THE HAMILTON-JACOBI EQUATION, INTEGRABILITY, AND NONHOLONOMIC SYSTEMS

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Abstract. By examining the linkage between conservation laws and symmetry, we explain why it appears there should not be an analogue of a complete integral for the Hamilton-Jacobi equation for integrable nonholonomic systems.

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1. Introduction

The Hamilton-Jacobi theory is at the heart of the symplectic and variational structure of Hamiltonian mechanics. There have been a large number of extensions of this theory to systems with nonholonomic constraints. A partial list includes [4, 5, 12, 15, 22, 23, 27, 28, 30]. In Hamiltonian mechanics there are two versions of the (time independent) Hamilton-Jacobi theory. The weaker version is related to the existence of an *integral* of the equation, and corresponds to the existence of an invariant Lagrangian submanifold. The stronger version is related to the existence of a *complete integral* of the equation, and corresponds to the complete integrability, in the sense of Liouville-Arnold, of the system. All the mentioned extensions to systems with nonholonomic constraints refer to the weaker of the two versions of the Hamilton-Jacobi theory, but several of these references point out, or at least mention, the interest of understanding the possible relations, in the nonholonomic context, between integrability and Hamilton-Jacobi theory. To our knowledge, however, there are no discussions of this relationship, nor clarifications of its very existence.

The aim of this work is to provide such an analysis. In terms that need to be made precise, our conclusion is that this relation fails. At the most primitive level, this is due to the failure of the equality of conservation laws and symmetries in nonholonomic mechanics. This equality is encoded in the interplay between dynamics and geometry so special to the Hamiltonian case. As a rule, outside the variational world, Noether theorem does not hold and symmetries need not provide integrals of motion, and outside the symplectic world, integrals of motion do not generate symmetries. The remnants of symplecticity that persist to the nonholonomic world are not enough to save this situation and this makes the link between symmetry and integrability, if any, different from that of the Hamiltonian case. The stronger version of the Hamilton-Jacobi theory embodies this very remarkable – and very

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special – link and implies that there may be no ‘nonholonomic Hamilton-Jacobi theory’ describing integrability of nonholonomic systems.

2. Nonholonomic systems

For a Lagrangian system with n-dimensional configuration manifold Q, a nonholonomic constraint is a nonintegrable distribution $\mathscr{D}$ on Q of rank $n-k$ for some $1 \le k \le n-2$. For our purposes it is however preferable to pass to the Hamiltonian formulation, where the nonholonomic system is given by a k-codimensional constraint manifold $M \subset P = T^*Q$ on which there is a symplectic distribution $H \subset TM$ of rank $2n-2k$ [9]. In the points of M, the restriction ω_H of the canonical 2-form ω to the distribution H has rank $2n-2k$. The *nonholonomic vector field* X_f^{nh} of a function $f : M \to \mathbb{R}$ is the unique vector field on M that lies in the distribution H and whose contraction with ω, in the points of M, agrees with the restriction $\mathrm{d}h_H$ of $\mathrm{d}h$ to H, in symbols

$$X_f^{nh} \lrcorner\, \omega_H = \mathrm{d} f_H \qquad (\text{on } M)\,.$$

If $h : P \to \mathbb{R}$ is the Hamiltonian of the uncostrained system, then the nonholonomic system is given by the vector field X_h^{nh} on M.

The equations of motion may be given in bracket form as well, as

$$\dot f = \{f,h\}^{nh} \qquad \forall f : M \to \mathbb{R}\,,$$

where $\{,\}^{nh}$ is the almost-Poisson bracket on M defined by

$$\{f,g\}^{nh} = \omega_H(X_f^{nh}, X_g^{nh})$$

for all functions $f, g : M \to \mathbb{R}$ [29, 6].

A function $f : M \to \mathbb{R}$ is an integral of motion of the nonholonomic system if and only if its almost-Poisson bracket with h vanishes,

$$\{f,h\}^{nh} = 0\,.$$

When $\{f,g\}^{nh} = 0$ we will say that f and g are in H-involution.

More details on the distributional symplectic formulation and on the almost-Poisson formulation of nonholonomic mechanics may be found in the quoted references. The important thing to note is that, in all cases, the almost-Poisson bracket fails to satisfy the Jacobi identity, which is satisfied if and only if the constraints are semi-holonomic. It is useful to have a measure of this failure, and it is quantified by a natural quadratic differential operator closely related to the Schouten bracket, which is totally antisymmetric and defined by

$$\Sigma(f,g,h) = \{f,\{g,h\}^{nh}\}^{nh} + \{g,\{h,f\}^{nh}\}^{nh} + \{h,\{f,g\}^{nh}\}^{nh}\,.$$

The failure of the Jacobi identity has the consequence that, at variance from the Hamiltonian case, in noholonomic mechanics there is no Lie algebra anti-homomorphism between functions and nonholonomic vector fields: $X_{\{f,g\}}^{nh}$ need not equal $[X_g^{nh}, X_f^{nh}]$. Hence, the nonholonomic vector fields of Poisson commuting functions need not commute. This fact is at the basis of the failure of the Hamilton-Jacobi method in nonholonomic mechanics.

3. The Hamilton-Jacobi equation

3.1. **A single invariant Lagrangian submanifold.** Consider a Hamiltonian system given by a Hamiltonian h on a $2n$-dimensional symplectic manifold (P, ω). Denote by X_f the Hamiltonian vector field of a function f on P. A submanifold $L \subset P$ is Lagrangian if ω vanishes on L. Tangency of X_h to a submanifold $N \subset P$ is a prerequisite for the invariance of N under the flow of X_h. The geometric-dynamical fact beyond the weaker version of the Hamilton-Jacobi equation is the following fact (see e.g. [31]):

Proposition 3.1. *Let L be a connected Lagrangian submanifold of P. X_h is tangent to L if and only if h is constant on L.*

Proof. The condition that X_h is tangent to L is $X_h \subset TL$, and the symplectic perpendicular of this is $X_h^\perp = \mathrm{d}h \supset TL^\perp = TL$ since L is Lagrangian. q.e.d.

The Hamilton-Jacobi equation provides a local description of this situation. Each point of a Lagrangian submanifold L has a neighbourhood U equipped with Darboux coordinates (q, p) (i.e., coordinates such that $\omega_{|U} = \mathrm{d}q \wedge \mathrm{d}p$) such that $L \cap U$ is the graph $p = \frac{\partial S}{\partial q}(q)$, with $q \mapsto S(q) \in \mathbb{R}$ a smooth function [3]. Conversely, a graph $p = \frac{\partial S}{\partial q}(q)$ is Lagrangian. Hence, at a local level, Proposition 3.1 can be restated as follows:

Corollary 3.2. *Let (q, p) be Darboux coordinates and $q \mapsto S(q)$ a smooth real functions.*

1. *X_h is tangent to the (Lagrangian) submanifold L given by $p = \frac{\partial S}{\partial q}(q)$ if and only if*
$$h\left(q, \frac{\partial S}{\partial q}(q)\right) = e \tag{1}$$
for some $e \in \mathbb{R}$.
2. *If condition 1 holds then, using the q as coordinates on L, the restriction of X_h to L is*
$$\dot{q} = \frac{\partial h}{\partial p}\left(q, \frac{\partial S}{\partial q}(q)\right). \tag{2}$$

In the classical literature, the function S is called an *integral* of the Hamilton-Jacobi equation (1). The first statement follows at once from Proposition 3.1, but it may as well be proven by an elementary computation: in the points of L, the Lie derivative
$$£_{X_h}\left(p - \frac{\partial S}{\partial q}\right) = -\frac{\partial h}{\partial q} - \frac{\partial^2 S}{\partial q \partial q}\frac{\partial h}{\partial p} = -\frac{\partial}{\partial q} h\left(q, \frac{\partial S}{\partial q}\right)$$
is itself a derivative. This fact, which reflects the interplay of geometry and dynamics peculiar to Hamiltonian systems, we refer to as the *lesser Hamilton-Jacobi miracle.* The second statement of the Corollary follows from Hamilton's equations, and the parameterization of L as $p = \frac{\partial S}{\partial q}(q)$.

Corollary 3.2 is local, but gets a geometric flavour in the case of a cotangent bundle ($P = T^*Q$ equipped with the canonical symplectic structure) and under the hypothesis of Lagrangian submanifolds transversal to the fibers of the cotangent bundle projection $\pi : T^*Q \to Q$. Such Lagrangian submanifolds are the images of closed, hence locally exact, 1-forms on Q. Thus, following [1], under these hypotheses, Corollary 3.2 may be equivalently restated in the following form, which also appears in many of the quoted references on nonholonomic extensions of the Hamilton-Jacobi equation:

Corollary 3.3. [1] *Given a smooth function $S : Q \to \mathbb{R}$, the following two conditions are equivalent:*

1. *S satisfies the Hamilton-Jacobi equation* $h \circ \mathrm{d}S = \mathrm{const.}$
2. *For every curve $c(t)$ in Q satisfying*

$$c'(t) = \pi_* X_h (\mathrm{d}S\,(c(t)))$$

the curve $t \to \mathrm{d}S\,(c(t))$ is an integral curve of X_h.

The restriction to the case of an exact 1-form is clearly not necessary in this statement. It retains its validity for a closed 1-form σ – that is, for any Lagrangian submanifold of a cotangent bundle that is transversal to the fibers of π. This point of view is taken e.g. in [11], where the system $\mathrm{d}(h \circ \sigma) = 0$, $\mathrm{d}\sigma = 0$ is called the geometric Hamilton-Jacobi equation. The transversality hypothesis is a source of well known analytic difficulties and gives these statements, compared to Proposition 3.1, a local character.

Results like Corollary 3.3 are sometimes considered as an integration tool: if one knows an integral $S(q)$ of the Hamilton-Jacobi equation, and if one is able to determine a solution of the restricted system of n equations (2), then lifting the latter gives a solution of the Hamiltonian system. In reality, this use of the Hamilton-Jacobi equation in Hamiltonian mechanics is seldom of any utility (with perhaps the exception of the so called weak KAM theory, where however viscosity solutions of the Hamilton-Jacobi equation are considered, see e.g. [20, 16]). On this point, see the comments on page 60 of [31].

There have been several generalizations of this type of result to systems with nonholonomic constraints, see the already mentioned [4, 5, 12, 15, 22, 23, 27, 28, 30]. Most of these generalizations are based on Corollary 3.3, with the search for a function $S : Q \to \mathbb{R}$ replaced by the search for an M-valued 1-form γ defined on Q whose differential vanishes when evaluated on $\mathscr{D} \times \mathscr{D}$. In some cases, this technique has been illustrated through the integration of simple nonholonomic systems, the solutions of which are known by direct integration of the equation of motion.

3.2. **A foliation by invariant Lagrangian submanifolds.** A *complete integral* of the Hamilton-Jacobi equation is a pair of real functions $S(q,a)$ and $e(a)$, defined for q and a in open subsets of $\mathbb{R}^n$, that satisfy (1) and are such that $\frac{\partial^2 S}{\partial q \partial a}$ is everywhere nonsingular. The previous analysis shows that the function $S(q,a)$ defines a foliation by Lagrangian submanifolds $p = \frac{\partial S}{\partial q}(q,a)$, each one contained in the level set $h = e(a)$ and having X_h tangent to it. However, there is more: the flow of X_h is conjugate to a linear flow on these submanifolds; this is the *greater Hamilton-Jacobi miracle.* Precisely, S is the generating function of a symplectic change of coordinates $(q,p)\mapsto(b,a)$ that conjugates the system to the system with Hamiltonian $e(a)$. This has various consequences, which all pertain to the Hamilton-Jacobi theory: the functions $a_1,\ldots,a_n$ are integrals of motion in involution, their joint level sets are the Lagrangian submanifolds $p=\frac{\partial S}{\partial q}$, and the coordinates b have the meaning of times along the flows of the Hamiltonian vector fields of the as; the significance, and the importance, of this last fact will be clear later.

Thus, the existence of a complete integral of the Hamilton-Jacobi equation is a local characterization of the 'complete' (or 'Liouville', or 'Liouville-Arnold') integrability of the Hamiltonian system: the existence of the maximal number of independent integrals of motion which are pairwise in involution, and are defined in some open and invariant subset of the phase space [2].[1] As is well known, complete integrability amounts to the flow being conjugate to a linear flow on either cylinders or tori. For the sake of definitness, we will consider here only the compact case, that is, quasi-periodicity of the dynamics.

As already mentioned, several treatments of nonholonomic Hamilton-Jacobi equation point out, or at least mention, the interest of understanding the possible relations, in the nonholonomic context, between integrability and Hamilton-Jacobi theory. Reference [12] goes so far as to define the notion of 'complete integral' of a nonholonomic Hamilton-Jacobi equation, without however relating it to integrability. As far as we can tell, there are no discussions of this relationship, nor clarifications of its possibility or limitations. We provide one such analysis in the next section.

4. Nonholonomic Hamilton-Jacobi and integrability

4.1. **Integrability as quasi-periodicity.** In order to do this analysis, we identify 'integrability' with quasi-periodicity (that is, the flow is conjugate to a linear flow on tori) and base our analysis on the following characterization of quasi-periodicity, that clarifies how many integrals and how much symmetry is needed for it. This would seem rather natural, but we know of no earlier clear formulation than the one due to Bogoyavlenskij [10] (see also [21])

Theorem 4.1. *Assume that on a d-dimensional manifold M there exist, for some $1\le k\le d$,*

1. *A submersion $f=(f_1,\ldots,f_{d-k}):M\to\mathbb{R}^{d-k}$ with compact and connected fibers.*
2. *k everywhere linearly independent vector fields $Y_1,\ldots,Y_k$ which pairwise commute and are such that $\pounds_{Y_j}f_i=0$ for all i, j.*

Then

1. *The fibers of f are k-tori.*

[1] The requirement of invariance is necessary to give a meaning to all this: locally, in a neighbourhood of every regular point, by the Hamiltonian flow box theorem (see e.g. [24]), every Hamiltonian system has n integrals of motion in involution.

2. *If a vector field X on M has* $f_1, \dots, f_{d-k}$ *as integrals of motion and* $Y_1, \dots, Y_k$ *as dynamical symmetries, then its flow is conjugate to a linear flow on the fibers of* f.

First integrals and symmetries with these properties always exist (at least at a semi-global level) when the dynamics is conjugate to a linear flow on tori, so in a sense this is more a characterization of quasi-periodicity than a criterion for quasi-periodicity. In this regard, Bogoyavlenskij's theorem is different from the Liouville-Arnol'd theorem [2] and its noncommutative generalizations for systems with invariant isotropic tori of any dimension (see [26, 25] and for a review [17]), that in the Hamiltonian case derive integrability from only one set of data – the integrals of motion. This is made possible by the Lie algebra anti-homomorphism between functions and Hamiltonian vector fields ($X_{\{f,g\}} = [X_g, X_f]$) specific to the Hamiltonian case, that provides a mechanism to produce (commuting) dynamical symmetries out of (commuting) integrals of motion. For instance, in the Liouville-Arnold case one has $d = 2n$ and the n commuting dynamical symmetries are the Hamiltonian vector fields of the n commuting integrals of motion.

It should be noted that the special link between integrals of motion and dynamical symmetries, peculiar to Hamiltonian mechanics, is present, at least at a local level, in the Hamilton-Jacobi method as well. In fact, as noticed above, a complete integral of the Hamilton-Jacobi equation provides two sets of data: the integrals of motion *and* the times along the flows of their Hamiltonian vector fields, namely the integrals of motion *and* their Hamiltonian vector fields. Thus, the greater Hamilton-Jacobi miracle is that a complete integral provides both the integrals of motion and the dynamical symmetries leading to integrability in the Liouville-Arnold sense. (This link is present in noncommutative integrability as well, but there is no Hamilton-Jacobi description for it: the Hamilton-Jacobi equation can describe only Lagrangian foliations, not other isotropic foliations).

4.2. **Commuting integrals of motion.** In order to illustrate this situation we investigate the implications of the existence of a number of almost–Poisson commuting integrals of motion of a nonholonomic system. Once again, assume that configuration space Q is n-dimensional, and that the constraint manifold M has codimension k.

LEMMA 4.2. *Consider* $p \geq 1$ *functionally independent functions* $f_1, \dots, f_p : M \to \mathbb{R}$ *and their nonholonomic vector fields* $X^{nh}_{f_1}, \dots, X^{nh}_{f_p}$.

1. *The* $X^{nh}_{f_i}$*s are tangent to the joint level sets of* $f_1, \dots, f_p$ *if and only if* $\{f_i, f_j\}^{nh} = 0$ *for all* i, j.
2. ω_H *vanishes on the distribution* $\mathrm{span}(X^{nh}_{f_1}, \dots, X^{nh}_{f_p}) \subset H$ *if and only if* $\{f_i, f_j\}^{nh} = 0$ *for all* i, j.
3. *Assume* $\{f_i, f_j\}^{nh} = 0$ *for all* i, j. *Then the vector fields* $X^{nh}_{f_i}$ *pairwise commute if and only if the matrix of vector fields whose* ij *component is* $\Sigma(f_i, f_j, \cdot)$ *vanishes identically.*

Proof. The first statement follows from $\pounds_{X^{nh}_{f_i}} f_j = \{f_i, f_j\}^{nh}$. The second from the fact that the vector fields $X^{nh}_{f_i}$s are sections of H and the definition of the almost-Poisson bracket. Given any three functions f_i, f_j, g, the identity $\pounds_{[X^{nh}_i, X^{nh}_j]} g = \{f_i, \{f_j, g\}^{nh}\}^{nh} - \{f_j, \{f_i, g\}^{nh}\}^{nh} = \Sigma(f_i, f_j, g) - \{g, \{f_i, f_j\}^{nh}\}^{nh}$ holds. The third statement follows from this immediately. q.e.d.

Note that the first two staments owe their validity to the fact that the Jacobi identity does not enter their proof; the failure of the Jacobi identity manifests itself in the third.

4.3. **Integrability *a là* Hamilton-Jacobi.** By this, we mean a mechanism that derives the dynamical symmetries needed for integrability from a set of integrals of motion in involution. Mimicking the Hamiltonian case, the obvious candidate would be the nonholonomic vector fields of the commuting integrals. In fact, this is more than just mimicking: no other way of deriving a vector field from a function, without any further information (e.g., symmetry groups, see below), has been devised so far in nonholonomic mechanics. So, this is likely the only general mechanism one might have. However, it obviously faces the difficulty that, due to the failure of the Jacobi identity, the nonholonomic vector fields of functions that almost-Poisson commute need not commute.

Let us anyway look into this possibility. Since ω_H has rank $2(n-k)$, there cannot be more than $n-k$ functionally independent functions on M which are pairwise in involution and whose nonholonomic vector fields are everywhere linearly independent. Let us thus assume that we have this maximum number of functionally independent integrals of motion $f_1, \ldots, f_{n-k} : M \to \mathbb{R}$ which are pairwise in H-involution, $\{f_i, f_j\}^{nh} = 0$ for all i, j, and whose joint level sets

$$N_c = f_1^{-1}(c_1) \cap \cdots \cap f_{n-k}^{-1}(c_{n-k}), \qquad c = (c_1, \ldots, c_{n-k}),$$

are compact and connected. We assume that the Hamiltonian h of the system, which is an integral of motion, is one of the f_is, say $f_1 = h$.

Since M has dimension $2n-k$, the invariant sets N_c have dimension n. We would like them to be tori and the dynamics to be quasi-periodic on them. According to Theorem 4.1 we need $\dim M - (n-k) = n$ commuting dynamical symmetries which are tangent to the N_cs. By Lemma 4.2, the $X^{nh}_{f_i}$s have some of the properties to be considered as candidates. However, even if these vector fields are tangent to the N_cs, there are some obstructions: not only are there too few vector fields, but neither is it granted that they are dynamical symmetries ($\{h, f_i\} = 0$ does not automatically implies $[X^{nh}_h, X^{nh}_{f_i}] = 0$) nor that they commute. And even if the $X^{nh}_{f_i}$s commute, more dynamical symmetries or more integrals of motion are needed, that were not obviously present initially and seem to definitely lie outside the Hamilton-Jacobi realm.

4.4. **Integrability without Hamilton-Jacobi.** We do not know any example of integrable nonholonomic systems that fits the previous, Hamilton-Jacobi-like setting. In known examples of integrable nonholonomic systems, the dynamical symmetries identifiable through Bogoyavlenskij's characterization do not arise in this way.

In certain cases, both the integrals of motion and the dynamical symmetries may be interpreted as generated by an underlying symmetry group, but in a 'gauge-like' way (see [8] or [18]). In this mechanism, integrals of motion are constructed as momenta of suitable varying linear combinations of the infinitesimal generators of the group action. It is enlighting to compare the two mechanisms in the simplest example.

Example 4.3. (The nonholonomic oscillator [7]) Consider the nonholonomic system with $Q = \mathbb{R}^3 \ni (x, y, z)$, $h = \frac{1}{2}(p_x^2+p_y^2+p_z^2)+\frac{1}{2}y^2$ and $M = \{(x, y, z, p_x, p_y, p_z) \in T^*\mathbb{R}^3 |\ p_z = y\,p_x\}$. In this case $H = \mathrm{span}_{\mathbb{R}}\left\{y\partial_z + \partial_x, \partial_y, \partial_{p_x}, \partial_{p_y}\right\}$ and the nonholonomic vector field of the Hamiltonian h is

$$X_h^{nh} = p_x\partial_x + p_y\partial_y + y\,p_x\partial_z - \frac{y}{1+y^2}\,p_x\,p_y\partial_{p_x} - y\,\partial_{p_y}.$$

By inspection, X_h^{nh} is invariant by translations in x and z. This $\mathbb{R}^2$ symmetry leads to the gauge momentum conservation law $k = p_x\sqrt{1+y^2}$. Together, h and k are two integrals of motion in H-involution. Their joint level sets are compact, but X_h^{nh} and

$$X_k^{nh} = \frac{1}{\sqrt{1+y^2}}(\partial_x + y\partial_z)$$

do not commute. However, the infinitesimal generators ∂_x and ∂_z of the $\mathbb{R}^2$-action are dynamical symmetries of X_h^{nh}, commute, and tangent to the joint level sets of h and k. Therefore, the system has the two independent integrals of motion h and k and the three dynamical symmetries X_h^{nh}, ∂_x and ∂_z and is integrable, with quasi-periodic dynamics on tori of dimension three.

A similar situation is met in other simple nonholonomic systems, such as a heavy ball rolling inside a vertical cylinder or inside a convex surface of revolution, and the rolling disk; for this aspect of these well known systems see [8, 18, 19]. It is not clear if the gauge method is a mechanism capable of explaining the integrability of some natural class of symmetric nonholonomic systems, but certainly it is not amenable to a Hamilton-Jacobi treatment.

4.5. **More general integrability scenarios.** It is not clear that quasi-periodicity is *the* characteristic property of integrable nonholonomic systems. Besides the obvious possibility of non-compact invariant sets, for which we could repeat the analysis done so far, with the same conclusions, a well known reason is that quasi-periodicity is often met only after a time reparameterization. At a deeper level, another possibility is that one need not insist that the dynamical symmetries commute. Instead, they may generate a finite dimensional solvable Lie algebra. Then it is still possible to integrate the differential equations. Perhaps structurally not as nice, but we are able to solve the differential equations when the vector fields still have a "solvable structure" but what they generate is not finite dimensional. The kind of thing we have in mind is illustrated by the following

EXAMPLE 4.4. Consider the problem of integrating the flow of the vector field $Z = aX + bY$ if a and b are constants, $X = \partial_x$ and $Y = f(x, y)\partial_y$. It is straightforward to find integral curves of Z, even though the algebra of vector fields generated by X and Y under the Lie bracket is not finite dimensional. In essence, this means that one still has to compute an integral, and can not do things by purely algebraic steps.

But these possibilities are even more remote from the Hamilton-Jacobi description.

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