On the ratio probability of the smallest eigenvalues in the Laguerre Unitary Ensemble

Max R. Atkin[*], Christophe Charlier[*], Stefan Zohren[†]

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Abstract

We study the probability distribution of the ratio between the second smallest and smallest eigenvalue in the $n \times n$ Laguerre Unitary Ensemble. The probability that this ratio is greater than $r > 1$ is expressed in terms of an $n \times n$ Hankel determinant with a perturbed Laguerre weight. The limiting probability distribution for the ratio as $n \to \infty$ is found as an integral over $(0, \infty)$ containing two functions $q_1(x)$ and $q_2(x)$. These functions satisfy a system of two coupled Painlevé V equations, which are derived from a Lax pair of a Riemann-Hilbert problem. We compute asymptotic behaviours of these functions as $rx \to 0_+$ and $(r-1)x \to \infty$, as well large n asymptotics for the associated Hankel determinants in several regimes of r and x.

1 Introduction and main results

The Laguerre Unitary Ensemble (LUE) consists of the space of $n \times n$ complex positive definite Hermitian matrices endowed with the distribution

$$\frac{1}{\widetilde{Z}_{n,\alpha}} (\det M)^{\alpha} e^{-n\mathrm{TrM}} dM, \qquad \alpha > -1, \tag{1.1}$$

where $\widetilde{Z}_{n,\alpha}$ is the normalisation constant and dM is the Lebesgue measure

$$dM = \prod_{i=1}^{n} dM_{ii} \prod_{1 \leq i < j \leq n} d\mathrm{Re}M_{ij} d\mathrm{Im}M_{ij}. \tag{1.2}$$

The probability measure (1.1) is invariant under unitary conjugation and induces a joint probability distribution on the eigenvalues $\lambda_1, ..., \lambda_n$ on $(\mathbb{R}^+)^n$ given by

$$\frac{1}{n!\widehat{Z}_{n,\alpha}} \Delta_n(\lambda)^2 \prod_{i=1}^{n} e^{-n\lambda_i} \lambda_i^{\alpha} \chi_{\mathbb{R}^+}(\lambda_i) d\lambda_i, \tag{1.3}$$

where $\Delta_n(\lambda)$ denotes the Vandermonde determinant

$$\Delta_n(\lambda) \equiv \Delta_n(\lambda_1, ..., \lambda_n) := \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \tag{1.4}$$

[*]Institut de Recherche en Mathématique et Physique, Université catholique de Louvain, Chemin du Cyclotron 2, B-1348 Louvain-La-Neuve, BELGIUM

[†]Quantum and Nanotechnology Theory Group, Department of Materials, and Machine Learning Research Group, Department of Engineering Science, University of Oxford, UK

and $\chi_{\mathbb{R}^+}$ is the characteristic or indicator function with support on the positive half line. The normalisation constant $\widehat{Z}_{n,\alpha}$, also known as the partition function, is given by

$$\widehat{Z}_{n,\alpha} = \frac{1}{n!} \int_0^\infty ... \int_0^\infty \Delta_n(\lambda)^2 \prod_{i=1}^n e^{-n\lambda_i} \lambda_i^\alpha d\lambda_i. \tag{1.5}$$

The theory of random matrices has enjoyed a growing interest and study for a number of decades in part due to the surprising number of connections between it and seemingly unrelated topics, see for example [18]. Some classical references in the fields are [1, 2, 21, 27]. Probably the earliest appearance of random matrix theory dates back to Wishart in 1928 in the context of multivariate data analysis [30]. He was studying matrices of the form X^*X, where X is an $m\times n$ matrix ($m \geq n$) whose entries X_{ij} are independent and identically distributed complex Gaussian variables

$$\operatorname{Re} X_{ij}, \operatorname{Im} X_{ij} \sim \mathcal{N}\left(0, \sigma^2 = \frac{1}{2n}\right), \tag{1.6}$$

and X^* is the conjugate transpose of X. Such positive semi-definite matrices, called Wishart matrices, possess eigenvalues which are also distributed according to (1.3), where $\alpha = m - n$ is an integer. Since then, the LUE has been studied a lot, and has found applications in different areas, for example in finance (see e.g. [5], Section 12.2).

Some known results

The limiting mean eigenvalue density function is given by the Marchenko-Pastur [20] law

$$d\mu(x) = \rho(x)dx = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}dx, \tag{1.7}$$

supported on the interval $[0, 4]$. Besides the eigenvalue density, other quantities of interest are related to the extreme value statistics of the eigenvalues. The most well-known extreme value statistics in the field of random matrix theory is the Tracy-Widom distribution [28] which describes the properly rescaled fluctuations of the largest (or smallest) eigenvalue of a matrix from the Gaussian Unitary Ensemble (GUE). This distribution is given by $\widetilde{F}_2(x) = \exp\left(-\int_x^\infty (s-x)u^2(s)ds\right)$ and u is the Hastings-McLeod [17] solution of the Painlevé II equation $u''(s) = su(s) + 2u(s)^3$ satisfying the boundary condition $u(s) \sim \operatorname{Ai}(s)$ as $s \to \infty$, where Ai is the Airy function. For the case of the LUE, the distribution of the rescaled fluctuations of the largest eigenvalue at the soft edge, at $x = 4$, is also given by the Tracy-Widom distribution $\widetilde{F}_2$.

In this paper we focus on the hard edge, at $x = 0$, where the distribution of the smallest eigenvalue (denoted by $\lambda_{\min}$) is given by another distribution $F_\alpha(x)$ in terms of a transcendent of the Painlevé V equation, also proved by Tracy and Widom [29]. More precisely, one has

$$F_\alpha(x) = \lim_{n\to\infty} \mathbb{P}_{n,\alpha}(4n^2\lambda_{\min} > x) = \exp\left(-\frac{1}{4}\int_0^x \log\left(\frac{x}{\xi}\right) q^2(\xi)d\xi\right) \tag{1.8}$$

where $q(x)$ is the solution of the equation

$$xq\big(1-q^2\big)\left(xqq'\right)' + x\left(\left(xq'\right)' + \frac{q}{4}\right)\left(1-q^2\right)^2 + x^2 q\left(qq'\right)^2 = \alpha^2\frac{q}{4}, \tag{1.9}$$

with boundary condition $q(\xi) \sim J_\alpha(\sqrt{\xi})$ for $\xi \to 0_+$, and J_α is the Bessel function of the first kind of order α (see [22, Section 10.2 and and Section 10.7] for definition and properties of this function).

That $q(\xi)$ is indeed a transcendent of the Painlevé V equation can be seen from the following transformation

$$q(\xi)=\frac{1+y(x)}{1-y(x)},\quad \xi=x^2,$$

from which it can be readily checked that $y(x)$ is a solution to the Painlevé V equation with suitable chosen coefficients.

Another quantity of interest is the gap probability

$$G_{n,\alpha}(d)=\mathbb{P}_{n,\alpha}\left(\lambda_{\text{smin}}-\lambda_{\text{min}}>d\right),\qquad d>0, \tag{1.10}$$

between the smallest and second smallest (denoted by λ_{smin}) eigenvalue of the LUE. Some results were obtained in [16], where it was shown that the density of $G_\alpha(x)=\lim_{n\to\infty}G_{n,\alpha}(\frac{x}{4n^2})$ exists and is characterised by the solution of a Painlevé III equation and its associated linear isomonodromic system. For results on the gap probability but at the soft edge, see [23] and [31]. In [24], using heuristics arguments and numerical simulations, the authors generalized the results obtained in [23] to a more general context.

The main focus of this paper is the distribution of the ratio

$$Q_{n,\alpha}(r)=\mathbb{P}_{n,\alpha}\left(\frac{\lambda_{\text{smin}}}{\lambda_{\text{min}}}>r\right),\qquad r>1, \tag{1.11}$$

between the second smallest and the smallest eigenvalue of the LUE. Note that the ratio distribution $Q_{n,\alpha}$ cannot be straightforwardly related to the gap $G_{n,\alpha}$ and the distribution of the smallest eigenvalue, since the three variables λ_{min}, $\lambda_{\text{smin}}-\lambda_{\text{min}}$ and $\frac{\lambda_{\text{smin}}}{\lambda_{\text{min}}}$ are not independent.

Our techniques differ from the techniques used in [16], [23] and [31]. We express the ratio probability in terms of a Hankel determinant and then apply well-known rigorous techniques from Riemann-Hilbert (RH) problems analysis to derive asymptotics. We obtain a description of this quantity in terms of a solution (q_1,q_2) to a system of two coupled Painlevé V equations arising from a Lax pair of a RH problem.

Statement of results

We begin the calculation of the ratio probability between the second smallest and smallest eigenvalue for general $\alpha>-1$ by writing the quantity $Q_{n,\alpha}(r)$ as an integral over a Hankel determinant. For $r>1$, by definition of (1.3) we have

$$Q_{n,\alpha}(r)=n\int_0^\infty e^{-ny}y^\alpha\left(\int_{yr}^\infty\dots\int_{yr}^\infty\frac{1}{n!\widehat{Z}_{n,\alpha}}\Delta_{n-1}^2(\lambda)\prod_{i=1}^{n-1}e^{-n\lambda_i}\lambda_i^\alpha(\lambda_i-y)^2d\lambda_i\right)dy,$$

where y can be interpreted as the smallest eigenvalue. By changing variables $n\lambda_i=(n-1)\tilde{\lambda}_i$, we can then write

$$Q_{n,\alpha}(r)=\frac{1}{\widehat{Z}_{n,\alpha}}\left(\frac{n-1}{n}\right)^{(n-1)(n+1+\alpha)}\int_0^\infty y^\alpha e^{-ny}Z_{n-1,\alpha}\left(\frac{n}{n-1}y;r\right)dy, \tag{1.12}$$

where we defined $Z_{n,\alpha}(y;r)$ by

$$Z_{n,\alpha}(y;r)=\frac{1}{n!}\int_{yr}^\infty\dots\int_{yr}^\infty\Delta_n(\lambda)^2\prod_{i=1}^n(\lambda_i-y)^2\lambda_i^\alpha e^{-n\lambda_i}d\lambda_i. \tag{1.13}$$

Note that $Z_{n,\alpha}(y;r)$ is the Hankel determinant ([25, equations (2.2.7) and (2.2.11)]) with respect to the weight $w(x)$ defined by

$$w(x) = (x-y)^2 x^\alpha e^{-nx} \chi_{[yr,\infty)}(x), \tag{1.14}$$

where $\chi_{[yr,\infty)}(x)$ is the characteristic function of $[yr, \infty)$, i.e. we have

$$Z_{n,\alpha}(y;r) = \det \left(\int_{yr}^{\infty} x^{i+j} w(x) dx \right)_{i,j=0,\ldots,n-1}. \tag{1.15}$$

Our main results concern asymptotics for the Hankel determinants $Z_{n,\alpha}(y;r)$ and the limiting distribution as $n \to \infty$ of the ratio probability $Q_{n,\alpha}(r)$ which can be expressed in a compact form through a solution (q_1, q_2) of a system of coupled Painlevé V equations.

Asymptotics for the Hankel determinant $Z_{n,\alpha}(y;r)$

Let us define $s := 4n^2 y$, which is a rescaling of y. We provide large n asymptotics for $Z_{n,\alpha}(y;r)$ in three different regimes:

$$\text{Case 1: } (s,r) \text{ are in a compact subset of } (0,\infty)\times(1,\infty), \tag{1.16}$$

$$\text{Case 2: } rs \to 0, \tag{1.17}$$

$$\text{Case 3: } \frac{rs}{n} \to 0 \text{ and } (r-1)s \to \infty. \tag{1.18}$$

Theorem 1.1 *Let $\alpha > -1$ be fixed. As $n \to \infty$ and simultaneously s and r satisfy one of the three cases presented in* (1.16), (1.17) *and* (1.18), *we have*

$$\log Z_{n,\alpha}\left(\frac{s}{4n^2}; r\right) - \log(\widehat{Z}_{n,\alpha+2}) = I(s;r) + \begin{cases} \mathcal{O}(n^{-1}), & \text{for Case } 1, \\ \mathcal{O}(n^{-1}), & \text{for Case } 2, \\ \mathcal{O}(\frac{(rs)^2}{n}), & \text{for Case } 3, \end{cases} \tag{1.19}$$

where

$$I(s;r) = -\frac{1}{4}\int_0^s (q_1^2(x;r) + r q_2^2(x;r)) \log\left(\frac{s}{x}\right) dx. \tag{1.20}$$

The functions $q_1^2(x;r)$ and $q_2^2(x;r)$ are real and analytic for $x \in (0,\infty)$ and $r \in (1,\infty)$, and they satisfy the following system of coupled Painlevé V equations:

$$\begin{aligned} &xq_1\left(1 - \sum_{j=1}^2 q_j^2\right)\sum_{j=1}^2 (xq_jq_j')' + \left[x\left((xq_1')' + \frac{q_1}{4}\right) + \frac{1}{q_1^3}\right]\left(1-\sum_{j=1}^2 q_j^2\right)^2 + x^2 q_1 \left(\sum_{j=1}^2 q_j q_j'\right)^2 = \frac{\alpha^2 q_1}{4}, \\ &xq_2\left(1 - \sum_{j=1}^2 q_j^2\right)\sum_{j=1}^2 (xq_jq_j')' + x\left((xq_2')' + \frac{rq_2}{4}\right)\left(1-\sum_{j=1}^2 q_j^2\right)^2 + x^2 q_2 \left(\sum_{j=1}^2 q_j q_j'\right)^2 = \frac{\alpha^2 q_2}{4}, \end{aligned} \tag{1.21}$$

where primes denote derivatives with respect to x. Furthermore, the functions q_1 and q_2 satisfy the following boundary conditions: as $(r-1)x \to \infty$, we have

$$q_1^2(x) = \frac{2}{\sqrt{(r-1)x}} + \mathcal{O}\left(\frac{1}{(r-1)x}\right), \tag{1.22}$$

$$q_2^2(x) = 1 - \frac{\alpha}{\sqrt{rx}} - \frac{2}{\sqrt{(r-1)x}} + \mathcal{O}\left(\frac{1}{(r-1)x}\right), \tag{1.23}$$

and as $rx \to 0$, we have

$$q_1(x) = \sqrt{\frac{2}{\alpha+2}}(1+\mathcal{O}(rx)), \tag{1.24}$$

$$q_2(x) = (1-r^{-1})J_{\alpha+2}(\sqrt{rx})(1+\mathcal{O}(rx)) = \frac{1-r^{-1}}{2^{\alpha+2}\Gamma(\alpha+3)}\sqrt{rx}^{\alpha+2}(1+\mathcal{O}(rx)). \tag{1.25}$$

Remark 1.2 We will prove in the present paper that the system (1.21) with boundary conditions (1.22), (1.23), (1.24) and (1.25) possesses at least one solution (q_1,q_2), but there is no guaranty of uniqueness of this solution. Therefore, q_1 and q_2 are not defined through this system, but they are explicitly constructed from the solution of a model Riemann-Hilbert problem, whose solution (denoted Φ) exists and is unique. This Riemann-Hilbert problem is presented in Section 3.

Remark 1.3 In the regime as $(r-1)x \to \infty$ in (1.23), there are different cases. For example, note that if $x \to \infty$ and $r \to 1$, then the $\mathcal{O}\big(((r-1)x)^{-1}\big)$ term is larger than $(rx)^{-1/2}$ if $(r-1)\sqrt{x} \to 0$. In this case, (1.23) can be rewritten as

$$q_2^2(x) = 1 - \frac{2}{\sqrt{(r-1)x}} + \mathcal{O}\left(\frac{1}{(r-1)x}\right), \qquad \text{as } x \to \infty, r \to 1 \text{ and } (r-1)\sqrt{x} \to 0. \tag{1.26}$$

Remark 1.4 The system (1.21) are two coupled Painlevé V equations. It is worth to compare it with the Painlevé V equation given by (1.9), and also to compare (1.20) with the Tracy-Widom distribution (1.8). This system is similar to the one obtained in [7], where the authors obtained a system of k ($k \in \mathbb{N}_0$) coupled Painlevé V equations. The main difference here lie in the q_1^{-3} extra term in the first equation of the system (1.21), and in the small rx asymptotics of q_1, which does not involve Bessel functions.

Corollary 1.5 *As $rs \to 0$, we have*

$$I(s;r) = \frac{-s}{2(\alpha+2)}(1+\mathcal{O}(rs)). \tag{1.27}$$

As $s \to \infty$ and r is in a compact subset of $(1,\infty)$, we have

$$I(s;r) = -\frac{rs}{4} + \alpha\sqrt{rs} + 2\sqrt{(r-1)s} + \mathcal{O}(\log s). \tag{1.28}$$

Proof. To obtain (1.27), it suffices to substitute asymptotics (1.24) and (1.25) into (1.20). To prove large s asymptotics of $I(s;r)$ given by (1.28), we decompose the integral into several parts as follows

$$\begin{aligned} I(s;r) &= I_1 + I_2 + I_3, \\ I_1 &= -\frac{1}{4}\int_0^{\frac{1}{rM}} (q_1^2(x;r) + rq_2^2(x;r))\log\left(\frac{s}{x}\right)dx, \\ I_2 &= -\frac{1}{4}\int_{\frac{1}{rM}}^{\frac{M}{r}} (q_1^2(x;r) + rq_2^2(x;r))\log\left(\frac{s}{x}\right)dx, \\ I_3 &= -\frac{1}{4}\int_{\frac{M}{r}}^{s} (q_1^2(x;r) + rq_2^2(x;r))\log\left(\frac{s}{x}\right)dx, \end{aligned} \tag{1.29}$$

where M is a sufficiently large but fixed constant. Asymptotics (1.24) and (1.25) allow us to write $|I_1| = \mathcal{O}(\log s)$, as $s \to \infty$. For I_2, the parameters (x,r) which appear in the functions q_1 and q_2

lie in a compact subset of $(0,\infty)\times(1,\infty)$, and thus we also have $|I_2| = \mathcal{O}(\log s)$ as $s\to\infty$. Over the domain of integration of I_3, the parameters x and r inside q_1 and q_2 satisfy $xr \geq M$, and thus we can use (1.22) and (1.23) to estimate it. We obtain

$$I_3 = -\frac{rs}{4} + \alpha\sqrt{rs} + 2\sqrt{(r-1)s} + \mathcal{O}(\log s), \qquad \text{as } s\to\infty, \tag{1.30}$$

which finishes the proof. □

Remark 1.6 Theorem 1.1 and Corollary 1.5 provide asymptotics for $Z_{n,\alpha}\left(\frac{s}{4n^2};r\right)$ and $I(s;r)$ in various regimes of n, s and r, which are useful to prove Theorem 1.7 below for the limiting distribution of the ratio. Note that asymptotics (1.28) hold for $s\to\infty$ and r in a compact subset of $(1,\infty)$, and not in the more general situation of $(r-1)x\to\infty$. The reason for that is, as it can be seen in the proof of Corollary 1.5, and more particularly in (1.29), to estimate I_2 we also need to find asymptotics for $q_1(x;r)$ and $q_2(x;r)$ in the two following cases: a) $xr = \mathcal{O}(1)$ and simultaneously $r\to 1$ and b) $xr = \mathcal{O}(1)$ and simultaneously $r\to\infty$. These asymptotics are also needed to obtain asymptotics as $r\to 1$ and as $r\to\infty$ for the limiting probability distribution of the ratio. These cases deserve another long and separate analysis and we intend to pursue this in another paper. We expect these asymptotics to be described in terms of a transcendental function, solution of a differential equation similar to the Painlevé V equation given by (1.9).

Limiting probability distribution of the ratio $\frac{\lambda_{\text{smin}}}{\lambda_{\min}}$

Theorem 1.7 *Let $\alpha > -1$ and $r > 1$ be fixed. As $n\to\infty$, the limit $Q_\alpha(r) := \lim_{n\to\infty} Q_{n,\alpha}(r)$ exists and is given by*

$$Q_\alpha(r) = \frac{1}{4^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^\infty x^\alpha e^{I(x;r)}dx, \tag{1.31}$$

where $I(x;r)$ given in Theorem 1.1.

Remark 1.8 Let $p_n(x_1,x_2)$ denote the joint density for the first two smallest eigenvalues at $\lambda_{\min} = x_1$ and $\lambda_{\text{smin}} = x_2$, which can be straightforwardly obtained by integrating (1.3), and is given by

$$p_n(x_1,x_2) = \frac{e^{-n(x_1+x_2)}(x_1x_2)^\alpha(x_2-x_1)^2}{(n-2)!\widehat{Z}_{n,\alpha}}\int_{x_2}^\infty\cdots\int_{x_2}^\infty\prod_{3\leq i<j\leq n}(\lambda_j-\lambda_i)^2\prod_{i=3}^n e^{-n\lambda_i}\lambda_i^\alpha(x_1-\lambda_i)^2(x_2-\lambda_i)^2 d\lambda_i.$$

The ratio probability $Q_{n,\alpha}(r)$ is expressed in terms of p_n by the relation

$$Q_{n,\alpha}(r) = \int_0^\infty\int_{yr}^\infty p_n(y,y_2)dy_2dy. \tag{1.32}$$

In [15], the authors expressed the density p_n in the special case where α is an integer, as a determinant involving the Laguerre polynomials, they obtained [15, formula (3.20)]

$$p_n(x_1,x_2) = n^4 e^{-n(x_1+(n-1)x_2)}\left(\frac{x_2}{x_1}\right)^\alpha (x_2-x_1)^2 D_n(x_1,x_2), \tag{1.33}$$

where

$$D_n(x_1,x_2) = (-1)^{\frac{(\alpha+1)(\alpha+2)}{2}}\det\begin{bmatrix}\left[\partial_t^{(j+k-2)}L_{\alpha+n-1}^{(-\alpha+1)}(t)\Big|_{t=-nx_2}\right]_{\substack{j=1,\dots,\alpha\\k=1,\dots,\alpha+2}}\\ \left[\partial_t^{(j+k-2)}L_{\alpha+n-1}^{(-\alpha+1)}(t)\Big|_{t=-n(x_2-x_1)}\right]_{\substack{j=1,2\\k=1,\dots,\alpha+2}}\end{bmatrix}, \tag{1.34}$$

and $L_j^{(\alpha)}$ is the generalized Laguerre polynomial of degree j and index α. These polynomials are defined for $\alpha \in \mathbb{R}$ (not necessarily for $\alpha > -1$) through the recursive relations

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = 1+\alpha-x, \text{ and } L_{k+1}^{(\alpha)}(x) = \frac{(2k+1+\alpha-x)L_k^{(\alpha)}(x) - (k+\alpha)L_{k-1}^{(\alpha)}(x)}{k+1}, \ k \geq 1.$$

By combining (1.32) and (1.33), this gives a determinantal representation for $Q_{n,\alpha}(r)$ if $\alpha \in \mathbb{N}$. Also, in [15, equations (3.34) and (3.35)], they obtain the following determinantal expression for the limiting density of the two smallest eigenvalues:

$$p(s_1, s_2) = \lim_{n\to\infty} \left(\frac{1}{4n^2}\right)^2 p_n\left(\frac{s_1}{4n^2}, \frac{s_2}{4n^2}\right) = \frac{e^{-\frac{s_2}{4}}}{16}\left(\frac{s_2}{s_1}\right)^\alpha D(s_1, s_2), \tag{1.35}$$

where

$$D(s_1, s_2) = \lim_{n\to\infty}\left(\frac{s_2 - s_1}{4n^2}\right)^2 D_n\left(\frac{s_1}{4n^2}, \frac{s_2}{4n^2}\right) = \det\begin{bmatrix} \left[I_{j-k+2}(\sqrt{s_2})\right]_{\substack{j=1,\dots,\alpha \\ k=1,\dots,\alpha+2}} \\ \left[\left(\frac{s_2-s_1}{s_2}\right)^{\frac{k-j}{2}} I_{j-k+2}(\sqrt{s_2-s_1})\right]_{\substack{j=1,2 \\ k=1,\dots,\alpha+2}} \end{bmatrix}$$

From the change of variables $y = \frac{s}{4n^2}$ and $y_2 = \frac{s_2}{4n^2}$ in (1.32) and then taking the limit $n \to \infty$, we have

$$Q_\alpha(r) = \lim_{n\to\infty}\left(\frac{1}{4n^2}\right)^2 \int_0^\infty \int_{rs}^\infty p_n\left(\frac{s}{4n^2}, \frac{s_2}{4n^2}\right) ds_2 ds = \int_0^\infty \int_{rs}^\infty p(s, s_2) ds_2 ds. \tag{1.36}$$

The fact that the limit exists and can be interchanged with the integrals is not direct, and can be justified as in [16, Proposition 5.11]. The formulas (1.35) and (1.36) give an explicit determinantal representation for $Q_\alpha(r)$ in terms of Bessel functions if $\alpha \in \mathbb{N}$.

Outline

In Section 2, we introduce a family of monic orthogonal polynomials in terms of which the Hankel determinant $Z_{n,\alpha}(y;r)$ can be expressed. We also use the RH problem for orthogonal polynomials introduced by Fokas, Its and Kitaev [14] and derive a differential identity in y for $Z_{n,\alpha}(y;r)$.

We apply the Deift/Zhou steepest descent method [12, 10, 11] on this RH problem in Section 6 to obtain large n asymptotics of $Z_{n,\alpha}(y;r)$ uniformly in y small enough.

In the analysis we will need a non standard model RH problem, which we introduce in Section 3. We derive a system of two coupled Painlevé V equations using a Lax pair in Section 4 and show asymptotic properties of certain solutions of these equations as $(r-1)x \to \infty$ and $rx \to 0$ in Section 5.

Finally we give a proof of Theorem 1.1 and Theorem 1.7 in Section 7 and 8 respectively, by integrating the differential identity and using equation (1.12).

2 Differential identity for the Hankel determinant $Z_{n,\alpha}(y, r)$

In this section we relate the Hankel determinant $Z_{n,\alpha}(y;r)$ to a RH problem by making use of orthogonal polynomials. We consider a family of monic orthogonal polynomials p_j of degree j characterised by the relations

$$\int_{yr}^\infty p_j(x)p_m(x)w(x)dx = h_j\delta_{jm}, \qquad j, m = 0, 1, 2, ..., \tag{2.1}$$

where the weight w is defined in (1.14) and h_j is the squared norm of p_j, which can be expressed in terms of Hankel determinants (see [25, equations (2.1.5) and (2.1.6)]) as follows:

$$h_j = \frac{Z_{j+1,\alpha}(y;r)}{Z_{j,\alpha}(y;r)}, \qquad Z_{0,\alpha}(y;r) := 1. \tag{2.2}$$

It will prove useful for the later analysis to consider $Y(z) = Y_n(z;y,r)$ the matrix valued function defined by,

$$Y(z) = \begin{pmatrix} p_n(z) & q_n(z) \\ -\frac{2\pi i}{h_{n-1}} p_{n-1}(z) & -\frac{2\pi i}{h_{n-1}} q_{n-1}(z) \end{pmatrix}, \tag{2.3}$$

where q_j is the Cauchy transform of p_j defined by

$$q_j(z) = \frac{1}{2\pi i} \int_{yr}^{\infty} \frac{p_j(x) w(x)}{x - z} dx. \tag{2.4}$$

The function Y can be characterised as the unique function satisfying a set of conditions [14, equations (3.19)-(3.21)], known as the RH problem for Y, which are as follows:

RH problem for Y

(a) $Y : \mathbb{C} \setminus [yr, \infty) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) The limits of $Y(z)$ as z approaches (yr, ∞) from above and below exist, are continuous on (yr, ∞) and are denoted by Y_+ and Y_- respectively. Furthermore they are related by

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix}, \qquad x \in (yr, \infty). \tag{2.5}$$

(c) $Y(z) = (I + \mathcal{O}(z^{-1})) z^{n\sigma_3}$ as $z \to \infty$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(d) $Y(z) = Y_{yr}(z) \begin{pmatrix} 1 & -\frac{w(z)}{2\pi i} \log(yr - z) \\ 0 & 1 \end{pmatrix}$ as $z \to yr$, where the principal branch of the logarithm is taken, and where $Y_{yr}(z)$ is analytic in a neighbourhood of yr.

Remark 2.1 Since Y is discontinuous on (yr, ∞), the function $z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right)$ is not analytic in a neighbourhood of ∞. Nevertheless, since $w(x)$ which appear in the jumps for Y becomes exponentially small for large x, the non-analytic part of $z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right)$ in a neighbourhood of ∞ is also exponentially small in z. In fact, from (2.3), we have

$$Y'(z) = \left(\frac{n}{z}\sigma_3 + \mathcal{O}(z^{-2})\right) z^{n\sigma_3}, \qquad \text{as } z \to \infty, \tag{2.6}$$

and thus

$$z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right) = 2n + \frac{c_1}{z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty. \tag{2.7}$$

for a certain $c_1 \in \mathbb{C}$. This constant will play a role in Lemma 2.2 below.

In [32, 33], the authors considered a similar but different RH problem, where they perturbed the classical Jacobi ensemble by adding n-dependent singularities to the weight. Having introduced the above objects we are now in a position to state the following lemma which is central for the asymptotic analysis of $Z_{n,\alpha}(y;r)$. The identity (2.8) and its proof are similar to the one performed in [4].

Lemma 2.2 *The following identity holds,*

$$\partial_y \log Z_{n,\alpha}(y;r) = \frac{n^2+(\alpha+2)n}{y} - \frac{nc_1}{2y}, \tag{2.8}$$

where c_1 is given in (2.7).

Proof. We begin by making the substitution $\lambda_i = y\xi_i$ in (1.13), which gives

$$Z_{n,\alpha}(y;r) = y^{n^2+(\alpha+2)n}\widetilde{Z}_{n,\alpha}(y;r), \tag{2.9}$$

where

$$\widetilde{Z}_{n,\alpha}(y;r) := \frac{1}{n!}\int_r^\infty ... \int_r^\infty \Delta_n(\xi)^2 \prod_{j=1}^n (\xi_j-1)^2 \xi_j^\alpha e^{-ny\xi_j} d\xi_j. \tag{2.10}$$

The above quantity may be computed by introducing monic orthogonal polynomials $\widetilde{p}_j$ satisfying

$$\int_r^\infty \widetilde{p}_\ell(x)\widetilde{p}_m(x)\widetilde{w}(x)dx = \widetilde{h}_\ell \delta_{\ell m}, \qquad \widetilde{w}(x) = (x-1)^2 x^\alpha e^{-nyx}. \tag{2.11}$$

Analogously to (2.2), for $j = 0, 1, 2, ...$ we have

$$\widetilde{h}_j = \frac{\widetilde{Z}_{j+1,\alpha}(y;r)}{\widetilde{Z}_{j,\alpha}(y;r)}, \qquad \widetilde{Z}_{0,\alpha}(y;r) := 1. \tag{2.12}$$

From the orthogonality conditions for p_j given in (2.1), we easily obtain

$$\widetilde{p}_j(x) = y^{-j}p_j(yx), \qquad \widetilde{h}_j = y^{-(2j+\alpha+3)}h_j. \tag{2.13}$$

A similar calculation for the Cauchy transform q_j appearing in (2.3) shows that

$$\widetilde{q}_j(z) = \frac{1}{2\pi i}\int_r^\infty \frac{\widetilde{p}_j(x)\widetilde{w}(x)}{x-z}dx = y^{-(j+\alpha+2)}q_j(yz). \tag{2.14}$$

Summarising, if we define

$$\widetilde{Y}(z) := \begin{pmatrix} \widetilde{p}_n(z) & \widetilde{q}_n(z) \\ -\frac{2\pi i}{\widetilde{h}_{n-1}}\widetilde{p}_{n-1}(z) & -\frac{2\pi i}{\widetilde{h}_{n-1}}\widetilde{q}_{n-1}(z) \end{pmatrix}, \tag{2.15}$$

we obtain the relationship

$$\widetilde{Y}(z) = y^{-n\sigma_3} y^{-\frac{\alpha+2}{2}\sigma_3} Y(yz) y^{\frac{\alpha+2}{2}\sigma_3}. \tag{2.16}$$

We now use the well known relation, which can be straightforwardly deduced from (2.12),

$$\widetilde{Z}_{n,\alpha}(y;r) = \prod_{i=0}^{n-1} \widetilde{h}_i, \tag{2.17}$$

from which it follows that

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \sum_{i=0}^{n-1} \frac{\partial_y \widetilde{h}_i}{\widetilde{h}_i}. \tag{2.18}$$

Note that

$$\partial_y \widetilde{h}_i = \int_r^\infty \widetilde{p}_i(x)^2 \partial_y \widetilde{w}(x) dx, \tag{2.19}$$

where in the above line we have used the fact that $\widetilde{p}_i$ and $\partial_y \widetilde{p}_i$ are orthogonal. Combining the above expressions then yields,

$$\partial_y \widetilde{h}_i = -n \int_r^\infty x \widetilde{p}_i(x)^2 \widetilde{w}(x) dx. \tag{2.20}$$

Using the above in (2.18) we have,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = -n \int_r^\infty x \widetilde{w}(x) \sum_{i=0}^{n-1} \frac{\widetilde{p}_i(x)^2}{\widetilde{h}_i} dx. \tag{2.21}$$

The summation can now be removed by use of the Christoffel-Darboux formula (see [25, equation (3.2.4)])

$$\sum_{i=0}^{n-1} \frac{\widetilde{p}_i(x)^2}{\widetilde{h}_i} = \frac{\widetilde{p}_n(x)' \widetilde{p}_{n-1}(x) - \widetilde{p}_{n-1}(x)' \widetilde{p}_n(x)}{\widetilde{h}_{n-1}}. \tag{2.22}$$

In order to simplify by a contour deformation the integral in the right-hand side of (2.21), we will use the formula

$$\widetilde{w}(x) \sum_{i=0}^{n-1} \frac{\widetilde{p}_i(x)^2}{\widetilde{h}_i} = -\frac{1}{4\pi i} \left(\mathrm{Tr}\left(\widetilde{Y}_+^{-1}(x) \widetilde{Y}_+'(x) \sigma_3 \right) - \mathrm{Tr}\left(\widetilde{Y}_-^{-1}(x) \widetilde{Y}_-'(x) \sigma_3 \right) \right), \qquad x \in (r,\infty), \tag{2.23}$$

which can be obtained from (2.22) and from the relation $\widetilde{Y}_+(x) = \widetilde{Y}_-(x) \begin{pmatrix} 1 & \widetilde{w}(x) \\ 0 & 1 \end{pmatrix}$ for $x \in (r,\infty)$, and where $\widetilde{Y}_\pm$ correspond to the limiting values of $\widetilde{Y}$ from above and below (r,∞). We now obtain,

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \frac{n}{4\pi i} \int_r^\infty x \left(\mathrm{Tr}\left(\widetilde{Y}_+^{-1}(x) \widetilde{Y}_+'(x) \sigma_3 \right) - \mathrm{Tr}\left(\widetilde{Y}_-^{-1}(x) \widetilde{Y}_-'(x) \sigma_3 \right) \right) dx. \tag{2.24}$$

Note also that (2.16) implies that

$$\mathrm{Tr}(\widetilde{Y}^{-1}(z) \widetilde{Y}'(z) \sigma_3) = y \mathrm{Tr}(Y^{-1}(yz) Y'(yz) \sigma_3). \tag{2.25}$$

Therefore, combining (2.24) with (2.25) gives after a change of variables

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = \frac{n}{4\pi i y} \int_{yr}^\infty x \left(\mathrm{Tr}\left(Y_+^{-1}(x) Y_+'(x) \sigma_3 \right) - \mathrm{Tr}\left(Y_-^{-1}(x) Y_-'(x) \sigma_3 \right) \right) dx. \tag{2.26}$$

Consider the integral of $z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right)$ over the contour $\mathcal{C}$ shown in Figure 1. As Y is analytic in $\mathbb{C} \setminus [yr,\infty)$ and $\mathcal{C}$ does not enclose any singularities of Y, this integral is zero. Therefore, we have

$$\int_{\mathcal{C}_+ \cup \mathcal{C}_-} z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right) dz = -\int_{\mathcal{C}_\epsilon + \mathcal{C}_R} z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right) dz. \tag{2.27}$$

Property (d) in the RH problem for Y implies that $Y^{-1}(z)Y'(z) = \mathcal{O}((\log(yr-z))^2)$ as $z \to yr$, and thus we have

$$\lim_{\epsilon \to 0} \int_{\mathcal{C}_\epsilon} z\mathrm{Tr}\left(Y^{-1}(z)Y'(z)\sigma_3\right) dz = 0. \tag{2.28}$$

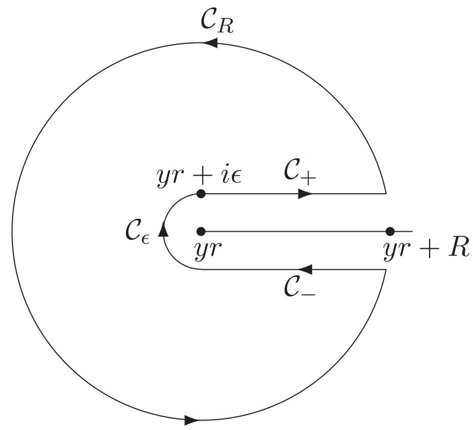

Figure 1: The contour $\mathcal{C} = \mathcal{C}_\epsilon \cup \mathcal{C}_+ \cup \mathcal{C}_R \cup \mathcal{C}_-$ used in establishing a differential identity for the Hankel determinant $Z_{n,\alpha}(y,r)$.

Thus, by Remark 2.1 and by taking first $\epsilon \to 0$ and then $R \to \infty$ in (2.27), only the term containing c_1 in (2.7) contributes to the limit, one has

$$\int_{yr}^{\infty} x \left(\mathrm{Tr}\left(Y_+^{-1}(x) Y_+'(x) \sigma_3 \right) - \mathrm{Tr}\left(Y_-^{-1}(x) Y_-'(x) \sigma_3 \right) \right) dx = -2\pi i c_1,$$

and equation (2.26) becomes

$$\partial_y \log \widetilde{Z}_{n,\alpha}(y;r) = -\frac{nc_1}{2y}. \tag{2.29}$$

The result follows from (2.9). □

3 A Riemann-Hilbert problem related to the system of ODEs

We first introduce some notations for the sake of convenience. We define the piecewise constant function

$$\theta(z) = \begin{cases} +1 & \text{if } \ \mathrm{Im} z > 0, \\ -1 & \text{if } \ \mathrm{Im} z < 0, \end{cases} \tag{3.1}$$

and for $t \in \mathbb{R}$, we define also

$$H_t(z) = \begin{cases} I, & \text{for } -\frac{2\pi}{3} < \arg(z-t) < \frac{2\pi}{3}, \\ \begin{pmatrix} 1 & 0 \\ -e^{\pi i \alpha} & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg(z-t) < \pi, \\ \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, & \text{for } -\pi < \arg(z-t) < -\frac{2\pi}{3}, \end{cases} \tag{3.2}$$

where the principal branch is chosen for the argument, such that $\arg(z-t) = 0$ if $z > t$. In the course of computing the asymptotics for the Hankel determinants $Z_{n,\alpha}(y;r)$, one is led to consider a model RH problem, which we will denote by Φ its unique solution. The matrix-valued function Φ depends on parameters $x > 0$ and $a > 1$. In the RH analysis of Y in Section 6, x and a will be related to y and r through the relations

$$x = n^2 f(y) \qquad \text{and} \qquad a = \frac{f(yr)}{f(y)}, \tag{3.3}$$

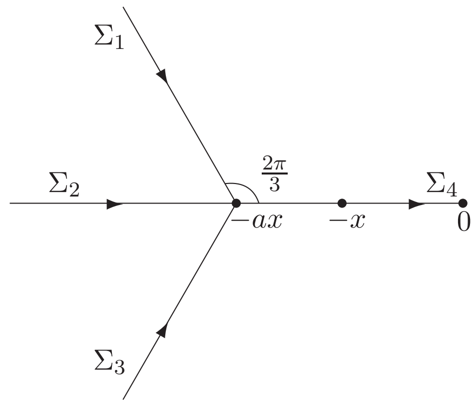

Figure 2: The jump contours appearing in the Riemann-Hilbert problem for Φ. The arcs Σ_j, $j = 1, 2, 3, 4$ depend on x and a. For convenience, they are chosen such that they do not contain $-ax$, $-x$ and 0.

where f is a conformal map from 0 to a neighbourhood of 0, satisfying $f'(0) = 4$, see (6.44) and (6.45). In particular, if we can write $y = \frac{s}{4n^2}$ for a fixed s, and if r is fixed, equation (3.3) implies that as $n \to \infty$, we have

$$x = s + \mathcal{O}(n^{-2}), \qquad a = r + \mathcal{O}(n^{-2}). \tag{3.4}$$

RH problem for $\Phi(z) = \Phi(z; x, a)$

(a) $\Phi : \mathbb{C} \setminus \Sigma_{x,ax} \to \mathbb{C}^{2\times 2}$ analytic, with $\Sigma_{x,ax} = \cup_{i=1}^{4} \Sigma_i \cup \{-ax, -x, 0\}$ as illustrated in Figure 2.

(b) Φ has continuous boundary values $\Phi_\pm(z)$ as $z \in \Sigma_{x,ax} \setminus \{-ax, -x, 0\}$ is approached from the left $(+)$ or right $(-)$ side of $\Sigma_{x,ax}$, and they are related by

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{3.5}$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{3.6}$$

$$\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{3.7}$$

$$\Phi_+(z) = \Phi_-(z) e^{\pi i \alpha \sigma_3}, \qquad z \in \Sigma_4. \tag{3.8}$$

(c) As $z \to \infty$, there exist functions $p(x), q(x)$ and $v(x)$ (these functions also depend on a), such that Φ has the asymptotic behaviour

$$\Phi(z) = \left(I + \frac{1}{z}\Phi_1(x) + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{\frac{1}{2}}\sigma_3}, \tag{3.9}$$

where $N = \frac{1}{\sqrt{2}}(I + i\sigma_1)$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$\Phi_1(x) = \begin{pmatrix} q(x) & iv(x) \\ ip(x) & -q(x) \end{pmatrix}.$$

(d) As $z \to -ax$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{1}{2\pi i}\log(z+ax) \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} H_{-ax}(z). \tag{3.10}$$

As $z \to -x$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} (z+x)^{\sigma_3}. \tag{3.11}$$

As $z \to 0$, Φ has the asymptotic behaviour

$$\Phi(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}}. \tag{3.12}$$

Remark 3.1 It can be verified by deleting the jumps around the singularities that the $\mathcal{O}(1)$ terms in asymptotics (3.10), (3.11) and (3.12) are analytic functions.

Remark 3.2 The uniqueness of the solution Φ follows by standard arguments, based on the fact that $\det \Phi \equiv 1$ (see e.g. [9, Theorem 7.18]). It is in general a more difficult task to prove existence of a given RH problem, this relies on showing a so-called "vanishing lemma". The existence of Φ has been proved for $\alpha \geq 0$ in [3, Lemma 2.6] (our situation corresponds to $I = (-\infty, -ax)$, $B = \{-x, 0\}$, $\hat{\alpha}_{-x} = 1$, $\hat{\alpha}_0 = \frac{\alpha}{2}$ and $\tau_{\infty,0} = -\frac{1}{2}$ in the language of [3]). Nevertheless, the proof of the vanishing lemma [3, Lemma 2.6] does not require the assumptions $\alpha \geq 0$ and holds more generally for $\alpha > -1$. Thus, Φ exists and is unique for $\alpha > -1$.

Remark 3.3 The fact that $\Phi_1(x)$ is traceless follows immediately from $\det \Phi \equiv 1$.

Remark 3.4 Since $\alpha \in \mathbb{R}$, we can check that $\sigma_3 \overline{\Phi(\overline{z})} \sigma_3$ is also a solution of the RH problem for Φ. Thus, by uniqueness of the solution (see Remark 3.2), we have

$$\Phi(z) = \sigma_3 \overline{\Phi(\overline{z})} \sigma_3. \tag{3.13}$$

In particular, this implies that all the functions $p(x), q(x)$ and $v(x)$ are real.

4 The Lax pair and two coupled Painlevé V equations

To derive the system of two coupled Painlevé V equations (1.21), we will use a well-known method of isomonodromic deformation theory [13]. We begin by making the transformation

$$\widetilde{\Phi}(z; x, a) := \begin{pmatrix} 1 & 0 \\ -\frac{v(x^2)}{x} & 1 \end{pmatrix} x^{\frac{\sigma_3}{2}} e^{\frac{\pi i}{4}\sigma_3} \Phi(x^2 z; x^2, a). \tag{4.1}$$

The jump contour for $\widetilde{\Phi}(z)$ is $\Sigma_{1,a}$ and is independent of x, while its asymptotic expansion as $z \to \infty$ is given by

$$\widetilde{\Phi}(z) = \begin{pmatrix} 1 & 0 \\ -\frac{v(x^2)}{x} & 1 \end{pmatrix} \left(I + \frac{1}{z}\widetilde{\Phi}_1(x) + \mathcal{O}(z^{-2}) \right) e^{\frac{\pi i}{4}\sigma_3} z^{-\frac{1}{4}\sigma_3} N e^{x z^{\frac{1}{2}}\sigma_3}, \tag{4.2}$$

where

$$\widetilde{\Phi}_1(x) = \begin{pmatrix} \frac{q(x^2)}{x^2} & -\frac{v(x^2)}{x} \\ \frac{p(x^2)}{x^3} & -\frac{q(x^2)}{x^2} \end{pmatrix}. \tag{4.3}$$

By standard arguments of RH analysis, $\tilde{\Phi}$ is analytic in $x \in (0,\infty)$. The Lax pair $(A,B) = (A(z;x,a), B(z;x,a))$ is defined by,

$$\partial_z \widetilde{\Phi}(z) = A(z)\widetilde{\Phi}(z), \tag{4.4}$$
$$\partial_x \widetilde{\Phi}(z) = B(z)\widetilde{\Phi}(z). \tag{4.5}$$

The fact that $\det \widetilde{\Phi}$ is constant implies that A and B are traceless. Also, since the jump matrices for $\widetilde{\Phi}(z)$ are independent of z and x, A and B are analytic on $\mathbb{C} \setminus \{-a,-1,0\}$. Using the asymptotic behaviour of $\widetilde{\Phi}(z)$ as $z \to 0$, $z \to -1$, $z \to -a$ and $z \to \infty$, it is easy to show that $A(z)$ is meromorphic on $\mathbb{C}$ with single poles at $-a$, -1 and 0 while $B(z)$ is an entire function. One has

$$A(z) = A_{\infty,0}(x) + A_{0,1}(x)z^{-1} + A_{1,1}(x)(z+1)^{-1} + A_{a,1}(x)(z+a)^{-1}, \tag{4.6}$$

$$B(z) = \begin{pmatrix} 0 & 1 \\ z+u(x) & 0 \end{pmatrix}, \qquad \text{where} \qquad u(x) = \frac{-2v'(x^2)x^2 + v(x^2)^2 - 2q(x^2) + v(x^2)}{x^2}, \tag{4.7}$$

the matrices $A_{0,1}(x)$, $A_{1,1}(x)$, $A_{a,1}(x)$ are analytic in $x \in (0,\infty)$ and $A_{\infty,0}(x) = \begin{pmatrix} 0 & 0 \\ \frac{x}{2} & 0 \end{pmatrix}$. There are infinitely many non-trivial relations between the functions appearing in (4.2). They can be found using the fact that B is entire. For example, by expending $B_{12}(z)$ as $z \to \infty$ using (4.2), we find

$$B_{12}(z) = 1 + \frac{-v(x^2)^2 + v(x^2) + 2q(x^2) - 2x^2 v'(x^2)}{x^2 z} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty, \tag{4.8}$$

from which we obtain the relation $q(x) = \frac{1}{2}(2xv'(x) + v^2(x) - v(x))$, and thus $u(x)$ can be rewritten more simply as $u(x) = -2\left(v(x^2)x^{-1}\right)'$. We now turn to the compatibility condition

$$\partial_z \partial_x \widetilde{\Phi} = \partial_x \partial_z \widetilde{\Phi}, \tag{4.9}$$

which upon rewriting the derivatives in terms of the Lax matrices becomes

$$\partial_x A - \partial_z B + AB - BA = 0. \tag{4.10}$$

If we parameterise A as

$$A(z;x,a) = \begin{pmatrix} d(z;x,a) & b(z;x,a) \\ c(z;x,a) & -d(z;x,a) \end{pmatrix}, \tag{4.11}$$

then (4.10) is equivalent to three coupled ODEs,

$$d = -\frac{b'}{2}, \tag{4.12}$$
$$c = (z+u)b - \frac{b''}{2}, \tag{4.13}$$
$$c' = 1 + 2(z+u)d, \tag{4.14}$$

where primes denote again derivatives with respect to x, and where the dependence of the functions in z, x and a have been omitted. The first two equations provide d and c in terms of b. Taking the determinant of A yields

$$\det A = -\frac{(b')^2}{4} - (z+u)b^2 + \frac{bb''}{2}. \tag{4.15}$$

From (4.6) we have that $b(z)$ is of the form

$$b(z) = b_0 z^{-1} + b_1(z+1)^{-1} + b_2(z+a)^{-1}, \tag{4.16}$$

where b_0, b_1 and b_2 only depend on x and a. Note that Remark 3.4 implies that $A(z) \in \mathbb{R}$ and $B(z) \in \mathbb{R}$ for $z \in \mathbb{R}$. In particular, b_0, b_1 and b_2 are real. Equation (4.16) together with (4.15), allow us to compute the asymptotics of $\det A$ at $z = 0$, $z = -1$, $z = -a$ and $z = \infty$ in terms of b_0, b_1 and b_2. Alternatively, we may compute $\det A$ at these four points using the asymptotic expansion of $\widetilde{\Phi}$. Equating these asymptotics with those expressed in terms of b_0, b_1 and b_2 we arrive at the equations

$$\frac{\alpha^2}{4} - b_0(x)^2 u(x) - \frac{1}{4} b_0'(x)^2 + \frac{1}{2} b_0(x) b_0''(x) = 0, \tag{4.17}$$

$$b_1(x)^2(1-u(x)) - \frac{1}{4} b_1'(x)^2 + \frac{1}{2} b_1(x) b_1''(x) + 1 = 0, \tag{4.18}$$

$$b_2(x)^2(a-u(x)) - \frac{1}{4} b_2'(x)^2 + \frac{1}{2} b_2(x) b_2''(x) = 0, \tag{4.19}$$

$$\frac{x^2}{4} - \left(b_0(x) + b_1(x) + b_2(x)\right)^2 = 0. \tag{4.20}$$

By expanding the expression $A_{12}(z) = b(z)$ in a Laurent series about $z = \infty$, we get the following identities between b_0, b_1, b_2 and v':

$$b_0(x) + b_1(x) + b_2(x) = \frac{x}{2}, \tag{4.21}$$

$$b_1(x) + a b_2(x) = -x v'(x^2). \tag{4.22}$$

Note that (4.21) is a better version of (4.20). If we express b_0 and u in terms of b_1 and b_2 from (4.17) and (4.21), and if we define

$$q_j^2(x) = \frac{2 b_j(\sqrt{x})}{\sqrt{x}}, \qquad j = 1, 2, \tag{4.23}$$

we obtain the system (1.21) with $a = r$ from (4.18) and (4.19). The same change of functions (4.23) was used in [7], where the authors obtained a system of k ($k \in \mathbb{N}_0$) coupled Painlevé V equations.

5 Further properties of the special solutions

Our main goal in this section is to get asymptotics for $b_0(x)$, $b_1(x)$ and $b_2(x)$ as $(a-1)x \to \infty$ and $ax \to 0_+$.

5.1 Asymptotic analysis when $(a-1)x \to \infty$

5.1.1 Re-scaling of the model problem

In order to have a jump contour independent of x, we make the transformation $C(z; x, a) = (ax)^{\frac{1}{4}\sigma_3} \Phi(axz; x, a)$. C is the solution of the following RH problem:

RH problem for U

(a) $C : \mathbb{C} \setminus \Sigma_{a^{-1},1} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) C has the following jumps

$$C_+(z) = C_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{5.1}$$

$$C_+(z) = C_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{5.2}$$

$$C_+(z) = C_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{5.3}$$

$$C_+(z) = C_-(z) e^{\pi i \alpha \sigma_3}, \qquad z \in \Sigma_4. \tag{5.4}$$

(c) As $z \to \infty$,

$$C(z) = \left(I + \frac{1}{z} C_1(x;a) + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N e^{\sqrt{ax} z^{\frac{1}{2}} \sigma_3}, \tag{5.5}$$

where

$$C_1(x;a) = \begin{pmatrix} \frac{q(x;a)}{ax} & \frac{iv(x;a)}{(ax)^{1/2}} \\ \frac{ip(x;a)}{(ax)^{3/2}} & -\frac{q(x;a)}{ax} \end{pmatrix}. \tag{5.6}$$

(d) As $z \to -1$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{1}{2\pi i} \log(z+1) \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} H_{-1}(z). \tag{5.7}$$

As $z \to -a^{-1}$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} \left(z + a^{-1} \right)^{\sigma_3}. \tag{5.8}$$

As $z \to 0$, C has the asymptotic behaviour

$$C(z) = \mathcal{O}(1) z^{\frac{\alpha \sigma_3}{2}}. \tag{5.9}$$

5.1.2 Normalisation at ∞ of the RH problem

We define the g-function by

$$g(z) = \sqrt{z+1} \tag{5.10}$$

where the principal branch is taken for the square root. As $z \to \infty$, g has the asymptotic behaviour

$$g(z) = z^{1/2} + g_1 z^{-1/2} + \mathcal{O}(z^{-3/2}), \qquad g_1 = \frac{1}{2}. \tag{5.11}$$

Now we define

$$W(z) = \begin{pmatrix} 1 & 0 \\ i g_1 \sqrt{ax} & 1 \end{pmatrix} C(z) e^{-\sqrt{ax} g(z) \sigma_3}. \tag{5.12}$$

RH problem for W

(a) $W : \mathbb{C} \setminus \Sigma_{a^{-1},1} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) W has the following jumps:

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} e^{-2\sqrt{ax}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{5.13}$$

$$W_+(z) = W_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{5.14}$$

$$W_+(z) = W_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} e^{-2\sqrt{ax}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{5.15}$$

$$W_+(z) = W_-(z) e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4. \tag{5.16}$$

(c) As $z \to \infty$,

$$W(z) = \left(I + \frac{1}{z} W_1(x;a) + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N, \tag{5.17}$$

where

$$(W_1(x;a))_{12} = \frac{iv(x;a)}{\sqrt{ax}} + ig_1\sqrt{ax}. \tag{5.18}$$

(d) As $z \to -1$,

$$W(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{\log(z+1)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} H_{-1}(z) e^{-\sqrt{ax}g(z)\sigma_3}. \tag{5.19}$$

As $z \to -a^{-1}$,

$$W(z) = \mathcal{O}(1) e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} \left(z + a^{-1}\right)^{\sigma_3}. \tag{5.20}$$

As $z \to 0$,

$$W(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}}. \tag{5.21}$$

For $z \in \Sigma_1 \cup \Sigma_3$, $\mathrm{Re}(g(z)) > 0$ and therefore the jumps of W on $\Sigma_1 \cup \Sigma_3$ are exponentially close to the identity matrix as $(a-1)x \to \infty$. Since $g(-1) = 0$, this convergence is not uniform as z approaches -1. Therefore we will construct a global parametrix which will be a good approximation of W as z stays away from a neighbourhood of -1, and a local parametrix around -1.

5.1.3 Global parametrix

Ignoring exponentially small entries in the jumps and a small neighbourhood of -1, we are led to consider the following RH problem.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus \mathbb{R}^-$ is analytic.

(b) $P^{(\infty)}$ has the following jumps on $\mathbb{R}^-$:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -1), \tag{5.22}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{\pi i \alpha \sigma_3}, \qquad z \in (-1, 0) \setminus \{-a^{-1}\}, \tag{5.23}$$

(c) As $z \to \infty$,

$$P^{(\infty)}(z) = \left(I + \frac{1}{z} P_1^{(\infty)}(a) + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N. \tag{5.24}$$

(d) As $z \to -1$,

$$P^{(\infty)}(z) = \mathcal{O}\left((z+1)^{-1/4} \right). \tag{5.25}$$

As $z \to -a^{-1}$,

$$P^{(\infty)}(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2} \theta(z) \sigma_3} \left(z + a^{-1} \right)^{\sigma_3}. \tag{5.26}$$

As $z \to 0$,

$$P^{(\infty)}(z) = \mathcal{O}(1) z^{\frac{\alpha \sigma_3}{2}}. \tag{5.27}$$

We can check that the solution of this RH problem is explicitly given by

$$P^{(\infty)}(z) = \begin{pmatrix} 1 & 0 \\ i(\alpha + 2\sqrt{1-a^{-1}}) & 1 \end{pmatrix} (z+1)^{-\frac{\sigma_3}{4}} N \times \left(\frac{\sqrt{z+1}+1}{\sqrt{z+1}-1} \right)^{-\frac{\alpha}{2}\sigma_3} \left(\frac{\sqrt{z+1}+\sqrt{1-a^{-1}}}{\sqrt{z+1}-\sqrt{1-a^{-1}}} \right)^{-\sigma_3}, \tag{5.28}$$

where the principal branch has been chosen for each root. Note that

$$(P_1^{(\infty)}(a))_{12} = i(\alpha + 2\sqrt{1-a^{-1}}). \tag{5.29}$$

5.1.4 Local parametrix near -1

We want to construct a function $P^{(-1)}$ defined in a open disk D_{-1} around -1 of radius $\frac{1}{3}(1-a^{-1})$ which satisfies the following RH conditions.

RH problem for $P^{(-1)}$

(a) $P^{(-1)} : D_{-1} \setminus \Sigma_{a^{-1},1}$ is analytic.

(b) $P^{(-1)}$ has the following jumps:

$$P_+^{(-1)}(z) = P_-^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} e^{-2\sqrt{ax}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_1 \cap D_{-1}, \tag{5.30}$$

$$P_+^{(-1)}(z) = P_-^{(-1)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2 \cap D_{-1}, \tag{5.31}$$

$$P_+^{(-1)}(z) = P_-^{(-1)}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} e^{-2\sqrt{ax}g(z)} & 1 \end{pmatrix}, \qquad z \in \Sigma_3 \cap D_{-1}, \tag{5.32}$$

$$P_+^{(-1)}(z) = P_-^{(-1)}(z) e^{\pi i\alpha\sigma_3}, \qquad z \in \Sigma_4 \cap D_{-1}. \tag{5.33}$$

(c) As $x \to \infty$,

$$P^{(-1)}(z) = \left(I + \mathcal{O}\left(((a-1)x)^{-1/2}\right)\right) P^{(\infty)}(z) \tag{5.34}$$

uniformly for $z \in \partial D_{-1}$.

(d) As $z \to -1$,

$$P^{(-1)}(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{\log(z+1)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} H_{-1}(z) e^{-\sqrt{ax}g(z)\sigma_3}. \tag{5.35}$$

The solution of this RH problem can be constructed in terms of the Bessel model RH problem with parameter $\alpha = 0$, which is presented in the appendix (see Subsection 9.2), and whose solution is denoted $\Upsilon^{(0)}$. The local parametrix is given by

$$P^{(-1)}(z) = E(z)\Upsilon^{(0)}(axf(z)) e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} e^{-\sqrt{ax}g(z)\sigma_3}, \tag{5.36}$$

where

$$f(z) = g(z)^2 = z+1 \quad \text{and} \quad E(z) = P^{(\infty)}(z) e^{-\frac{\pi i\alpha}{2}\theta(z)\sigma_3} N^{-1} f(z)^{\frac{\sigma_3}{4}} (ax)^{\frac{1}{4}\sigma_3}. \tag{5.37}$$

It can be verified that E is analytic in D_{-1}.

5.1.5 Small norm RH problem

Define

$$R(z) = \begin{cases} W(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus (\overline{D_{-1}} \cup \Sigma_1 \cup \Sigma_3), \\ W(z)P^{(-1)}(z)^{-1}, & \text{for } z \in D_{-1}. \end{cases} \tag{5.38}$$

Since W and $P^{(-1)}$ have the same jumps inside D_{-1} and the same behaviour near -1, R is analytic inside D_{-1}. Also, W and $P^{(\infty)}$ have the same jumps on $\mathbb{R}^-$, and the same behaviour near $-a^{-1}$ and 0. Therefore, R is analytic on $\mathbb{C}\setminus((\partial D_{-1} \cup \Sigma_1 \cup \Sigma_3) \setminus D_{-1})$. Let us put the clockwise orientation on ∂D_{-1}. On ∂D_{-1} by (5.34), we have $R_-(z)^{-1}R_+(z) = I + \mathcal{O}\left(((a-1)x)^{-1/2}\right)$ and by (5.13) and (5.15), on $(\Sigma_1 \cup \Sigma_3) \setminus D_{-1}$, $R_-(z)^{-1}R_+(z) = I + \mathcal{O}(e^{-c\sqrt{(a-1)x}})$ where $c > 0$ is a constant. From (5.17) and (5.24), as $z \to \infty$ one has $R(z) = I + \mathcal{O}(z^{-1})$. By small norm theory for RH problems, it follows that R exists for sufficiently large $(a-1)x$ and satisfies $R(z) = I + \mathcal{O}(((a-1)x)^{-1/2})$ uniformly in $z \in \mathbb{C} \setminus ((\partial D_{-1} \cup \Sigma_1 \cup \Sigma_3) \setminus D_{-1})$ and as $z \to \infty$,

$$R(z) = I + \frac{R_1(x;a)}{z} + \mathcal{O}(z^{-2}), \qquad R'(z) = \mathcal{O}\left(((a-1)x)^{-1/2}\right) \tag{5.39}$$

We have in particular that $R_1(x;a) = \mathcal{O}(((a-1)x)^{-1/2})$. For $z \in \mathbb{C} \setminus (\overline{D_{-1}} \cup \Sigma_1 \cup \Sigma_3)$, we have $W(z) = R(z)P^{(\infty)}(z)$ and therefore

$$W_1(x;a) = R_1(x;a) + P_1^{(\infty)}(a) = P_1^{(\infty)}(a) + \mathcal{O}(((a-1)x)^{-1/2}), \tag{5.40}$$

Using (5.18), (5.29) and (5.40), we obtain

$$v(x) = -\frac{1}{2}ax + (\alpha + 2\sqrt{1-a^{-1}})\sqrt{ax} + \mathcal{O}((1-a^{-1})^{-1/2}), \qquad \text{as } (a-1)x \to \infty. \tag{5.41}$$

For the asymptotics for b_0 and b_1, from (4.4), (4.11) and (4.16), we can use

$$b_0(\sqrt{x}) = \lim_{z\to 0} i\sqrt{x}z \left[\partial_z \Phi(axz;x,a)\Phi^{-1}(axz;x,a)\right]_{12}, \tag{5.42}$$

$$b_1(\sqrt{x}) = \lim_{z\to -a^{-1}} i\sqrt{x}(z+a^{-1}) \left[\partial_z \Phi(axz;x,a)\Phi^{-1}(axz;x,a)\right]_{12}. \tag{5.43}$$

To compute these limits, we will need the global parametrix. For $z \in \mathbb{C} \setminus (\overline{D_{-1}} \cup \Sigma_1 \cup \Sigma_3)$, by the definition of C given at the beginning of Subsection 5.1.1, (5.12) and (5.38), we have

$$\Phi(axz;x,a) = (ax)^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -\frac{i}{2}\sqrt{ax} & 1 \end{pmatrix} R(z)P^{(\infty)}(z)e^{\sqrt{ax}g(z)\sigma_3}. \tag{5.44}$$

Using the definition of the global parametrix given by (5.28), together with the asymptotics for R (5.39) and the above equation, the limits (5.42) and (5.43) are straightforward to compute. As $(a-1)x \to \infty$, we find

$$b_0(\sqrt{x}) = \frac{\alpha}{2\sqrt{a}}\left(1 + \mathcal{O}\left(((a-1)x)^{-1/2}\right)\right), \tag{5.45}$$

$$b_1(\sqrt{x}) = \frac{1}{\sqrt{a-1}}\left(1 + \mathcal{O}\left(((a-1)x)^{-1/2}\right)\right). \tag{5.46}$$

Asymptotics for b_2 can be obtained directly from the relation (4.21), as $(a-1)x \to \infty$ we have

$$b_2(\sqrt{x}) = \frac{\sqrt{x}}{2} - \frac{\alpha}{2\sqrt{a}} - \frac{1}{\sqrt{a-1}} + \mathcal{O}\left(\frac{1}{(a-1)\sqrt{x}}\right). \tag{5.47}$$

Large $(a-1)x$ asymptotics for $q_1(x)^2$ and $q_2(x)^2$ are immediate to obtain from (4.23), (5.46) and (5.47) and are given in (1.22) and (1.23) with $a = r$.

We will need later the asymptotics for $\Phi(z;x,a)$ when $z \to \infty$ and simultaneously $(a-1)x \to \infty$. Note that the global parametrix $P^{(\infty)}$ defined in (5.28) only depends on a, and is such that its behaviour at ∞ (5.24) has the form

$$P^{(\infty)}(z) = (I + \mathcal{O}(z^{-1}))z^{-\frac{1}{4}\sigma_3}N, \qquad \text{as } z \to \infty \quad \text{and} \quad (a-1)x \to \infty. \tag{5.48}$$

Thus from (5.12), (5.38) and the definition of C given at the beginning of Subsection 5.1.1, we have as $\frac{z}{ax} \to \infty$ and simultaneously $(a-1)x \to \infty$ that

$$\begin{aligned}\Phi(z;x,a) &= (ax)^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -\frac{i}{2}\sqrt{ax} & 1 \end{pmatrix} \left(I + \mathcal{O}\left(\tfrac{ax}{z}\right)\right)\left(\tfrac{z}{ax}\right)^{-\frac{\sigma_3}{4}} Ne^{\sqrt{z+ax}\sigma_3}, \\ &= z^{-\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ -\frac{i}{2}\frac{ax}{\sqrt{z}} & 1 \end{pmatrix} \left(I + \begin{pmatrix} \mathcal{O}(\frac{ax}{z}) & \mathcal{O}(\sqrt{\frac{ax}{z}}) \\ \mathcal{O}((\frac{ax}{z})^{\frac{3}{2}}) & \mathcal{O}(\frac{ax}{z}) \end{pmatrix}\right) Ne^{\sqrt{z+ax}\sigma_3}.\end{aligned} \tag{5.49}$$

Furthermore, if we assume that $\frac{\sqrt{z}}{ax} \to \infty$, we have

$$Ne^{\sqrt{z+ax}\sigma_3} = \left(I + \begin{pmatrix} \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^2\right) & \mathcal{O}\left(\frac{ax}{\sqrt{z}}\right) \\ \frac{i}{2}\frac{ax}{\sqrt{z}} + \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^3\right) & \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^2\right) \end{pmatrix} \right) Ne^{\sqrt{z}\sigma_3}. \tag{5.50}$$

Thus, if $z \to \infty$, $(a-1)x \to \infty$ and simultaneously $\frac{\sqrt{z}}{ax} \to \infty$, (5.49) becomes

$$\Phi(z; x, a) = z^{-\frac{\sigma_3}{4}} \left(I + \begin{pmatrix} \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^2\right) & \mathcal{O}\left(\frac{ax}{\sqrt{z}}\right) \\ \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^3\right) & \mathcal{O}\left(\left(\frac{ax}{\sqrt{z}}\right)^2\right) \end{pmatrix} \right) Ne^{\sqrt{z}\sigma_3}. \tag{5.51}$$

5.2 Asymptotic analysis when $ax \to 0$

In order to have the rays Σ_1 and Σ_3 of the jump contour independent of a and x, we make the following transformation on Φ:

$$\widetilde{W}(z) = \Phi(z) H_{-ax}(z)^{-1} H_0(z). \tag{5.52}$$

It is easy to verify that $\widetilde{W}$ satisfies the following RH problem.

RH problem for $\widetilde{W}$

(a) $\widetilde{W} : \mathbb{C} \setminus \Sigma_{0,0} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $\widetilde{W}$ has the following jumps

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad \arg(z) = \frac{2\pi}{3}, \tag{5.53}$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -ax), \tag{5.54}$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad \arg(z) = -\frac{2\pi}{3}, \tag{5.55}$$

$$\widetilde{W}_+(z) = \widetilde{W}_-(z) \begin{pmatrix} e^{\pi i \alpha} & 0 \\ -2 & e^{-\pi i \alpha} \end{pmatrix}, \qquad z \in (-ax, 0) \setminus \{-x\}. \tag{5.56}$$

(c) As $z \to \infty$,

$$\widetilde{W}(z) = \left(I + \frac{1}{z}\Phi_1(x) + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{\frac{1}{2}}\sigma_3}. \tag{5.57}$$

(d) As $z \to -ax$, $\widetilde{W}$ has the asymptotic behaviour

$$\widetilde{W}(z) = \mathcal{O}(1) \begin{pmatrix} 1 & \frac{\log(z+ax)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} H_0(z). \tag{5.58}$$

As $z \to -x$,

$$\widetilde{W}(z) = \mathcal{O}(1) e^{\frac{\pi i \alpha}{2}\theta(z)\sigma_3} (z+x)^{\sigma_3} H_0(z). \tag{5.59}$$

As $z \to 0$,

$$\widetilde{W}(z) = \mathcal{O}(1) z^{\frac{\alpha\sigma_3}{2}} H_0(z). \tag{5.60}$$

In equations (5.58), (5.59) and (5.60), the $\mathcal{O}(1)$ are analytic functions in a neighbourhood of their respective point.

5.2.1 Global parametrix

As $ax \to 0$, the length of $(-ax, 0)$ tends to 0 and the pole at $-x$, the algebraic singularity at 0, as well as the logarithmic singularity at $-ax$, are merging together. Therefore, for z outside of a neighbourhood of 0, we expect that the Bessel model RH problem of order $\alpha+2$ (presented in the appendix, see Subsection 9.2) will be relevant to construct the global parametrix $P^{(\infty)}$. In a small neighbourhood of 0 and we will construct a new local parametrix around the origin.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus \Sigma_{0,0}$ is analytic.

(b) $P^{(\infty)}$ has the following jumps on $\Sigma_{0,0}$:

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathbb{R}^-, \tag{5.61}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad \arg(z) = \frac{2\pi}{3}, \tag{5.62}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad \arg(z) = -\frac{2\pi}{3}. \tag{5.63}$$

(c) As $z \to \infty$,

$$P^{(\infty)}(z) = \left(I + \frac{P_1^{(\infty)}}{z} + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{\frac{1}{2}}\sigma_3}. \tag{5.64}$$

(d) As $z \to 0$,

$$P^{(\infty)}(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}} \\ |z|^{\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}} \end{pmatrix}, & \text{for } -\frac{2\pi}{3} < \arg(z) < \frac{2\pi}{3}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}} \\ |z|^{-\frac{\alpha+2}{2}} & |z|^{-\frac{\alpha+2}{2}} \end{pmatrix}, & \text{for } \arg(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi). \end{cases} \tag{5.65}$$

The only solution of this RH problem is well-known and given by

$$P^{(\infty)}(z) = \Upsilon^{(\alpha+2)}(z), \tag{5.66}$$

where $\Upsilon^{(\alpha+2)}$ is the solution of the Bessel model RH problem, presented in Subsection 9.2. Note that if we don't specify condition (d) in the RH problem for $P^{(\infty)}$, the solution is not unique. From a mathematical point of view, we remark that $\Upsilon^{(\alpha)}$ or $\Upsilon^{(\alpha+4)}$ for example could have also been a suitable choice for $P^{(\infty)}$, but $\Upsilon^{(\alpha+2)}$ is the only one which satisfies condition (d) and which allows us to create a local parametrix around 0 respecting the matching condition (5.74). By (9.12), we have

$$(P_1^{(\infty)})_{12} = \frac{i}{8}(4(\alpha+2)^2 - 1). \tag{5.67}$$

In the construction of the local parametrix in a neighbourhood of 0, we will need a more explicit knowledge of the behaviour of $P^{(\infty)}$ at the origin. It can be verified (see [4]) that $P^{(\infty)}$ can be written as

$$P^{(\infty)}(z) = P_0^{(\infty)}(z) z^{\frac{\alpha+2}{2}\sigma_3} \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z), \qquad z \in \mathbb{C} \setminus \Sigma_{0,0}, \tag{5.68}$$

where $P_0^{(\infty)}(z) = P_{0,\alpha}^{(\infty)}(z)$ is an entire function in z for every α while

$$h(z) = \begin{cases} \frac{1}{2i\sin(\pi\alpha)}, & \alpha \notin \mathbb{Z}, \\ \frac{(-1)^\alpha}{2\pi i}\log z, & \alpha \in \mathbb{Z}. \end{cases} \tag{5.69}$$

5.2.2 Local parametrix near 0

We want to construct a function $P^{(0)}$ defined in a fixed open disk D_0 around 0 which satisfies exactly the same RH conditions than $\widetilde{W}$ on D_0 and matches with $P^{(\infty)}$ on ∂D_0.

RH problem for $P^{(0)}$

(a) $P^{(0)} : D_0 \setminus \Sigma_{0,0}$ is analytic.

(b) $P^{(0)}$ has the following jumps

$$P_+^{(0)}(z) = P_-^{(0)}(z)\begin{pmatrix} 1 & 0 \\ e^{\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \left\{\arg(z) = \frac{2\pi}{3}\right\} \cap D_0, \tag{5.70}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in (-\infty, -ax) \cap D_0, \tag{5.71}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z)\begin{pmatrix} 1 & 0 \\ e^{-\pi i\alpha} & 1 \end{pmatrix}, \qquad z \in \left\{\arg(z) = -\frac{2\pi}{3}\right\} \cap D_0, \tag{5.72}$$

$$P_+^{(0)}(z) = P_-^{(0)}(z)\begin{pmatrix} e^{\pi i\alpha} & 0 \\ -2 & e^{-\pi i\alpha} \end{pmatrix}, \qquad z \in (-ax, 0) \setminus \{-x\}. \tag{5.73}$$

(c) As $ax \to 0$,

$$P^{(0)}(z) = (I + \mathcal{O}(ax))P^{(\infty)}(z) \tag{5.74}$$

uniformly for $z \in \partial D_0$.

(d) As $z \to -ax$, $P^{(0)}$ has the asymptotic behaviour

$$P^{(0)}(z) = \mathcal{O}(1)\begin{pmatrix} 1 & \frac{\log(z+ax)}{2\pi i} \\ 0 & 1 \end{pmatrix} e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3}H_0(z). \tag{5.75}$$

As $z \to -x$,

$$P^{(0)}(z) = \mathcal{O}(1)e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3}(z+x)^{\sigma_3}H_0(z). \tag{5.76}$$

As $z \to 0$,

$$P^{(0)}(z) = \mathcal{O}(1)z^{\frac{\alpha\sigma_3}{2}}H_0(z). \tag{5.77}$$

We can check that the solution of this RH problem is explicitly given by:

$$P^{(0)}(z) = P_0^{(\infty)}(z)\left(\frac{z+x}{z}\right)^{\sigma_3}\begin{pmatrix} 1 & f(z;x) \\ 0 & 1 \end{pmatrix} z^{\frac{\alpha+2}{2}\sigma_3}\begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix} H_0(z) \tag{5.78}$$

where

$$f(z;x) = \frac{-z^2}{2\pi i}\int_{-ax}^{0} \frac{|s|^\alpha}{s-z}ds. \tag{5.79}$$

5.2.3 Small norm RH problem

Define

$$R(z) = \begin{cases} \widetilde{W}(z)P^{(\infty)}(z)^{-1}, & \text{for } z \in \mathbb{C} \setminus \overline{D_0}, \\ \widetilde{W}(z)P^{(0)}(z)^{-1}, & \text{for } z \in D_0. \end{cases} \tag{5.80}$$

By definition of $\widetilde{W}$, $P^{(\infty)}$ and $P^{(0)}$, R is analytic on $\mathbb{C} \setminus \partial D_0$. Let us put the clockwise orientation on ∂D_0. The jumps of R on ∂D_0 are given by

$$R_-(z)^{-1}R_+(z) = P^{(0)}(z)P^{(\infty)}(z)^{-1} = I + \mathcal{O}(ax), \qquad \text{as } ax \to 0, \tag{5.81}$$

where we have used (5.74). Also, from (5.57) and (5.64), as $z \to \infty$ we have $R(z) = I + \mathcal{O}(z^{-1})$. Thus, by standard theory for small norm RH problems, R exists for sufficiently small ax and satisfies

$$R(z) = I + \mathcal{O}(ax), \qquad R'(z) = \mathcal{O}(ax), \tag{5.82}$$

as $ax \to 0$ uniformly in $z \in \mathbb{C} \setminus \partial D_0$. Also, as $z \to \infty$, we have

$$R(z) = I + \frac{R_1(x;a)}{z} + \mathcal{O}(z^{-2}), \tag{5.83}$$

where $R_1(x;a) = \mathcal{O}(ax)$ as $ax \to 0$. For $z \in \mathbb{C} \setminus \overline{D_0}$, we have $\widetilde{W}(z) = R(z)P^{(\infty)}(z)$ and therefore we obtain

$$\Phi_1(x;a) = P_1^{(\infty)} + R_1(x;a). \tag{5.84}$$

In particular, from (5.67) and (5.84), as $ax \to 0$ we obtain

$$v(x;a) = v(0) + \mathcal{O}(ax), \tag{5.85}$$

with $v(0) := \frac{1}{8}(4(\alpha+2)^2 - 1)$. To obtain asymptotics for b_1 and b_2, we will proceed similarly as done in Subsection 5.1.5, but instead of the global parametrix, we will need the local parametrix. From (4.4), (4.11) and (4.16), we have

$$b_1(\sqrt{x}) = \lim_{z\to -1} i\sqrt{x}(z+1)\left[\partial_z \Phi(xz;x,a)\Phi^{-1}(xz;x,a)\right]_{12}, \tag{5.86}$$

$$b_2(\sqrt{x}) = \lim_{z\to -a} i\sqrt{x}(z+a)\left[\partial_z \Phi(xz;x,a)\Phi^{-1}(xz;x,a)\right]_{12}. \tag{5.87}$$

On the other hand, from (5.52), (5.78) and (5.80), we have as $ax \to 0$ and for $z \in D_0$

$$\Phi(xz;x,a) = R(xz)P_0^{(\infty)}(xz)\left(\frac{z+1}{z}\right)^{\sigma_3}\begin{pmatrix}1 & f(xz;x)\\ 0 & 1\end{pmatrix}(xz)^{\frac{\alpha+2}{2}\sigma_3}\begin{pmatrix}1 & h(xz)\\ 0 & 1\end{pmatrix}H_{-ax}(z). \tag{5.88}$$

Thus, by the estimate (5.82) and a direct calculation, we have

$$b_2(\sqrt{x}) = i\sqrt{x}(1+\mathcal{O}(ax))\left[P_0^{(\infty)}(0)\begin{pmatrix}0 & \star\\ 0 & 0\end{pmatrix}P_0^{(\infty)}(0)^{-1}\right]_{12}, \qquad \text{as } ax \to 0, \tag{5.89}$$

where in the above equation

$$\star = \lim_{z\to -a}(z+a)\frac{(z+1)^2}{z^2}\partial_z f(xz;x) \tag{5.90}$$

$$= \frac{-(a-1)^2x^2}{2\pi i}\lim_{z\to -a}(z+a)\int_{-ax}^{0}\frac{x|s|^\alpha}{(s-xz)^2}ds = \frac{(a-1)^2x^2}{2\pi i}(ax)^\alpha. \tag{5.91}$$

Therefore, (5.89) becomes

$$b_2(\sqrt{x}) = \frac{(1-a^{-1})^2}{2\pi}\sqrt{x}(ax)^{\alpha+2}P^{(\infty)}_{0,11}(0)^2(1+\mathcal{O}(ax)), \qquad \text{as } ax\to 0. \tag{5.92}$$

To compute $P^{(\infty)}_{0,11}(0)$, we can use (5.68) and (9.11). For $z\in\{z\in\mathbb{C}: |\arg(z)|<\frac{2\pi}{3}\}$ we have

$$P^{(\infty)}_{0,11}(z) = \sqrt{\pi}I_{\alpha+2}(\sqrt{z})z^{-\frac{\alpha+2}{2}}. \tag{5.93}$$

Taking the limit $z\to 0$ in (5.93), and using the small z expansion of $I_{\alpha+2}(z)$ (see [22, formula 10.30.1]) we obtain $P^{(\infty)}_{0,11}(0) = \frac{\sqrt{\pi}}{2^{\alpha+2}\Gamma(\alpha+3)}$. Inserting this value in (5.92), we have as $ax\to 0$ that

$$b_2(\sqrt{x}) = \frac{(1-a^{-1})^2\sqrt{x}(ax)^{\alpha+2}}{2^{2\alpha+5}\Gamma(\alpha+3)^2}(1+\mathcal{O}(ax)) \tag{5.94}$$

$$= \frac{(1-a^{-1})^2\sqrt{x}}{2}I^2_{\alpha+2}(\sqrt{ax}))(1+\mathcal{O}(ax)). \tag{5.95}$$

Similarly, from (5.86) and (5.88), we have as $ax\to 0$ that

$$b_1(\sqrt{x}) = i\sqrt{x}\left[P^{(\infty)}_0(0)\sigma_3P^{(\infty)}_0(0)^{-1}\right]_{12}(1+\mathcal{O}(ax)) = -2i\sqrt{x}P^{(\infty)}_{0,11}(0)P^{(\infty)}_{0,12}(0)(1+\mathcal{O}(ax)). \tag{5.96}$$

Again, from (9.11) and (5.68), we obtain for $z\in\{z\in\mathbb{C}: |\arg(z)|<\frac{2\pi}{3}\}$ that

$$P^{(\infty)}_{0,12}(z) = \frac{i}{\sqrt{\pi}}K_{\alpha+2}(\sqrt{z})z^{\frac{\alpha+2}{2}} - P_{0,11}(z)h(z)z^{\alpha+2}. \tag{5.97}$$

By taking the limit $z\to 0$ and using [22, formulas 10.30.2], this gives $P^{(\infty)}_{0,12}(0) = \frac{i}{\sqrt{\pi}}2^{\alpha+1}\Gamma(\alpha+2)$, and by (5.96) we have

$$b_1(\sqrt{x}) = \frac{\sqrt{x}}{\alpha+2}(1+\mathcal{O}(ax)), \qquad \text{as } ax\to 0. \tag{5.98}$$

With the change of functions (4.23), we obtain from (5.95) and (5.98) the small ax asymptotics for $q_1(x;a)$ and $q_2(x;a)$ given in (1.24) and (1.25). We will also need later the asymptotics of $\Phi(z;x,a)$ as $z\to\infty$ and simultaneously $ax\to 0$. This can be obtained from (5.52), (5.64), (5.80) and (5.83), and by the fact that the global parametrix (5.66) is independent of a and x, we have

$$\begin{aligned}\Phi(z;x,a) &= \left(I+\mathcal{O}(\tfrac{ax}{z})\right)(I+\mathcal{O}(z^{-1}))z^{-\frac{1}{4}\sigma_3}Ne^{z^{\frac{1}{2}}\sigma_3}, && \text{as } z\to\infty \text{ and } ax\to 0,\\ &= (I+\mathcal{O}(z^{-1}))z^{-\frac{1}{4}\sigma_3}Ne^{z^{\frac{1}{2}}\sigma_3}, && \text{as } z\to\infty \text{ and } ax\to 0.\end{aligned} \tag{5.99}$$

6 Steepest descent analysis of Y as $nyr\to 0$

An essential ingredient in the steepest descent analysis is the equilibrium measure μ, which in our case is the unique probability measure which minimizes

$$\int_0^\infty\int_0^\infty \log\frac{1}{|x-y|}d\tilde{\mu}(x)d\tilde{\mu}(y) + \int_0^\infty y\,d\tilde{\mu}(y), \tag{6.1}$$

among all Borel probability measures $\tilde{\mu}$ on $(0,\infty)$. The unique solution μ of the minimization problem (6.1) is supported on $\overline{\mathcal{S}}$, where $\mathcal{S}:=(0,4)$, and its density is known as the Marchenko-Pastur law:

$$\frac{d\mu(x)}{dx} = \rho(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}. \tag{6.2}$$

The equilibrium measure μ satisfies the following identities [26], known as the Euler-Lagrange variational conditions:

$$2\int_{\mathcal{S}} \log|x-y|\rho(y)dy - x = \ell, \quad x \in \overline{\mathcal{S}}, \tag{6.3}$$

$$2\int_{\mathcal{S}} \log|x-y|\rho(y)dy - x < \ell, \quad x \in (4,\infty), \tag{6.4}$$

where $\ell = -2$. We define the g-function by

$$g(z) = \int_{\mathcal{S}} \log(z-x)d\mu(x), \tag{6.5}$$

where the principal branch of the logarithm is taken, meaning g is analytic on $\mathbb{C}\setminus(-\infty,4]$. The g-function possesses the following properties

$$g_+(x) + g_-(x) - x - \ell = 0, \qquad x \in \mathcal{S}, \tag{6.6}$$

$$2g(x) - x - \ell < 0, \qquad x \in (4,\infty), \tag{6.7}$$

$$g_+(x) - g_-(x) = 2\pi i \int_x^4 \rho(s)ds, \qquad x \in \mathcal{S}, \tag{6.8}$$

$$g_+(x) - g_-(x) = 2\pi i, \qquad x \in (-\infty,0). \tag{6.9}$$

Let us also define

$$\xi(z) = -\pi \int_4^z \tilde{\rho}(s)ds \qquad \text{for } z \in \mathbb{C}\setminus(-\infty,4], \tag{6.10}$$

where the integration path does not cross $(-\infty,4]$, and where $\tilde{\rho}(z) = \frac{1}{2\pi}\sqrt{\frac{z-4}{z}}$ is analytic in $\mathbb{C}\setminus\overline{\mathcal{S}}$ and such that $\tilde{\rho}_\pm(x) = \pm i\rho(x)$ for $x \in \mathcal{S}$. By (6.6) and (6.8) we have,

$$2\xi_\pm(x) = \pm(g_+(x) - g_-(x)) = 2g_\pm(x) - x - \ell, \qquad x \in \mathcal{S}. \tag{6.11}$$

By analytically continuing $\xi - g$ on the whole complex plane from the above expression, we obtain the identity

$$2\xi(z) = 2g(z) - z - \ell, \qquad z \in \mathbb{C}\setminus(-\infty,4]. \tag{6.12}$$

The jumps of ξ follow from those of g, we have

$$\xi_+(x) + \xi_-(x) = 0, \qquad x \in \mathcal{S}, \tag{6.13}$$

$$2\xi(x) < 0, \qquad x \in (4,\infty), \tag{6.14}$$

$$\xi_+(x) - \xi_-(x) = 2\pi i \int_x^4 \rho(s)ds, \qquad x \in \mathcal{S}, \tag{6.15}$$

$$\xi_+(x) - \xi_-(x) = 2\pi i, \qquad x \in (-\infty,0). \tag{6.16}$$

6.1 Transformation to constant jumps

The weight (1.14) is defined on (yr,∞). We consider its natural extension

$$w(z) = (z-y)^2 z^\alpha e^{-nz}, \qquad z \in \mathbb{C}\setminus(-\infty,0], \tag{6.17}$$

where the principal branch is taken for the root, and we define $\Psi(z) = Y(z)w(z)^{\frac{\sigma_3}{2}}$. The matrix function Ψ satisfies the following RH problem.

RH problem for Ψ

(a) $\Psi : \mathbb{C} \setminus ((-\infty, 0] \cup \{y\} \cup [yr, \infty)) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Let $j_\Psi(z) := \Psi_-(z)^{-1}\Psi_+(z)$. Then,

$$j_\Psi(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (yr, \infty). \tag{6.18}$$

$$j_\Psi(z) = e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0). \tag{6.19}$$

(c) $\Psi(z) = (I + \mathcal{O}(z^{-1}))z^{(n+\frac{\alpha+2}{2})\sigma_3} e^{-\frac{nz}{2}\sigma_3}$ as $z \to \infty$.

(d) Ψ has the following behaviour near 0, y and yr:

$$\Psi(z) = \mathcal{O}(1) z^{\frac{\alpha}{2}\sigma_3}, \qquad \text{as } z \to 0, \tag{6.20}$$

$$\Psi(z) = \mathcal{O}(1)(z - y)^{\sigma_3}, \qquad \text{as } z \to y, \tag{6.21}$$

$$\Psi(z) = \mathcal{O}(1) \begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix}, \qquad \text{as } z \to yr, \tag{6.22}$$

where in the three above asymptotics it can be verified that the $\mathcal{O}(1)$ terms are analytic in a neighbourhood of their respective point.

6.2 Opening of the lenses

We now perform the step of opening the lenses. Since the jumps for Ψ are constant, the lens contours are unconstrained. In a subsequent transformation we will use the g-function to normalise the RH problem at infinity, at which point the lens contours will be required to stay within a region in which they converge to the identity matrix as $n \to \infty$. Let D_0 and D_4 denote small but fixed open discs centred at 0 and 4 respectively and $U = D_0 \cup D_4$. Note that since $nyr \to 0$ as $n \to \infty$, the points 0, y and yr lie in D_0 for sufficiently large n. Let us define $\gamma = (yr, 4)$. We now make the transformation

$$S(z) := \Psi(z)K(z) \tag{6.23}$$

where K is a piecewise function designed to open the lens,

$$K(z) := \begin{cases} I, & \text{for } z \in \mathbb{C} \setminus (\Omega_+^\gamma \cup \Omega_-^\gamma), \\ \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } z \in \Omega_+^\gamma, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } z \in \Omega_-^\gamma. \end{cases} \tag{6.24}$$

The regions Ω_+^γ and Ω_-^γ are shown in Figure 3, as well as their boundaries $\partial\Omega_+^\gamma = \Sigma_+^\gamma \cup \gamma$ and $\partial\Omega_-^\gamma = \Sigma_-^\gamma \cup \gamma$. The function S satisfies the following RH problem.

RH problem for S

(a) $S : \mathbb{C} \setminus ((-\infty, 0] \cup \{y\} \cup [yr, \infty) \cup \Sigma^\gamma) \to \mathbb{C}^{2\times 2}$ is analytic, where $\Sigma^\gamma = \Sigma_+^\gamma \cup \Sigma_-^\gamma$ is shown in Figure 3.

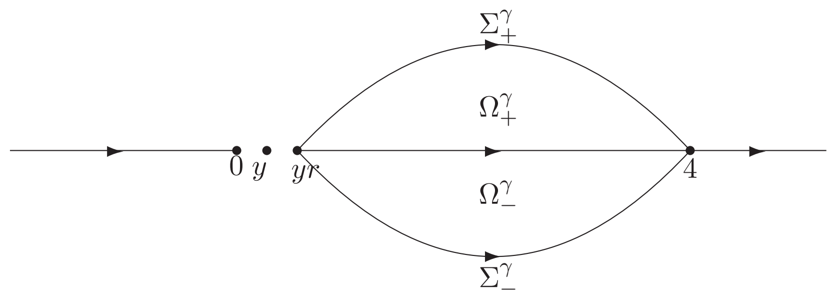

Figure 3: Jump contours for the RH problem for S. The lens contours are labelled Σ_+^γ and Σ_-^γ while the upper and lower lens regions are labelled Ω_+^γ and Ω_-^γ.

(b) Let $j_S(z) := S_-(z)^{-1}S_+(z)$. We have,

$$j_S(z) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \gamma, \tag{6.25}$$

$$j_S(z) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad z \in (4,\infty), \tag{6.26}$$

$$j_S(z) = e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0), \tag{6.27}$$

$$j_S(z) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad z \in \Sigma^\gamma. \tag{6.28}$$

(c) $S(z) = (I + \mathcal{O}(z^{-1}))z^{(n+\frac{\alpha+2}{2})\sigma_3}e^{-\frac{nz}{2}\sigma_3}$ as $z \to \infty$.

(d) Near 0, y, yr and 4, the behaviour of S takes the form

$$\begin{aligned} S(z) &= \mathcal{O}(1)z^{\frac{\alpha}{2}\sigma_3}, && \text{as } z \to 0, \\ S(z) &= \mathcal{O}(1)(z-y)^{\sigma_3}, && \text{as } z \to y, \\ S(z) &= \mathcal{O}(1)\begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix} K(z), && \text{as } z \to yr, \\ S(z) &= \mathcal{O}(1)K(z), && \text{as } z \to 4, \end{aligned} \tag{6.29}$$

where in the above asymptotics the $\mathcal{O}$ terms are analytic in a neighbourhood of their respective point.

6.3 Normalisation at infinity

The next transformation takes the form,

$$T(z) = e^{-\frac{n\ell\sigma_3}{2}}S(z) \times \begin{cases} e^{-n\xi(z)\sigma_3}, & \text{if } z \in \mathbb{C} \setminus \overline{U}, \\ I, & \text{if } z \in U. \end{cases} \tag{6.30}$$

The above transformation has the effect of normalising the problem at infinity.

RH problem for T

(a) $T : \mathbb{C} \setminus ((-\infty, 0] \cup \{y\} \cup [yr, \infty) \cup \Sigma^\gamma \cup \partial U) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) Let $j_T(z) := T_-(z)^{-1} T_+(z)$. Then,

$$\begin{aligned}
j_T(z) &= j_S(z), && z \in ((-\infty,0) \cup (yr,\infty) \cup \Sigma^\gamma) \cap U,\\
j_T(z) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && z \in \gamma \setminus \overline{U},\\
j_T(z) &= \begin{pmatrix} 1 & e^{2n\xi(z)} \\ 0 & 1 \end{pmatrix}, && z \in (4,\infty) \setminus \overline{U},\\
j_T(z) &= \begin{pmatrix} 1 & 0 \\ e^{-2n\xi(z)} & 1 \end{pmatrix}, && z \in \Sigma^\gamma \setminus \overline{U},\\
j_T(z) &= e^{\pi i \alpha \sigma_3}, && z \in (-\infty, 0),\\
j_T(z) &= e^{-n\xi(z)\sigma_3}, && z \in \partial U,
\end{aligned}$$

where the orientation of ∂U is clockwise.

(c) As $z \to \infty$,

$$T(z) = (1 + \mathcal{O}(z^{-1})) z^{\frac{\alpha+2}{2}\sigma_3}. \tag{6.31}$$

(d) Near the endpoints, the behaviour of T takes the form

$$\begin{aligned}
T(z) &= \mathcal{O}(1) z^{\frac{\alpha}{2}\sigma_3}, && \text{as } z \to 0,\\
T(z) &= \mathcal{O}(1)(z-y)^{\sigma_3}, && \text{as } z \to y,\\
T(z) &= \mathcal{O}(1) \begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix} K(z), && \text{as } z \to yr,\\
T(z) &= \mathcal{O}(1), && \text{as } z \to 4.
\end{aligned} \tag{6.32}$$

6.4 Global parametrix

Finally we define the global parametrix as a function $P^{(\infty)}$ satisfying the following RH problem.

RH problem for $P^{(\infty)}$

(a) $P^{(\infty)} : \mathbb{C} \setminus (-\infty, 4] \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(\infty)}$ has the jump relations

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \mathcal{S} \tag{6.33}$$

$$P_+^{(\infty)}(z) = P_-^{(\infty)}(z) e^{\pi i \alpha \sigma_3}, \qquad z \in (-\infty, 0). \tag{6.34}$$

(c) As $z \to \infty$,

$$P^{(\infty)}(z) = (I + \mathcal{O}(z^{-1})) z^{\frac{\alpha+2}{2}\sigma_3}. \tag{6.35}$$

(d) As $z \to \hat{z} \in \partial\mathcal{S} = \{0, 4\}$, we have

$$P^{(\infty)}(z) = \mathcal{O}((z-\hat{z})^{-\frac{1}{4}}). \tag{6.36}$$

The solution is explicitly given by

$$P^{(\infty)}(z) = N^{-1}\left(\frac{z-4}{z}\right)^{-\frac{\sigma_3}{4}} N\varphi\left(\frac{z}{2}-1\right)^{\frac{\alpha+2}{2}\sigma_3}, \tag{6.37}$$

where $\varphi(z) = z + \sqrt{z^2-1}$ is analytic in $\mathbb{C}\setminus[-1,1]$. We now need to construct local parametricies valid in the fixed open discs D_0 and D_4 around 0 and 4.

6.5 Local parametrix near 4

RH problem for $P^{(4)}$

(a) $P^{(4)} : D_4 \setminus (\Sigma^\gamma \cup \mathbb{R}) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(4)}$ has the jump relations

$$\begin{aligned} P_+^{(4)}(z) &= P_-^{(4)}(z)\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, && \text{on } (-\infty,4)\cap D_4,\\ P_+^{(4)}(z) &= P_-^{(4)}(z)\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}, && \text{on } (4,\infty)\cap D_4,\\ P_+^{(4)}(z) &= P_-^{(4)}(z)\begin{pmatrix} 1 & 0 \\ 1 & 1\end{pmatrix}, && \text{on } \Sigma^\gamma\cap D_4. \end{aligned} \tag{6.38}$$

(c) As $z\to 4$, $P^{(4)}(z) = \mathcal{O}(1)$.

(d) As $n\to\infty$, $P^{(4)}(z) = \left(I + \mathcal{O}(n^{-1})\right) P^{(\infty)}(z)e^{n\xi(z)\sigma_3}$ uniformly for $z \in \partial D_4$.

The solution $P^{(4)}$ can be constructed in term of the solution $\widetilde{\Upsilon}$ of the Airy model RH problem parametrix, which is presented in the appendix, see Subsection 9.1. The local parametrix inside D_4 can then be written as

$$P^{(4)}(z) = \widetilde{E}(z)\widetilde{\Upsilon}(n^{2/3}\widetilde{f}(z)), \tag{6.39}$$

where $\widetilde{f}(z)$ is given by

$$\widetilde{f}(z) := \left(-\frac{3}{2}\xi(z)\right)^{2/3}. \tag{6.40}$$

From the definition of ξ given by (6.10), $\widetilde{f}$ is a conformal map from a neighbourhood of 4 to a neighbourhood of 0. The matrix function $\widetilde{E}$ is defined in D_4 by

$$\widetilde{E}(z) = P^{(\infty)}(z)N^{-1}\widetilde{f}(z)^{\frac{\sigma_3}{4}}n^{\frac{\sigma_3}{6}}, \tag{6.41}$$

where the principal branch is taken for $(\cdot)^{\frac{1}{4}}$. It can be directly verified from the RH problem for $P^{(\infty)}$ and the definition of $\widetilde{f}$ that $\widetilde{E}$ is analytic in D_4, and from the properties of $\widetilde{\Upsilon}$ (presented in Subsection 9.1) that $P^{(4)}$ given by (6.39) satisfies indeed the above RH problem.

6.6 Local parametrix near 0

Inside D_0 we require a local parametrix satisfying the following RH problem.

RH problem for $P^{(0)}$

(a) $P^{(0)} : D_0 \setminus ((-\infty,0] \cup \{y\} \cup [yr,\infty) \cup \Sigma^\gamma) \to \mathbb{C}^{2\times 2}$ is analytic.

(b) $P^{(0)}$ has the jump relations

$$\begin{aligned} P^{(0)}_+(z) &= P^{(0)}_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, && \text{on } (yr,\infty)\cap D_0, \\ P^{(0)}_+(z) &= P^{(0)}_-(z)e^{\pi i\alpha\sigma_3}, && \text{on } (-\infty,0)\cap D_0, \\ P^{(0)}_+(z) &= P^{(0)}_-(z)\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, && \text{on } \Sigma^\gamma\cap D_0. \end{aligned} \tag{6.42}$$

(c) Near 0, y and yr, the behaviour of $P^{(0)}$ takes the form

$$\begin{aligned} P^{(0)}(z) &= \mathcal{O}(1)z^{\frac{\alpha}{2}\sigma_3}, && \text{as } z\to 0, \\ P^{(0)}(z) &= \mathcal{O}(1)(z-y)^{\sigma_3}, && \text{as } z\to y, \\ P^{(0)}(z) &= \mathcal{O}(1)\begin{pmatrix} 1 & -\frac{\log(yr-z)}{2\pi i} \\ 0 & 1 \end{pmatrix}K(z), && \text{as } z\to yr, \end{aligned} \tag{6.43}$$

where in the above asymptotics the $\mathcal{O}$ terms are analytic in a neighbourhood of their respective point.

(d) As $n\to\infty$, $P^{(0)}(z) = (I+o(1))P^{(\infty)}(z)e^{n\xi(z)\sigma_3}$ uniformly for $z\in\partial D_0$.

The solution $P^{(0)}$ uses the model RH problem Φ presented in Section 3, with the parameters x and a chosen such that

$$x = n^2 f(y) \qquad \text{and} \qquad a = \frac{f(yr)}{f(y)}, \tag{6.44}$$

where f is given by

$$f(z) = -(\xi(z)-\xi(0))^2. \tag{6.45}$$

From the definition of ξ given by (6.10), f is a conformal map from a neighbourhood of 0 to a neighbourhood of 0, and satisfies $f'(0)=4$. Note that since $nyr\to 0$ as $n\to\infty$, this implies that $x = 4n^2y(1+\mathcal{O}(y))$ and $a = r(1+\mathcal{O}(yr))$ as $n\to\infty$.

Lemma 6.1 *As $n\to\infty$ and simultaneously $nyr\to 0$, the matrix function*

$$P^{(0)}(z) = E(z)\sigma_3\Phi(-n^2f(z);n^2f(y),f(yr)/f(y))\sigma_3 e^{\frac{\pi i\alpha}{2}\theta(z)\sigma_3} \tag{6.46}$$

satisfies the RH problem for $P^{(0)}$, where E is the analytic function in D_0 given by

$$E(z) = (-1)^n P^{(\infty)}(z)e^{-\frac{\pi i\alpha}{2}\theta(z)\sigma_3}N(-f(z))^{\frac{\sigma_3}{4}}n^{\frac{\sigma_3}{2}}, \tag{6.47}$$

the function f is given by (6.45), and $\Phi(z;x,a)$ is the model RH problem introduced in Section 3. The $o(1)$ in the condition (d) of the RH problem for $P^{(0)}$ can be specified as

$$\mathcal{O}(\max\{n^{-1},nry\}) = \begin{cases} \mathcal{O}(n^{-1}), & \textit{if } (n^2y,r) \textit{ are in a compact subset of } (0,\infty)\times(1,\infty), \\ \mathcal{O}(n^{-1}), & \textit{if } n^2ry\to 0, \\ \mathcal{O}(nry), & \textit{if } nry\to 0 \textit{ and } (r-1)n^2y\to\infty. \end{cases} \tag{6.48}$$

Proof. The analyticity of E inside D_0 follows from the RH problem for $P^{(\infty)}$ and the definition of f. By definition of Φ (see Section 3), $P^{(0)}$ satisfies the RH problem for P^0. The explicit forms of $o(1)$ given by (6.48) follow from (3.9) (for (x,a) in a compact subset of $(0,\infty)\times(1,\infty)$), (5.99) (for $ax\to 0$) and (5.51) (for $(a-1)x\to\infty$ and $\frac{ax}{n}\to 0$). □

Finally, we define $R(z)$ as follows

$$R(z)=\begin{cases}T(z)P^{(\infty)}(z)^{-1}, & \text{for } z\in\mathbb{C}\setminus\overline{U},\\ T(z)P^{(0)}(z)^{-1}, & \text{for } z\in D_0,\\ T(z)P^{(4)}(z)^{-1}, & \text{for } z\in D_4.\end{cases} \tag{6.49}$$

Using the above definition we can derive the following RH problem for R.

RH problem for R

(a) $R:\mathbb{C}\setminus\Sigma_R\to\mathbb{C}^{2\times 2}$ is analytic, where $\Sigma_R=((4,\infty)\cup\Sigma^\gamma\cup\partial U)\setminus U$.

(b) Let $j_R(z):=R_-(z)^{-1}R_+(z)$. We have

$$j_R(z)=P^{(\infty)}(z)\begin{pmatrix}1 & 0\\ e^{-2n\xi(z)} & 1\end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z\in\Sigma^\gamma\setminus\overline{U}, \tag{6.50}$$

$$j_R(z)=P^{(\infty)}(z)\begin{pmatrix}1 & e^{2n\xi(z)}\\ 0 & 1\end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z\in(4,\infty)\setminus\overline{D_4}, \tag{6.51}$$

$$j_R(z)=P^{(4)}(z)\begin{pmatrix}e^{-n\xi(z)} & 0\\ 0 & e^{n\xi(z)}\end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z\in\partial D_4, \tag{6.52}$$

$$j_R(z)=P^{(0)}(z)\begin{pmatrix}e^{-n\xi(z)} & 0\\ 0 & e^{n\xi(z)}\end{pmatrix}P^{(\infty)}(z)^{-1}, \qquad z\in\partial D_0. \tag{6.53}$$

(c) As $z\to\infty$, we have $R(z)=I+\dfrac{R_1}{z}+\mathcal{O}(z^{-2})$. (6.54)

(d) As $z\to b\in\{0,y,yr,4\}$, $R(z)=\mathcal{O}(1)$.

From (6.37), (6.39) and (6.46), as $n\to\infty$ we have

$$j_R(z)=I+\mathcal{O}(\max\{n^{-1},nry\}), \qquad \text{uniformly for } z\in\partial D_0, \tag{6.55}$$

$$j_R(z)=I+\mathcal{O}(n^{-1}), \qquad \text{uniformly for } z\in\partial D_4, \tag{6.56}$$

$$j_R(z)=I+\mathcal{O}(e^{-cn}), \qquad \text{uniformly for } z\in\Sigma_R\setminus\partial U. \tag{6.57}$$

where $c>0$ is a constant independent of n. It follows from standard theory for small norm RH problem that R exists for n sufficiently large and satisfies

$$R(z)=I+\mathcal{O}(\max\{n^{-1},nry\}) \qquad \text{and} \qquad \partial_z R(z)=\mathcal{O}(\max\{n^{-1},nry\}), \tag{6.58}$$

uniformly for z in compact subsets of $\mathbb{C}\setminus\Sigma_R$.

6.7 Computation of R_1

The quantity R_1 can be computed via a perturbative calculation. From (6.55) and (6.56) we can write for $z \in \Sigma_R$,

$$j_R(z) = I + \frac{1}{n} J_1(z) + \mathcal{O}(n^{-2}), \qquad n \to \infty, \tag{6.59}$$

where the matrix $J_1(z)$ is non-zero only on ∂U, satisfies $J_1(z) = \mathcal{O}(1)$ uniformly for $z \in \partial D_4$ and

$$J_1(z) = \mathcal{O}(\max\{1, n^2 ry\}), \qquad \text{uniformly for } z \in \partial D_0. \tag{6.60}$$

Therefore, by a perturbative analysis of R, we have

$$R(z) = I + R^{(1)}(z) n^{-1} + \mathcal{O}(\max\{n^{-2}, (nry)^2\}), \qquad z \in \mathbb{C} \setminus \Sigma_R, \tag{6.61}$$

where

$$R^{(1)}(z) = \mathcal{O}(\max\{1, n^2 ry\}), \qquad \text{uniformly for } z \in \mathbb{C} \setminus \Sigma_R. \tag{6.62}$$

The quantity $R^{(1)}(z)$ may be expressed in terms of $J_1(z)$ by substituting (6.59) into the jump relation $R_+(z) = R_-(z) j_R(z)$, from which we obtain the following RH problem for $R^{(1)}$.

RH problem for $R^{(1)}$

(a) $R^{(1)} : \mathbb{C} \setminus \partial U \to \mathbb{C}^{2\times 2}$ is analytic,

(b) $R^{(1)}_+(z) = R^{(1)}_-(z) + J_1(z)$ for $z \in \partial U$,

(c) $R^{(1)}(z) \to 0$ as $z \to \infty$.

The above RH problem can be solved explicitly in terms of a Cauchy transform,

$$R^{(1)}(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{J_1(\xi)}{\xi - z} d\xi, \tag{6.63}$$

where the integral is taken entry-wise. Explicit computations using (3.9), (6.46) and (6.53) gives for $z \in \partial D_0$

$$J_1(z) = \frac{v(n^2 f(y); f(ry)/f(y))}{2\sqrt{-f(z)}} P^{(\infty)}(z) \begin{pmatrix} -1 & -ie^{-i\pi\alpha\theta(z)} \\ -ie^{i\pi\alpha\theta(z)} & 1 \end{pmatrix} P^{(\infty)}(z)^{-1}. \tag{6.64}$$

The function v is the special function appearing in the model problem Φ. By using (6.39), (6.52) and (9.2), the term $J_1(z)$ on ∂D_4 is given by,

$$J_1(z) = \frac{1}{8\widetilde{f}(z)^{3/2}} P^{(\infty)}(z) \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix} P^{(\infty)}(z)^{-1}. \tag{6.65}$$

Putting (6.64) and (6.65) in (6.63) gives (after a residue calculation)

$$\begin{aligned} R^{(1)}(z) &= \frac{v(n^2 f(y); f(ry)/f(y))}{2z} \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} + \frac{5}{12(z-4)^2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix} \\ &+ \frac{1}{16(z-4)} \begin{pmatrix} 1 - 4(\alpha+2)^2 & \frac{i}{3}\left(12(\alpha+2)^2 + 24(\alpha+2) + 11\right) \\ \frac{i}{3}\left(12(\alpha+2)^2 - 24(\alpha+2) + 11\right) & 4(\alpha+2)^2 - 1 \end{pmatrix}. \end{aligned} \tag{6.66}$$

Therefore, we have

$$R_1 = \lim_{z\to\infty} z(R(z)-I) = \frac{v(n^2 f(y); f(ry)/f(y))}{2n}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix} +$$
$$\frac{1}{16n}\begin{pmatrix}1-4(\alpha+2)^2 & \frac{i}{3}\left(12(\alpha+2)^2+24(\alpha+2)+11\right)\\ \frac{i}{3}\left(12(\alpha+2)^2-24(\alpha+2)+11\right) & 4(\alpha+2)^2-1\end{pmatrix} + \mathcal{O}(\max\{n^{-2},(nry)^2\}). \tag{6.67}$$

In particular, one has

$$n\mathrm{Tr}(R_1\sigma_3) = v(n^2 f(y); f(ry)/f(y)) - v(0) + \mathcal{O}(\max\{n^{-1}, n^{-1}(rn^2y)^2\}) \qquad \text{as } n\to\infty, \tag{6.68}$$

where $v(0) = \frac{1}{8}(4(\alpha+2)^2-1)$.

7 Proof of Theorem 1.1

Let us define $s := 4n^2 y$, which is a rescaling of y. In order to use Lemma 2.2, we need to compute large n asymptotics for $\mathrm{Tr}(Y^{-1}(z)\partial_z Y(z)\sigma_3)$ uniformly for z in a neighbourhood of ∞. The large n analysis for Y done in Section 6 is valid when the parameters y and r satisfy fall in one of the three cases presented in (1.16), (1.17) and (1.18). For large z, by (6.23), (6.30), (6.37) and (6.49), we have

$$Y(z) = e^{\frac{n\ell\sigma_3}{2}} R(z) P^{(\infty)}(z) e^{n\xi(z)\sigma_3} w(z)^{-\frac{\sigma_3}{2}}. \tag{7.1}$$

Using the above expression for Y, we obtain

$$\begin{aligned}\mathrm{Tr}(Y^{-1}(z)\partial_z Y(z)\sigma_3) \quad = \quad & -\partial_z \log w(z) + 2n\partial_z\xi(z)\\ & +\mathrm{Tr}(P^{(\infty)}(z)^{-1}\partial_z P^{(\infty)}(z)\sigma_3)\\ & +\mathrm{Tr}(P^{(\infty)}(z)^{-1}R(z)^{-1}\partial_z R(z)P^{(\infty)}(z)\sigma_3).\end{aligned}$$

Using (6.37), as $z\to\infty$ we have

$$\mathrm{Tr}(P^{(\infty)}(z)^{-1}\partial_z P^{(\infty)}(z)\sigma_3) = \frac{\alpha+2}{z} + \frac{2\alpha+4}{z^2} + \mathcal{O}(z^{-3}) \tag{7.2}$$

Similarly, (6.10) and (6.17) give

$$-\partial_z \log w(z) + 2n\partial_z\xi(z) = 2n\left(\frac{1}{z}+\frac{1}{z^2}\right) - \frac{\alpha+2}{z} - \frac{2y}{z^2} + \mathcal{O}(z^{-3}), \quad \text{as } z\to\infty. \tag{7.3}$$

Also, by (6.37) and (6.54), we have

$$\mathrm{Tr}(P^{(\infty)}(z)^{-1}R(z)^{-1}\partial_z R(z)P^{(\infty)}(z)\sigma_3) = -\frac{\mathrm{Tr}(R_1\sigma_3)}{z^2} + \mathcal{O}(z^{-3}), \qquad \text{as } z\to\infty. \tag{7.4}$$

Using the above expressions and Lemma 2.2 gives,

$$z\mathrm{Tr}(Y^{-1}(z)\partial_z Y(z)\sigma_3) = 2n + \frac{2(\alpha+2)+2n-2y-\mathrm{Tr}(R_1\sigma_3)}{z} + \mathcal{O}(z^{-2}), \quad \text{as } z\to\infty, \tag{7.5}$$

$$\partial_y \log Z_{n,\alpha}(y;r) = n + \frac{n}{2y}\mathrm{Tr}(R_1\sigma_3). \tag{7.6}$$

By using (6.68) in (7.6) and rewriting it in terms of $s = 4n^2y$, we have as $n \to \infty$ that

$$\begin{aligned}
\partial_s \log Z_{n,\alpha}\left(\frac{s}{4n^2};r\right) &= \frac{1}{4n} + \frac{n}{2s}\mathrm{Tr}(R_1\sigma_3), \\
&= \frac{1}{2s}\left(v\left(n^2 f(\tfrac{s}{4n^2}); \frac{f(\frac{rs}{4n^2})}{f(\frac{s}{4n^2})}\right) - v(0)\right) + \frac{1}{s}\mathcal{O}(\max\{n^{-1}, \tfrac{(rs)^2}{n}\}), \\
&= \frac{1}{2s}\left(v(s;r) - v(0)\right) + \frac{1}{s}\mathcal{O}(\max\{n^{-1}, \tfrac{(rs)^2}{n}\}).
\end{aligned} \tag{7.7}$$

Let us fix r. By integrating the left-hand side of (7.7) from $\epsilon > 0$ to a certain $s > \epsilon$ we obtain

$$\log\left(\frac{Z_{n,\alpha}(\frac{s}{4n^2};r)}{Z_{n,\alpha}(\frac{\epsilon}{4n^2};r)}\right) = \int_\epsilon^s \partial_x \log Z_{n,\alpha}\left(\frac{x}{4n^2};r\right)dx. \tag{7.8}$$

Since the function $y \in [0,\infty) \mapsto Z_{n,\alpha}(y;r)$ is continuous, and since

$$Z_{n,\alpha}(0;r) = \widehat{Z}_{n,\alpha+2} > 0, \tag{7.9}$$

the left-hand-side of (7.8) is bounded as $\epsilon \to 0$, and thus the same is true for the right-hand side. In order to use (7.7) in (7.8), it is important to note that $\mathcal{O}$ term of (7.7) is uniform when s is in a compact subset of $(0,\infty)$ and also as $s \to 0$. Thus, we obtain

$$\log\left(\frac{Z_{n,\alpha}(\frac{s}{4n^2};r)}{\widehat{Z}_{n,\alpha+2}}\right) = \frac{1}{2}\int_0^s [v(x;r) - v(0)]\frac{dx}{x} + \mathcal{O}\big(\max\{n^{-1}, \tfrac{(rs)^2}{n}\}\big). \tag{7.10}$$

By an integration by parts, we have

$$I(s;r) := \int_0^s [v(x;r)-v(0)]\frac{dx}{2x} = \frac{1}{2}\int_0^s v'(x;r)\log\left(\frac{s}{x}\right)dx = -\frac{1}{4}\int_0^s \left(q_1^2(x;r)+rq_2^2(x;r)\right)\log\left(\frac{s}{x}\right)dx,$$

where for the last equality we have used (4.22) and (4.23). This completes the proof of Theorem 1.1. Note that (7.10) can be rewritten as

$$Z_{n,\alpha}\left(\frac{s}{4n^2};r\right) = \widehat{Z}_{n,\alpha+2}e^{I(s;r)}\left(1 + \mathcal{O}\big(\max\{n^{-1}, \tfrac{(rs)^2}{n}\}\big)\right). \tag{7.11}$$

8 Proof of Theorem 1.7

From (1.12) by changing variables $x := 4(n-1)ny$, we obtain

$$Q_{n,\alpha}(r) = \frac{\widehat{Z}_{n-1,\alpha+2}}{\widehat{Z}_{n,\alpha}}\left(\frac{n-1}{n}\right)^{(n-1)(n+1+\alpha)}(4(n-1)n)^{-1-\alpha}\int_0^\infty \frac{x^\alpha e^{-\frac{x}{n}}}{\widehat{Z}_{n-1,\alpha+2}} Z_{n-1,\alpha}\left(\frac{x}{4(n-1)^2};r\right)dx. \tag{8.1}$$

It is known that (see [21, formula 17.6.5])

$$\widehat{Z}_{n,\alpha} = \frac{1}{n!}n^{-n^2-\alpha n}\prod_{j=1}^n j!\Gamma(j+\alpha), \tag{8.2}$$

and therefore we have

$$\begin{aligned}
\frac{\widehat{Z}_{n-1,\alpha+2}}{\widehat{Z}_{n,\alpha}} &= \frac{(n-1)^{\alpha+1}\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2)}\left(\frac{n}{n-1}\right)^{n^2+\alpha n}, \\
&= \frac{n^{2\alpha+2}}{\Gamma(\alpha+1)\Gamma(\alpha+2)}\left(\frac{n}{n-1}\right)^{n^2+\alpha n}\left(1+\mathcal{O}\left(n^{-1}\right)\right), \qquad \text{as } n \to \infty.
\end{aligned} \tag{8.3}$$

We can thus simplify expression (8.1) as

$$Q_{n,\alpha}(r)=\frac{1+\mathcal{O}(n^{-1})}{4^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2)}(I_1+I_2), \tag{8.4}$$

where

$$I_1=\int_0^M x^\alpha e^{-\frac{x}{n}}e^{I(x;r)}dx, \qquad I_2=\int_M^\infty \frac{x^\alpha e^{-\frac{x}{n}}}{\widehat{Z}_{n-1,\alpha+2}}Z_{n-1,\alpha}\left(\frac{x}{4(n-1)^2};r\right)dx, \tag{8.5}$$

and $M>0$ is a constant. The asymptotics (1.28) implies that for any $\epsilon>0$, we have

$$\frac{Z_{n-1,\alpha}(\frac{x}{4(n-1)^2};r)}{\widehat{Z}_{n-1,\alpha+2}}=\mathcal{O}\big(e^{-(\frac{1}{4}-\epsilon)rx}\big), \qquad \text{as} \qquad \frac{rx}{n}\to 0, \quad (r-1)x\to\infty. \tag{8.6}$$

Note that because of the restriction $\frac{rx}{n}\to 0$, this asymptotic formula alone is not sufficient to estimate I_2. Nevertheless, we can derive the following inequality

$$\begin{aligned}\frac{Z_{n-1,\alpha}(\frac{rx}{4(n-1)^2};r)}{\widehat{Z}_{n-1,\alpha+2}} &= \frac{(n-1)!^{-1}}{\widehat{Z}_{n-1,\alpha+2}}\int_{\frac{rx}{4(n-1)^2}}^\infty \cdots \int_{\frac{rx}{4(n-1)^2}}^\infty \Delta_{n-1}(\lambda)^2\prod_{i=1}^{n-1}\left(\lambda_i-\frac{x}{4(n-1)^2}\right)^2\lambda_i^\alpha e^{-(n-1)\lambda_i}d\lambda_i\\ &\le \frac{1}{(n-1)!\widehat{Z}_{n-1,\alpha+2}}\int_{\frac{rx}{4(n-1)^2}}^\infty \cdots \int_{\frac{rx}{4(n-1)^2}}^\infty \Delta_{n-1}(\lambda)^2\prod_{i=1}^{n-1}\lambda_i^{\alpha+2}e^{-(n-1)\lambda_i}d\lambda_i\\ &= \mathbb{P}_{n-1,\alpha+2}\left(\lambda_{\min}>\frac{rx}{4(n-1)^2}\right).\end{aligned} \tag{8.7}$$

It is well-known [29] that the above quantity is bounded by e^{-Cxr} for sufficiently large x as $n\to\infty$ (this is a large deviation principle), and where $C>0$ is a constant. Combining (8.6) and (8.7), we thus have

$$I_2 \;\le\; \int_M^\infty x^\alpha e^{-\frac{x}{n}}e^{-Cxr}dx\le e^{-\frac{C}{2}Mr}, \tag{8.8}$$

if M is chosen big enough. Therefore we obtain,

$$\lim_{n\to\infty}Q_{n,\alpha}(r)=\frac{1}{4^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+2)}\int_0^M x^\alpha e^{I(x;r)}dx+\mathcal{O}\big(e^{-\frac{C}{2}Mr}\big). \tag{8.9}$$

Letting $M\to\infty$ in (8.9) finishes the proof of Theorem 1.7.

9 Appendix

9.1 Airy model RH problem

(a) $\widetilde{\Upsilon}(z):\mathbb{C}\setminus\Sigma_A\to\mathbb{C}^{2\times 2}$ is analytic where Σ_A is shown in Figure 4.

(b) $\widetilde{\Upsilon}$ has the jump relations

$$\begin{aligned}\widetilde{\Upsilon}_+(z)&=\widetilde{\Upsilon}_-(z)\begin{pmatrix}0&1\\-1&0\end{pmatrix}, && \text{on } \mathbb{R}^-,\\ \widetilde{\Upsilon}_+(z)&=\widetilde{\Upsilon}_-(z)\begin{pmatrix}1&1\\0&1\end{pmatrix}, && \text{on } \mathbb{R}^+,\\ \widetilde{\Upsilon}_+(z)&=\widetilde{\Upsilon}_-(z)\begin{pmatrix}1&0\\1&1\end{pmatrix}, && \text{on } e^{\frac{2\pi}{3}i}\mathbb{R}^+,\\ \widetilde{\Upsilon}_+(z)&=\widetilde{\Upsilon}_-(z)\begin{pmatrix}1&0\\1&1\end{pmatrix}, && \text{on } e^{-\frac{2\pi}{3}i}\mathbb{R}^+.\end{aligned} \tag{9.1}$$

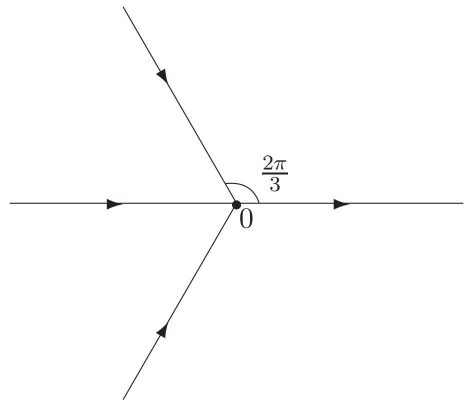

Figure 4: The jump contour Σ_A for $\tilde{\Upsilon}$.

(c) As $z \to \infty$, we have

$$\widetilde{\Upsilon}(z) = z^{-\frac{\sigma_3}{4}} N \left(I + \frac{1}{z^{3/2}} \widetilde{\Upsilon}_1 + \mathcal{O}(z^{-3}) \right) e^{-\frac{2}{3} z^{3/2} \sigma_3}, \tag{9.2}$$

with

$$\widetilde{\Upsilon}_1 = \frac{1}{8} \begin{pmatrix} \frac{1}{6} & i \\ i & -\frac{1}{6} \end{pmatrix}. \tag{9.3}$$

(d) As $z \to 0$, $\widetilde{\Upsilon}(z) = \mathcal{O}(1)$.

The following matrix-valued function solves the above Airy model RH problem (see [8, 6]):

$$\widetilde{\Upsilon}(z) := M_A \times \begin{cases} \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } 0 < \arg z < \frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & \mathrm{Ai}(\omega^2 z) \\ \mathrm{Ai}'(z) & \omega^2 \mathrm{Ai}'(\omega^2 z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, & \text{for } \frac{2\pi}{3} < \arg z < \pi, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } -\pi < \arg z < -\frac{2\pi}{3}, \\ \begin{pmatrix} \mathrm{Ai}(z) & -\omega^2 \mathrm{Ai}(\omega z) \\ \mathrm{Ai}'(z) & -\mathrm{Ai}'(\omega z) \end{pmatrix} e^{-\frac{\pi i}{6}\sigma_3}, & \text{for } -\frac{2\pi}{3} < \arg z < 0, \end{cases} \tag{9.4}$$

with $\omega = e^{\frac{2\pi i}{3}}$, Ai the Airy function and

$$M_A := \sqrt{2\pi} e^{\frac{\pi i}{6}} \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}. \tag{9.5}$$

9.2 Bessel model RH problem

This RH problem depends on a parameter $\alpha \in \mathbb{R}$.

(a) $\Upsilon = \Upsilon^{(\alpha)} : \mathbb{C} \setminus \Sigma_{0,0}$ is analytic.

(b) Υ has the following jumps on $\Sigma_{0,0} \setminus \{0\}$:

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in \Sigma_2, \tag{9.6}$$

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 1 & 0 \\ e^{\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_1, \tag{9.7}$$

$$\Upsilon_+(z) = \Upsilon_-(z) \begin{pmatrix} 1 & 0 \\ e^{-\pi i \alpha} & 1 \end{pmatrix}, \qquad z \in \Sigma_3, \tag{9.8}$$

(c) As $z \to \infty$,

$$\Upsilon(z) = \left(I + \frac{1}{z}\Upsilon_1 + \mathcal{O}(z^{-2}) \right) z^{-\frac{1}{4}\sigma_3} N e^{z^{1/2}\sigma_3}. \tag{9.9}$$

(d) As $z \to 0$,

(i) If $\alpha < 0$, $\Upsilon(z) = \mathcal{O}\begin{pmatrix} |z|^{\frac{\alpha}{2}} & |z|^{\frac{\alpha}{2}} \\ |z|^{\frac{\alpha}{2}} & |z|^{\frac{\alpha}{2}} \end{pmatrix}$.

(ii) If $\alpha = 0$, $\Upsilon(z) = \mathcal{O}\begin{pmatrix} \log|z| & \log|z| \\ \log|z| & \log|z| \end{pmatrix}$.

(iii) If $\alpha > 0$,

$$\Upsilon(z) = \begin{cases} \mathcal{O}\begin{pmatrix} |z|^{\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \\ |z|^{\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \end{pmatrix}, & \text{for } -\frac{2\pi}{3} < \arg(z) < \frac{2\pi}{3}, \\ \mathcal{O}\begin{pmatrix} |z|^{-\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \\ |z|^{-\frac{\alpha}{2}} & |z|^{-\frac{\alpha}{2}} \end{pmatrix}, & \text{for } \arg(z) \in (-\pi, -\frac{2\pi}{3}) \cup (\frac{2\pi}{3}, \pi). \end{cases} \tag{9.10}$$

The solution of this RH problem is explicitly given in terms of the Bessel functions (see [19] or [4]), one has

$$\Upsilon^{(\alpha)}(z) = \begin{pmatrix} 1 & 0 \\ \frac{i}{8}(4\alpha^2+3) & 1 \end{pmatrix} \pi^{\sigma_3/2} \begin{pmatrix} I_\alpha(z^{1/2}) & \frac{i}{\pi}K_\alpha(z^{1/2}) \\ \pi i z^{1/2} I'_\alpha(z^{1/2}) & -z^{1/2}K'_\alpha(z^{1/2}) \end{pmatrix} H_0(z), \tag{9.11}$$

and

$$(\Upsilon_1)_{12} = \frac{i}{8}(2\alpha-1)(2\alpha+1). \tag{9.12}$$

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References

[1] G. Akemann, J. Baik and P. D. Francesco, *The Oxford Handbook of Random Matrix Theory*, Oxford Univ. Press, Oxford, 2011.

[2] G.W. Anderson, A. Guionnet, and O. Zeitouni, *An introduction to random matrices*, Cambridge studies in advanced mathematics **118** (2010).

[3] M. Atkin, A Riemann-Hilbert problem for equations of Painlevé type in the one matrix model with semi-classical potential, to appear in *Comm. Math. Phys.*

[4] M. Atkin, T. Claeys, F. Mezzadri, Random matrix ensembles with singularities and a hierarchy of Painlevé III equations, *Int. Math. Res. Notices* 2015, doi: 10.1093/imrn/rnv195.

[5] Z. Bai, J.W. Silverstein. *Spectral Analysis of Large Dimensional Random Matrices. Second Edition.* **20** (2010), Springer, New York.

[6] P. Bleher, Lectures on random matrix models. The Riemann-Hilbert approach. Random matrices, random processes and integrable systems (2011), 251–349, *CRM Ser. Math. Phys.*, Springer, New York.

[7] C. Charlier and A. Doeraene, The generating function for the Bessel point process and a system of coupled Painlevé V equations, preprint.

[8] T. Claeys, T. Grava, K. T-R McLaughlin, Asymptotics for the partition function in two-cut random matrix models, *Comm. Math. Phys.* **339** (2015), no. 2, 513–587.

[9] P. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach*, Amer. Math. Soc. **3** (2000).

[10] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.

[11] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.

[12] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. *Bull. Amer. Math. Soc. (N.S.)* **26** (1992), no. 1, 119–123.

[13] A.S. Fokas, A.R. Its, A.A. Kapaev and V.Y. Novokshenov, *Painlevé Transcendents: The Riemann-Hilbert Approach*, American Mathematical Soc. **128**, 2006.

[14] A.S. Fokas, A.R. Its, and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* **147** (1992), no. 2, 395–430.

[15] P. J. Forrester, T. D. Hughes, Complex Wishart matrices and conductance in mesoscopic systems: exact results, *J. Math. Phys.* **35** (1994), no. 12, 6736–6747.

[16] P. J. Forrester, N. S. Witte, The distribution of the first eigenvalue spacing at the hard edge of the Laguerre unitary ensemble, *Kyushu J. Math.* **61** (2007), no. 2, 457–526.

[17] S. P. Hastings, J. B. McLeod, A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Arch. Rational Mech. Anal.* **73** (1980), no. 1, 31–51.

[18] J.P. Keating and N.C. Snaith, Random matrix theory and $\zeta(1/2+it)$, *Comm. Math. Phys.* **214** (2001), 57–89.

[19] A.B.J. Kuijlaars, K. T.-R. McLaughlin, W. Van Assche, M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$. *Adv. Math.* **188** (2004), no. 2, 337–398.

[20] V. A. Marchenko, L. A. Pastur, Distribution of eigenvalues for some sets of random matrices, *Mat. Sb. (N.S.)* **72** (1967), 507–536.

[21] Mehta M.L., Random matrices, Third Edition, *Pure and Applied Mathematics Series* **142**, Elsevier Academic Press, 2004.

[22] F.W.J. Olver, A.B. Olde Daalhuis, D.W. Lozier, B.I. Schneider, R.F. Boisvert, C.W. Clark, B.R. Miller, and B.V. Saunders, NIST Digital Library of Mathematical Functions. http://dlmf.nist.gov/, Release 1.0.13 of 2016-09-16.

[23] A. Perret, G. Schehr, Near-extreme eigenvalues and the first gap of Hermitian random matrices, *J. Stat. Phys.* **156** (2014), no. 5, 843–876.

[24] A. Perret, G. Schehr, The density of eigenvalues seen from the soft edge of random matrices in the Gaussian β-ensembles, *Acta Phys. Polon. B* **46** (2015), no. 9, 1693–1707.

[25] G. Szegő, *Orthogonal polynomials*, AMS Colloquium Publ. **23** (1959), New York: AMS.

[26] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, Springer-Verlag (1997).

[27] T. Tao, *Topics in random matrix theory*, Amer. Math. Soc. **132** (2012).

[28] C. A. Tracy, H. Widom. Level-spacing distributions and the Airy kernel. *Phys. Lett. B* **305** (1993), no. 1-2, 115–118.

[29] C. A. Tracy, H. Widom. Level spacing distributions and the Bessel kernel. *Comm. Math. Phys.* **161** (1994), no. 2, 289–309.

[30] J. Wishart, The generalized product moment distribution in samples from a normal multivariate population, *Biometrika A* **20** (1928), 32–52.

[31] N.S. Witte, F. Bornemann, P.J. Forrester, Joint distribution of the first and second eigenvalues at the soft edge of unitary ensembles, *Nonlinearity* **26** (2013), no. 6, 1799–1822.

[32] S.-X. Xu, Y.-Q. Zhao, Critical edge behavior in the modified Jacobi ensemble and the Painlevé V transcendents, *J. Math. Phys.* **54** (2013), no. 8, 083304, 29 pp.

[33] S.-X. Xu, Y.-Q. Zhao, Critical edge behavior in the modified Jacobi ensemble and Painlevé equations, *Nonlinearity* **28** (2015), no. 6, 16331674.