\partial h}{\partial x_j}(x - y) \varphi(y) dy \right| \\ &\quad + \left| \int_{\Omega} \frac{\partial}{\partial x_j} \{g_\delta(x - y)\} h(x - y) \varphi(y) dy \right| \\ &\leq (1 + \|g\|_{L^\infty(\mathbb{R})}) \|\varphi\|_{L^\infty(\Omega)} 2^n \left\| \frac{\partial h}{\partial x_j} \right\|_{A_{q, \lambda+1}^0} \int_{\delta \tilde{Q}} |y|^{-\lambda-1} dy \\ &\quad + \frac{1}{\delta} \sup |g'| \|\varphi\|_{L^\infty(\Omega)} 2^n \|h\|_{A_{q, \lambda}^0} \int_{\delta \tilde{Q}} |y|^{-\lambda} dy \end{aligned}$$

for all $\delta \in]0, \min\{1, \min_{j=1, \dots, n} q_{jj}\}[$ and $j \in \{1, \dots, n\}$. Next we note that

$$\int_{\delta \tilde{Q}} |y|^{-\lambda-1} dy = \delta^{n-\lambda-1} \int_{\tilde{Q}} |y|^{-\lambda-1} dy, \quad \frac{1}{\delta} \int_{\delta \tilde{Q}} |y|^{-\lambda} dy = \frac{1}{\delta} \delta^{n-\lambda} \int_{\tilde{Q}} |y|^{-\lambda} dy.$$

Since $n - \lambda - 1 > 0$, we conclude that the limiting relation (3.20) holds. Hence, we have $\mathcal{P}^+[h, \varphi] \in C_q^1(\mathbb{R}^n)$.

The proof of statement (iv) follows the lines of that of statement (ii) by replacing the integration in Ω with that on $Q \setminus \text{cl}\Omega$, and it is accordingly omitted. \square

Next we prove the following lemma.

Lemma 3.21. *Let $\lambda \in]0, n - 1[$. Let Ω be a bounded open Lipschitz subset of \mathbb{R}^n such that $\text{cl}\Omega \subseteq Q$. Then the following statements hold.*

(i) Let $\Omega \neq \emptyset$. If $(h, \varphi) \in A_{q,\lambda}^1 \times C^1(\text{cl}\Omega)$, then

$$\frac{\partial}{\partial x_j} \mathcal{P}^+[h, \varphi](x) = \mathcal{P}^+[h, \frac{\partial \varphi}{\partial x_j}](x) - \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y, \quad (3.22)$$

for all $x \in \mathbb{R}^n$.

(ii) Let $\Omega \neq \emptyset$. If $(h, \varphi) \in A_{q,\lambda}^1 \times C^1(\text{cl}Q \setminus \Omega)$, then

$$\begin{aligned} \frac{\partial}{\partial x_j} \mathcal{P}^-[h, \varphi](x) &= \mathcal{P}^-[h, \frac{\partial \varphi}{\partial x_j}](x) \\ &+ \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y - \int_{\partial Q} h(x-y)\varphi(y)(\nu_Q)_j(y) d\sigma_y, \end{aligned} \quad (3.23)$$

for all $x \in \mathbb{R}^n$.

(iii) Let $\Omega = \emptyset$. If $(h, \varphi) \in A_{q,\lambda}^1 \times C^1(\text{cl}Q)$, then

$$\frac{\partial}{\partial x_j} \mathcal{P}^-[h, \varphi](x) = \mathcal{P}^-[h, \frac{\partial \varphi}{\partial x_j}](x) - \int_{\partial Q} h(x-y)\varphi(y)(\nu_Q)_j(y) d\sigma_y, \quad (3.24)$$

for all $x \in \mathbb{R}^n$.

Proof. We first consider statement (i). Again, we proceed by a classical argument. By the previous proposition, we have

$$\begin{aligned} \frac{\partial}{\partial x_j} \mathcal{P}^+[h, \varphi](x) &= \int_{\Omega} \frac{\partial h}{\partial x_j}(x-y)\varphi(y) dy \\ &= - \int_{\Omega} \frac{\partial}{\partial y_j}(h(x-y))\varphi(y) dy \\ &= - \int_{\Omega} \frac{\partial}{\partial y_j}(h(x-y)\varphi(y)) dy + \int_{\Omega} h(x-y)\frac{\partial \varphi}{\partial y_j}(y) dy \quad \forall x \in \mathbb{R}^n. \end{aligned} \quad (3.25)$$

Since the singularities of the kernel $h(x-y)$ are weak and h is continuous in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and $\varphi(\cdot)(\nu_\Omega)_j(\cdot)$ is integrable on $\partial\Omega$, the Vitali Convergence Theorem implies that the integral $\int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y$ is continuous and q -periodic in $x \in \mathbb{R}^n$. Then Propositions 3.6 (ii) and 3.16 (ii) imply that both the left and the right hand sides of (3.22) are continuous and q -periodic in $x \in \mathbb{R}^n$. Hence, it suffices to verify (3.22) on $\Omega \cup (Q \setminus \text{cl}\Omega)$. We first consider case $x \in \Omega$. Thus we now fix $x \in \Omega$, and we take $\epsilon_x \in]0, \text{dist}(x, \partial\Omega)[$. Then $\text{cl}\mathbb{B}_n(x, \epsilon) \subseteq \Omega$ and the set

$$\Omega_\epsilon \equiv \Omega \setminus \text{cl}\mathbb{B}_n(x, \epsilon)$$

is an open Lipschitz set for all $\epsilon \in]0, \epsilon_x[$. By the Divergence Theorem, we have

$$\begin{aligned} &\int_{\Omega} \frac{\partial}{\partial y_j}(h(x-y)\varphi(y)) dy \\ &= \int_{\Omega_\epsilon} \frac{\partial}{\partial y_j}(h(x-y)\varphi(y)) dy + \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial}{\partial y_j}(h(x-y)\varphi(y)) dy \\ &= \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y + \int_{\partial\mathbb{B}_n(x, \epsilon)} h(x-y)\varphi(y)\frac{x_j - y_j}{|x-y|} d\sigma_y \\ &\quad - \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial h}{\partial x_j}(x-y)\varphi(y) dy + \int_{\mathbb{B}_n(x, \epsilon)} h(x-y)\frac{\partial \varphi}{\partial y_j}(y) dy, \end{aligned} \quad (3.26)$$

for all $\epsilon \in]0, \epsilon_x[$. By the inclusion $\text{cl}\mathbb{B}_n(x, \epsilon_x) \subseteq \Omega \subseteq Q$, we have $\text{cl}\mathbb{B}_n(0, \epsilon_x) \subseteq \tilde{Q}$ and

$$\begin{aligned} & \left| \int_{\partial\mathbb{B}_n(x, \epsilon)} h(x-y)\varphi(y) \frac{x_j - y_j}{|x-y|} d\sigma_y \right| \\ & \leq \left(\sup_{x \in \text{cl}\mathbb{B}_n(0, \epsilon_x) \setminus \{0\}} |h(x)| |x|^\lambda \right) \|\varphi\|_{L^\infty(\Omega)} \int_{\partial\mathbb{B}_n(0, \epsilon)} |y|^{-\lambda} d\sigma_y \\ & \leq \left(\sup_{x \in \tilde{Q} \setminus \{0\}} |h(x)| |x|^\lambda \right) \|\varphi\|_{L^\infty(\Omega)} s_n \epsilon^{n-1-\lambda}, \end{aligned} \quad (3.27)$$

for all $\epsilon \in]0, \epsilon_x[$. By Proposition 3.16 (i) the functions $\frac{\partial h}{\partial x_j}(x-y)\varphi(y)$, $h(x-y)\frac{\partial \varphi}{\partial x_j}(y)$ are integrable in the variable $y \in \Omega$. Then we have

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial h}{\partial x_j}(x-y)\varphi(y) dy &= 0, \\ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{B}_n(x, \epsilon)} h(x-y)\frac{\partial \varphi}{\partial y_j}(y) dy &= 0. \end{aligned} \quad (3.28)$$

By (3.27) and (3.28), we can take the limit as ϵ tends to 0 in (3.26) and deduce that

$$\int_{\Omega} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) dy = \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y. \quad (3.29)$$

Then equality (3.25) implies that formula (3.22) holds.

We now turn to consider case $x \in Q \setminus \text{cl}\Omega$. Since $h(x-y)\varphi(y)$ is continuously differentiable in $y \in \text{cl}\Omega$, the Divergence Theorem implies that (3.29) holds. Hence, equality (3.25) implies that equality (3.22) holds.

We now prove statement (ii). By the previous proposition, we have

$$\begin{aligned} \frac{\partial}{\partial x_j} \mathcal{P}^- [h, \varphi](x) &= \int_{Q \setminus \text{cl}\Omega} \frac{\partial h}{\partial x_j}(x-y)\varphi(y) dy \\ &= - \int_{Q \setminus \text{cl}\Omega} \frac{\partial}{\partial y_j} (h(x-y))\varphi(y) dy \\ &= - \int_{Q \setminus \text{cl}\Omega} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) dy + \int_{Q \setminus \text{cl}\Omega} h(x-y)\frac{\partial \varphi}{\partial y_j}(y) dy, \end{aligned} \quad (3.30)$$

for all $x \in \mathbb{R}^n$. Since the singularities of the kernel $h(x-y)$ are weak and h is continuous in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and $\varphi(\cdot)(\nu_\Omega)_j(\cdot)$, $\varphi(\cdot)(\nu_Q)_j(\cdot)$ are integrable on $\partial\Omega$ and on ∂Q , respectively, the Vitali Convergence Theorem implies that the integrals on $\partial\Omega$ and on ∂Q in the right hand side of (3.23) are continuous and q -periodic in $x \in \mathbb{R}^n$. Then Propositions 3.6 (v) and 3.16 (iv) imply that both the left and the right hand sides of (3.23) are continuous in $x \in \mathbb{R}^n$. Hence, it suffices to verify that equality (3.23) holds for $x \in \Omega \cup (Q \setminus \text{cl}\Omega)$. We first consider case $x \in \Omega$. Since $h(x-y)\varphi(y)$ is continuously differentiable in $y \in \text{cl}Q \setminus \Omega$, the Divergence Theorem implies that

$$\begin{aligned} & \int_{Q \setminus \text{cl}\Omega} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) dy \\ &= \int_{\partial Q} h(x-y)\varphi(y)(\nu_Q)_j(y) d\sigma_y - \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) d\sigma_y. \end{aligned} \quad (3.31)$$

Hence, equality (3.30) implies that equality (3.23) holds. Next we consider the case when $x \in Q \setminus \text{cl}\Omega$. Let $\epsilon_x \in]0, \text{dist}(x, \partial(Q \setminus \text{cl}\Omega))]$. Then $\text{cl}\mathbb{B}_n(x, \epsilon) \subseteq Q \setminus \text{cl}\Omega$ and the set

$$V_\epsilon \equiv (Q \setminus \text{cl}\Omega) \setminus \text{cl}\mathbb{B}_n(x, \epsilon)$$

is an open Lipschitz set for all $\epsilon \in]0, \epsilon_x[$. By the Divergence Theorem, we have

$$\begin{aligned} & \int_{Q \setminus \text{cl}\Omega} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) \, dy & (3.32) \\ &= \int_{V_\epsilon} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) \, dy + \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial}{\partial y_j} (h(x-y)\varphi(y)) \, dy \\ &= \int_{\partial Q} h(x-y)\varphi(y)(\nu_Q)_j(y) \, d\sigma_y - \int_{\partial\Omega} h(x-y)\varphi(y)(\nu_\Omega)_j(y) \, d\sigma_y \\ & \quad + \int_{\partial\mathbb{B}_n(x, \epsilon)} h(x-y)\varphi(y) \frac{x_j - y_j}{|x-y|} \, d\sigma_y \\ & \quad - \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial h}{\partial x_j} (x-y)\varphi(y) \, dy + \int_{\mathbb{B}_n(x, \epsilon)} h(x-y) \frac{\partial \varphi}{\partial y_j} (y) \, dy \end{aligned}$$

for all $\epsilon \in]0, \epsilon_x[$. Then by arguing precisely as in the proof of statement (i), we can prove that

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \int_{\partial\mathbb{B}_n(x, \epsilon)} h(x-y)\varphi(y) \frac{x_j - y_j}{|x-y|} \, dy = 0, \\ & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{B}_n(x, \epsilon)} \frac{\partial h}{\partial x_j} (x-y)\varphi(y) \, dy = 0, \\ & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{B}_n(x, \epsilon)} h(x-y) \frac{\partial \varphi}{\partial y_j} (y) \, dy = 0. \end{aligned}$$

Then by taking the limit in (3.32) as ϵ tends to 0, we deduce that equality (3.31) holds. Then equality (3.30) implies that formula (3.23) holds. The proof of statement (iii) can be effected by a simplification of the proof of statement (ii) and it is accordingly omitted. \square

Remark. Under the same assumptions of Lemma 3.21 (ii), (iii), if φ is the restriction of an element of $C^1_q(\text{cl}\mathbb{S}[\Omega]^-)$, then we have

$$\int_{\partial Q} h(x-y)\varphi(y)(\nu_Q)_j(y) \, d\sigma_y = 0, \quad (3.33)$$

for all $j \in \{1, \dots, n\}$, and the second integral in the right hand side of (3.23) and the integral in the right hand side of (3.24) vanish.

Next we introduce a class of weakly singular q -periodic kernels, which we consider in our main results.

Definition 3.34. Let G be an open subset of \mathbb{R}^n such that $\text{cl}G$ is bounded and contained in $\mathbb{R}^n \setminus q\mathbb{Z}^n$. Let $\rho \in]0, +\infty[$. Then we set

$$H_q^{\lambda, \rho}(G) \equiv \{h \in A_{q, \lambda}^1 : h|_{\text{cl}G} \in C_{\omega, \rho}^0(\text{cl}G)\},$$

and we set

$$\|h\|_{H_q^{\lambda, \rho}(G)} \equiv \|h\|_{A_{q, \lambda}^1} + \|h\|_{C_{\omega, \rho}^0(\text{cl}G)} \quad \forall h \in H_q^{\lambda, \rho}(G).$$

Here we understand that if $G = \emptyset$, then $H_q^{\lambda, \rho}(G) = A_{q, \lambda}^1$ and $\|\cdot\|_{H_q^{\lambda, \rho}(G)} = \|\cdot\|_{A_{q, \lambda}^1}$.

Since both $A_{q, \lambda}^1$ and $C_{\omega, \rho}^0(\text{cl}G)$ are Banach spaces, also $(H_q^{\lambda, \rho}(G), \|\cdot\|_{H_q^{\lambda, \rho}(G)})$ is a Banach space.

We are now ready to prove here below in Theorems 3.35 and 3.40 our main results on the continuity of the bilinear maps which take the pair consisting of a kernel and of a density function to the volume potentials. In Theorem 3.35 we consider the restriction of the volume potential to a domain $\mathbb{S}[\Omega_1]$, with $\text{cl}\Omega_1 \subseteq \Omega \subseteq \text{cl}\Omega \subseteq Q$, while in Theorems 3.40 we study the restriction to a domain $\mathbb{S}[\Omega_2]^-$, with $\text{cl}\Omega \subseteq \Omega_2 \subseteq \text{cl}\Omega_2 \subseteq Q$.

Theorem 3.35. Let $\rho \in]0, +\infty[$, $\lambda \in]0, n-1[$. Let Ω be a bounded open Lipschitz subset of \mathbb{R}^n such that $\text{cl}\Omega \subseteq Q$. Let Ω_1 be a nonempty open subset of \mathbb{R}^n such that $\text{cl}\Omega_1 \subseteq \Omega$. Let G be an open relatively compact neighborhood of the set

$$\{t - s : t \in \text{cl}\Omega_1, s \in \partial\Omega\},$$

such that $(\text{cl}G) \cap q\mathbb{Z}^n = \emptyset$. Then the following statements hold.

- (i) The restriction of $\mathcal{P}^+[h, \varphi]$ to $\text{cl}\mathbb{S}[\Omega_1]$ belongs to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_1])$ for all pairs $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$ and the map from $H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$ to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_1])$ which takes (h, φ) to $\mathcal{P}^+[h, \varphi]|_{\text{cl}\mathbb{S}[\Omega_1]}$ is bilinear and continuous.
- (ii) The restriction of $\mathcal{P}^-[h, \varphi|_{Q \setminus \text{cl}\Omega}]$ to $\text{cl}\mathbb{S}[\Omega_1]$ belongs to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_1])$ for all pairs $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega]^-)$ and the map from $H_q^{\lambda, \rho}(G) \times C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega]^-)$ to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_1])$ which takes (h, φ) to $\mathcal{P}^-[h, \varphi|_{Q \setminus \text{cl}\Omega}]|_{\text{cl}\mathbb{S}[\Omega_1]}$ is bilinear and continuous.

Proof. We first consider statement (i), and we prove that if $m \in \mathbb{N} \setminus \{0\}$ and if $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C^m(\text{cl}\Omega)$, then $\mathcal{P}^+[h, \varphi]|_{\text{cl}\mathbb{S}[\Omega_1]} \in C_q^m(\text{cl}\mathbb{S}[\Omega_1])$ and

$$\begin{aligned} \partial^\beta \mathcal{P}^+[h, \varphi](x) &= \mathcal{P}^+[h, \partial^\beta \varphi](x) \\ &= \sum_{k=1}^n \sum_{l_k=0}^{\beta_k-1} \partial_{x_n}^{\beta_n} \dots \partial_{x_{k+1}}^{\beta_{k+1}} \partial_{x_k}^{l_k} \left\{ \int_{\partial\Omega} h(x-y) \right. \\ &\quad \left. \times \left(\partial_{y_k}^{\beta_k-1-l_k} \partial_{y_{k-1}}^{\beta_{k-1}} \dots \partial_{y_1}^{\beta_1} \varphi(y) \right) (\nu_\Omega)_k(y) d\sigma_y \right\}, \end{aligned} \tag{3.36}$$

for all $x \in \text{cl}\mathbb{S}[\Omega_1]$ and for all $\beta \in \mathbb{N}^n$ such that $|\beta| \leq m$, where we understand that $\sum_{l_k=0}^{\beta_k-1}$ is omitted if $\beta_k = 0$. Since h is q -periodic, it suffices to prove (3.36) for $x \in \text{cl}\Omega_1$. If $m = 1$, then the statement follows by Lemma 3.21 (i).

Next we assume that the statement holds for m and we prove it for $m + 1$. Let $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C^{m+1}(\text{cl}\Omega)$. By the inductive assumption, $\mathcal{P}^+[h, \frac{\partial \varphi}{\partial x_j}]_{|\text{cl}\mathbb{S}[\Omega_1]} \in C_q^m(\text{cl}\mathbb{S}[\Omega_1])$ for all $j \in \{1, \dots, n\}$.

Since $h \in C^m(\text{cl}G)$ and $\varphi, (\nu_\Omega)_j \in C^0(\partial\Omega)$, the classical differentiability theorem for integrals depending on a parameter implies that the second term in the right hand side of formula (3.22) defines a function of class $C_q^m(\text{cl}\mathbb{S}[\Omega_1])$. Then formula (3.22) implies that $\frac{\partial}{\partial x_j} \mathcal{P}^+[h, \varphi]$ belongs to $C_q^m(\text{cl}\mathbb{S}[\Omega_1])$. Hence, $\mathcal{P}^+[h, \varphi]_{|\text{cl}\mathbb{S}[\Omega_1]} \in C_q^{m+1}(\text{cl}\mathbb{S}[\Omega_1])$.

Next one proves the formula for the derivatives by following the lines of the corresponding argument of [22, p. 856]: first one proves the formula for $\partial^\beta = \partial_{x_j}^{\beta_j}$ by finite induction on β_j , then one proves the formula for $\partial^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_j}^{\beta_j}$ by finite induction on $j \in \{1, \dots, n\}$, and finally one deduces that the formula holds for $|\beta| \leq m + 1$.

If $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C^\infty(\text{cl}\Omega)$, then by applying the above statement for all $m \in \mathbb{N} \setminus \{0\}$ we deduce that $\mathcal{P}^+[h, \varphi]_{|\text{cl}\mathbb{S}[\Omega_1]}$ belongs to $C_q^\infty(\text{cl}\mathbb{S}[\Omega_1])$ and that formula (3.36) holds for all order derivatives.

We now assume that $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$ and we turn to estimate the sup-norm in $\text{cl}\Omega_1$ of the sum in the right hand side of (3.36), which we denote by I .

We abbreviate by $I(k, l_k)$ the (k, l_k) -th term in the sum I , and we now estimate the supremum of $I(k, l_k)$ in $\text{cl}\Omega_1$. We can clearly assume that $\beta_k > 0$. Then we have

$$\begin{aligned} & \sup_{\text{cl}\Omega_1} |I(k, l_k)| \tag{3.37} \\ &= \sup_{x \in \text{cl}\Omega_1} \left| \partial_{x_n}^{\beta_n} \dots \partial_{x_{k+1}}^{\beta_{k+1}} \partial_{x_k}^{l_k} \left\{ \int_{\partial\Omega} h(x-y) \partial_{y_k}^{\beta_k-1-l_k} \partial_{y_{k-1}}^{\beta_{k-1}} \dots \partial_{y_1}^{\beta_1} \varphi(y) (\nu_\Omega)_k(y) d\sigma_y \right\} \right| \\ &\leq \int_{\partial\Omega} \sup_{\xi \in G} \left| \partial_{\xi_n}^{\beta_n} \dots \partial_{\xi_{k+1}}^{\beta_{k+1}} \partial_{\xi_k}^{l_k} h(\xi) \right| \left| \partial_{y_k}^{\beta_k-1-l_k} \partial_{y_{k-1}}^{\beta_{k-1}} \dots \partial_{y_1}^{\beta_1} \varphi(y) \right| d\sigma_y. \end{aligned}$$

Since $h \in H_q^{\lambda, \rho}(G)$, we have

$$\sup_{\xi \in G} \left| \partial_{\xi_n}^{\beta_n} \dots \partial_{\xi_{k+1}}^{\beta_{k+1}} \partial_{\xi_k}^{l_k} h(\xi) \right| \leq \|h\|_{H_q^{\lambda, \rho}(G)} \frac{(\beta_n + \dots + \beta_{k+1} + l_k)!}{\rho^{\beta_n + \dots + \beta_{k+1} + l_k}}.$$

Moreover,

$$\left| \partial_{y_k}^{\beta_k-1-l_k} \partial_{y_{k-1}}^{\beta_{k-1}} \dots \partial_{y_1}^{\beta_1} \varphi(y) \right| \leq \|\varphi\|_{C_{\omega, \rho}^0(\text{cl}\Omega)} \frac{(\beta_1 + \dots + \beta_{k-1} + \beta_k - 1 - l_k)!}{\rho^{\beta_1 + \dots + \beta_{k-1} + \beta_k - 1 - l_k}},$$

for all $y \in \text{cl}\Omega$. Then we have

$$\begin{aligned} \sup_{\text{cl}\Omega_1} |I(k, l_k)| &\leq m_{n-1}(\partial\Omega) \|h\|_{H_q^{\lambda, \rho}(G)} \|\varphi\|_{C_{\omega, \rho}^0(\text{cl}\Omega)} \frac{1}{\rho^{|\beta|-1}} \\ &\quad \times (\beta_n + \dots + \beta_{k+1} + l_k)! (\beta_1 + \dots + \beta_{k-1} + \beta_k - 1 - l_k)!, \end{aligned}$$

where $m_{n-1}(\partial\Omega)$ denotes the $(n-1)$ dimensional Lebesgue measure of $\partial\Omega$. Next we note that

$$m_1! m_2! \leq (m_1 + m_2)!,$$

for all $m_1, m_2 \in \mathbb{N}$. Then we have

$$\sup_{\text{cl}\Omega_1} |I(k, l_k)| \leq m_{n-1}(\partial\Omega) \|h\|_{H_q^{\lambda, \rho}(G)} \|\varphi\|_{C_{\omega, \rho}^0(\text{cl}\Omega)} \frac{(|\beta| - 1)!}{\rho^{|\beta| - 1}}.$$

Hence,

$$\sup_{\text{cl}\Omega_1} |I| \leq n\rho m_{n-1}(\partial\Omega) \|h\|_{H_q^{\lambda, \rho}(G)} \|\varphi\|_{C_{\omega, \rho}^0(\text{cl}\Omega)} \frac{|\beta|!}{\rho^{|\beta|}}. \quad (3.38)$$

On the other hand, Proposition 3.6 (iii) implies that

$$\|\mathcal{P}^+[h, \partial^\beta \varphi]\|_{L^\infty(\Omega)} \leq 2^n \int_{\tilde{Q}} |y|^{-\lambda} dy \|h\|_{A_{q, \lambda}^0} \|\partial^\beta \varphi\|_{L^\infty(\Omega)}. \quad (3.39)$$

Then equality (3.36) and inequalities (3.38) and (3.39) imply that there exists $C > 0$ such that

$$\|\partial^\beta \mathcal{P}^+[h, \varphi]_{|\Omega_1}\|_{L^\infty(\Omega_1)} \leq C \|h\|_{H_q^{\lambda, \rho}(G)} \|\varphi\|_{C_{\omega, \rho}^0(\text{cl}\Omega)} \frac{|\beta|!}{\rho^{|\beta|}} \quad \forall \beta \in \mathbb{N}^n,$$

for all $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$. Hence, statement (i) holds true.

The proof of statement (ii) follows the lines of the proof of statement (i). We only point out that formula (3.36) holds if we replace \mathcal{P}^+ by \mathcal{P}^- , and the minus sign in the right hand side by a plus sign (see also (3.30)). \square

Then, if $\text{cl}\Omega \subseteq \Omega_2 \subseteq \text{cl}\Omega_2 \subseteq Q$, we have the following result in $\mathbb{S}[\Omega_2]^-$.

Theorem 3.40. *Let $\rho \in]0, +\infty[$, $\lambda \in]0, n - 1[$. Let Ω be a bounded open Lipschitz subset of \mathbb{R}^n such that $\text{cl}\Omega \subseteq Q$. Let Ω_2 be an open subset of \mathbb{R}^n such that*

$$\text{cl}\Omega \subseteq \Omega_2 \subseteq \text{cl}\Omega_2 \subseteq Q.$$

Let G be an open relatively compact subset of \mathbb{R}^n such that $(\text{cl}G) \cap q\mathbb{Z}^n = \emptyset$.

If $\Omega \neq \emptyset$, then we assume that G is an open neighborhood of the set

$$\{t - s : t \in \text{cl}Q \setminus \Omega_2, s \in \partial\Omega\}.$$

If instead $\Omega = \emptyset$, then we assume that $G = \emptyset$. Then the following statements hold.

- (i) Let $\Omega \neq \emptyset$. The restriction of $\mathcal{P}^+[h, \varphi]$ to $\text{cl}\mathbb{S}[\Omega_2]^-$ belongs to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_2]^-)$ for all pairs $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$ and the map from $H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\Omega)$ to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_2]^-)$ which takes (h, φ) to $\mathcal{P}^+[h, \varphi]_{|\text{cl}\mathbb{S}[\Omega_2]^-}$ is bilinear and continuous.
- (ii) The restriction of $\mathcal{P}^-[h, \varphi]_{|Q \setminus \text{cl}\Omega}$ to $\text{cl}\mathbb{S}[\Omega_2]^-$ belongs to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_2]^-)$ for all pairs $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega]^-)$ and the map from $H_q^{\lambda, \rho}(G) \times C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega]^-)$ to $C_{q, \omega, \rho}^0(\text{cl}\mathbb{S}[\Omega_2]^-)$ which takes (h, φ) to $\mathcal{P}^-[h, \varphi]_{|Q \setminus \text{cl}\Omega}|_{\text{cl}\mathbb{S}[\Omega_2]^-}$ is bilinear and continuous. (We note that if $G = \emptyset$, then by Definition 3.34 we have $H_q^{\lambda, \rho}(G) = A_{q, \lambda}^1$.)

Proof. We first prove statement (ii). For the sake of simplicity, we write just φ instead of $\varphi|_{Q \setminus \text{cl}\Omega}$, and we prove that if $m \in \mathbb{N} \setminus \{0\}$ and if $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_q^m(\text{cl}\mathbb{S}[\Omega]^-)$, then $\mathcal{P}^-[h, \varphi]_{|\text{cl}\mathbb{S}[\Omega_2]^-} \in C_q^m(\text{cl}\mathbb{S}[\Omega_2]^-)$ and

$$\begin{aligned} \partial^\beta \mathcal{P}^-[h, \varphi](x) &= \mathcal{P}^-[h, \partial^\beta \varphi](x) \\ &+ \sum_{k=1}^n \sum_{l_k=0}^{\beta_k-1} \partial_{x_n}^{\beta_n} \dots \partial_{x_{k+1}}^{\beta_{k+1}} \partial_{x_k}^{l_k} \left\{ \int_{\partial\Omega} h(x-y) \right. \\ &\quad \left. \times \left(\partial_{y_k}^{\beta_k-1-l_k} \partial_{y_{k-1}}^{\beta_{k-1}} \dots \partial_{y_1}^{\beta_1} \varphi(y) \right) (\nu_\Omega)_k(y) d\sigma_y \right\}, \end{aligned} \quad (3.41)$$

for all $x \in \text{cl}\mathbb{S}[\Omega_2]^-$ and for all $\beta \in \mathbb{N}^n$ such that $|\beta| \leq m$, where we understand that $\sum_{l_k=0}^{\beta_k-1}$ is omitted if $\beta_k = 0$, and where the integral in the right hand side is omitted if $\Omega = \emptyset$. Now let V be a bounded open connected subset of \mathbb{R}^n of class C^∞ such that

$$\text{cl}Q \subseteq V, \quad \text{cl}V \cap (qz + \text{cl}\Omega) = \emptyset, \quad \forall z \in \mathbb{Z}^n \setminus \{0\}.$$

Possibly shrinking V , we can assume that

$$\{t - s : t \in \text{cl}V \setminus \Omega_2, s \in \partial\Omega\} \subseteq G,$$

where we understand that the set in the left hand side is empty if $\partial\Omega = \emptyset$. Let $W \equiv V \setminus \text{cl}\Omega_2$. Since h is q -periodic, it suffices to show that $\mathcal{P}^-[h, \varphi]_{|\text{cl}W} \in C^m(\text{cl}W)$ and that (3.41) holds for all $x \in \text{cl}W$. If $m = 1$, then the statement follows by Lemma 3.21 (ii), (iii), and (3.33).

Next we assume that the statement holds for m and we prove it for $m+1$. We will confine ourself to the case where $\Omega \neq \emptyset$. The case where $\Omega = \emptyset$ can be treated by a simplification of the argument used for $\Omega \neq \emptyset$.

Let $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_q^{m+1}(\text{cl}\mathbb{S}[\Omega]^-)$. By the inductive assumption, $\mathcal{P}^-[h, \frac{\partial \varphi}{\partial x_j}]_{|\text{cl}\mathbb{S}[\Omega_2]^-} \in C_q^m(\text{cl}\mathbb{S}[\Omega_2]^-)$ for all $j \in \{1, \dots, n\}$. Since $h \in C^{m+1}(\text{cl}G)$ and $\varphi, (\nu_\Omega)_j \in C^0(\partial\Omega)$, the classical differentiability theorem for integrals depending on a parameter implies that the second term in the right hand side of formula (3.23) defines a function of class $C^m(\text{cl}W)$ (see also (3.30) and following comment). Then formula (3.23) implies that $\frac{\partial}{\partial x_j} \mathcal{P}^-[h, \varphi]$ belongs to $C^m(\text{cl}W)$. Hence, $\mathcal{P}^-[h, \varphi]_{|\text{cl}W} \in C^{m+1}(\text{cl}W)$.

Next one proves the formula for the derivatives by following the lines of the corresponding argument of [22, p. 856]: first one proves the formula for $\partial^\beta = \partial_{x_j}^{\beta_j}$ by finite induction on β_j , then one proves the formula for $\partial^\beta = \partial_{x_1}^{\beta_1} \dots \partial_{x_j}^{\beta_j}$ by finite induction on $j \in \{1, \dots, n\}$, and finally one deduces that the formula holds for $|\beta| \leq m+1$.

If $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C^\infty(\text{cl}Q \setminus \Omega)$, then by applying the above statement for all $m \in \mathbb{N} \setminus \{0\}$ we deduce that $\mathcal{P}^-[h, \varphi]_{|\text{cl}W}$ belongs to $C^\infty(\text{cl}W)$ and that formula (3.41) holds for all order derivatives.

We now assume that $(h, \varphi) \in H_q^{\lambda, \rho}(G) \times C_{\omega, \rho}^0(\text{cl}\mathbb{S}[\Omega]^-)$ and we turn to estimate the sup-norm in $\text{cl}W$ of the sum in the right hand side of (3.41), which we denote by I .

We abbreviate by $I(k, l_k)$ the (k, l_k) -th term in the sum I , and we now estimate the supremum of $I(k, l_k)$ in $\text{cl}W$. We can clearly assume that $\beta_k > 0$. Then we obtain the

inequality in (3.37) with W in place of Ω_1 . Then we can argue precisely as in the proof of Theorem 3.35 by replacing Ω_1 by W and Ω by $Q \setminus \text{cl}\Omega$ and obtain

$$\sup_{\text{cl}W} |I| \leq n\rho m_{n-1}(\partial\Omega) \|h\|_{H_q^{\lambda,\rho}(G)} \|\varphi\|_{C_{\omega,\rho}^0(\text{cl}Q \setminus \Omega)} \frac{|\beta|!}{\rho^{|\beta|}}. \quad (3.42)$$

On the other hand, Proposition 3.6 (vi) implies that

$$\|\mathcal{P}^- [h, \partial^\beta \varphi]\|_{L^\infty(\mathbb{S}[\Omega]^-)} \leq 2^n \int_{\bar{Q}} |y|^{-\lambda} dy \|h\|_{A_{q,\lambda}^0} \|\partial^\beta \varphi\|_{L^\infty(\text{cl}Q \setminus \Omega)}. \quad (3.43)$$

Then formula (3.41) and inequalities (3.42) and (3.43) imply that there exists $C > 0$ such that

$$\|\partial^\beta \mathcal{P}^- [h, \varphi]_{|W}\|_{L^\infty(W)} \leq C \|h\|_{H_q^{\lambda,\rho}(G)} \|\varphi\|_{C_{\omega,\rho}^0(\text{cl}W)} \frac{|\beta|!}{\rho^{|\beta|}} \quad \forall \beta \in \mathbb{N}^n.$$

Hence, statement (ii) holds true.

The proof of statement (i) follows the lines of the proof of statement (ii). We only point out that formula (3.41) holds if we replace \mathcal{P}^- by \mathcal{P}^+ , and the plus sign in the right hand side by a minus sign. \square

4 A remark on the class $H_q^{\lambda,\rho}(G)$ of weakly singular q -periodic kernels

In the present section, we show that if q is a real $n \times n$ positive definite diagonal matrix (such as (2.1)), then there exists a class of q -periodic kernels which is actually contained in the class $H_q^{\lambda,\rho}(G)$ as in the assumptions of Theorem 3.35 or of Theorem 3.40 and which is relevant in the analysis of q -periodic non-homogeneous boundary value problems for elliptic equations. Let $a_\alpha \in \mathbb{C}$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$ and let $P(x)$ be the polynomial

$$P(x) \equiv \sum_{|\alpha| \leq 2} a_\alpha x^\alpha \quad \forall x \in \mathbb{R}^n.$$

Then $P(D)$ denotes the partial differential operator

$$P(D) \equiv \sum_{|\alpha| \leq 2} a_\alpha D^\alpha.$$

We also assume that $P(D)$ is of second order and strongly elliptic, namely that

$$\text{Re} \left(\sum_{|\alpha|=2} a_\alpha \xi^\alpha \right) \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Then, a q -periodic distribution F_q is a q -periodic fundamental solution of $P(D)$ if

$$P(D)F_q = \sum_{z \in \mathbb{Z}^n} \delta_{qz}, \quad (4.1)$$

where δ_{qz} denotes the Dirac measure with mass at qz , for all $z \in \mathbb{Z}^n$. Unfortunately however, not all operators $P(D)$ admit q -periodic fundamental solutions: not even well known operators, such as the Laplace operator Δ , do have one.

Instead, if we denote by $E_{2\pi iq^{-1}z}$, the function defined by

$$E_{2\pi iq^{-1}z}(x) \equiv e^{2\pi i(q^{-1}z) \cdot x} \quad \forall x \in \mathbb{R}^n,$$

for all $z \in \mathbb{Z}^n$, then one can show that the set

$$\mathbb{Z}(P) \equiv \{z \in \mathbb{Z}^n : P(2\pi iq^{-1}z) = 0\}$$

is finite and that the q -periodic distribution

$$S_q \equiv \sum_{z \in \mathbb{Z}^n \setminus \mathbb{Z}(P)} \frac{1}{m_n(Q)} \frac{1}{P(2\pi iq^{-1}z)} E_{2\pi iq^{-1}z} \quad (4.2)$$

satisfies the equality

$$P(D)S_q = \sum_{z \in \mathbb{Z}^n} \delta_{qz} - \sum_{z \in \mathbb{Z}(P)} \frac{1}{m_n(Q)} E_{2\pi iq^{-1}z} \quad (4.3)$$

(cf. *e.g.*, Ammari and Kang [2, p. 53], [23, §3]). Equality (4.3) can be considered as an effective substitute of equality (4.1), and the distribution S_q can be exploited to introduce either layer or volume potentials, which can be employed to analyze boundary value problems on q -periodic domains. We note that Lin and Wang [28], Mityushev and Adler [31], and Mamode [29] have proved the validity of a constructive formula for the q -periodic analog of the fundamental solution for the Laplace equation in case $n = 2$ via elliptic functions.

As it is well known (see, *e.g.*, [23, Thm. 3.5]), if S_q is the distribution in (4.2) and if S is a classical (non-periodic) fundamental solution of the same operator P , then $S_q - S$ is a real analytic function in $(\mathbb{R}^n \setminus q\mathbb{Z}^n) \cup \{0\}$. Accordingly, by John [16], one deduces that

$$\sup_{x \in \tilde{Q} \setminus \{0\}} |S_q(x)| |x|^{n-2} < +\infty, \quad \sup_{x \in \tilde{Q} \setminus \{0\}} \left| \frac{\partial S_q}{\partial x_j}(x) \right| |x|^{n-1} < +\infty,$$

for all $j \in \{1, \dots, n\}$, if $n > 2$, and that

$$\sup_{x \in \tilde{Q} \setminus \{0\}} |S_q(x)| |x|^{1/2} < +\infty, \quad \sup_{x \in \tilde{Q} \setminus \{0\}} \left| \frac{\partial S_q}{\partial x_j}(x) \right| |x|^{3/2} < +\infty,$$

for all $j \in \{1, \dots, n\}$, if $n = 2$. Moreover, S_q is analytic in $\mathbb{R}^n \setminus q\mathbb{Z}^n$, and the classical Cauchy inequalities for the derivatives of S_q on a compact set imply that $S_q \in C_{\omega, \rho}^0(\text{cl}G)$ for all open bounded subsets G of \mathbb{R}^n such that $(\text{cl}G) \cap q\mathbb{Z}^n = \emptyset$ and for $\rho \in]0, +\infty[$ sufficiently small (cf. *e.g.*, John [17, p. 65]). Hence,

$$S_q \in H_q^{\max\{n-2, \frac{1}{2}\}, \rho}(G),$$

and thus our class $H_q^{\max\{n-2, \frac{1}{2}\}, \rho}(G)$ contains the q -periodic analogs of the fundamental solutions of second order elliptic operators.

Also if we have a one-parameter analytic family of q -periodic analogs of the fundamental solution in the space $H_q^{\max\{n-2, \frac{1}{2}\}, \rho}(G)$, we can apply Theorems 3.35 and 3.40 and deduce results of analytic dependence for the corresponding volume potentials upon the parameter and densities (or moments) of the volume potentials (see, *e.g.*, [10], where the authors have obtained similar results for the non-periodic case).

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