On the trajectory generation of the hydrodynamic Chaplygin sleigh

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***Abstract*— In this paper we consider the asymptotic behaviour and the trajectory generation problem for the Chaplygin sleigh interacting with a potential fluid. We investigate which trajectories can be obtained, at least asymptotically as t tents to infinity, by controlling some of the coordinates (shape–control variables) and using the theory of reconstruction. Moreover we support our conclusions via numerical simulations.**

I. INTRODUCTION

The control of nonholonomic mechanical systems is a well-studied field of research, extremely active in the last forty years, with strong relations with other fields of mathematics and with applications to control, trajectory generation and to robotics (a non-exhaustive list of references that show the huge research in the field includes [26], [21], [29], [3], [24], [32], [7], [20], [33]). Nonholonomic systems are mechanical systems, in which not all the velocity directions are permitted. Simple examples are a penny that rolls without sliding on a table, the snakeboard, the so–called Chaplygin sleigh and many others (see e.g. [3], [15]).

In this work we investigate the trajectory generation problem of the so–called hydrodynamical Chaplygin sleigh introduced in [17]. The Chaplygin sleigh is a nonholonomic system introduced by Chaplygin in 1905 [13], that models a platform, supported on two points and on a blade, that moves on a horizontal plane. The blade is free to rotate about the axis orthogonal to the plane and passing through the contact point of the blade with the plane, under the nonholonomic constraint that displacements in the direction orthogonal to the blade are forbidden. While in [17] the authors analyze only the dynamics of the hydrodynamic Chapligin sleigh, here we investigate the trajectory generation by adding a moving mass on the platform whose coordinates play the role of shape control variables. To the best of our knowledge, besides the research efforts, the controllability or even the small time local controllability by using external forces as controls has not been proved yet, even for the classical Chaplygin sleigh, (see e.g. [5], [3], [28], [6], [33]). Some authors (see e.g. [11], [30], [31]) have investigated the controllability, motion planning and trajectory tracking exploiting the interaction with different kind of surfaces. Controllability results have been recently proved by using several shape control variables [33],[1] while we use only one additional mass. The dynamics of the Chaplygin sleigh with addition of masses has been deeply investigated in [8], [10] and in the presence of friction in [9], but without taking into account the interaction with a potential fluid and without the use of reconstruction from periodic orbits. A first investigation on the control of a Chaplygin sleigh in an ideal fluid has been carried out in [2], but with only one shape parameter. In this work we investigate the more interesting case of two shape parameters.

Inspired by [16], we show that we are able to predict the kind of trajectory attainable after a periodic control loop, using reconstruction techniques from reduced periodic orbits, and exploiting the symmetry properties of the system (see [19], [22], [1], [15] for basic aspects on reconstruction theory and [16] for a first link of reconstruction techniques and trajectory generation). In [16], the authors apply reconstruction techniques to systems whose equations of motion depend linearly on the controls. Our hydrodynamic Chaplygin sleigh is a neat example where the equations are more involved, because of affine and polynomial dependance on the controls and the techniques in [16] cannot be directly applied. The use of tools of reconstruction theory is ensured by proving the existence of periodic solutions of a certain affine system of ODE's with periodic coefficients, and the (controlled) reduced equations of the hydrodynamic Chaplygin sleigh turn out to be of this type for some choice of the controls.

The article is organised as follows. In Section II we introduce the notation and basic tools of nonholonomic systems with symmetry and geometric control theory, specifying the particular case of control systems on principal bundles. Then we review the main results on reconstruction from (reduced–)periodic orbits. In Section III we introduce an example of nonholonomic control system on a principle bundle: the hydrodynamic Chaplygin sleigh that is the classical Chaplygin sleigh interacting with a potential fluid. In Section IV, we show which kind of trajectories can be obtained by using periodic controls and the reconstruction techniques outlined above. We also support our theoretical predictions by numerical simulations. A short Section of Conclusions and perspectives for future works follows.

Unless differently said, all mathematical objects are assumed to be smooth, all vector fields are assumed to be complete, every Lie group is assumed to be connected, and every group action to be free, proper and, for the sake of simplicity, with trivial isotropy. All numerical simulations and graphs were made with the software Mathematica©.

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[1]In [33] shape variables are obtained by adding moving masses and rotors to the sleigh.

II. Basic notions on nonholonomic systems, controllability and reconstruction theory

Definition 1: A *nonholonomic mechanical system with symmetry* is a quadruple $(Q, L, \mathcal{D}, G)$, where $(Q, L, \mathcal{D})$ is a nonholonomic mechanical system, in which Q is an n–dimensional configuration manifold, $L : TQ \longrightarrow \mathbb{R}$ $L = T - V$ is a mechanical Lagrangian, with T and V the kinetic and potential energy, respectively, and $\mathcal{D}$ a constant rank non–integrable distribution on Q. G is a Lie group that acts on Q leaving either the Lagrangian and the constraint invariant.

We consider the notion of nonholonomic mechanical shape control system, in which the control input are given in terms in term of an internal 'shape' variable

Definition 2: A nonholonomic mechanical shape–control system is a sextuple $(Q, L, \mathcal{D}, \mathcal{S}, U, G)$, where $(Q, L, \mathcal{D}, G)$ is a nonholonomic mechanical system with symmetry. $\mathcal{S} := Q/G$, the so–called *shape space*, represents the controlled variables, and is assumed to be diffeomorphic to $\mathbb{R}^h$, for some positive integer h and U diffeomorphic to $\mathbb{R}^h$ as well, is the control space.

Let $(g, p, s, \dot{s})$ be coordinates on $\mathcal{D}$ adapted to the constraint and to the symmetry. Following [29], the equations of motion are then

$$\begin{cases} \ddot{s} = f(s, \dot{s}, p) + u \\ \dot{g} = q(J(s)p + A(s)\dot{s}) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, \dot{s}\rangle + \langle C(s)\dot{s}, \dot{s}\rangle \end{cases} \tag{1}$$

where $u : t \longrightarrow U$ an admissible control, and define a dynamical system on $\mathcal{D}$. The invariance properties of the constraint and of the Lagrangian, guarantee that equations of motion (1) define a dynamical system on the principal bundle $\mathcal{D} \longrightarrow \mathcal{D}/G$. If one assumes to directly control the shape variables, the equations of motion read:

$$\begin{cases} \dot{s} = u \\ \dot{g} = g(J(s)p + A(s)u) \\ \dot{p} = \langle M(s)p, p\rangle + \langle N(s)p, u\rangle + \langle C(s)u, u\rangle \end{cases} \tag{2}$$

with $u : t \longmapsto u(t)$ is an admissible control. As above equations (2) define a dynamical system on the principal bundle $\mathcal{D} \to \mathcal{D}/G$.

Remark 3: In Section III we will consider the case in which $Q = G \times \mathcal{S}$ and G acts on Q leaving L, $\mathcal{D}$ and $\mathcal{S}$ invariant. Precisely for the hydrodynamic Chaplygin system $G = SE(2)$ and $\mathcal{S}$ is the product of two intervals of $\mathbb{R}$.

A. Basic reconstruction techniques

The reconstruction of the dynamics from reduced equilibria and reduced periodic orbits has been well studied in [19], [22], when the symmetry group is compact and in [1] in the non–compact case. In this subsection we shortly review the basic results of reconstruction theory in the simplest framework, of free and proper group actions. We consider a Lie group G that acts on a manifold M. The freeness and properness of the action guarantee that the quotient space M/G has a manifold structure and $\pi : M \longrightarrow M/G$ is a principal bundle with structural group G. Let X be a G–equivariant vector field on M, then there exists a vector field $\hat{X}$ on M/G, π–related to X.

Definition 4:
- Let $m_0 \in M$. A G–orbit $\mathcal{O}_{m_0} = G{\cdot}m_0$ is a relative equilibrium for X, if it is invariant with respect to the flow of X.
- A G-invariant subset $\mathcal{P}$ of M is called a relative periodic orbit for X, if its projection by $\pi_{\mathcal{P}}$ on the quotient manifold M/G is a periodic orbit of $\hat{X}$. We call a periodic orbit on M/G a loop.

Let $\mathcal{P}$ be a relative periodic orbit and γ a curve in $\mathcal{P}$. By the periodicity of the reduced dynamics, the integral curves of the complete system, that pass through $\gamma(0)$, returns periodically, with period $\tau > 0$, to the G–orbit through $\gamma(0)$. The freeness of the action of G on M guarantees that $\forall\gamma$ in $\mathcal{P}$ there exists a unique $p(\hat{\gamma})$ in G such that

$$\phi^X_\tau(\gamma) = \psi_{p(\hat{\gamma})}(\gamma)\,,$$

where ϕ^X_τ is the flow of X at time τ, ψ_g is the action of G on M, $\hat{\gamma}$ is the projection of γ on M/G with respect to π, and the map $p : \mathcal{P} \to G$, $\gamma \mapsto p = p(\hat{\gamma})$ is the so–called *phase* [16]. The phase p is a piecewise smooth map, constant along the orbits of X (i.e. $p \circ \phi^X_t = p$, $\forall t$) and it is equivariant with respect to conjugation, that is $p(h \cdot \gamma) = h\, p(\hat{\gamma})h^{-1}$, $\forall h \in G, \forall \gamma \in \mathcal{P}$.

Let us now introduce the following

Definition 5:
- A flow of a vector field is called *quasi-periodic* with k frequencies if there exist a differentiable map which conjugates it to a linear flow on a torus $\mathbb{T}^k$.
- [16] A flow of a vector field is called *spiral flow* if it is an action of $\mathbb{R}$ on $\mathbb{T}^k \times \mathbb{R}^n$ as

$$(t, \alpha, z) \to (\alpha + tw\,(mod\ 1), z + tv)$$

with $w \in \mathbb{R}^k$ and $v \in \mathbb{R}$, $v \neq 0$.

Then the following Proposition holds.

Proposition 6: [19], [22], [1] Let $\mathcal{P}$ be a relative periodic orbit of X and let $\hat{\mathcal{P}}$ the projection of $\mathcal{P}$ on $\mathcal{D}/G$ with respect to $\pi_{\mathcal{P}}$. Then

i) if the group G is compact, the flow of X over $\hat{\mathcal{P}}$ is quasi–periodic with at most $rankG + 1$ frequencies;
ii) if G is non–compact, the flow of X over $\hat{\mathcal{P}}$ is either quasi–periodic, or a spiral flow.

The non–compact case is the most frequent and also the most interesting. For example one can say more on which of the two behaviours (quasi–periodic or spiral) is "generic", by investigating the group G. A behaviour is called *generic* depending on the *codimension* of the two sets

$$\mathfrak{g}_{\mathfrak{T}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a torus}\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{\xi \in \mathfrak{g} \quad | \quad K(\xi) \text{ is a subgroup isomorphic to } \mathbb{R}\}\,.$$

More precisely, according to [1], if the codimension of $\mathfrak{g}_{\mathfrak{T}}$ (respectively $\mathfrak{g}_{\mathfrak{R}}$) is low, the corresponding quasi–periodic flow (respectively spiral flow) is *generic*, on the other hand, if its codimension is high,[2] we call the corresponding flow *special*. (For more details and examples see [1]).

[2]Here *low* and *high* depend on the relations between the dimensions of $\mathfrak{g}_{\mathfrak{T}}$ and $\mathfrak{g}_{\mathfrak{R}}$.

III. **The model**

The hydrodynamic Chaplygin sleigh is a planar rigid body that slides on a horizontal plane immersed in an ideal incompressible and irrotational fluid. The body is supported at three points, two of which slide freely without friction while the third is a blade, that cannot move transversely with respect to itself (see [25] for the classical Chaplygin sleigh and [17] for the hydrodynamical one).

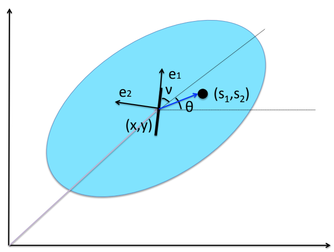

Fig. 1. The hydrodynamic Chaplygin sleigh

A. *Hydrodynamics*

Following [17], we consider an elliptic planar rigid body of mass M, major semi–axis A and minor semi–axis B, immersed in a potential incompressible fluid, with a mass m free to move on the platform. Let (e_x, e_y) denote an orthonormal inertial reference frame fixed in space and (e_1, e_2) an orthonormal 'body' reference frame attached to the platform. We assume, without any restriction, that the body frame is centered in the center of mass (x, y) of the platform and that the axis e_1 is oriented as the blade (see Figure 1). θ denotes the angle that the major semi-axes of the platform forms with the x–axis. The coordinates (θ, x, y) on $S^1 \times \mathbb{R}^2$ identify the position and the orientation of the rigid body with respect to the fixed inertial frame. Since a configuration of the platform is given by an element of $SE(2)$, (θ, x, y) identifies an element $g \in SE(2)$. Let Ω and $V = (v_1, v_2)$ denote, respectively, the angular and the translational velocity of the platform in the the body frame representation, and $\dot\theta$ and $(\dot x, \dot y)$, the same velocities in the space representation. The velocity in the body representation in a configuration g of $SE(2)$ is related to the one in the space representation by multiplication of an element of $SE(2)$, precisely:

$$(\dot\theta \ \dot x \ \dot y)^T = g(\Omega \ v_1 \ v_2)^T \, . \tag{3}$$

Therefore $(\theta, x, y, \Omega, v_1, v_2)$ are left trivialized coordinates on $SE(2) \times \mathbb{R}^3$, where we identify $\mathfrak{se}(2)$ with $\mathbb{R}^3$ as vector spaces.

Let (s_1, s_2) be the coordinates of the mass m (see Figure 1) and (v_{s_1}, v_{s_2}) its velocity in the body frame representation. Thus the configuration space of the system formed by the platform and the mass is a Lie group diffeomorphic to $SE(2) \times \mathbb{R}^2$ and the tangent bundle to the configuration space is diffeomorphic to $SE(2) \times \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^2$. We equip $T(SE(2) \times \mathbb{R}^2)$ with local coordinates $(\theta, x, y, s_1, s_2, \Omega, v_1, v_2, v_{s_1}, v_{s_2})$ in the body frame representation, and $(\theta, x, y, s_1, s_2, \dot\theta, \dot x, \dot y, \dot s_1, \dot s_2)$ in the space frame representation.

By denoting with $\Gamma = (\Omega, v_1, v_2, v_{s_1}, v_{s_2})^T$ the velocity of the system in the body frame representation, the kinetic energy of the body reads

$$T^B = \frac{1}{2}\Gamma^T \begin{pmatrix} I + m(s_1^2 + s_2^2) & -ms_2 & ms_1 & -ms_2 & ms_1 \\ -ms_2 & m+M & 0 & m & 0 \\ ms_1 & 0 & m+M & 0 & m \\ -ms_2 & m & 0 & m & 0 \\ ms_1 & 0 & m & 0 & m \end{pmatrix} \Gamma$$

By taking into account the motion of the potential fluid that surrounds the body (see e.g. the classical book of Lamb [23] for details), it can be shown that the pressure forces exerted by the fluid are actually kinetic terms [34]. Therefore the system given by the body and the fluid is a geodesic dynamic with Lagrangian given by the sum of the kinetic energy of the body and the one of the fluid. Denoting by $u = u(x, t)$ the fluid's velocity at the point x, the kinetic energy of the fluid is

$$T^f = \int_{\mathbb{R}^2 \setminus \mathcal{B}} \frac{|u(x)|^2}{2} \, dx \, ,$$

where $\mathcal{B}$ represents the domain occupied by the elliptic platform. It is well known that the velocity field of an incompressible potential fluid can be determined by setting $u = \nabla\Phi$, for some function Φ called stream function and by solving the Neumann problem in the exterior domain

$$\begin{cases} \Delta\Phi = 0 \quad x \in \mathbb{R}^2 \setminus \mathcal{B} \\ \frac{\partial\Phi}{\partial n} = (V + \Omega \times x) \cdot n \, , \quad x \in \partial\mathcal{B} \\ |\Phi| \to 0 \quad |x| \to \infty \end{cases}$$

where n the unit exterior normal to $\partial\mathcal{B}$. Moreover, since the rigid body is an ellipsis, by applying Kirchhoff decomposition and added masse theory [26], the kinetic energy of the fluid can be expressed as the quadratic form

$$T^f = \frac{1}{2}\Gamma^T \begin{pmatrix} \frac{\pi\rho}{4}K & 0 & 0 & 0 & 0 \\ 0 & \pi\rho H & \frac{\pi\rho}{2}K\sin 2\nu & 0 & 0 \\ 0 & \frac{\pi\rho}{2}K\sin 2\nu & \pi\rho H & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Gamma \, , \tag{4}$$

where we set $K = A^2 - B^2$ and $H = B^2 \sin^2\nu + A^2 \cos^2\nu$, where ν is the angle that the blade possibly forms with the major semi-axes (see Figure 1), ρ is the density of the fluid, and as above A is the major and B the minor semi–axis. The Lagrangian of the system is then given by the sum of the kinetic energy of the body and of the fluid:

$$L = T^b + T^f \, . \tag{5}$$

Kirchhoff's equations for a (planar) rigid body on a potential fluid are Lie–Poisson equations on the dual of the Lie algebra of $SE(2)$ (see e.g. [27], [20] for details on that) or Euler–Poincaré equations on the Lie algebra, if one considers their Lagrangian formulation.

B. The nonholonomic constraint and the symmetry of the system

The hydrodynamic Chaplygin sleigh is an LL–system, see [17] for details. In the case under study, according to Definition 2, the group action is given by the lift of left multiplication by $SE(2)$ and represents the invariance of the Lagrangian under translations and rotations in space. By assuming that the axes of the body frame are oriented as the principal axes of inertia of the platform and that the blade is oriented as the major principal axis of inertia (i.e $\nu = 0$), the nonholonomic constraint, that forces the blade to slide only along the e_1 direction, is

$$-\dot{x}\sin\theta + \dot{y}\cos\theta = 0, \text{ or in body coordinates } v_2 = 0\,. \quad (6)$$

C. Equations of motion

The conjugate momenta are,

$$p_\Omega = \frac{\partial L}{\partial \Omega} \qquad p_1 = \frac{\partial L}{\partial v_1}\,. \quad (7)$$

By expressing the momenta in function of Ω and v_1 and by taking into account the nonholonomic constraint, we get

$$\Omega = \frac{m\Xi s_2 + (p_\Omega + mv_{s_2}s_1 - mv_{s_1}s_2)(m + M + B^2\pi\rho)}{m^2 s_2^2 - (m + M + B^2\pi\rho)(T + \frac{K^2\pi\rho}{4})}$$

$$v_1 = \frac{[\Xi(I + ms_1^2) + m(p_\Omega - mv_{s_2}s_1)s_2 + mp_1 s_2^2] + K^2\Xi\pi\rho}{2m^2 s_2^2 - (m + M + B^2\pi\rho)(T + \frac{K^2\pi\rho}{4})}$$

$$v_2 = 0 \quad (8)$$

where we set: $\Xi = p_1 - mv_{s_1}$, $T = I + ms_1^2 + ms_2^2$ and as above $K = A^2 - B^2$. The reduced equations are

$$\dot{p}_\Omega = -v_1 m\left(\frac{v_{s_2}}{2} + s_1\Omega\right), \quad \dot{p}_1 = \Omega m\left(\frac{v_{s_2}}{2} + s_1\Omega\right) \quad (9)$$

$$\dot{s}_1 = u_1, \qquad \dot{s}_2 = u_2 \quad (10)$$

where the equations for p_Ω and p_{v_1} are the so called *momentum equations*. If one integrates the momentum equations obtains the dynamics of the group configuration variables by the reconstruction equations (3) and the nonholonomic constraint (6). Moreover observe that equations (8), up to a premultiplication for the group variables, and (9),(10), are exactly of the type (2) and are affine and polynomial in the controls.

IV. Trajectory generation of the hydrodynamic Chaplygin sleigh

In this section we assume to be able to assign the velocities, $\dot{s}_1$, $\dot{s}_2$, of the moving mass as functions of time, that is as shape control functions. We then show that using periodic controls it is possible to make the sleigh moving along a circle or spiraling away in a certain direction as t tends to infinity.

A. The strategy

In the spirit of Proposition 6 in Section II , the control strategy uses periodic controls to produce periodic solutions of the reduced equation (9), whose reconstructed trajectories are either quasi–periodic or spirals (see Definition 5). More precisely we give conditions under which the infinitesimal generator of the phase, coming from a periodic (loop) solution of the reduced equations, generates a circular dynamic or a spiral flow in certain directions.

As already mentioned the symmetry group of the hydrodynamic Chaplygin sleigh is non–compact and the (semi–algebraic) sets $\mathfrak{g}_{\mathfrak{T}}$ and $\mathfrak{g}_{\mathfrak{R}}$ are

$$\mathfrak{g}_{\mathfrak{T}} := \{(\xi_{v_1}, \xi_\Omega) \in \mathbb{R}\times\mathbb{R} \quad | \quad \xi_\Omega \neq 0\} \cup \{(0,0)\}$$
$$\mathfrak{g}_{\mathfrak{R}} := \{(\xi_{v_1}, \xi_\Omega) \in \mathbb{R}\times\mathbb{R} \quad | \quad \xi_\Omega = 0, v_1 \neq 0\}\,,$$

where $\xi = (\xi_{v_1}, \xi_\Omega)$ is the infinitesimal generator of the phase, and codim$(\mathfrak{g}_{\mathfrak{T}}) = 0$, codim$(\mathfrak{g}_{\mathfrak{R}}) = 1$.[3]

By Proposition 6, the generic reconstructed behaviour is then the quasi–periodic flow and the special one is the spiral flow. Given a loop in the reduced space, the values of ξ_{v_1} and ξ_Ω determine the type of the reconstructed dynamics. Precisely: if $\xi_\Omega \neq 0$, the hydrodynamic Chaplygin sleigh moves along a circle. On the other hand, if $\xi_\Omega = 0$, it moves spiralling along a certain direction. Given periodic controls, conditions under which there exists at least one periodic solution of the momentum equation (9), are not straightforward. See Section below for a proof of this fact.

B. Extended Floquet theory

Here we show why for the hydrodynnamic Chaplygin sleigh system we are allowed to suppose the existence of a periodic solution of the reduced equation choosing periodic controls. We exploit the following general theorem. Let us consider the following ordinary differential equation

$$\dot{q} = A(t)q + b(t) \quad (11)$$

with $t \mapsto A(t)$ is a T–periodic matrix function and $t \mapsto b(t)$ a T–periodic vector function.

Theorem 7: [14] If one is not an eigenvalue of the monodromy matrix of the T–periodic homogeneous system $\dot{q} = A(t)q$, then (11) has at least one T–periodic solution.

Now consider the reduced equations of the hydrodynamic-Chaplygin sleigh given by (9) (10), with Ω and v_1 given by equations (8). Let us assign $u_1(t) = 0$, and $s_1(0) = 0$ for all t, thus integrating $\dot{s}_1 = u_1$ we obtain $s_1(t) = 0$ for all t. This assumption guarantees that the quadratic term in p in the momentum equation vanishes. Moreover suppose that the control $u_2(t)$ is periodic of period T. Using these controls $u_1(t)$ and $u_2(t)$ in the reduced equations, we have that clearly equations (10) have a periodic solution and the momentum equations are of type (11). Indeed Ω and v_1 are affine functions of p_Ω and p_1 with coefficients depending on u_2 and s_2 which are T-periodic. Hence we can use the previous Theorem which gives us conditions on the matrix $A(t)$ and the vector $b(t)$ to have the existence of at least one T–periodic solution. Thus, under these conditions, we have a periodic solution of the reduced equations and we can exploit the reconstruction techniques from periodic orbits.

[3]For the definition of a semi–algebraic set and its codimension see [1].

C. Trajectory generation

We now show, that we are always able to find control inputs that starting from a certain point in the plane put the system on a trajectory which asymptotically approaches a straight line with the desired direction.

Theorem 8: Let us suppose to start from a certain group configuration (x_A, y_A, θ_A) in the plane and suppose that, there exist two control loops $\hat{\mathcal{P}}^\tau = (s_1^\tau(t), s_2^\tau(t))$, $\hat{\mathcal{P}}^\Re = (s_1^\Re(t), s_2^\Re(t))$, such that the flow of the vector field defined by (8) (9) and (10) over $\hat{\mathcal{P}}^\tau$ is a circular motion in (x, y) and the flow over $\hat{\mathcal{P}}^\Re$ is a spiral flow along a certain direction. Then it is possible to steer the Chaplygin sleigh on a trajectory that in mean, asymptotically approaches a straight line motion in a certain direction of the (x, y)–plane (see also Figure 3 in Subsection IV-D) .

Proof: Without loss of generality assume that at $t = 0$, $\theta_A = 0$. Let α be the angle between the asymptotic direction of the spiral flow generated by $\hat{\mathcal{P}}^\Re$ as $t \to \infty$ and the x–axis, and β be the angle between the desired direction and the positive direction of the x–axis. The strategy is the following: use the control loop $\hat{\mathcal{P}}^\tau$ until a time $T_1 > 0$ such that $\theta(T_1) = \beta - \alpha$. Then consider the following control functions

$$f_1(t) = \begin{cases} s_1^\tau(T_1) & \text{if } t < 0 \\ s_1^\tau(T_1) + \frac{2s_1^\Re(0)}{T^2}t^2 & \text{if } 0 \le t \le \frac{T}{2} \\ \frac{4s_1^\tau(T_1)}{T^2}(t-T)^2 - \frac{s_1^\Re(0)(2t^2-5tT+T^2)}{2T^2} & \text{if } \frac{T}{2} \le t \le T \\ s_1^\Re(t-T) & \text{if } t \ge T \end{cases}$$

$$f_2(t) = \begin{cases} s_2^\tau(T_1) & \text{if } t < 0 \\ s_2^\tau(T_1) + \frac{2s_2^\Re(0)}{T^2}t^2 & \text{if } 0 \le t \le \frac{T}{2} \\ \frac{4s_2^\tau(T_1)}{T^2}(t-T)^2 - \frac{s_2^\Re(0)(2t^2-5tT+T^2)}{2T^2} & \text{if } \frac{T}{2} \le t \le T \\ s_2^\Re(t-T) & \text{if } t \ge T \end{cases}$$

Using these two functions as controls for $t \ge T$ the trajectory in mean, as $t \to \infty$ asymptotically approaches a straight line that forms an angle α with the x-axis. Thus setting

$$s_1^*(t) = \begin{cases} s_1^\tau(t) & \text{if } 0 \le t \le T_1 \\ f_1(t - T_1) & \text{if } t > T_1 \end{cases}$$

$$s_2^*(t) = \begin{cases} s_2^\tau(t) & \text{if } 0 \le t \le T_1 \\ f_2(t - T_1) & \text{if } t > T_1 \end{cases}$$

we have that the corresponding trajectory is the one obtained with the previous controls affected by a rotation about the center of the initial circular trajectory by the angle $\beta - \alpha$. Therefore the controls (s_1^*, s_2^*) satisfy the statement of the theorem. □

The simulations in Section IV-D show that we are able to produce both rotation (quasi–periodic flow) and translation (spiral flow) behaviours.

D. Numerical simulations using periodic controls

In this subsection we show some numerical simulations that support the validity of the first control strategy proposed above to use periodic controls to steer the hydrodynamic Chaplygin sleigh in the plane.

We refer to equations (8) and (9), and use the following parameters: $A = 2$, $B = \frac{2}{\sqrt{3}}$, $\rho = 1$, $M = 1$, $m = 0.01$ and $J = A^2 + B^2 = \frac{16}{3}$. At first we chose the following periodic controls $v_{s_1}(t) = 3\cos t$, $v_{s_2}(t) = 10\sin t$.
Integrating the momentum equation (9) with initial data: $s_1(0) = 0$, $s_2(0) = 1$, $p_1(0) = 1$, $p_\Omega(0) = 0$, we end up with the following periodic reduced solutions

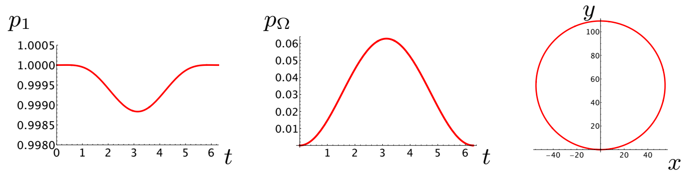

Fig. 2. Reduced periodic solutions of (9) on the left and reconstructed periodic trajectory on the right.

Integrating the reconstruction equations (3) over the periodic orbit corresponding to the solutions in Figure 2 on the left, and iterating the control loop for several periods, we get, as expected, a periodic trajectory (Figure 2 on the right) in the (x, y)–plane. This matches the predictions from the theory, since the generic behaviour for a nonholonomic mechanical control system with symmetry group $SE(2)$ is the quasi–periodic one, i.e a circle. We now exhibit a choice of periodic controls that produces the so–called *special* behaviour of the reconstructed trajectory, i.e. the spiral flow. Given the periodic controls $v_{s_1}(t) = \cos\left(t + \frac{\pi}{2}\right)$, $v_{s_2}(t) = \sin 2t$, the corresponding periodic solutions of the momentum equations with initial data $s_1(0) = 0$, $s_2(0) = 1$, $p_1(0) = 1$, $p_\Omega(0) = 0$ are the first two of Figure 3, and the reconstructed trajectory in the (x, y)–plane is the last one in Figure 3 and turns out to be horizontal.

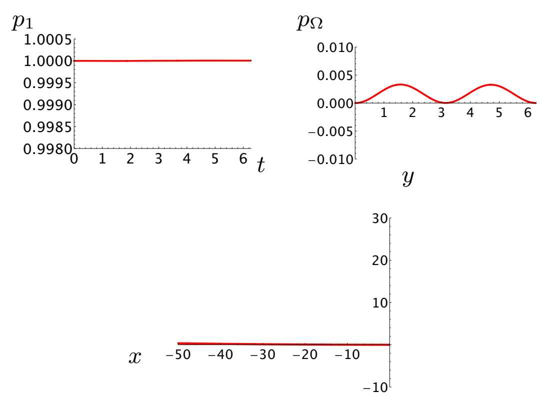

Fig. 3. Reduced periodic solutions of (9), that give rise to a *special* reconstructed behaviour.

Remark 9: We notice that the idea for proving the asymptotic behaviour of the system are similar to the ones used for the proof without hydrodynamics [28], in which only reconstruction from equilibria is taken into account. This means that the system of the Chaplygin sleigh preserves its dynamical properties, when immersed in an ideal incompressible irrotational fluid. Furthermore we observe that out result extends to the case in which also circulation is present as in [18].

V. Conclusions and perspectives

In this paper we consider the so–called hydrodynamic Chaplygin sleigh and analyze the trajectories attainable using periodic shape deformations, via reconstruction techniques. More precisely, we prove that combining periodic shape controls, it is possible to steer the hydrodynamic Chaplygin sleigh on a trajectory that in mean, asymptotically approaches a straight line motion in a certain direction in the plane. Moreover, through numerical simulations we show that we are able to produce both rotation and translation behaviours.

The reconstruction techniques are an important tool that can be used to predict and analyze the kind of trajectories performed by a non-holonomic system which is controlled by periodic shape actuators. Here we applied these techniques to the paradigmatic example of the hydrodynamic Chaplygin sleigh, but we conjecture that the strategy proposed here, can be used for more general non-holonomic shape control systems and, up to our knowledge, it is an important step forward in the study of the controllability of these kind of systems. Moreover we point out that our techniques could be adapted and extended to the case of obstacle avoidance problem. A rigorous proof of the extension of our results to a wider class of systems will be the subject of further studies.

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