Making sense of deep mathematical learning: A review of some literature

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This piece of work was commissioned by Mathematics Mastery – a UK-based non-profit organisation using and advocating a pedagogical approach that seeks to enable all pupils to experience deep mathematical learning. This literature review contributes to the knowledge about deep mathematical learning. The review explores three mathematics education papers that contain 'deep' in their titles to consider the authors' uses of 'deep' in the context of mathematical learning and any analytical or theoretical frameworks upon which they drew. We reveal that 'deep' was not always clearly defined in the research. Rather, the authors' interpretations were left implicit in the terminology they used. A word frequency analysis revealed common words used across the articles. These data were used to offer a *'lexicon for deep mathematical learning'* to assist teachers to describe the quality of pupils' mathematical understanding.

Deep mathematical learning, conceptual understanding, procedural understanding, primary, secondary, elementary,

Introduction

When teachers comment on how their pupils appear to have a 'deep' understanding of some mathematics, they find it difficult to explain what it is their pupils do or say that leads them to make this evaluation. This literature review aims to contribute to the development of a teaching approach called Mathematics Mastery and has the potential to inform mathematics teachers and mathematics education researchers when seeking to understanding how to provide better learning experiences. A goal for all pupils taught using the Mathematics Mastery principles is that they all experience deep mathematical understanding. The purpose of this review is to inform the development of a common language that teachers can explicitly use when monitoring and evaluating the quality of their pupils' mathematical learning. The review attempts to answer the question: *is there a consensus view in the literature on the meaning of 'deep' (and its derivatives) in the context of mathematical learning or understanding?*

Deep Learning and Documentation for Teachers

At the end of KS1 (age 6/7 years) in England, pupils are assessed against three standards: *working towards the expected standard, working at expected standard* and *working at greater depth within the expected standard*. Teachers use the Teacher Assessment Framework (TAF) (STA, 2018) to assist them in making judgements about the quality of pupils' mathematical learning. Also at the end of KS1 and KS2 (age 10/11) pupils take written tests. These tests are designed using the Test Frameworks (STA, 2015a, 2015b). The test items are selected from 'content domains' (e.g. *multiply and divide using written methods*) and 'cognitive domains'. One such cognitive domain is *depth of understanding*. (STA (2015b), p.16). The table below defines the four levels of depth that are used to categorise test items.

Strand	Rating scale			
	(low) 1	2	3	4 (high)
Depth of understanding	recall of facts	application of learned facts and procedures	use facts to solve simple problems	understand and use facts and procedures to solve more complex problems

Table 1 Cognitive domain strand for depth of understanding (STA, 2015b)

Both the TAF for KS1 (STA, 2018) and the Test Frameworks (STA, 2015a, 2015b) are used to measure the depth of mathematical knowledge but there appears to be a lack of clarity in these documents about how depth of learning may present itself. The second and third descriptors from the TAF for KS1 (STA, 2018), which are used to assess pupils who may be working at 'greater depth', state that pupils should be able to:

- recall and use multiplication and division facts for 2, 5 and 10 and make deductions outside known multiplication facts
- use reasoning about numbers and relationships to solve more complex problems and explain their thinking (e.g. 29 + 17 = 15 + 4 + □; 'together Jack and Sam have £14. Jack has £2 more than Sam. How much money does Sam have? etc.) (STA, 2018, p. 9)

These descriptors in the TAF for KS1 (STA, 2018) require pupils to be able to *reason* and *deduce*. There are two points to note here. Firstly, although *reasoning* appears once in the descriptors used to assess pupils who may be working *at* the expected standard, deduction does not. Neither of these terms appear in the definitions for 'depth of understanding' cognitive domain of the test framework (STA, 2015a, 2015b) (see table 1) which, instead, is associated with the complexity of problem solving.

Therefore, in the national documentation for teachers, 'deep' understanding of mathematics appears to be a measure of high achievement rather than a goal for all pupils. This is at odds with one of the principles of Mathematics Mastery. Furthermore a quick word-search in the National Curriculum Programme of Study for Mathematics (DfE, 2013) reveals that there is no mention of the term 'deep' or its derivatives in the document despite the statutory assessment attempting to measure depth. Such confusing messages explains why it is not surprising that many teachers find it difficult to clearly articulate how they know when pupils have a deep understanding of a piece of mathematics. In the next section we explore the language that can be used to define depth of mathematical learning in research.

Literature concerned with deep mathematical learning

Methodology

An initial search was performed using the University of London's *Explore* database which provides access to a vast range of academic journals. Search queries involved titles containing the word 'deep' (or its derivatives) along with 'mathematics' or 'mathematical' as well as 'learning' and 'understanding'. Journal articles were limited to the date parameters of 1st January 2000 to 1st September 2018. The abstracts of the search result articles were each read and determined by the authors to be included or excluded in the review based on a set of specified criteria.

In total three articles were identified for the review (Roschelle, Vahey, Hoyles, and Noss (2013); Shield and Dole (2013) and Watson and De Geest (2005)). Each article was then read to identify definitions of 'deep' (or its derivatives) in the context of mathematical learning and the usage of any common terminology between the articles. We also sought to determine the analytical and theoretical frameworks chosen by the authors of the articles that they associated with deep learning.

Defining 'deep' in the context of mathematical learning or understanding

In each of the three papers, definitions of deep learning varied both in the detail and perspective. Roschelle et al. (2013) provided their own standpoint for their research project, stating that,

> For deep understanding, students must see how familiar ideas connect with formal ideas. They must see how the same mathematical idea is represented in different forms: in a graph, in a table, in an algebraic expression, in a story. They must also see how concepts they learned earlier connect with more advanced concepts." (Roschelle et al., 2013, p. 16)

Their research promoted mathematical learning through a "dynamic representation" (p.16) technology platform call Cornerstone, used by teachers and pupils in KS3. Roschelle et al. (2013) implied their teaching approaches promote *conceptual understanding* and that this was an attribute of 'deep' learning. The pupils in the project in the US made better progress using the Cornerstone materials compared with other resources used. This evaluation was based on test items given before and after the use of the teaching modules. What is unclear, from this short summary of the project, is how the team ensured that the test items they used reflected their clear definition of 'deep understanding', as quoted above. They state that they used test items *"that stress mathematical meaning, such as the types of items found on the PISA international test"*. (p.16)

Shield and Dole (2013) analysed junior secondary mathematics textbooks in Australia to assess their potential to promote teaching and learning for deep mathematical knowledge. However, they did not provide a clear articulation of their interpretation or definition of 'deep'. Their interest in 'deep' learning was rooted in an objective from the most recent Australian National Curriculum that seeks to *"encourage the development of important ideas in more depth, and the interconnectedness of mathematics concepts"* (National Curriculum Board, 2009) p.8*)*. Shield and Dole (2013) referred to curriculum objectives that emphasise mathematics as the *study of patterns and relationships* and the importance of students learning mathematics through an *investigative approach* and *problem-solving*; echoing those of the USA's National Council of Teachers of Mathematics (NCTM, 2000). They do not, however, make an explicit link between these curricular features and 'deep' learning. Instead they leave the reader to infer such a relationship. They do allude later, however, to the promotion of *connected knowledge* and *rich conceptual understanding* (Bell, 1993) which leads to an awareness of *mathematical structure* but it is only implicit that these were associated with 'deep' learning.

Watson and De Geest (2005) report on a longitudinal study that supported teachers working with low attaining secondary pupils. Whilst 'deep', 'learning' and 'mathematical' appear in this paper's title, deep is not used to describe the quality of learning but to describe 'progress'. They consider what contributes to 'deep progress' in mathematics and define it as a goal whereby *students learn more mathematics, get better at learning mathematics, feel better about themselves as mathematics students.* This interpretation differs from the other two papers (Roschelle et al. (2013), and Shield et al (2012)) in that there is no reference to features that are associated with 'deep learning' such as concepts and structures; the focus is instead on affective qualities.

Analytical/Theoretical Frameworks associated with Deep Learning in Mathematics

In this section, we consider the presence of analytical or theoretical frameworks that the authors of the three review papers identified in their research. As mentioned earlier Shield and Dole (2013) do not make explicit a definition for 'deep learning'. They do however refer to Vergnaud's (1983) 'conceptual fields' theory arguing that multiplicative structures should be taught for *connectedness*. It is left for the reader to assume that Shield and Dole (2013) consider *connectedness* to have some association with 'deep learning'. They offer their own analytical framework to evaluate Australian mathematics textbooks for lower secondary pupils to understand how deep learning is promoted in the chapters relating to multiplicative structures. Their indicators describe anticipated features for multiplicative reasoning that are necessary for 'deep learning' i.e. the promotion of comparisons of structures, the availability of multiple representations and prompting connections. Shield and Dole's (2013) framework, drawn from the literature, considers the promotion of deep learning but they do not qualify their choices of the identified features and how they are associated with 'deep' learning.

Roschelle et al. (2013) provided their own clear definition of what deep learning means for the purpose of their research study, but do not, in the space available to them, allude to any theoretical framework of learning mathematics from which this standpoint was drawn. Although, Watson and De Geest (2005) provided a very general definition of 'deep progress', their theoretical framework for considering mathematical learning was evident from their reference to the work of Krutetskii (1976) who characterised 'gifted' mathematicians' mathematical behaviour as "*grasping formal structures, logic, generalisation, flexibility, and so on.*" (Watson and De Geest (2005), p.215). Although Kruteskii (1976) described these 'behaviours' for a particular group of pupils, Watson and De Geest (2005) argued that all learners of mathematics, even those judged as being 'low attaining in mathematics' have an entitlement to experience mathematics in this way; a principle aligned to that of Mathematics Mastery. Since neither Roschelle et al. (2013) nor Shield and Dole (2013) mentioned 'deep' learning or understanding as a specific goal for particular groups of pupils, one assumes that this principle is related to all learners of mathematics.

What is apparent thus far is that 'deep', as a definition of quality of mathematical learning or understanding, appears to be subjective and that mathematics educators, holding strong beliefs in the meaning of 'depth' themselves, may consequently assume that their interpretation is shared with others. In the absence, thus far, of a theoretical definition of 'deep' learning, we now seek to identify any commonalities between the papers in regard to the terminology used to discuss 'deep' mathematical learning.

The use of common terminology to describe 'deep' learning

In order to investigate the use of common terminology across the three articles we used Wordcloud[1]. This online tool produces graphics from texts by counting the frequency of each word; the more frequent the word, the larger it appears in the graphic. Using the background data generated to create the graphic, we sorted the words in a spreadsheet by frequency and by alphabetical order. Sorting by alphabetical order enabled groups of derivative words to be aggregated. For example, frequencies for representations and representation (which had been counted separately), were counted as *representation*. The frequency data for each article was then manually trawled to filter out subject specific vocabulary, such as authors' names. This methodology is recognisably crude as it does not pick up the semantics of the words used in the text (e.g.'comparison' might have been used in the text to when research data was being compared or when pupils were using comparison for mathematical learning) and the data is also skewed towards the contribution that Shield and Dole's (2013) article makes because of the length of the article. However, in defence of this methodology, it gives insight into the possible terms used when reporting on research associated with depth of mathematical learning.

Fig I: Wordcloud for proposed 'lexicon for deep mathematical learning'

This interrogation of the literature explicitly revealed the previously implied interpretations for 'deep' mathematical learning in the three articles. *Representation*, *relationships* and *structures* are key terms that are used commonly across the three papers. As a result, we claim that from the literature, depth of mathematical learning is derived largely from mathematical representations, relationships and structures along with other factors that are presented in a Wordcloud in Fig I, defined as a '*lexicon for deep mathematical learning*'.

Conclusion

In this review sought to understand whether there is a common understanding in the literature of 'deep' when referring to mathematical learning. By interrogating three research reports with regards to definitions and uses of analytical/ theoretical frameworks, along with a scrutiny of terms used across these articles, we proposed a '*lexicon for deep mathematical learning*'. This lexicon illuminates the meaning of 'deep' learning more precisely than England's official documents used to assess the depth of learning in national assessments. The next part of this project will use the '*lexicon for deep mathematical learning*' to develop assessment tasks to use with

[1] https://www.wordclouds.com/

pupils to assist teachers to describe the quality of their pupils' mathematical learning in a more meaningful way.

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