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SPATIAL GROWTH PROCESSES WITH LONG RANGE DISPERSION: MICROSCOPICS, MESOSCOPICS AND DISCREPANCY IN SPREAD RATE

BY VIKTOR BEZBORODOV[1,*], LUCA DI PERSIO[2], TYLL KRUEGER[1,**] AND PASHA TKACHOV[3]

[1]*Faculty of Electronics, Wroclaw University of Science and Technology,* [*]*viktor.bezborodov@pwr.edu.pl;* [**]*tyll.krueger@pwr.wroc.pl*

[2]*Department of Computer Science, University of Verona, luca.dipersio@univr.it*

[3]*Gran Sasso Science Institute, pasha.tkachov@gssi.it*

We consider the speed of propagation of a continuous-time continuous-space branching random walk with the additional restriction that the birth rate at any spatial point cannot exceed 1. The dispersion kernel is taken to have density that decays polynomially as $|x|^{-2\alpha}$, $x \to \infty$. We show that if $\alpha > 2$, then the system spreads at a linear speed, while for $\alpha \in (\frac{1}{2}, 2]$ the spread is faster than linear. We also consider the mesoscopic equation corresponding to the microscopic stochastic system. We show that in contrast to the microscopic process, the solution to the mesoscopic equation spreads exponentially fast for every $\alpha > \frac{1}{2}$.

1. Introduction. We analyze the truncated pure birth model introduced in [6] on the subject of the speed of space propagation. Our aim is to approach the question from the microscopic probabilistic as well as the mesoscopic point of views. It turns out that the scaling significantly changes the behavior of the system: while the microscopic model grows linearly in time provided the exponent is larger than four, the mesoscopic model spreads exponentially fast.

The limiting behavior of the branching random walk has been extensively studied. For an overview of branching random walks and related topics, see for example, [48]. The asymptotic behavior of the position of the rightmost particle of the branching random walk under different assumptions are given in [17] and [16], see also references therein. A shape theorem for a one-dimensional discrete-space supercritical branching random walk with an exponential moment can be found in [7]; [8] contains further comments and extensions, in particular for a multidimensional branching random walk. Further results and references on the branching random walk with the focus on the position of rightmost particle can be found in [9]. More refined limiting properties have been obtained recently, such as the limiting law of the minimum or the limiting process seen from its tip or the asymptotics of the position of the minima of a branching random walk, see [1–4]. For maximal displacement of branching random walks in an environment see for example, [20, 38] and references therein. A branching random walk with a fixed number of particles is treated in [5], where asymptotic properties are obtained both in time and in the number of particles. In [19], conditions for the survival and extinction of different versions of the Bolker–Pacala model are given.

Among asymptotic results for other stochastic models, Blondel [10] proves a shape result and an ergodic theorem for the process viewed from the tip for the East model. A continuous-space set-valued stochastic growth model with the related shape theorem was given in [15]. The results have been extended in [28]. The agent based model we treat in the present manuscript shares some features with this set based models.

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The transition from the microscopic probabilistic models to macroscopic deterministic evolutions is a subject of several works, see for example, [13, 24]. Equations similar to those considered in the present paper appear in [12] during the analysis of the rightmost particle of the Branching random walk. Convolution with a probability density is often considered in biological and ecological models to describe a nonlocal interaction [14, 36]. Evolution equations involving convolution terms naturally appear as a limiting behavior of rescaled stochastic processes [18, 21, 35, 41, 45]. We do not give a formal derivation of the macroscopic model here, however we show that the microscopic and macroscopic models may have qualitatively different asymptotic growth rate when the underlying geographic space is not compact. This phenomenon can also be deduced for other models (see Remark 2.10).

The main results are Theorems 2.1, 2.7 and 2.8. Theorem 2.1 states that the birth process with the birth rate given by (1) and (2) below propagates not faster than linearly if $\alpha > 2$. We give a proof for the negative direction only as the proof for the opposite direction is identical due to symmetricity. Of course, Theorem 2.1 also applies to any stochastic process dominated by the birth process defined in Section 2, see Remark 2.6 for more detail. Theorem 2.7 shows that when $\alpha < 2$ the birth process does in fact spread faster than linearly. In combination with Theorem 2.1 it allows us to conclude that $\alpha = 2$ is a critical value for the birth proces defined by (1) and (2). On page 1097 two heuristic arguments are given on why one could expect the critical value to be two. In contrast to the linear speed in the stochastic microscopic model for $\alpha > 2$, Theorem 2.8 shows that the solution to the respective mesoscopic equation propagates exponentially fast. Let us note that the effect is different for the models without restriction: a dispersion kernel with polynomially decaying tails gives exponentially fast propagation for both the rightmost particle of the branching random walk (as shown in [17]) and the unique solution to the corresponding mesoscopic equation (see [11, 23, 26]).

The paper is organized as follows. The models we consider, assumptions and results are collected in Section 2. Proofs of the main results, Theorems 2.1, 2.7, and 2.8, are contained in Sections 3 and 5, 4 and 5, and 6, respectively. Sections 3 and 4 are devoted to the discrete-space version of the birth process. Section 6 also contains a remark on heuristic connection between the microscopic and mesoscopic models.

2. The model, assumptions and results. Let Γ_0 be the collection of subsets of finite number of points in $\mathbb{R}^1$,

$$\Gamma_0(\mathbb{R}^1) = \{\eta \subset \mathbb{R}^1 : |\eta| < \infty\},$$

where $|\eta|$ is the number of elements in η. Let also $b : \mathbb{R}^1 \times \Gamma_0 \to \mathbb{R}_+$ be the birth rate

$$b(x,\eta) = 1 \wedge \left(\sum_{y\in\eta} a(x-y)\right), \quad x \in \mathbb{R}, \eta \in \Gamma_0(\mathbb{R}^1) \tag{1}$$

with

$$a(z) = \frac{c_\alpha}{(1+|z|^2)^\alpha}, \quad z \in \mathbb{R}, \tag{2}$$

where $\alpha > \frac{1}{2}$ and $c_\alpha > 0$ is such that $\int_{\mathbb{R}} a(z)\,dz = 1$. The time evolution can be imagined as follows. We denote the state of the process at time t by $\eta_t \in \Gamma_0$. If the state of the system is $\eta \in \Gamma_0$, then the rate at which a birth occurs in a bounded Borel set B is $\int_B b(x,\eta)\,dx$, that is, the probability that a new particle appears (a "birth") in a bounded set $B \in \mathscr{B}(\mathbb{R}^1)$ over time interval $[t; t+\Delta t]$ is

$$\Delta t \int_B b(x,\eta)\,dx + o(\Delta t).$$

More details can be found in [6]. Note that the birth rate without restriction

$$\bar{b}(x,\eta) = \sum_{y \in \eta} a(x-y)$$

corresponds to a continuous-space branching random walk.

THEOREM 2.1. *Assume that $\alpha > 2$. For the continuous-space birth process $(\eta_t)_{t \geq 0}$ with birth rate (1) and initial condition $\eta_0 = \{0\}$ there exists a constant $\mathrm{C}_\alpha > 0$ such that a.s. for sufficiently large t,*

$$\eta_t \subset [-\mathrm{C}_\alpha t, \mathrm{C}_\alpha t]. \tag{3}$$

REMARK 2.2. As is the case for many shape theorems for growth models, Theorem 2.1 holds true for any initial condition $\eta_0 \in \Gamma_0(\mathbb{R}^1)$. Also, the upper bound in (1) does not have to be 1, it can be any positive constant.

REMARK 2.3. In fact, analyzing the proof of the shape theorem in [6], we can obtain a stronger result for the one dimensional continuous-space birth process with birth rate satisfying

$$b(x,\eta) \leq C_b \wedge \left(C_b \sum_{y \in \eta} a(x-y) \right),$$

for some constant $C_b > 0$, provided that certain additional conditions are satisfied (monotonicity, translation and rotation invariance, and nondegeneracy as defined in [6]). Specifically, there exists a constant $\lambda > 0$ such that for every $\varepsilon > 0$ a.s. for sufficiently large t both

$$\eta_t \subset \left[-\lambda(1+\varepsilon)t, \lambda(1+\varepsilon)t \right] \tag{4}$$

and

$$\bigcup_{x \in \eta_t} [x-1, x+1] \supset \left[-\lambda(1-\varepsilon)t, \lambda(1-\varepsilon)t \right] \tag{5}$$

hold true. In particular, (4) and (5) hold for b defined in (1) and (2). Note that such b does not satisfy Condition 2.1 from [6], however Condition 2.1 from that paper is only used to establish that the growth is at most linear, which we do in a different way in Theorem 2.1.

REMARK 2.4. Theorem 2.1 can be compared with the result of Durrett [17], which shows that we observe an exponential growth for the maximal displacement of a branching random walk with polynomially decreasing dispersion kernel. A related result for a branching random walk with dispersion kernel satisfying certain semiexponential conditions can be found in [25]. Semiexponential kernels in [25] satisfy

$$\mathbb{P}\{Y \geq t\} = l(t) \exp\bigl(-L(t)t^r\bigr)$$

for t sufficiently large, where Y is a random variable distributed as displacement of the offspring from the parent, $r \in (0,1)$, l and L are slowly varying functions, and $L(t)/t^{1-r}$ is nonincreasing for large t. The spread rate for a branching random walk with such a displacement kernel is given in [25] explicitly. The system grows faster than linearly; for some choices of L the spread rate is polynomial. For Deijfen's model of a randomly growing set, Gouéré and Marchand [28] give a sharp condition on the distribution of the outburst radii for linear or superlinear growth (i.e., faster than linear).

REMARK 2.5. In the language of statistical physics, Theorem 2.1 means that our model exhibits the directed percolation (DP) class properties while having longe-range interaction, see for example, [40], Section 6.7 and elsewhere, [27, 30, 43].

REMARK 2.6. As noted in the Introduction, Theorem 2.1 also applies to any stochastic process dominated by the birth process with birth rate (1). In particular, the statement holds true if every particle is removed after an exponential time with mean δ^{-1}, that is, if each particle also has a death rate equal to δ.

The next result shows that the condition $\alpha > 2$ in Theorem 2.1 is sharp. The system exhibits a superlinear spread rate when $\alpha \leq 2$.

THEOREM 2.7. *Assume that* $\alpha \in (\frac{1}{2}, 2]$. *Then* (η_t) *grows faster than linearly in the sense that for any* $K_0, K_1 > 0$,

$$\mathbb{P}\{\eta_t \subset [-K_0 - K_1 t, K_0 + K_1 t] \textit{ for sufficiently large } t\} = 0. \tag{6}$$

Put differently, (6) means that any $K_0, K_1 > 0$ the set

$$\{t : \eta_t \setminus [-K_0 - K_1 t, K_0 + K_1 t] \neq \varnothing\} \subset [0, \infty)$$

is a.s. unbounded.

A mesoscopic approximation of the point process $(\eta_t)_{t\geq 0}$ is given by the following evolution equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \min\left\{\displaystyle\int_{\mathbb{R}} a(x-y)u(y,t)\,dx, 1\right\} & x \in \mathbb{R}, t \in (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{7}$$

where a is defined by (2).

It turns out that the mesoscopic model shows a very different behavior. No matter how large $\alpha > \frac{1}{2}$ is in (2), the speed of propagation is faster than linear as we see in Theorem 2.8 which states that the solution to (7) propagates exponentially fast. Moreover, solutions with roughly speaking "monotone" initial conditions (case 2) propagate faster than solutions with "integrable" initial conditions (case 1).

THEOREM 2.8. *Let* $0 \leq u_0 \in L^\infty(\mathbb{R})$ *and* $u = u(x,t)$ *be the corresponding classical solution to* (7) *with* $a(x)$ *defined by* (2). *Then, for any* $\varepsilon \in (0,1)$, $n \geq 1$, *there exists* $\tau = \tau(\varepsilon, n)$ *such that the following inclusions hold*:

1. *If there exists* $C > 0$ *such that* $u_0(x) \leq \mathrm{Ca}(x)$, $x \in \mathbb{R}$, *and there exist* $\mu > 0$, $x_0 \in \mathbb{R}$, *such that* $u_0(x) \geq \mu$, $x \in [x_0 - \mu, x_0 + \mu]$, *then for all* $t \geq \tau$,

$$\left\{x : u(x,t) \in \left[\frac{1}{n}, n\right]\right\} \subset \left\{x : e^{\frac{1-\varepsilon}{2\alpha}t} \leq |x| \leq e^{\frac{1+\varepsilon}{2\alpha}t}\right\}. \tag{8}$$

2. *If there exists* $C > 0$ *such that* $u_0(x) \leq C\int_x^\infty a(y)\,dy$, $x \in \mathbb{R}$, *and there exist* $\mu > 0$, $\rho \in \mathbb{R}$, *such that* $u_0(x) \geq \mu$, $x \leq \rho$, *then for all* $t \geq \tau$,

$$\left\{x : u(x,t) \in \left[\frac{1}{n}, n\right]\right\} \subset \left\{x : e^{\frac{1-\varepsilon}{(2\alpha-1)}t} \leq x \leq e^{\frac{1+\varepsilon}{(2\alpha-1)}t}\right\}. \tag{9}$$

REMARK 2.9. We use the term "mesoscopic approximation" here instead of "macroscopic approximation", even though some authors might use the latter to describe (7). We follow here [45]; see also [34], [47] for discussions of microscopic, mesoscopic and macroscopic descriptions of complex systems.

REMARK 2.10. Remark 2.6 can also be contrasted with the spread rate of the system driven by the equation

$$u_t = J * u - u + f(u), \tag{10}$$

where J is the dispersion kernel, $\|J\|_{L^1} = 1$, and $f : [0, 1] \to \mathbb{R}_+$ is some differentiable function with $f(0) = f(1) = 0$ and $f'(0) > 0$, and certain other mild conditions. It is shown in [26] that the solution to (10) has level sets moving faster than linearly. We note that since the solution to (10) takes values between 0 and 1 (provided that the initial condition lies between 0 and 1; see [26]), we have $J * u \le 1$, and hence (10) can be written as

$$u_t = \min\{1, J * u\} - u + f(u). \tag{11}$$

Notation and conventions. Let $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{Z}_+ = \{m \in \mathbb{Z} : m \ge 0\}$. For processes indexed by $\mathbb{R}_+$ (which represents time) we will use (X_t) as a shorthand for $(X_t)_{t\ge 0}$ or $\{X_t, t \ge 0\}$. For a Poisson process (N_t), $0 < a \le b$, $N(a, b] = N_b - N_a$ and $N(\{a\}) = N_a - N_{a-}$. For $a, b \in \mathbb{R}$, $a_+ = \max\{a, 0\}$, $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$. Concening the operation order, we take for $a, b, c \in \mathbb{R}$, $-a \wedge b = -(a \wedge b)$, $ab \wedge c = (ab) \wedge c$, and the same rules for $\vee$. $\mathrm{Cov}(X, Y)$ and $\mathrm{Var}(X)$ denote the covariance between X and Y and variance of X, respectively. $\mathbb{1}$ is an indicator, for example

$$\mathbb{1}\{x \ge 0\} = \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Throughout the paper, C denotes different universal constants whose exact values are irrelevant. Even in the concatenation

$$F \le CG \le CH,$$

where F, G, and H are some expressions, two occurrences of C may have different values. We set $B_r(x) = \{y \in \mathbb{R} | |x - y| \le r\}$ and $B_r = B_r(0)$. For simplicity of notation we will write "$x \in \mathbb{R}$" instead of "*a.e.* $x \in \mathbb{R}$" for the elements of $L^\infty(\mathbb{R})$. We denote

$$\begin{aligned} L^\infty_+(\mathbb{R}) = \bigl\{ & f \in L^\infty(\mathbb{R}) | f(x) \ge 0, x \in \mathbb{R}; \\ & \exists \delta > 0, x_0 \in \mathbb{R} : f(x) \ge \delta, x \in B_\delta(x_0)\bigr\}. \end{aligned} \tag{12}$$

We will write for $f_1, f_2 \in L^\infty(\mathbb{R})$, $A \subset \mathbb{R}$,

$$f_1(x) \lesssim f_2(x), \quad x \in A,$$

if there exists $c > 0$ such that $f_1(x) \le c f_2(x)$, $x \in A$. For $p \in [1,\infty]$, $\|\cdot\|_p := \|\cdot\|_{L^p(\mathbb{R})}$.

A very brief outline of the proof of Theorem 2.1. The proof of Theorem 2.1 is split across Sections 3 and 5. The main bulk of the proof is carried out in Section 3, where we prove the equivalent of Theorem 2.1 for the case when the underlying "geographical" space is discrete $\mathbb{Z}^1$ rather than continuous $\mathbb{R}^1$. This equivalent is given in Theorem 3.12, and Sections 3 is entirely devoted to the proof of Theorem 3.12. The main idea of the proof is a coupling of the process seen from its tip with a simpler process. Some of the ingredients are the strong law of large numbers for dependent random variables, a form of the strong law for martingales, and Novikov's inequalities, or Bichteler–Jacod's inequalities, for discontinuous martingales. A brief outline of the proof of Theorem 3.12 and Section 3 can be found on Page 1098. In Section 5 we finally prove Theorem 2.1 by coupling the continuous-space process with the discrete-space process from Section 3.

3. Lattice truncated process. Linear growth for $\alpha > 2$. In this section we introduce a discrete-space equivalent defined by (13) and (14) for our continuous-space process defined by (1) and (2). We prove in this section that this discrete-space process spreads not faster than linearly (Theorem 3.12).

We consider the birth process on $\mathbb{Z}_+^{\mathbb{Z}}$ with the birth rate

$$b^{(d)}(x,\eta) = 1 \wedge \left(\sum_{y\in\mathbb{Z}} \eta(y) a^{(d)}(x-y)\right), \quad x \in \mathbb{Z}^1, \eta \in \mathbb{Z}_+^{\mathbb{Z}}, \tag{13}$$

where

$$a^{(d)}(x) = \frac{1}{(1 \vee |x|^2)^\alpha} \tag{14}$$

(for convenience we consider a slightly modified a in this section compared to (2)) and the initial condition

$$\eta_0(k) = \mathbb{1}\{k=0\}, \quad k \in \mathbb{Z}.$$

Thus, if the state of the system is η, the birth at $x \in \mathbb{Z}$ (i.e., the increase by 1 of the value at x) occurs at rate $b^{(d)}(x,\eta)$. In this section we denote the resulting birth process by (η_t). The process is constructed from a Poisson point process as a unique solution to a certain stochastic equation as described below.

Note that since $a^{(d)}(x) \le 1$ for all x,

$$b^{(d)}(x,\eta) = 1 \quad \text{if } \eta(x) \ge 1. \tag{15}$$

Our aim now is to show that the process propagates not faster than at a finite speed if $\alpha > 2$. Throughout this session we assume $\alpha > 2$. To this end we introduce the process $(\xi_t)_{t\ge 0}$ as $(\eta_t)_{t\ge 0}$ seen from its left tip.

DEFINITION 3.1. Define $\xi_t(k) = \eta_t(\mathrm{tip}(\eta_t)+k)$, $k \ge 0$, where

$$\mathrm{tip}(\eta) = \min\{n : \eta(n) > 0\}.$$

Note that (ξ_t) takes values in $\mathbb{Z}_+^{\mathbb{Z}_+}$. Now we introduce another process taking values in $\mathbb{Z}_+^{\mathbb{Z}_+}$. We will see later that this process dominates (ξ_t) in a certain sense specified below.

DEFINITION 3.2. Let (ζ_t) be a process on $\mathbb{Z}_+^{\mathbb{Z}_+}$ evolving as follows. The process starts from $\zeta_0(x) = \mathbb{1}\{x \ge 0\}$ and

- at rate 1 the configuration is shifted to the right by 1 and a particle is added at zero; that is, if a shift occurs at t and $\zeta_{t-} \in \mathbb{Z}_+^{\mathbb{Z}_+}$ is the state before the shift, then

$$\zeta_t(k) = \zeta_{t-}(k-1), \quad k \in \mathbb{N},$$

  and $\zeta_t(0) = 1$.
- between the shifts, $\zeta_t(k)$, $k \in \mathbb{Z}_+$, evolves as a Poisson process. The Poisson processes are independent for different k and of shift times.

Some heuristics on why the critical value α_c is two. As was noted above, Theorem 2.1 and Theorem 2.7 allow us to conclude that for the birth process with rate given by (1) the critical value of α is two. In this section we prove that the growth is linear for the discrete-space equivalent model and $\alpha > 2$. Before describing the proof, we take a brief pause and give a few heuristic arguments on why the critical value is 2. We give here two arguments, the first one being shorter and possibly more straightforward, while the second one relying on a heuristic comparison to other models.

We start from the following observation. Since we take a minimum with 1 in (13), it is to be expected that, provided that the spread is linear, η_t seen from its tip satisfies for some $c > 0$

$$\mathbb{E}\xi_t(k) \approx ck, \quad k \in \mathbb{N}, \tag{16}$$

or at least

$$\frac{1}{t}\int_0^t \mathbb{E}\xi_s(k) \approx ck, \quad k \in \mathbb{N}, t \to \infty. \tag{17}$$

We now proceed with the first shorter argument. If the system spreads linearly in time, then we can expect that (16) holds. Let $X_t = -\operatorname{tip}(\eta_t)$ be the distance from the leftmost occupied site to the origin. The rate at which X_t jumps by k is

$$j_k(t) = j_k = \sum_{i=0}^{\infty} a^{(d)}(-k-i)\xi_{t-}(i).$$

For the speed of propagation to be finite we need the sum $\sum_{k=1}^{\infty} kj_k$ to be finite (more precisely, the time averages of $\sum_{k=1}^{\infty} kj_k$ need to be finite and growing not faster than linearly in time; note that $X_t - \int_0^t \sum_{k=1}^{\infty} kj_k(s-)\,ds$ is a martingale). Substituting $\xi_{t-}(i)$ by ci as in (16), we get

$$\sum_{k=1}^{\infty} kj_k = c\sum_{k=1}^{\infty}\sum_{i=0}^{\infty} \frac{ik}{(i+k)^{2\alpha}} = c\sum_{m=1} \frac{1}{m^{2\alpha}}\sum_{k=1}^{m} k(m-k) \sim \sum_{m=1} \frac{m^3}{m^{2\alpha}}, \tag{18}$$

where $\sim$ means that two series have the same convergence/divergence properties. We see that the sum in (18) is finite if and only if $\alpha > 2$, hence one could expect that the critical value $\alpha_c = 2$.

To make the first heuristic argument rigorous we would have to prove something like (16) or (17). However, to prove (16) or (17) we would probably need to prove the linear spread rate first. In the actual proof that $\alpha = 2$ is critical we dominate (ξ_t) by another process satisfying a weaker version of (16). This auxiliary process helps us derive an inequality giving an upper bound for certain time averages of (ξ_t), see Proposition 3.9.

The second argument is of purely heuristic nature. We introduce two more birth rates,

$$b^{(d,1)}(x,\eta) = a^{(d)}\bigl(x - \operatorname{tip}(\eta)\bigr), \quad x \in \mathbb{Z}^1, \eta \in \mathbb{Z}_+^{\mathbb{Z}}, \tag{19}$$

and

$$b^{(d,2)}(x,\eta) = \sum_{y=\operatorname{tip}(\eta)}^{0} a^{(d)}(x-y), \quad x \in \mathbb{Z}^1, \eta \in \mathbb{Z}_+^{\mathbb{Z}}. \tag{20}$$

Denote by $(\eta_t^{(d,i)})$ the respective birth processes, $i = 1, 2$, and by $X_t^{(d,i)} = -\operatorname{tip}(\eta^{(d,i)})$ the distance from the leftmost occupied site to the origin. For $\eta \in \mathbb{Z}_+^{\mathbb{Z}}$ with $\operatorname{tip}(\eta)$ well defined, let the "essential parts" of the configuration be

$$\tilde{\eta}^{(d,1)}(k) = \mathbb{1}\{k = \operatorname{tip}(\eta)\},$$

$$\tilde{\eta}^{(d,2)}(k) = \mathbb{1}\{k \in \{0, -1, \dots, \operatorname{tip}(\eta)\}\}.$$

Note that

$$b^{(d,i)}(x,\eta) = b^{(d,i)}(x, \tilde{\eta}^{(d,i)}), \quad x \in \mathbb{Z}^1, \eta \in \mathbb{Z}_+^{\mathbb{Z}}, i = 1, 2, \tag{21}$$

so to determine the spread rate of $(\eta_t^{d,i})$ it is sufficient to know only $(\tilde{\eta}_t^{(d,i)})$.

From the definition of $b^{(d,1)}(x,\eta)$ we see that $X^{(d,1)}$ is a continuous-time discrete-space random walk with jumps by $n \in \mathbb{N}$ occuring at rate $a^{(d)}(n)$. Therefore, for $b^{(d,1)}$ the critical value separating linear and superlinear growth is $\alpha = \alpha_c^{(1)} = 1$ in (14).

Now, it is not as straightforward to determine the critical value $\alpha_c^{(2)}$ for $(\eta_t^{(d,2)})$. We note however that $(\tilde{\eta}_t^{(d,2)})$ is a discrete-space equivalent of the Deijfen's model [15, 28]. It was shown in [28] that in one dimension the critical exponent in the kernel is three. Hence it should hold $\alpha_c^{(2)} = \frac{3}{2}$.

Let us come back to (η_t) with birth rate (13) and compare $(\eta_t^{(d,1)})$, $(\eta_t^{(d,2)})$, and (η_t). We start by noting that all three processes are related because they are defined in terms of $a^{(d)}$. The essential part of $(\eta_t^{(d,1)})$ is a single site $\tilde{\eta}_t^{(d,1)}$. We can roughly say that the essential part of $(\eta_t^{(d,1)})$ has dimension zero. The critical exponent for $(\eta_t^{(d,1)})$ is two, which corresponds to the critical value $\alpha_c^{(1)} = 1$. The essential part of $(\eta_t^{(d,2)})$, namely $\tilde{\eta}_t^{(d,2)}$, can be thought of as a growing interval. Thus, informally, the essential part of $(\eta_t^{(d,2)})$ has dimension one. The respective critical exponent is three, corresponding to the critical value $\alpha_c^{(2)} = \frac{3}{2}$.

Now, the essential part of η_t is η_t itself, since every site affects the birth rates beyond the tip. The number of occupied sites for (η_t) grows at least linearly with time. According to (16), the number of particles at each occupied site also grows linearly. Thus, roughly speaking, the essential part of (η_t) has two dimensions. We can then conjecture that the critical exponent for (η_t) should be one more than that for $(\eta_t^{(d,2)})$, to compensate for the one more dimension (see also Table 1), and hence $\alpha_c = 2$.

Of course, for the above heuristic argument to work it is necessary also to assume that the restriction in b given by (13), that is, taking minimum with 1, does not affect the sites beyond the tip too much. This seems to be plausible, at least for the sites far away from the tip, while the sites near the tip should not affect the critical value too much.

Knowing that the guess $\alpha_c = 2$ is correct, we can go a little bit further and conjecture that for this kind of model

$$\text{Critical exponent} = \text{Dimension of the essential part} + 2. \tag{22}$$

Note that this is compatible with the results of [28] as the "essential part" of their d-dimensional model would have dimension d as well. Let us add that (22) is also compatible with the discussion of the DP regime for the one-dimensional long-range contact process in [27], Page 6 and elsewhere, because the "essential part" of the contact process conditioned on nonextinction has dimension one. For the birth process in d dimensions with birth rate as in (13), (22) would mean that the critical value is

$$\alpha = \frac{d+3}{2}.$$

TABLE 1
The critical exponents and the essential dimension

The process	Dimension of the essential part	Critical exponent	Critical value of α
$(\eta_t^{(d,1)})$	0	2	1
$(\eta_t^{(d,2)})$	1	3	$\frac{3}{2}$
(η_t)	2	???	???

A brief summary of the section. As mentioned above, this section is devoted to proving that the distance (X_t) from the origin to the leftmost particle of (η_t) does not grow faster than linearly in time, as formulated in Theorem 3.12. We rely on the representation $X_t = Q_t + M_t$, where (Q_t) is a suitable increasing process, and (M_t) is a local martingale later shown to be a true martingale, see (36), (37), and Lemma 3.8.

We then proceed to show that a.s. (Q_t) grows not faster than linearly in time as stated Proposition 3.9. To prove Proposition 3.9, we introduce in (40) a sequence of random variables $\{Y_n\}_{n\in\mathbb{N}}$ dominating $\{Q_n\}_{n\in\mathbb{N}}$. The sequence $\{Y_n\}_{n\in\mathbb{N}}$ is closely related to (ζ_t) while $\{Q_n\}_{n\in\mathbb{N}}$ is related to (ξ_t), and we make use of the fact that the process (ζ_t) stochastically dominates (ξ_t) in the sense made precise below, see Definition 3.3 and Proposition 3.5. We then proceed to show that $\{Y_n\}_{n\in\mathbb{N}}$ grows not faster than linearly with n. A key point in this step is a certain decorrelation property (43), which we establish using properties of (ζ_t). Thanks to (43) we are able to apply to $\{Y_n\}_{n\in\mathbb{N}}$ a strong law of large numbers for dependent random variables, concluding the proof of Proposition 3.9.

Then, using representation (35) for (X_t) and Novikov's inequality for discontinuous martingales, we obtain a moment estimate for (M_t) in Proposition 3.10. This moment estimate allows us to apply a strong law of large numbers for martingales formulated in Theorem 3.11.

By that point we have practically shown that (Q_t) grows at most linearly in time and $\frac{M_n}{n} \to 0$, $n \in \mathbb{N}$. This allows us to conclude in Theorem 3.12 that $X_t = Q_t + M_t$ does not grow faster than linearly either.

DEFINITION 3.3. We say that a random element R_2 taking values in $\mathbb{Z}_+^{\mathbb{Z}_+}$ *stochastically dominates* a random (again $\mathbb{Z}_+^{\mathbb{Z}_+}$-valued) element R_1 if a.s. for every $k = 0, 1, \dots$

$$\sum_{i=0}^{k} R_1(i) \le \sum_{i=0}^{k} R_2(i). \tag{23}$$

We will say that a process $(\hat{\zeta}_t)$ *stochastically dominates* another process $(\hat{\xi}_t)$ if a.s. for every t and every $k = 0, 1, \dots$

$$\sum_{i=0}^{k} \hat{\xi}_t(i) \le \sum_{i=0}^{k} \hat{\zeta}_t(i). \tag{24}$$

The following lemma is a straightforward consequence of Definition 3.3.

LEMMA 3.4. *Let $\{a_i\}_{i\in\mathbb{Z}_+}$ be a nonincreasing sequence of nonnegative numbers. If R_2 stochastically dominates R_1, both are $\mathbb{Z}_+^{\mathbb{Z}_+}$-valued random elements, then*

$$\mathbb{E} \sum_{i\in\mathbb{Z}_+} a_i R_1(i) \le \mathbb{E} \sum_{i\in\mathbb{Z}_+} a_i R_2(i). \tag{25}$$

In particular, if the right hand side of (25) is finite, then so is the left hand side.

Construction and coupling of (η_t), (ξ_t) and (ζ_t). Here we construct the processes (η_t), (ξ_t) and (ζ_t) in such a way that (ζ_t) stochastically dominates (ξ_t). We start with (η_t), which in this section is the discrete space birth process with birth rate given by (13) and (14), and in whose behavior we are interested in. The processes (ξ_t) and (ζ_t) are auxiliary processes we need to analyze the position of the leftmost occupied site of (η_t).

Let $\mathbf{N}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{Z} \times [0, 1]$ with mean measure $ds \times \# \times du$, where # is the counting measure on $\mathbb{Z}$. Then (η_t) can be defined as the unique solution to the equation (see [6], Section 5)

$$\eta_t(k) = \int_{(0,t]\times\{k\}\times[0,1]} \mathbb{1}_{[0,b^{(d)}(i,\eta_{s-})]}(u)\mathbf{N}(ds\,di\,du) + \eta_0(k). \tag{26}$$

Define a filtration of σ-algebras $\{\mathscr{F}_t, t \geq 0\}$ as the completion of

$$\mathscr{F}_t^0 = \sigma\{\mathbf{N}(B_1 \times \{k\} \times B_2), B_1 \in \mathscr{B}([0,t]), k \in \mathbb{Z}, B_2 \in \mathscr{B}([0,1])\}. \tag{27}$$

The filtration $\{\mathscr{F}_t, t \geq 0\}$ is right-continuous and complete. All the stopping times we consider in this section are with respect to this filtration.

Let $\{N^{(j)}\}_{j\in\mathbb{Z}}$ be a collection of independent Poisson processes indexed by $\mathbb{Z}$ defined by

$$N_t^{(j)} = \mathbf{N}([0,t] \times \{j\} \times [0,1])$$

and let $\{u_i^{(j)}\}_{j,i\in\mathbb{N}}$ be a two-dimensional array of independent uniformly distributed on [0, 1] random variables uniquely defined by

$$\mathbf{N}(\{t_i^{(j)}\} \times \{j\} \times \{u_i^{(j)}\}) = 1, \tag{28}$$

where $t_i^{(j)} = \inf\{t > 0 : \mathbf{N}([0,t] \times \{j\} \times [0,1]) = i\}$. Note that the processes $\{N^{(j)}\}_{j\in\mathbb{Z}}$ and $\{u_i^{(j)}\}_{j,i\in\mathbb{N}}$ are mutually independent.

The evolution of (ξ_t) can be described in terms of $\{N^{(j)}\}_{j\in\mathbb{Z}}$ and $\{u_i^{(j)}\}_{j,i\in\mathbb{N}}$ as follows. Shifts by $m \in \mathbb{N}$ to the right occur at moments t when

$$N^{(\mathrm{tip}(\eta)-m)}\left(\{t\} \times \left[0, \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}}\right]\right) = 1,$$

and a particle at zero is added. Between the shift times, the number of particles at a site j grows according to $N^{(\mathrm{tip}(\eta)+j)}$ for (ξ_t); however, an increment by 1 at time t at the site j actually occurs if not only $N_t^{(\mathrm{tip}(\eta)+j)} - N_{t-}^{(\mathrm{tip}(\eta)+j)} = 1$, but also additionally

$$u^{(\mathrm{tip}(\eta)+j)}_{N_t^{(\mathrm{tip}(\eta)+j)}} \leq \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(1 \vee |k-j|)^{2\alpha}}. \tag{29}$$

If (29) is not satisfied, then the value stays the same: $\xi_t^{(\mathrm{tip}(\eta)+j)} = \xi_{t-}^{(\mathrm{tip}(\eta)+j)}$. Thus, (ξ_t) is a $\mathbb{Z}_+^{\mathbb{Z}_+}$-valued process started from $\xi_0(k) = \mathbb{1}\{k=0\}$, $k \in \mathbb{Z}_+$, that can be described by the following list of events:

- for $m \in \mathbb{N}$, shifts by m occur at rate $\sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}}$. Whenever a shift occurs, a single particle is added at the origin (this event occurs at moments t when $N^{(\mathrm{tip}(\eta)-m)}(\{t\} \times [0, \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}}]) = 1$);
- the number of particles at a site j increases by 1 at rate $1 \wedge \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(1\vee|k-j|)^{2\alpha}}$ (the increase by 1 occurs at the jump times of $N^{(\mathrm{tip}(\eta)+j)}$ provided that additionally $u^{(\mathrm{tip}(\eta)+j)}_{N_t^{(\mathrm{tip}(\eta)+j)}} \leq \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(1\vee|k-j|)^{2\alpha}}$);
- The above events happen independently, and no two events occur at the same time.

Let us now define (ζ_t) in terms of $\{N^{(j)}\}_{j\in\mathbb{Z}}$. Recall that the initial configuration is $\zeta_0(k) = 1$, $k \in \mathbb{Z}_+$. A shift by 1 occurs at time moments t when $N^{(\mathrm{tip}(\eta)-1)}(\{t\}) = 1$. Between the shift times, the number of particles at a site j grows according to $N^{(\mathrm{tip}(\eta_{t-})+j)}$ for (ζ_t), that is, $\zeta_t(j) - \zeta_{t-}(j) = 1$ if and only if $N_t^{(\mathrm{tip}(\eta_{t-})+j)} - N_{t-}^{(\mathrm{tip}(\eta_{t-})+j)} = 1$.

Let us now list some of the properties of the processes (ζ_t) and (ξ_t) which are used later on. They follow from definitions and construction of (ζ_t) and (ξ_t).

1. A.s. for all $t \geq 0$, $\xi_t(0) \geq 1$ and $\zeta_t(0) \geq 1$.

2. Every shift for (ζ_t) is a shift for (ξ_t) too, since for $m = 1$,

$$\sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}} \geq \xi_{t-}(0) \geq 1.$$

3. If a shift occurs for (ζ_t) $((\xi_t))$ at time t, then $\zeta_t(0) = 1$ ($\xi_t(0) = 1$ respectively).

4. If $(\xi_t(j))$ is increased by 1 at time t, $j \in \mathbb{Z}_+$, then so is $(\zeta_t(j))$ (but not necessarily vice versa by (29)).

5. The processes (η_t), (ξ_t), (ζ_t) are Markov processes with respect to $\{\mathscr{F}_t, t \geq 0\}$.

Let $(\varphi_k)_{k\in\mathbb{N}}$ be the shift times of (ξ_t), that is, $t \in \{\varphi_k\}_{k\in\mathbb{N}}$ if and only if for some $m \in \mathbb{N}$

$$\mathbf{N}\left(\{t\} \times \{\operatorname{tip}(\eta_t) - m\} \times \left[0, \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}}\right]\right) = 1,$$

or alternatively if for some $m \in \mathbb{N}$

$$N_t^{(\operatorname{tip}(\eta_t)-m)} - N_{t-}^{(\operatorname{tip}(\eta_t)-m)} = 1 \quad \text{and} \quad u^{(\operatorname{tip}(\eta_t)-m)}_{N_t^{(\operatorname{tip}(\eta_t)-m)}} \leq \sum_{k\geq 0} \frac{\xi_{t-}(k)}{(k+m)^{2\alpha}}.$$

Denote by $(\tilde{N}_t)$ the Poisson process such that $\tilde{N}_t - \tilde{N}_{t-} = 1$ for those t when $N_t^{\operatorname{tip}(\eta_t)-1} - N_{t-}^{\operatorname{tip}(\eta_t)-1} = 1$, so that $(\tilde{N}_t)$ is the Poisson process whose jumps are exactly the shift times for (ζ_t). Let $\sigma_k = \inf\{t > 0 : \tilde{N}_t = k\}$ be the jump times of the process $(\tilde{N}_t)$, that is, $t \in \{\sigma_k\}_{k\in\mathbb{N}}$ if and only if $\tilde{N}_t - \tilde{N}_{t-} = 1$. Let also $\varphi_k = \sigma_k = 0$ for $k = 0, -1, -2, \ldots$. Note that $\{\sigma_k\}_{k\in\mathbb{N}} \subset \{\varphi_k\}_{k\in\mathbb{N}}$ since every shift for (ζ_t) is a shift for (ξ_t) too. The process (ζ_t) has the following representation (let us stress here that we do not use this representation in the proofs): for $t \geq 0$ let $n \in \mathbb{N}$ be such that $t \in [\varphi_n, \varphi_{n+1})$, then

$$\zeta_t(j) = 1 + \sum_{\substack{k\in\{0,1,\ldots,n\}: \\ \tilde{N}_{\varphi_k}+j\geq \tilde{N}_{\varphi_n}}}^{n} N^{(\operatorname{tip}(\eta_{\varphi_k})+j+\tilde{N}_{\varphi_k}-\tilde{N}_{\varphi_n})}(\varphi_k, \varphi_{k+1} \wedge t], \qquad j \in \mathbb{Z}_+.$$

PROPOSITION 3.5. *(ζ_t) stochastically dominates (ξ_t).*

PROOF. Let us show that (24) is satisfied for every $k = 0, 1, \ldots$ if we take $\hat{\xi}_t = \xi_t$ and $\hat{\zeta}_t = \zeta_t$.

We use induction on k. For $k = 0$ (24) is clear since by construction every shift of $(\zeta_t)_{t\geq 0}$ is a shift for $(\xi_t)_{t\geq 0}$ too, while every time $(\xi_t(0))_{t\geq 0}$ is increased by 1 $(\zeta_t(0))_{t\geq 0}$ is increased too.

Fix $n \in \mathbb{N}$ and assume that (24) holds for $k = 0, \ldots, n-1$. At $t = 0$ (24) with $k = n$ holds. Let $\theta < \infty$ be the first moment when (24) with $k = n$ does not hold; note that θ is well defined since a.s. there are only finitely many shifts up to any time moment, and finitely many increments at sites $0, 1, \ldots, n$ took place. Thus we have

$$\sum_{i=0}^{n} \xi_{\theta-}(i) \leq \sum_{i=0}^{n} \zeta_{\theta-}(i) \tag{30}$$

but

$$\sum_{i=0}^{n} \xi_{\theta}(i) > \sum_{i=0}^{n} \zeta_{\theta}(i). \tag{31}$$

If (ξ_t) got shifted by m at θ, then, at θ, (ζ_t) got shifted by 1 or did not change; in either case

$$\sum_{i=0}^{n} \xi_\theta(i) \leq 1 + \sum_{i=0}^{n-1} \xi_{\theta-}(i) \leq 1 + \sum_{i=0}^{n-1} \zeta_{\theta-}(i) \leq \sum_{i=0}^{n} \zeta_\theta(i).$$

If on the other hand (ξ_t) got increased by 1 at a site j, $0 \leq j \leq n$, at θ, then (ζ_t) got increased by 1 at the same site too. So, (30) and (31) cannot both be satisfied for a finite θ, and thus we have a contradiction. $\square$

We now introduce another $\mathbb{Z}_+^{\mathbb{Z}_+}$-valued process defined by

$$\bar{\zeta}_t(k) = 1 + N^{(n-k)}(\sigma_{n-k}, t], \quad t \in (\sigma_n, \sigma_{n+1}], \tag{32}$$

which is equal in distribution to (ζ_t) by the strong Markov property of a Poisson point process, see the appendix in [6]. It is a little bit easier to work with, so we will use it in the estimates below.

Denote the distance from the leftmost occupied site for (η_t) to the origin by X_t, so that

$$X_t := -\operatorname{tip}(\eta_t).$$

Note that (X_t) allows the representation

$$X_t = \sum_{m \in \mathbb{N}} m \int_{(0,t]\times[0,1]} \mathbb{1}_{[0, b^{(d)}(\operatorname{tip}(\eta_{s-})-m, \eta_{s-})]}(u) N^{(\operatorname{tip}(\eta_{s-})-m)}(ds\, du), \tag{33}$$
$$t \geq 0.$$

To represent X_t as an integral with respect to a Poisson point process, for $0 < a < b$ and $m \in \mathbb{N}$ define the set

$$T(a, b, m) = \{(s, k) \in \mathbb{R}_+ \times \mathbb{Z} | a < s \leq b, \operatorname{tip}(\eta_{s-}) + m = k\}$$

and the point process

$$\begin{aligned} &N^{(X)}((a, b] \times \{m\} \times U) \\ &\quad = \mathbf{N}(T(a, b, m) \times U), \quad 0 < a < b, m \in \mathbb{N}, U \in \mathscr{B}[0, 1]. \end{aligned} \tag{34}$$

Note that for $0 < a < b$ a.s.

$$\begin{aligned} &N^{(X)}((a, b] \times \{m\} \times U)\mathbb{1}\{\operatorname{tip}(\eta_a) = \operatorname{tip}(\eta_b)\} \\ &\quad = \mathbf{N}((a, b] \times \{\operatorname{tip}(\eta_a) + m\} \times U)\mathbb{1}\{\operatorname{tip}(\eta_a) = \operatorname{tip}(\eta_b)\}. \end{aligned}$$

It follows from the strong Markov property for a Poisson point process (as formulated in the appendix in [6]) that $N^{(X)}$ is a Poisson point process; also, $N^{(X)}$ is equal in distribution to $\mathbf{N}$. It follows from (33) and (34) that

$$X_t = \int_{(0,t]\times\mathbb{N}\times[0,1]} m \mathbb{1}_{[0, b^{(d)}(\operatorname{tip}(\eta_{s-})-m, \eta_{s-})]}(u) N^{(X)}(ds\, dm\, du). \tag{35}$$

The process

$$\begin{aligned} M_t &:= X_t - \int_0^t \sum_{m \in \mathbb{N}} m b^{(d)}(\operatorname{tip}(\eta_{s-}) - m, \eta_{s-})\, ds \\ &= X_t - \int_0^t \sum_{m \in \mathbb{N}} m \left(1 \wedge \sum_{k=0}^{\infty} \frac{\xi_{s-}(k)}{(m+k)^{2\alpha}}\right) ds, \quad t \geq 0, \end{aligned} \tag{36}$$

is therefore a local martingale with respect to $\{\mathscr{F}_t, t \ge 0\}$, see for example, (3.8) in Section 3, Chapter 2 in [32]. We will see in Lemma 3.8 below that (M_t) is a (true) martingale. We denote by Q_t the second summand on the right hand side of (36), so that

$$M_t = X_t - Q_t. \tag{37}$$

In the remaining part of this section we prove that (X_t) grows at most linearly (Theorem 3.12). First we prove that (Q_t) grows at most linearly (Proposition 3.9), then we show that the martingale (M_t) has some nice properties (Proposition 3.10) which allow us to apply a strong law of large numbers for martingales in the proof of Theorem 3.12. The following lemma collects some relatively straightforward properties which are used multiple times in the rest of this section.

LEMMA 3.6. *Let β, X and Y be nonnegative random variables with finite third moment.*

(i) *if $\beta \perp (X, Y)$ (β is independent to (X, Y)), then*

$$\mathrm{Cov}(\beta X, Y) = \mathbb{E}\beta\,\mathrm{Cov}(X, Y);$$

(ii) *if $X \mid \beta \perp Y \mid \beta$ (that is, X and Y are conditionally independent given β) and $\mathbb{E}[X|\beta] = \mathbb{E}[Y|\beta] = \beta$, then*

$$\mathrm{Cov}(X, Y) = \mathrm{Var}(\beta);$$

(iii) *if $\mathbb{E}(X \mid \beta) = \beta$, then*

$$\mathrm{Cov}(X, \beta) = \mathrm{Var}(\beta), \qquad E\beta X = E\beta^2;$$

(iv) *if $\mathbb{E}(X \mid \beta) = \mathbb{E}(Y \mid \beta) = \beta$ and $X \mid \beta \perp Y \mid \beta$, then*

$$\mathrm{Cov}(\beta X, Y) = \mathbb{E}\beta^3 - \mathbb{E}\beta^2\mathbb{E}\beta;$$

(v) *if $\mathbb{E}(X \mid \beta) = \beta$, $\mathbb{E}(X^2 \mid \beta) = \beta^2 + \beta$ and $Y \perp (X, \beta)$, then*

$$\mathrm{Cov}\big(\beta(X+Y), (X+Y)\big) = \mathbb{E}\beta^2 + \mathbb{E}\beta^3 - \mathbb{E}\beta^2\mathbb{E}\beta + \mathbb{E}Y\,\mathrm{Var}(\beta) + \mathbb{E}\beta\,\mathrm{Var}(Y);$$

(vi) *if N is a Poisson process independent of β, then*

$$\mathbb{E}\big[N(\beta)|\beta\big] = \beta.$$

PROOF. The proof is based on the properties of conditional expectation. The proofs of (i)–(v) are done by conditioning on β. We give the proofs for (ii), (iv) and (vi) only; the others are similar to (ii) and (iv). For (ii),

$$\begin{aligned}\mathrm{Cov}(X, Y) &= \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y = \mathbb{E}\mathbb{E}[XY|\beta] - (\mathbb{E}\beta)^2 \\ &= \mathbb{E}\big(\mathbb{E}[X|\beta]\mathbb{E}[Y|\beta]\big) - (\mathbb{E}\beta)^2 = \mathbb{E}\beta^2 - (\mathbb{E}\beta)^2 = \mathrm{Var}(\beta).\end{aligned}$$

For (iv),

$$\begin{aligned}\mathrm{Cov}(\beta X, Y) &= \mathbb{E}\beta XY - \mathbb{E}\beta X\mathbb{E}Y \\ &= \mathbb{E}\mathbb{E}[\beta XY|\beta] - \mathbb{E}\beta\mathbb{E}\mathbb{E}[\beta X|\beta] \\ &= \mathbb{E}\beta\mathbb{E}[XY|\beta] - \mathbb{E}\beta\mathbb{E}\big(\beta\mathbb{E}[X|\beta]\big) \\ &= \mathbb{E}\big(\beta\mathbb{E}[X|\beta]\mathbb{E}[Y|\beta]\big) - \mathbb{E}\beta\mathbb{E}\beta^2 = \mathbb{E}\beta^3 - E\beta\mathbb{E}\beta^2.\end{aligned}$$

To prove (vi) we use the disintegration theorem for regular conditional probability distribution, see for example, Kallenberg [33], Theorem 6.4. To adapt to the notation in the preceding reference, let $S = D([0, +\infty), \mathbb{R})$ (the Skorokhod space) equipped with the cylindrical σ-algebra, and $T = \mathbb{R}_+$ equipped with the Borel σ-algebra, and consider N and β as random elements in S and T respectively. Note that since N and β are independent, the regular conditional probability distribution of N given β is simply the distribution of N in S, which we denote by ν. Define $f(s,t) = s(t)$, $s \in S$, $t \in T$. For every $q \geq 0$,

$$\int_S \nu(ds)s(q) = \mathbb{E}N(q) = q,$$

hence by the disintegration theorem a.s.

$$\mathbb{E}[f(N,\beta)|\beta] = \int_S \nu(ds) f(s,\beta) = \int_S \nu(ds)s(\beta) = \beta. \qquad \square$$

REMARK 3.7. Concerning item (vi), note that the conditional distribution of $N(\beta)$ given β is $\mathcal{P}\text{ois}(\beta)$, where $\mathcal{P}\text{ois}(q)$ is the Poisson distribution with parameter $q \geq 0$.

LEMMA 3.8. *The process (M_t) is a true martingale.*

PROOF. By Lemma 3.6 for every $t \geq 0$,

$$\begin{aligned}
\mathbb{E}Q_t &= \mathbb{E}\int_0^t \sum_{m\in\mathbb{N}} m\left(1 \wedge \sum_{k=0}^{\infty} \frac{\xi_{s-}(k)}{(m+k)^{2\alpha}}\right) ds \\
&\leq \mathbb{E}\int_0^t \sum_{m\in\mathbb{N}} m \sum_{k=0}^{\infty} \frac{\bar{\zeta}_{s-}(k)}{(m+k)^{2\alpha}}\, ds \\
&= \int_0^t \sum_{m\in\mathbb{N}} m \sum_{k=0}^{\infty} \frac{1+\mathbb{E}N^{(n-k)}(\sigma_{n-k}, s]\mathbb{1}\{\sigma_{n-k} \leq t\}}{(m+k)^{2\alpha}}\, ds \\
&= \int_0^t \sum_{m\in\mathbb{N}} m \sum_{k=0}^{\infty} \frac{1+\mathbb{E}(s-\sigma_{n-k})_+}{(m+k)^{2\alpha}}\, ds \\
&\leq \int_0^t \sum_{m\in\mathbb{N}} m \sum_{k=0}^{\infty} \frac{1+s}{(m+k)^{2\alpha}}\, ds \\
&< t(t+1)\sum_{m\in\mathbb{N}} \frac{m}{m^\alpha} \sum_{k=0}^{\infty} \frac{1}{k^\alpha} \\
&= t(t+1)\sum_{m\in\mathbb{N}} \frac{1}{m^{\alpha-1}} \sum_{k=0}^{\infty} \frac{1}{k^\alpha},
\end{aligned}$$

and hence for every $t \geq 0$

$$\mathbb{E}\sup_{s\leq t}|M_s| \leq \mathbb{E}|X_t| + \mathbb{E}|Q_t| = 2\mathbb{E}|Q_t| < \infty.$$

The statement of the lemma now follows from Theorem 51 in Protter [46]. $\square$

The following proposition is a key step in the proof of the main result of this section, Theorem 3.12. We establish here that (Q_t) grows at most linearly with t.

PROPOSITION 3.9. (i) *There exists $C > 0$ such that a.s. for sufficiently large t,*

$$\int_{s=0}^t ds \sum_{k\in\mathbb{Z}_+} \frac{\xi_s(k)}{k^\alpha} < Ct. \tag{38}$$

(ii) *There exists* $C > 0$ *such that a.s. for sufficiently large* t,

$$Q_t \leq Ct.$$

PROOF. First we show that (i) implies (ii). Indeed,

$$\begin{aligned} Q_t &= \int_0^t \sum_{m\in\mathbb{N}} m\left(1 \wedge \sum_{k=0}^{\infty} \frac{\xi_{s-}(k)}{(m+k)^{2\alpha}}\right) ds \\ &\leq \int_0^t \sum_{m\in\mathbb{N}} m \sum_{k=0}^{\infty} \frac{\xi_{s-}(k)}{m^\alpha k^\alpha} ds \\ &= \sum_{m\in\mathbb{N}} \frac{1}{m^{\alpha-1}} \int_0^t \sum_{k=0}^{\infty} \frac{\xi_{s-}(k)}{k^\alpha} ds, \end{aligned}$$

so that (i) yields (ii).

The rest is devoted to the proof of (i). By Lemma 3.4 and Proposition 3.5,

$$\int_{s=0}^t ds \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \xi_s(k) \leq \int_{s=0}^t ds \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \zeta_s(k). \tag{39}$$

Define $\sigma(-i) = 0$, $i \in \mathbb{N}$, and

$$Y_n = \sum_{k\in\mathbb{Z}_+} \frac{1 + N^{(n-k)}(\sigma_{n-k}, \sigma_{n+1}]}{k^\alpha}. \tag{40}$$

Recall that the process $(\bar{\zeta}_t)$ was defined in (32). Clearly

$$Y_n \geq \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \bar{\zeta}_t(k), \quad t \in (\sigma_n, \sigma_{n+1}]. \tag{41}$$

Combining (39) and (41) and recalling that $(\zeta_t) \stackrel{d}{=} (\bar{\zeta}_t)$ result in the observation that it is sufficient to show that the strong law of large numbers holds for $(Z_n)_{n\in\mathbb{N}}$, where

$$Z_n := (\sigma_{n+1} - \sigma_n) Y_n.$$

As jump times of a Poisson process, $\sigma_{n+1} - \sigma_n$ are independent unit exponentials, in particular

$$\mathbb{E}\big((\sigma_{n+1} - \sigma_n)^k\big) = k!, \quad k \in \mathbb{N}.$$

Note that for every $n \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}Z_n &= \mathbb{E}\bigg[(\sigma_{n+1} - \sigma_n) \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} N^{(n-k)}(\sigma_{n-k}, \sigma_{n+1}]\bigg] + \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \\ &= \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \mathbb{E}[(\sigma_{n+1} - \sigma_n) N^{(n-k)}(\sigma_{n-k}, \sigma_n]] \\ &\quad + \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} \mathbb{E}[(\sigma_{n+1} - \sigma_n) N^{(n-k)}(\sigma_n, \sigma_{n+1}]] \\ &\quad + \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha} = \sum_{k\in\mathbb{Z}_+} \frac{k}{k^\alpha} + 3 \sum_{k\in\mathbb{Z}_+} \frac{1}{k^\alpha}, \end{aligned} \tag{42}$$

and the last two sums are finite. Thus $\mathbb{E}Z_n$ is bounded in n. In (42) we applied Lemma 3.6(iii). In this proof we make use of Lemma 3.6 in multiple places.

The random variables $\{Z_n\}_{n\in\mathbb{N}}$ are not independent, however the covariance is small for distant elements: we are going to show that there exists a constant $C_Z > 0$ such that for $n, m \in \mathbb{N}$.

$$\operatorname{Cov}(Z_n, Z_{n+m}) \le \frac{C_Z}{m^{\alpha-1}}. \tag{43}$$

We have

$$\begin{aligned}
&\operatorname{Cov}(Z_n, Z_{n+m})\\
&\quad = \operatorname{Cov}\Bigg(\sum_{i\in\mathbb{Z}_+} \frac{\sigma_{n+1}-\sigma_n}{i^\alpha} N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}],\\
&\qquad \sum_{j\in\mathbb{Z}_+} \frac{\sigma_{n+m+1}-\sigma_{n+m}}{j^\alpha} N^{(n+m-j)}(\sigma_{n+m-j}, \sigma_{n+m+1}]\Bigg)\\
&\quad = \sum_{i,j\in\mathbb{Z}_+} \frac{1}{i^\alpha j^\alpha} \operatorname{Cov}\big((\sigma_{n+1}-\sigma_n) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}],\\
&\qquad (\sigma_{n+m+1}-\sigma_{n+m}) N^{(n+m-j)}(\sigma_{n+m-j}, \sigma_{n+m+1}]\big).
\end{aligned} \tag{44}$$

Let us denote by $\mathcal{COV}(i, j)$ the covariance in the last sum of (44). Recall that we defined $\sigma_k = 0$ for $k = 0, -1, -2, \dots$. We can split the interval $(\sigma_{n+m-j}, \sigma_{n+m+1}]$ as follows:

$$\begin{aligned}
&(\sigma_{n+m-j}, \sigma_{n+m+1}]\\
&\quad = \begin{cases}
(\sigma_{n+m-j}, \sigma_{n-i}] \cup (\sigma_{n-i}, \sigma_n] \cup (\sigma_n, \sigma_{n+1}] \cup (\sigma_{n+1}, \sigma_{n+m+1}] \\
\quad \text{if } j > m+i, \\
(\sigma_{n+m-j}, \sigma_n] \cup (\sigma_n, \sigma_{n+1}] \cup (\sigma_{n+1}, \sigma_{n+m+1}] \\
\quad \text{if } m+i \ge j > m, \\
(\sigma_n, \sigma_{n+1}] \cup (\sigma_{n+1}, \sigma_{n+m+1}] & \text{if } j = m, \\
(\sigma_{n+m-j}, \sigma_{n+m+1}] & \text{if } j < m,
\end{cases}
\end{aligned}$$

or, alternatively,

$$\begin{aligned}
&(\sigma_{n+m-j}, \sigma_{n+m+1}]\\
&\quad = \begin{cases}
(\sigma_{n+m-j}, \sigma_{n-i}] \cup (\sigma_{(n+m-j)\vee(n-i)}, \sigma_n] \cup (\sigma_n, \sigma_{n+1}] \\
\quad \cup (\sigma_{(n+m-j)\vee(n+1)}, \sigma_{n+m+1}] & \text{if } j > m+i, \\
(\sigma_{(n+m-j)\vee(n-i)}, \sigma_n] \cup (\sigma_n, \sigma_{n+1}] \\
\quad \cup (\sigma_{(n+m-j)\vee(n+1)}, \sigma_{n+m+1}] & \text{if } m+i \ge j > m, \\
(\sigma_n, \sigma_{n+1}] \cup (\sigma_{(n+m-j)\vee(n+1)}, \sigma_{n+m+1}] & \text{if } j = m, \\
(\sigma_{(n+m-j)\vee(n+1)}, \sigma_{n+m+1}] & \text{if } j < m,
\end{cases}
\end{aligned} \tag{45}$$

and hence (with convention that $(a, b] = \varnothing$ if $a > b$)

$$\begin{aligned}
&\big((\sigma_{n+m-j}, \sigma_{n-i}] \ne \varnothing \\
&\quad \text{and } (\sigma_{n+m-j}, \sigma_{n-i}] \subset (\sigma_{n+m-j}, \sigma_{n+m+1}]\big) \quad \Leftrightarrow \quad j > m+i, \\
&\big((\sigma_{(n+m-j)\vee(n-i)}, \sigma_n] \ne \varnothing \\
&\quad \text{and } (\sigma_{(n+m-j)\vee(n-i)}, \sigma_n] \subset (\sigma_{n+m-j}, \sigma_{n+m+1}]\big) \quad \Leftrightarrow \quad j > m, \\
&(\sigma_n, \sigma_{n+1}] \subset (\sigma_{n+m-j}, \sigma_{n+m+1}] \quad \Leftrightarrow \quad j \ge m.
\end{aligned} \tag{46}$$

We now proceed to estimate $\mathcal{COV}(i,j)$. Using (46) we get

$$
\begin{aligned}
&\mathcal{COV}(i,j)\\
&\quad= \operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)\{N^{(n-i)}(\sigma_{n-i},\sigma_n]+N^{(n-i)}(\sigma_n,\sigma_{n+1}]\},\\
&\qquad(\sigma_{n+m+1}-\sigma_{n+m})\{\mathbb{1}\{j>m+i\}N^{(n+m-j)}(\sigma_{n+m-j},\sigma_{n-i}]\\
&\qquad+\mathbb{1}\{j>m\}N^{(n+m-j)}(\sigma_{(n+m-j)\vee(n-i)},\sigma_n]\\
&\qquad+\mathbb{1}\{j\ge m\}N^{(n+m-j)}(\sigma_n,\sigma_{n+1}]\\
&\qquad+N^{(n+m-j)}(\sigma_{(n+m-j)\vee(n+1)},\sigma_{n+m+1}]\}\big)\\
&\quad= s_{11}+s_{12}+s_{13}+s_{14}+s_{21}+s_{22}+s_{23}+s_{24},
\end{aligned}\tag{47}
$$

where s_{uv}, $u\in\{1,2\}$, $v\in\{1,2,3,4\}$, stands for the covariance of uth and vth summands in the decomposition in (47), for example

$$
\begin{aligned}
s_{23}=\ &\operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)N^{(n-i)}(\sigma_n,\sigma_{n+1}],\\
&(\sigma_{n+m+1}-\sigma_{n+m})\mathbb{1}\{j\ge m\}N^{(n+m-j)}(\sigma_n,\sigma_{n+1}]\big).
\end{aligned}
$$

Let us estimate each of s_{uv}. To start off, $s_{11}=s_{21}=s_{14}=s_{24}=s_{22}=0$ as the covariance of independent random variables. In particular,

$$
\begin{aligned}
s_{22}=\ &\mathbb{1}\{j>m\}\operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)N^{(n-i)}(\sigma_n,\sigma_{n+1}],\\
&(\sigma_{n+m+1}-\sigma_{n+m})N^{(n+m-j)}(\sigma_{(n+m-j)\vee(n-i)},\sigma_n]\big)=0.
\end{aligned}
$$

To other terms we apply Lemma 3.6. Assume first that $n-i\neq n+m-j$. We have by Lemma 3.6(i), (ii), and (vi),

$$
\begin{aligned}
s_{12}&=\mathbb{1}\{j>m\}\operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)N^{(n-i)}(\sigma_{n-i},\sigma_n],\\
&\qquad(\sigma_{n+m+1}-\sigma_{n+m})N^{(n+m-j)}(\sigma_{(n+m-j)\vee(n-i)},\sigma_n]\big)\\
&=\mathbb{1}\{j>m\}\mathbb{E}(\sigma_{n+1}-\sigma_n)\mathbb{E}(\sigma_{n+m+1}-\sigma_{n+m})\\
&\qquad\times\operatorname{Cov}\big(N^{(n-i)}(\sigma_{n-i},\sigma_n],N^{(n+m-j)}(\sigma_{(n+m-j)\vee(n-i)},\sigma_n]\big)\\
&\le\mathbb{1}\{j>m\}\operatorname{Cov}\big(N^{(n-i)}(\sigma_{n-i},\sigma_n],N^{(n+m-j)}(\sigma_{n-i},\sigma_n]\big)\\
&=\mathbb{1}\{j>m\}\operatorname{Var}(\sigma_n-\sigma_{n-i})=i\mathbb{1}\{j>m\}.
\end{aligned}
$$

Applying Lemma 3.6(iii), we continue

$$
\begin{aligned}
s_{13}&=\mathbb{1}\{j\ge m\}\operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)N^{(n-i)}(\sigma_{n-i},\sigma_n],\\
&\qquad(\sigma_{n+m+1}-\sigma_{n+m})N^{(n+m-j)}(\sigma_n,\sigma_{n+1}]\big)\\
&=\mathbb{1}\{j\ge m\}\mathbb{E}(\sigma_{n+m+1}-\sigma_{n+m})\mathbb{E}N^{(n-i)}(\sigma_{n-i},\sigma_n]\\
&\qquad\operatorname{Cov}((\sigma_{n+1}-\sigma_n),N^{(n+m-j)}(\sigma_n,\sigma_{n+1})\\
&=\mathbb{1}\{j\ge m\}i\operatorname{Var}(\sigma_{n+1}-\sigma_n)=\mathbb{1}\{j\ge m\}i.
\end{aligned}
$$

In the same spirit by Lemma 3.6(iv)

$$
\begin{aligned}
s_{23}=\ &\mathbb{1}\{j\ge m\}\operatorname{Cov}\big((\sigma_{n+1}-\sigma_n)N^{(n-i)}(\sigma_n,\sigma_{n+1}],\\
&(\sigma_{n+m+1}-\sigma_{n+m})N^{(n+m-j)}(\sigma_n,\sigma_{n+1}]\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{1}\{j \geq m\}\mathbb{E}(\sigma_{n+m+1} - \sigma_{n+m}) \operatorname{Cov}\big((\sigma_{n+1} - \sigma_n) N^{(n-i)}(\sigma_n, \sigma_{n+1}], \\
&\quad N^{(n+m-j)}(\sigma_n, \sigma_{n+1}]\big) \\
&= \mathbb{1}\{j \geq m\}\big[\mathbb{E}(\sigma_{n+1} - \sigma_n)^3 - \mathbb{E}(\sigma_{n+1} - \sigma_n)^2\big] \\
&= (3! - 2!)\mathbb{1}\{j \geq m\} = 4\mathbb{1}\{j \geq m\}.
\end{aligned}
$$

The computations started from (47) imply that

$$
\mathcal{COV}(i, j) \leq (2i + 4)\mathbb{1}\{j \geq m\} \tag{48}
$$

provided $j \neq m + i$.

If $j = m + i$ then

$$
\begin{aligned}
\mathcal{COV}(i, j) &= \operatorname{Cov}\big((\sigma_{n+1} - \sigma_n) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}], \\
&\qquad (\sigma_{n+m+1} - \sigma_{n+m}) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+m+1}]\big) \\
&= \operatorname{Cov}\big((\sigma_{n+1} - \sigma_n) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}], \\
&\qquad (\sigma_{n+m+1} - \sigma_{n+m}) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}]\big) + 0 \\
&= \operatorname{Cov}\big((\sigma_{n+1} - \sigma_n) N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}], N^{(n-i)}(\sigma_{n-i}, \sigma_{n+1}]\big) \\
&= \operatorname{Cov}\big((\sigma_{n+1} - \sigma_n)\{N^{(n-i)}(\sigma_{n-i}, \sigma_n] + N^{(n-i)}(\sigma_n, \sigma_{n+1}]\}, \\
&\qquad \{N^{(n-i)}(\sigma_{n-i}, \sigma_n] + N^{(n-i)}(\sigma_n, \sigma_{n+1}]\}\big) \\
&= 2 + 6 - 2 + i + 2i = 3i + 6
\end{aligned} \tag{49}
$$

by Lemma 3.6(v) where we can take $\beta = \sigma_{n+1} - \sigma_n$, $X = N^{(n-i)}(\sigma_n, \sigma_{n+1}]$ and $Y = N^{(n-i)}(\sigma_{n-i}, \sigma_n]$. Note that $\mathbb{E}N^{(n-i)}(\sigma_{n-i}, \sigma_n] = \mathbb{E}(\sigma_n - \sigma_{n-i}) = i$,

$$
\mathbb{E}\big[\big(N^{(n-i)}(\sigma_n, \sigma_{n+1}]\big)^2 | (\sigma_n, \sigma_{n+1}]\big] = (\sigma_{n+1} - \sigma_n)^2 + \sigma_{n+1} - \sigma_n,
$$

and

$$
\begin{aligned}
&\operatorname{Var}\big(N^{(n-i)}(\sigma_{n-i}, \sigma_n]\big) \\
&\quad = \mathbb{E}\big[\big(N^{(n-i)}(\sigma_{n-i}, \sigma_n]\big)^2 | \sigma_n - \sigma_{n-i}\big] - \big(\mathbb{E}\big(N^{(n-i)}(\sigma_{n-i}, \sigma_n]\big)\big)^2 \\
&\quad = \mathbb{E}\big((\sigma_n - \sigma_{n-i})^2 + \sigma_n - \sigma_{n-i}\big) - \big(\mathbb{E}(\sigma_n - \sigma_{n-i})\big)^2 \\
&\quad = \mathbb{E}(\sigma_n - \sigma_{n-i}) + \operatorname{Var}(\sigma_n - \sigma_{n-i}) = 2i.
\end{aligned}
$$

In conjunction with (48), (49) allows us to estimate $\operatorname{Cov}(Z_n, Z_{n+m})$. Recalling (44), we get

$$
\begin{aligned}
\operatorname{Cov}(Z_n, Z_{n+m}) &= \sum_{i,j \in \mathbb{N}} \frac{1}{i^\alpha j^\alpha} \mathcal{COV}(i, j) \\
&\leq \sum_{i,j \in \mathbb{N}, i \neq j} \frac{(2i + 4)\mathbb{1}\{j \geq m\}}{i^\alpha j^\alpha} + \sum_{i \in \mathbb{N}} \frac{3i + 6}{i^\alpha (i + m)^\alpha} \\
&\leq \sum_{i \in \mathbb{N}} \frac{2i + 4}{i^\alpha} \sum_{j=m}^{\infty} \frac{1}{j^\alpha} + \frac{1}{m^\alpha} \sum_{i \in \mathbb{N}} \frac{3i + 6}{i^\alpha}.
\end{aligned} \tag{50}
$$

Since $\sum_{j=m}^{\infty} \frac{1}{j^\alpha} = O(\frac{1}{m^{\alpha-1}})$ as $m \to \infty$, (50) implies (43). The statement of (i) follows from (43) and the strong law of large numbers for dependent random variables, see for example, Hu, Rosalsky, and Volodin [31], or Corollary 11 of Lyons [37]. □

Let $\Delta M_n = M_{n+1} - M_n$, and let ΔX_n and ΔQ_n be defined in the same way. In the following proposition we establish finiteness of a moment of the martingale difference ΔM_n. Later on this allows us to apply a strong law of large numbers for martingales to M_n.

PROPOSITION 3.10. *Let $p \in (1, (\alpha - 1)) \cap (1, 2]$. Then $\mathbb{E}|\Delta M_n|^p$ is bounded uniformly in n.*

PROOF. By (35),

$$\Delta X_n = \int_{s\in(n,n+1],m\in\mathbb{N},u\in\mathbb{R}_+} m\mathbb{1}\left\{u \le 1 \wedge \sum_{k\in\mathbb{Z}_+} \frac{\xi_{s-}(k)}{(k+m)^{2\alpha}}\right\} \times N^{(X)}(ds\,dm\,du). \tag{51}$$

Note that for every $k \in \mathbb{Z}$ and $s \ge 0$, $\mathbb{E}\eta_s(k) \le k+1$ because $(\eta_t(k) - \eta_0(k))_{t\ge 0}$ is dominated by a Poisson process, and consequently also

$$\mathbb{E}\xi_{s-}(k) \le k+1.$$

Novikov's inequalities for discontinuous martingales (also known as "Bichteler–Jacod's inequalities"; see Novikov [42], or Marinelli and Röckner [39] for generalizations and historical discussions) give

$$\begin{aligned}
&\mathbb{E}|\Delta M_n|^p \\
&\quad = \mathbb{E}\left|\Delta X_n - \int_{\substack{s\in(n,n+1],m\in\mathbb{N},\\ u\in\mathbb{R}_+}} m\mathbb{1} \right.\\
&\qquad \left. \times \left\{u \le 1 \wedge \sum_{k\in\mathbb{Z}_+} \frac{\xi_{s-}(k)}{(k+m)^{2\alpha}}\right\} ds\#(dm)\,du\right|^p \\
&\quad \le C\mathbb{E}\int_{\substack{s\in(n,n+1],m\in\mathbb{N},\\ u\in\mathbb{R}_+}} m^p\mathbb{1}\left\{u \le 1 \wedge \sum_{k\in\mathbb{Z}_+} \frac{\xi_{s-}(k)}{(k+m)^{2\alpha}}\right\} ds\#(dm)\,du \\
&\quad = C\int_n^{n+1} ds \sum_{m\in\mathbb{N}} m^p\left(1 \wedge \sum_{k\in\mathbb{Z}_+} \frac{\mathbb{E}\xi_{s-}(k)}{(k+m)^{2\alpha}}\right) \\
&\quad \le C\int_n^{n+1} ds \sum_{m\in\mathbb{N}} m^p \sum_{k\in\mathbb{Z}_+} \frac{k+1}{(k+1)^\alpha m^\alpha} \\
&\quad = C\sum_{m\in\mathbb{N}} \frac{1}{m^{\alpha-p}} \times \sum_{k\in\mathbb{N}} \frac{1}{k^{\alpha-1}}.
\end{aligned} \tag{52}$$

Hence

$$\mathbb{E}|\Delta M_n|^p < C < \infty,$$

where C does not depend on n. □

We are now ready to prove the main result of this section. We will need the following form of the strong law of large numbers for martingales, which is an abridged version of [29], Theorem 2.18.

THEOREM 3.11. *Let* $\{S_n = \sum_{i=1}^{n} x_i, n \in \mathbb{N}\}$ *be an* $\{\mathscr{F}_n\}$*-martingale and* $\{U_n\}_{n\in\mathbb{N}}$ *be a nondecreasing sequence of positive real numbers*, $\lim_{n\to\infty} U_n = \infty$. *Then for* $p \in [1,2]$ *we have*

$$\lim_{n\to\infty} U_n^{-1} S_n = 0$$

a.s. on the set $\{\sum_{i=1}^{\infty} U_n^{-p} \mathbb{E}[|x_i|^p | \mathscr{F}_{i-1}] < \infty\}$.

THEOREM 3.12 (Linear speed). *There exists* $\bar{C} > 0$ *such that a.s.*

$$X_t \le \bar{C} t \tag{53}$$

for sufficiently large t.

PROOF. Note that a.s.

$$\sum \frac{\mathbb{E}(|\Delta M_n|^p | \mathcal{F}_{n-1})}{n^p} < \infty,$$

since by Proposition 3.10

$$\mathbb{E} \sum \frac{\mathbb{E}(|\Delta M_n|^p | \mathcal{F}_{n-1})}{n^p} = \sum \frac{\mathbb{E}|\Delta M_n|^p}{n^p} < \infty. \tag{54}$$

Then Proposition 3.10 and Theorem 3.11, where we take $S_n = M_n$, $U_n = n$, and $p = \frac{\alpha}{2} \wedge 2$, imply that a.s.

$$\frac{M_n}{n} \to 0, \quad n \in \mathbb{N}. \tag{55}$$

Hence Proposition 3.9(ii), yields that a.s. for large n

$$\frac{X_n}{n} = \frac{M_n}{n} + \frac{Q_n}{n} \le C_X, \tag{56}$$

where $C_X > 0$ is independent of n.

Since X_t is nondecreasing, (56) holds for continuous parameter too if we increase the constant: a.s. for large t,

$$\frac{X_t}{t} \le C_X + 1. \tag{57}$$

□

4. Superlinear growth for $\alpha \in (\frac{1}{2}, 2]$ in the discrete-space settings. Our aim in this section is to prove the discrete-space equivalent of Theorem 2.7. This is done in Theorem 4.3. In Section 5 we use Theorem 4.3 to prove Theorem 2.7. The idea of the proof of Theorem 4.3 is to find a certain system growing slower than our system, and then estimate the probability of births outside an interval linearly growing with time.

Let (η_t) be the birth process on $\mathbb{Z}_+^{\mathbb{Z}}$ with birth rate (13), (14), but with $\alpha \in (\frac{1}{2}, 2]$. As in Section 3, (η_t) can be obtained as the unique solution to (26). We focus here on the positive half line because it is sufficient for our purposes.

The next lemma has an auxiliary character and is a straightforward application of the large deviations principle.

LEMMA 4.1. *Let* χ *be a Poisson random variable with mean* $\lambda > 0$. *Then for large* λ,

$$\mathbb{P}\left(\chi \le \frac{\lambda}{3}\right) \le e^{-\frac{\lambda}{6}}. \tag{58}$$

PROOF. Assume first that $\lambda \in \mathbb{N}$. Then

$$\chi \overset{d}{=} \chi_1 + \cdots + \chi_\lambda,$$

where $\chi_1, \chi_2, \dots$ are i.i.d. Poisson random variables with mean 1. The cumulant-generating function of χ_1 is $\Lambda(u) = e^u - 1$, and the corresponding rate function

$$\Lambda^*(x) = \sup_{u \in \mathbb{R}} \bigl(ux - \Lambda(u)\bigr) = x \ln x - x + 1, \quad x \geq 0.$$

By the large deviations principle, see for example, [33], Theorem 27.5,

$$\limsup_{\lambda \to \infty} \lambda^{-1} \ln \mathbb{P}\left(\frac{1}{\lambda} \sum_{i=1}^{\lambda} \chi_i \leq \frac{1}{3}\right) \leq - \inf_{x \in [0, \frac{1}{3}]} \Lambda^*(x) = -\frac{2 - \ln 3}{3}.$$

Hence for large λ

$$\ln \mathbb{P}\left(\chi \leq \frac{\lambda}{3}\right) \leq -\frac{2 - \ln 3}{3} \lambda + o(\lambda) < -\frac{\lambda}{4},$$

which gives the desired result for $\lambda \in \mathbb{N}$. The statement for $\lambda \notin \mathbb{N}$ follows by considering a Poisson random variable with mean $\lfloor \lambda \rfloor$ and noting that for large λ, $\frac{\lfloor \lambda \rfloor}{4} > \frac{\lambda}{6}$. $\square$

In the next lemma it is shown that η_t dominates a "rectangle-like" configuration, at least for large t.

LEMMA 4.2. *A.s. for sufficiently large t*

$$\eta_t(x) \geq \frac{t}{10}, \tag{59}$$

for all $x \in \mathbb{Z} \cap [0, \frac{t}{4}]$.

PROOF. Let (γ_t) be another birth process with birth rate

$$b^{(\gamma)}(x, \eta) = 1 \wedge \bigl(\eta(x) + \eta(x-1) - 1\bigr)_+, \quad x \in \mathbb{Z}^1, \eta \in \mathbb{Z}_+^{\mathbb{Z}}, \tag{60}$$

where $\kappa_+ = \max(\kappa, 0)$, and the initial condition $\gamma_0(k) = \mathbb{1}\{k = 0\}$, $k \in \mathbb{Z}$. Alternatively,

$$b^{(\gamma)}(x, \eta) = \begin{cases} 1 & \text{if } \eta(x) + \eta(x-1) > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{61}$$

The process (γ_t) can be obtained as a unique solution to (26) with the birth rate $b^{(\gamma)}$ instead of $b^{(d)}$.

We have

$$b^{(\gamma)}(x, \eta) \leq b^{(d)}(x, \eta), \quad x \in \mathbb{Z}, \eta \in \mathbb{Z}_+^{\mathbb{Z}}. \tag{62}$$

Using (62), it is not difficult to show that a.s. for all $t \geq 0$,

$$\gamma_t \leq \eta_t. \tag{63}$$

In the continuous-space settings, the fact that (62) implies (63) is proven in [6], Lemma 5.1. In our case here we can take exactly the same proof.

Let $\tau(n) := \inf\{t : \gamma_t(n) > 0\}$ be the time when $n \in \mathbb{Z}_+$ becomes occupied for (γ_t). Note that a.s. $\tau(1) < \tau(2) < \cdots$. Let $X_t^{(\gamma)} := \max\{n : \gamma_t(n) > 0\}$ be the position of the rightmost occupied site for (γ_t). The process $(X_t^{(\gamma)})$ is a counting Markov process (that is, having unit jumps only) and with jump rate constantly being 1. Therefore $(X_t^{(\gamma)})$ is a Poisson process whose nth jump time coincides with $\tau(n)$. By the law of large numbers, a.s. for large n,

$$\tau(n) < \frac{5}{4}n.$$

Therefore, a.s. for large n for $x \in \{0, 1, \dots, n\}$, $t \in [\frac{5}{4}n, 2n]$, we have $b^{(\gamma)}(x, \eta_t) = 1$. By (26) (recall that the birth rate for (γ_t) is $b^{(\gamma)}$ instead of $b^{(d)}$)

$$\gamma_{2n}(x) - \gamma_{\frac{5}{4}n}(x) = \mathbf{N}\biggl(\biggl[\frac{5}{4}n, 2n\biggr] \times \{x\} \times [0, 1]\biggr), \quad x = 1, \dots, n,$$

hence a.s. for large n

$$\gamma_{2n}(x) \geq \mathbf{N}\biggl(\biggl[\frac{5}{4}n, 2n\biggr] \times \{x\} \times [0, 1]\biggr), \quad x = 1, \dots, n. \tag{64}$$

The random variables $\omega_x^{(n)} := \mathbf{N}([\frac{5}{4}n, 2n] \times \{x\} \times [0, 1])$, $x = 1, \dots, n$, are i.i.d Poisson with mean $\frac{3}{4}n$. By Lemma 4.1 for $x = 0, 1, \dots, n$,

$$\mathbb{P}\biggl\{\omega_x^{(n)} < \frac{n}{4}\biggr\} \leq e^{-\frac{n}{8}},$$

hence

$$\mathbb{P}\biggl\{\omega_x^{(n)} < \frac{n}{4} \text{ for some } x \in \{1, \dots, n\}\biggr\} \leq ne^{-\frac{n}{8}}. \tag{65}$$

Since $\sum_{n\in\mathbb{N}} ne^{-\frac{n}{8}} < \infty$, by the Borel–Cantelli lemma the event in (65) happens a.s. finitely many times only, therefore a.s. for sufficiently large n

$$\omega_x^{(n)} \geq \frac{n}{4}, \quad x \in \{1, \dots, n\}.$$

By (63), (64), and the definition of $\omega_x^{(n)}$, a.s. for sufficiently large n,

$$\eta_{2n}(x) \geq \gamma_{2n}(x) \geq \frac{n}{4}, \quad x \in \{1, \dots, n\}. \tag{66}$$

By taking $n = \lfloor \frac{t}{2} \rfloor - 1$ in (66) we get a.s. for large t

$$\eta_t(x) \geq \frac{t}{10}, \quad x \in \biggl\{1, \dots, \biggl\lfloor \frac{t}{2} \biggr\rfloor - 1\biggr\}, \tag{67}$$

and the statement of the lemma follows. $\square$

Now we are ready to prove the main result of the section.

THEOREM 4.3. *For every $K_0, K_1 > 0$ the set*

$$\biggl\{t : \sum_{\substack{x\in\mathbb{Z}, \\ x > K_0 + K_1 t}} \eta_t(x) > 0\biggr\}$$

is a.s. unbounded.

PROOF. Without loss of generality we assume that $K_1 > 1$. Let ζ_t be defined by $\zeta_t(k) = \lfloor \frac{t}{10} \rfloor \mathbb{1}\{0 \leq k \leq \frac{t}{4}\}$, hence by Lemma 4.2 a.s. for large t,

$$\zeta_t \leq \eta_t. \tag{68}$$

Provided that t is sufficiently large, the rate of a birth occuring inside $[K_0+K_1t+K_1,\infty)$ at time t is

$$\sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} b^{(d)}(x,\eta_t) \geq \sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} b^{(d)}(x,\zeta_t)$$
$$\geq 1 \wedge \sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} \sum_{k=0}^{\lfloor\frac{t}{4}\rfloor} \zeta(k)a^{(d)}(x-k). \tag{69}$$

Note that

$$\sum_{k=0}^{\lfloor\frac{t}{4}\rfloor} a^{(d)}(x-k) = \sum_{k=0}^{\lfloor\frac{t}{4}\rfloor} \frac{1}{|x-k|^{2\alpha}} \geq \sum_{k=0}^{\lfloor\frac{t}{4}\rfloor} \frac{1}{|x|^{2\alpha}} = \left\lfloor\frac{t}{4}\right\rfloor \frac{1}{|x|^{2\alpha}} \tag{70}$$

hence for large t

$$\sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} \sum_{k=0}^{\lfloor\frac{t}{4}\rfloor} a^{(d)}(x-k)$$
$$\geq \left\lfloor\frac{t}{4}\right\rfloor \sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} \frac{1}{|x|^{2\alpha}} \tag{71}$$
$$\geq \frac{t}{5} \times \frac{1}{2(2\alpha-1)|K_0+K_1t+K_1+1|^{2\alpha-1}} \geq \frac{C}{t^{2\alpha-2}}.$$

Since $\zeta(k) = \lfloor\frac{t}{10}\rfloor$, $k\in 0,1,\dots,\lfloor\frac{t}{4}\rfloor$, by (69) and (71),

$$\sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1t+K_1}} b^{(d)}(x,\eta_t) \geq 1 \wedge \left(\left\lfloor\frac{t}{10}\right\rfloor \frac{C}{t^{2\alpha-2}}\right) \geq 1 \wedge \frac{c}{t^{2\alpha-3}}, \tag{72}$$

where $c>0$ is a constant depending on K_0, K_1, α, but not on time t.

Let L_t be the number of jumps for (η_t) that have occured prior t to the right of a growing interval $[0, K_0+K_1s]$ for some $s\leq t$, that is,

$$\mathrm{L}_t = \#\{(s,k): s\in(0,t], k\in(K_0+K_1s,\infty)\cap\mathbb{Z}, \eta_s(k)-\eta_{s-}(k)=1\}$$
$$= \int_{\substack{\{(s,k,u): s\in(0,t],\\ k\in(K_0+K_1s,\infty)\cap\mathbb{Z}, u\in[0,1]\}}} \mathbb{1}_{[0,b^{(d)}(k,\eta_{s-})]}(u)\mathbf{N}(ds\,dk\,du). \tag{73}$$

Let $n\in\mathbb{N}$. We have

$$\mathrm{L}_{n+1}-\mathrm{L}_n = \int_{\substack{\{(s,k,u): s\in(n,n+1],\\ k\in(K_0+K_1s,\infty)\cap\mathbb{Z}, u\in[0,1]\}}} \mathbb{1}_{[0,b^{(d)}(k,\eta_{s-})]}(u)\mathbf{N}(ds\,dk\,du)$$
$$\geq \int_{\substack{\{(s,k,u): s\in(n,n+1],\\ k\in(K_0+K_1n+K_1,\infty)\cap\mathbb{Z}, u\in[0,1]\}}} \mathbb{1}_{[0,b^{(d)}(k,\eta_n)]}(u)\mathbf{N}(ds\,dk\,du). \tag{74}$$

Define the sequence of independent random variables $\{F_n\}_{n\in\mathbb{N}}$,

$$F_n := \int_{\substack{\{(s,k,u): s\in(n,n+1],\\ k\in(K_0+K_1n+K_1,\infty)\cap\mathbb{Z}, u\in[0,1]\}}} \mathbb{1}_{[0,b^{(d)}(k,\zeta_n)]}(u)\mathbf{N}(ds\,dk\,du). \tag{75}$$

By (68) and (74), a.s. for large n

$$\mathrm{L}_{n+1}-\mathrm{L}_n \geq F_n. \tag{76}$$

Since ζ_n is a nonrandom element of $\mathbb{Z}_+^{\mathbb{Z}}$, F_n is a Poisson random variable with mean

$$m_n := \sum_{\substack{x\in\mathbb{Z}:\\ x>K_0+K_1n+K_1}} b^{(d)}(x,\zeta_n).$$

As we saw in (72), $m_n \geq 1 \wedge cn^{-(2\alpha-3)}$ large n. Hence, at least for large n,

$$\mathbb{P}\{F_n \geq 1\} = 1 - e^{-m_n} \geq 1 - \exp\{-1 \wedge cn^{-(2\alpha-3)}\}. \tag{77}$$

Recall that for $c_1, c_2 \in \mathbb{R}$, $-c_1 \wedge c_2 = -(c_1 \wedge c_2)$. The series

$$\sum_{n\in\mathbb{N}} \left(1 - \exp\{-1 \wedge cn^{-(2\alpha-3)}\}\right) \tag{78}$$

diverges since $2\alpha - 3 \leq 1$. Hence by the Borel–Cantelli lemma and (77),

$$\mathbb{P}\{F_n \geq 1 \text{ for infinitely many } n \in \mathbb{N}\} = 1. \tag{79}$$

Finally, by (76) and (79),

$$\mathbb{P}\{\mathrm{L}_{n+1} - \mathrm{L}_n \geq 1 \text{ for infinitely many } n \in \mathbb{N}\} = 1. \tag{80}$$

Recalling the definition of L_n in (73), we see that our theorem is proven. □

5. Continuous-space model. We now return to the continuous-space model with the birth rate (1) described in the Introduction. To prove Theorem 2.1 and Theorem 2.7, we couple the continuous-space process with the discrete-space process from Sections 3 and 4 and make use of Theorem 3.12 and Theorem 4.3.

The continuous-space birth process defined by (1) and (2) can be obtained as a unique solution to the stochastic equation

$$\begin{aligned} |\eta_t \cap B| = &\int_{(0,t]\times B\times[0,\infty)} \mathbb{1}_{[0,b^{(c)}(x,\eta_{s-})]}(u) N^{(c)}(ds,dx,du) \\ &+ |\eta_0 \cap B|, \quad t \geq 0, B \in \mathscr{B}(\mathbb{R}^1), \end{aligned} \tag{81}$$

where $(\eta_t)_{t\geq 0}$ is a cadlag Γ_0-valued solution process, $N^{(c)}$ is a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^1 \times \mathbb{R}_+$, the mean measure of $N^{(c)}$ is $ds \times dx \times du$, and $\eta_0 = \{0\}$. Equation (81) is understood in the sense that the equality holds a.s. for every bounded $B \in \mathscr{B}(\mathbb{R}^1)$ and $t \geq 0$. In the integral on the right-hand side of (81), x is the location and s is the time of birth of a new particle. Thus, the integral over B from 0 to t represents the number of births inside B which occurred before t (see [6] for more details). The birth rate $b^{(c)}$ is as in (1) with a defined in (2).

In this section we denote the solution to (81) by $(\eta_t^{(c)})$ with the upper index "(c)" standing for "continuous". We compare $(\eta_t^{(c)})$ to the solution $(\eta_t^{(d)})$ ((d) for "discrete") of another equation

$$\eta_t(k) = \int_{(0,t]\times\{k\}\times[0,1]} \mathbb{1}_{[0,C_\alpha b^{(d)}(i,\eta_{s-})]}(u) N^{(d)}(ds\,di\,du) + \eta_0^{(d)}(k), \tag{82}$$

which is of the form (26) but with the birth rate multiplied by $C_\alpha > 0$:

$$C_\alpha b^{(d)}(x,\eta) = C_\alpha \wedge \left(C_\alpha \sum_{y\in\mathbb{Z}} \eta(y) a^{(d)}(x-y)\right), \quad x \in \mathbb{Z}, \eta \in \mathbb{Z}_+^{\mathbb{Z}}, \tag{83}$$

with $a^{(d)}$ as in (14) and $\eta_0^{(d)}(k) = \mathbb{1}\{k=0\}$, and with the driving Poisson point process

$$N^{(d)}([0,t]\times\{k\}\times[0,u]) = N^{(c)}\left([0,t]\times\left(k-\frac{1}{2}, k+\frac{1}{2}\right]\times[0,u]\right).$$

Note that $(\eta_t^{(d)})$ is the process from the previous section evolving C_α times faster in time (or slower if $C_\alpha < 1$), and Theorem 3.12 applies to $(\eta_t^{(d)})$ too.

Define also the discretization of the continuous-space process $(\eta_t^{(c)})$ as the process $(\eta_t^{(dc)})$ taking values in $Z_+^{\mathbb{Z}}$ and

$$\eta_t^{(dc)}(k) = \left|\eta_t^{(c)} \cap \left(k - \frac{1}{2}, k + \frac{1}{2}\right]\right|, \quad k \in \mathbb{Z}. \tag{84}$$

Recall that for $c_1, c_2, c_3 \in \mathbb{R}$, $c_1c_2 \vee c_3 = (c_1c_2) \vee c_3$, and the same for $\wedge$.

PROPOSITION 5.1.

(i) *Let $C_\alpha \geq c_\alpha 2^\alpha \vee 2$. Then a.s. for all $t \geq 0$*

$$\eta_t^{(dc)}(k) \leq \eta_t^{(d)}(k), \quad k \in \mathbb{Z}. \tag{85}$$

(ii) *Let $C_\alpha \leq c_\alpha 4^{-\alpha} \wedge \frac{1}{2}$. Then a.s. for all $t \geq 0$*

$$\eta_t^{(dc)}(k) \geq \eta_t^{(d)}(k), \quad k \in \mathbb{Z}. \tag{86}$$

PROOF. We start with (i). The proof will be done by induction on the birth moments of $(\eta_t^{(c)})$. Let $\{\theta_k\}$ be the moment of kth birth for $(\eta_t^{(c)})$, $\theta_0 = 0$. For $t = \theta_0$, (85) is satisfied. For $x \in \mathbb{R}$, let here round(x) is the closest integer to x, with convention that round$(m + \frac{1}{2}) = m$, $m \in \mathbb{Z}$. It is sufficient to show that if a birth occurs for $(\eta_t^{(c)})$ at time θ at $x \in \mathbb{R}$, then a birth also occurs for $(\eta_t^{(d)})$ at θ at round(x). Assume (85) holds for $k < n \in \mathbb{N}$ and let x_n be the place of birth at time θ_n. Since $(\eta_t^{(c)})$ solves (81), we have a.s.

$$N^{(c)}\big(\{\theta_k\} \times \{x_n\} \times \big[0, b^{(c)}\big(x_n, \eta_{\theta_k-}^{(c)}\big)\big)\big) = 1.$$

Since $\frac{1 \vee |\mathrm{round}(x)|^{2\alpha}}{(1+|x|^2)^\alpha} \leq 2^\alpha$ for $x \in \mathbb{R}$, we have

$$a(x) \leq c_\alpha 2^\alpha a^{(d)}\big(\mathrm{round}(x)\big), \quad x \in \mathbb{R}, \tag{87}$$

and hence by the induction assumption a.s.

$$b^{(c)}\big(x_n, \eta_{\theta_k-}^{(c)}\big) \leq C_\alpha b^{(d)}\big(\mathrm{round}(x_n), \eta_{\theta_k-}^{(dc)}\big).$$

Consequently, we also have a.s.

$$N^{(d)}\big(\{\theta_k\} \times \big\{\mathrm{round}(x_n)\big\} \times \big[0, C_\alpha b^{(d)}\big(\mathrm{round}(x_n), \eta_{\theta_k}^{(d)}\big)\big)\big) = 1,$$

and so we also have a birth for $(\eta_t^{(d)})$ at time θ_k at round(x_n) since $\eta_{\theta_k-}^{(dc)} \leq \eta_{\theta_k-}^{(d)}$ and thus (85) holds at θ_n as well.

The proof of (ii) can be done by induction on the birth moments of $(\eta_t^{(d)})$, following exactly the same steps as the proof of (i), so we omit it. We just point out that the counterpart of (87):

$$a(x) \geq c_\alpha 4^{-\alpha} a^{(d)}\big(\mathrm{round}(x)\big), \quad x \in \mathbb{R}. \qquad \square$$

PROOF OF THEOREM 2.1. The statement of the theorem follows from Theorem 3.12 and Proposition 5.1(i). $\square$

PROOF OF THEOREM 2.7. The statement of the theorem is a consequence of Theorem 4.3 and Proposition 5.1(ii). $\square$

6. Mesoscopic equation. In this section we study the long time behavior of nonnegative bounded solutions to the following nonlinear nonlocal evolution equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \min\{(a * u)(x,t), 1\} & x \in \mathbb{R}, t \in (0,\infty), \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{88}$$

Here $u \in C(\mathbb{R}_+, L^\infty(\mathbb{R})) \cap C^1((0,\infty), L^\infty(\mathbb{R}))$ is a classical solution to (88), $u_0 \in L^\infty(\mathbb{R}, \mathbb{R}_+)$ is an initial condition; the function $a \in L^1(\mathbb{R}) := L^1(\mathbb{R}, dx)$ is a probability density, that is, $a(x) \geq 0$ for a.a. (almost all) $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} a(x)\, dx = 1; \tag{89}$$

the symbol $*$ stands for the convolution in x on $\mathbb{R}$, that is,

$$(a * u)(x,t) := \int_{\mathbb{R}} a(x-y) u(y,t)\, dx.$$

An informal scaling and link between the microscopic and mesoscopic models. *Here we describe the heuristic arguments which connect the birth process defined by* (1) *and* (2) *and the solution to the equation* (88). *We follow here the line of thought from* [24], *Theorem 5.3. Let us stress that we do not in any way give a rigorous proof of the link.*

For a bounded measurable function $\phi : \Gamma_0 \to \mathbb{R}$ consider the birth rate

$$b^n(x,\eta) = n \wedge \left(\sum_{y \in \eta} a(x-y) \right), \tag{90}$$

and the corresponding spatial birth process $(\eta_t^n)_{t \geq 0}$.

For $t \geq 0$, let ν_t^n be a random purely atomic measure on $\mathbb{R}$ defined by

$$\nu_t^n(A) = |\eta_t^n \cap A|.$$

The intuition is that considering $(\eta_t^n)_{t\geq 0}$ and $(\nu_t^n)_{t \geq 0}$ we increase the birth rate but then we are going to rescale the process by multiplying by $\frac{1}{n}$ to compensate for the increase in the number of particles. Let $\mathscr{M}(\mathbb{R})$ be the space of finite nonnegative measures equipped with the vague topology. Assume that if $\frac{1}{n}\nu_0^n(dx)$ converges in law to a deteministic measure $\mu_0(dx)$, then the measure valued function $\frac{1}{n}\nu_t^n(dx)$ converges in law in the Skorokhod space $\mathrm{D}([0,T], \mathscr{M}(\mathbb{R}))$ to a deterministic $\mathscr{M}(\mathbb{R})$-valued function $t \mapsto \mu_t$. Since (92) *below is a martingale with a vanishing quadratic variation, this limiting measure-valued function should then be a unique solution to the integral equation written in the weak form:*

$$\langle \mu_t, f \rangle = \langle \mu_0, f \rangle + \int_0^t ds \int_{x \in \mathbb{R}} f(x) \min\left\{1, \int_{y \in \mathbb{R}} a(x-y) \mu_t(dy)\right\} dx. \tag{91}$$

Assume furthermore that μ_t has a density with respect to the Lebesgue measure provided that the initial condition does: $\mu_0(x) = u_0(x)\,dx$. We denote the density of μ_t by $u(t,x)$, so that $\mu_t(dx) = u(t,x)\,dx$. Denote $u_t = u(t,\cdot)$, (u_t is a function on $\mathbb{R}$). Then we have

$$\frac{1}{n} \sum_{y \in \eta_t^n} a(x-y) \to (a * u_t)(x)$$

and hence, assuming that u is differentiable,

$$\frac{\partial u(t,x)}{\partial t}(t,x) = \lim_n \frac{1}{n} b^{(n)}(x, \eta_t^n)$$

$$= \lim_n \frac{1}{n}\Big[n \wedge \Big(\sum_{y\in\eta_t^n} a(x-y)\Big)\Big]$$

$$= \lim_n 1 \wedge \Big(\frac{1}{n}\sum_{y\in\eta_t^n} a(x-y)\Big) = 1 \wedge \big((a * u_t)(x)\big),$$

which coincides with (88).

The proof that the limiting measure is indeed the unique solution to (91) *would have to rely on the martingale properties of the spatial birth processes. The generator of the birth process with the rate* (90) *is*

$$(L^n \phi)(\eta) = \int_{\mathbb{R}} b^n(x,\eta)\big[\phi(\eta \cup x) - \phi(\eta)\big]\,dx.$$

As in [24], *one could show that for any bounded measurable* $f : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} M_t^{n,f} :=& \frac{1}{n}\int_{\mathbb{R}} f(x)\nu_t^n(dx) - \frac{1}{n}\int_{\mathbb{R}} f(x)\nu_0(dx) \\ &- \frac{1}{n}\int_0^t \int_{\mathbb{R}}\Big[n \wedge \int_{\mathbb{R}} a(x-y)\nu_t^n(dy)\Big] f(x)\,dx\,ds \end{aligned} \tag{92}$$

is a càdlàg martingale with the quadratic variation

$$\langle M^{n,f}\rangle_t = \frac{1}{n^2}\int_0^t \int_{\mathbb{R}}\Big[n \wedge \int_{\mathbb{R}} a(x-y)\nu_t^n(dy)\Big] f^2(x)\,dx\,ds.$$

Hence

$$\mathbb{E}|M_t^{n,f}|^2 = \mathbb{E}\langle M^{n,f}\rangle_t \le \frac{c_\alpha \|f\| \mathbb{E}|\eta_t^n|}{n},$$

where $\|f\| = \sup_{x\in\mathbb{R}} f(x)$. *Thus* $\mathbb{E}|M_t^{n,f}|^2 \to 0$ *a.s. uniformly on any finite interval* $[0,T]$, $n \to \infty$.

The proof of Theorem 2.8 falls naturally into two parts. First, we obtain an estimate of the solution u from above (see Proposition 6.11), which implies that u propagates at most exponentially. Second, we construct subsolutions (114) to (88) in order to estimate “small” level-sets of the solution from below. Then, locally uniform convergence of u to infinity (Lemma 6.6) demonstrates that the solution does not propagate slower than exponentially.

We start with general properties of the solutions to (88).

DEFINITION 6.1. We call an operator G in $L^\infty(\mathbb{R})$ monotone, if for all $h_1, h_2 \in L^\infty(\mathbb{R})$,

$$h_1(x) \le h_2(x), \quad x \in \mathbb{R} \quad \Rightarrow \quad Gh_1(x) \le Gh_2(x), \quad x \in \mathbb{R}.$$

We call an operator G in $L^\infty(\mathbb{R})$ Lipschitz continuous, if there exists $K > 0$, such that for all $h_1, h_2 \in L^\infty(\mathbb{R})$,

$$\|Gh_2 - Gh_1\|_{L^\infty(\mathbb{R})} \le K\|h_2 - h_1\|_{L^\infty(\mathbb{R})}.$$

REMARK 6.2. Gu $= \min\{a * u, 1\}$ is a monotone and Lipschitz continous operator in $L^\infty(\mathbb{R})$ with the Lipschitz constant $K = 1$.

Since G is Lipschitz-continuous in the Banach space $L^\infty(\mathbb{R})$, well-posedness of (88) is easily shown by a Picard iteration scheme (see, e.g., [44], Chapter 6, Theorem 1.2, Theorem 1.7). For completeness we provide the details (cf. [49]).

PROPOSITION 6.3. *Let G be Lipschitz continuous on $L^\infty(\mathbb{R})$ and $u_0 \in L^\infty(\mathbb{R})$. Then for any $T > 0$ there exists a unique classical solution $u \in C(\mathbb{R}_+, L^\infty(\mathbb{R})) \cap C^1((0,\infty), L^\infty(\mathbb{R}))$ to the equation,*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = (\mathrm{Gu})(x,t) & t \in (0,\infty), x \in \mathbb{R}^1, \\ u(x,0) = u_0(x) & x \in \mathbb{R}^1. \end{cases} \tag{93}$$

PROOF. For $0 \le \tau < \Upsilon < \infty$, $v \in C([\tau,\Upsilon], L^\infty(\mathbb{R}))$, $w \in L^\infty(\mathbb{R})$, we define,

$$(\Phi_w v)(x,t) := w(x) + \int_\tau^t (\mathrm{Gv})(x,s)\,ds, \quad t \in [\tau,\Upsilon], x \in \mathbb{R}. \tag{94}$$

Let $\|v\|_{\tau,\Upsilon} := \sup_{t\in[\tau,\Upsilon]} \|v(\cdot,t)\|_\infty$. Then, one easily gets, that $\|\Phi_w v\|_{\tau,\Upsilon} < \infty$ and

$$\|\Phi_w v_1 - \Phi_w v_2\|_{\tau,\Upsilon} \le K(\Upsilon - \tau)\|v_1 - v_2\|_{\tau,\Upsilon},$$

where K is the Lipschitz constant of G. Therefore, Φ_w is a contraction mapping on $C([\tau,\Upsilon], L^\infty(\mathbb{R}))$, provided that $\Upsilon - \tau < \frac{1}{K}$. Fixing any $\delta \in (0, \frac{1}{K})$, one gets that there exists the limit u of $(\Phi_w)^n v$, $n \to \infty$, for any v, on time intervals $[k\delta, (k+1)\delta]$, $k \in \mathbb{N} \cup \{0\}$, with the corresponding $w(x) = u(x, k\delta)$. Therefore, for any $0 \le \tau < \Upsilon$, we have that $u \in C([\tau,\Upsilon], L^\infty(\mathbb{R}))$ and

$$u(x,t) = (\Phi_{u(\cdot,\tau)} u)(x,t), \quad t \in [\tau,\Upsilon].$$

Since G is Lipschitz continuous, then it follows that $u \in C(\mathbb{R}_+, L^\infty(\mathbb{R})) \cap C^1((0,\infty), L^\infty(\mathbb{R}))$ and it solves (93). The proof is completed. □

We introduce the following operators:

$$Z_y v(x) = v(x-y), \quad v \in L^\infty(\mathbb{R}), y \in \mathbb{R}, \tag{95}$$

$$Q_t v(x) = u(x,t), \quad t \ge 0, x \in \mathbb{R}, \tag{96}$$

where $u(x,0) = v(x)$ and u solves (93). Thus Z_y is a shift operator in $\mathbb{R}$, and Q_t is the semiflow generated by (93). The following important property follows form the proof of Proposition (6.3).

COROLLARY 6.4. *If Z_y and G are commutative for all $y \in \mathbb{R}$, then the operators Z_y and Q_t are commutative, namely,*

$$Z_y Q_t = Q_t Z_y, \quad y \in \mathbb{R}, t \ge 0. \tag{97}$$

PROOF. Following the notation of the proof of Propostion 6.3, we have for $v \in C([0,\delta], L^\infty(\mathbb{R}))$, $u_0 \in L_\infty(\mathbb{R})$, $y \in \mathbb{R}$,

$$(Z_y \Phi_{u_0} v)(x,t) = (\Phi_{Z_y u_0} Z_y v)(x,t), \quad x \in \mathbb{R}, t \in [0,\delta].$$

Hence, we have, for $t \in [0,\delta]$, $y \in \mathbb{R}$,

$$Z_y Q_t u_0 = Z_y \lim_{n\to\infty} \Phi_{u_0}^n v = \lim_{n\to\infty} \Phi_{Z_y u_0}^n Z_y v = Q_t Z_y u_0.$$

Repeating the same argument on $[\delta, 2\delta], \dots, [k\delta, (k+1)\delta], \dots$, finishes the proof. □

We denote, for $u \in C([0,T], L^\infty(\mathbb{R})) \cap C^1((0,T], L^\infty(\mathbb{R}))$,

$$\mathcal{F}u(x,t) := \frac{\partial u}{\partial t}(x,t) - \mathrm{Gu}(x,t), \quad x \in \mathbb{R}, t > 0. \tag{98}$$

PROPOSITION 6.5 (Comparison principle). *Let G be monotone and Lipschitz on $L^\infty(\mathbb{R})$, $T \in (0,\infty)$ be fixed and functions $u_1, u_2 \in C([0,T], L^\infty(\mathbb{R})) \cap C^1((0,T], L^\infty(\mathbb{R}))$, be such that, for any $(x,t) \in \mathbb{R}^1 \times (0,T]$,*

$$\mathcal{F}u_1(x,t) \le \mathcal{F}u_2(x,t), \tag{99}$$

$$0 \le u_1(x,t), \qquad 0 \le u_2(x,t) \le c, \qquad u_1(x,0) \le u_2(x,0). \tag{100}$$

Then $u_1(x,t) \le u_2(x,t)$, for all $(x,t) \in \mathbb{R}^1 \times [0,T]$. In particular, $u_1 \le c$.

PROOF. Define the following functions for $x \in \mathbb{R}^1$, $t \in (0,T]$, $w \in L^\infty(\mathbb{R})$:

$$f(x,t) := \mathcal{F}u_2(x,t) - \mathcal{F}u_1(x,t) \ge 0, \tag{101}$$

$$F(x,t,w) := G(w+u_1)(x,t) - Gu_1(x,t) + f(x,t), \tag{102}$$

$$v(x,t) := u_2(x,t) - u_1(x,t). \tag{103}$$

Clearly, $v \in C([0,T], L^\infty(\mathbb{R})) \cap C^1((0,T], L^\infty(\mathbb{R}))$, and it is straightforward to check that

$$\frac{\partial}{\partial t} v(x,t) = F\big(x,t,v(x,t)\big), \tag{104}$$

for all $x \in \mathbb{R}^1$, $t \in (0,T]$. Therefore, v solves the following integral equation in $L^\infty(\mathbb{R})$:

$$\begin{cases} v(x,t) = v(x,0) + \displaystyle\int_0^t F\big(x,s,v(x,s)\big)\,ds & (x,t) \in \mathbb{R}^1 \times (0,T], \\ v(x,0) = u_2(x,0) - u_1(x,0) & x \in \mathbb{R}^1, \end{cases} \tag{105}$$

where $v(x,0) \ge 0$, by (100).

Consider also another integral equation in $L^\infty(\mathbb{R})$:

$$\tilde{v}(x,t) = (\Psi\tilde{v})(x,t), \quad (x,t) \in \mathbb{R}^1 \times (0,T], \tag{106}$$

where

$$\begin{gathered} (\Psi w)(x,t) := v(x,0) + \int_0^t \max\{F\big(x,s,w(x,s)\big), 0\}\,ds, \\ w \in C\big([0,T], L^\infty(\mathbb{R})\big). \end{gathered} \tag{107}$$

It is easily seen that $0 \le w \in C([0,T], L^\infty(\mathbb{R}))$ yields

$$0 \le \Psi w \in C\big([0,T], L^\infty(\mathbb{R})\big).$$

Next, for any $\tilde{T} < T$ and for any w_1, w_2 from $C([0,\tilde{T}], L^\infty(\mathbb{R}, \mathbb{R}_+))$, one gets by (107) that

$$\|\Psi w_1 - \Psi w_2\|_{\tilde{T}} \le \tilde{T} K \|w_2 - w_1\|_{\tilde{T}}, \tag{108}$$

where $K > 0$ is the Lipschitz constant of G and we used the elementary inequality $|\max\{a,0\} - \max\{b,0\}| \le |a-b|$, $a, b \in \mathbb{R}$. Therefore, for $\tilde{T} < K^{-1}$, Ψ is a contraction on $C([0,\tilde{T}], L^\infty(\mathbb{R}, \mathbb{R}_+))$. Thus, there exists a unique solution to (106) on $[0,\tilde{T}]$. In the same way, the solution can be extended on $[\tilde{T}, 2\tilde{T}]$, $[2\tilde{T}, 3\tilde{T}]$, …, and therefore, on the whole $[0,T]$. By (106), (107),

$$\tilde{v}(x,t) \ge v(x,0) \ge 0, \tag{109}$$

hence, by (107),

$$\tilde{v}(x,t) = v(x,0) + \int_0^t F\big(s, \tilde{v}(x,s)\big)\,ds =: \Xi(\tilde{v})(x,t). \tag{110}$$

Since $0 \le \tilde{v} \in C([0,T], L^\infty(\mathbb{R}))$ and G is monotone, (110) implies that $\tilde{v}$ is a solution to (105) as well. The same estimate as in (108) shows that Ξ is a contraction on $C([0,\tilde{T}], L^\infty(\mathbb{R}))$, for small enough $\tilde{T}$. Thus $\tilde{v} = v$ on $\mathbb{R}^1 \times [0,\tilde{T}]$, and one continues this consideration as before on the whole $[0,T]$. Then, by (109), $v(x,t) \ge 0$ on $\mathbb{R}^1 \times [0,T]$, and the statement of the proposition follows. $\square$

Let us recall that B_σ denotes the interval $[-\sigma,\sigma]$ and $L^\infty_+(\mathbb{R})$ is defined by (12).

LEMMA 6.6. *Suppose there exists $\sigma > 0$ such that $a(x) \ge \sigma$, $x \in B_\sigma$. Suppose also that $u_0 \in L^\infty_+(\mathbb{R})$ and u be the corresponding solution to* (88).

Then for any $r > 0$, the following limit holds:

$$\lim_{t\to\infty} \inf_{x\in B_r} u(x,t) \to \infty. \tag{111}$$

PROOF. By assumptions of the lemma,

$$d(x) := \sigma \mathbb{1}_{B_\sigma}(x) \le a(x), \quad x \in \mathbb{R}.$$

Since $u_0 \in L^\infty_+(\mathbb{R})$, there exist $\delta > 0$, $x_0 \in \mathbb{R}$, such that $u_0(x) \ge v_0(x) := \delta \mathbb{1}_{B_\delta(x_0)}(x)$, $x \in \mathbb{R}$. Let v satisfy

$$\frac{\partial v}{\partial t}(x,t) = (d * v)(x,t), \quad x \in \mathbb{R}, t > 0; \qquad v(x,0) = v_0(x) \le u_0(x).$$

We define $Df := d * f$. Since for any $r_1 \le r_2$,

$$(\mathbb{1}_{B_{2r_1}} * \mathbb{1}_{B_{2r_2}})(x) \ge r_1 \mathbb{1}_{B_{2r_2+r_1}}(x), \quad x \in \mathbb{R},$$

the following estimate holds:

$$\delta \sum_{j\ge 0} (\min\{\delta,\sigma\})^j \frac{t^j \sigma^j}{2^j j!} \mathbb{1}_{B_{\delta+\sigma j/2}}(x) \le \sum_{j \ge 0} \frac{t^j D^j v_0(x)}{j!} = v(x,t), \quad x \in \mathbb{R}, t \ge 0.$$

Hence, for any $t > 0$, $r > 0$,

$$\nu_t := \inf_{x \in B_{r+\sigma}} v(x,t) > 0.$$

Let us define,

$$T := \inf\{t > 0, \|v(\cdot,t)\|_\infty \ge 1\} > 0.$$

By Proposition 6.5, applied with $\mathrm{Gu} = \min\{a * u, 1\}$,

$$u(x,t_0) \ge v(x,t_0) \ge \nu_{t_0}, \quad x \in B_{r+\sigma}, t_0 \in (0,T).$$

Since $u \ge 0$, then by (88), $u(x,t)$ is nondecreasing in t. Thus for all $t \ge t_0$, $x \in B_r$,

$$\frac{\partial u}{\partial t}(x,t) = \min\{(a*u)(x,t), 1\} \ge \min\{(a*u)(x,t_0), 1\} \ge \min\left\{\frac{\sigma \nu_{t_0}}{2}, 1\right\} > 0.$$

As a result, (111) holds. The proof is completed. $\square$

From now on we study the case when $a(x)$ is defined by (2), with $\alpha > \frac{1}{2}$.

LEMMA 6.7. *Let $a(x)$ be defined by* (2) *with $\alpha > \frac{1}{2}$ and $u_0 \in L^\infty_+(\mathbb{R})$. Then there exists $R > 0$ such that the following statements hold*:

1. *For all* $|x| \geq R$,

$$|x|^{-2\alpha} \lesssim a(x) \lesssim (a * u_0)(x). \tag{112}$$

2. *If there exist* $\mu > 0$, $\rho \in \mathbb{R}$, *such that* $u_0(x) \geq \mu$, $x \leq \rho$, *then for all* $x \geq R$,

$$x^{-2\alpha+1} \lesssim \int_x^\infty a(y)\, dy \lesssim (a * u_0)(x). \tag{113}$$

PROOF. We start with the first part of the lemma. Without loss of generality we may assume that $u_0 \in L^1(\mathbb{R})$.

By (12), there exist $\delta > 0$ and $x_0 \in \mathbb{R}$, such that $u_0(x) \geq \delta$, $x \in B_\delta(x_0)$. Since for any $r \geq |x_0|$, $a(x) \sim a(|x| + r)$ as $|x| \to \infty$, then there exists $R > 0$ such that the following estimate holds, for all $|x| \geq R$,

$$\begin{aligned} |x|^{-2\alpha} &\lesssim a(x) \\ &\lesssim a(|x| + r) \int_{|y| \leq r} u_0(y)\, dy \\ &\leq \int_{|y| \leq r} a(x - y) u_0(y)\, dy \leq (a * u_0)(x). \end{aligned}$$

Now we prove the second part of the lemma. By the assumptions on u_0, there exists decreasing smooth $v_0 \in L^\infty_+(\mathbb{R})$ such that $v_0(x) \to 0$ as $x \to \infty$, $v_0 \leq u_0$ and $\frac{\partial v_0(x)}{\partial x} \leq 0$ is compactly supported. Then by the first part of the lemma applied to $-\frac{\partial v_0(x)}{\partial x}$ instead of u_0, there exists $R > 0$ such that

$$x^{-2\alpha} \lesssim a(x) \lesssim -\left(a * \frac{\partial v_0}{\partial x}\right)(x), \quad x \geq R.$$

Hence, for all $x \geq R$,

$$x^{-2\alpha+1} \lesssim \int_x^\infty a(y)\, dy \lesssim -\int_x^\infty \left(a * \frac{\partial v_0}{\partial y}\right)(y)\, dy = (a * v_0)(x) \leq (a * u_0)(x).$$

The proof is completed. $\square$

LEMMA 6.8. *Let a be defined by* (2) *with* $\alpha > \frac{1}{2}$, *and we define*

$$\begin{aligned} h(x,t) &= \mathbb{1}_{\mathbb{R}_-}(x) + \min\{1, x^{-2\alpha+1} e^{(1-\varepsilon)t} \mathbb{1}_{(0,\infty)}(x)\}, \\ g(x,t) &= \min\{1, |x|^{-2\alpha} e^{(1-\varepsilon)t}\}. \end{aligned} \tag{114}$$

Then, for any $\varepsilon \in (0,1)$ *there exists* $\tau_0 = \tau_0(\varepsilon) > 0$ *such that for all* $l > 0$ *the functions*

$$\begin{aligned} H(x,t,l) &:= \frac{1}{l} \int_t^{t+l} h(x,s)\, ds, \\ G(x,t,l) &:= \frac{1}{l} \int_t^{t+l} g(x,s)\, ds, \end{aligned} \tag{115}$$

are sub-solutions to $\partial_t u = a * u$ *on* $[\tau_0, \infty)$, *namely* (*cf.* (98)), *for all* $l > 0$,

$$\begin{aligned} &\frac{\partial G}{\partial t}(x,t,l) \leq (a * G)(x,t,l), \\ &\frac{\partial H}{\partial t}(x,t,l) \leq (a * H)(x,t,l), \quad x \in \mathbb{R}, t \geq \tau_0. \end{aligned}$$

In this case one can understand g and h as "weak" sub-solutions to $\partial_t u = a * u$.

PROOF. We denote $r_t = \exp(\frac{t-\varepsilon t}{2\alpha-1})$. Note that $h(x,t) = 1 \Leftrightarrow x \le r_t$. Since $t \to h(x,t)$ is absolutely continuous, then for all $x \in \mathbb{R}$ and almost all $t > 0$, we have

$$-\frac{\partial h}{\partial t}(x,t) + (a * h)(x,t) = -(1-\varepsilon)h(x,t)\mathbb{1}_{x \ge r_t} + (a * h)(x,t). \tag{116}$$

Note that

$$\frac{\partial H}{\partial t}(x,t,l) = \frac{h(x,t+l) - h(x,t)}{l} = \frac{1}{l}\int_t^{l+t} \frac{\partial h}{\partial t}(x,s)\,ds, \tag{117}$$

$$(a * H)(x,t,l) = \frac{1}{l}\int_t^{l+t} (a * h)(x,s)\,ds. \tag{118}$$

Hence, by (117), $H \in C(\mathbb{R}_+, L^\infty(\mathbb{R})) \cap C^1((0,\infty), L^\infty(\mathbb{R}))$. Moreover, by (117) and (118), $\partial_t h \le a * h$, for all $x \in \mathbb{R}$ and almost all $t > 0$, yields $\partial_t H \le a * H$, for H as a vector valued function. Thus, it is sufficient to check that the right-hand side of (116) is nonnegative.

Take $\delta \in (0,1)$. There exists $x_0 = x_0(\delta) > 0$, such that

$$\sup_{|y| \le \sqrt{x}} \left(\frac{x+y}{x}\right)^{2\alpha-1} \ge 1 - \delta, \quad x \ge x_0. \tag{119}$$

Let $\tau > 0$ be such that $r_t \ge x_0$, $t \ge \tau$. By (116), in order to show that h is a subsolution, it is sufficient to prove that there exists $t_0 = t_0(\varepsilon, \delta) > \tau$, such that

$$\frac{(a * h)(x,t)}{h(x,t)} \ge (1-\delta)\int_{-\sqrt{r_t}}^{r_t} a(y)\,dy \tag{120}$$

for all $x \in \mathbb{R}$ and $t \ge t_0$. Note that,

$$(a * h)(x,t) \ge \int_{-\sqrt{r_t}}^{r_t} a(y)h(x-y,t)\,dy \tag{121}$$

for $x \in \mathbb{R}$ and $t > \tau$.

1. Let $x \in (-\infty, r_t - \sqrt{r_t})$, $t > \tau$. Since $h(x,t) = 1$, for $x \le r_t$, then we have

$$\begin{aligned} \frac{(a * h)(x,t)}{h(x,t)} &= \frac{\int_{\mathbb{R}} a(y)h(x-y,t)\,dy}{h(x,t)} \\ &\ge \int_{\mathbb{R}} a(y)\mathbb{1}_{x-y \le r_t}(y)\,dy \\ &= \int_{x-r_t}^{\infty} a(y)\,dy \\ &\ge \int_{-\sqrt{r_t}}^{\infty} a(y)\,dy, \end{aligned} \tag{122}$$

and (120) holds.

2. Let $x \in [r_t - \sqrt{r_t}, r_t)$, $t > \tau$. Note that $h(x,t) = 1$, and $h(x-y,t) = 1$ for $y \ge x - r_t$. Then (121) yields, that

$$\frac{(a * h)(x,t)}{h(x,t)} \ge \int_{x-r_t}^{r_t} a(y)\,dy + \int_{-\sqrt{r_t}}^{x-r_t} a(y)\left(\frac{r_t}{x-y}\right)^{2\alpha-1} dy. \tag{123}$$

Next, for the considered x, $-\sqrt{r_t} \le y \le x - r_t$ yields $0 \le x - y - r_t < \sqrt{r_t}$, and hence, by (119), there exists $t_1 > \tau$ such that for all $t \ge t_1$ and $x \in [r_t - \sqrt{r_t}, r_t)$

$$\begin{aligned} \left(\frac{r_t}{x-y}\right)^{2\alpha-1} &= \left(\frac{r_t}{r_t + (x-y-r_t)}\right)^{2\alpha-1} \\ &\ge \left(\frac{r_t}{r_t + \sqrt{r_t}}\right)^{2\alpha-1} \ge 1 - \delta, \end{aligned}$$

that, together with (123), implies (120).

3. Let $x \geq r_t$, $t > \tau$. Then, by (121),

$$\frac{(a * h)(x,t)}{h(x,t)} \geq \frac{x^{2\alpha-1}}{e^{(1-\varepsilon)t}} \int_{x-r_t}^{r_t} a(y)\,dy + \int_{-\sqrt{r_t}}^{x-r_t} a(y)\Big(\frac{x}{x-y}\Big)^{2\alpha-1} dy. \tag{124}$$

Next, $e^{(1-\varepsilon)t} = r_t^{2\alpha-1} \leq x^{2\alpha-1}$ for $t > \tau$. The latter also implies that $(x-y)^{2\alpha-1} \leq x^{2\alpha-1}$ if $0 \leq y \leq x - r_t$. Finally, by (119), there exists $t_2 > t_1$, such that $x^{2\alpha-1} \geq (1-\delta)(x-y)^{2\alpha-1}$, if only $-\sqrt{r_t} \leq y < 0$, $x \geq r_t$, $t \geq t_2$. As a result, (124) implies (120), which is proved hence for all $x \in \mathbb{R}$ and $t \geq t_2$. The proof for $g(x,t)$ with $r_t = \exp(\frac{t-\varepsilon t}{2\alpha})$ is similar. $\square$

LEMMA 6.9. *Let a be defined by (2) with $\alpha > \frac{1}{2}$. Then for any $\gamma \in (\frac{1}{2\alpha}, 1)$ the following limit holds*:

$$\frac{a * a^\gamma(x)}{a^\gamma(x)} \to 1, \quad |x| \to \infty. \tag{125}$$

PROOF. Take arbitrary $\delta \in (0,1)$, $\gamma \in (\frac{1}{2\alpha}, 1)$. Let us consider, for x such that $|x| > 2|x|^\delta$, a disjoint decomposition $\mathbb{R} = D_1(x) \sqcup D_2(x) \sqcup D_3(x)$, where

$$\begin{aligned} D_1(x) &:= [-|x|^\delta, |x|^\delta], \\ D_2(x) &:= \Big(-\frac{|x|}{2}, -|x|^\delta\Big) \cup \Big(|x|^\delta, \frac{|x|}{2}\Big), \\ D_3(x) &= \Big(-\infty, -\frac{|x|}{2}\Big] \cup \Big[\frac{|x|}{2}, \infty\Big). \end{aligned}$$

Then, $\frac{(a*a^\gamma)(x)}{a^\gamma(x)} = I_1(x) + I_2(x) + I_3(x)$, where

$$I_j(x) := \int_{D_j(x)} a(y)\Big(\frac{1+|x|^2}{1+|x-y|^2}\Big)^{\alpha\gamma} dy, \quad j = 1,2,3.$$

Using the inequality $|x-y| \geq |x| - |y| \geq |x| - |x|^\delta$ for $y \in D_1(x)$, $|x| > 2^{1-\delta}$, one has

$$I_1(x) \leq \Big(\frac{1+|x|^2}{1+(|x|-|x|^\delta)^2}\Big)^{\alpha\gamma} \int_{D_1(x)} a(y)\,dy \to 1, \quad |x| \to \infty.$$

Next, we evidently have, for any $|y| < \frac{|x|}{2}$, that $1 + |x-y|^2 \geq 1 + (|x| - |y|)^2 \geq \frac{1}{4}(1+|x|^2)$; therefore,

$$I_2(x) \leq 4^{\alpha\gamma} \int_{\{|y| \geq |x|^\delta\}} a(y)\,dy \to 0, \quad |x| \to \infty.$$

Finally, $a(y) \leq \frac{c_\alpha}{(1+\frac{x^2}{4})^\alpha}$ for $y \in D_3(x)$, hence

$$\begin{aligned} I_3(x) &\leq c_\alpha 4^\alpha \frac{(1+|x|^2)^{\alpha\gamma}}{(4+|x|^2)^\alpha} \int_{D_3(x)} \frac{1}{(1+|x-y|^2)^{\alpha\gamma}} dy \\ &\leq c_\alpha c_{\alpha\gamma} 4^\alpha \Big(\frac{(1+|x|^2)^\gamma}{4+|x|^2}\Big)^\alpha \to 0, \quad |x| \to \infty, \end{aligned}$$

where c_α is the normalising constant defined in (2). As a result (125) holds. The proof is completed. $\square$

LEMMA 6.10. *Let a be defined by* (2) *with $\alpha > \frac{1}{2}$, $\gamma \in (\frac{1}{2\alpha}, 1)$. Then, for any $\delta \in (0, 1)$, there exists $\lambda = \lambda(\delta, \gamma) > 0$, such that*

$$(a * \omega_\lambda)(x) \le (1+\delta)\omega_\lambda(x), \quad x \in \mathbb{R},$$

where

$$\omega_\lambda(x) := \min\{\lambda, a^\gamma(x)\}, \quad x \in \mathbb{R}^1. \tag{126}$$

PROOF. For any $\lambda > 0$, we define the set

$$\Omega_\lambda := \Omega_\lambda(\gamma) := \{x \in \mathbb{R}^1 : a^\gamma(x) < \lambda\}. \tag{127}$$

By (126), for an arbitrary $\lambda > 0$, we have $\omega_\lambda(x) \le \lambda$, $x \in \mathbb{R}^1$; then $(a * \omega_\lambda)(x) \le \lambda$, $x \in \mathbb{R}^1$, as well. In particular (cf. (126)),

$$(a * \omega_\lambda)(x) \le \omega_\lambda(x), \quad x \in \mathbb{R}^1 \setminus \Omega_\lambda. \tag{128}$$

Next, by Lemma 6.9, for any $\delta > 0$ there exists $\lambda = \lambda(\delta) \in (0, 1)$ such that

$$\sup_{x \in \Omega_\lambda} \frac{(a * a^\gamma)(x)}{a^\gamma(x)} \le 1 + \delta,$$

in particular,

$$(a * a^\gamma)(x) \le (1+\delta)a^\gamma(x) = (1+\delta)\omega_\lambda(x), \quad x \in \Omega_\lambda.$$

Therefore, for all $x \in \Omega_\lambda$,

$$(a * \omega_\lambda)(x) = (a * a^\gamma)(x) - (a * (a^\gamma - \omega_\lambda))(x) \le (1+\delta)\omega_\lambda(x), \tag{129}$$

where we used the obvious inequality: $a^\gamma \ge \omega_\lambda$. By (128) and (129), one gets the statement. □

For a function $\omega : \mathbb{R}^1 \to (0, +\infty)$, we define, for any $f : \mathbb{R}^1 \to \mathbb{R}$,

$$\|f\|_\omega := \sup_{x \in \mathbb{R}^1} \frac{|f(x)|}{\omega(x)} \in [0, \infty]. \tag{130}$$

PROPOSITION 6.11 (cf. [22], Propostion 3.1). *Let a be defined by* (2) *with $\alpha > \frac{1}{2}$, function $\omega : \mathbb{R}^1 \to (0, +\infty)$ be such that $a * \omega$ is well-defined* (*for example, let ω be bounded*) *and, for some $\nu \in (0, \infty)$,*

$$\frac{(a * \omega)(x)}{\omega(x)} \le \nu, \quad x \in \mathbb{R}^1. \tag{131}$$

Let $0 \le u_0 \in L^\infty(\mathbb{R}^1)$ and $\|u_0\|_\omega < \infty$; let $u = u(x, t)$ be the corresponding solution to (88). *Then*

$$\|u(\cdot, t)\|_\omega \le \|u_0\|_\omega e^{\nu t}, \quad t \ge 0. \tag{132}$$

PROOF. For any $f : \mathbb{R}^1 \to \mathbb{R}_+$, with $\|f\|_\omega < \infty$, we have

$$\begin{aligned} \frac{\min\{(a * f)(x), 1\}}{\omega(x)} &\le \frac{(a * f)(x)}{\omega(x)} \\ &\le \int_{\mathbb{R}^1} \frac{a(y)\omega(x-y)}{\omega(x)} \frac{|f(x-y)|}{\omega(x-y)}\,dy \\ &\le \frac{a * \omega(x)}{\omega(x)} \|f\|_\omega. \end{aligned} \tag{133}$$

By Proposition 6.3 and (94), for any $0 \le \tau < \Upsilon$, we have that

$$u(x,t) = (\Phi u)(x,t), \quad t \in [\tau, \Upsilon],$$

where $\Phi = \Phi_{u(\cdot,\tau)}$. Suppose that for some $\tau = (N-1)\delta$, $\delta \in (0,1)$, $N \in \mathbb{N}$, we have $\|u_\tau\|_\omega \le \|u_0\|_\omega e^{\nu\tau}$. Take any $v \in C([\tau,\Upsilon], L^\infty(\mathbb{R},\mathbb{R}_+))$, $t \in [\tau,\Upsilon]$, $\Upsilon := \tau+\delta$, $0 \le u_\tau \in L^\infty(\mathbb{R})$ such that

$$\big\|v(\cdot,t)\big\|_\omega \le \|u_0\|_\omega e^{\nu t}, \quad t \in [\tau,\Upsilon]. \tag{134}$$

We will check the following inequality:

$$\big\|(\Phi v)(\cdot,t)\big\|_\omega \le \|u_0\|_\omega e^{\nu t}, \quad t \in [\tau,\Upsilon].$$

By (94), (133), (134), one gets, for $t \in [\tau,\Upsilon]$,

$$\begin{aligned} 0 &\le \frac{(\Phi v)(x,t)}{\omega(x)} \\ &\le \frac{u_\tau(x)}{\omega(x)} + \int_\tau^t \frac{(a*v)(x,s)}{\omega(x)}\,ds \\ &\le \|u_0\|_\omega e^{\nu\tau} + \|u_0\|_\omega \int_\tau^t \nu e^{\nu s}\,ds = \|u_0\|_\omega e^{\nu t}. \end{aligned}$$

Since, by the proof of Proposition 6.3, u is the limiting function for the sequence $\Phi^n v$, $n \in \mathbb{N}$, and $u_\tau(x) = u(x,\tau)$, one gets the statement. □

PROPOSITION 6.12. *Let a be defined by (2) with $\alpha > \frac{1}{2}$, $u_0 \in L^\infty_+(\mathbb{R})$, and u is the corresponding solution to (88). Then for any $\varepsilon > 0$ the following statements hold:*

1. *If $u_0(x) \lesssim a(x)$ for $x \in \mathbb{R}$, then there exists t_0, such that for all $t \ge t_0$,*

$$u(x,t) \lesssim e^{-\frac{\varepsilon t}{2}}, \quad x \in \big(-\infty, -e^{\frac{1+\varepsilon}{2\alpha}t}\big) \cup \big(e^{\frac{1+\varepsilon}{2\alpha}t}, \infty\big). \tag{135}$$

2. *If $u_0(x) \lesssim \int_x^\infty a(y)\,dy$ for $x \in \mathbb{R}$, then there exists t_0, such that for all $t \ge 0$,*

$$u(x,t) \lesssim e^{-\frac{\varepsilon t}{2}}, \quad x \in \big(e^{\frac{1+\varepsilon}{2\alpha-1}t}, \infty\big). \tag{136}$$

PROOF. We start with proving the first statement. Recall that $\omega_\lambda(x) = \min\{a^\gamma(x), \lambda\}$, $x \in \mathbb{R}$, for $\gamma \in (\frac{1}{2\alpha}, 1)$. By Lemma 6.10 and Proposition 6.11, for any $\delta \in (0,1)$ there exists $\lambda > 0$ such that, for $\omega := \omega_\lambda$,

$$u(x,t) \le \|u_0\|_\omega e^{(1+\delta)t} \min\{a^\gamma, \lambda\}, \quad x \in \mathbb{R}, t \ge 0.$$

Then for t_0, such that $a^\gamma(e^{\frac{1+\varepsilon}{2\alpha}t_0}) \le \lambda$, and for all $t \ge t_0$, $|x| \ge e^{\frac{1+\varepsilon}{2\alpha}t}$,

$$u(x,t) \le c_\alpha \|u_0\|_\omega \frac{e^{(1+\delta)t}}{(1+e^{\frac{1+\varepsilon}{\alpha}t})^{\alpha\gamma}} \le c_\alpha \|u_0\|_\omega e^{(1+\delta-\varepsilon\gamma-\gamma)t},$$

where the first inequality holds by (2). Hence it suffices to choose

$$\gamma \in \Big(\frac{1}{\min\{2,2\alpha\}}, 1\Big), \qquad \delta \in \Big(0, \varepsilon\Big(\gamma - \frac{1}{2}\Big)\Big),$$

and redefine t_0 such that $c_\alpha \|u_0\|_\omega e^{(1+\delta-\varepsilon\gamma-\gamma+\frac{\varepsilon}{2})t_0} \le 1$.

To prove the second statement we note that, by Lemma 6.10, for any $\delta \in (0,1)$, there exists $\lambda > 0$, such that for $\omega_\lambda(x) = \min\{\lambda, a^\gamma(y)\}$, $\omega(x) = \int_x^\infty \omega_\lambda(y)\,dy$,

$$\begin{aligned}(a * \omega)(x) &= \int_x^\infty (a * \omega_\lambda)(y)\,dy \\ &\le (1+\delta)\int_x^\infty \omega_\lambda(y)\,dy = (1+\delta)\omega(x), \quad x \in \mathbb{R}.\end{aligned}$$

Hence Proposition 6.11 may be applied. The rest of the proof is analogous to the first part. The proof is completed. □

Now we can prove the main result.

PROOF OF THEOREM 2.8. We prove the first part of the theorem. Let v solve (88) with $v(x,0) = v_0(x) = \min\{u_0, \frac{1}{2}\}$. By Proposition 6.5, for fixed $t_0 \in (0,T)$, $T := \inf\{t : \|v(\cdot,t)\|_\infty \ge 1\}$,

$$(a * v_0)(x) \lesssim \sum_{j \ge 0} \frac{t_0^j A^j}{j!} v_0(x) = v(x,t_0) \le u(x,t_0), \quad x \in \mathbb{R},$$

where $Af := a * f$. Hence, by the first part of Lemma 6.7 applied to v_0, and since $u(x,t)$ is increasing in t, there exists $R > 0$ such that

$$|x|^{-2\alpha} \lesssim u(x,t), \quad |x| \ge R, t \ge t_0. \tag{137}$$

By (137) and Lemma 6.6, there exists $\tau_1 \ge t_0$ such that

$$\min\{1, |x|^{-2\alpha} e^{(1-\frac{\varepsilon}{2})(\tau_0+1)}\} \lesssim u(x,\tau_1), \quad x \in \mathbb{R},$$

where τ_0 is defined in Lemma 6.8. Hence, by Proposition 6.5 and Lemma 6.8, there exits $\lambda \in (0,1)$, such that

$$\begin{aligned}\lambda g(x, t+\tau_0) &= \lim_{l \to 0} \frac{1}{l} \int_t^{t+l} \lambda g(x, s+\tau_0)\,ds \\ &= \lim_{l \to 0} \lambda G(x, t+\tau_0, l) \\ &\le u(x, t+\tau_1), \quad x \in \mathbb{R}, t \ge 0,\end{aligned}$$

where g is defined by (114) with $\frac{\varepsilon}{2}$ instead of ε and we used, by the monotonicity of g in t, $\lambda G(x,\tau_0,l) \le \lambda g(x,\tau_0+1) \le u(x,\tau_1)$, $x \in \mathbb{R}$, $l \in (0,1)$.

By Lemma 6.6 and (97), for any $n > 0$ there exists t_n such that $u_0(x) \ge \lambda$, for $x \in B_1(x_0)$, yields $u(x, t+t_n) \ge n$, for $x \in B_1(x_0)$, $t \ge 0$. Hence, for $t \ge \frac{2-2\varepsilon}{\varepsilon}(\tau_1 + t_n)$,

$$u(x, t+\tau_1+t_n) \ge n, \quad x \in \{x : |x|^{-2\alpha} e^{(1-\varepsilon)(t+\tau_1+t_n)} \ge 1\},$$

since $\{x : |x|^{-2\alpha} e^{(1-\varepsilon)(t+\tau_1+t_n)} \ge 1\} \subset \{x : \lambda g(x,t+\tau_0) \ge \lambda\} = \{x : |x|^{-2\alpha} e^{(1-\frac{\varepsilon}{2})t} \ge 1\}$. On the other hand by Proposition 6.12 there exits $\tau \ge t_n + \tau_1$ such that

$$u(x,t) \le \frac{1}{n}, \quad x \in \{x : |x|^{-2\alpha} e^{(1+\varepsilon)t} \le 1\}.$$

As a result (8) is proved.

Let us prove (9). Let v solve (88) with $v_0(x) := v(x,0) \not\equiv 0$ such that $v_0 \in C^\infty(\mathbb{R})$ is decreasing and $v_0 \le \min\{u_0, \frac{1}{2}\}$. As before,

$$(a * v_0)(x) \lesssim v(x,t_0) \le u(x,t_0), \quad x \in \mathbb{R}.$$

Similarly to (137), by the second part of Lemma 6.7,

$$x^{-2\alpha+1} \lesssim u(x,t), \quad x \geq R, t \geq t_0. \tag{138}$$

By Corollary 6.4 and since v_0 is decreasing, then $v(\cdot,t)$ is decreasing in x, for all $t \geq 0$. Therefore by Proposition 6.5 and Lemma 6.6, for any $r \in \mathbb{R}$,

$$\infty = \lim_{t\to\infty} \inf_{x\leq r} v(x,t) \leq \lim_{t\to\infty} \inf_{x\leq r} u(x,t). \tag{139}$$

By (138) and (139) there exists $\tau_1 \geq t_0$, such that

$$\mathbb{1}_{\mathbb{R}_-}(x) + \min\{1, x^{-2\alpha+1} e^{(1-\frac{\varepsilon}{2})(\tau_0+1)} \mathbb{1}_{\mathbb{R}_+}(x)\} \lesssim u(x,\tau_1), \quad x \in \mathbb{R}.$$

Hence,

$$\lambda h(x, t+\tau_0) \leq u(x, t+\tau_1), \quad x \in \mathbb{R}, t \geq 0,$$

where h is defined by (114) with $\frac{\varepsilon}{2}$ instead of ε. The rest of the proof runs as before. □

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