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TABLE DES MATIÈRES

WAVE-FRONT SETS IN NON-QUASIANALYTIC SETTING FOR FOURIER-LEBESGUE AND MODULATION SPACES

NENAD TEOFANOV

Dedicated to Professor Bogoljub Stanković (1924–2018)

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A b s t r a c t. *We define and study wave-front sets for weighted Fourier-Lebesgue spaces when the weights are moderate with respect to associated functions for general sequences* $\{M_p\}$ *which satisfy Komatsu's conditions* $(M.1) - (M.3)'$. *In particular, when* $\{M_p\}$ *is the Gevrey sequence* $(M_p = p!^s,\ s > 1)$ *we recover some previously observed results. Furthermore, we consider wave-front sets for modulation spaces in the same setting, and prove the invariance property related to the Fourier-Lebesgue type wave-front sets.*

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Key Words: Wave-front sets, weighted Fourier-Lebesgue spaces, Gelfand-Shilov spaces, ultradistributions.

1. *Introduction*

Wave front sets in the context of Fourier-Lebesgue spaces, together with the study of corresponding pseudodifferential operatros, were first considered in [35], see also [36, 37, 38]. They are recently used in [7] for a mathematical explanation of phenomena related to the interferences in the Born-Jordan distribution. The conic neighborhoods in the definition of such wave-front sets are replaced in [14] by a filter of

neighborhoods for the study of propagation of singularities of Fourier-Lebesgue type for partial (pseudo)differential equations, whose symbol satisfies generalized elliptic properties. We refer to [10, 21, 24] for discrete characterization of the wave front sets. An important extension of investigations from [36, 37] to general weighted Fourier Banach spaces is given in [2, 3].

The above mentioned results are performed in the framework of weights of polynomial growth and, consequently, within the realm of tempered distributions. Spaces of ultradistributions in the context of weighted Fourier-Lebesgue type spaces were first observed in [22], see also [23]. The sequences of the form $M_p = p!^s$, $s > 1$, are used there to define the corresponding test function spaces. This in turn leads to the analysis of weighted Fourier-Lebesgue spaces such that the growth of the weight function at infinity is bounded by $e^{k|\cdot|^{1/s}}$, for some $k > 0$.

In this paper we extend the results from [22] to a more general context when the spaces of test functions are given by the means of $\{M_p\}$ sequences which satisfy Komatsu's conditions $(M.1) - (M.3)'$, see Section 2. Note that this allows "fine tuning" between the two Gevrey type sequences, see Remark 2.1.

The paper is organized as follows. We end introduction with the basic notation and a brief account on weight functions. Section 2. contains a discussion on sequences and corresponding associate functions, which are the basic notions in our analysis. We proceed with an exposition of Gelfand-Shilov spaces and other test function spaces, and their dual spaces of ultradistributions. Section 3. contains the definition of wave-front sets for weighted Fourier-Lebesgue spaces when the weights are submultiplicative with respect to the associated function of a given non-quasianalytic sequence $\{M_p\}$. We study its basic properties, convolution relations, and discuss its relation to some other types of wave-front sets. In Section 4. we first study the short-time Fourier transform in the context of test function spaces and their duals from Section 2., and then define modulation spaces and recall their basic properties. Finally, in Section 5. we introduce wave-front sets for modulation spaces and show that they coincide with appropriate wave-front sets from Section 3.. Since we consider general non-quasianalytic sequences $\{M_p\}$, we recover the main results from [22, 23] where the particular case $M_p = p!^s$, $s > 1$, is observed.

1.1. Basic notation

We put $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\langle x \rangle = (1 + |x|^2)^{1/2}$, $x \in \mathbb{R}^d$, $xy = x \cdot y$ denotes the scalar product on $\mathbb{R}^d$ and

$$\langle (x, \omega) \rangle^s = \langle z \rangle^s = (1 + x^2 + \omega^2)^{s/2}, \quad z = (x, \omega) \in \mathbb{R}^{2d}, \quad s \in \mathbb{R}.$$

The partial derivative of a vector $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ with respect to x_j is denoted

by $\partial_j = \frac{\partial}{\partial x_j}$. Given a multi-index $p = (p_1, \ldots, p_d) \geq 0$, i.e., $p \in \mathbb{N}_0^d$ and $p_j \geq 0$, we write $\partial^p = \partial_1^{p_1} \cdots \partial_d^{p_d}$ and $x^p = (x_1, \ldots, x_d)^{(p_1, \ldots, p_d)} = \prod_{i=1}^d x_i^{p_i}$. Similarly, $h \cdot |x|^{1/\alpha} = \sum_{i=1}^d h_i |x_i|^{1/\alpha_i}$. Moreover, for $p \in \mathbb{N}_0^d$ and $\alpha \in \mathbb{R}_+^d$, we set $(p!)^\alpha = (p_1!)^{\alpha_1} \ldots (p_d!)^{\alpha_d}$. In the sequel, a real number $r \in \mathbb{R}_+$ may play the role of the vector with constant components $r_j = r$, so for $\alpha \in \mathbb{R}_+^d$, by writing $\alpha > r$ we mean $\alpha_j > r$ for all $j = 1, \ldots, d$. By X we denote an open set in $\mathbb{R}^d$, and $K \Subset X$ means that K is compact subset in X.

The Fourier transform is normalized to be

$$\hat{f}(\omega) = \mathcal{F}f(\omega) = \int f(t) e^{-2\pi i t \omega} dt.$$

We use the brackets $\langle f, g \rangle$ to denote the extension of the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$ to the dual pairing between a test function space $\mathcal{A}$ and its dual $\mathcal{A}'$: $\langle \cdot, \cdot \rangle = {}_{\mathcal{A}'}\langle \cdot, \overline{\cdot} \rangle_{\mathcal{A}}$. We use the standard notation for usual spaces of functions and distributions, e.g. $L^p(\mathbb{R}^d)$, $L^p_{loc}(\Omega)$, $1 \leq p \leq \infty$, denote Lebesgue spaces and their local versions respectively, $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz space of rapidly decreasing test functions, etc.

Translation and modulation operators, T and M respectively, when acting on $f \in L^2(\mathbb{R}^d)$ are defined by

$$T_x f(\cdot) = f(\cdot - x) \quad \text{and} \quad M_\xi f(\cdot) = e^{2\pi i \xi \cdot} f(\cdot), \quad x \in \mathbb{R}^d. \tag{1.1}$$

Then for $f, g \in L^2(\mathbb{R}^d)$ the following relations hold:

$$M_y T_x = e^{2\pi i x \cdot y} T_x M_y, \quad (T_x f)\hat{} = M_{-x} \hat{f}, \quad (M_x f)\hat{} = T_x \hat{f}, \quad x, y \in \mathbb{R}^d.$$

These operators are extended to other spaces of functions and distributions in a natural way.

Throughout the paper, $A \lesssim B$ denotes $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means that $c^{-1}A \leq B \leq cA$ for some $c \geq 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous and dense embedding of the topological vector space B_1 into B_2.

1.2. Weights

In general, a weight function is a non-negative function in L^∞_{loc}.

Definition 1.1. Let ω, v be weight functions. Then

1. v is called submultiplicative if

$$v(x + y) \leq v(x) v(y), \qquad \forall\, x, y \in \mathbb{R}^d;$$

2. ω is called v-moderate if

$$\omega(x + y) \lesssim v(x)\omega(y), \qquad \forall\, x, y \in \mathbb{R}^d.$$

For a given submultiplicative weight v the set of all v-moderate weights will be denoted by $\mathcal{M}_v$.

If v is even and $\omega \in \mathcal{M}_v$, then $1/v \lesssim \omega \lesssim v$, $\omega \neq 0$ everywhere and $1/\omega \in \mathcal{M}_v$.

In the sequel we assume that v is an even submultiplicative function. Submultiplicativity implies that v is dominated by an exponential function, i.e.

$$v \leq Ce^{k|\cdot|} \quad \text{for some} \quad C, k > 0.$$

For example, every weight of the form

$$v(z) = e^{s\|z\|^b}(1 + \|z\|)^a \log^r(e + \|z\|)$$

for parameters $a, r, s \geq 0$, $0 \leq b \leq 1$ satisfies the above conditions.

Let $s > 1$. By $\mathcal{M}_{\{s\}}(\mathbb{R}^d)$ we denote the set of all weights which are moderate with respect to a weight v which satisfies $v \leq Ce^{k|\cdot|^{1/s}}$ for some positive constants C and k. The weight v satisfy the Beurling-Domar non-quasi-analyticity condition which takes the form

$$\sum_{n=0}^{\infty} \frac{\log v(nx)}{n^2} < \infty, \quad x \in \mathbb{R}^d.$$

We refer to [17] for a detailed account on weights in time-frequency analysis.

2. *Spaces of test functions and their duals*

Let $(M_p)_{p \in \mathbb{N}_0}$ be a sequence of positive numbers monotonically increasing to infinity which satisfies:

$(M.1)$ $M_p^2 \leq M_{p-1}M_{p+1}$, $p \in \mathbb{N}$;

$(M.2)$ There exist positive constants A, H such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_{p-q}M_q, \; p, q \in \mathbb{N}_0,$$

or, equivalently, there exist positive constants A, H such that

$$M_{p+q} \leq AH^{p+q}M_p M_q, \; p, q \in \mathbb{N}_0;$$

$(M.3)'$ $\sum_{p=1}^{\infty} M_{p-1}/M_p < \infty$.

We assume that $M_0 = 1$, and that $M_p^{1/p}$ is bounded below by a positive constant.

The condition $(M.3)'$ provides the existence of nontrivial compactly supported smooth functions (and therefore partitions of unity) in the corresponding spaces of test functions. It is therefore known as the non-quasianalyticity condition.

The Gevrey sequences $M_p = p!^s$, $p \in \mathbb{N}$, $s > 1$, are basic examples of sequences which satisfy $(M.1) - (M.3)'$.

Let $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ be sequences which satisfy $(M.1)$. We write $M_p \subset N_q$ ($(M_p) \prec (N_q)$), respectively) if there are constants $H, C > 0$ (for any $H > 0$ there is a constant $C > 0$, respectively) such that $M_p \leq CH^p N_p$, $p \in \mathbb{N}_0$. Also, $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ are said to be equivalent if $M_p \subset N_q$ and $N_q \subset M_p$ hold.

Remark 2.1. The conditions (M.1) and (M.2) can be described as follows. Let $(s_p)_{p \in \mathbb{N}_0}$ be a sequence of positive numbers monotonically increasing to infinity ($s_p \nearrow \infty$) so that for every $p, q \in \mathbb{N}_0$ there exist $A, H > 0$ such that

$$\prod_{j=1}^{q} s_{p+j} = s_{p+1} \cdots s_{p+q} \leq AH^p s_1 \cdots s_q = AH^p \prod_{j=1}^{q} s_j. \qquad (2.1)$$

Then the sequence $(S_p)_{p \in \mathbb{N}_0}$ given by $S_p = \prod_{j=1}^{p} s_j$, $S_0 = 1$, satisfies $(M.1)$ and $(M.2)$.

Conversely, if $(S_p)_{p \in \mathbb{N}_0}$ given by $S_p = \prod_{j=1}^{p} s_j$, $s_j > 0$, $j \in \mathbb{N}$, $S_0 = 1$, satisfies $(M.1)$ then the sequence $(s_p)_{p \in \mathbb{N}_0}$ increases to infinity. If, in addition, it satisfies $(M.2)$ then (2.1) holds.

Furthermore, if $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ are given by

$$M_p := p!^{\frac{1}{2}} \prod_{k=0}^{p} l_k = p!^{\frac{1}{2}} L_p, \quad p \in \mathbb{N}_0, \quad N_q := q!^{\frac{1}{2}} \prod_{k=0}^{q} r_k = q!^{\frac{1}{2}} R_q, \quad q \in \mathbb{N}_0,$$

where $(r_p)_{p \in \mathbb{N}_0}$ and $(l_p)_{p \in \mathbb{N}_0}$ are sequences of positive numbers monotonically increasing to infinity such that (2.1) holds with the letter s replaced by r and l respectively, and which satisfy: For every $\alpha \in (0, 1]$ and every $k > 1$ so that $kp \in \mathbb{N}$, $p \in \mathbb{N}$,

$$\max \left\{ \left(\frac{r_{kp}}{r_p} \right)^2, \left(\frac{l_{kp}}{l_p} \right)^2 \right\} \leq k^\alpha, \quad p \in \mathbb{N}.$$

Then $p! \prec M_p N_p$ and the sequences $(R_p)_{p \in \mathbb{N}_0}$ and $(L_p)_{p \in \mathbb{N}_0}$ ($R_p = r_1 \cdots r_p$, $L_p = l_1 \cdots l_p$, $p \in \mathbb{N}$ $R_0 = 1$, and $L_0 = 1$) satisfy $(M.1)$ and $(M.2)$. Moreover,

$$\max\{R_p, L_p\} \leq p!^{\alpha/2}, p \in \mathbb{N},$$

for every $\alpha \in (0, 1]$. (For $p, q, k \in \mathbb{N}_0^d$ we have $L_{|p|} = \prod_{|k| \leq |p|} l_{|k|}$, and $R_{|q|} = \prod_{|k| \leq |q|} r_{|q|}$.) Such sequences are used in the study of localization operators in the context of quasianalytic spaces in [9].

The *associated function* for a given sequence (M_p) is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \ln_+ \frac{\rho^p M_0}{M_p}, \quad 0 < \rho < \infty,$$

where $\ln_+ t := \max\{\ln t, 0\}$, $t > 0$. It is a non-negative monotonically increasing function which vanishes for sufficiently small ρ, and tends to infinity faster than $\ln \rho^p$, as $\rho \to \infty$. Moreover, if (M_p) satisfies $(M.1)$ and $(M.3)'$, then $k^p p!/M_p \to 0$ as $p \to \infty$.

For example, the associated function for the Gevrey sequence $M_p = p!^s$, $p \in \mathbb{N}_0$, $s > 1$, behaves at infinity as $|\cdot|^{1/s}$, cf. [30]. In fact, the interplay between the defining sequence and its associated function plays an important role in the theory of ultradistributions.

The following result will be intensively used in this paper. We refer to [1] for its proof.

Lemma 2.1. *Let there be given sequence* (M_p) *which satisfies* $(M.1)$. *Then*

$$M\left(\sum_{k=1}^n \rho_k\right) \le \sum_{k=1}^n M(\rho_k), \quad \rho_k > 0, \ k = 1, \dots, n.$$

If, in addition, (M_p) *satisfies* $(M.2)$, *then*

$$2M(\rho) \le M(H\rho) + \ln_+(A), \quad \rho > 0,$$

where A and H are the constants in $(M.2)$. *Furthermore, if $L \ge 1$, then there is a constant $C > 0$ such that*

$$M(L\rho) \le \frac{3}{2}LM(\rho) + C, \quad \rho > 0,$$

and there is a constant $B > 0$ and a constant $K_L > 0$ which depends on L, such that

$$LM(\rho) \le M(B^{L-1}\rho) + K_L, \quad \rho > 0.$$

Remark 2.2. By Lemma 2.1, it follows that estimates of the form $|f(\cdot)| \lesssim e^{M(h|\cdot|)}$ for some/every $h > 0$ and $|f(\cdot)| \lesssim e^{kM(|\cdot|)}$ for some/every $k > 0$ are equivalent. This observation will be often used in proofs.

2.1. Gelfand-Shilov spaces

We give here only the basic properties and refer to [15, 29] for a more detailed discussion and applications in partial differential equations.

Definition 2.1. Let there be given sequences of positive numbers $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ which satisfy $(M.1)$ and $(M.2)$. Let $\mathcal{S}_{M_p,A}^{N_q,B}(\mathbb{R}^d)$ be defined by

$$\mathcal{S}_{M_p,A}^{N_q,B}(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \mid \|x^\alpha \partial^\beta f\|_{L^\infty} \leq C A^\alpha M_{|\alpha|} B^\beta N_{|\beta|}, \ \forall \alpha, \beta \in \mathbb{N}_0^d\},$$

for some positive constant C, and $A = (A_1, \ldots, A_d)$, $B = (B_1, \ldots, B_d)$, $A, B > 0$.

Gelfand-Shilov spaces $\Sigma_{M_p}^{N_q}(\mathbb{R}^d)$ and $\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ are projective and inductive limits of (Fréchet) spaces $\mathcal{S}_{M_p,A}^{N_q,B}(\mathbb{R}^d)$ with respect to A and B:

$$\Sigma_{M_p}^{N_q}(\mathbb{R}^d) := \operatorname{proj} \lim_{A>0,B>0} \mathcal{S}_{M_p,A}^{N_q,B}(\mathbb{R}^d); \quad \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d) := \operatorname{ind} \lim_{A>0,B>0} \mathcal{S}_{M_p,A}^{N_q,B}(\mathbb{R}^d).$$

The corresponding dual spaces of $\Sigma_{M_p}^{N_q}(\mathbb{R}^d)$ and $\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ are the spaces of ultra-distributions of Beurling and Roumieu type respectively:

$$(\Sigma_{M_p}^{N_q})'(\mathbb{R}^d) := \operatorname{ind} \lim_{A>0,B>0} (\mathcal{S}_{M_p,A}^{N_q,B})'(\mathbb{R}^d);$$

$$(\mathcal{S}_{M_p}^{N_q})'(\mathbb{R}^d) := \operatorname{proj} \lim_{A>0,B>0} (\mathcal{S}_{M_p,A}^{N_q,B})'(\mathbb{R}^d).$$

Gelfand-Shilov spaces are closed under translation, dilation, multiplication with $x \in \mathbb{R}^d$, and differentiation. Moreover, they are closed under the action of certain differential operators of infinite order (ultradifferentiable operators in the terminology of Komatsu).

Whenever nontrivial, Gelfand-Shilov spaces contain "enough functions" in the following sense. A test function space Φ is "rich enough" if

$$\int f(x)\varphi(x)dx = 0, \quad \forall \varphi \in \Phi \Rightarrow f(x) \equiv 0 \ (a.e.).$$

The following theorem enlightens the fundamental properties of Gelfand-Shilov spaces implicitly contained in their definition. Among other things, it states that the decay and regularity estimates of $f \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ can be studied separately.

Theorem 2.1. *Let there be given sequences of positive numbers $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ which satisfy $(M.1)$, $(M.2)$ and $p! \subset M_p N_p$ ($p! \prec M_p N_p$, respectively). Moreover, let $M(\cdot)$ and $N(\cdot)$ denote the associated functions for $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ respectively. Then the following conditions are equivalent:*

1. $f \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$ ($f \in \Sigma_{M_p}^{N_q}(\mathbb{R}^d)$, respectively).

2. *There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$ respectively), and there exist $C > 0$ such that*

$$\|e^{M(|Ax|)} \partial^q f(x)\|_{L^\infty} \leq CB^q N_{|q|}, \quad \forall p, q \in \mathbb{N}_0^d.$$

3. *There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively), and there exist $C > 0$ such that*

$$\|x^p f(x)\|_{L^\infty} \leq CA^p M_{|p|} \quad and \quad \|\partial^q f(x)\|_{L^\infty} \leq CB^q N_{|q|}, \quad \forall p, q \in \mathbb{N}_0^d.$$

4. *There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively), and there exist $C > 0$ such that*

$$\|x^p f(x)\|_{L^\infty} \leq CA^p M_{|p|} \quad and \quad \|\omega^q \hat{f}(\omega)\|_{L^\infty} \leq CB^q N_{|q|}, \quad \forall p, q \in \mathbb{N}_0^d.$$

5. *There exist constants $A, B \in \mathbb{R}^d$, $A, B > 0$ (for every $A, B \in \mathbb{R}^d$, $A, B > 0$, respectively), such that*

$$\|f(x)e^{M(|Ax|)}\|_{L^\infty} < \infty \quad and \quad \|\hat{f}(\omega) \, e^{N(|B\omega|)}\|_{L^\infty} < \infty.$$

Theorem 2.1 is proved in [5] and reinvented many times afterwards, see e.g. [8, 19, 25, 29, 33, 45].

By the above characterization $\mathcal{F}\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d) = \mathcal{S}_{N_q}^{M_p}(\mathbb{R}^d)$. When $M_p = N_q$ we put $\mathcal{S}_{M_p}^{M_p}(\mathbb{R}^d) = \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, and $\Sigma_{M_p}^{M_p}(\mathbb{R}^d) = \mathcal{S}^{(M_p)}(\mathbb{R}^d)$. Moreover, the Fourier transform $\mathcal{F}$ extends to a homeomorphism on $(\mathcal{S}^{\{M_p\}})'(\mathbb{R}^d)$ and on $(\mathcal{S}^{(M_p)})'(\mathbb{R}^d)$ in a usual way.

Next we discuss the important case when $(M_p)_{p \in \mathbb{N}_0}$ and $(N_q)_{q \in \mathbb{N}_0}$ are chosen to be the Gevrey sequences $M_p = p!^r$, $p \in \mathbb{N}_0$ and $N_q = q!^s$, $q \in \mathbb{N}_0$, for some $r, s \geq 0$, then we use the notation

$$\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d) = \mathcal{S}_r^s(\mathbb{R}^d) \quad \text{and} \quad \Sigma_{M_p}^{N_q}(\mathbb{R}^d) = \Sigma_r^s(\mathbb{R}^d).$$

If, in addition, $s = r$, then we put

$$\mathcal{S}^{\{s\}}(\mathbb{R}^d) = \mathcal{S}_s^s(\mathbb{R}^d) \quad \text{and} \quad \Sigma^{(s)}(\mathbb{R}^d) = \Sigma_s^s(\mathbb{R}^d).$$

The choice of Gevrey sequences is the most often used choice in the literature since it serves well in different contexts. For example, when discussing nontriviality of Gelfand-Shilov spaces we have the following:

1. the space $\mathcal{S}_r^s(\mathbb{R}^d)$ is nontrivial if and only if $s+r > 1$, or $s+r = 1$ and $sr > 0$,

2. if $s+r \geq 1$ and $s < 1$, then every $f \in \mathcal{S}_r^s(\mathbb{R}^d)$ can be extended to the complex domain as an entire function,

3. if $s+r \geq 1$ and $s = 1$, then every $f \in \mathcal{S}_r^s(\mathbb{R}^d)$ can be extended to the complex domain as a holomorphic function in a strip $\{x + iy \in \mathbb{C}^d \; : \; |y| < T\}$ some $T > 0$

4. the space $\Sigma_r^s(\mathbb{R}^d)$ is nontrivial if and only if $s + r > 1$, or, if $s + r = 1$ and $sr > 0$ and $(s,r) \neq (1/2, 1/2)$.

We refer to [15] or [29] for the proof in the case of $\mathcal{S}_r^s(\mathbb{R}^d)$, and to [31] for the spaces $\Sigma_r^s(\mathbb{R}^d)$, see also [46].

The discussion here above shows that Gelfand-Shilov classes $\mathcal{S}_r^s(\mathbb{R}^d)$ consist of quasi-analytic functions when $s \in (0, 1)$. This is in a sharp contrast with e.g. Gevrey classes $G^s(\mathbb{R}^d)$, $s > 1$, another family of functions commonly used in regularity theory of partial differential equations, whose elements are always non-quasi-analytic. Recall, for $1 < s < \infty$ and an open set $X \in \mathbb{R}^d$ the Gevrey class $G^s(X)$ is given by

$$G^s(X) = \{\phi \in C^\infty(X) \mid (\forall K \Subset X)(\exists C > 0)(\exists h > 0)$$
$$\sup_{x \in K} |\partial^\alpha \phi(x)| \leq Ch^{|\alpha|}|\alpha|!^s\}.$$

We refer to [39] for microlocal analysis in Gervey classes and note that

$$G_0^s(\mathbb{R}^d) \hookrightarrow \mathcal{S}_s^s(\mathbb{R}^d) \hookrightarrow G^s(\mathbb{R}^d), \quad s > 1.$$

When the spaces are nontrivial we have the inclusions:

$$\Sigma_r^s(\mathbb{R}^d) \hookrightarrow \mathcal{S}_r^s(\mathbb{R}^d) \hookrightarrow \mathcal{S}(\mathbb{R}^d),$$

and $\mathcal{S}(\mathbb{R}^d)$ can be revealed as the limiting case of spaces $S_r^s(\mathbb{R}^d)$, i.e.

$$\mathcal{S}(\mathbb{R}^d) = \mathcal{S}_\infty^\infty(\mathbb{R}^d) = \lim_{s,r \to \infty} \mathcal{S}_r^s(\mathbb{R}^d),$$

when the passage to the limit when s and r tend to infinity is interpreted correctly, see [15, page 169].

Remark 2.3. Note that $\Sigma_{1/2}^{1/2}(\mathbb{R}^d) = \{0\}$ and $\Sigma_s^s(\mathbb{R}^d)$ is dense in the Schwartz space whenever $s > 1/2$. One may consider a "fine tuning", that is the spaces $\Sigma_{M_p}^{N_q}(\mathbb{R}^d)$ such that

$$\{0\} = \Sigma_{1/2}^{1/2}(\mathbb{R}^d) \hookrightarrow \Sigma_{M_p}^{N_q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d) \hookrightarrow \Sigma_s^s(\mathbb{R}^d), \quad s > 1/2,$$

see also Remark 2.1.

We refer to [47] where it is shown how to overcome the minimality condition $(\Sigma^{1/2}_{1/2}(\mathbb{R}^d) = 0)$ by transferring the estimates for $\|x^\alpha \partial^\beta f\|_{L^\infty}$ into the estimates of the form $\|H^N f\|_{L^\infty} \lesssim h^N (N!)^{2s}$, for some (for every) $h > 0$, where $H = |x|^2 - \Delta$ is the harmonic oscillator.

We also mention that the Gelfand-Shilov space of analytic functions $\mathcal{S}^{(1)}(\mathbb{R}^d) := \Sigma^1_1(\mathbb{R}^d)$ plays a prominent role in the theory since it is isomorphic to the Sato test function space for the space of Fourier hyperfunctions. More precisely, if $f \in \mathcal{S}^{(1)}(\mathbb{R}^d)$ then it can be extended to a holomorphic function $f(x + iy)$ in the strip $\{x + iy \in \mathbb{C}^d \ : \ |y| < T\}$ for some $T > 0$. According to Theorem 2.1, we have

$$f \in \mathcal{S}^{(1)}(\mathbb{R}^d) \iff \sup_{x \in \mathbb{R}^d} |f(x) e^{h \cdot |x|}| < \infty \ \text{ and } \ \sup_{\omega \in \mathbb{R}^d} |\hat{f}(\omega) e^{h \cdot |\omega|}| < \infty,$$

for every $h > 0$. This representation is used to establish an isomorphism between its dual space $(\mathcal{S}^{(1)})'(\mathbb{R}^d)$ and the space of Fourier hyperfunctions, see [4] for details.

Already in [15] it is shown that the Fourier transform is a topological isomorphism between $\mathcal{S}^s_r(\mathbb{R}^d)$ and $\mathcal{S}^r_s(\mathbb{R}^d)$ $(\mathcal{F}(\mathcal{S}^s_r) = \mathcal{S}^r_s)$, which extends to a continuous linear transform from $(\mathcal{S}^s_r)'(\mathbb{R}^d)$ onto $(\mathcal{S}^r_s)'(\mathbb{R}^d)$. In particular, if $s = r$ and $s \geq 1/2$ then $\mathcal{F}(\mathcal{S}^s_s)(\mathbb{R}^d) = \mathcal{S}^s_s(\mathbb{R}^d)$, and $\mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d)$ is the smallest non-empty Gelfand-Shilov space invariant under the Fourier transform, cf. [45, Remark 1.2]. Similar assertions hold for $\Sigma^s_r(\mathbb{R}^d)$.

2.2. Test function spaces on open sets

Since we are interested in non-quasianalytic classes, we restrict our intention to the sequences which satisfy $(M.1) - (M.3)'$, and refer to [26] for a more general setting.

Definition 2.2. Let there be given a sequence (M_p), $p \in \mathbb{N}^d$, which satisfies $(M.1) - (M.3)'$ and let X be an open set in $\mathbb{R}^d$. For a given compact set $K \subset X$ and a constant $A > 0$ we denote by $\mathcal{E}^{M_p}_{A,K}(X)$ the space of all $\varphi \in C^\infty(X)$ such that the norm

$$\|\varphi\|_{M_p,A,K} = \sup_{p \in \mathbf{N}^n_0} \sup_{x \in K} \frac{A^{|p|}}{M_p} |\varphi^{(p)}(x)| < \infty. \tag{2.2}$$

Note that $\| \cdot \|_{M_p,A,K}$ is a norm in $\mathcal{E}^{M_p}_{A,K}(X)$.

The space of functions $\varphi \in C^\infty(X)$ such that (2.2) holds and $\text{supp}\,\varphi \subseteq K$ is denoted by $\mathcal{D}^{M_p}_A(K)$.

Let $(K_n)_n$ be a sequence of compact sets such that $K_n \subset\subset K_{n+1}$ and $\bigcup K_n = X$. Then

$$\mathcal{E}^{(M_p)}(X) = \text{proj} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{E}^{M_p}_{A,K_n})(X),$$

$$\mathcal{E}^{\{M_p\}}(X) = \text{proj} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{E}^{M_p}_{A,K_n})(X),$$

$$\mathcal{D}^{(M_p)}(X) = \text{ind} \lim_{n\to\infty} (\text{proj} \lim_{A\to\infty} \mathcal{D}^{M_p}_A(K_n))$$

$$= \text{ind} \lim_{n\to\infty} (\mathcal{D}^{(M_p)}_{K_n}),$$

$$\mathcal{D}^{\{M_p\}}(X) = \text{ind} \lim_{n\to\infty} (\text{ind} \lim_{A\to 0} \mathcal{D}^{M_p}_A(K_n))$$

$$= \text{ind} \lim_{n\to\infty} (\mathcal{D}^{\{M_p\}}_{K_n}).$$

Obviously, $\mathcal{D}^{(M_p)}(X)$ ($\mathcal{D}^{\{M_p\}}(X)$ resp.) is the subspace of $\mathcal{E}^{(M_p)}(X)$ (of $\mathcal{E}^{\{M_p\}}(X)$ resp.) whose elements are compactly supported.

Remark 2.4. Let $*$ denote (M_p) or $\{M_p\}$. Then $\mathcal{D}^*$, $\mathcal{S}^*$ and $\mathcal{E}^*$ correspond to C_0^∞, $\mathcal{S}$ and C^∞, respectively, and

$$\mathcal{D}^* \subseteq C_0^\infty, \quad \mathcal{S}^* \subseteq \mathcal{S} \quad \text{and} \quad \mathcal{E}^* \subseteq C^\infty.$$

The spaces of linear functionals over $\mathcal{D}^{(M_p)}(X)$ and $\mathcal{D}^{\{M_p\}}(X)$, denoted by $(\mathcal{D}^{(M_p)})'(X)$ and $(\mathcal{D}^{\{M_p\}})'(X)$ respectively, are called the spaces of *ultradistributions* of Beurling and Roumieu type respectively, while the spaces of linear functionals over $\mathcal{E}^{(M_p)}(X)$ and $\mathcal{E}^{\{M_p\}}(X)$, denoted by $(\mathcal{E}^{(M_p)})'(X)$ and $(\mathcal{E}^{\{M_p\}})'(X)$, respectively are called the spaces of *ultradistributions of compact support* of Beurling and Roumieu type respectively. Clearly,

$$(\mathcal{E}^{\{M_p\}})'(X) \subseteq (\mathcal{E}^{(M_p)})'(X), \quad (\mathcal{E}^{(M_p)})'(X) \subseteq (\mathcal{E}^{(M_p)})'(\mathbb{R}^d) \quad \text{and}$$

$$(\mathcal{E}^{\{M_p\}})'(X) \subseteq (\mathcal{E}^{\{M_p\}})'(\mathbb{R}^d).$$

Moreover,

$$(\mathcal{E}^{\{M_p\}})'(\mathbb{R}^d) \subseteq (\mathcal{S}^{\{M_p\}})'(\mathbb{R}^d) \subseteq (\mathcal{D}^{\{M_p\}})'(\mathbb{R}^d)$$

and

$$(\mathcal{E}^{(M_p)})'(\mathbb{R}^d) \subseteq (\mathcal{S}^{(M_p)})'(\mathbb{R}^d) \subseteq (\mathcal{D}^{(M_p)})'(\mathbb{R}^d).$$

Any ultra-distribution with compact support can be viewed as an element of $(\mathcal{S}^{(1)})'(\mathbb{R}^d)$. More generally, by using similar reasoning as in the case of distributions (see [20]), it follows that $\mathcal{E}^*$ are exactly those elements in $\mathcal{S}^*$ or $\mathcal{D}^*$ with compact support.

The following fact follows from the Paley-Wiener type theorems which can be found e.g. in [26].

Theorem 2.2. *Let there be given a sequence* (M_p), $p \in \mathbb{N}^d$, *which satisfies* $(M.1) - (M.3)'$ *and let* K *be a compact convex set in* $\mathbb{R}^d$. *Then* $\varphi \in \mathcal{D}_K^{(M_p)}$ *(* $\varphi \in \mathcal{D}_K^{\{M_p\}}$ *resp.) if and only if for every* $h > 0$ *there is a constant* $C > 0$ *(there are constants* $h > 0$ *and* $C > 0$ *resp.) such that*

$$|\hat{\varphi}(\xi)| \le Ce^{-hM(|\xi|)}, \quad \xi \in \mathbb{R}^d.$$

3. *Wave-front sets in weighted Fourier-Lebesgue spaces*

Although in principle both Beurling and Roumieu cases could be treated simultaneously (as we did in Section 2.), in order to simplify the exposition, from now on we will treat the Beurling case only. See also [23] for a discussion related to a slight difference between the cases.

Throughout the section $\{M_p\}$ will always denote a sequence satisfies $(M.1) - (M.3)'$ and $M(\rho)$ denotes its associated function. For the notational convenience, the set of weights ω moderated with respect to the weight $e^{M(\rho)}$ will be denoted by $\mathcal{M}_{M(\rho)}(\mathbb{R}^d)$ (instead of a more cumbersome notation $\mathcal{M}_{e^{M(\rho)}}(\mathbb{R}^d)$).

Let $q \in [1, \infty]$ and let $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$. The (weighted) Fourier Lebesgue space $\mathcal{F}L_{(\omega)}^q(\mathbb{R}^d)$ is the inverse Fourier image of $L_{(\omega)}^q(\mathbb{R}^d)$, i. e. $\mathcal{F}L_{(\omega)}^q(\mathbb{R}^d)$ consists of all $f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{F}L_{(\omega)}^q} \equiv \|\widehat{f} \cdot \omega\|_{L^q}.$$

is finite. If $\omega = 1$, then the notation $\mathcal{F}L^q$ is used instead of $\mathcal{F}L_{(\omega)}^q$. We note that if $\omega(\xi) = \langle \xi \rangle^s$, then $\mathcal{F}L_{(\omega)}^q$ is the Fourier image of the Bessel potential space H_s^p.

Remark 3.1. We may permit an x dependency for the weight ω in the definition of Fourier Lebesgue spaces. More precisely, for each $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$ we let $\mathcal{F}L_{(\omega)}^q$ be the set of all ultradistributions f such that

$$\|f\|_{\mathcal{F}L_{(\omega)}^q} \equiv \|\widehat{f}\, \omega(x, \cdot)\|_{L^q}$$

is finite. Since ω is $e^{M(\rho)}$-moderate it follows that different choices of x give rise to equivalent norms, hence $\|f\|_{\mathcal{F}L_{(\omega)}^q} < \infty$ is independent of x. Therefore, a $\mathcal{F}L_{(\omega)}^q(\mathbb{R}^d)$ is independent of x although $\| \cdot \|_{\mathcal{F}L_{(\omega)}^q}$ might depend on x.

Next we introduce local Fourier-Lebesgue spaces of ultradistributions related to the given sequence $\{M_p\}$. Let X be an open set in $\mathbb{R}^d$ and let $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$. The *local* Fourier Lebesgue space $\mathcal{F}L^q_{(\omega),loc}(X)$ consists of all $f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ such that $\varphi f \in \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$ for each $\varphi \in \mathcal{D}^{(M_p)}(X)$. It is a Fréchet space under the topology given by the family of seminorms $f \mapsto \|\varphi f\|_{\mathcal{F}L^q_{(\omega)}}$, where $\varphi \in \mathcal{D}^{(M_p)}(X)$, and the following simple properties hold.

Lemma 3.1. *Let there be given a sequence $\{M_p\}$ with the associate function $M(\rho)$, $\rho > 0$. Let X be an open set in $\mathbb{R}^d$ and $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$. Then*

$$\mathcal{F}L^q_{(\omega)}(\mathbb{R}^d) \subseteq \mathcal{F}L^q_{(\omega),loc}(\mathbb{R}^d) \subseteq \mathcal{F}L^q_{(\omega),loc}(X). \tag{3.1}$$

Furthermore, let $q_1, q_2 \in [1, \infty]$ and $\omega_1, \omega_2 \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$. Then

$$\mathcal{F}L^{q_1}_{(\omega_1),loc}(X) \subseteq \mathcal{F}L^{q_2}_{(\omega_2),loc}(X), \quad when \quad q_1 \le q_2 \quad and \quad \omega_2 \lesssim \omega_1. \tag{3.2}$$

PROOF. If $f \in \mathcal{F}L^q_{(\omega)}(\mathbb{R}^d)$ and if $\varphi \in \mathcal{D}^{(M_p)}(X)$, then Young's inequality gives

$$\|\varphi f\|_{\mathcal{F}L^q_{(\omega)}} = \|\mathcal{F}(\varphi f)\,\omega\|_{L^q} = (2\pi)^{-d/2}\|(\widehat{\varphi} * \widehat{f})\,\omega\|_{L^q}$$

$$\lesssim \||\widehat{\varphi}\,e^{M(\cdot)}| * |\widehat{f}\,\omega|\|_{L^q} \lesssim \|\widehat{f}\,\omega\|_{L^q} = \|f\|_{\mathcal{F}L^q_{(\omega)}},$$

if $\|\widehat{\varphi}\,e^{M(\cdot)}\|_{L^1}$ is finite. Since $\varphi \in \mathcal{D}^{(M_p)}(X)$, from Theorem 2.2 and Remark 2.2 it follows that for every $N > 0$ we have

$$|\widehat{\varphi}(\xi)|e^{M(\xi)} \lesssim e^{-(N+1)M(\xi)}e^{M(\xi)} = e^{-NM(\xi)}. \tag{3.3}$$

Therefore, $\|\widehat{\varphi}\,e^{M(\cdot)}\|_{L^p} < \infty$ for every $p \in [1, \infty]$, and (3.1) is proved.

It remains to prove (3.2). The inclusion in (3.2) is clear when $q_1 = q_2$ and $\omega_2 \lesssim \omega_1$. It remains to show that $\mathcal{F}L^q_{(\omega),loc}$ increases with respect to q. Assume, without any loss of generality, that $f \in (\mathcal{E}^{(M_p)})'(X)$, and that $\varphi \in \mathcal{D}^{(M_p)}(\mathbb{R}^d)$ is such that $\varphi \equiv 1$ in the neighborhood of $\operatorname{supp} f$. Choose $p \in [1, \infty]$ such that $1/q_1 + 1/p = 1/q_2 + 1$. Then, for a $e^{M(\cdot)}$-moderate weight ω, it follows from Young's inequality that

$$\|f\|_{\mathcal{F}L^{q_2}_{(\omega)}} \lesssim \|(\widehat{\varphi} * \widehat{f})\omega\|_{L^{q_2}} \lesssim \|\widehat{\varphi}e^{M(\cdot)}\|_{L^p}\|\widehat{f}\,\omega\|_{L^{q_1}} = C\|f\|_{\mathcal{F}L^{q_1}_{(\omega)}},$$

for some constant C, and the result follows.

Next we extend the definition of wave-front sets of Fourier-Lebesgue type given in [22, 35, 36].

Let $\{M_p\}$ satisfy $(M.1) - (M.3)'$ and let $M(\rho)$ denote its associated function. Furthermore, let $q \in [1, \infty]$, and $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone. If $f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ and $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$, then we define

$$|f|_{\mathcal{F}L_{(\omega)}^{q,\Gamma}} = |f|_{\mathcal{F}L_{(\omega),x}^{q,\Gamma}} \equiv \Big(\int_\Gamma |\widehat{f}(\xi)\omega(x,\xi)|^q \, d\xi \Big)^{1/q} \tag{3.4}$$

(with obvious interpretation when $q = \infty$). We note that $| \cdot |_{\mathcal{F}L_{(\omega),x}^{q,\Gamma}}$ defines a semi-norm on $(\mathcal{S}^{(1)})'(\mathbb{R}^d)$ which might attain the value $+\infty$. Since ω is $e^{M(\rho)}$-moderate it follows that different $x \in \mathbb{R}^d$ gives rise to equivalent semi-norms $|f|_{\mathcal{F}L_{(\omega),x}^{q,\Gamma}}$, see Remark 3.1. Furthermore, if $\Gamma = \mathbb{R}^d \setminus 0$, $f \in \mathcal{F}L_{(\omega)}^q(\mathbb{R}^d)$ and $q < \infty$, then $|f|_{\mathcal{F}L_{(\omega),x}^{q,\Gamma}}$ agrees with the Fourier Lebesgue norm $\|f\|_{\mathcal{F}L_{(\omega),x}^q}$ of f.

For the sake of notational convenience we set

$$\mathcal{B} = \mathcal{F}L_{(\omega)}^q = \mathcal{F}L_{(\omega)}^q(\mathbb{R}^d), \quad \text{and} \quad | \cdot |_{\mathcal{B}(\Gamma)} = | \cdot |_{\mathcal{F}L_{(\omega),x}^{q,\Gamma}}. \tag{3.5}$$

We let $\Theta_\mathcal{B}(f) = \Theta_{\mathcal{F}L_{(\omega)}^q}(f)$ be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that $|f|_{\mathcal{B}(\Gamma)} < \infty$, for some open conical neighborhood $\Gamma = \Gamma_\xi$ of ξ. We also let $\Sigma_\mathcal{B}(f)$ be the complement of $\Theta_\mathcal{B}(f)$ in $\mathbb{R}^d \setminus 0$. Then $\Theta_\mathcal{B}(f)$ and $\Sigma_\mathcal{B}(f)$ are open respectively closed subsets in $\mathbb{R}^d \setminus 0$, which are independent of the choice of $x \in \mathbb{R}^d$ in (3.4).

Definition 3.1. Let there be given a sequence $\{M_p\}$ which satisfies $(M.1) - (M.3)'$ and let $M(\rho)$ be its associated function. Furthermore, let $q \in [1, \infty]$, $\mathcal{B}$ be as in (3.5), and let X be an open subset of $\mathbb{R}^d$. If $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$, then the wave-front set of $f \in (\mathcal{D}^*)'(X)$, $\mathrm{WF}_\mathcal{B}(f) \equiv \mathrm{WF}_{\mathcal{F}L_{(\omega)}^q}(f)$ with respect to $\mathcal{B}$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbb{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_\mathcal{B}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(M_p)}(X)$ such that $\varphi(x_0) \neq 0$.

The set $\mathrm{WF}_\mathcal{B}(f)$ is a closed set in $\mathbb{R}^d \times (\mathbb{R}^d \setminus 0)$, since it is obvious that its complement is open. We also note that if $x \in \mathbb{R}^d$ is fixed and $\omega_0(\xi) = \omega(x, \xi)$, then $\mathrm{WF}_\mathcal{B}(f) = \mathrm{WF}_{\mathcal{F}L_{(\omega_0)}^q}(f)$, since $\Sigma_\mathcal{B}$ is independent of x.

The following theorem shows that wave-front sets with respect to $\mathcal{F}L_{(\omega)}^q$ satisfy appropriate micro-local properties. It also shows that such wave-front sets are decreasing with respect to the parameter q, and increasing with respect to the weight ω.

Theorem 3.1. *Let there be given a sequence $\{M_p\}$ which satisfy $(M.1) - (M.3)'$ and let $M(\rho)$ be its associated function. Furthermore, let $q, r \in [1, \infty]$, X be an open*

set in $\mathbb{R}^d$ and $\omega, \vartheta \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$ be such that

$$r \leq q, \quad \text{and} \quad \omega(x, \xi) \lesssim \vartheta(x, \xi).$$

Also let $\mathcal{B}$ be as in (3.5) and put $\mathcal{B}_0 = \mathcal{F}L^r_{(\vartheta)}(\mathbb{R}^d)$. If $f \in (\mathcal{D}^{(M_p)})'(X)$ and $\varphi \in \mathcal{D}^{(M_p)}(X)$ then

$$\mathrm{WF}_{\mathcal{B}}(\varphi f) \subseteq \mathrm{WF}_{\mathcal{B}_0}(f).$$

PROOF. When $M_p = p!^s$, $s > 1$, we recover [22, Theorem 2.1]. In fact, the more general situation when $\{M_p\}$ is an arbitrary sequence which satisfies $(M.1) - (M.3)'$ can be proved by using the idea of the proof of [22, Theorem 2.1] as follows.

By the definition it is sufficient to prove

$$\Sigma_{\mathcal{B}}(\varphi f) \subseteq \Sigma_{\mathcal{B}_0}(f)$$

when $\varphi \in \mathcal{D}^{(M_p)}(X)$, $\vartheta = \omega$ and $f \in (\mathcal{E}^{(M_p)})'(\mathbf{R}^d)$, since the statement only involves local assertions. For the same reasons we may assume that $\omega(x, \xi) = \omega(\xi)$ is independent of x. We prove the assertion for $r \in [1, \infty)$, and leave the case $r = \infty$ to the reader.

By using the idea of the proof of [39, Theorem 1.6.1] we conclude that if $f \in (\mathcal{E}^{(M_p)})'(\mathbf{R}^d)$ then there exists $N_0 > 0$ such that $|\widehat{f}(\xi)\omega(\xi)| \lesssim e^{N_0 M(|\xi|)}$.

Choose open cones Γ_1 and Γ_2 in $\mathbb{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. It is enough to prove that for every $N > 0$, there exist $C_N > 0$ such that

$$|\varphi f|_{\mathcal{B}(\Gamma_2)} \leq C_N \Big(|f|_{\mathcal{B}_0(\Gamma_1)} + \sup_{\xi \in \mathbb{R}^d} \big(|\widehat{f}(\xi)\omega(\xi)|e^{-NM(|\xi|)} \big) \Big) \tag{3.6}$$

when $\overline{\Gamma}_2 \subseteq \Gamma_1$.

Since $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$ by letting $F(\xi) = |\widehat{f}(\xi)\omega(\xi)|$ and $\psi(\xi) = |\widehat{\varphi}(\xi)|e^{M(|\xi|)}$ we have

$$\begin{aligned}
|\varphi f|_{\mathcal{B}(\Gamma_2)} &= \Big(\int_{\Gamma_2} |\mathcal{F}(\varphi f)(\xi)\omega(\xi)|^q \, d\xi \Big)^{1/q} \\
&\lesssim \Big(\int_{\Gamma_2} \Big(\int_{\mathbf{R}^d} \psi(\xi - \eta)F(\eta) \, d\eta \Big)^q d\xi \Big)^{1/q} \lesssim J_1 + J_2,
\end{aligned}$$

where

$$J_1 = \Big(\int_{\Gamma_2} \Big(\int_{\Gamma_1} \psi(\xi - \eta)F(\eta) \, d\eta \Big)^q d\xi \Big)^{1/q},$$

$$J_2 = \Big(\int_{\Gamma_2} \Big(\int_{\complement \Gamma_1} \psi(\xi - \eta)F(\eta) \, d\eta \Big)^q d\xi \Big)^{1/q}.$$

Let q_0 be chosen such that $1/r_0 + 1/r = 1 + 1/q$, and let χ_{Γ_1} be the characteristic function of Γ_1. Then Young's inequality gives

$$J_1 \leq \left(\int_{\mathbf{R}^d} \left(\int_{\Gamma_1} \psi(\xi - \eta) F(\eta)\, d\eta \right)^q d\xi \right)^{1/q}$$

$$= \|\psi * (\chi_{\Gamma_1} F)\|_{L^q} \leq \|\psi\|_{L^{r_0}} \|\chi_{\Gamma_1} F\|_{L^r} = C_\psi |f|_{\mathcal{B}_0(\Gamma_1)},$$

where $C_\psi = \|\psi\|_{L^{q_0}} < \infty$.

To estimate J_2, we note that since $\varphi \in \mathcal{D}^{(M_p)}(X)$, then by Theorem 2.2 it follows that for every $N > 0$ there exist $C_N > 0$ such that

$$\psi(\xi) = |\widehat{\varphi}(\xi)|\, e^{M(|\xi|)} \leq C_N e^{-(N+1)M(|\xi|)} e^{M(|\xi|)} \leq C_N e^{-NM(|\xi|)}. \qquad (3.7)$$

Furthermore, $\overline{\Gamma_2} \subseteq \Gamma_1$ implies that

$$|\xi - \eta| > 2c \max(|\xi|, |\eta|)$$

$$\geq c(|\xi| + |\eta|), \qquad \xi \in \Gamma_2, \ \eta \notin \Gamma_1 \qquad (3.8)$$

holds for some constant $c > 0$, since this is true when $1 = |\xi| \geq |\eta|$. Now, a combination of Lemma 2.1, (3.7) and (3.8) (together with the monotone increasing property of $M(\rho)$) implies that for every $N_1 > 0$ we have

$$\psi(\xi - \eta) \lesssim C e^{-2N_1(M(|\xi|) + M(|\eta|))},$$

which gives

$$J_2 \lesssim \left(\int_{\Gamma_2} \left(\int_{\complement\Gamma_1} e^{-2N_1(M(|\xi|)+M(|\eta|))} F(\eta)\, d\eta \right)^r d\xi \right)^{1/r}$$

$$\lesssim \left(\int_{\Gamma_2} \left(\int_{\complement\Gamma_1} e^{-2N_1(M(|\xi|)+M(|\eta|))} e^{N_1 M(|\eta|)} (e^{-N_1 M(|\eta|)} F(\eta))\, d\eta \right)^r d\xi \right)^{1/r}$$

$$\lesssim \sup_{\eta \in \mathbb{R}^d} |e^{-N_1 M(|\eta|)} F(\eta))|.$$

This implies (3.6) and the proof is finished.

3.1. Comparisons to other types of wave-front sets

Let $\omega \in \mathcal{M}_v(\mathbb{R}^{2d})$ be moderated with respect to the weight v of a polynomial growth at infinity, and let $f \in \mathcal{D}'(X)$. Then the wave frpont set $\mathrm{WF}_{\mathcal{F}L^q_{(\omega)}}(f)$ in Definition 3.1 agrees with the wave-front set introduced in [36, Definition 3.1]. Therefore, the information on regularity in the background of wave-front sets of Fourier-Lebesgue type in Definition 3.1 might be compared to the information obtained from the classical wave-front sets, cf. Example 4.9 in [36].

Next we compare the wave-front sets introduced in Definition 3.1 to the wave-front sets in spaces of ultradistributions given in [20, 32, 39].

Let $s > 1$ and let X be an open subset of $\mathbb{R}^d$. The ultradistribution $f \in (\mathcal{D}^{(s)})'(X)$ ($f \in (\mathcal{D}^{\{s\}})'(X)$) is (s)-micro-regular ($\{s\}$-micro-regular) at (x_0, ξ_0) if there exists $\varphi \in \mathcal{D}^{(s)}(X)$ ($\varphi \in \mathcal{D}^{\{s\}}(X)$) such that $\varphi(x) = 1$ in a neighborhood of x_0 and an open cone Γ which contains ξ_0 such that

$$|\mathcal{F}(\varphi f)(\xi)| \lesssim e^{-N|\xi|^{1/s}}, \quad \xi \in \Gamma,$$

for each $N > 0$ (for some $N > 0$). The (s)-wave-front set ($\{s\}$-wave-front set) of f, $\mathrm{WF}_{(s)}(f)$ ($\mathrm{WF}_{\{s\}}(f)$) is defined as the complement in $X \times \mathbb{R}^d \setminus 0$ of the set of all (x_0, ξ_0) where f is (s)-micro-regular ($\{s\}$-micro-regular), cf. [39, Definition 1.7.1].

The $\{s\}$-wave-front set $\mathrm{WF}_{\{s\}}(f)$ can be found in [32] and it coincides with certain wave-front set $\mathrm{WF}_L(f)$ introduced in [20, Chapter 8.4].

Next we modify the definitions from [36, 22].

Let there be given a sequence $\{M_p\}$ which satisfy $(M.1) - (M.3)'$ and let $M(\rho)$ be its associated function. Furthermore, let $\omega_j \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$, $q_j \in [1, \infty]$ when j belongs to some index set J, and let $\mathcal{B}$ be the array of spaces, given by

$$(\mathcal{B}_j) \equiv (\mathcal{B}_j)_{j \in J}, \quad \text{where} \quad \mathcal{B}_j = \mathcal{F}L_{(\omega_j)}^{q_j} = \mathcal{F}L_{(\omega_j)}^{q_j}(\mathbb{R}^d), \quad j \in J. \qquad (3.9)$$

If $f \in (\mathcal{D}^{(M_p)})'(\mathbb{R}^d)$, and $(\mathcal{B}_j)$ is given by (3.9), then we let $\Theta_{(\mathcal{B}_j)}^{\sup}(f)$ be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and each $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. We also let $\Theta_{(\mathcal{B}_j)}^{\inf}(f)$ be the set of all $\xi \in \mathbb{R}^d \setminus 0$ such that for some $\Gamma = \Gamma_\xi$ and some $j \in J$ it holds $|f|_{\mathcal{B}_j(\Gamma)} < \infty$. Finally we let $\Sigma_{(\mathcal{B}_j)}^{\sup}(f)$ and $\Sigma_{(\mathcal{B}_j)}^{\inf}(f)$ be the complements in $\mathbb{R}^d \setminus 0$ of $\Theta_{(\mathcal{B}_j)}^{\sup}(f)$ and $\Theta_{(\mathcal{B}_j)}^{\inf}(f)$ respectively.

Definition 3.2. Let there be given a sequence $\{M_p\}$ which satisfy $(M.1) - (M.3)'$ and let $M(\rho)$ be its associated function. Furthermore, let J be an index set, $q_j \in [1, \infty]$, $\omega_j \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$ when $j \in J$, $(\mathcal{B}_j)$ be as in (3.9), and let X be an open subset of $\mathbb{R}^d$.

1. The wave-front set of $f \in (\mathcal{D}^{(M_p)})'(X)$, of *sup-type* with respect to $(\mathcal{B}_j)$, $\mathrm{WF}_{(\mathcal{B}_j)}^{\sup}(f)$, consists of all pairs (x_0, ξ_0) in $X \times (\mathbb{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(\mathcal{B}_j)}^{\sup}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(M_p)}(X)$ such that $\varphi(x_0) \neq 0$;

2. The wave-front set of $f \in (\mathcal{D}^{(M_p)})'(X)$, of *inf-type* with respect to $(\mathcal{B}_j)$, $\mathrm{WF}_{(\mathcal{B}_j)}^{\inf}(f)$ consists of all pairs (x_0, ξ_0) in $X \times (\mathbb{R}^d \setminus 0)$ such that $\xi_0 \in \Sigma_{(\mathcal{B}_j)}^{\inf}(\varphi f)$ holds for each $\varphi \in \mathcal{D}^{(M_p)}(X)$ such that $\varphi(x_0) \neq 0$.

Now we are ready to rewrite the classical Gevrey wave-front sets $\mathrm{WF}_{\{s\}}(f)$ and $\mathrm{WF}_{(s)}(f)$ in terms of wave-front sets introduced in Definition 3.2.

Proposition 3.1. [22] *Let $s > 1$, and let $\mathcal{B}_j$ be the same as in (3.9) with $q_j \in [1, \infty]$ and $\omega_j(\xi) \equiv e^{j|\xi|^{1/s}}$. Then the following is true:*

1. if $f \in (\mathcal{D}^{\{s\}})'(\mathbb{R}^d)$, then

$$\mathrm{WF}_{(\mathcal{B}_j)}^{\inf}(f) = \bigcap_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) = \mathrm{WF}_{\{s\}}(f) \subseteq \mathrm{WF}_{(s)}(f);$$

2. if $f \in (\mathcal{D}^{(s)})'(\mathbb{R}^d)$, then

$$\mathrm{WF}_{(s)}(f) = \bigcup_{j>0} \mathrm{WF}_{\mathcal{B}_j}(f) \subseteq \mathrm{WF}_{(\mathcal{B}_j)}^{\sup}(f).$$

Remark 3.2. We recall that if $f \in \mathcal{D}'(\mathbb{R}^d)$, and $\omega_j(x, \xi) = \langle \xi \rangle^j$ for $j \in J = \mathbf{N}$, then it follows that $\mathrm{WF}_{(\mathcal{B}_j)}^{\sup}(f)$ in Definition 3.2 is equal to the standard wave front set $\mathrm{WF}(f)$ in Chapter VIII in [20].

3.2. Convolution

We finish the section by recalling that the convolution properties, valid for standard wave-front sets of Hörmander type, also hold for the wave-front sets of Fourier Lebesgue types, see [37, 38] for related results in the framework of tempered distributions. More generally, the following convolution result holds true.

Theorem 3.2. *Let there be given a sequence $\{M_p\}$ which satisfy $(M.1) - (M.3)'$ and let $M(\rho)$ be its associated function. Furthermore, let $q, q_1, q_2 \in [1, \infty]$ and let $\omega, \omega_1, \omega_2 \in \mathcal{M}_{M(\rho)}(\mathbb{R}^d)$ satisfy*

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q} \quad and \quad \omega(\xi) \lesssim \omega_1(\xi)\omega_2(\xi).$$

Then the convolution map $(f_1, f_2) \mapsto f_1 * f_2$ from $\mathcal{S}^{(1)}(\mathbb{R}^d) \times \mathcal{S}^{(1)}(\mathbb{R}^d)$ to $\mathcal{S}^{(1)}(\mathbb{R}^d)$ extends to a continuous mapping from $\mathcal{F}L_{(\omega_1)}^{q_1}(\mathbb{R}^d) \times \mathcal{F}L_{(\omega_2)}^{q_2}(\mathbb{R}^d)$ to $\mathcal{F}L_{(\omega)}^{q}(\mathbb{R}^d)$. This extension is unique if $q_1 < \infty$ or $q_2 < \infty$.

If $f_1 \in \mathcal{F}L_{(\omega_1), loc}^{q_1}(\mathbb{R}^d)$, $f_2 \in (\mathcal{D}^{(M_p)})'(\mathbb{R}^d)$ and f_1 or f_2 have compact supports, then

$$\mathrm{WF}_{\mathcal{F}L_{(\omega)}^{q}}(f_1 * f_2) \subseteq \{(x + y, \xi) \, ; \, x \in \operatorname{supp} f_1 \text{ and } (y, \xi) \in \mathrm{WF}_{\mathcal{F}L_{(\omega_2)}^{q_2}}(f_2)\}.$$

The proof is omitted, since the arguments for the first part of Theorem are the same as in the proof of [37, Lemma 2.1], taking into account that $\mathcal{S}^{(1)}$ is dense in $\mathcal{F}L^q_{(\omega)}$ when $q < \infty$. The second part of Theorem 3.2 can be proved in the same way as [23, Theorem 2.2].

4. *Modulation Spaces*

In this section we first recall the action of the short-time Fourier transform on Gelfand-Shilov spaces and their dual spaces, and then proceed with modulation spaces and their properties. Since the short-time Fourier transform gives a phase-space description of a function or distribution, we first extend Definition 2.1.

Definition 4.1. Let there be given sequences of positive numbers $(M_p)_{p \in \mathbb{N}_0}$, $(N_q)_{q \in \mathbb{N}_0}$, $(\tilde{M}_p)_{p \in \mathbb{N}_0}$, $(\tilde{N}_q)_{q \in \mathbb{N}_0}$ which satisfy $(M.1)$ and $(M.2)$.

We define $\mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A}(\mathbb{R}^{2d})$ to be the set of smooth functions $f \in C^\infty(\mathbb{R}^{2d})$ such that

$$\|x^{\alpha_1} \omega^{\alpha_2} \partial_x^{\beta_1} \partial_\omega^{\beta_2} f\|_{L^\infty} \leq C A^{|\alpha_1 + \alpha_2|} M_{|\alpha_1|} \tilde{M}_{|\alpha_2|} B^{|\beta_1 + \beta_2|} N_{|\beta_1|} \tilde{N}_{|\beta_2|},$$

$$\forall \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{N}_0^d\},$$

and for some $A, B, C > 0$. *Gelfand-Shilov spaces* are projective and inductive limits of $\mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A}(\mathbb{R}^{2d})$:

$$\Sigma^{N_q, \tilde{N}_q}_{M_p, \tilde{M}_p}(\mathbb{R}^{2d}) := \operatorname{proj} \lim_{A > 0, B > 0} \mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A}(\mathbb{R}^{2d});$$

$$\mathcal{S}^{N_q, \tilde{N}_q}_{M_p, \tilde{M}_p}(\mathbb{R}^{2d}) := \operatorname{ind} \lim_{A > 0, B > 0} \mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A}(\mathbb{R}^{2d}).$$

Clearly, the corresponding dual spaces are given by

$$(\Sigma^{N_q, \tilde{N}_q}_{M_p, \tilde{M}_p})'(\mathbb{R}^{2d}) := \operatorname{ind} \lim_{A > 0, B > 0} (\mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A})'(\mathbb{R}^{2d});$$

$$(\mathcal{S}^{N_q, \tilde{N}_q}_{M_p, \tilde{M}_p})'(\mathbb{R}^{2d}) := \operatorname{proj} \lim_{A > 0, B > 0} (\mathcal{S}^{N_q, \tilde{N}_q, B}_{M_p, \tilde{M}_p, A})'(\mathbb{R}^{2d}).$$

By Theorem 2.1, the Fourier transform is a homeomorphism from $\Sigma^{N_q, \tilde{N}_q}_{M_p, \tilde{M}_p}(\mathbb{R}^{2d})$ to $\Sigma^{M_p, \tilde{M}_p}_{N_q, \tilde{N}_q}(\mathbb{R}^{2d})$ and, if $\mathcal{F}_1 f$ denotes the partial Fourier transform of $f(x, \omega)$ with respect to the x variable, and if $\mathcal{F}_2 f$ denotes the partial Fourier transform of $f(x, \omega)$

with respect to the ω variable, then $\mathcal{F}_1$ and $\mathcal{F}_2$ are homeomorphisms from $\Sigma_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q}(\mathbb{R}^{2d})$ to $\Sigma_{M_p,\tilde{N}_q}^{N_q,\tilde{M}_p}(\mathbb{R}^{2d})$ and $\Sigma_{M_p,\tilde{N}_q}^{N_q,\tilde{M}_p}(\mathbb{R}^{2d})$, respectively. Similar facts hold when $\Sigma_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q}(\mathbb{R}^{2d})$ is replaced by $\mathcal{S}_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q}(\mathbb{R}^{2d})$, $(\Sigma_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q})'(\mathbb{R}^{2d})$ or $(\mathcal{S}_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q})'(\mathbb{R}^{2d})$.

When $M_p = \tilde{M}_p$ and $N_q = \tilde{N}_q$ we use usual abbreviated notation: $\mathcal{S}_{M_p}^{N_q}(\mathbb{R}^{2d}) = \mathcal{S}_{M_p,\tilde{M}_p}^{N_q,\tilde{N}_q}(\mathbb{R}^{2d})$ and similarly for other spaces.

4.1. Short-time Fourier transform

Let $(M_p)_{p\in\mathbb{N}_0}$ satisfy $(M.1)$ and $(M.2)$. For any given $f,g \in \mathcal{S}_{M_p}^{M_p}(\mathbb{R}^d)$ ($f,g \in \Sigma_{M_p}^{M_p}(\mathbb{R}^d)$, respectively) the short-time Fourier transform (STFT) of f with respect to the window g is given by

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(y)\,\overline{g(y-x)}\,e^{-2\pi i\langle \xi,y\rangle}\,dy \;=\; \langle f, M_\xi T_x g\rangle,$$

where the translations T_x and modulations M_ξ are given by (1.1).

The following theorem (and its variations) is a folklore, in particular in the framework of the duality between $\mathcal{S}(\mathbb{R}^{2d})$ and $\mathcal{S}'(\mathbb{R}^{2d})$. For Gelfand-Shilov spaces we refer to e.g. [19, 42, 44, 46].

Theorem 4.1. *Let there be given sequences $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ which satisfy $(M.1)$, $(M.2)$ and*

$$\{N.1\}: \quad (\exists H > 0)(\exists A > 0) \;\; p!^{1/2} \leq A H^p M_p, \;\; p \in \mathbb{N}_0.$$

If $f,g \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$, then $V_\phi f \in \mathcal{S}_{M_p,N_q}^{N_q,M_p}(\mathbb{R}^d d)$ and extends uniquely to a continuous map from $(\mathcal{S}_{M_p}^{N_q})'(\mathbb{R}^d) \times (\mathcal{S}_{N_q}^{M_p})'(\mathbb{R}^d)$ into $(\mathcal{S}_{M_p,N_q}^{N_q,M_p})'(\mathbb{R}^{2d})$.

Conversely, if $V_\phi f \in \mathcal{S}_{M_p,N_q}^{N_q,M_p}(\mathbb{R}^d d)$ then $f,g \in \mathcal{S}_{M_p}^{N_q}(\mathbb{R}^d)$.

Next, assume that $(M_p)_{p\in\mathbb{N}_0}$ and $(N_q)_{q\in\mathbb{N}_0}$ satisfy (M.1), (M.2) and

$$(N.1): \quad (\forall H > 0)(\exists A > 0) \;\; p!^{1/2} \leq A H^p M_p, \;\; p \in \mathbb{N}_0.$$

If $f,g \in \Sigma_{M_p}^{N_q}(\mathbb{R}^d)$, then $V_\phi f \in \Sigma_{M_p,N_q}^{N_q,M_p}(\mathbb{R}^d)$ and extends uniquely to a continuous map from $(\Sigma_{M_p}^{N_q})'(\mathbb{R}^d) \times (\Sigma_{N_q}^{M_p})'(\mathbb{R}^d)$ into $(\Sigma_{M_p,N_q}^{N_q,M_p})'(\mathbb{R}^{2d})$.

Conversely, if $V_\phi f \in \Sigma_{M_p,N_q}^{N_q,M_p}(\mathbb{R}^d d)$ then $f,g \in \Sigma_{M_p}^{N_q}(\mathbb{R}^d)$.

The conditions $\{N.1\}$ and $(N.1)$ are taken from [28] where they are called *non-triviality conditions* for the spaces $\mathcal{S}_{M_p}^{M_p}(\mathbb{R}^d)$ and $\Sigma_{M_p}^{M_p}(\mathbb{R}^d)$ respectively, see also [27].

We will also need the following proposition when proving that the wave-front sets of Fourier-Lebesgue and modulation space types are the same. The first part is an extension of [8, Proposition 4.2].

Proposition 4.1. *Let $\{M_p\}$ satisfies $(M.1) - (M.3)'$ and let $M(\rho)$ denotes its associated function. Then the following is true:*

1. *if $f \in (\mathcal{E}^{(M_p)})'(\mathbb{R}^d)$ and $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$, then*

$$|V_\phi f(x,\xi)| \lesssim e^{-hM(|x|)}e^{\varepsilon M(|\xi|)},$$

   for some $\varepsilon > 0$ and for every $h > 0$;

2. *if $f \in (\mathcal{D}^{(M_p)})'(\mathbb{R}^d)$ and $\phi \in \mathcal{D}^{(M_p)}(\mathbb{R}^d) \setminus 0$, then $f \in (\mathcal{E}^{(M_p)})'(\mathbb{R}^d)$, if and only if $\operatorname{supp} V_\phi f \subseteq K \times \mathbb{R}^d$ for some compact set K, and then*

$$|V_\phi f(x,\xi)| \lesssim e^{\varepsilon M(|\xi|)},$$

   for some $\varepsilon > 0$.

PROOF. We only prove (1) and (3). The other statements follow by similar arguments and are left for the reader. As before, we will use Remark 2.2 in our calculations. Recall, $f \in (\mathcal{E}^{(M_p)})'(\mathbf{R}^d)$ implies that

$$|\widehat{f}(\xi)| \lesssim e^{\varepsilon M(|\xi|)},$$

for some $\varepsilon > 0$, cf. [39, Theorem 1.6.1].

For $\phi \in \mathcal{S}^{(M_p)}(\mathbf{R}^d)$ and $\psi \in \mathcal{D}^{(M_p)}(\mathbf{R}^d)$ such that $\psi = 1$ in $\operatorname{supp} f$ by Theorem 4.1, Lemma 2.1 and Remark 2.2 it follows that

$$|V_\psi \phi(x,\xi)| \lesssim e^{-hM(|x|)-kM(|\xi|)},$$

for every $h, k > 0$. Now straight-forward calculations give

$$|V_\phi f(x,\xi)| = |(V_\phi(\psi f))(x,\xi)| \lesssim (|V_\psi \phi(x,\cdot)| * |\widehat{f}|)(\xi)$$

$$= \int |V_\psi \phi(x,\xi-\eta)||\widehat{f}(\eta)|\, d\eta \lesssim \int e^{-hM(|x|)-2\varepsilon M(|\xi-\eta|)}e^{\varepsilon M(|\eta|)}\, d\eta$$

$$\leq e^{-hM(|x|)} \int e^{-2\varepsilon M(|\eta|)+2\varepsilon M(|\xi|)+\varepsilon M(|\eta|)}\, d\eta \lesssim e^{-hM(|x|)+2\varepsilon M(|\xi|)},$$

and (1) follows.

Next we prove (3). First assume that $\phi \in \mathcal{D}^{(M_p)}(\mathbf{R}^d) \setminus 0$ and $f \in (\mathcal{E}^{(M_p)})'(\mathbf{R}^d)$. Since both ϕ and f have compact support, it follows that $\operatorname{supp}(V_\phi f) \subseteq K \times \mathbf{R}^d$. Furthermore, by slightly modifying the proof of [46, Theorem 2.5] we conclude thay

$$|V_\phi f(x, \xi)| \lesssim e^{\varepsilon(M(|x|) + M(|\xi|))},$$

for some $\varepsilon > 0$, see also [22, Proposition 3.2]. Since $V_\phi f(x, \xi)$ has compact support in the x-variable, it follows that

$$|V_\phi f(x, \xi)| \lesssim e^{\varepsilon M(|\xi|)}.$$

For the opposite direction, assume that $\operatorname{supp} V_\phi f \subseteq K \times \mathbf{R}^d$, for some compact set K. Assume that $\operatorname{supp} \phi \subseteq K$ and choose $\varphi \in \mathcal{D}^{(s)}(\mathbf{R}^d)$ such that $\operatorname{supp} \varphi \cap 2K = \emptyset$. Then

$$(f, \varphi) = (\|\phi\|_{L^2})^{-2} (V_\phi f, V_\phi \varphi) = 0,$$

which implies that f has compact support. Here the first equality is the Moyal's identity (cf. [16]). This implies that f has compact support and the condition $f \in (\mathcal{D}^{(M_p)})'(\mathbf{R}^d)$ now gives $f \in (\mathcal{E}^{(M_p)})'(\mathbf{R}^d)$.

4.2. Modulation spaces and their basic properties

The modulation space norms traditionally measure the joint time-frequency distribution of $f \in \mathcal{S}'$, we refer, for instance, to [11], [16, Ch. 11-13] and the original literature quoted there for various properties and applications. It is usually sufficient to observe modulation spaces with weights which admit at most polynomial growth at infinity. However the study of ultra-distributions requires a more general approach that includes the weights of exponential or even superexponential growth, cf. [8, 47]. Note that the general approach introduced already in [11] includes the weights of sub-exponential growth. We refer to [12, 13] for related but even more general constructions, based on the general theory of coorbit spaces.

Depending on the growth of the weight function m, different Gelfand-Shilov classes may be chosen as fitting test function spaces for modulation spaces, see [8, 42, 47]. The widest class of weights allowing to define modulation spaces is the weight class $\mathcal{N}$. A weight function m on $\mathbb{R}^d$ belongs to $\mathcal{N}$ if it is a continuous, positive function such that

$$m(z) = o(e^{cz^2}), \quad \text{for } |z| \to \infty, \quad \forall c > 0,$$

with $z \in \mathbb{R}^d$. For instance, every function $m(z) = e^{s|z|^b}$, with $s > 0$ and $0 \le b < 2$, is in $\mathcal{N}$. Thus, the weight m may grow faster than exponentially at infinity. For example, the choice $m \in \mathcal{N} \setminus \cup_v \mathcal{M}_v$, when the weights v satisfy the Beurling-Domar

condition from Introduction, is related to the spaces of quasianalytic functions, [9]. We notice that there is a limit in enlarging the weight class for modulation spaces, imposed by Hardy's theorem: if $m(z) \geq Ce^{cz^2}$, for some $c > \pi/2$, then the corresponding modulation spaces are trivial [18].

Definition 4.2. Let $m \in \mathcal{N}$, and g a non-zero *window* function in $\mathcal{S}_{1/2}^{1/2}(\mathbb{R}^d)$. For $1 \leq p, q \leq \infty$ the *modulation space* $M_m^{p,q}(\mathbb{R}^d)$ consists of all $f \in (\mathcal{S}_{1/2}^{1/2})'(\mathbb{R}^d)$ such that $V_g f \in L_m^{p,q}(\mathbb{R}^{2d})$ (weighted mixed-norm spaces). The norm on $M_m^{p,q}$ is

$$\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right)^{1/q}$$

(with obvious changes if either $p = \infty$ or $q = \infty$). If $p, q < \infty$, the modulation space $M_m^{p,q}$ is the norm completion of $\mathcal{S}_{1/2}^{1/2}$ in the $M_m^{p,q}$-norm. If $p = \infty$ or $q = \infty$, then $M_m^{p,q}$ is the completion of $\mathcal{S}_{1/2}^{1/2}$ in the weak* topology.

When $f, g \in \mathcal{S}^{(1)}(\mathbb{R}^d)$, the above integral is convergent thanks to Theorem 4.1. Namely, for a given $m \in \mathcal{M}_v$ there exist $l > 0$ such that $m(x,\omega) \leq Ce^{l\|(x,\omega)\|}$ and therefore

$$\left| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p m(x,\omega)^p \, dx \right)^{q/p} d\omega \right|$$

$$\leq C \left| \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p e^{lp\|(x,\omega)\|} \, dx \right)^{q/p} d\omega \right| < \infty$$

since by Theorems 4.1 and Theorem 2.1 we have $|V_g f(x,\omega)| < Ce^{-s\|(x,\omega)\|}$ for every $s > 0$. This implies $\mathcal{S}^{(1)} \subset M_m^{p,q}$.

In particular, when m is a polynomial weight of the form $m(x,\omega) = \langle x \rangle^t \langle \omega \rangle^s$ we will use the notation $M_{s,t}^{p,q}(\mathbb{R}^d)$ for the modulation spaces which consists of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M_{s,t}^{p,q}} \equiv \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_\phi f(x,\omega) \langle x \rangle^t \langle \omega \rangle^s|^p \, dx \right)^{q/p} d\omega \right)^{1/q} < \infty$$

(with obvious interpretation of the integrals when $p = \infty$ or $q = \infty$).

If $p = q$, we write M_m^p instead of $M_m^{p,p}$, and if $m(z) \equiv 1$ on $\mathbb{R}^{2d}$, then we write $M^{p,q}$ and M^p for $M_m^{p,q}$ and $M_m^{p,p}$, and so on.

In the next proposition we show that $M_m^{p,q}(\mathbb{R}^d)$ are Banach spaces whose definition is independent of the choice of the window $g \in M_v^1 \setminus \{0\}$. In order to do so, we need the adjoint of the short-time Fourier transform.

For given window $g \in \mathcal{S}^{(1)}$ and a function $F(x, \xi) \in L_m^{p,q}(\mathbb{R}^{2d})$ we (formally) define $V_g^* F$ by

$$\langle V_g^* F, f \rangle := \langle F, V_g f \rangle.$$

Proposition 4.2. *Let v be a submultiplicative weight. Fix $m \in \mathcal{M}_v$ and $g, \psi \in \mathcal{S}^{(1)}$, with $\langle g, \psi \rangle \neq 0$. Then*

1. *$V_g^* : L_m^{p,q}(\mathbb{R}^{2d}) \to M_m^{p,q}(\mathbb{R}^d)$, and*

$$\|V_g^* F\|_{M_m^{p,q}} \leq C \|V_\psi g\|_{L_v^1} \|F\|_{L_m^{p,q}}.$$

2. *The inversion formula holds: $I_{M_m^{p,q}} = \langle g, \psi \rangle^{-1} V_g^* V_\psi$, where $I_{M_m^{p,q}}$ stands for the identity operator.*

3. *$M_m^{p,q}(\mathbb{R}^d)$ are Banach spaces whose definition is independent on the choice of $g \in \mathcal{S}^{(1)} \setminus \{0\}$.*

4. *The space of admissible windows can be extended from $\mathcal{S}^{(1)}$ to M_v^1.*

PROOF. We refer to [8] for the proof which is based on the proof of [16, Proposition 11.3.2.]. Note that in (4) the density of $\mathcal{S}^{(1)}$ in $M_m^{p,q}$ is essential. This fact is not obvious, and we refer to [6] for the proof. Then we may proceed by using the standard arguments, cf. [16, Theorem 11.3.7].

The following theorem lists some basic properties of modulation spaces. We refer to [11, 16, 19, 33, 34, 43, 46] for the proof.

Theorem 4.2. *Let $p, q, p_j, q_j \in [1, \infty]$ and $s, t, s_j, t_j \in \mathbb{R}$, $j = 1, 2$. Then:*

1. *$M_{s,t}^{p,q}(\mathbb{R}^d)$ are Banach spaces, independent of the choice of $\phi \in \mathcal{S}(\mathbb{R}^d) \setminus 0$;*

2. *if $p_1 \leq p_2$, $q_1 \leq q_2$, $s_2 \leq s_1$ and $t_2 \leq t_1$, then*

$$\mathcal{S}(\mathbb{R}^d) \subseteq M_{s_1,t_1}^{p_1,q_1}(\mathbb{R}^d) \subseteq M_{s_2,t_2}^{p_2,q_2}(\mathbb{R}^d) \subseteq \mathcal{S}'(\mathbb{R}^d);$$

3. *$\cap_{s,t} M_{s,t}^{p,q}(\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d)$, $\quad \cup_{s,t} M_{s,t}^{p,q}(\mathbb{R}^d) = \mathcal{S}'(\mathbb{R}^d)$;*

4. *Let $1 \leq p, q \leq \infty$, and let $w_s(x, \omega) = e^{s\|(x,\omega)\|}$, $x, \omega \in \mathbb{R}^d$. Then*

$$\Sigma_1^1(\mathbb{R}^d) = \mathcal{S}^{(1)}(\mathbb{R}^d) = \bigcap_{s \geq 0} M_{w_s}^{p,q}(\mathbb{R}^d), \quad (\Sigma_1^1)'(\mathbb{R}^d) = \bigcup_{s \geq 0} M_{1/w_s}^{p,q}(\mathbb{R}^d),$$

$$\mathcal{S}_1^1(\mathbb{R}^d) = \mathcal{S}^{\{1\}}(\mathbb{R}^d) = \bigcup_{s > 0} M_{w_s}^{p,q}(\mathbb{R}^d), \quad (\mathcal{S}_1^1)'(\mathbb{R}^d) = \bigcap_{s > 0} M_{1/w_s}^{p,q}(\mathbb{R}^d).$$

5. *For $p, q \in [1, \infty)$, the dual of $M^{p,q}_{s,t}(\mathbb{R}^d)$ is $M^{p',q'}_{-s,-t}(\mathbb{R}^d)$, where $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$.*

Remark 4.1. In the context of quasianalytic Gelfand-Shilov spaces, we recall (a special case of) [46, Theorem 3.9]: Let $s, t > 1/2$ and set

$$w_h(x, \omega) \equiv e^{h(|x|^{1/t} + |\omega|^{1/s})}, \quad h > 0, \ x, \omega \in \mathbb{R}^d.$$

Then

$$\Sigma^s_t(\mathbb{R}^d) = \bigcap_{h>0} M^{p,q}_{w_h}(\mathbb{R}^d), \quad (\Sigma^s_t)'(\mathbb{R}^d) = \bigcup_{h>0} M^{p,q}_{1/w_h}(\mathbb{R}^d),$$

$$\mathcal{S}^s_t(\mathbb{R}^d) = \bigcup_{h>0} M^{p,q}_{w_h}(\mathbb{R}^d), \quad (\mathcal{S}^s_t)'(\mathbb{R}^d) = \bigcap_{h>0} M^{p,q}_{1/w_h}(\mathbb{R}^d).$$

Modulation spaces include the following well-know function spaces:

1. $M^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and $M^2_{t,0}(\mathbb{R}^d) = L^2_t(\mathbb{R}^d)$;

2. The Feichtinger algebra: $M^1(\mathbb{R}^d) = S_0(\mathbb{R}^d)$;

3. Sobolev spaces: $M^2_{0,s}(\mathbb{R}^d) = H^2_s(\mathbb{R}^d) = \{ f \mid \hat{f}(\omega)\langle\omega\rangle^s \in L^2(\mathbb{R}^d) \}$;

4. Shubin spaces: $M^2_s(\mathbb{R}^d) = L^2_s(\mathbb{R}^d) \cap H^2_s(\mathbb{R}^d) = Q_s(\mathbb{R}^d)$, cf. [41].

5. *The invariance property of Wave-front sets*

Next we define wave-front sets with respect to modulation spaces and show that they agree with corresponding wave-front sets of Fourier Lebesgue types. More precisely, we prove that [36, Theorem 6.1] holds if the weights of polynomial growth are replaced by more general submultiplicative weights.

Let there be given a sequence $\{M_p\}$ which satisfies $(M.1) - (M.3)'$ and let $M(\rho)$ denote its associated function. Furthermore, let $p, q \in [1, \infty]$, and $\Gamma \subseteq \mathbb{R}^d \setminus 0$ be an open cone. If $f \in (\mathcal{S}^{(1)})'(\mathbb{R}^d)$ and $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$, then we define

$$|f|_{\mathcal{B}(\Gamma)} = |f|_{\mathcal{B}(\phi,\Gamma)} \equiv \left(\int_\Gamma \left(\int_{\mathbb{R}^d} |V_\phi f(x, \xi)\omega(x, \xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q}$$

$$\text{when} \quad \mathcal{B} = M^{p,q}_{(\omega)} = M^{p,q}_{(\omega)}(\mathbb{R}^d). \quad (5.1)$$

We note that $|f|_{\mathcal{B}(\Gamma)} = \|f\|_{M^{p,q}_{(\omega)}}$ when $\Gamma = \mathbb{R}^d \setminus 0$ and $\phi \in \mathcal{S}^{(s)}(\mathbb{R}^d)$, and that $|f|_{\mathcal{B}(\phi,\Gamma)}$ might attain $+\infty$.

Furthermore, when $\mathcal{B} = M_{(\omega)}^{p,q}$, the sets $\Theta_{\mathcal{B}}(f)$, $\Sigma_{\mathcal{B}}(f)$ and $\mathrm{WF}_{\mathcal{B}}(f)$ with respect to the modulation space $\mathcal{B}$ are defined in the same way as in Section 3., after replacing the semi-norms of Fourier Lebesgue types in (3.4) with the semi-norms in (5.1).

Proposition 5.1. *Let there be given a sequence of positive numbers* $(M_p)_{p \in \mathbb{N}_0}$ *which satisfies* $(M.1) - (M.3)'$, *and let* $M(\rho)$, $\rho > 0$, *be its associated function. If* $f \in (\mathcal{D}^{(M_p)})'(\mathbb{R}^d)$ *then* $\mathrm{WF}_{M_{(\omega)}^{p,q}}(f)$ *is independent of* p *and* $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d) \setminus 0$ *in* (5.1).

PROOF. We may assume that $f \in (\mathcal{E}^{(M_p)})'(\mathbb{R}^d)$ and that $\omega(x, \xi) = \omega(\xi)$ since the statements only concern local assertions.

We follow the idea of the proof of [22, Theorem 3,1], and in order to prove that $\mathrm{WF}_{M_{(\omega)}^{p,q}}(f)$ is independent of $\phi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d) \setminus 0$, we assume that $\phi, \phi_1 \in \mathcal{S}^{(M_p)}(\mathbb{R}^d) \setminus 0$ and let $| \cdot |_{\mathcal{C}_1(\Gamma)}$ be the semi-norm in (5.1) after ϕ has been replaced by ϕ_1. Let Γ_1 and Γ_2 be open cones in $\mathbb{R}^d$ such that $\overline{\Gamma_2} \subseteq \Gamma_1$. The asserted independency of ϕ follows if we prove that

$$|f|_{\mathcal{C}(\Gamma_2)} \leq C(|f|_{\mathcal{C}_1(\Gamma_1)} + 1), \tag{5.2}$$

for some positive constant C. Let

$$\Omega_1 = \{ (x, \xi) \, ; \, \xi \in \Gamma_1 \} \subseteq \mathbb{R}^{2d} \quad \text{and} \quad \Omega_2 = \complement \Omega_1 \subseteq \mathbb{R}^{2d},$$

with characteristic functions χ_1 and χ_2 respectively, and set

$$F_k(x, \xi) = |V_{\phi_1} f(x, \xi)| \omega(\xi) \chi_k(x, \xi), \quad k = 1, 2,$$

and $G = |V_\phi \phi_1(x, \xi) e^{M(|\xi|)}|$. Since ω is v-moderate, it follows from [16, Lemma 11.3.3] that

$$|V_\phi f(x, \xi) \omega(x, \xi)| \lesssim \big((F_1 + F_2) * G\big)(x, \xi),$$

which implies that

$$|f|_{\mathcal{C}(\Gamma_2)} \lesssim J_1 + J_2,$$

where

$$J_k = \left(\int_{\Gamma_2} \left(\int_{\mathbb{R}^d} |(F_k * G)(x, \xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q}, \quad k = 1, 2.$$

By Young's inequality

$$J_1 \leq \|F_1 * G\|_{L_1^{p,q}} \leq \|G\|_{L^1} \|F_1\|_{L_1^{p,q}} = C|f|_{\mathcal{C}_1(\Gamma_1)},$$

where $C = \|G\|_{L^1} = \|V_\phi \phi_1(x,\xi) e^{M(|\xi|)}\|_{L^1} < \infty$, in view of Proposition 4.1.

Next we consider J_2. For $\xi \in \Gamma_2$ fixed and integrating over $\eta \in \complement\Gamma_1$, it follows from Propositon 4.1 and Lemma 2.1 that for some $\varepsilon > 0$ and every $N, h > 0$ we have that $|(F_2 * G)(x,\xi)|$ is bounded by

$$
C \iint_{\mathbb{R}^{2d}} e^{-NM(|y|)} e^{\varepsilon M(|\eta|)} e^{-h(M(|x-y|)+M(|\xi-\eta|))} e^{M(|\xi-\eta|)} \, dy d\eta,
$$

for some constant $C > 0$. Therefore, there exist a constant $c > 0$ such that

$$
|(F_2 * G)(x,\xi)|
$$
$$
\lesssim \iint_{\mathbb{R}^{2d}} e^{-NM(|y|)} e^{\varepsilon M(|\eta|)} e^{-hM(|x-y|)-hc(M(|\xi|)+M(|\eta|))} e^{(M(|\xi|)+M(|\eta|))} \, dy d\eta
$$
$$
\lesssim e^{(-N+h)M(|x|)} e^{(1-hc)M(|\xi|)} \iint_{\mathbb{R}^{2d}} e^{-hM(|y|)} e^{(1+\varepsilon-hc)M(|\eta|)} \, dy d\eta,
$$
$$
\lesssim e^{(-N+h)M(|x|)} e^{(1-hc)M(|\xi|)} < \infty,
$$

since $N > 0$ and h can be chosen arbitrarily. Therefore

$$
J_2 = \left(\int_{\Gamma_2} \left(\int_{\mathbb{R}^d} |(F_2 * G)(x,\xi)|^p \, dx \right)^{q/p} d\xi \right)^{1/q}
$$
$$
\lesssim \left(\int_{\Gamma_2} \left(\int_{\mathbb{R}^d} \left(e^{(-N+h)M(|x|)} e^{(1-hc)M(|\xi|)} \right)^p dx \right)^{q/p} d\xi \right)^{1/q} < \infty.
$$

This proves that (5.2), and hence $\mathrm{WF}_{\mathcal{C}}(f)$ is independent of $\phi \in \mathcal{S}^{(s)}(\mathbb{R}^d) \setminus 0$.

The main result of this section, Theorem 5.1, now follows from Proposition 5.1 and calculations given in the proof of [22, Theorem 3.1]. For that reason we omit the proof.

Theorem 5.1. *Let there be given a sequence of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ which satisfies $(M.1) - (M.3)'$, and let $M(\rho)$, $\rho > 0$ be its associated function. Let $p, q \in [1,\infty]$ and $\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$. If $f \in (\mathcal{D}^{(M_p)})'(\mathbb{R}^d)$ then*

$$
\mathrm{WF}_{\mathcal{F}L^q_{(\omega)}}(f) = \mathrm{WF}_{M^{p,q}_{(\omega)}}(f).
$$

Finally, note that for a given sequence of positive numbers $(M_p)_{p\in\mathbb{N}_0}$ which satisfies $(M.1) - (M.3)'$, and its associated function $M(\rho)$, $\rho > 0$, when $p, q \in [1,\infty]$,

$\omega \in \mathcal{M}_{M(\rho)}(\mathbb{R}^{2d})$ and $f \in (\mathcal{E}^{(M_p)})'(\mathbb{R}^d)$, then it follows from the definition of wave-front sets that then

$$f \in \mathcal{B} \quad \Longleftrightarrow \quad \mathrm{WF}_{\mathcal{B}}(f) = \emptyset,$$

when $\mathcal{B}$ is equal to $\mathcal{F}L^q_{(\omega)}$ or $M^{p,q}_{(\omega)}$. In particular, by Theorem 5.1 we obtain

$$\mathcal{F}L^q_{(\omega)} \cap (\mathcal{E}^{(M_p)})'(\mathbb{R}^d) = M^{p,q}_{(\omega)} \cap (\mathcal{E}^{(M_p)})'(\mathbb{R}^d),$$

and we recover Corollary 6.2 in [36], Theorem 2.1 and Remark 4.6 in [40].

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Department for Mathematics and Informatics
University of Novi Sad
Trg D. Obradovića 4, 21000
Novi Sad, Serbia
e-mail: nenad.teofanov@dmi.uns.ac.rs