

# Almost partitioning the hypercube into copies of a graph

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## Abstract

Let  $H$  be an induced subgraph of the hypercube  $Q_k$ , for some  $k$ . We show that for some  $c = c(H)$ , the vertices of  $Q_n$  can be partitioned into induced copies of  $H$  and a remainder of at most  $O(n^c)$  vertices. We also show that the error term cannot be replaced by anything smaller than  $\log n$ .

## 1 introduction

Given graphs  $G$  and  $H$ , an  $H$ -packing of  $G$  is a collection of vertex-disjoint copies of  $H$  in  $G$ . A *perfect  $H$ -packing* (also known as an  $H$ -factor) is an  $H$ -packing that covers all the vertices of the ground graph  $G$  (so, in order for  $G$  to have a perfect  $H$ -packing,  $|H|$  must divide  $|G|$ ). A natural question asks for conditions on  $G$  that imply the existence of an  $H$ -factor. For example, a well researched question asks for the smallest minimum degree that implies the existence of an  $H$ -factor. If  $H$  is an edge (and more generally if  $H$  is a path), then, by Dirac's theorem [4], if  $G$  has  $n$  vertices and minimum degree at least  $n/2$  (and  $|H|$  divides  $|G|$ ), then  $G$  has a perfect  $H$ -packing. Corrádi and Hajnal [3] showed that  $\delta(G) \geq 2n/3$  guarantees the existence of a perfect  $K_3$ -packing and Hajnal and Szemerédi [8] extended this result by showing that if  $\delta(G) \geq (1 - 1/r)n$  then  $G$  has a perfect  $K_r$ -packing. We remark that these conditions on  $\delta(G)$  are best possible.

After a series of papers by Alon and Yuster [1, 2] and by Komlós, Sárközy and Szemerédi [9], Kuhn and Osthus [10] found the smallest minimum degree condition that guarantees the existence of an  $H$ -factor, up to an additive constant error term, and for all  $H$ .

We consider a different problem, where instead of looking for  $H$ -packings in graphs of large minimum degree, we focus on  $H$ -packings of the hypercube  $Q_n$ . There are two obvious conditions for the

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existence of a perfect  $H$ -packing in  $Q_n$ :  $H$  has to be a subgraph of  $Q_n$ ; and the order of  $H$  has to be a power of 2. Gruslys [5] showed that these two conditions are sufficient for large  $n$ , thus confirming a conjecture of Offner [12]. In fact, he showed that if  $H$  is an induced subgraph of  $Q_k$  for some  $k$  and  $|H|$  is a power of 2, then there is a perfect packing of  $G$  into *induced* copies of  $H$ .

A similar problem concerns packings of the Boolean lattice  $2^{[n]}$  into induced copies of a poset  $P$ . Note that here, if we drop the induced condition, we reduce to the case where  $P$  is a chain, thus in the case of posets we only consider induced copies of  $P$ . Again, there are two obvious necessary conditions:  $P$  must have a minimum and maximum elements; and the order of  $P$  has to be a power of 2. Lonc [11] conjectured that for large enough  $n$ , these conditions are also sufficient, and verified the conjecture for the case where  $P$  is a chain. This conjecture was recently solved by Gruslys, Leader and Tomon [7].

It is natural to ask what can be said when the divisibility condition does not hold. Gruslys, Leader and Tomon [7] conjectured that if  $P$  is a poset with a maximum and a minimum, then there is a  $P$ -packing of  $Q_n$  that covers all but at most  $c$  elements, where  $c = c(P)$  is a constant that depends on  $P$ . This conjecture was recently proved by Tomon [14].

In light of this result, it is natural to ask if a similar phenomenon holds in the case of  $H$ -packings of the hypercube  $Q_n$ . Namely, if  $H$  is a subgraph of  $Q_k$  for some  $k$ , how large an  $H$ -packing of  $Q_n$  can we find? As our first main result, we show that if  $H$  is a subgraph of  $Q_k$  then there is an  $H$ -packing of  $Q_n$  that covers all but at most  $O(n^c)$  vertices, where  $c = c(H)$ .

**Theorem 1.** *Let  $H$  be an induced subgraph of  $Q_k$  for some  $k$ . Then there exists a packing of  $G$  by induced copies of  $H$ , such that at most  $O(n^c)$  vertices remain uncovered, where  $c = c(H)$ .*

It is natural to wonder if the number of uncovered vertices can be reduced to be at most  $c(H)$ . Perhaps surprisingly, it turns out that this is not always the case. As our second main result, we show that a  $(P_3)^3$ -packing of  $Q_n$  misses at least  $\log n$  vertices ( $P_3$  is the path on three vertices).

**Theorem 2.** *In every  $(P_3)^3$ -packing of  $Q_n$ , at least  $\log n$  points are uncovered.*

## 1.1 Notation

We denote the path on  $l$  vertices by  $P_l$ . When we say that  $G$  can be partitioned into copies of  $H$ , we mean that there exists a perfect  $H$ -packing of  $G$ . For graphs  $G_1$  and  $G_2$ , we denote by  $G_1 \times G_2$  the Cartesian product of  $G_1$  and  $G_2$ , which has vertex set  $V(G_1) \times V(G_2)$  and  $((u_1, v_1), (u_2, v_2))$  is an edge iff  $u_1 = v_1$  and  $u_2 v_2 \in E(G_2)$  or  $u_2 = v_2$  and  $u_1 v_1 \in E(G_1)$ . The  $n$ -th power  $G^n$  of  $G$  is defined to be  $G \times \dots \times G$  (where  $G$  appears  $n$  times).

## 1.2 Structure of the paper

This paper consists of three parts. In the first part (see Section 3) we prove that if  $H$  is an induced subgraph of  $Q_k$  for some  $k$ , then for sufficiently large  $n$ , there is a perfect packing of  $(P_{2|H|})^n$  into induced copies of  $H$  (see Theorem 3). In the second part (Section 4), we prove that there is a packing of  $Q_n$  by induced copies of  $(P_l)^t$  which leaves at most  $O(n^{t-1})$  vertices uncovered. These two parts easily combine to form a proof of Theorem 1. Finally, in the third part (Section 5) we prove Theorem 2, thus showing that the error term  $O(n^{t-1})$  cannot be replaced by something smaller than  $\log n$ .

Before proceeding to the proofs, we give an overview of them in Section 2. We finish the paper with concluding remarks and open problems in Section 6.

## 2 Overview of the proofs

In this section we give an overview of the proofs in this paper.

### 2.1 Partitioning $(P_{2l})^n$

Our first aim in this paper is to prove Theorem 3.

**Theorem 3.** *Let  $H$  be an induced subgraph of  $Q_k$  for some  $k$ . Then  $(P_{2|H|})^n$  can be partitioned into induced copies of  $H$ , whenever  $n$  is sufficiently large.*

Our proof follows the footsteps of Gruslys [5] who proved that if  $H$  is an induced subgraph of a hypercube whose order is a power of 2, then for large  $n$  there is a perfect packing of  $Q_n$  into induced copies of  $H$ .

An important tool in the proof of Theorem 3 is a result of Gruslys, Leader and Tomon [7] (introduced by Gruslys, Leader and Tan [6] for tiling of  $\mathbb{Z}^n$ ) which gives a general method for proving the existence of perfect packings of a product space  $A^n$  into copies of a subset  $S$  of  $A$ . Given a subset  $S$  of  $A$ , a collection of copies of  $S$  in  $A^n$  (which may contain a certain copy several times) is called an  $l$ -partition ( $(r \bmod l)$ -partition) if every vertex in  $A^n$  is covered by exactly  $l$  ( $(r \bmod l)$ ) copies of  $S$ . We note that a 1-partition is simply a perfect packing. Trivially, the existence of a perfect packing implies the existence of an  $l$ -partition and a  $(1 \bmod l)$ -partition. Remarkably, the aforementioned result of Gruslys, Leader and Tomon shows that, roughly speaking, the opposite is true. Namely, they showed that if there exists  $l$  for which  $A^m$  admits an  $l$ -partition and a  $(1 \bmod l)$ -partition into copies of  $S$ , then, for large  $n$ ,  $A_n$  admits a perfect packing into copies of  $S$ . The precise statement of this result is given in Theorem 5.

The existence of an  $|H|$ -partition of  $(P_{2|H|})^n$  into induced copies of  $H$  is a simple observation (see Observation 6). The existence of a  $(1 \bmod |H|)$ -partition of  $(P_{2|H|})^n$  into copies of  $H$  is more difficult to prove, but it is quite straightforward to adapt the methods of Gruslys [5] to work in our setting. These two facts, together the aforementioned result [7], form the proof of Theorem 3.

## 2.2 Almost partitioning $Q_n$ into powers of a path

Our second aim is to prove Theorem 4.

**Theorem 4.** *For any  $l$  and  $t$ , there is a packing of  $Q_n$  into induced copies of  $(P_l)^t$ , for which at most  $O(n^{t-1})$  vertices are uncovered.*

The fact that  $Q_n$  is Hamiltonian shows that there is a  $P_l$ -packing of  $Q_n$  missing fewer than  $l$  vertices. In fact, if  $l$  divides  $2^n - 1$ , then exactly one vertex remains uncovered. This observation allows us to prove the existence of a  $(P_l)^t$ -packing of  $Q_n$  with at most  $O(n^{t-1})$  uncovered vertices, whenever  $l$  is odd (see Observation 8). It is then not hard to conclude that the same holds for all  $l$  (see Corollary 9) using the observation that  $(P_{2l})^t$  is a subgraph of  $(P_l)^t \times Q_t$ .

Note that this does not imply Theorem 4, since we require that the copies of  $(P_l)^t$  are induced. We notice that if  $H$  is a graph on  $l$  vertices with a Hamilton path, then  $H \times P_{l-1}$  has a perfect packing into induced  $P_l$ 's (see Observation 11). This fact, with a little more work, allows us to use the packing of  $Q_n$  into (not necessarily induced) copies of  $(P_{l'})^t$  to obtain a packing into induced copies of  $(P_l)^t$  (where  $l'$  is suitably chosen).

We note that Theorem 1 follows from Theorems 3 and 4.

**Proof of Theorem 1.** Let  $H$  be a subgraph of  $Q_k$ . Then by Theorem 3, there exists  $m$  for which there is a perfect packing of  $(P_{2|H|})^m$  into induced copies of  $H$ . By Theorem 4, there is a packing of  $Q_n$  into induced copies of  $(P_{2|H|})^m$ , such that at most  $O(n^{m-1})$  vertices are uncovered. Hence there exists a packing of  $Q_n$  into induced copies of  $H$  with at most  $O(n^{m-1})$  uncovered vertices (note that  $m$  depends only on  $H$ ).  $\square$

## 2.3 Lower bound on the number of uncovered vertices

Our final aim is to prove Theorem 2.

**Theorem 2.** *In every  $(P_3)^3$ -packing of  $Q_n$ , at least  $\log n$  points are uncovered.*

We use the properties of  $Q_n$  and of  $(P_3)^3$  to conclude that the size of the intersection of any co-dimension-2 subcube of  $Q_n$  with any copy of  $(P_3)^3$  is divisible by 3. In fact, we deduce this from a similar statement for  $(P_3)^t$  (see Proposition 13). We conclude that the set of uncovered vertices in a  $(P_3)^3$ -packing of  $Q_n$  forms a *separating family* for  $[n]$ , implying that it has size at least  $\log n$ .

### 3 Perfect $H$ -packings of $(P_{2|H|})^n$

Our main aim in this section is to prove Theorem 3.

**Theorem 3.** *Let  $H$  be an induced subgraph of  $Q_k$  for some  $k$ . Then  $(P_{2|H|})^n$  can be partitioned into induced copies of  $H$ , whenever  $n$  is sufficiently large.*

Recall that a result of Gruslys, Leader and Tomon [7] implies that it suffices to find  $l$ - and  $(1 \bmod l)$ -partitions into copies of  $H$ . Before stating their result precisely, we introduce some notation.

Let  $A$  be a set. We identify  $A^n \times A^m$  with  $A^{n+m}$  (whenever  $m$  and  $n$  are positive integers). Thus, for any  $x \in A^n$  and  $y \in A^m$ , we treat  $(x, y)$  as an element of  $A^{n+m}$ . Given a set  $X$  in  $A^n$ , and a permutation  $\pi : [n] \rightarrow [n]$ , we define  $\pi(X)$  to be the image of  $X$  under the permutation of the coordinates according to  $\pi$ . In other words,  $\pi(X) = \{(x_{\pi(1)}, \dots, x_{\pi(n)}) : (x_1, \dots, x_n) \in X\}$ . Finally, given sets  $X$  in  $A^m$  and  $Y$  in  $A^n$  where  $m \leq n$ , we say that  $Y$  is a *copy* of  $X$  if  $Y = \pi(X \times \{y\})$  for some  $y \in A^{n-m}$ .

**Theorem 5** (Gruslys, Leader, Tomon [7]). *Let  $\mathcal{F}$  be a family of subsets of a finite set  $A$ . If there exists  $l$  for which  $\mathcal{F}$  contains an  $l$ -partition and a  $(1 \bmod l)$ -partition of  $S$ , then there exists  $n$  for which  $S^n$  admits a partition into copies of elements in  $\mathcal{F}$ .*

The task of finding an  $l$ -partition is quite simple. In fact, it follows directly from the analogous result in [5] and the fact that  $(P_{2l})^n$  can be partitioned into copies of  $Q_n$ . For the sake of completeness, we include the proof here.

**Observation 6.** *Let  $H$  be an induced subgraph of  $Q_k$  for some  $k$ . Then there is an  $|H|$ -partition of  $(P_{2|H|})^n$  into induced copies of  $H$ , for any  $n \geq k$ .*

**Proof.** Denote  $l = |H|$ . Note that, since the path  $P_{2l}$  can be partitioned into  $l$  edges, its  $n$ -th power  $(P_{2l})^n$  can be partitioned into  $l^n$  induced copies of  $Q_n$ . Thus, it suffices to exhibit an  $l$ -partition of  $Q_n$  into induced copies of  $H$ . Let  $X$  be the vertex set of some induced copy of  $H$  in  $Q_n$ . We consider the set of all *shifts* of  $X$ . For every  $u \in Q_n$ , we note that the set  $X + u = \{x + u : x \in X\}$  (addition is done coordinate-wise and modulo 2) is an induced copy of  $H$  in  $Q_n$ . Consider the collection  $\{X + u : u \in Q_n\}$ . By symmetry, every vertex in  $Q_n$  is covered by the same number of sets. Furthermore, there are  $2^n$  such sets, each covers  $l$  points, so the number of times each vertex is covered is  $\frac{2^n l}{2^n} = l$ . So, we found an  $l$ -partition of  $Q_n$  into induced copies of  $H$ .  $\square$

The next task, of finding a  $(1 \bmod l)$ -partition of  $(P_{2l})^n$  into copies of  $H$ , is significantly harder. Unlike Observation 6, we cannot directly apply the analogous result of Gruslys [5]. Instead, we adapt his method to our setting.

**Theorem 7.** *Let  $H$  be a non-empty induced subgraph of  $Q_k$  for some  $k$ . Then there is a  $(1 \bmod l)$ -partition of  $(P_{2l})^k$  into induced copies of  $H$ .*

We note that in Theorem 3, unlike Observation 6, there is no restriction on the order of  $H$ .

Before proceeding to the proof of Theorem 7, we show how to prove Theorem 3 using Observation 6 and theorem 7.

**Proof of Theorem 3.** Denote  $l = |H|$ . Note that it suffices to show that for some  $n$  the graph  $(P_{2l})^n$  can be partitioned into induced copies of  $H$ . Recall that  $H$  is an induced subgraph of  $Q_k$ , for some  $k$ . By Theorem 7, there is a  $(1 \bmod l)$ -partition of  $A = (P_{2l})^k$  into induced copies of  $H$ . By Observation 6 there is an  $l$ -partition of  $A$  into induced copies of  $H$ . Hence, by Theorem 5 there exists  $n$  for which there is a perfect  $H$ -packing of  $A^n = (P_{2l})^{kn}$ , as required.  $\square$

We now proceed to the proof of Theorem 7.

**Proof of Theorem 7.** We shall prove the following slightly stronger claim: if  $H$  is a non-empty induced subgraph of  $Q_k$ , then there is a  $(1 \bmod r)$  partition of  $(P_{2l})^k$  into isometric copies of  $H$ .

Let us first explain briefly what we mean by an isometric copy of  $H$  in a graph  $G$ . We consider the graphs  $Q_k$  and  $G$  together with the metric coming from the graph distance. An isometric copy of  $H$  is the image of  $H$  under an isometry  $f : Q_k \rightarrow G$  (here we fix a particular embedding of  $H$  in  $Q_k$ ).

Define  $H_-$  and  $H_+$  as follows.

$$H_- = \{u \in Q_{k-1} : (u, 0) \in H\}$$

$$H_+ = \{u \in Q_{k-1} : (u, 1) \in H\}.$$

We prove the claim by induction on  $k$ . It is trivial for  $k = 1$  (then  $H$  is either a single vertex or an edge), so suppose that  $k \geq 2$  and the claim holds for  $k - 1$ . Note that we may assume that  $H_-$  and  $H_+$  are both non-empty. We shall show that  $(P_{2l})^k$  has a  $(1 \bmod l)$ -partition into isometric copies of  $H$ .

We denote the vertices of  $P_{2l}$  by  $\{0, 1, \dots, 2l - 1\}$ . Let  $p \in [2l - 2]$ . By induction, there is a collection of isometric copies of  $H_-$  in  $(P_{2l})^{k-1} \times \{p\}$  such that each point is covered  $(1 \bmod l)$  times. Let  $A$  be the vertex set of such a copy of  $H_-$ . Then there is an isometric copy of  $H$  in  $(P_{2l})^{k-1} \times \{p, p + 1\}$  whose intersection with  $(P_{2l})^{k-1} \times \{p\}$  is  $A$ . It follows that there exists a collection  $\mathcal{H}$  of isometric copies of  $H$  in  $(P_{2l})^{k-1} \times \{p, p + 1\}$  for which every point in  $(P_{2l})^{k-1} \times \{p\}$  is covered  $(1 \bmod l)$  times. Let  $\mathcal{H}'$  be the collection of isometric copies of  $H$  in  $(P_{2l})^{k-1} \times \{p - 1, p\}$ , which is the image

of  $\mathcal{H}$  under the map from  $(P_{2l})^{k-1} \times \{p, p+1\}$  to  $(P_{2l})^{k-1} \times \{p-1, p\}$  obtained by changed the last coordinate from  $p+1$  to  $p-1$ .

Denote by  $\mathcal{H}_p$  the collection  $(l-1)\mathcal{H} + \mathcal{H}'$  (i.e. each copy of  $H$  in  $\mathcal{H}$  is taken  $l-1$  times).  $\mathcal{H}$  is a collection of copies of  $H$  in  $(P_{2l})^{k-1} \times \{p-1, p, p+1\}$  which we view as a collection of copies of  $H$  in  $(P_{2l})^k$ . For every  $x \in (P_{2l})^k$ , the number of times  $x$  is covered is

$$w_p(x) = \begin{cases} (1 \bmod l) & x \in (P_{2l})^{k-1} \times \{p+1\} \\ (-1 \bmod l) & x \in (P_{2l})^{k-1} \times \{p-1\} \\ (0 \bmod l) & \text{otherwise} \end{cases}$$

Let  $\mathcal{G}$  be the collection of isometric copies of  $H$  obtained by taking  $i$  copies of  $\mathcal{H}_{2i}$  and  $\mathcal{H}_{2i-1}$  for each  $i \in [l-1]$ . We show that every vertex in  $(P_{2l})^n$  is covered  $(1 \bmod l)$  time by  $\mathcal{G}$ . Let  $x \in (P_{2l})^{k-1}$ . Then  $(x, 0)$  and  $(x, 1)$  are covered  $(1 \bmod l)$  times (they get non zero weight only in  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively). The vertices  $(x, 2i)$  and  $(x, 2i+1)$  (where  $i \in [l-2]$ ) are covered  $(-i \bmod l)$  times by  $\mathcal{H}_{2i-1}$  and  $\mathcal{H}_{2i}$  respectively, and  $(i+1 \bmod l)$  times by  $\mathcal{H}_{2i+1}$  and  $\mathcal{H}_{2i+2}$ , so in total they are covered  $(1 \bmod l)$  times. Finally,  $(x, 2l-2)$  and  $(x, 2l-1)$  are covered  $l-1$  times by  $\mathcal{H}_{2l-3}$  and  $\mathcal{H}_{2l-2}$  respectively, so the number of times they are covered is  $(-(l-1) \bmod l) = (1 \bmod l)$ .  $\square$

## 4 Almost partitioning the hypercube into powers of a path

Our main aim in this section is to prove Theorem 4.

**Theorem 4.** *For any  $l$  and  $t$ , there is a packing of  $Q_n$  into induced copies of  $(P_l)^t$ , for which at most  $O(n^{t-1})$  vertices are uncovered.*

Before proceeding to the proof, we make several observations and mention a result that we shall use. The following observation makes use of the fact that  $Q_n$  is Hamiltonian to prove the special case of Theorem 4 where  $l$  is odd and the requirement that the copies are induced is dropped.

**Observation 8.** *Let  $l$  be odd, and let  $t$  be a positive integer. Then there exists a  $(P_l)^t$ -packing of  $Q_n$ , that covers all but at most  $O(n^{t-1})$  vertices.*

**Proof.** It is well known that  $Q_m$  is Hamiltonian. Therefore, if  $l$  divides  $2^m - 1$ , then  $Q_m$  may be partitioned into copies of  $P_l$  and a single vertex. Note that by the Fermat-Euler theorem, there exists  $m$  such that  $l$  divides  $2^m - 1$  (take  $m = \phi(l)$ , where  $\phi(n)$  is Euler's totient function, counting the number of integers  $p < n$  for which  $p$  and  $n$  are coprime). It follows that all but at most  $2^{m(t-1)} \cdot \lfloor \lceil r \rceil^{<t} \rfloor$  vertices of  $(Q_m)^r$  can be partitioned into copies of  $(P_l)^t$ . Given any  $n$ , write  $n = rm + a$  where  $a < r$ . Since we may view  $Q_n$  as  $(Q_m)^r \times Q_a$ , there is a collection of pairwise

disjoint induced copies of  $(P_l)^t$  that covers all but at most  $2^{a+m(t-1)} \cdot |[r]^{<t}| \leq 2^m \cdot n^{t-1} = O(n^{t-1})$  vertices.  $\square$

The following corollary allows us to extend Observation 8 to all  $l$ .

**Corollary 9.** *Let  $l$  and  $t$  be integers. Then all but  $O(n^{t-1})$  vertices of  $Q_n$  may be partitioned into copies of  $(P_l)^t$ .*

**Proof.** We prove the statement by induction on  $i$ , the maximum power of 2 that divides  $l$ . If  $i = 0$ ,  $l$  is odd, and the statement follows from Observation 8. Now suppose that  $i \geq 1$ . Write  $l = 2k$  and  $Q_n = Q_{n-t} \times Q_t$ . By induction, there is a collection of pairwise disjoint copies of  $(P_k)^t$  that covers all but at most  $O(n^{t-1})$  vertices in  $Q_{n-t}$ . Note that the product  $P_k \times Q_1$  is Hamiltonian, i.e. it spans a  $P_{2k} = P_l$ . We conclude that  $Q_n$  may be covered by pairwise disjoint copies of  $(P_i)^t$  and a remainder of at most  $O(n^{t-1} \cdot 2^t) = O(n^{t-1})$  vertices.  $\square$

Let  $\mathcal{H}_l$  be the collection of graphs on  $l$  vertices which have a Hamilton path, and let  $(\mathcal{H}_l)^t = \{H_1 \times \dots \times H_t : H_i \in \mathcal{H}\}$ . We note that the proofs of Observation 8 and Corollary 9 actually give the following slightly stronger statement.

**Corollary 10.** *Let  $l$  and  $t$  be integers. Then there is a collection of pairwise disjoint copies of graphs in  $(\mathcal{H}_l)^t$  that covers all but at most  $O(n^{t-1})$  vertices of  $Q_n$ .*

In order to obtain an almost-partition into induced copies of  $(P_l)^t$ , we need two more ingredients. One is the following observation.

**Observation 11.** *Let  $H \in \mathcal{H}_l$ . Then  $H \times P_{l-1}$  may be partitioned into induced copies of  $P_l$ .*

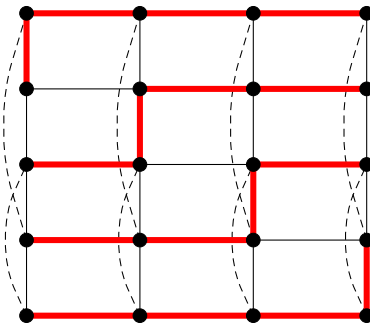
**Proof.** Denote the vertices of  $H$  by  $[l]$  and suppose that  $(1, \dots, l)$  is a path in  $H$ . Similarly, we denote the vertices of  $P_{l-1}$  by  $[l-1]$ . Let  $Q_i$  be the path  $((i, 1), \dots, (i, l-i), (i+1, l-i), \dots, (i+1, l-1))$ , for  $i \in [l-1]$  (see Figure 1). It is easy to see that each  $Q_i$  is an induced  $P_l$ .  $\square$

The second ingredient is a result of Ramras [13], which states that if  $n+1$  is a power of 2, then  $Q_n$  may be partitioned into antipodal paths. In particular, we have the following corollary.

**Corollary 12** (Ramras [13]). *If  $n+1$  is a power of 2, then  $Q_n$  may be partitioned into induced copies of  $P_{n+1}$ .*

We are now ready to prove Theorem 4.





**Figure 1:** Illustration of the paths in Observation 11 (vertical lines denote copies of  $H$ )

**Proof of Theorem 4.** Let  $m$  be minimal such that  $2^m \geq l^2$ . Suppose that  $2^m = a \pmod{l}$  where  $0 \leq a < l$  and write  $2^m = (l - 1 - a)(l - 1) + b$ . Note that  $(l - 1 - a)(l - 1) < l^2$  so  $b > 0$ , and by choice of  $m$ ,  $b \leq 2l^2$ . Furthermore,  $b = -1 \pmod{l}$ . The path  $P_{2^m}$  can be partitioned into  $l - 1 - a$  copies of  $P_{l-1}$  and one copy of  $P_b$ . It follows from Corollary 12 that  $Q_{2^m-1}$  may be partitioned into induced copies of  $P_{l-1}$  and  $P_b$ , implying that  $(Q_{2^m-1})^{2^t}$  may be partitioned into induced copies of  $(P_{l-1})^t$  and  $(P_b)^t$ .

Write  $Q_n = Q_{n-t(2^m-1)} \times (Q_{2^m-1})^{2^t}$ . Recall that by Corollary 9,  $Q_{n-t(2^m-1)}$  may be partitioned into copies of graphs in  $(\mathcal{H}_{b+1})^t$  and a remainder of at most  $O(n^{t-1})$  vertices.

Combining these two facts, we conclude that  $Q_n$  can be partitioned into copies of graphs isomorphic to  $\prod_{i \in [t]} (H_i \times P_x)$  (where  $H_i \in \mathcal{H}_{b+1}$  and  $x \in \{l-1, b\}$ ), and a remainder of at most  $O(n^{t-1} \cdot 2^{t(2^m-1)}) = O(n^{t-1})$ . We claim that if  $H \in \mathcal{H}_{b+1}$  and  $x \in \{l-1, b\}$ , then  $H \times P_x$  may be partitioned into induced copies of  $P_l$ . Indeed, if  $x = b$  then by Observation 11,  $H \times P_x$  may be partitioned into induced  $P_{b+1}$ 's, which may in turn be partitioned into induced  $P_l$ 's (since, by choice of  $b$ ,  $l$  divides  $b+1$ ). A similar argument holds if  $x = l-1$ : first note that  $H$  may be partitioned into graphs in  $\mathcal{H}_l$  (since  $l$  divides  $b+1$ ), and the product of these graphs with  $P_{l-1}$  may be partitioned into induced copies of  $P_l$ , by Observation 11. It follows that  $Q_n$  may be partitioned into induced copies of  $(P_l)^t$  and a remainder of order at most  $O(n^{t-1})$ .  $\square$

## 5 A lower bound on the number of uncovered vertices

In this section we prove Theorem 2.

**Theorem 2.** *In every  $(P_3)^3$ -packing of  $Q_n$ , at least  $\log n$  points are uncovered.*

We start by proving the following propositions that characterises the intersection of a copy of  $(P_3)^t$  in  $Q_n$  with a subcube of co-dimension 1.

**Proposition 13.** *Let  $H$  be a copy of  $(P_3)^k$  in  $Q_n$ . Then the intersection of  $H$  with any subcube  $S$  of co-dimension 1 is a copy of one of the following graphs:  $\emptyset$ ,  $(P_3)^{k-1}$ ,  $P_2 \times (P_3)^{k-1}$  or  $(P_3)^k$ .*

**Proof.** Let  $S$  be the vertex set of a subcube of  $Q_n$  of co-dimension 1. Write  $H = H' \times P_3$ , where every  $H'$  is a copy of  $(P_3)^{k-1}$ , and denote  $H_i = H' \times \{i\}$  (where  $V(P_3) = \{1, 2, 3\}$ ). We prove the statement by induction on  $k$ .

Let  $k = 1$ , then each  $H_i$  is a single vertex. Without loss of generality,  $H_2$  is in  $S$  (otherwise consider the complement of  $S$ ). But then at least one of  $H_1$  and  $H_3$  also are in  $S$  (because every vertex in  $S$  has exactly one neighbour outside of  $S$ ). So, without loss of generality,  $H_1$  is in  $S$ . It follows that  $V(H) \cap S$  is either  $H$  or  $H_1 \times H_2$ , as claimed.

Now suppose that  $k \geq 2$ . Then by induction, and without loss of generality, the intersection of  $S$  with  $H_1$  is either  $H_1$  or a copy of  $P_2 \times (P_3)^{k-1}$ .

Suppose that the first case holds, i.e. the intersection of  $S$  with  $H_1$  is  $H_1$ . Then, if any vertex in  $H_2$  is in  $S$ , all vertices of  $H_1$  are in  $S$  (since every vertex in  $S$  has exactly one neighbour outside of  $S$ ). In other words,  $H_2$  is either contained in  $S$  or it is contained in  $\bar{S}$ , the complement of  $S$ . If the former holds, then, similarly,  $H_2$  is contained in either  $S$  or  $\bar{S}$ , and if the latter holds then  $H_2$  is contained in  $\bar{S}$  (since every vertex in  $H_2$  is in  $\bar{S}$  and has a neighbour in  $S \cap H_1$ ). It follows that the intersection of  $S$  with  $H$  in this case is  $H_1$ ,  $H_1 \times H_2$ , or  $H$ , as required.

Now suppose that the second case holds, i.e. the intersection of  $H_1$  with  $S$  is a copy of  $P_2 \times (P_3)^{k-1}$ . Then we may write  $H_1 = H'' \times P_3$  where  $H''$  is a copy of  $(P_3)^{k-2}$ , and  $H'' \times \{1, 2\}$  (where  $V(P_3) = \{1, 2, 3\}$ ) is the intersection of  $H_1$  with  $S$ . Denote  $H_{i,j} = H'' \times \{i\} \times \{j\}$ . So  $H_{1,1}$  and  $H_{1,2}$  are in  $S$  and  $H_{1,3}$  is in  $\bar{S}$ . It follows that  $H_{2,2}$  is in  $S$  (otherwise some vertex in  $H_{1,2}$  would have two neighbours in  $\bar{S}$ );  $H_{2,1}$  is in  $S$  (otherwise a vertex of  $\bar{S} \cap H_{2,1}$  would have two neighbours in  $S$ ); and  $H_{2,3}$  is in  $\bar{S}$ . Similarly,  $H_{3,1}$  and  $H_{3,2}$  are contained in  $S$  and  $H_{3,3}$  is in  $\bar{S}$ . It follows that the intersection of  $H$  with  $S$  is a copy of  $P_2 \times (P_3)^{k-1}$ .  $\square$

We are now ready for the proof of Theorem 2

**Proof of Theorem 2.** Let  $\mathcal{H}$  be a collection of pairwise disjoint copies of  $(P_3)^3$  in  $Q_n$  and let  $S$  be a subcube of co-dimension 2 in  $Q_n$ , and let  $S'$  be a subcube of co-dimension 1 in  $Q_n$  that contains  $S$ . Let  $H \in \mathcal{H}$ . Then, by Proposition 13, the intersection of  $H$  with  $S'$  is either the empty set or it is the disjoint union of up to three copies of  $(P_3)^2$ . It follows from Proposition 13 that the intersection of  $H$  with  $S$  is the disjoint union of copies of  $P_3$ . In particular, since 3 does not divide the order of  $S$ , at least one vertex in  $S$  is not covered by  $\mathcal{H}$ .

Let  $\mathcal{P}$  be the collection of subsets of  $[n]$  that correspond to vertices of  $Q_n$  that are not covered by  $\mathcal{H}$  (where we consider the usual map between  $Q_n$  and  $\mathcal{P}([n])$  that sends a vertex  $u$  in  $Q_n$  to the set of

elements in  $[n]$  whose coordinates in  $u$  is 1). We claim that the collection  $\mathcal{P}$  is a *separating family* for  $[n]$ , namely, for every distinct elements  $i$  and  $j$  in  $[n]$ , there is a set  $A \in \mathcal{P}$  that contains  $i$  but not  $j$ . Indeed, given distinct  $i$  and  $j$  in  $[n]$ , let  $S$  be the subcube of co-dimension 2 of vertices whose  $i$ -th coordinate is 1 and whose  $j$ -th coordinate is 0. Then  $S$  contains a vertex which is uncovered by  $\mathcal{H}$ . This vertex corresponds to a set in  $\mathcal{P}$  that contains  $i$  but not  $j$ . It is a well known fact that a family that separates  $[n]$  has size at least  $\log n$ . It follows  $\mathcal{P}$  has size at least  $\log n$ , implying that at least  $\log n$  vertices of  $Q_n$  are not covered by  $\mathcal{H}$ .  $\square$

We remark that by considering  $(P_3)^{2k+1}$  packings of  $Q_n$ , the number of missing vertices can be shown to be at least  $(1 + o(1))k \log n$  (since the subsets corresponding to the missing vertices form a separating system for  $[n]^{(k)}$ ).

## 6 Concluding remarks

We showed that if  $H$  is an induced subgraph of  $Q_k$  then there exists a packing of  $Q_n$  into induced copies of  $H$ , which misses at most  $O(n^c)$ , for  $c = c(H)$ . On the other hand, we showed that the error term cannot be replaced by anything smaller than  $\log n$  (or, as we remarked in Section 5 by  $c \log n$  for any  $c$ ). It would be very interesting to close the gap between the two bounds.

We believe that the upper bound, of  $O(n^c)$  is closer to the truth, i.e., we believe that there exist graphs  $H$  for which at least  $\Omega(n^c)$  vertices remain uncovered in any  $H$ -packing of  $Q_n$ . More specifically, it seems plausible to believe that every  $(P_3)^k$  packing of  $Q_n$  leaves at least  $\Omega(n^{k-1})$  vertices uncovered. We thus state the following question.

**Question 14.** *Is there a  $(P_3)^k$  packing of  $Q_n$  for which the number of uncovered points is at most  $o(n^{k-1})$ ?*

In this paper we are interested in  $H$ -packings of  $Q_n$ , which can be viewed as  $(P_2)^n$ . It would be interesting to consider the more general setting of  $H$ -packings of  $G^n$ . We mention a conjecture of Gruslys [5].

**Conjecture 15** (Gruslys [5]). *Let  $G$  be a finite vertex-transitive graph, and let  $H$  be an induced subgraph of  $G$ . Suppose further that  $|H|$  divides  $|G|$ . Then for some  $n$  there is a perfect  $H$ -packing of  $G^n$ .*

We note that the conjecture does not hold if we drop the vertex-transitivity (see Proposition 9 in [5]).

In Section 1, we mentioned a recent result of Tomon [14] who proved that if  $P$  is a poset with a minimum and a maximum, then the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$  and a

remainder of at most  $c$  elements, where  $c = c(P)$ . It would be interesting to generalise his result to all posets  $P$ , dropping the requirement of the existence of a minimum and a maximum. This would resolve a conjecture of Gruslys, Leader and Tomon [7].

**Conjecture 16** (Gruslys [7]). *Let  $P$  be a poset. Then the Boolean lattice  $2^{[n]}$  can be partitioned into copies of  $P$  and a remainder of at most  $c = c(P)$  elements.*

Finally, we mention a question about Hamilton paths of  $Q_n$ . Recall that in order to prove that  $Q_n$  can be almost partitioned into induced copies of  $(P_l)^t$ , we first proved this statement without requiring the copies to be induced. That followed easily from the fact that  $Q_n$  is Hamiltonian. We then used such a partition to obtain a partition of  $Q_n$  into induced copies of  $(P_l)^t$ . A more direct approach could be to find a Hamilton path  $P$  in  $Q_n$  for which every  $l$  consecutive vertices induced a  $P_l$ . We were unable to determine if such a Hamilton path exists. We thus conclude the paper with the following question.

**Question 17.** *Let  $l$  be integer. Is it true that for sufficiently large  $n$ , there is a Hamilton path  $Q_n$  for which every  $l$  consecutive vertices induce a  $P_l$ ?*

## References

- [1] N. Alon and R. Yuster, *Almost  $h$ -factors in dense graphs*, Graphs and Combinatorics **8** (1992), no. 2, 95–102.
- [2] ———,  *$H$ -factors in dense graphs*, J. Combin. Ser. B **66** (1996), no. 2, 269–282.
- [3] K. Corrádi and A. Hajnal, *On the maximal number of independent circuits in a graph*, Acta Math. Acad. Sci. Hungar. **14** (1963), 423–439.
- [4] G.A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952), 69–81.
- [5] V. Gruslys, *Decomposing the vertex set of a hypercube into isomorphic subgraphs*, preprint, arXiv:1611.02021.
- [6] V. Gruslys, I. Leader, and T. S. Tan, *Tiling with arbitrary tiles*, Proc. Lond. Math. Soc. **112** (2016), no. 6, 1019–1039.
- [7] V. Gruslys, I. Leader, and I. Tomon, *Partitioning the boolean lattice into copies of a poset*, preprint, arXiv:1609.02520.
- [8] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfu red, 1969), North Holland, Amsterdam.

- [9] Komlós, J. and Sárközy, G. and Szemerédi, E., *Proof of the alonyuster conjecture*, *Discr. Math.* **235** (2001), no. 1, 255–269.
- [10] D. Kühn and D Osthus, *The minimum degree threshold for perfect graph packings*, *Combinatorica* **29** (2009), no. 1, 65–107.
- [11] Z. Lonc, *Proof of a conjecture on partitions of a boolean lattice*, *Order* **8** (1991), no. 1, 17–27.
- [12] D. Offner, *Packing the hypercube*, *Discuss. Math. Graph Theory* **34** (2014), no. 1, 85–93.
- [13] M. Ramras, *Symmetric vertex partitions of hypercubes by isometric trees*, *J. Combin. Theory Ser. B* **54** (1992), no. 2, 239 – 248.
- [14] I. Tomon, *Almost tiling of the Boolean lattice with copies of a poset*, preprint, arXiv:1611.06842.