10-2017

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# Neutrosophic $\mathcal{N}$-Structures Applied to $BCK/BCI$-Algebras

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Received: 12 September 2017; Accepted: 6 October 2017; Published: 16 October 2017

**Abstract:** Neutrosophic $\mathcal{N}$-structures with applications in $BCK/BCI$-algebras is discussed. The notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a $BCK/BCI$-algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are considered, and relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are stated. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal are provided.

**Keywords:** neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-subalgebra; (closed) neutrosophic $\mathcal{N}$-ideal

**MSC:** 06F35, 03G25, 03B52

## 1. Introduction

$BCK$-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean $D$-posets ($MV$-algebras). Additionally, Iséki introduced the notion of a $BCI$-algebra, which is a generalization of a $BCK$-algebra (see [2]).

A (crisp) set $A$ in a universe $X$ can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set $A$ and the value 0 for elements excluded from the set $A$. So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

$$(t, i, f) = (\text{truth, indeterminacy, falsehood})$$

For more details, refer to the following site:

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In this paper, we discuss a neutrosophic $\mathcal{N}$-structure with an application to $BCK/BCI$-algebras. We introduce the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a $BCK/BCI$-algebra, and investigate related properties. We consider characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We discuss relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal.

## 2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2, 0)$. A *BCI-algebra* refers to a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = \theta$,
(II) $(x * (x * y)) * y = \theta$,
(III) $x * x = \theta$,
(IV) $x * y = y * x = \theta \;\Rightarrow\; x = y$.

for all $x, y, z \in X$. If a BCI-algebra $X$ satisfies $\theta * x = \theta$ for all $x \in X$, then we say that $X$ is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \;\Rightarrow\; x * y = \theta)$$

In a BCK/BCI-algebra $X$, the following hold:

$$(\forall x \in X)\ (x * \theta = x) \tag{1}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y) \tag{2}$$

A non-empty subset $S$ of a $BCK/BCI$-algebra $X$ is called a *subalgebra* of $X$ if $x * y \in S$ for all $x, y \in S$.

A subset $I$ of a $BCK/BCI$-algebra $X$ is called an *ideal* of $X$ if it satisfies the following:

(I1) $0 \in I$,
(I2) $(\forall x, y \in X)(x * y \in I,\ y \in I \;\Rightarrow x \in I)$.

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

We denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set $X$ to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from $X$ to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on $X$). An $\mathcal{N}$*-structure* refers to an ordered pair $(X, f)$ of $X$ and an $\mathcal{N}$-function $f$ on $X$ (see [3]). In what follows, we let $X$ denote the nonempty universe of discourse unless otherwise specified.

A *neutrosophic* $\mathcal{N}$*-structure* over $X$ (see [8]) is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3}$$

where $T_N$, $I_N$ and $F_N$ are $\mathcal{N}$-functions on $X$, which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on $X$.

We note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over $X$ satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

### 3. Application in *BCK/BCI*-Algebras

In this section, we take a $BCK/BCI$-algebra $X$ as the universe of discourse unless otherwise specified.

**Definition 1.** *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over $X$ is called a neutrosophic $\mathcal{N}$-subalgebra of $X$ if the following condition is valid:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \end{pmatrix} \tag{4}$$

**Example 1.** *Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.*

| $*$ | $\theta$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $\theta$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ |
| $a$ | $a$ | $\theta$ | $\theta$ | $a$ |
| $b$ | $b$ | $a$ | $\theta$ | $b$ |
| $c$ | $c$ | $c$ | $c$ | $\theta$ |

*The neutrosophic $\mathcal{N}$-structure*

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.7,-0.2,-0.6)}, \frac{a}{(-0.5,-0.3,-0.4)}, \frac{b}{(-0.5,-0.3,-0.4)}, \frac{c}{(-0.3,-0.8,-0.5)} \right\}$$

*over $X$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$.*

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$T_N^{\alpha} := \{x \in X \mid T_N(x) \leq \alpha\}$$
$$I_N^{\beta} := \{x \in X \mid I_N(x) \geq \beta\}$$
$$F_N^{\gamma} := \{x \in X \mid F_N(x) \leq \gamma\}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the $(\alpha, \beta, \gamma)$-*level set* of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$$

**Theorem 1.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$, then the nonempty $(\alpha, \beta, \gamma)$-level set of $X_{\mathbf{N}}$ is a subalgebra of $X$.*

**Proof.** Let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$ and $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$. If $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (4) that

$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha$,
$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$, and
$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma$.

Hence, $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subalgebra of $X$. □

**Theorem 2.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ and assume that $T_N^\alpha$, $I_N^\beta$ and $F_N^\gamma$ are subalgebras of $X$ for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$.*

**Proof.** Assume that there exist $a, b \in X$ such that $T_N(a * b) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(a * b) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $a * b \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(a * b) < \bigwedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then

$$I_N(a * b) < t_\beta < \bigwedge\{I_N(a), I_N(b)\}$$

where $t_\beta := \frac{1}{2}\{I_N(a * b) + \bigwedge\{I_N(a), I_N(b)\}\}$. Thus $a, b \in I_N^{t_\beta}$ and $a * b \notin I_N^{t_\beta}$, which is a contradiction. Therefore

$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(a * b) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}$$

Then $a, b \in F_N^{t_\gamma}$ and $a * b \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$. □

Because $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

**Theorem 3.** *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subalgebras of $X$, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

**Proposition 1.** *If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over $X$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$, then $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$.*

**Proof.** Straightforward. □

**Theorem 4.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of $X$. If there exists a sequence $\{a_n\}$ in $X$ such that $\lim_{n\to\infty} T_N(a_n) = -1$, $\lim_{n\to\infty} I_N(a_n) = 0$ and $\lim_{n\to\infty} F_N(a_n) = -1$, then $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$.*

**Proof.** By Proposition 1, we have $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Hence $T_N(\theta) \leq T_N(a_n)$, $I_N(a_n) \leq I_N(\theta)$ and $F_N(\theta) \leq F_N(a_n)$ for every positive integer $n$. It follows that

$$-1 \leq T_N(\theta) \leq \lim_{n\to\infty} T_N(a_n) = -1$$
$$0 \geq I_N(\theta) \geq \lim_{n\to\infty} I_N(a_n) = 0$$
$$-1 \leq F_N(\theta) \leq \lim_{n\to\infty} F_N(a_n) = -1$$

Hence $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$. □

**Proposition 2.** *If every neutrosophic $\mathcal{N}$-subalgebra $X_{\mathbf{N}}$ of $X$ satisfies:*

$$T_N(x*y) \leq T_N(y), I_N(x*y) \geq I_N(y), F_N(x*y) \leq F_N(y) \tag{5}$$

*for all $x, y \in X$, then $X_{\mathbf{N}}$ is constant.*

**Proof.** Using Equations (1) and (5), we have $T_N(x) = T_N(x*\theta) \leq T_N(\theta)$, $I_N(x) = I_N(x*\theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x*\theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta)$, $I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore $X_{\mathbf{N}}$ is constant. □

**Definition 2.** *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over $X$ is called a neutrosophic $\mathcal{N}$-ideal of $X$ if the following assertion is valid:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(\theta) \leq T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(\theta) \geq I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(\theta) \leq F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \end{pmatrix} \tag{6}$$

**Example 2.** *The neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over $X$ in Example 1 is a neutrosophic $\mathcal{N}$-ideal of $X$.*

**Example 3.** *Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, *, \theta)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ given by*

$$X_{\mathbf{N}} = \left\{ \tfrac{x}{(\alpha,0,\gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \tfrac{x}{(0,\beta,0)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}$$

*where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$.*

**Proposition 3.** *Every neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of $X$ satisfies the following assertions:*

$$(x, y \in X)\,(x \preceq y \;\Rightarrow\; T_N(x) \leq T_N(y), I_N(x) \geq I_N(y), F_N(x) \leq F_N(y)) \tag{7}$$

**Proof.** Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} = \bigvee\{T_N(\theta), T_N(y)\} = T_N(y)$$
$$I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} = \bigwedge\{I_N(\theta), I_N(y)\} = I_N(y)$$
$$F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} = \bigvee\{F_N(\theta), F_N(y)\} = F_N(y)$$

This completes the proof. □

**Proposition 4.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of $X$. Then*

(1) $T_N(x*y) \leq T_N((x*y)*y) \;\Leftrightarrow\; T_N((x*z)*(y*z)) \leq T_N((x*y)*z)$
(2) $I_N(x*y) \geq I_N((x*y)*y) \;\Leftrightarrow\; I_N((x*z)*(y*z)) \geq I_N((x*y)*z)$
(3) $F_N(x*y) \leq F_N((x*y)*y) \;\Leftrightarrow\; F_N((x*z)*(y*z)) \leq F_N((x*y)*z)$

*for all $x, y, z \in X$.*

**Proof.** Note that

$$((x*(y*z))*z)*z \preceq (x*y)*z \tag{8}$$

for all $x, y, z \in X$. Assume that $T_N(x*y) \leq T_N((x*y)*y)$, $I_N(x*y) \geq I_N((x*y)*y)$ and $F_N(x*y) \leq F_N((x*y)*y)$ for all $x, y \in X$. It follows from Equation (2) and Proposition 3 that

$$\begin{aligned} T_N((x*z)*(y*z)) &= T_N((x*(y*z))*z) \\ &\leq T_N(((x*(y*z))*z)*z) \\ &\leq T_N((x*y)*z) \end{aligned}$$

$$\begin{aligned} I_N((x*z)*(y*z)) &= I_N((x*(y*z))*z) \\ &\geq I_N(((x*(y*z))*z)*z) \\ &\geq I_N((x*y)*z) \end{aligned}$$

and

$$\begin{aligned} F_N((x*z)*(y*z)) &= F_N((x*(y*z))*z) \\ &\leq F_N(((x*(y*z))*z)*z) \\ &\leq F_N((x*y)*z) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose

$$\begin{aligned} T_N((x*z)*(y*z)) &\leq T_N((x*y)*z) \\ I_N((x*z)*(y*z)) &\geq I_N((x*y)*z) \\ F_N((x*z)*(y*z)) &\leq F_N((x*y)*z) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. If we substitute $z$ for $y$ in Equation (9), then

$$\begin{aligned} T_N(x*z) &= T_N((x*z)*\theta) = T_N((x*z)*(z*z)) \leq T_N((x*z)*z) \\ I_N(x*z) &= I_N((x*z)*\theta) = I_N((x*z)*(z*z)) \geq I_N((x*z)*z) \\ F_N(x*z) &= F_N((x*z)*\theta) = F_N((x*z)*(z*z)) \leq F_N((x*z)*z) \end{aligned}$$

for all $x, z \in X$ by using (III) and Equation (1). □

**Theorem 5.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$, then the nonempty $(\alpha, \beta, \gamma)$-level set of $X_{\mathbf{N}}$ is an ideal of $X$.*

**Proof.** Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Clearly, $\theta \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Let $x, y \in X$ be such that $x*y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$ and $y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x*y) \leq \alpha$, $I_N(x*y) \geq \beta$, $F_N(x*y) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x*y), T_N(y)\} \leq \alpha \\ I_N(x) &\geq \bigwedge\{I_N(x*y), I_N(y)\} \geq \beta \\ F_N(x) &\leq \bigvee\{F_N(x*y), F_N(y)\} \leq \gamma \end{aligned}$$

so that $x \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is an ideal of $X$. □

**Theorem 6.** *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ and assume that $T_N^{\alpha}$, $I_N^{\beta}$ and $F_N^{\gamma}$ are ideals of $X$ for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$.*

**Proof.** If there exist $a, b, c \in X$ such that $T_N(\theta) > T_N(a)$, $I_N(\theta) < I_N(b)$ and $F_N(\theta) > F_N(c)$, respectively, then $T_N(\theta) > a_t \geq T_N(a)$, $I_N(\theta) < b_i \leq I_N(b)$ and $F_N(\theta) > c_f \geq F_N(c)$ for some $a_t, c_f \in [-1, 0)$ and $b_i \in (-1, 0]$. Then $\theta \notin T_N^{a_t}$, $\theta \notin I_N^{b_i}$ and $\theta \notin F_N^{c_f}$. This is a contradiction. Hence, $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_N(a_t) > \bigvee\{T_N(a_t * b_t), T_N(b_t)\}$, $I_N(a_i) < \bigwedge\{I_N(a_i * b_i), I_N(b_i)\}$ and $F_N(a_f) > \bigvee\{F_N(a_f * b_f), F_N(b_f)\}$. Then there exist $s_t, s_f \in [-1, 0)$ and $s_i \in (-1, 0]$ such that

$$\begin{aligned} &T_N(a_t) > s_t \geq \bigvee\{T_N(a_t * b_t), T_N(b_t)\} \\ &I_N(a_i) < s_i \leq \bigwedge\{I_N(a_i * b_i), I_N(b_i)\} \\ &F_N(a_f) > s_f \geq \bigvee\{F_N(a_f * b_f), F_N(b_f)\} \end{aligned}$$

It follows that $a_t * b_t \in T_N^{s_t}$, $b_t \in T_N^{s_t}$, $a_i * b_i \in I_N^{s_i}$, $b_i \in I_N^{s_i}$, $a_f * b_f \in F_N^{s_f}$ and $b_f \in F_N^{s_f}$. However, $a_t \notin T_N^{s_t}$, $a_i \notin I_N^{s_i}$ and $a_f \notin F_N^{s_f}$. This is a contradiction, and so

$$\begin{aligned} &T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \\ &I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \\ &F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$. □

**Proposition 5.** *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of $X$, we have*

$$(\forall x, y, z \in X) \left( x * y \preceq z \ \Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right) \tag{10}$$

**Proof.** Let $x, y, z \in X$ be such that $x * y \preceq z$. Then $(x * y) * z = \theta$, and so

$$\begin{aligned} &T_N(x * y) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} = \bigvee\{T_N(\theta), T_N(z)\} = T_N(z) \\ &I_N(x * y) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} = \bigwedge\{I_N(\theta), I_N(z)\} = I_N(z) \\ &F_N(x * y) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} = \bigvee\{F_N(\theta), F_N(z)\} = F_N(z) \end{aligned}$$

It follows that

$$\begin{aligned} &T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\} \\ &I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\} \\ &F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\} \end{aligned}$$

This completes the proof. □

**Theorem 7.** *In a BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra.*

**Proof.** Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a *BCK*-algebra $X$. For any $x, y \in X$, we have

$$
\begin{aligned}
T_N(x*y) &\leq \bigvee\{T_N((x*y)*x), T_N(x)\} = \bigvee\{T_N((x*x)*y), T_N(x)\} \\
&= \bigvee\{T_N(\theta*y), T_N(x)\} = \bigvee\{T_N(\theta), T_N(x)\} \\
&\leq \bigvee\{T_N(x), T_N(y)\}
\end{aligned}
$$

$$
\begin{aligned}
I_N(x*y) &\geq \bigwedge\{I_N((x*y)*x), I_N(x)\} = \bigwedge\{I_N((x*x)*y), I_N(x)\} \\
&= \bigwedge\{I_N(\theta*y), I_N(x)\} = \bigwedge\{I_N(\theta), I_N(x)\} \\
&\geq \bigwedge\{I_N(y), I_N(x)\}
\end{aligned}
$$

and

$$
\begin{aligned}
F_N(x*y) &\leq \bigvee\{F_N((x*y)*x), F_N(x)\} = \bigvee\{F_N((x*x)*y), F_N(x)\} \\
&= \bigvee\{F_N(\theta*y), F_N(x)\} = \bigvee\{F_N(\theta), F_N(x)\} \\
&\leq \bigvee\{F_N(x), F_N(y)\}
\end{aligned}
$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of a $BCK$-algebra $X$. □

The converse of Theorem 7 may not be true in general, as seen in the following example.

**Example 4.** *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.*

| $*$ | $\theta$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\theta$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ | $\theta$ |
| 1 | 1 | $\theta$ | $\theta$ | $\theta$ | $\theta$ |
| 2 | 2 | 1 | $\theta$ | 1 | $\theta$ |
| 3 | 3 | 3 | 3 | $\theta$ | $\theta$ |
| 4 | 4 | 4 | 4 | 3 | $\theta$ |

*Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$, which is given as follows:*

$$
X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.8, 0, -1)}, \frac{1}{(-0.8, -0.2, -0.9)}, \frac{2}{(-0.2, -0.6, -0.5)}, \frac{3}{(-0.7, -0.4, -0.7)}, \frac{4}{(-0.4, -0.8, -0.3)} \right\}
$$

*Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of $X$, but it is not a neutrosophic $\mathcal{N}$-ideal of $X$ as $T_N(2) = -0.2 > -0.7 = \bigvee\{T_N(2*3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge\{I_N(4*3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee\{F_N(4*3), F_N(3)\}$.*

Theorem 7 is not valid in a $BCI$-algebra; that is, if $X$ is a $BCI$-algebra, then there is a neutrosophic $\mathcal{N}$-ideal that is not a neutrosophic $\mathcal{N}$-subalgebra, as seen in the following example.

**Example 5.** *Consider the neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of $X$ in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (\theta, 0) * (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence*

$$
\begin{aligned}
T_N(x*y) &= 0 > \alpha = \bigvee\{T_N(x), T_N(y)\} \\
I_N(x*y) &= \beta < 0 = \bigwedge\{I_N(x), I_N(y)\} \text{ or} \\
F_N(x*y) &= 0 > \gamma = \bigvee\{F_N(x), F_N(y)\}
\end{aligned}
$$

*Therefore $X_{\mathbf{N}}$ is not a neutrosophic $\mathcal{N}$-subalgebra of $X$.*

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$X_{\mathbf{N}}^{\omega_t} := \{x \in X \mid T_N(x) \leq T_N(\omega_t)\}$$
$$X_{\mathbf{N}}^{\omega_i} := \{x \in X \mid I_N(x) \geq I_N(\omega_i)\}$$
$$X_{\mathbf{N}}^{\omega_f} := \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}$$

Clearly, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

**Theorem 8.** *Let $\omega_t$, $\omega_i$ and $\omega_f$ be any elements of $X$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$.*

**Proof.** Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t}$, $\theta \in X_{\mathbf{N}}^{\omega_i}$ and $\theta \in X_{\mathbf{N}}^{\omega_f}$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$T_N(x * y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t)$$
$$I_N(x * y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i)$$
$$F_N(x * y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f)$$

It follows from Equation (6) that

$$T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq T_N(\omega_t)$$
$$I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq I_N(\omega_i)$$
$$F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq F_N(\omega_f)$$

Hence $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$. □

**Theorem 9.** *Let $\omega_t, \omega_i, \omega_f \in X$ and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$. Then*

(1) *If $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$, then the following assertion is valid:*

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\} \Rightarrow T_N(x) \geq T_N(y) \\ I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\} \Rightarrow I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\} \Rightarrow F_N(x) \geq F_N(y) \end{pmatrix} \tag{11}$$

(2) *If $X_{\mathbf{N}}$ satisfies Equation (11) and*

$$(\forall x \in X)\ (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}$$

*then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$ for all $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$.*

**Proof.** (1) Assume that $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$ for $\omega_t, \omega_i, \omega_f \in X$. Let $x, y, z \in X$ be such that $T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\}$, $I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\}$. Then $y * z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, where $\omega_t = \omega_i = \omega_f = x$. It follows from (I2) that $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for $\omega_t = \omega_i = \omega_f = x$. Hence $T_N(y) \leq T_N(\omega_t) = T_N(x)$, $I_N(y) \geq I_N(\omega_i) = I_N(x)$ and $F_N(y) \leq F_N(\omega_f) = F_N(x)$.

(2) Let $\omega_t \in \text{Im}(T_N)$, $\omega_i \in \text{Im}(I_N)$ and $\omega_f \in \text{Im}(F_N)$ and suppose that $X_{\mathbf{N}}$ satisfies Equations (11) and (12). Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ by Equation (12). Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$T_N(x*y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t)$$
$$I_N(x*y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i)$$
$$F_N(x*y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f)$$

which implies that $\bigvee\{T_N(x*y), T_N(y)\} \leq T_N(\omega_t)$, $\bigwedge\{I_N(x*y), I_N(y)\} \geq I_N(\omega_i)$, and $\bigvee\{F_N(x*y), F_N(y)\} \leq F_N(\omega_f)$. It follows from Equation (11) that $T_N(\omega_t) \geq T_N(x)$, $I_N(\omega_i) \leq I_N(x)$ and $F_N(\omega_f) \geq F_N(x)$. Thus, $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of $X$. □

**Definition 3.** *A neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of $X$ is said to be closed if it is a neutrosophic $\mathcal{N}$-subalgebra of $X$.*

**Example 6.** *Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table.*

| $*$ | $\theta$ | $1$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|
| $\theta$ | $\theta$ | $\theta$ | $a$ | $b$ | $c$ |
| $1$ | $1$ | $\theta$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $\theta$ | $c$ | $b$ |
| $b$ | $b$ | $b$ | $c$ | $\theta$ | $a$ |
| $c$ | $c$ | $c$ | $b$ | $a$ | $\theta$ |

*Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over $X$ which is given as follows:*

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.9,-0.3,-0.8)}, \frac{1}{(-0.7,-0.4,-0.7)}, \frac{a}{(-0.6,-0.8,-0.3)}, \frac{b}{(-0.2,-0.6,-0.3)}, \frac{c}{(-0.2,-0.8,-0.5)} \right\}$$

*Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of $X$.*

**Theorem 10.** *Let $X$ be a BCI-algebra, For any $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [-1, 0)$ and $\beta_1, \beta_2 \in (-1, 0]$ with $\alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\beta_1 > \beta_2$, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over $X$ given as follows:*

$$T_N : X \to [-1, 0],\ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in X_+ \\ \alpha_2 & \text{otherwise} \end{cases}$$
$$I_N : X \to [-1, 0],\ x \mapsto \begin{cases} \beta_1 & \text{if } x \in X_+ \\ \beta_2 & \text{otherwise} \end{cases}$$
$$F_N : X \to [-1, 0],\ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X_+ \\ \gamma_2 & \text{otherwise} \end{cases}$$

*where $X_+ = \{x \in X \mid \theta \preceq x\}$. Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of $X$.*

**Proof.** Because $\theta \in X_+$, we have $T_N(\theta) = \alpha_1 \leq T_N(x)$, $I_N(\theta) = \beta_1 \geq I_N(x)$ and $F_N(\theta) = \gamma_1 \leq F_N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$T_N(x) = \alpha_1 \leq \bigvee\{T_N(x*y), T_N(y)\}$$
$$I_N(x) = \beta_1 \geq \bigwedge\{I_N(x*y), I_N(y)\}$$
$$F_N(x) = \gamma_1 \leq \bigvee\{F_N(x*y), F_N(y)\}$$

Suppose that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we have

$$T_N(x) = \alpha_2 = \bigvee\{T_N(x * y), T_N(y)\}$$
$$I_N(x) = \beta_2 = \bigwedge\{I_N(x * y), I_N(y)\}$$
$$F_N(x) = \gamma_2 = \bigvee\{F_N(x * y), F_N(y)\}$$

For any $x, y \in X$, if any one of $x$ and $y$ does not belong to $X_+$, then

$$T_N(x * y) \leq \alpha_2 = \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) \geq \beta_2 = \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) \leq \gamma_2 = \bigvee\{F_N(x), F_N(y)\}$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence

$$T_N(x * y) = \alpha_1 = \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) = \beta_1 = \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) = \gamma_1 = \bigvee\{F_N(x), F_N(y)\}$$

Therefore $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of $X$. □

**Proposition 6.** *Every closed neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of a $BCI$-algebra $X$ satisfies the following condition:*

$$(\forall x \in X)\ (T_N(\theta * x) \leq T_N(x),\ I_N(\theta * x) \geq I_N(x),\ F_N(\theta * x) \leq F_N(x)) \tag{13}$$

**Proof.** Straightforward. □

We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be closed.

**Theorem 11.** *Let $X$ be a $BCI$-algebra. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of $X$ that satisfies the condition of Equation (13), then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and hence is a closed neutrosophic $\mathcal{N}$-ideal of $X$.*

**Proof.** Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Equations (10) and (13), we have

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(\theta * y)\} \leq \bigvee\{T_N(x), T_N(y)\}$$
$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(\theta * y)\} \geq \bigwedge\{I_N(x), I_N(y)\}$$
$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(\theta * y)\} \leq \bigvee\{F_N(x), F_N(y)\}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and is therefore a closed neutrosophic $\mathcal{N}$-ideal of $X$. □

**Author Contributions:** In this paper, Y. B. Jun conceived and designed the main idea and wrote the paper, H. Bordbar performed the idea, checking contents and finding examples, F. Smarandache analyzed the data and checking language.

**Conflicts of Interest:** The authors declare no conflict of interest.

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