

City University of New York (CUNY)

## CUNY Academic Works

---

Open Educational Resources

Queensborough Community College

---

2020

### Trigonometry: A Brief Conversation

Carolyn D. King PhD

*CUNY Queensborough Community College*

Tam Evelyn

*QCC*

Fei Ye PhD

*QCC*

Beata Ewa Carvajal

*QCC*

[How does access to this work benefit you? Let us know!](#)

More information about this work at: [https://academicworks.cuny.edu/qb\\_oers/167](https://academicworks.cuny.edu/qb_oers/167)

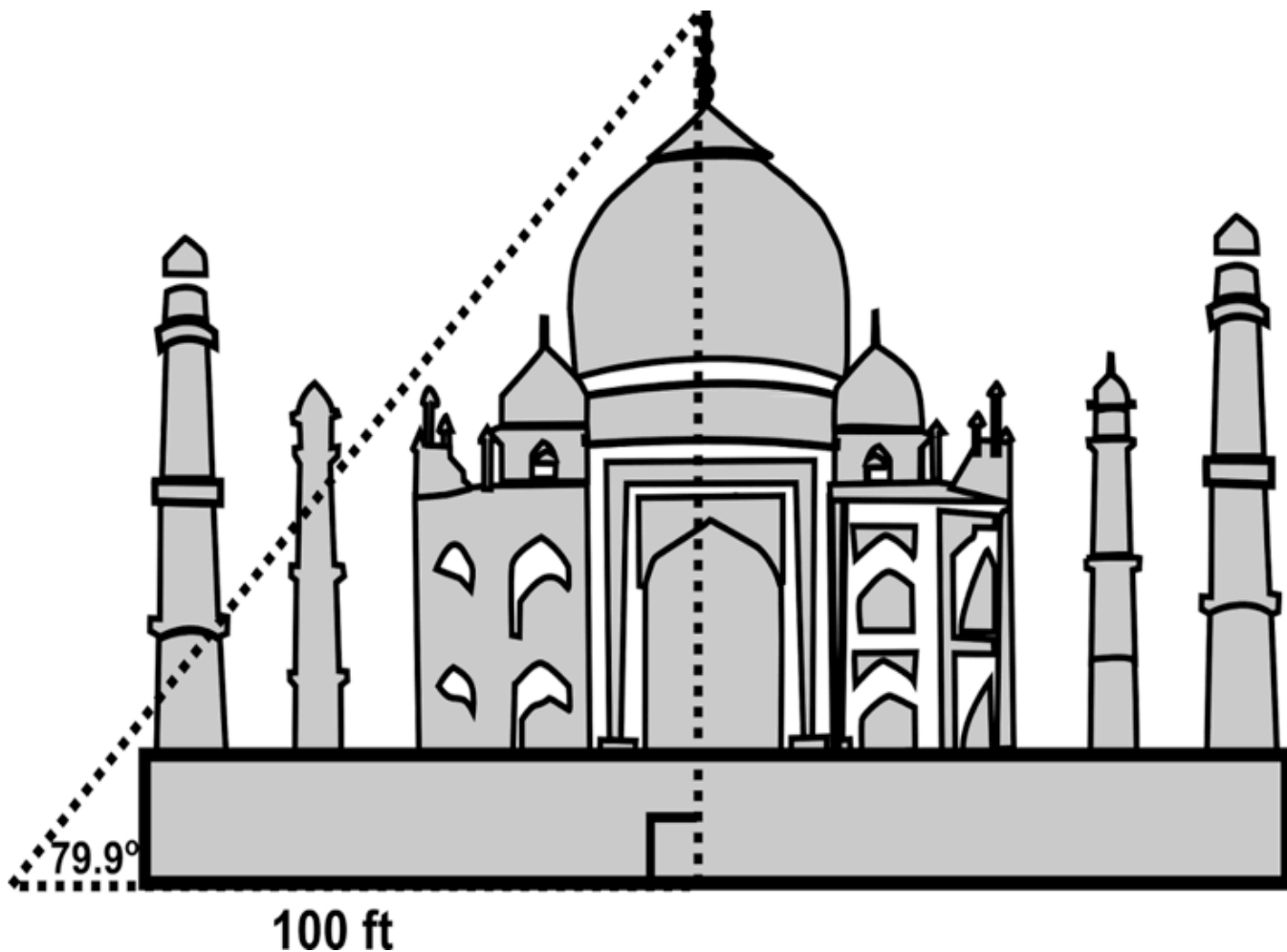
Discover additional works at: <https://academicworks.cuny.edu>

---

This work is made publicly available by the City University of New York (CUNY).

Contact: [AcademicWorks@cuny.edu](mailto:AcademicWorks@cuny.edu)

# *Trigonometry: A Brief Conversation*



Carolyn King

Evelyn Tam

Fei Ye

Beata Ewa Carvajal

2020 Edition

These five units are specifically tailored to foster the mastery of a few selected trigonometry topics that comprise the one credit MA-121 Elementary Trigonometry course. Each unit introduces the topic, provides space for practice, but more importantly, provides opportunities for students to reflect on the work in order to deepen their conceptual understanding.

These units have also been assigned to students of other courses such as pre-calculus and calculus as a review of trigonometric basics essential to those courses. This is the **second** edition of the materials. **Units 4 and 5 have been edited to reflect suggestions from instructors who have used the materials. We have also added answers to the odd numbered practice exercises.**

We are grateful for the support we received from the Open Educational Research (OER) initiative of the City University of New York (CUNY). This second edition was part of the OER Workshop-Cohort III.

We thank ALL of the MA-121 instructors for their invaluable input.

The image on the cover is a Clone of “Trig applied to the Taj Mahal” copied from ThingLink. Retrieved August 11, 2020, from <https://www.thinglink.com/scene/1350598687734104067>

This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License. To view a copy of this license, visit <http://creativecommons.org/licenses/by-nc-sa/4.0/>.



## ***Contents/ Learning Objectives***

### ***Unit 1: Angle Measure (page 5)***

- 1.1 Measure in Degrees
- 1.2 Measure in Radians
- 1.3 Conversion from Degrees to Radians
- 1.4 Conversion from Radians to Degrees

*Students will be able to draw angles in standard position, find co-terminal angles, and express angles in both degree and radian measurement.*

### ***Unit 2: Trigonometry of Right Triangles (page 25)***

- 2.1 Sides of a Right Triangle
- 2.2 Definitions of the Trigonometric Functions
- 2.3 The Pythagorean Identity
- 2.4 Applications

*Students will identify the adjacent and opposite sides of an angle, learn the definitions of the six trigonometric functions, recognize the co-functions, construct the two special right triangles, learn fundamental trigonometric identities, and use right triangle trigonometry to solve word problems.*

### ***Unit 3: Trigonometric Functions of Any Angle (page 49)***

- 3.1 Definitions of the Trigonometric Functions of Angles
- 3.2 Reference Angles
- 3.3 Using Reference Angles to Rewrite Trigonometric Functions
- 3.4 Finding the Exact Values of Trigonometric Functions

*Students will learn how to find the six trigonometric functions of any angle, identify reference angles, rewrite trigonometric functions in terms of positive acute angles, and find the exact values of trigonometric functions.*

### ***Unit 4: Graphs of the Sine Curve (page 68)***

4.1 The Properties and Graphs of the Sine Curve

4.2 Variation I:  $y = A \sin x$

4.3 Variation II:  $y = \sin(Bx)$

4.4 Variation III:  $y = A \sin(Bx)$

*Students will learn the basic properties of the sine curve, understand the effect of the parameters  $A$  and  $B$  on the sine curve and graph these curves.*

### ***Unit 5: Graphs of the Cosine Curve (page 89)***

5.1 The Properties and Graphs of the Cosine Curve

5.2 Variation I:  $y = A \cos x$

5.3 Variation II  $y = \cos(Bx)$

5.4 Variation III:  $y = A \cos(Bx)$

*Students will use their knowledge of co-functions to graph the cosine curve, learn the properties of cosine curves, and understand the effect of the parameters  $A$  and  $B$  on the cosine curve.*

### ***Answers to Odd Numbered Practice Exercises (page 105)***

## Unit 1: Angle Measure

The study of trigonometry begins with the study of angles. **An angle** consists of two rays,  $R_1$  and  $R_2$ , with a common endpoint called **the vertex**. We can interpret an angle as a rotation of  $R_1$  onto  $R_2$ . In this case,  $R_1$  is called **the initial side** and  $R_2$  is called **the terminal side**.

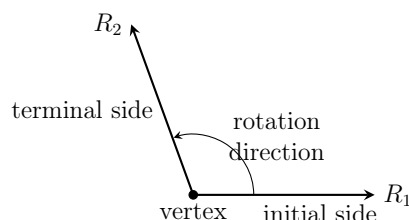


Figure 1.1

### 1.1 Measure in Degrees

The amount of rotation from the initial side to the terminal side determines the measure of the angle. We use the  $^\circ$  symbol to show that the angle is measured in degrees. A circle is formed when the terminal side is rotated counterclockwise and ends at the initial side. This circle is considered one complete rotation and the angle is defined to be  $360^\circ$ . Some angles are classified by the following special names:

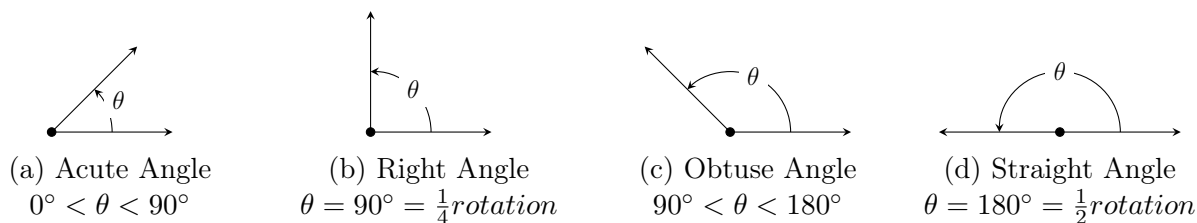
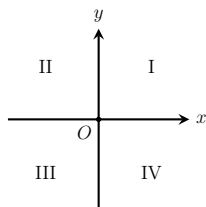


Figure 1.2

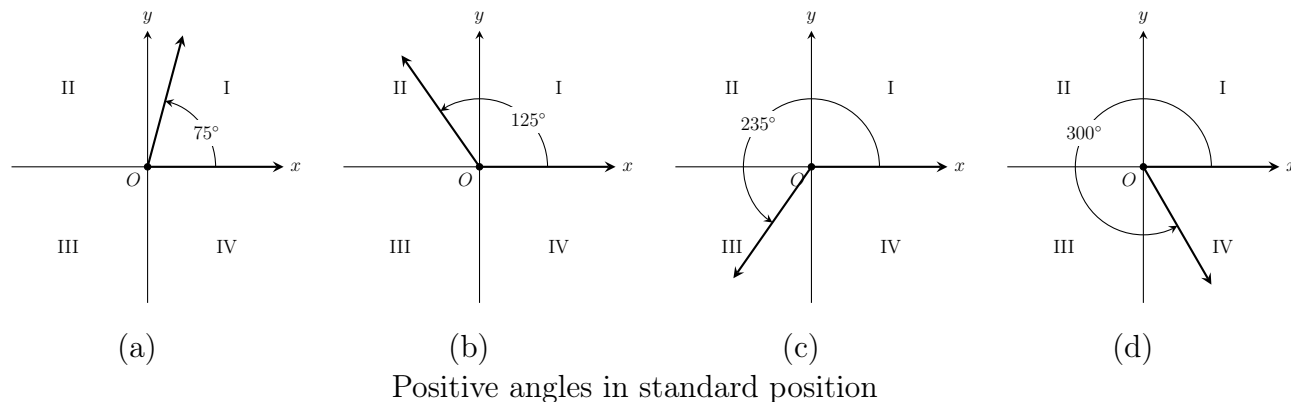
Angles can be positive, negative, and can have multiple rotations. To draw these angles, we use the Cartesian plane,  $x$ - $y$  plane. Each quarter of this plane is called a quadrant and is represented by the Roman numerals, I, II, III, IV.



Positions of the four quadrants

Figure 1.3

An angle is in **standard position** if its vertex is at the origin  $(0, 0)$  of the  $x$ - $y$  plane and its initial side is fixed along the positive  $x$ -axis. We say an angle **lies in a quadrant** if its terminal side is in that quadrant. A **positive angle** is generated by a counterclockwise rotation of the terminal side.



**Figure 1.4**

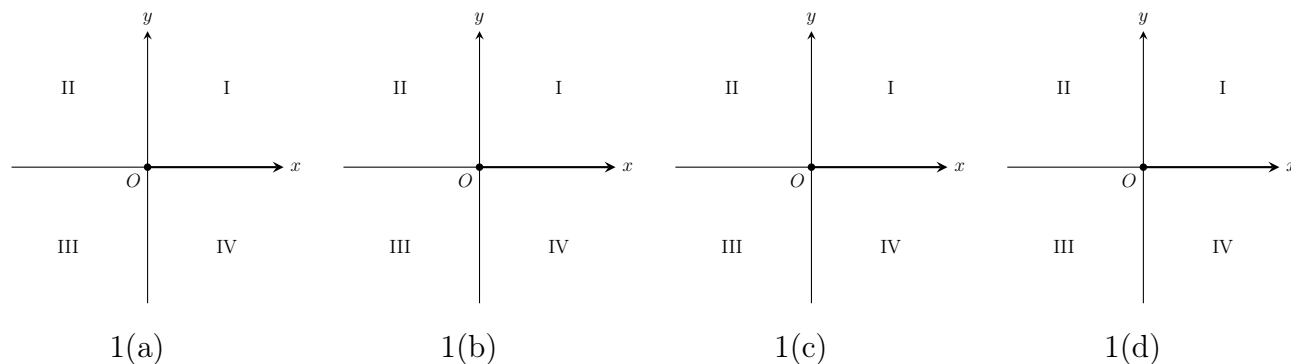
**Example 1.** Draw each of the angles in standard position.

(a)  $310^\circ$

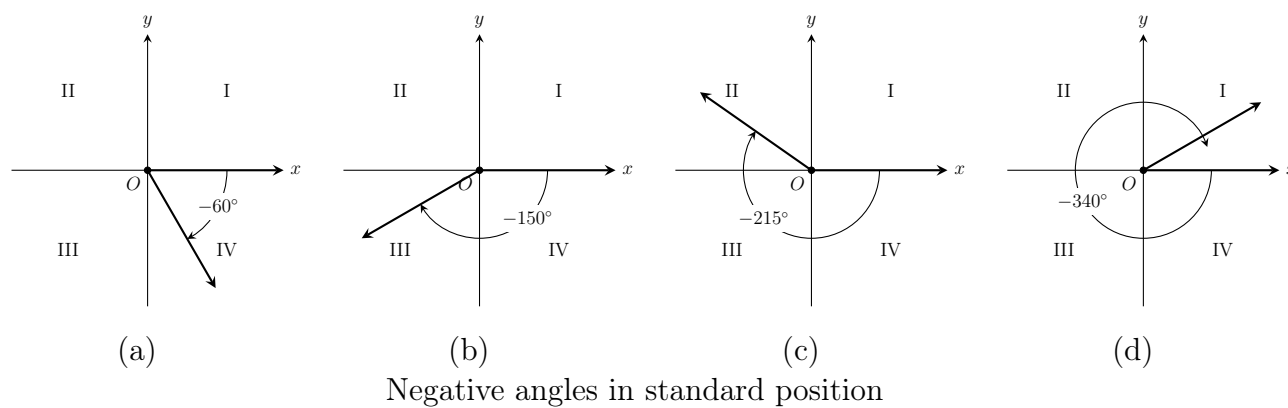
(b)  $165^\circ$

(c)  $214^\circ$

(d)  $78^\circ$



A *negative angle* is generated by a clockwise rotation of the terminal side.



**Figure 1.5**

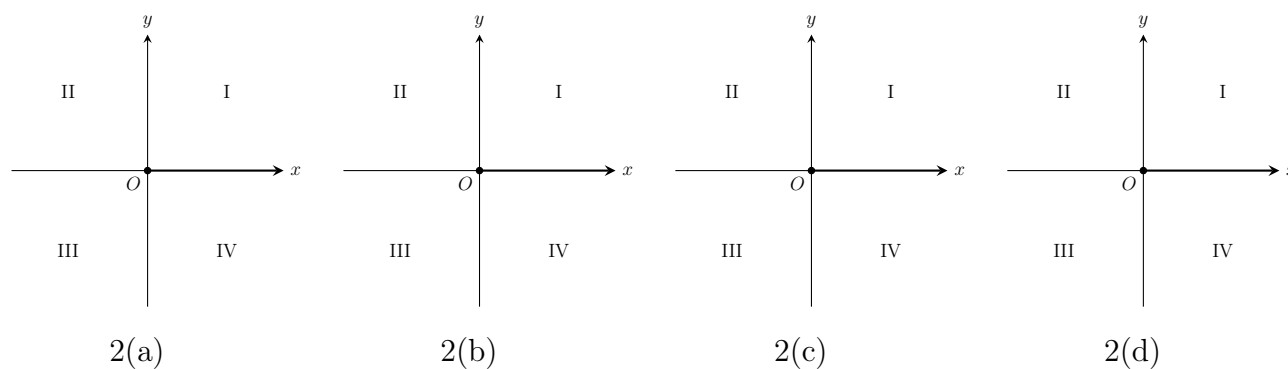
**Example 2.** Draw each of the angles in standard position.

(a)  $-160^\circ$

(b)  $-250^\circ$

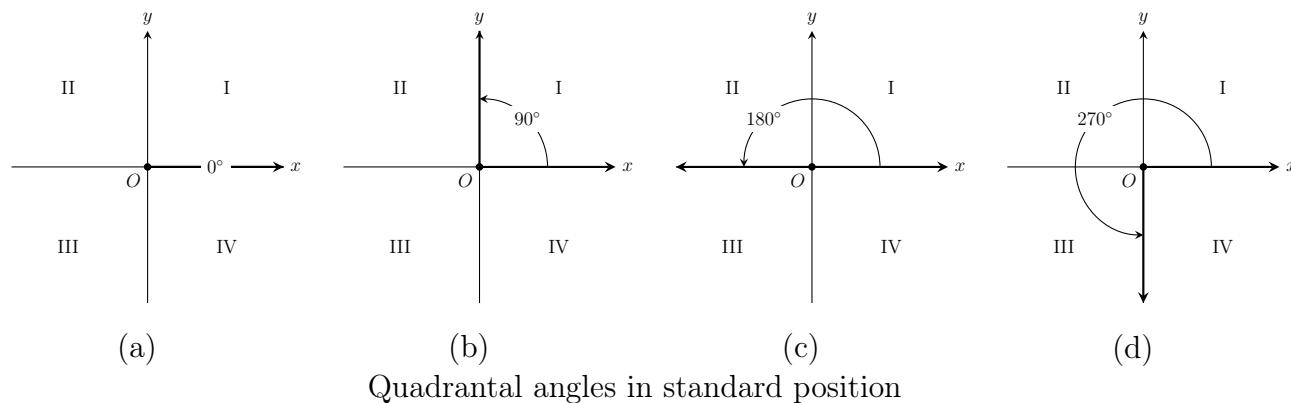
(c)  $-45^\circ$

(d)  $-315^\circ$





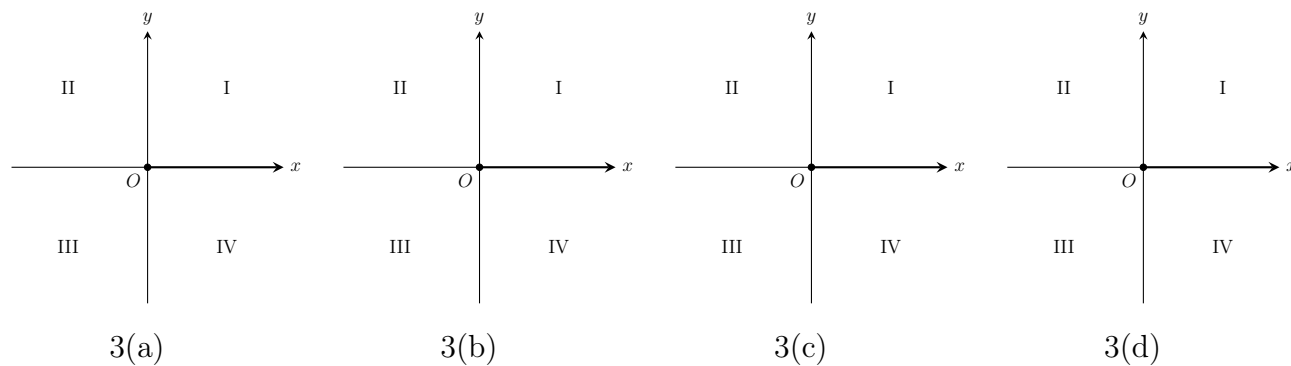
An angle is called *quadrantal* if its terminal side lies on the  $x$ -axis or  $y$ -axis.



**Figure 1.6**

**Example 3.** Draw each of the angles in standard position.

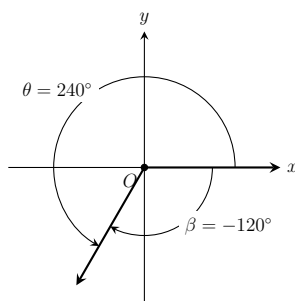
- (a)  $360^\circ$
- (b)  $-180^\circ$
- (c)  $-90^\circ$
- (d)  $-270^\circ$



### Observations from Example 3

$0^\circ$  and  $360^\circ$  share the same initial and terminal sides. Which other quadrantal angles drawn in Figure 1.6 and Example 3 above share the same initial and terminal sides?

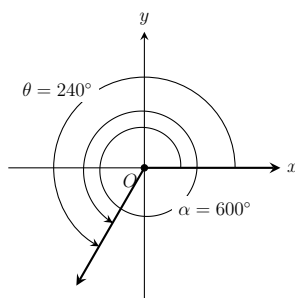
Two angles with the same initial and terminal sides, but with different rotations, are called *coterminal angles*.



**Figure 1.7**

Figure 1.7 shows that  $240^\circ$  and  $-120^\circ$  are coterminal angles because they have the same initial and terminal sides.

Suppose we take the terminal side of the angle which measures  $240^\circ$  and rotate it counter-clockwise one full rotation, or  $360^\circ$ .



**Figure 1.8**

The terminal side returns to the same position. Therefore,  $240^\circ$  and  $600^\circ$  are coterminal angles. We can generate coterminal angles by adding multiples of  $360^\circ$ .

$$240^\circ + 360^\circ = 600^\circ.$$

Similarly, if we take the terminal side of the angle which measures  $240^\circ$  and rotate clockwise one full rotation, the terminal side returns to the same position (Figure 1.7). This means we can also generate coterminal angles by subtracting multiples of  $360^\circ$ .

$$240^\circ - 360^\circ = -120^\circ.$$

**Example 4.** (a) *How many coterminal angles are there for each angle in standard position?*

(b) *Find one other positive and one other negative angle that is coterminal with  $240^\circ$ .*

**Example 5.** *Find an angle between  $0^\circ$  and  $360^\circ$  that is coterminal with the angle  $1280^\circ$ . We can subtract  $360^\circ$  as many times as we need from  $1280^\circ$  to obtain the coterminal angle that we are trying to find.*

$$1280^\circ - 360^\circ = 920^\circ$$

$$1280^\circ - (2)360^\circ = 1280^\circ - 720^\circ = 560^\circ$$

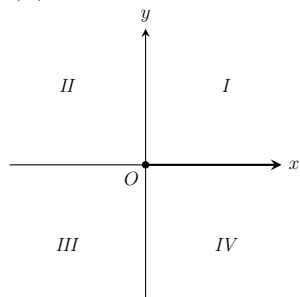
$$1280^\circ - (3)360^\circ = 1280^\circ - 1080^\circ = 200^\circ$$

The angle between  $0^\circ$  and  $360^\circ$  that is coterminal with the angle  $1280^\circ$  is  $200^\circ$ .

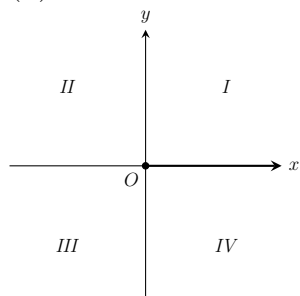
A more efficient method would be to find how many times  $360^\circ$  divides into  $1280^\circ$ . Since  $360^\circ$  divides into  $1280^\circ$  three times (3 complete rotations) with a remainder of  $200^\circ$ , the remainder is the angle we are looking for.

**Example 6.** Draw each of the angles in standard position. Find an angle between  $0^\circ$  and  $360^\circ$  which is coterminal with each of the given angles.

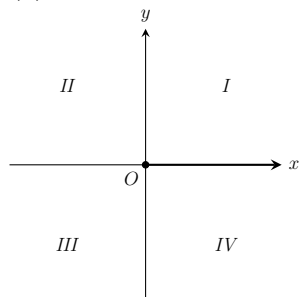
(a)  $-300^\circ$



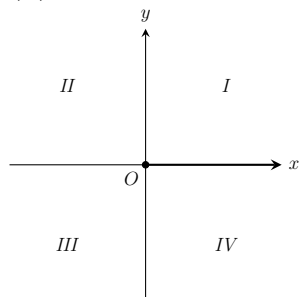
(b)  $540^\circ$



(c)  $-837^\circ$



(d)  $1215^\circ$



## 1.2 Measure in Radians

An angle whose vertex is at the center of a circle is called a *central angle*. The *radian measure* of a central angle  $\theta$ , is defined as the ratio of the arc length  $s$ , and the length of the radius,  $r$ . In formula, the radian measure  $\theta$  of a central angle is  $\theta = \frac{s}{r}$ .

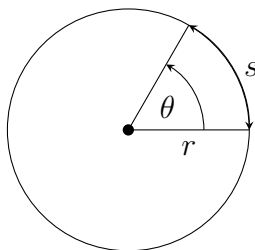


Figure 1.9

When the radius of the circle and the arc length are equal,  $s = r$ , we have

$$\theta = \frac{r}{r} = 1$$

and we say the angle  $\theta$  measures 1 radian.

In each of the circles in Figure 1.10,  $\theta$  is considered to measure 1 radian.

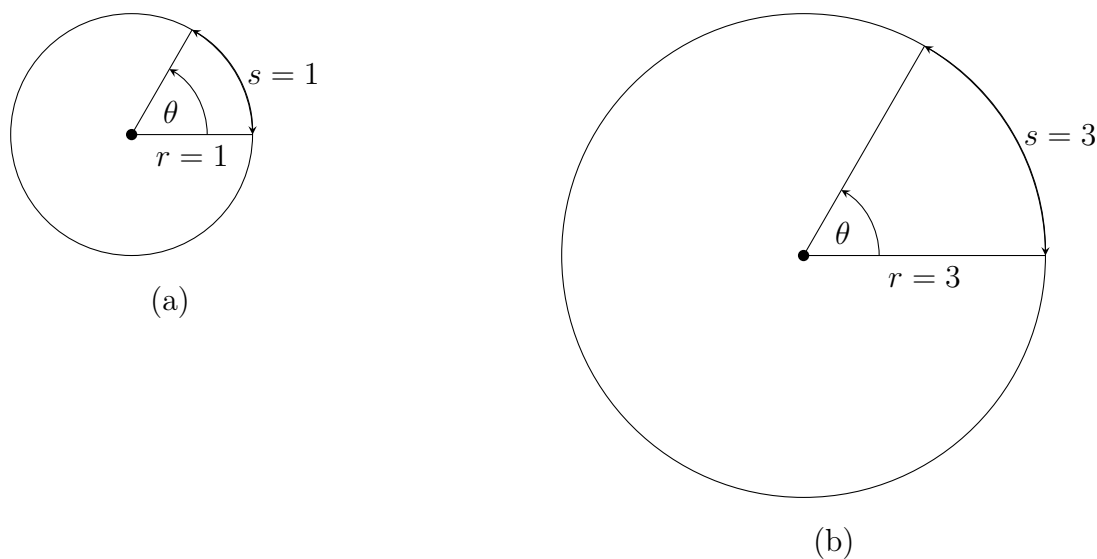


Figure 1.10

We have expressed the ratio of the arc length,  $s$ , to radius,  $r$  as :

$$\theta = \frac{s}{r}$$

The arc length of a  $360^\circ$  central angle is the circumference of a circle of radius  $r$ . Therefore, the radian measure of a  $360^\circ$  angle is

$$\theta = \frac{2\pi r}{r} = 2\pi.$$

We now have the following relationship between degrees and radians

$$360^\circ = 2\pi \text{ radians.}$$

Equivalently,

$$180^\circ = \pi \text{ radians.}$$

### 1.3 Conversion from Degrees to Radians

We can convert from degrees to radians by using or creating a proportion. However, for numbers that are factors of 180, we can simply use the equivalence  $\pi \text{ radians} = 180^\circ$ .

**Example 7.** (a) Write down ALL the factors of 180.

$\{1, 2, 3,$

(b) Complete the table:

<i>Degrees</i>	<i>Radians</i>
$90^\circ$	$\frac{\pi}{2}$
	$\frac{\pi}{3}$
$45^\circ$	
	$\frac{\pi}{5}$
$30^\circ$	
$20^\circ$	
	$\frac{\pi}{10}$
$15^\circ$	
$12^\circ$	
	$\frac{\pi}{18}$

**Example 8.** *What is  $50^\circ$  in radian measure? We use proportions to do this conversion:*

$$180^\circ = \pi \text{ radians}$$

$$50^\circ = x \text{ radians}$$

$$\frac{50^\circ}{180^\circ} = \frac{x \text{ radians}}{\pi \text{ radians}}$$

*If we solve for  $x$*

$$x = \frac{50^\circ}{180^\circ} \pi \text{ radians}$$

$$x = \frac{5}{18} \pi \text{ radians}$$

$$50^\circ = \frac{5}{18} \pi \text{ radians}$$

The answer  $\frac{5}{18} \pi$  radians is considered an exact value. If want an approximate value to the nearest thousandth (rounded to three decimal places) we have  $50^\circ \approx 0.873$  radians.

**Example 9.** *Find the radian measure of  $-212^\circ$ .*

$$180^\circ = \pi \text{ radians}$$

$$-212^\circ = x \text{ radians}$$

$$\frac{-212^\circ}{180^\circ} = \frac{x \text{ radians}}{\pi \text{ radians}}$$

*If we solve for  $x$*

$$x = \frac{-212^\circ}{180^\circ} \pi \text{ radians}$$

$$x = -\frac{53}{45} \pi \text{ radians}$$

$$-212^\circ = -\frac{53}{45} \pi \text{ radians}$$

$$-212^\circ \approx -3.7 \text{ radians.}$$

From our use of proportions, we have arrived at the conversion formula:

**To convert an angle from degrees to radians we multiply the angle by  $\frac{\pi}{180^\circ}$  radians.**

**Example 10.** *Convert each angle from degrees to radians. Reduce the fraction and express your result in terms of  $\pi$  (exact value).*

(a)  $54^\circ$

(b)  $240^\circ$

(c)  $315^\circ$

(d)  $-144^\circ$

**Example 11.** *Convert each angle from degrees to radians. Round your answer to three decimal places.*

(a)  $153^\circ$

(b)  $-311^\circ$



### 1.4 Conversion from Radians to Degrees

We can similarly convert from radians to degrees by proportion. However, if the radian measure contains  $\pi$ , we can simply use the equivalence  $\pi$  radians =  $180^\circ$ .

**Example 12.** Convert  $\frac{5\pi}{4}$  radians to degrees.

$$\frac{5\pi}{4} \text{ radians} = \frac{5}{4}(180^\circ) = 225^\circ$$

**Example 13.** Convert 2 radians to degrees. We use proportions to do this conversion:

$$180^\circ = \pi \text{ radians}$$

$$x^\circ = 2 \text{ radians}$$

$$\frac{x^\circ}{180^\circ} = \frac{2 \text{ radians}}{\pi \text{ radians}}$$

If we solve for  $x$

$$x = \frac{2}{\pi}(180^\circ) = \frac{360^\circ}{\pi}$$

This is an exact answer, but what quadrant does the angle lie in? We can find the approximate value by dividing by  $\pi$ .

$$x = \frac{2}{\pi}(180^\circ) = \frac{360^\circ}{\pi} \approx 114.591^\circ$$

This angle lies in quadrant II.

From our use of proportions, we have arrived at the following conversion formula:

**To convert an angle from radians to degrees we multiply the angle by  $\frac{180^\circ}{\pi}$ .**

**Example 14.** *Convert each angle from radians to degrees.*

(a)  $\frac{7\pi}{4}$

(b)  $\frac{4\pi}{3}$

(c)  $-4\pi$

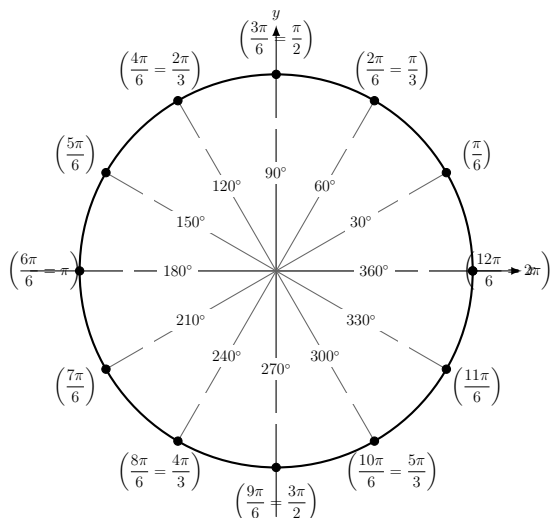
(d)  $\frac{17\pi}{6}$

**Example 15.** *Convert each angle from radians to degrees. Round to the nearest hundredth.*

(a)  $3.7$  radians.

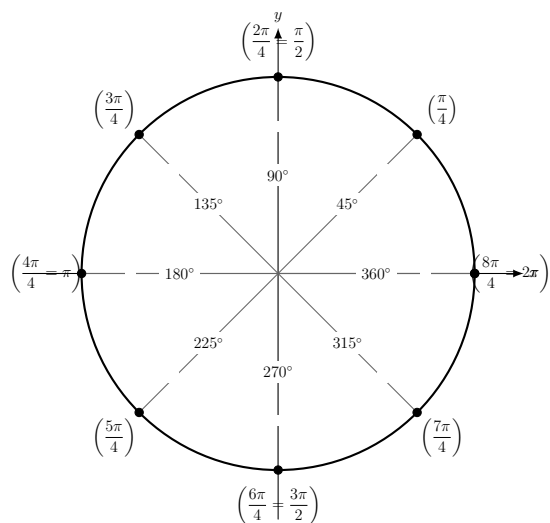
(b)  $5$  radians.

The circles in Figure 1.11 and Figure 1.12 below illustrate the relationship between angles measured in radians and degrees.



**Figure 1.11**

In the Figure 1.11, the circle is divided into increments of  $30^\circ$  or  $\frac{\pi}{6}$  radians.



**Figure 1.12**

In Figure 1.12, the circle is divided into increments of  $45^\circ$  or  $\frac{\pi}{4}$  radians.

Coterminal Angles in Radians To find angles that are coterminal with an angle in radian measure, we add or subtract multiples of  $2\pi$ . One convention for expressing the process of creating coterminal angles is the formula,  $\theta + 2\pi k$ , where  $k$  is an integer.

**Example 16.** Find angles that are coterminal with  $\theta = \frac{\pi}{3}$  radians

When  $k = 1$  we have:

$$\theta = \frac{\pi}{3} + 2\pi(1) = \frac{\pi}{3} + 2\pi = \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3} \text{ radians}$$

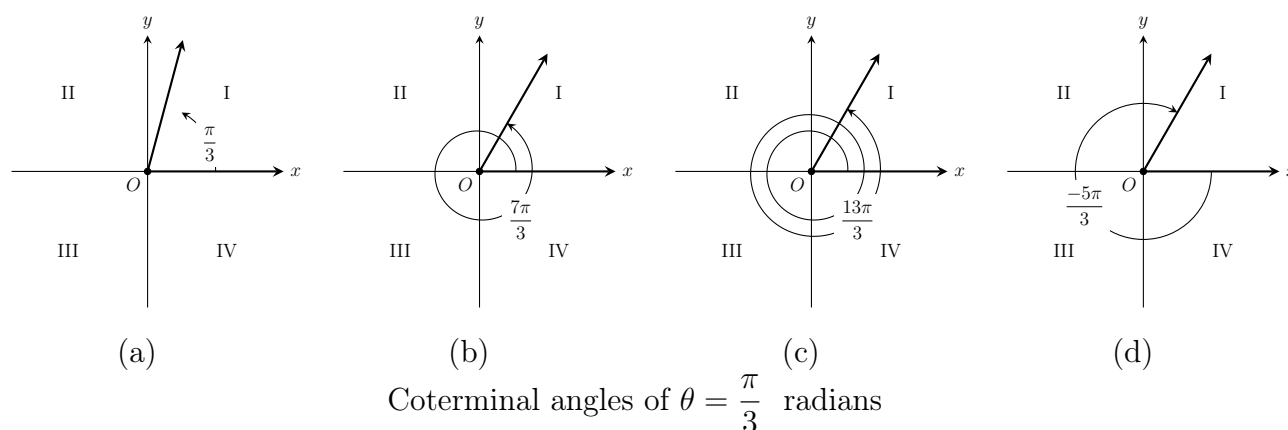
When  $k = 2$  we have:

$$\theta = \frac{\pi}{3} + 2\pi(2) = \frac{\pi}{3} + 4\pi = \frac{\pi}{3} + \frac{12\pi}{3} = \frac{13\pi}{3} \text{ radians}$$

To generate some negative coterminal angles, we let  $k$  be a negative integer:

When  $k = -1$  we have:

$$\theta = \frac{\pi}{3} + 2\pi(-1) = \frac{\pi}{3} - 2\pi = \frac{\pi}{3} - \frac{6\pi}{3} = \frac{-5\pi}{3} \text{ radians}$$



**Figure 1.13**

**Example 17.** Find the angle  $\theta$  between 0 radians and  $2\pi$  radians that is coterminal with the angle  $-\frac{11\pi}{3}$  radians.

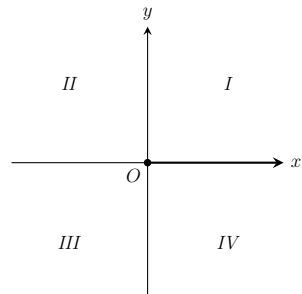
We can add  $2\pi$  radians as many times as we need to  $-\frac{11\pi}{3}$  radians to obtain the coterminal angle that we are trying to find.

$$\begin{aligned} \theta &= -\frac{11\pi}{3} + 2\pi(1) = -\frac{11\pi}{3} + \frac{6\pi}{3} = -\frac{5\pi}{3} \\ \theta &= -\frac{11\pi}{3} + 2\pi(2) = -\frac{11\pi}{3} + \frac{12\pi}{3} = \frac{\pi}{3} \end{aligned}$$

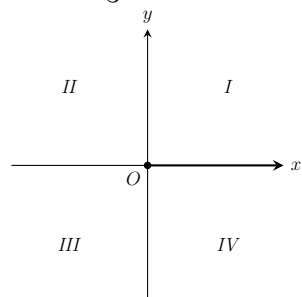
The angle between 0 radians and  $2\pi$  radians that is coterminal with the angle  $-\frac{11\pi}{3}$  radians is  $\frac{\pi}{3}$  radians.

**Example 18.** Draw each of the angles in standard position. Find an angle between 0 radians and  $2\pi$  radians which is coterminal with each of the given angles.

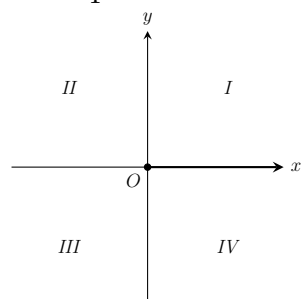
(a)  $\frac{12\pi}{5}$



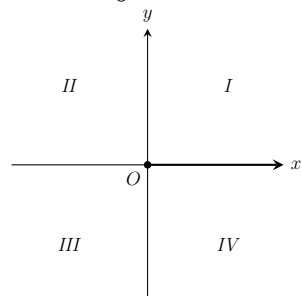
(b)  $-\frac{7\pi}{3}$



(c)  $\frac{17\pi}{4}$



(d)  $-\frac{25\pi}{6}$



## EXERCISES

**Practice 1.** Convert each angle from degrees to radians. Reduce the fraction and express your result in terms of  $\pi$ .

(a)  $230^\circ$

(b)  $160^\circ$

(c)  $-330^\circ$

(d)  $-150^\circ$

(e)  $-270^\circ$

(f)  $135^\circ$

**Practice 2.** Convert each angle from degrees to radians. Round your answer to three decimal places.

(a)  $-123^\circ$

(b)  $327^\circ$

**Practice 3.** Convert each angle from radians to degrees.

(a)  $\frac{7\pi}{3}$

(b)  $-3\pi$

(c)  $\frac{\pi}{9}$

(d)  $-\frac{3\pi}{4}$

(e)  $\frac{3\pi}{10}$

**Practice 4.** Convert each angle from radians to degrees. Round to two decimal places.

(a)  $-1.4$  radians.

(b)  $4$  radians.

**Practice 5.** Find a positive angle less than  $360^\circ$  that is coterminal with each angle.

(a)  $-115^\circ$

(b)  $823^\circ$

(c)  $400^\circ$

(d)  $-742^\circ$

**Practice 6.** Find a positive angle less than  $2\pi$  radians that is coterminal with each angle.

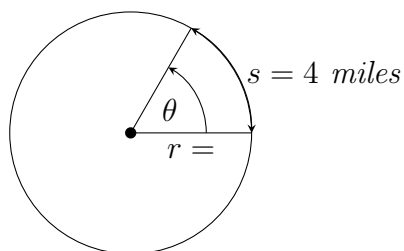
(a)  $\frac{11\pi}{5}$

(b)  $-\frac{23\pi}{6}$



**Practice 7.**

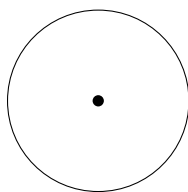
You walk 4 miles around a circular lake. You may recall that  $\theta = \frac{s}{r}$ .



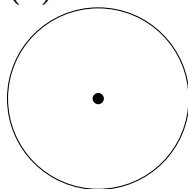
What is the measure of the angle  $\theta$ , in radians that represents your final position relative to your starting position if the radius of the lake is:

(Note: Diagrams not drawn to scale.)

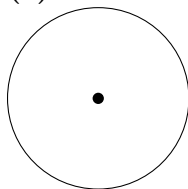
(a) 1 mile?



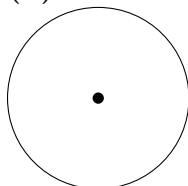
(b) 2 miles?



(c) 3 miles?



(d) 4 miles?

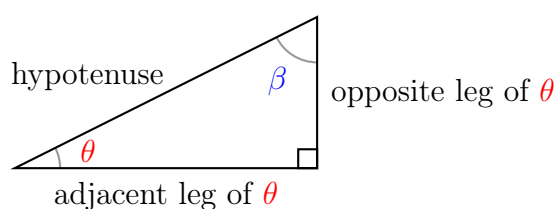


## Unit 2: Trigonometry of Right Triangles

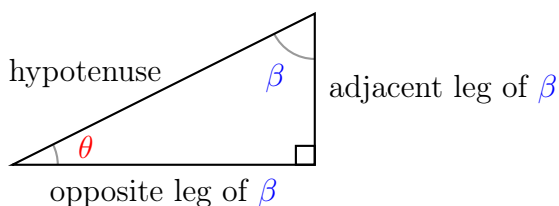
### 2.1 Sides of a Right Triangle

Trigonometry is the study of the measurements of a triangle. We start out by concentrating on right triangles and the ratios of the sides of a right triangle

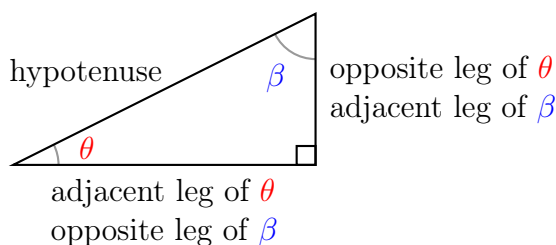
A right triangle has one right angle ( $90^\circ$  or  $\frac{\pi}{2}$  radians) and two acute angles that are **complementary** to each other. Let's name the two acute angles  $\theta$  and  $\beta$ . We define the sides (or legs) of the right triangle in relation to angle  $\theta$  as follows:



Similarly, the sides of the right triangle in relation to angle  $\beta$  are as follows:



As you can see, the opposite side of  $\theta$  is the adjacent side of  $\beta$ . And the adjacent side of  $\theta$  is the opposite side of  $\beta$ .

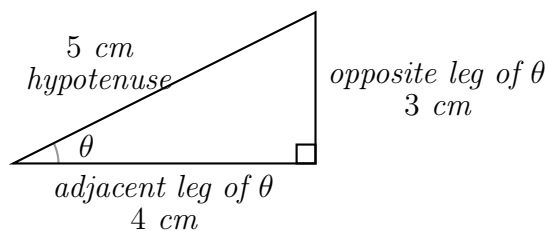


## 2.2 Definitions of the Trigonometric Functions

Now that the sides of a right triangle are defined in relation to an acute angle  $\theta$ , we can look at trigonometric functions defined as the *ratios* of the lengths of the sides of the right triangle in relation to angle  $\theta$ . For example, sine of  $\theta$ , denoted by  $\sin \theta$ , is defined as:

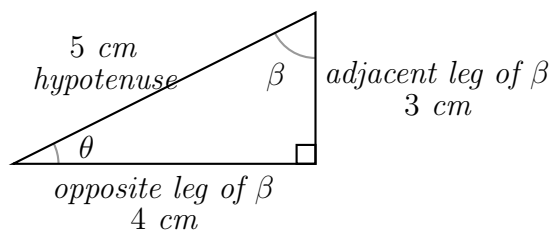
$$\sin \theta = \frac{\text{length of the leg opposite } \theta}{\text{length of the hypotenuse}}$$

**Example 19.** For the right triangle below, we can determine  $\sin \theta$ :



$$\sin \theta = \frac{\text{length of the leg opposite } \theta}{\text{length of the hypotenuse}} = \frac{3 \text{ cm}}{5 \text{ cm}} = \frac{3}{5}.$$

Similarly we can determine  $\sin \beta$ :



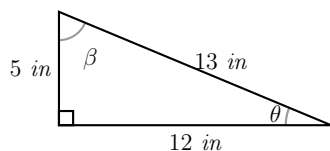
$$\sin \beta = \frac{\text{length of the leg opposite } \beta}{\text{length of the hypotenuse}} = \frac{4 \text{ cm}}{5 \text{ cm}} = \frac{4}{5}.$$

We notice that the value of a trigonometric function is a number without units, since the function is a ratio of lengths.

There are six trigonometric functions of the angle  $\theta$  : sine of  $\theta$  ( $\sin \theta$ ), cosine of  $\theta$  ( $\cos \theta$ ), tangent of  $\theta$  ( $\tan \theta$ ), cosecant of  $\theta$  ( $\csc \theta$ ), secant of  $\theta$  ( $\sec \theta$ ), and cotangent of  $\theta$  ( $\cot \theta$ ). The six trigonometric functions are defined in the following chart.

$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$		$\csc \theta = \frac{\text{hypotenuse}}{\text{opposite}}$
$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}}$		$\sec \theta = \frac{\text{hypotenuse}}{\text{adjacent}}$
$\tan \theta = \frac{\text{opposite}}{\text{adjacent}}$		$\cot \theta = \frac{\text{adjacent}}{\text{opposite}}$

**Example 20.** Based on the right triangle below, find the values of all six functions of both acute angles  $\theta$  and  $\beta$ .



$$\sin \theta = \frac{5}{13}$$

$$\csc \theta =$$

$$\sin \beta =$$

$$\csc \beta =$$

$$\cos \theta = \frac{12}{13}$$

$$\sec \theta =$$

$$\cos \beta =$$

$$\sec \beta =$$

$$\tan \theta =$$

$$\cot \theta =$$

$$\tan \beta =$$

$$\cot \beta =$$

### Observations from Example 20

(a)  $\sin \theta$  and  $\csc \theta$  are reciprocals. We can see this relationship from the definitions of the functions. Are there other pairs of reciprocals?

(b) *List all the pairs of trigonometric functions of  $\theta$  that are reciprocals of each other.*

(c) *Compare the values of  $\sin \theta$  and  $\cos \beta$ . These values are the same. What is the reason?*

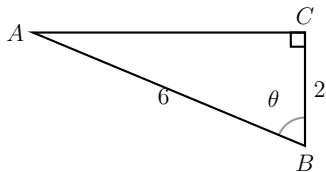
(d) *Are there other functions of  $\theta$  that are equal to functions of  $\beta$ ? Why do these functions of  $\theta$  and  $\beta$  have the same value? Do they come in pairs?*

(e) *List all the pairs of functions of  $\theta$  that are equal to the functions of  $\beta$ .*

Function of  $\theta$  = Function of  $\beta$

(f) *The functions sine and cosine are **cofunctions**. The name cosine is short for the complement of sine, because  $\theta$  and  $\beta$  are complementary angles. Examine the names of the pairs of cofunctions on your list in (e) above.*

**Example 21.** Consider the following triangle. Find the values of all six functions of the acute angle  $\theta$ .



(a) In order to find the values of all six trigonometric functions of  $\theta$ , we need the lengths of all the sides of the right triangle. We can find the unknown length  $AC$  using the Pythagorean Theorem:

$$a^2 + b^2 = c^2,$$

where  $a$  and  $b$  are the lengths of the two legs and  $c$  is the length of the hypotenuse. Let's say the length of  $AC = b$ , and apply the Pythagorean Theorem:

$$\begin{aligned} 2^2 + b^2 &= 6^2 \\ b^2 &= 32 \\ b &= \sqrt{32} \\ b &= 4\sqrt{2} \end{aligned}$$

(b) We now have the lengths of all the sides of the right triangle and we can find the values of the trigonometric functions.

$$\sin \theta =$$

$$\csc \theta =$$

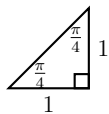
$$\cos \theta =$$

$$\sec \theta =$$

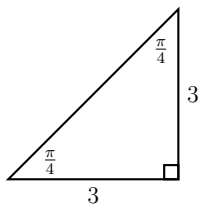
$$\tan \theta =$$

$$\cot \theta =$$

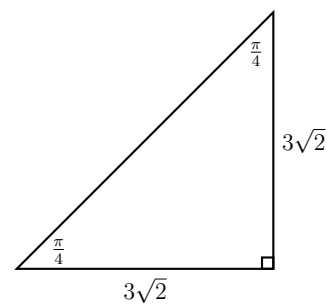
**Example 22.** *Special triangle  $45^\circ - 45^\circ - 90^\circ$  (or  $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ ) Let's draw three of these isosceles right triangles.*



Triangle A



Triangle B



Triangle C

*These triangles are all of the same shape but they are of different sizes. They are similar triangles and their corresponding sides are proportional.*

*Determine the values of the six trigonometric functions for the acute angle of  $45^\circ$  ( $\frac{\pi}{4}$ ).*

*Function Values Based on Triangle A*

(a) *First determine the length of the hypotenuse using the Pythagorean theorem.*

$$1^2 + 1^2 = c^2$$

$$2 = c^2$$

$$\sqrt{2} = c$$

(b)

$$\sin \frac{\pi}{4} =$$

$$\csc \frac{\pi}{4} =$$

$$\cos \frac{\pi}{4} =$$

$$\sec \frac{\pi}{4} =$$

$$\tan \frac{\pi}{4} =$$

$$\cot \frac{\pi}{4} =$$



*Function Values Based on Triangle B*

(a) *First determine the length of the hypotenuse using the Pythagorean theorem.*

$$3^2 + 3^2 = c^2$$

$$18 = c^2$$

$$3\sqrt{2} = c$$

(b)

$$\sin \frac{\pi}{4} =$$

$$\csc \frac{\pi}{4} =$$

$$\cos \frac{\pi}{4} =$$

$$\sec \frac{\pi}{4} =$$

$$\tan \frac{\pi}{4} =$$

$$\cot \frac{\pi}{4} =$$

*Function Values Based on Triangle C*

(a) *First determine the length of the hypotenuse using the Pythagorean theorem.*

$$(3\sqrt{2})^2 + (3\sqrt{2})^2 = c^2$$

$$36 = c^2$$

$$6 = c$$

(b)

$$\sin \frac{\pi}{4} =$$

$$\csc \frac{\pi}{4} =$$

$$\cos \frac{\pi}{4} =$$

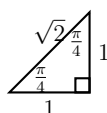
$$\sec \frac{\pi}{4} =$$

$$\tan \frac{\pi}{4} =$$

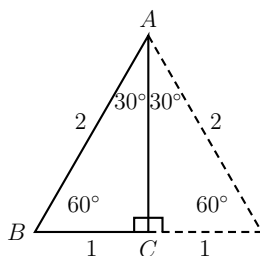
$$\cot \frac{\pi}{4} =$$

**Observations from Example 22 - the special triangle  $45^\circ - 45^\circ - 90^\circ$  (or  $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ )**

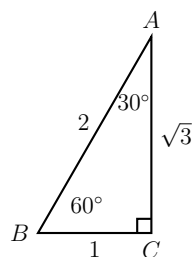
- (a)  $\sin 45^\circ$  is equal to  $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$  based on all three triangles. What is the reason?
- (b) In general, the values of the trigonometric functions are dependent on the measure of the angle  $\theta$  and not on the size of the triangle in which angle  $\theta$  lies.
- (c) There are many problems that ask us to determine the exact values of functions based on special triangles. We should memorize the angles and their corresponding side of this special triangle  $45^\circ - 45^\circ - 90^\circ$  (or  $\frac{\pi}{4} - \frac{\pi}{4} - \frac{\pi}{2}$ ).



**Example 23.** *Special triangle  $30^\circ - 60^\circ - 90^\circ$  (or  $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ )  
This triangle is half of an equilateral triangle.*



*Using the Pythagorean Theorem, we can determine the third side  $AC$  and we have the following special  $30^\circ - 60^\circ - 90^\circ$  (or  $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ ) triangle. We should memorize the angles and their corresponding side of this special triangle.*



Determine the six functions for  $30^\circ(\frac{\pi}{6})$  and  $60^\circ(\frac{\pi}{3})$ .

$\sin 30^\circ = \frac{1}{2}$	$\csc 30^\circ =$	$\sin 60^\circ =$	$\csc 60^\circ =$
$\cos 30^\circ = \frac{\sqrt{3}}{2}$	$\sec 30^\circ =$	$\cos 60^\circ =$	$\sec 60^\circ =$
$\tan 30^\circ =$	$\cot 30^\circ =$	$\tan 60^\circ =$	$\cot 60^\circ =$

**Observations from Example 23 - the special triangle  $30^\circ - 60^\circ - 90^\circ$  (or  $\frac{\pi}{6} - \frac{\pi}{3} - \frac{\pi}{2}$ )**

(a) Compare  $\tan \frac{\pi}{3}$  to  $\cot \frac{\pi}{6}$ . Why are they the same?

(b) List all the pairs of cofunctions.

(c) List all the pairs of functions that are reciprocals of each other.

(d) Check that

$$\frac{\sin \frac{\pi}{3}}{\cos \frac{\pi}{3}} = \tan \frac{\pi}{3}.$$

(e) Does the relationship in (d) above hold for the angle  $\frac{\pi}{6}$ ?

(f) Does the relationship in (d) above hold for the angle  $\frac{\pi}{4}$ ?

(g) Does the relationship in (d) above hold for any angle  $\theta$ ? Examine the definitions of  $\sin \theta$ ,  $\cos \theta$ , and  $\tan \theta$ .

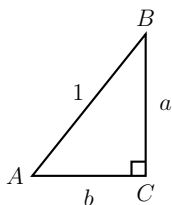
### 2.3 The Pythagorean Identity

The Pythagorean Theorem  $a^2 + b^2 = c^2$  is in terms of the lengths of the sides of a right triangle. This theorem may also be expressed in terms of trigonometric functions of an acute angle  $\theta$  of a right triangle.

$$\sin^2 \theta + \cos^2 \theta = 1$$

This is an identity because this equation is true for all values of  $\theta$ , for which each of the trigonometric functions is defined.

We can easily derive this Pythagorean Identity. Let's consider this right triangle:



where the length of the two legs are  $a$  and  $b$  and the hypotenuse is  $1$ . The Pythagorean Theorem as applied to this triangle is  $a^2 + b^2 = 1$ . From the triangle, we have

$$\sin \theta = \underline{\hspace{2cm}}$$

$$\cos \theta = \underline{\hspace{2cm}}$$

Let's rewrite the Pythagorean Theorem in terms of  $\sin \theta$  and  $\cos \theta$ .

$$\begin{aligned} a^2 + b^2 &= 1 \\ \sin^2 \theta + \cos^2 \theta &= 1 \end{aligned}$$

**Variations of The Pythagorean Identity** There are two variations of the Pythagorean Identity:

$$\begin{aligned} \tan^2 \theta + 1 &= \sec^2 \theta \\ 1 + \cot^2 \theta &= \csc^2 \theta \end{aligned}$$

Let's derive  $\tan^2 \theta + 1 = \sec^2 \theta$ . The other variation is left as an exercise.

To derive this variation, we start out with the Pythagorean Identity we just learned.

$$\sin^2 \theta + \cos^2 \theta = 1$$

We'll divide both sides of the equation by  $\cos^2 \theta$ . The result is:

$$\frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

We have just derived the identity :  $\tan^2 \theta + 1 = \sec^2 \theta$ . The chart below summarizes the trigonometric identities that we've learned so far.

### Fundamental Trigonometric Identities

#### Reciprocal Identities

$$\begin{array}{lll} \sin \theta = \frac{1}{\csc \theta} & \cos \theta = \frac{1}{\sec \theta} & \tan \theta = \frac{1}{\cot \theta} \\ \csc \theta = \frac{1}{\sin \theta} & \sec \theta = \frac{1}{\cos \theta} & \cot \theta = \frac{1}{\tan \theta} \end{array}$$

#### Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \cot \theta = \frac{\cos \theta}{\sin \theta}$$

#### Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan^2 \theta + 1 = \sec^2 \theta \quad 1 + \cot^2 \theta = \csc^2 \theta$$

#### Cofunction Identities

In radians

$$\begin{array}{ll} \sin \left( \frac{\pi}{2} - \theta \right) = \cos \theta & \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta \\ \tan \left( \frac{\pi}{2} - \theta \right) = \cot \theta & \cot \left( \frac{\pi}{2} - \theta \right) = \tan \theta \\ \sec \left( \frac{\pi}{2} - \theta \right) = \csc \theta & \csc \left( \frac{\pi}{2} - \theta \right) = \sec \theta \end{array}$$

In degrees

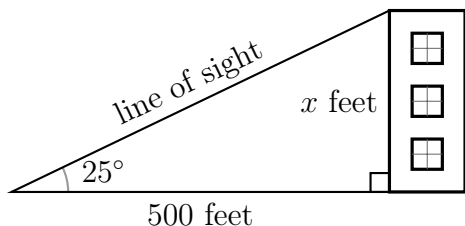
$$\begin{array}{ll} \sin (90^\circ - \theta) = \cos \theta & \cos (90^\circ - \theta) = \sin \theta \\ \tan (90^\circ - \theta) = \cot \theta & \cot (90^\circ - \theta) = \tan \theta \\ \sec (90^\circ - \theta) = \csc \theta & \csc (90^\circ - \theta) = \sec \theta \end{array}$$

## 2.4 Applications

Right triangle trigonometry is often used to measure distance and height.

**Example 24.** *From a point on the ground 500 feet from the base of a building, a surveyor measures the angle of elevation from the point on the ground to the top of the building to be  $25^\circ$ . Determine the height of the building, to the nearest foot.*

*Solution.* (A) We first draw a picture based on the information given by the problem.



The angle of elevation is the angle formed by the horizontal ground and the line of sight from the point on the ground to the top of the building.

(B) We now have a right triangle. We are looking for the height of the building and so we let it be  $x$  feet.

(C) In the triangle, we want to relate the given angle of  $25^\circ$  to the sides that measure 500 feet and  $x$  feet. For the  $25^\circ$  angle, the side measuring 500 feet is the adjacent side and the side measuring  $x$  feet is the opposite side. Now, what is the trigonometric function that is defined by the opposite and adjacent sides?

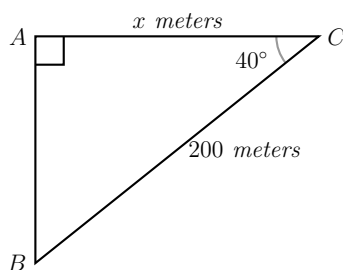
(D)

$$\tan 25^\circ = \frac{x}{500}$$

$$x = 500 \tan 25^\circ \quad \text{use your calculator to find } \tan 25^\circ$$

$$x \approx 233 \text{ feet} \quad \text{rounded to the nearest foot}$$

**Example 25.** Determine the length of  $AC$  to the nearest tenth of a meter.



*Solution.* (A) Relate the sides measuring 200 meters and  $x$  meters to the  $40^\circ$  angle.

The side  $BC$  measuring 200 meters is the \_\_\_\_\_ of the  $40^\circ$  angle.

The side  $AC$  measuring  $x$  meters is the \_\_\_\_\_ of the  $40^\circ$  angle.

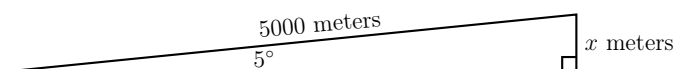
What is the appropriate trigonometric function?

(B)

$$\begin{aligned}\cos 40^\circ &= \frac{x}{200} \\ x &= 200 \cos 40^\circ \\ x &\approx 153.2 \text{ meters}\end{aligned}$$

**Example 26.** A road is inclined at an angle of  $5^\circ$ . A bicyclist travels 5000 meters along the road. How much has her altitude increased? Round to the nearest meter.

*Solution.* (A) Draw a picture



(B) Relate 5000 meters and  $x$  meters to the angle of  $5^\circ$ . What is the appropriate function?

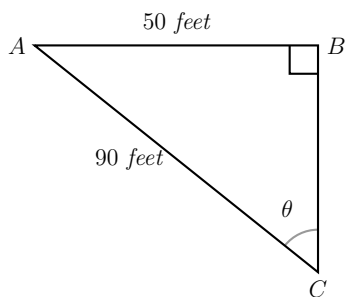
$$\frac{x}{5000} = \sin 5^\circ$$

(C) Solve the equation

$$x = 5000 \sin 5^\circ \approx 436 \text{ meters.}$$



**Example 27.** Determine the angle  $\theta$  to the nearest degree.



*Solution.* (A) Relate the side  $AB$  to angle  $\theta$  and the side  $AC$  to right angle  $B$ .

Side  $AB$  is 50 feet long and is the \_\_\_\_\_ side of angle  $\theta$ .

Side  $AC$  is 90 feet long is the \_\_\_\_\_ of the triangle.

What is the appropriate trigonometric function?

(B)

$$\sin \theta = \frac{50}{90}$$

$$\sin \theta = 0.556$$

We are looking for the acute angle,  $\theta$  whose *sine* value is 0.556. The mathematical notation is:

$$\theta = \sin^{-1} 0.556, \text{ or}$$

$$\theta = \sin^{-1} \frac{50}{90}$$

On the calculator, the second function (2nd) of  $\sin$  is  $\sin^{-1}$  which gives the acute angle

$$\theta = 34^\circ$$

when rounded to the nearest degree.

## EXERCISES

**Practice 8.** *List all the pairs of functions that are reciprocals of each other.*

**Practice 9.** *Find the cofunction with the same value as the given function.*

	<u>Function</u>	=	<u>Cofunction</u>
a)	$\sin 25^\circ$	=	
b)	$\cot 78^\circ$	=	
c)	$\cos 17.6^\circ$	=	
d)	$\sec \frac{\pi}{8}$	=	
e)	$\csc \frac{2\pi}{5}$	=	
f)	$\tan \frac{3\pi}{7}$	=	

**Practice 10.** Use a calculator to find the values of trigonometric functions and round your answer to 4 decimal places.

(a)  $\sin 37^\circ$

(b)  $\cos 59^\circ$

(c)  $\tan 12.8^\circ$

(d)  $\sec 35^\circ$

(e)  $\csc 1.2$  radians

(f)  $\cot 0.8$  radians

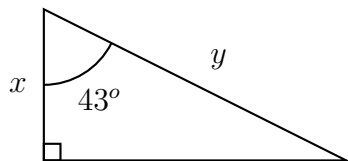
**Practice 11.** Use a calculator to find the acute angles to the nearest tenth of a degree.

(a)  $\sin^{-1}(0.3416)$

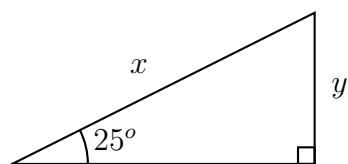
(b)  $\tan^{-1}(12.3451)$

(c)  $\cos^{-1}(0.7618)$

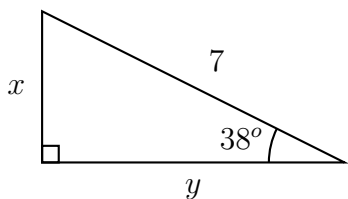
**Practice 12.** Find the unknown sides  $x$  and  $y$ , to the nearest thousandth. Please note that the triangles are not drawn to scale.



(a)



(b)



(c)

**Practice 13.** Angle  $\theta$  is an acute angle and  $\cos \theta = \frac{\sqrt{21}}{5}$ . Find the values of the other five trigonometric functions.

**Practice 14.** A house is on the top of a hill with a road that makes an angle of inclination of  $10^\circ$ . The road is 1000 feet long from the bottom of the hill to the house. What is altitude of the house, to the nearest foot?

**Practice 15.** At a certain time of day, an 85 foot tall tree casts a 35 foot long shadow. Find the angle of elevation of the sun to the nearest tenth of a degree.

**Practice 16.** For acute angle  $A$ ,  $\sin A = 0.7$  and  $\cos A = \sqrt{0.51}$ .

(a) Use a quotient identity to find  $\tan A$ .

(b) Use a reciprocal identity to find  $\sec A$ .

**Practice 17.** From the top of a 65 foot-high building, an observer measures the angle of depression of a bicycle on the sidewalk as  $25^\circ$ . Find the distance from the bicycle to the base of the building to the nearest foot.

**Practice 18.** John wants to measure the height of the tree he planted many years ago. He walks exactly 18 meters from the base of the tree, turns around and looks up to the top of the tree. The angle of elevation from the ground to the top of the tree is  $27^\circ$ . How tall is the tree to the nearest tenth of a meter?

**Practice 19.** Angle  $\theta$  is an acute angle and  $\sin \theta = \frac{6}{7}$ .

(a) Find  $\cos \theta$  using a Pythagorean identity.

(b) Find the values of all the other trigonometric functions using quotient identities and reciprocal identities.

(c) Are there others ways to solve this problem? Compare this exercise with Practice 13.

(d) Determine the measure of  $\theta$  to the nearest degree.

**Practice 20.** *A 25-foot ladder leans against a wall. The bottom of the ladder is 18 feet from the base of the wall.*

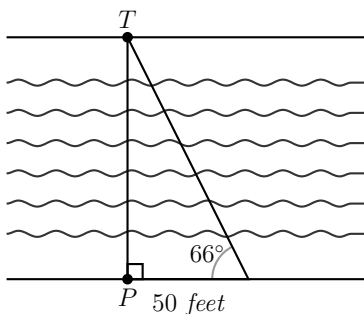
(a) *Determine the angle the bottom of the ladder makes with the ground. Round to the nearest degree.*

(b) *Determine the angle the top of the ladder makes with the wall. Round to the nearest degree. Determine this angle using two different methods.*

(c) *How high, to the nearest tenth of a foot, is the top of the ladder from the ground? Is there more than one method to determine the height?*



**Practice 21.** A group of students wants to measure the width of a river. They set up a post  $P$  on their side of the river directly across from a tree  $T$ . Then they walked downstream 50 feet and measured the angle formed by the line of sight to the post  $P$  and the line of sight to tree  $T$ . They found the angle is  $66^\circ$ . Determine the width of the river. Round your answer to the nearest foot.



**Practice 22.** The straight line  $y = 2x$  forms an angle with the positive  $x$ -axis in the first quadrant. Let's call this angle  $\theta$ . Determine the value of  $\tan \theta$ . Compare  $\tan \theta$  to the slope of the line  $y = 2x$ . Why are they the same?

**Practice 23.** Based on the Pythagorean Identity:  $\sin^2 \theta + \cos^2 \theta = 1$  derive the Pythagorean identity  $1 + \cot^2 \theta = \csc^2 \theta$

### Unit 3: Trigonometric Functions of Any Angle

To extend the definition of the six trigonometric functions, we return to the Cartesian Plane and angles in standard position.

#### 3.1 Definition of The Trigonometric Functions of Angles

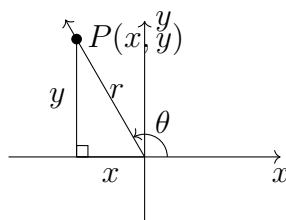
##### Trigonometric Functions of Angles

Let  $\theta$  be an angle in standard position and  $P = (x, y)$  be a point on the terminal side of  $\theta$ . Using the Pythagorean Theorem, the distance between  $P$  and the origin is given by  $r = \sqrt{x^2 + y^2}$ . The six trigonometric functions are defined as follows.

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}, (x \neq 0)$$



$$\csc \theta = \frac{r}{y}, (y \neq 0)$$

$$\sec \theta = \frac{r}{x}, (x \neq 0)$$

$$\cot \theta = \frac{x}{y}, (y \neq 0)$$

We can use the definitions of the sine and cosine functions to verify the Pythagorean Identity:

$$\sin^2 \theta + \cos^2 \theta = 1$$

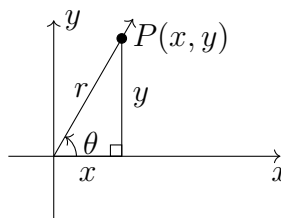
Since  $\sin \theta = \frac{y}{r}$  and  $\cos \theta = \frac{x}{r}$

$$\left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \left(\frac{y^2 + x^2}{r^2}\right) = \frac{r^2}{r^2} = 1$$

The value of each of the six trigonometric functions depends on the values of  $x$ ,  $y$  and  $r$ . The distance,  $r$ , between the origin and Point  $P$  is always a positive value. The quadrant that  $\theta$  lies in determines whether the value of the six trigonometric functions is positive or negative.

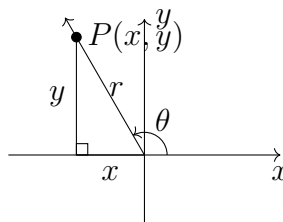
#### Quadrant I

In this quadrant,  $x > 0$  and  $y > 0$ . Since  $x$ ,  $y$  and  $r$  are all positive, all six trigonometric functions are positive.



Quadrant II

In this quadrant,  $x < 0$  and  $y > 0$ .



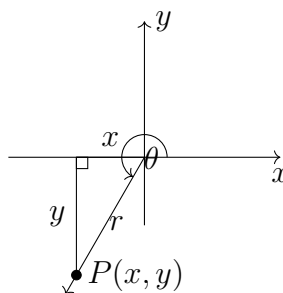
$$\begin{aligned}\sin \theta &= \frac{y}{r} > 0 \\ \cos \theta &= \frac{x}{r} < 0 \\ \tan \theta &= \frac{y}{x} < 0\end{aligned}$$

$$\begin{aligned}\csc \theta &= \frac{r}{y} > 0 \\ \sec \theta &= \frac{r}{x} < 0 \\ \cot \theta &= \frac{x}{y} < 0\end{aligned}$$

So in quadrant II,  $\sin \theta$  and its reciprocal  $\csc \theta$  are both positive. The other four trigonometric functions are negative.

Quadrant III

In this quadrant,  $x < 0$  and  $y < 0$ .



$$\begin{aligned}\sin \theta &= \frac{y}{r} < 0 \\ \cos \theta &= \frac{x}{r} < 0 \\ \tan \theta &= \frac{y}{x}\end{aligned}$$

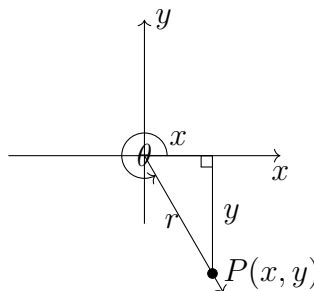
$$\begin{aligned}\csc \theta &= \frac{r}{y} \\ \sec \theta &= \frac{r}{x} \\ \cot \theta &= \frac{x}{y}\end{aligned}$$

Which trigonometric functions are positive?

Which trigonometric functions are negative?

## Quadrant IV

In this quadrant,  $x > 0$  and  $y < 0$ .



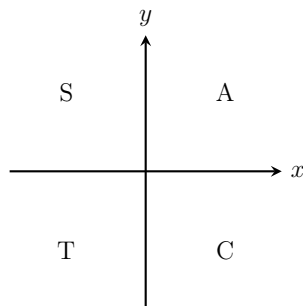
$$\begin{aligned}\sin \theta &= \frac{y}{r} < 0 \\ \cos \theta &= \frac{x}{r} > 0 \\ \tan \theta &= \frac{y}{x}\end{aligned}$$

$$\begin{aligned}\csc \theta &= \frac{r}{y} \\ \sec \theta &= \frac{r}{x} \\ \cot \theta &= \frac{x}{y}\end{aligned}$$

Which trigonometric functions are positive?

Which trigonometric functions are negative?

We can use the mnemonic All-Students-Take-Calculus (ASTC) to help us recall the signs of the values of the six trigonometric functions in each of the quadrants. The "A" is in quadrant I and means that all six trigonometric functions are positive. The "S" is in quadrant II and means that *sin* and its reciprocal are positive. The "T" is in quadrant III and means that *tan* and its reciprocal are positive. The "C" is in quadrant IV and means that *cos* and its reciprocal are positive.



ASTC

**Example 28.** *What quadrant does  $\theta$  lie in if  $\cos \theta > 0$  and  $\sin \theta < 0$ ?*

**Step 1.** *Find the quadrants in which  $\cos \theta > 0$ .*

*$\cos$  is positive in Quadrants I and IV.*

**Step 2.** *Find the quadrants in which  $\sin \theta < 0$ .*

*$\sin$  is negative in Quadrants III and IV.*

**Step 3.** *Find the quadrant that satisfies both conditions*

*$\theta$  lies in Quadrant IV*

**Example 29.**

*Find the quadrant in which  $\theta$  lies.*

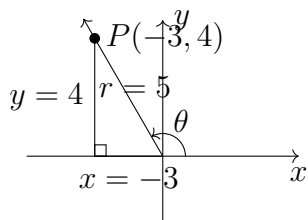
(a)  $\cos \theta > 0$  and  $\tan \theta > 0$

(b)  $\csc \theta < 0$  and  $\sec \theta < 0$

**Example 30.** Let  $P = (-3, 4)$  be a point on the terminal side of an angle  $\theta$ . Find the exact value of the six trigonometric functions of  $\theta$ .

**Step 1.** Find the radius  $r$  using the Pythagorean Theorem:  $r^2 = x^2 + y^2$ .

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = 5.$$



**Step 2.** Evaluate the functions using definitions.

$$\sin \theta = \frac{y}{r} = \frac{4}{5}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{5}{4}$$

$$\cos \theta = \frac{x}{r} = -\frac{3}{5}$$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{5}{3}$$

$$\tan \theta = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

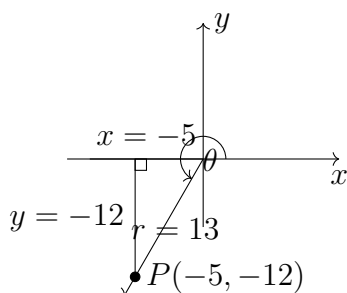
$$\cot \theta = \frac{1}{\tan \theta} = -\frac{3}{4}$$

**Example 31.** Given  $\tan \theta = \frac{12}{5}$  and  $\sin \theta < 0$ , find the exact values of the ALL six trigonometric functions of  $\theta$ .

**Step 1.** Find the quadrant in which  $\theta$  lies. Since  $\tan \theta > 0$  and  $\sin \theta < 0$ ,  $\theta$  is in Quadrant III. In this quadrant,  $x < 0$  and  $y < 0$  so we express tangent as the ratio of two negative numbers and write:  $\tan \theta = \frac{-12}{-5}$ .

**Step 2.** Apply the Pythagorean theorem with appropriate signs to find  $r$ .

$$r = \sqrt{x^2 + y^2} = \sqrt{(-5)^2 + (-12)^2} = 13.$$



**Step 3.** Evaluate the functions using definitions.

$$\sin \theta = \frac{y}{r} = \frac{-12}{13} = -\frac{12}{13}$$

$$\cos \theta = \frac{x}{r} = \frac{-5}{13} = -\frac{5}{13}$$

$$\tan \theta = \frac{y}{x} = \frac{12}{5}$$

$$\csc \theta = \frac{1}{\sin \theta} = -\frac{13}{12}$$

$$\sec \theta = \frac{1}{\cos \theta} = -\frac{13}{5}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{5}{12}$$

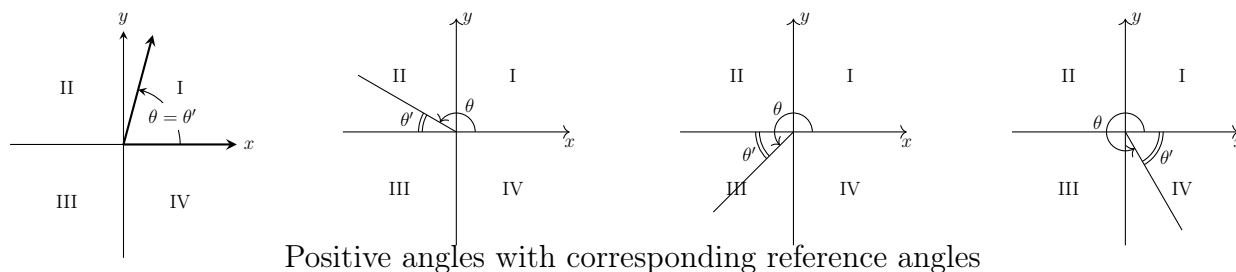
**Example 32.** Find the exact value of the All six trigonometric functions.

(a)  $\tan \theta = \frac{4}{3}$  and  $\cos \theta < 0$ .

(b) Given the point  $(3, -3)$  on the terminal side of an angle.

### 3.2 Reference Angles

We use reference angles to simplify the calculation of trigonometric functions of non-acute angles. The trigonometric function for a non-acute angle has the same value, except possibly for the sign, as the corresponding function of the acute angle. The acute angle is called the reference angle of the non-acute angle. Let  $\theta$  be an angle in standard position. The reference angle  $\theta'$  associated with  $\theta$  is the acute angle formed by the terminal side of  $\theta$  and the  $x$ -axis.



#### Finding Reference Angles

Method 1 Draw the angle in standard position and find the acute angle formed by the terminal side of  $\theta$  and the  $x$ -axis.

Method 2 (Without drawing the angle.)

**Step 1.** Find the quadrant in which  $\theta$  lies.

**Step 2.** Use the table below to calculate the reference angle,  $\theta'$ . Recall that  $360^\circ = 2\pi$  radians and that  $180^\circ = \pi$  radians.

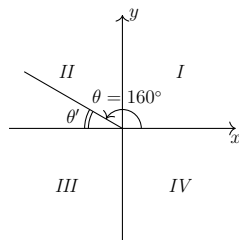
**Step 3.** If  $\theta$  is negative or larger than  $2\pi$  radians (or  $360^\circ$ ) find the coterminal angle that is between 0 and  $2\pi$  (or between  $0^\circ$  and  $360^\circ$ ). Recall that coterminal angles have the same initial and terminal sides, but possibly different rotations.

Quadrant Containing $\theta$	Reference angle $\theta'$ (degrees)	Reference angle $\theta'$ (radians)
I	$\theta' = \theta$	$\theta' = \theta$
II	$\theta' = 180^\circ - \theta$	$\theta' = \pi - \theta$
III	$\theta' = \theta - 180^\circ$	$\theta' = \theta - \pi$
IV	$\theta' = 360^\circ - \theta$	$\theta' = 2\pi - \theta$

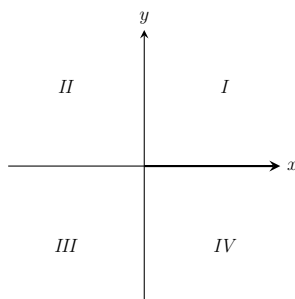


**Example 33.** Find the reference angle  $\theta'$ , for each given angle,  $\theta$ . Draw each angle in standard position and find the acute angle formed by the terminal side of  $\theta$  and the  $x$ -axis.

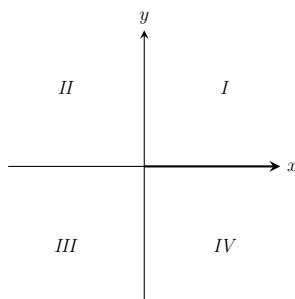
1. If  $\theta = 160^\circ$ , then  $\theta' =$



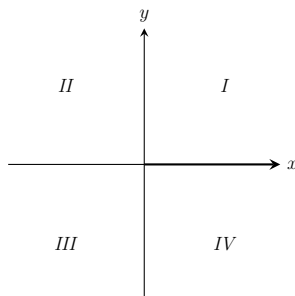
2. If  $\theta = 314^\circ$ , then  $\theta' =$



3. If  $\theta = 27^\circ$ , then  $\theta' =$



4. If  $\theta = 239^\circ$ , then  $\theta' =$



**Example 34.** Find the reference angle  $\theta'$  for the angle  $\theta$ .

(a)  $\theta = \frac{\pi}{5}$ .

Since  $\theta$  is in the first quadrant, the reference angle is  $\theta' = \theta = \frac{\pi}{5}$ .

(b)  $\theta = 150^\circ$ .

Since  $\theta$  is in the second quadrant, the reference angle is  $\theta' = 180^\circ - 150^\circ = 30^\circ$ . Converting  $\theta = 150^\circ$  to radians, we obtain  $\theta' = \pi - \frac{5}{6}\pi = \frac{\pi}{6}$ .

(c)  $\theta = 225^\circ$ .

Since  $\theta$  is in the third quadrant, the reference angle is  $\theta' = 225^\circ - 180^\circ = 45^\circ$ . Converting  $\theta = 225^\circ$  to radians, we obtain  $\theta' = \frac{5}{4}\pi - \pi = \frac{\pi}{4}$ .

(d)  $\theta = 300^\circ$ .

Since  $\theta$  is in the fourth quadrant, the reference angle is  $\theta' = 360^\circ - 300^\circ = 60^\circ$ . Converting  $\theta = 300^\circ$  to radians, we obtain  $\theta' = 2\pi - \frac{5}{3}\pi = \frac{\pi}{3}$ .

(e)  $\theta = \frac{7}{6}\pi$  radians.

Since  $\theta$  is in the third quadrant, the reference angle is  $\theta' = \frac{7}{6}\pi - \pi = \frac{\pi}{6}$ .

(f)  $\theta = \frac{3}{4}\pi$  radians.

Since  $\theta$  is in the second quadrant, the reference angle is  $\theta' = \pi - \frac{3}{4}\pi = \frac{\pi}{4}$ .

**Example 35.** Let  $\theta$  be an angle in standard position. Find the reference angle  $\theta'$  for each.

(a)  $\theta = 18^\circ$

(b)  $\theta = 275^\circ$

(c)  $\theta = -131^\circ$

(d)  $\theta = 520^\circ$

(e)  $\theta = \frac{3\pi}{4}$

(f)  $\theta = -\frac{7\pi}{4}$

(g)  $\theta = -\frac{17\pi}{4}$

### 3.3 Using Reference Angles to rewrite trigonometric functions as functions of positive acute angles.

**Example 36.** Rewrite  $\tan 125^\circ$  as a function of a positive acute angle. *DO NOT EVALUATE.*

**Step 1.** Find the quadrant in which  $125^\circ$  lies.

$125^\circ$  in Quadrant II

**Step 2.** Find the sign of  $\tan$  in that Quadrant.

$\tan$  is negative in Quadrant II.

**Step 3.** Find the reference angle in Quadrant II,

$$\theta' = 180^\circ - \theta = 180^\circ - 125^\circ = 55^\circ$$

**Step 4.** Finally, we can we write:

$$\tan 125^\circ = -\tan 55^\circ$$

**Example 37.** Rewrite the expression as a function of a positive acute angle. DO NOT EVALUATE.

(a)  $\sin 311^\circ =$

(b)  $\cos 142^\circ =$

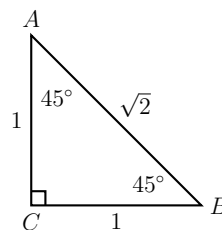
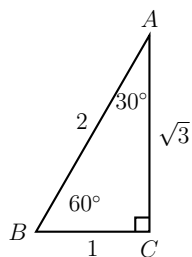
(c)  $\csc 152^\circ =$

(d)  $\tan 609^\circ =$

(e)  $\sin\left(\frac{7\pi}{12}\right) =$

(f)  $\cot\left(\frac{13\pi}{9}\right) =$

### 3.4 Using reference angles and special right triangles to FIND THE EXACT VALUE of Trigonometric Functions.



You may recall:

$$30^\circ = \frac{\pi}{6} \text{ radians}$$

$$45^\circ = \frac{\pi}{4} \text{ radians}$$

$$60^\circ = \frac{\pi}{3} \text{ radians}$$

**Example 38.** Find the EXACT VALUE of  $\sin\left(\frac{4\pi}{3}\right)$ .

**Step 1.** Find the quadrant in which  $\frac{4\pi}{3}$  lies.

$$\frac{4\pi}{3} \text{ in Quadrant III}$$

**Step 2.** Find the sign of  $\sin$  in that Quadrant III.

$\sin$  is negative in Quadrant III.

**Step 3.** Find the reference angle in Quadrant III.

$$\theta' = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

**Step 4.** We use the special triangle to evaluate the function.

$$\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}.$$

**Example 39.** Find the EXACT VALUE of each trigonometric function.

(a)  $\tan 120^\circ =$

(b)  $\cot 315^\circ =$

(c)  $\csc 150^\circ =$

(d)  $\cot 240^\circ =$

(e)  $\cot \left( \frac{5\pi}{4} \right) =$

(f)  $\sin \left( \frac{7\pi}{4} \right) =$

## EXERCISES

**Practice 24.** Find the quadrant in which  $\theta$  lies.

(a)  $\tan \theta < 0$  and  $\sec \theta > 0$ .

(b)  $\cos \theta < 0$  and  $\csc \theta > 0$ .

**Practice 25.** Find the exact value of the All six trigonometric functions.

(a)  $\cos \theta = \frac{1}{3}$  and  $\theta$  is in quadrant IV.

(b)  $\sin \theta = \frac{8}{17}$  and  $\cos \theta > 0$ .

(c) *Given the point  $(-1, 2)$  on the terminal side of an angle.*



**Practice 26.** Let  $\theta$  be an angle in standard position. Find the reference angle  $\theta'$  for each.

(a)  $\theta = 147^\circ$

(b)  $\theta = 213^\circ$

(c)  $\theta = -232^\circ$

(d)  $\theta = -665^\circ$

(e)  $\theta = \frac{7\pi}{6}$

(f)  $\theta = \frac{11\pi}{6}$

(g)  $\theta = \frac{6\pi}{5}$

**Practice 27.** Rewrite the expression as a function of a positive acute angle. DO NOT EVALUATE.

(a)  $\tan 280^\circ =$

(b)  $\cot 195^\circ =$

(c)  $\sin 394^\circ =$

(d)  $\tan\left(\frac{6\pi}{5}\right) =$

(e)  $\cos\left(\frac{4\pi}{5}\right) =$

(f)  $\csc\left(\frac{11\pi}{9}\right) =$

**Practice 28.** Find the EXACT VALUE of each trigonometric function.

(a)  $\sin 210^\circ =$

(b)  $\cos 330^\circ =$

(c)  $\sec 300^\circ =$

(d)  $\sin\left(\frac{3\pi}{4}\right) =$

(e)  $\cos\left(\frac{2\pi}{3}\right) =$

(f)  $\tan\left(\frac{4\pi}{3}\right) =$

(g)  $\sin\left(\frac{5\pi}{3}\right) =$

**Practice 29.** Find the equivalent expressions without using your calculator!

(a) Which expression(s) is(are) equivalent to  $\sin 200^\circ$ ?

(i)  $-\sin 20^\circ$

(ii)  $\cos 20^\circ$

(iii)  $\cos 70^\circ$

(iv)  $-\sin 70^\circ$

(b) Which expression(s) is(are) equivalent to  $\cos 150^\circ$ ?

(i)  $-\cos 30^\circ$

(ii)  $-\sin 60^\circ$

(iii)  $\cos(-210^\circ)$

(iv)  $-\cos 210^\circ$

(c) Which expression(s) is(are) equivalent to  $\tan 230^\circ$ ?

(i)  $\cot 40^\circ$

(ii)  $-\cot 40^\circ$

(iii)  $\tan 50^\circ$

(iv)  $-\tan 50^\circ$

## Unit 4: Graphs of the Sine Curve

### 4.1 The Properties and Graphs of the Sine Curve: $y = \sin x$

- (a) Let  $x$  be an angle in terms of **radians**.
- (b) Because the values of  $\sin x$  repeat every  $2\pi$  radians, the sine function is a **periodic function**.
- (c) The **period** of the function is the length of the interval of  $x$  that is needed to produce one complete cycle of the curve.
- (d) The sine curve **oscillates** about a **horizontal line** (let's call it the **center-line**). For the basic sine curve, it is the  **$x$ -axis**.
- (e) The **amplitude** of the function is the distance from the **center-line** of the function to the **maximum  $y$**  value. (Or from the **center-line** of the function to the **minimum  $y$**  value.) (Or one-half the difference between the maximum  $y$  value and the minimum  $y$  value.) Since the amplitude is a distance, it is a positive number.
- (f) The tables of values of the basic sine curve  $y = \sin x$ .

#### $x$ in Quadrant I

Degrees	Radians	Reference Angle	$y = \sin x$
$0^\circ$	0	$0^\circ$	0
$30^\circ$	$\frac{\pi}{6}$	$30^\circ$	$\frac{1}{2}$
$45^\circ$	$\frac{\pi}{4}$	$45^\circ$	$\frac{\sqrt{2}}{2} \approx .71$
$60^\circ$	$\frac{\pi}{3}$	$60^\circ$	$\frac{\sqrt{3}}{2} \approx .87$
$90^\circ$	$\frac{\pi}{2}$	$90^\circ$	1

**$x$  in Quadrant II**

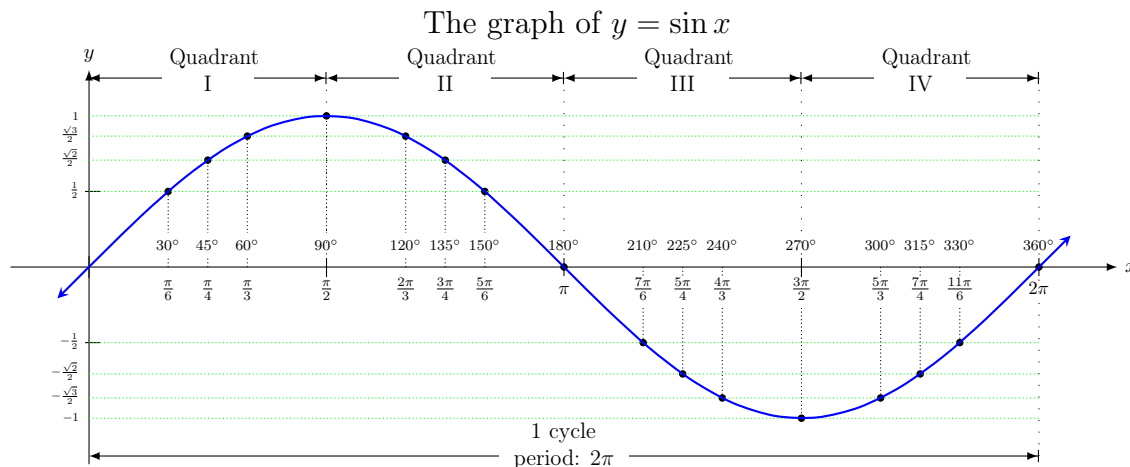
Degrees	Radians	Reference Angle	$y = \sin x$
$120^\circ$	$\frac{2\pi}{3}$	$60^\circ$	$\sin 60^\circ = \frac{\sqrt{3}}{2} \approx .87$
$135^\circ$	$\frac{3\pi}{4}$	$45^\circ$	$\sin 45^\circ = \frac{\sqrt{2}}{2} \approx .71$
$150^\circ$	$\frac{5\pi}{6}$	$30^\circ$	$\sin 30^\circ = \frac{1}{2}$
$180^\circ$	$\pi$	$0^\circ$	$\sin 0^\circ = 0$

 **$x$  in Quadrant III**

Degrees	Radians	Reference Angle	$y = \sin x$
$210^\circ$	$\frac{7\pi}{6}$	$30^\circ$	$-\sin 30^\circ = -\frac{1}{2}$
$225^\circ$	$\frac{5\pi}{4}$	$45^\circ$	$-\sin 45^\circ = -\frac{\sqrt{2}}{2} \approx -.71$
$240^\circ$	$\frac{4\pi}{3}$	$60^\circ$	$-\sin 60^\circ = -\frac{\sqrt{3}}{2} \approx -.87$
$270^\circ$	$\frac{3\pi}{2}$	$90^\circ$	$-\sin 90^\circ = -1$

 **$x$  in Quadrant IV**

Degrees	Radians	Reference Angle	$y = \sin x$
$300^\circ$	$\frac{5\pi}{3}$	$60^\circ$	$-\sin 60^\circ = -\frac{\sqrt{3}}{2} \approx -.87$
$315^\circ$	$\frac{7\pi}{4}$	$45^\circ$	$-\sin 45^\circ = -\frac{\sqrt{2}}{2} \approx -.71$
$330^\circ$	$\frac{11\pi}{6}$	$30^\circ$	$-\sin 30^\circ = -\frac{1}{2}$
$360^\circ$	$2\pi$	$0^\circ$	$\sin 0^\circ = 0$



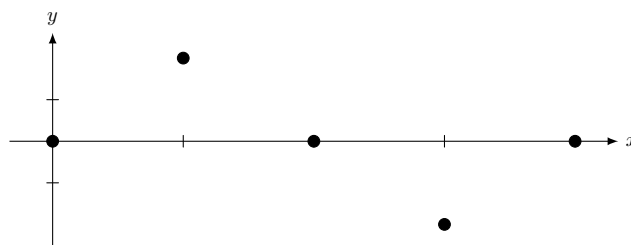
**Observations from the graph of  $y = \sin x$ :**

- (a) The curve of Quadrant I is symmetrical to Quadrant II about the vertical line  $x = \frac{\pi}{2}$ .
- (b) The curve of Quadrant III is symmetrical to Quadrant IV about the vertical line  $x = \frac{3\pi}{2}$ .
- (c) What is the horizontal line about which the sine curve oscillates (the center-line)?
- (d) Compare the curve of Quadrants I and II to the curve of Quadrants III and IV relative to the center-line.
- (e) The graph of the sine curve highlights the fact that the values of sine are positive in Quadrants I and II, and the values are negative in Quadrants III and IV.

**Questions on the graph:**

- (a) What is the period of the graph?
- (b) What is the amplitude of the graph?
- (c) What are the coordinates of the maximum and minimum points?
- (d) What is the center-line of the graph? What are the coordinates of the points of the graph that lie on the center-line?
- (e) Why are the values of the sine curve positive in Quadrants I and II?
- (f) Why are the values of the sine curve negative in Quadrants III and IV?
- (g) In this course, we are asked to roughly sketch a complete cycle of a sine curve. To do this, we just consider the following 5 points of a sine curve: the maximum and the minimum points above and below the center-line and the three points that are on the center-line of the curve. These points are given in the table below. List the coordinates of these 5 points, plot them on the graph and also label these points on the graph. Do you see how these points give you a rough sketch of the sine curve?
- (h) What is the relationship between the  $x$ -coordinates of the five points and the period of the curve?

$x$	$y = \sin x$	$(x, y)$
0	$y = \sin 0 = 0$	
$\frac{\pi}{2}$	$y = \sin \frac{\pi}{2} = 1$ (max)	
$\pi$	$y = \sin \pi = 0$	
$\frac{3\pi}{2}$	$y = \sin \frac{3\pi}{2} = -1$ (min)	
$2\pi$	$y = \sin 2\pi = 0$	





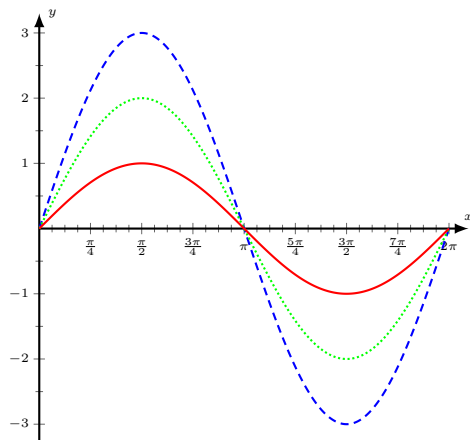
## 4.2 Variation I: $y = A \sin x$

Consider the function  $y = A \sin x$ , where  $A$  is a constant. In this case, all the  $y$  values of the basic sine curve are multiplied by the constant  $A$ .

**Example 40.** Compare the basic graph  $y = \sin x$  to the graphs  $y = 2 \sin x$  and  $y = 3 \sin x$ . To roughly sketch these graphs, we are plotting the 5 points. The  $y$ -value of every point is multiplied by the constant  $A$ .

$y = \sin x$		$y = 2 \sin x$		$y = 3 \sin x$	
$x$	$y$	$x$	$y$	$x$	$y$
0	0	0	$2 \cdot 0 = 0$	0	?
$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	$2 \cdot 1 = 2$	$\frac{\pi}{2}$	?
$\pi$	0	$\pi$	$2 \cdot 0 = 0$	$\pi$	?
$\frac{3\pi}{2}$	-1	$\frac{3\pi}{2}$	$2 \cdot (-1) = -2$	$\frac{3\pi}{2}$	?
$2\pi$	0	$2\pi$	$2 \cdot 0 = 0$	$2\pi$	?

Write the correct equation next to each graph.



**Observations from the graph of  $y = A \sin x$ :**

How does multiplying  $\sin x$  by  $A$  change the following characteristics of  $\sin x$ :

Period: \_\_\_\_\_

Amplitude: \_\_\_\_\_

Center-Line: \_\_\_\_\_

**Example 41.** For each of the three curves:  $y = \sin x$ ,  $y = 2 \sin x$ , and  $y = 3 \sin x$ , state the period, amplitude and the center-line.

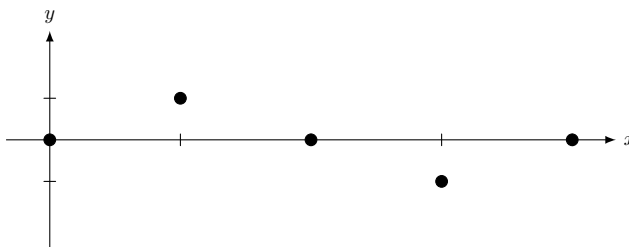
$y = \sin x$	$y = 2 \sin x$	$y = 3 \sin x$
<i>Period:</i>	<i>Period:</i>	<i>Period:</i>
<i>Amplitude:</i>	<i>Amplitude:</i>	<i>Amplitude:</i>
<i>Center-Line:</i>	<i>Center-Line:</i>	<i>Center-Line:</i>

**Example 42.** (a) Roughly sketch one complete cycle of  $y = \frac{1}{2} \sin x$ . On the graph, label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

(b) The  $x$ -coordinates of the 5 points are related to the period of the graph. State the  $x$ -coordinates in terms of the period.

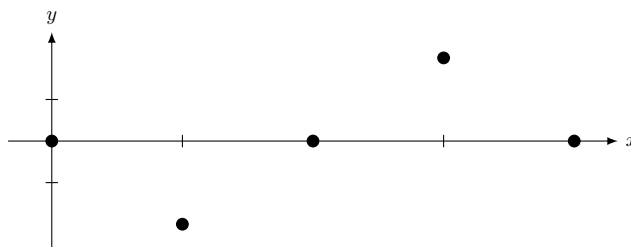
(c) The  $y$ -coordinates of the 5 points are related to another feature of the graph. State the  $y$ -coordinates in terms of that feature.

$x$	$y = \frac{1}{2} \sin x$	$(x, y)$



**Example 43.** Roughly sketch one complete cycle of  $y = -4 \sin x$ . On the graph, label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve. How does a negative constant  $A$  change the sine curve?

$x$	$y = -4 \sin x$	$(x, y)$



**Example 44.** Briefly describe what  $A$  in  $y = A \sin x$  does to the basic sine curve. How is the graph of the basic sine curve changed when it is multiplied by  $A$ ? What features are not changed by the multiplication by  $A$ ?

**Example 45.** Match each equation with its graph. Also write the period, amplitude and center-line next to each graph.

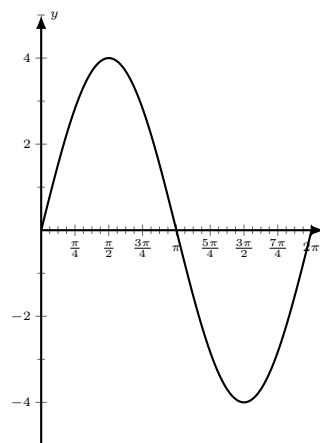
(a)  $y = -\sin x$

(b)  $y = 4 \sin x$

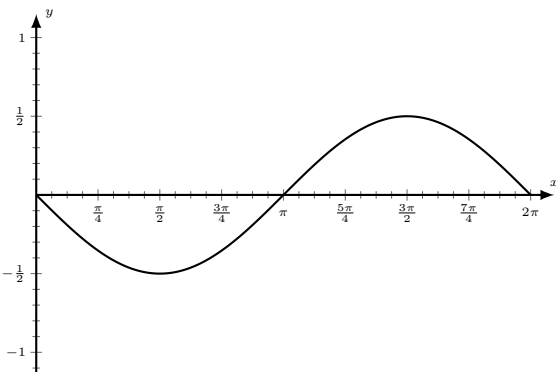
(c)  $y = -2 \sin x$

(d)  $y = -\frac{1}{2} \sin x$

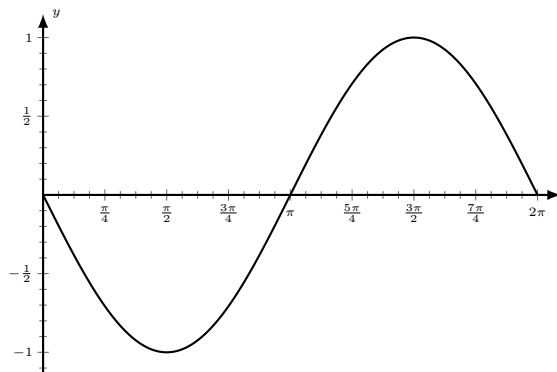
(e)  $y = 2 \sin x$



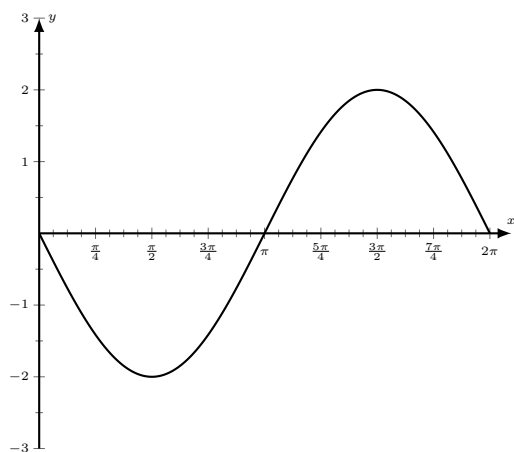
A



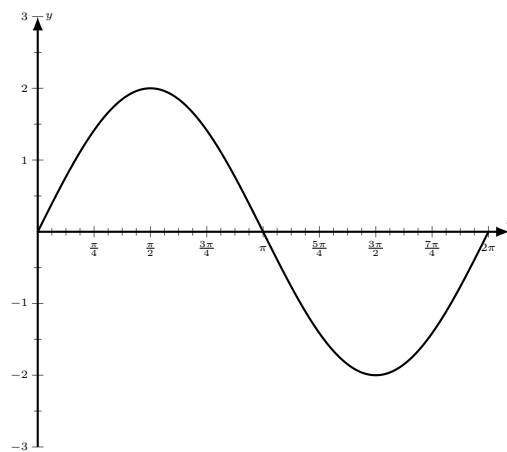
B



C



D



E

### 4.3 Variation II: $y = \sin Bx$

Consider the function  $y = \sin Bx$ , where  $B$  is a constant.

**Example 46.** Graph the function  $y = \sin 2x$ .

Let's look at some values of the graph.

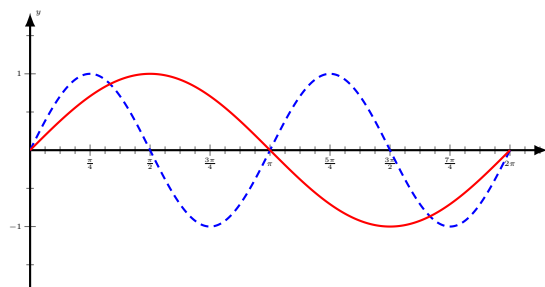
<u><math>x</math> in radians</u>	<u><math>y = \sin 2x</math></u>
0	$\sin 0 = 0$
$\frac{\pi}{12}$	$\sin(2 \cdot \frac{\pi}{12}) = \sin \frac{\pi}{6} = \frac{1}{2}$
$\frac{\pi}{6}$	$\sin(2 \cdot \frac{\pi}{6}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\sin(2 \cdot \frac{\pi}{4}) = \sin \frac{\pi}{2} = 1$
$\frac{\pi}{2}$	$\sin(2 \cdot \frac{\pi}{2}) = \sin \pi = 0$
$\pi$	$\sin 2\pi = 0$

We observe that when  $x$  goes from 0 to  $\pi$ , the corresponding  $y$  value of  $\sin 2x$  goes from  $\sin 0$  to  $\sin 2\pi$ , forming one complete cycle of the sine curve.

This observation can be stated in other ways:

- (a) Because we are considering the sine of 2 times the angle  $x$ , the sine values go twice as fast. A complete cycle from  $\sin 0$  to  $\sin 2\pi$  is achieved in half of a regular period of  $2\pi$ .
- (b) The period of the curve  $y = \sin 2x$  is half of the regular period of  $2\pi$  or  $\frac{2\pi}{2} = \pi$ .
- (c) Because the sine values go twice as fast, there are 2 complete cycles in a regular period of  $2\pi$ .

Write the correct equation next to each graph.



**Observations from the graph of  $y = \sin 2x$ :**

(a) The graph of  $y = \sin 2x$  completes 2 full cycles in  $2\pi$ . The graph of  $y = \sin x$  completes one full cycle in  $2\pi$ .

(b) How does multiplying the angle  $x$  by 2 change the following characteristics of  $\sin x$ :

Period: \_\_\_\_\_

Amplitude: \_\_\_\_\_

Center-Line: \_\_\_\_\_

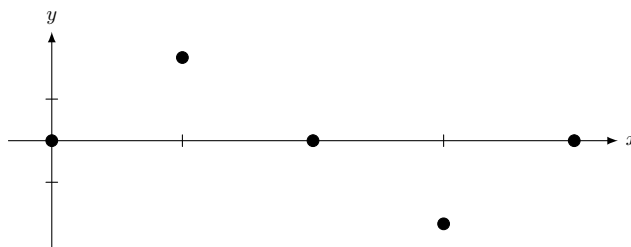
**Example 47.** Graph the function  $y = \sin 4x$ .

The  $y$  value is the sine value of 4 times the angle  $x$ . Therefore one complete cycle of sine values is reached 4 times as fast.

Let's use the following questions to help us graph the function.

- What is the period of  $y = \sin 4x$ ?
- What is the amplitude?
- What is the center-line?
- How many complete cycles of  $y = \sin 4x$  are there in  $2\pi$ ?
- What are the  $x$ -coordinates of the 5 points?
- Roughly sketch one complete cycle of  $y = \sin 4x$ . Notice how the  $x$ -coordinates of the 5 points are related to the period and how the  $y$ -coordinates are related to the amplitude.

$x$	$y = \sin 4x$	$(x, y)$
$0$		
$\frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$		
$\frac{2}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$		
$\frac{3}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi}{8}$		
$\frac{4}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$		





**Example 48.** Briefly describe how  $B$  in  $y = \sin Bx$  affects the

Period: \_\_\_\_\_

Amplitude: \_\_\_\_\_

Center-Line: \_\_\_\_\_

We've observed that the period of the curve  $y = \sin 2x$  is half of the regular period of  $2\pi$  or  $\frac{2\pi}{2} = \pi$ . Similarly, we observed that the period of the curve  $y = \sin 4x$  is one-quarter of the regular period of  $2\pi$  or  $\frac{2\pi}{4} = \frac{\pi}{2}$ . We can extend this concept to the curve  $y = \sin Bx$ , and arrive at the following formula to find the period,  $P$ :

$$P = \frac{2\pi}{B}$$

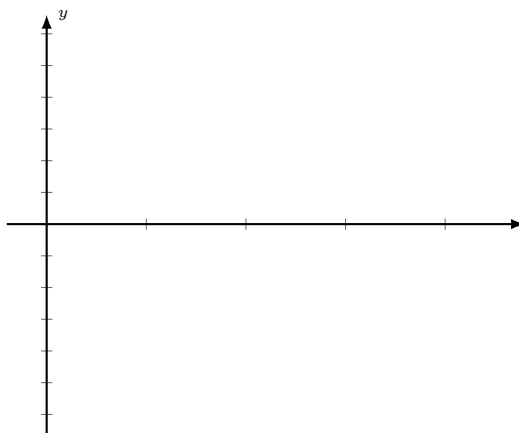
The constant  $B$  is called the angular frequency and represents the number of cycles of the curve that is completed between 0 and  $2\pi$ .

**Example 49.** (a) Roughly sketch one complete cycle of  $y = \sin \frac{1}{2}x$ . On the graph, label the coordinates of the 5 points. Determine the period, amplitude and the center-line of the curve.

(b) Write the  $x$ -coordinates of the 5 points in terms of the period.

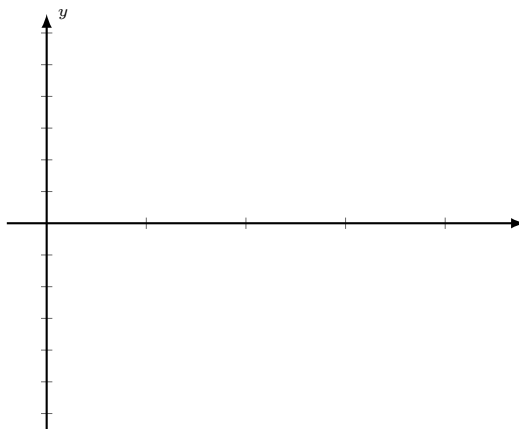
(c) Write the  $y$ -coordinates in terms of the amplitude.

$x$	$y = \sin \frac{1}{2}x$	$(x, y)$



**Example 50.** Roughly sketch one complete cycle of  $y = \sin \frac{1}{4}x$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve. Notice how the coordinates of the 5 points are related to the period and the amplitude.

$x$	$y = \sin \frac{1}{4}x$	$(x, y)$



#### 4.4 Variation III: $y = A \sin Bx$

Consider the function  $y = A \sin Bx$ , where  $A$  and  $B$  are both constants.

In this case, we are combining two variations together in the same graph. First the angle  $x$  is multiplied by the constant  $B$ . Then, all the  $y$  values of the  $y = \sin Bx$  curve are multiplied by the constant  $A$ .

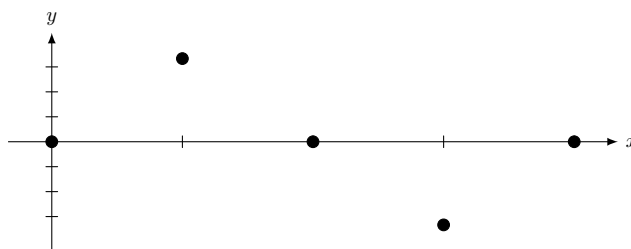
**Example 51.** Graph the function  $y = 3 \sin 2x$

First, we consider the graph of  $y = \sin 2x$ . This is a sine curve with a period of  $\pi$ , and an amplitude of 1. What are the coordinates of the 5 points of  $y = \sin 2x$ ?

Next, we multiply the  $y$ -values of  $y = \sin 2x$  by 3 to obtain the  $y$ -values of the curve  $y = 3 \sin 2x$ .

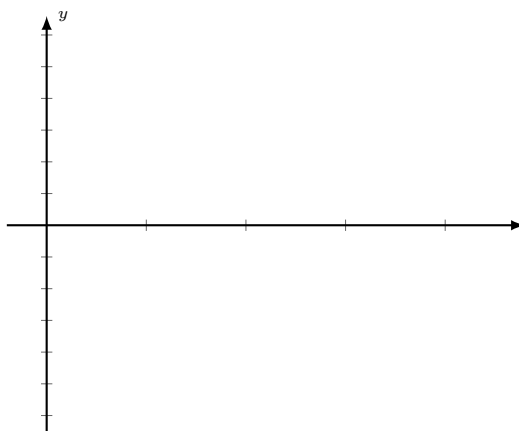
Roughly sketch one complete cycle of  $y = 3 \sin 2x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = 3 \sin 2x$	$(x, y)$



**Example 52.** Roughly sketch one complete cycle of  $y = -3 \sin \frac{1}{2}x$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve. Notice how the coordinates of the 5 points are related to the period and the amplitude.

$x$	$y = -3 \sin \frac{1}{2}x$	$(x, y)$



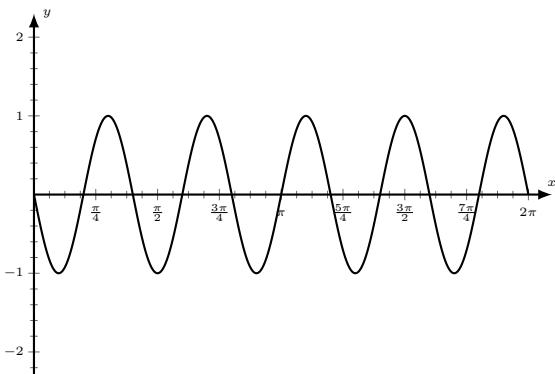
**Example 53.** Match each equation with its graph. Also write the period, amplitude and center-line next to each graph.

(a)  $y = -3 \sin \frac{x}{8}$

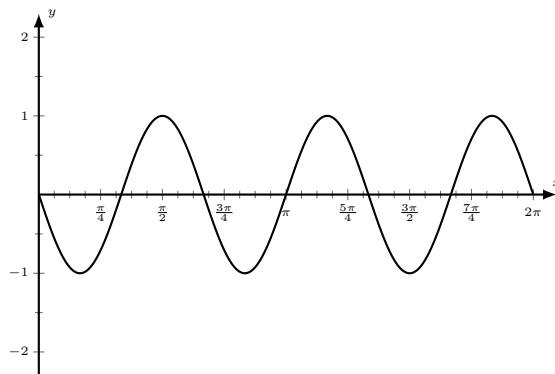
(b)  $y = -\sin 3x$

(c)  $y = 2 \sin \frac{x}{3}$

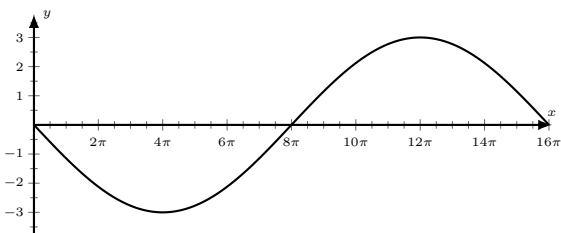
(d)  $y = -\sin(5x)$



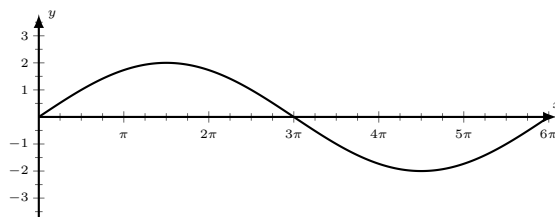
A



B



C

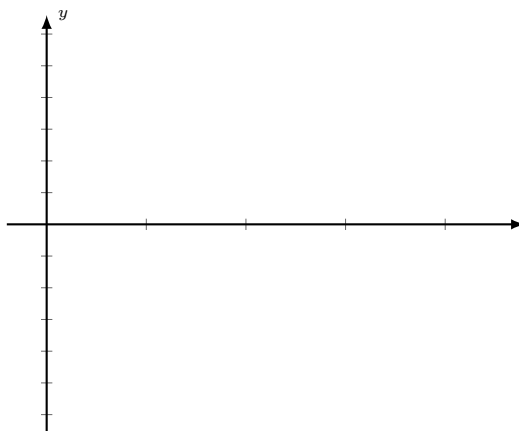


D

## EXERCISES

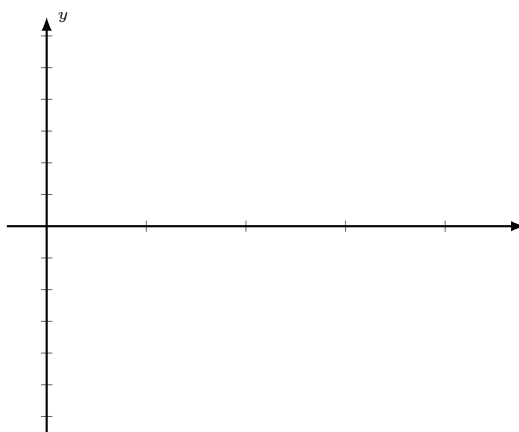
**Practice 30.** Roughly sketch one complete cycle of  $y = 5 \sin x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = 5 \sin x$	$(x, y)$



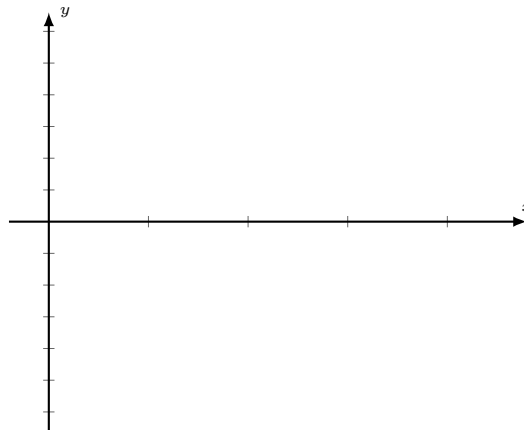
**Practice 31.** Roughly sketch one complete cycle of  $y = -\frac{3}{4} \sin x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = -\frac{3}{4} \sin x$	$(x, y)$



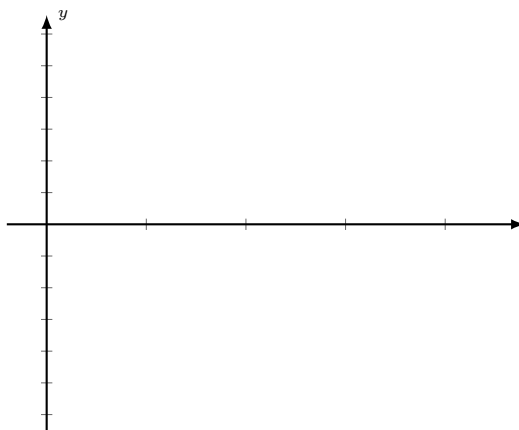
**Practice 32.** Roughly sketch one complete cycle of  $y = \sin \pi x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = \sin \pi x$	$(x, y)$



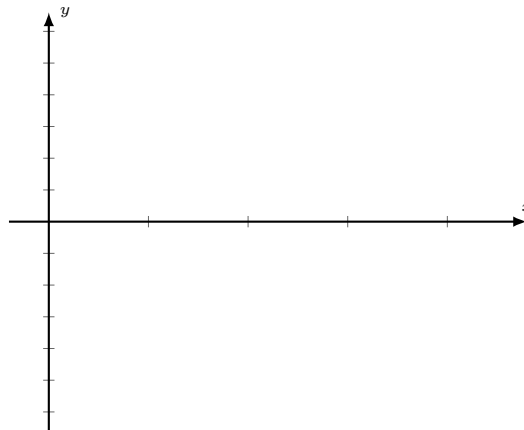
**Practice 33.** Roughly sketch one complete cycle of  $y = \sin 3x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = \sin 3x$	$(x, y)$



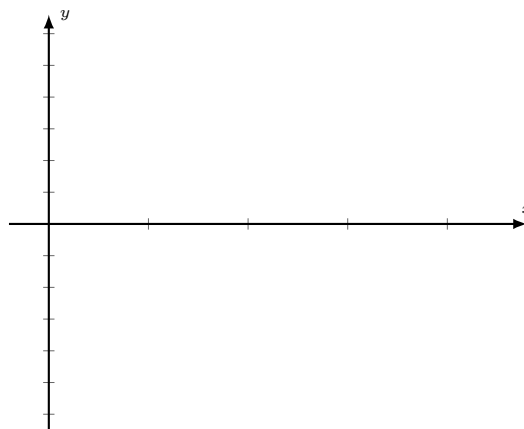
**Practice 34.** Roughly sketch one complete cycle of  $y = -4 \sin\left(\frac{\pi}{4}x\right)$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = -4 \sin\left(\frac{\pi}{4}x\right)$	$(x, y)$



**Practice 35.** Roughly sketch one complete cycle of  $y = -\frac{1}{5} \sin 4x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = -\frac{1}{5} \sin 4x$	$(x, y)$





## Unit 5: Graphs of the Cosine Curve

### 5.1 The Properties and Graphs of the Cosine Curve: $y = \cos x$

Recall that cosine and sine are cofunctions. That is,  $\cos x$  is equal to the sine of the complement of  $x$ . Or,  $\cos x = \sin(\frac{\pi}{2} - x)$ . This means that the cosine curve is the sine curve, shifted by  $\frac{\pi}{2}$  radians.

Let's look at a table of values of the basic cosine curve  $y = \cos x$ .

#### $x$ in Quadrant I

Degree	Radians	Reference Angle	$y = \cos x$
$0^\circ$	0	$0^\circ$	1
$30^\circ$	$\frac{\pi}{6}$	$30^\circ$	$\frac{\sqrt{3}}{2} \approx .87$
$45^\circ$	$\frac{\pi}{4}$	$45^\circ$	$\frac{\sqrt{2}}{2} \approx .71$
$60^\circ$	$\frac{\pi}{3}$	$60^\circ$	$\frac{1}{2}$
$90^\circ$	$\frac{\pi}{2}$	$90^\circ$	0

#### $x$ in Quadrant II

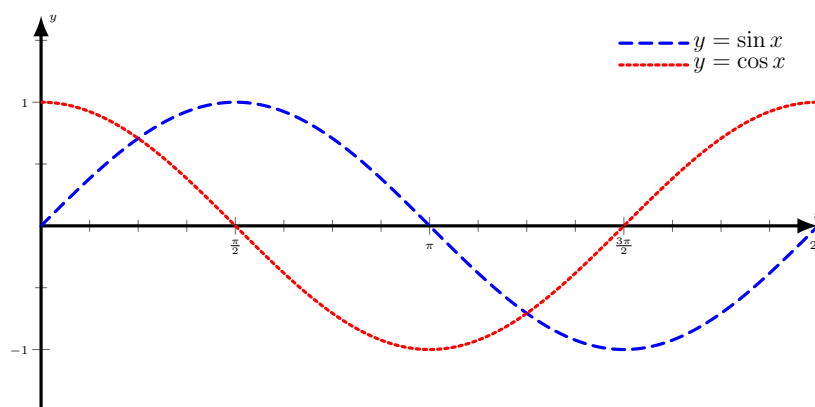
Degree	Radians	Reference Angle	$y = \cos x$
$120^\circ$	$\frac{2\pi}{3}$	$60^\circ$	$-\cos 60^\circ = -\frac{1}{2}$
$135^\circ$	$\frac{3\pi}{4}$	$45^\circ$	$-\cos 45^\circ = -\frac{\sqrt{2}}{2} \approx -.71$
$150^\circ$	$\frac{5\pi}{6}$	$30^\circ$	$-\cos 30^\circ = -\frac{\sqrt{3}}{2} \approx -.87$
$180^\circ$	$\pi$	$0^\circ$	$-\cos 0^\circ = -1$

**$x$  in Quadrant III**

Degree	Radians	Reference Angle	$y = \cos x$
$210^\circ$	$\frac{7\pi}{6}$	$30^\circ$	$-\cos 30^\circ = -\frac{\sqrt{3}}{2} \approx -.87$
$225^\circ$	$\frac{5\pi}{4}$	$45^\circ$	$-\cos 45^\circ = -\frac{\sqrt{2}}{2} \approx -.71$
$240^\circ$	$\frac{4\pi}{3}$	$60^\circ$	$-\cos 60^\circ = -\frac{1}{2}$
$270^\circ$	$\frac{3\pi}{2}$	$90^\circ$	$\cos 90^\circ = 0$

 **$x$  in Quadrant IV**

Degree	Radians	Reference Angle	$y = \cos x$
$300^\circ$	$\frac{5\pi}{3}$	$60^\circ$	$\cos 60^\circ = \frac{1}{2}$
$315^\circ$	$\frac{7\pi}{4}$	$45^\circ$	$\cos 45^\circ = \frac{\sqrt{2}}{2} \approx .71$
$330^\circ$	$\frac{11\pi}{6}$	$30^\circ$	$\cos 30^\circ = \frac{\sqrt{3}}{2} \approx .87$
$360^\circ$	$2\pi$	$0^\circ$	$\cos 0^\circ = 1$



**Observations from the graph of  $y = \cos x$ :**

- (a) Compare the values of  $\cos x$  in Quadrant I to the values of  $\sin x$  in Quadrant II.
- (b) Compare the values of  $\cos x$  in Quadrant II to the values of  $\sin x$  in Quadrant III.
- (c) Do you think this pattern will continue to other quadrants?
- (d) Just like the basic sine curve, the basic cosine curve is also a periodic function with

Period: \_\_\_\_\_

Amplitude: \_\_\_\_\_

Center-Line: \_\_\_\_\_

Coordinates of the 5 points: \_\_\_\_\_

- (e) The graph of the basic cosine curve highlights the fact that the values of cosine are positive in Quadrants I and IV, and the values of cosine are negative in Quadrants II and III.
- (f) What are the values of  $x$  when  $\cos x = \sin x$ , for  $0 \leq x \leq 2\pi$ ?

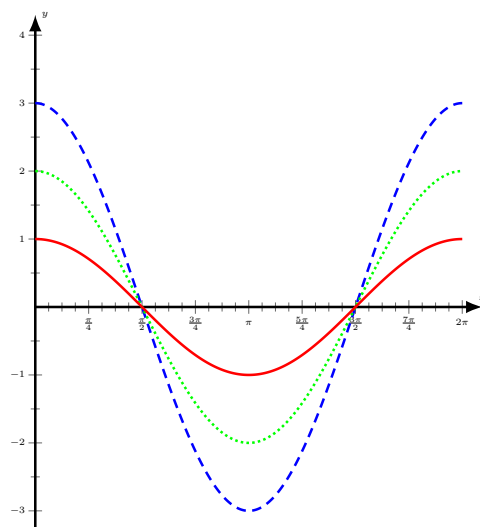
## 5.2 Variation I: $y = A \cos x$

Consider the function  $y = A \cos x$ , where  $A$  is a constant. Just like the graph  $y = A \sin x$ , all the  $y$  values of the basic cosine curve are multiplied by the constant  $A$ . The period ( $2\pi$ ) and the center-line ( $x$ -axis) are unchanged.

**Example 54.** Compare the basic graph  $y = \cos x$  to the graphs  $y = 2 \cos x$  and  $y = 3 \cos x$ .

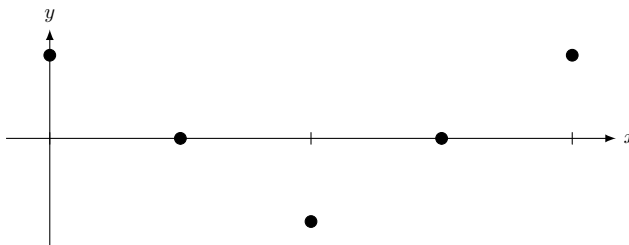
$y = \cos x$		$y = 2 \cos x$		$y = 3 \cos x$	
$x$	$y$	$x$	$y$	$x$	$y$
0	1	0	$2 \cdot 1 = 2$	0	?
$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	$2 \cdot 0 = 0$	$\frac{\pi}{2}$	?
$\pi$	-1	$\pi$	$2 \cdot (-1) = -2$	$\pi$	?
$\frac{3\pi}{2}$	0	$\frac{3\pi}{2}$	$2 \cdot 0 = 0$	$\frac{3\pi}{2}$	?
$2\pi$	1	$2\pi$	$2 \cdot 1 = 2$	$2\pi$	?

Write the correct formula next to each graph.



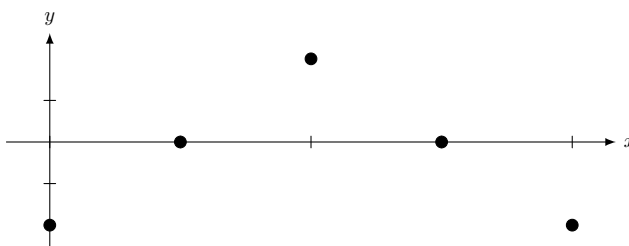
**Example 55.** Roughly sketch one complete cycle of  $y = \frac{1}{2} \cos x$ . On the graph, label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = \frac{1}{2} \cos x$	$(x, y)$



**Example 56.** Roughly sketch one complete cycle of  $y = -2 \cos x$ . On the graph, label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = -2 \cos x$	$(x, y)$



How does a negative constant  $A$  change the cosine curve?

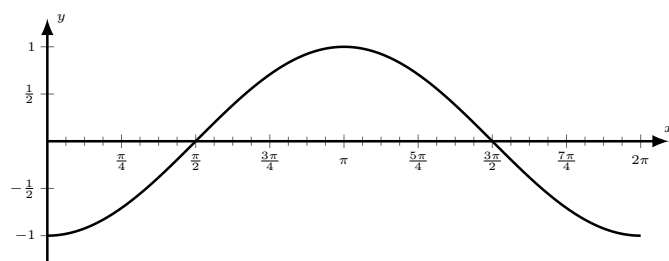
**Example 57.** Match each equation with its graph. Also write the period, amplitude and center-line next to each graph.

(a)  $y = 3 \cos x$

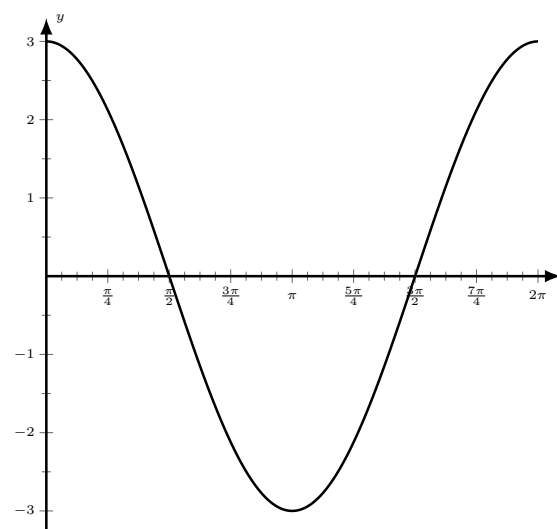
(b)  $y = -4 \cos x$

(c)  $y = -\frac{1}{2} \cos x$

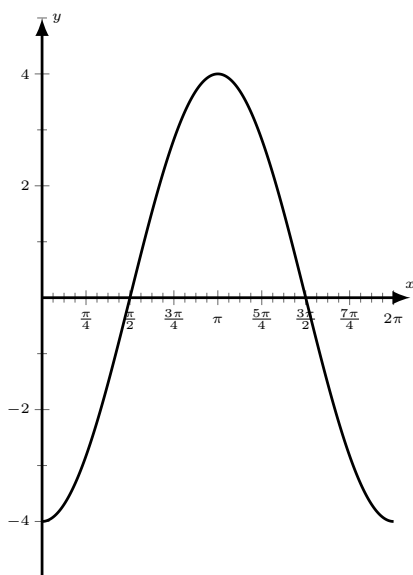
(d)  $y = -\cos x$



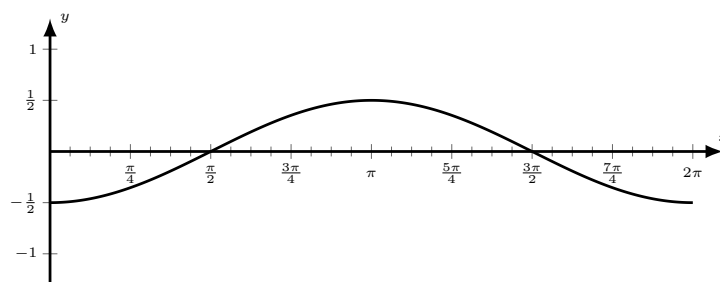
A



B



C



D

### 5.3 Variation II: $y = \cos Bx$

Consider the function  $y = \cos Bx$ , where  $B$  is a constant. As in the graph of  $y = \sin Bx$ , the angle  $x$  is multiplied by  $B$ .

**Example 58.**  $y = \cos 2x$

Let's look at some values of the graph.

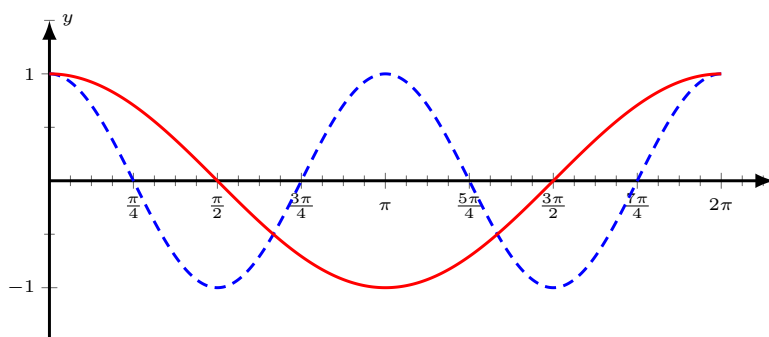
<u><math>x</math> in radians</u>	<u><math>y = \cos 2x</math></u>
0	$\cos 0 = 1$
$\frac{\pi}{12}$	$\cos(2 \cdot \frac{\pi}{12}) = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$\frac{\pi}{6}$	$\cos(2 \cdot \frac{\pi}{6}) = \cos \frac{\pi}{3} = \frac{1}{2}$
$\frac{\pi}{4}$	$\cos(2 \cdot \frac{\pi}{4}) = \cos \frac{\pi}{2} = 0$
$\frac{\pi}{2}$	$\cos(2 \cdot \frac{\pi}{2}) = \cos \pi = -1$
$\pi$	$\cos 2\pi = 1$

We observe that as  $x$  goes from 0 to  $\pi$ , the corresponding  $y$  values of  $\cos 2x$  goes from  $\cos 0$  to  $\cos 2\pi$ , forming one complete cycle of the cosine curve.

This observation can be stated in other ways:

- (a) Because we are considering the cosine of 2 times the angle  $x$ , a complete cycle from  $\cos 0$  to  $\cos 2\pi$  is achieved in half of a regular period of  $2\pi$ .
- (b) The period of the curve  $y = \cos 2x$  is half of the regular period of  $2\pi$  or  $\frac{2\pi}{2} = \pi$ .
- (c) There are 2 complete cycles in a regular period of  $2\pi$ .

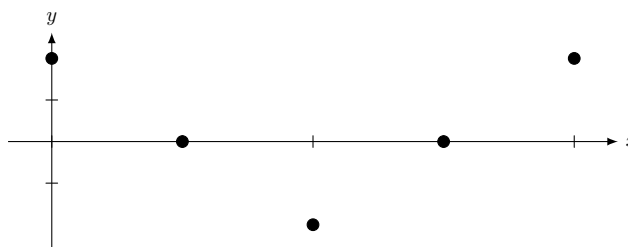
Write the correct formula next to each graph.



**Example 59.** Consider the function  $y = \cos 4x$ .

The  $y$  value is the cosine value of 4 times the angle  $x$ . Therefore one complete cycle of cosine values is reached in one-quarter of a regular period of  $2\pi$ .

$x$	$y = \cos 4x$	$(x, y)$
$0$		
$\frac{1}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{8}$		
$\frac{2}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$		
$\frac{3}{4}\left(\frac{\pi}{2}\right) = \frac{3\pi}{8}$		
$\frac{4}{4}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$		





**Example 60.** Briefly describe how  $B$  in  $y = \cos Bx$  affects:

Period: \_\_\_\_\_

Amplitude: \_\_\_\_\_

Center-Line: \_\_\_\_\_

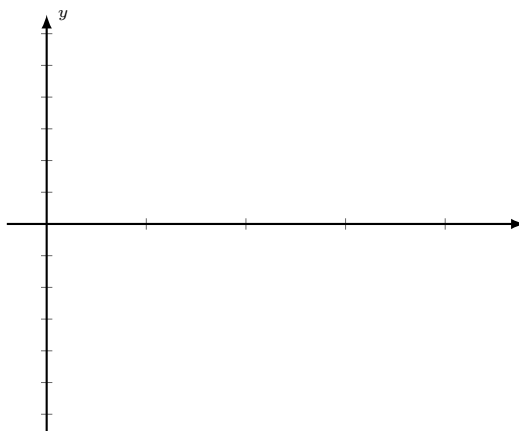
We've observed that the period of the curve  $y = \cos 2x$  is half of the regular period of  $2\pi$  or  $\frac{2\pi}{2} = \pi$ . Similarly, we observed that the period of the curve  $y = \cos 4x$  is one-quarter of the regular period of  $2\pi$  or  $\frac{2\pi}{4} = \frac{\pi}{2}$ . We can extend this concept to the curve  $y = \cos Bx$ , and arrive at the following formula to find the period,  $P$

$$P = \frac{2\pi}{B}$$

The constant  $B$  is called the angular frequency and represents the number of cycles of the curve that is completed between 0 and  $2\pi$ .

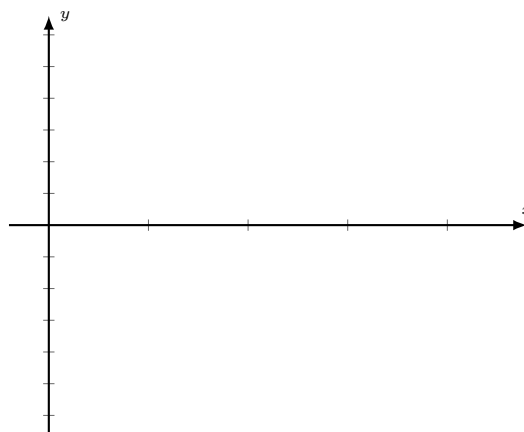
**Example 61.** Roughly sketch one complete cycle of  $y = \cos \frac{1}{2}x$ . On the graph, label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = \cos \frac{1}{2}x$	$(x, y)$



**Example 62.** Roughly sketch one complete cycle of  $y = \cos \pi x$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = \cos \pi x$	$(x, y)$

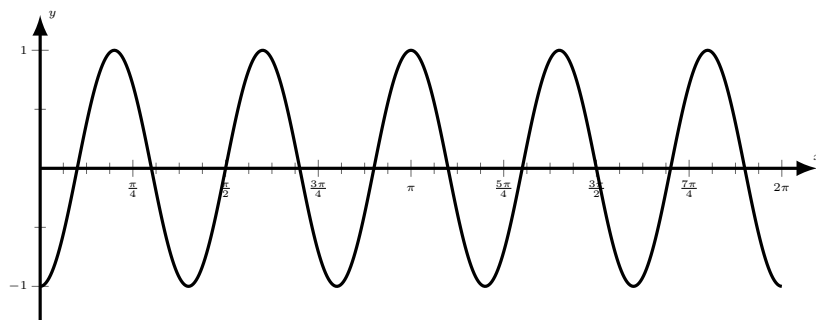


**Example 63.** Match each equation with its graph. Also write the period, amplitude and center-line next to each graph.

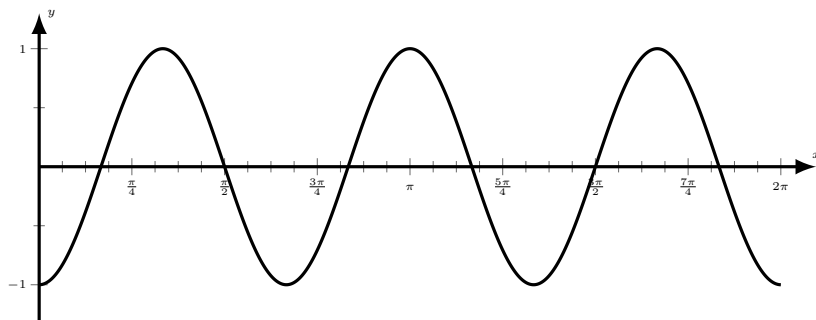
(a)  $y = -\cos 3x$

(b)  $y = \cos \frac{\pi}{3}$

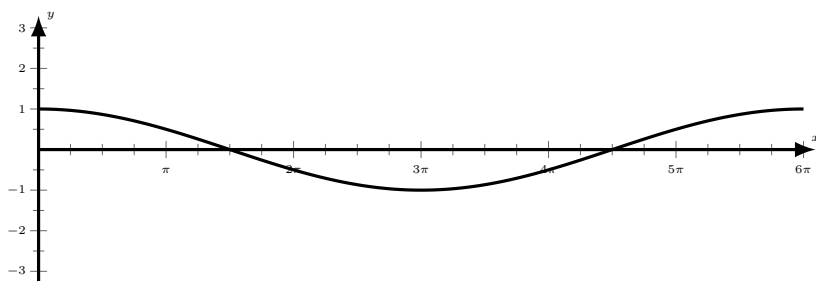
(c)  $y = -\cos 5x$



A



B



C

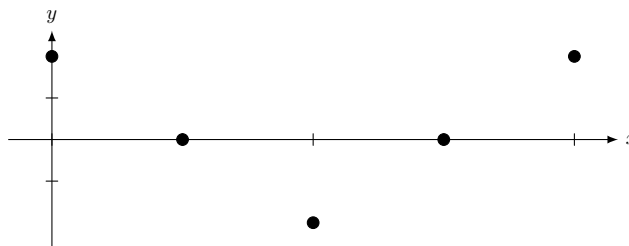
### 5.4 Variation III: $y = A \cos Bx$

Consider the function  $y = A \cos Bx$ , where  $A$  and  $B$  are both constants.

In this case, we are combining two variations together in the same graph. First the angle  $x$  is multiplied by the constant  $B$ . Then, all the  $y$  values of the  $y = \cos Bx$  curve are multiplied by the constant  $A$ .

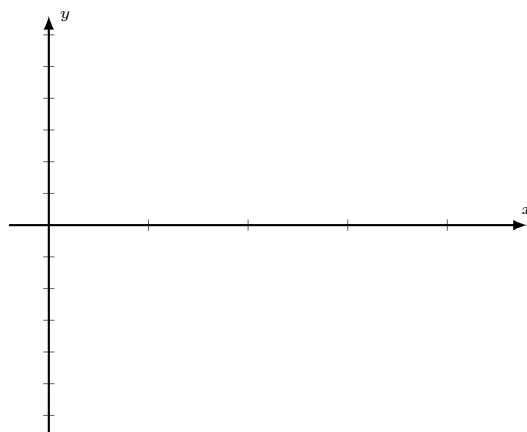
**Example 64.** Roughly sketch one complete cycle of the function  $y = 3 \cos\left(\frac{x}{2}\right)$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = 3 \cos \frac{x}{2}$	$(x, y)$



**Example 65.** Roughly sketch one complete cycle of the function  $y = -4 \cos(\pi x)$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = -4 \cos(\pi x)$	$(x, y)$



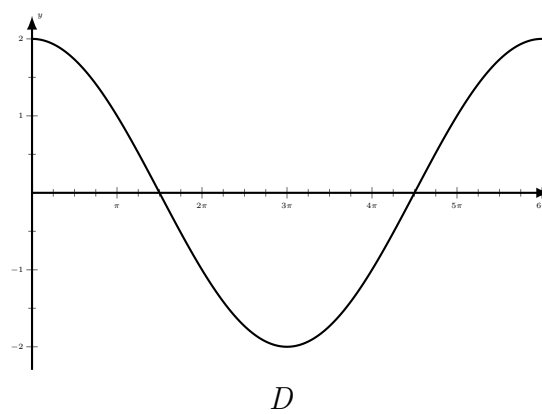
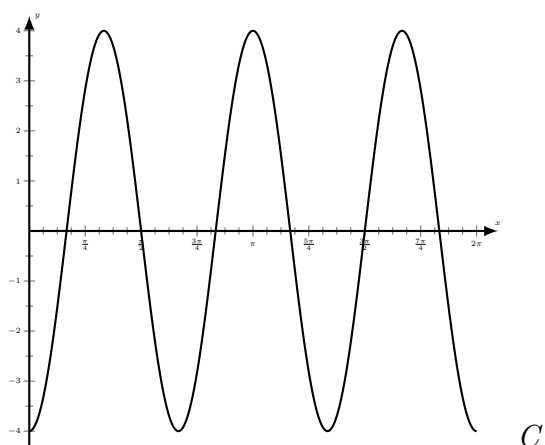
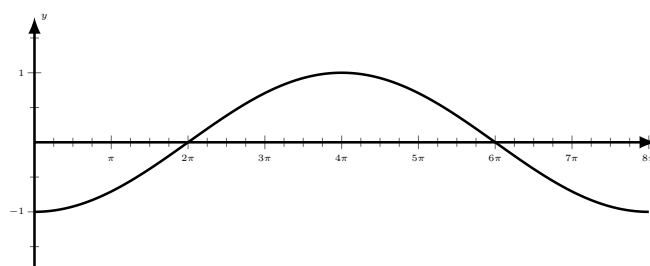
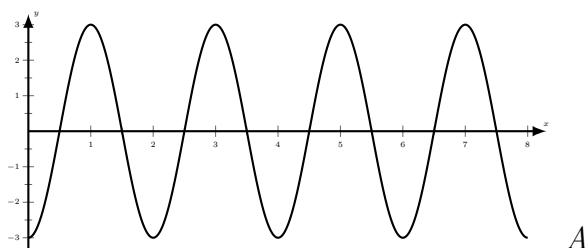
**Example 66.** Match each equation with its graph. Also write the period, amplitude and center-line next to each graph.

(a)  $y = -3 \cos \pi x$

(b)  $y = 2 \cos \frac{x}{3}$

(c)  $y = -\cos \frac{x}{4}$

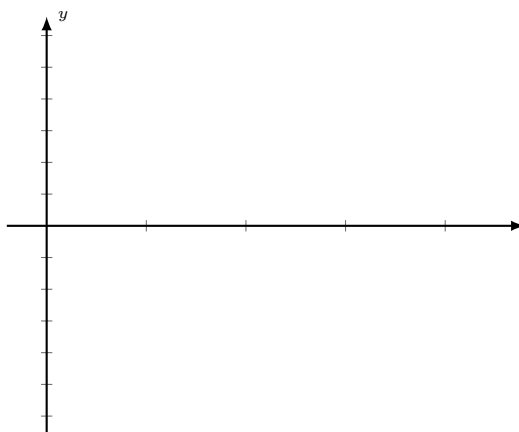
(d)  $y = -4 \cos 3x$



## EXERCISES

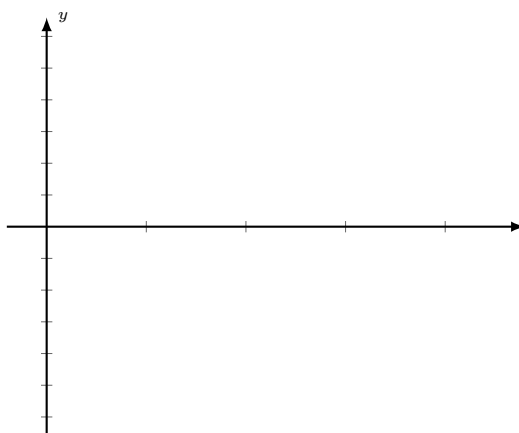
**Practice 36.** Roughly sketch one complete cycle of  $y = \frac{3}{5} \cos x$ . Label the coordinates of the 5 points. Determine the period, the amplitude and the center-line of the curve.

$x$	$y = \frac{3}{5} \cos x$	$(x, y)$



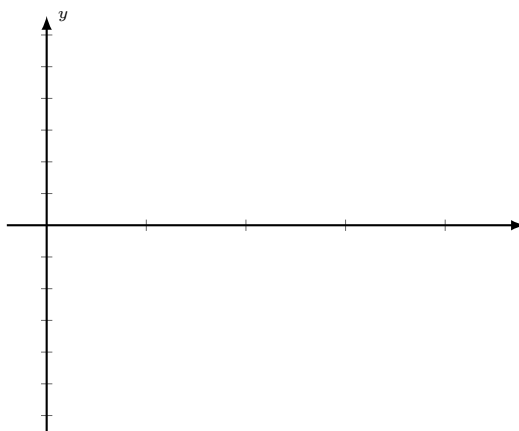
**Practice 37.** Roughly sketch one complete cycle of  $y = -\frac{1}{4} \cos x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points.

$x$	$y = -\frac{1}{4} \cos x$	$(x, y)$



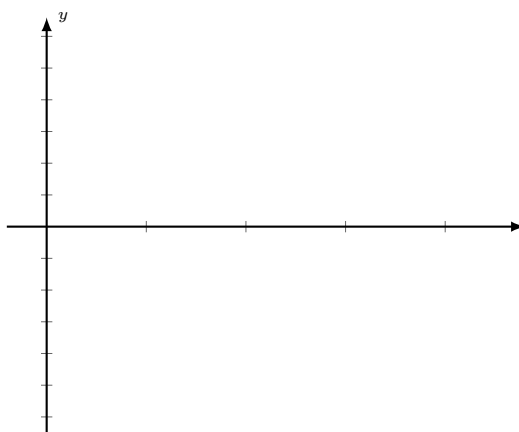
**Practice 38.** Roughly sketch one complete cycle of  $y = \cos 3x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = \cos 3x$	$(x, y)$



**Practice 39.** Roughly sketch one complete cycle of  $y = \cos \frac{1}{4}x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

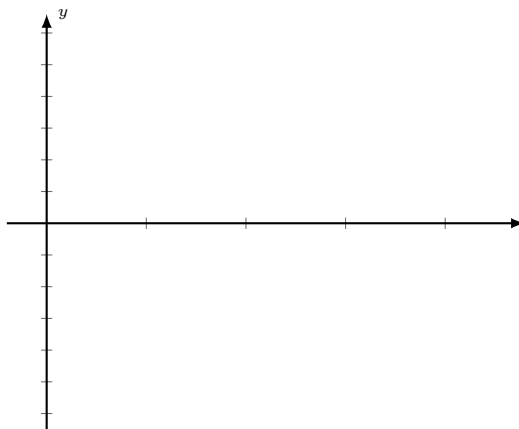
$x$	$y = \cos \frac{1}{4}x$	$(x, y)$





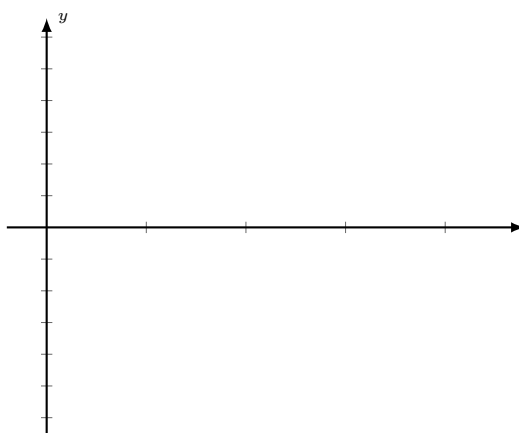
**Practice 40.** Roughly sketch one complete cycle of  $y = -2 \cos \frac{\pi}{8}x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = -2 \cos \frac{\pi}{8}x$	$(x, y)$



**Practice 41.** Roughly sketch one complete cycle of  $y = 3 \cos 4x$ . Determine the period, the amplitude and the center-line of the curve. Label the coordinates of the 5 points. How are the coordinates of these points related to the period and the amplitude of the curve?

$x$	$y = 3 \cos 4x$	$(x, y)$



## Answers to Odd Numbered Practice Exercises

**Unit 1: Angle Measure**

**Practice 1.** (a)  $\frac{23\pi}{18}$  (b)  $\frac{8\pi}{9}$  (c)  $-\frac{11\pi}{6}$  (d)  $-\frac{5\pi}{6}$  (e)  $-\frac{3\pi}{2}$  (f)  $\frac{3\pi}{4}$

**Practice 3.** (a)  $420^\circ$  (b)  $-540^\circ$  (c)  $20^\circ$  (d)  $-135^\circ$  (e)  $54^\circ$

**Practice 5.** (a)  $245^\circ$  (b)  $103^\circ$  (c)  $40^\circ$  (d)  $338^\circ$

**Practice 7.** (a) 4 radians (b) 2 radians (c)  $\frac{4}{3}$  radians (d) 1 radian

**Unit 2: Trigonometry of Right Triangles**

**Practice 9.** (a)  $\cos 65^\circ$  (b)  $\tan 12^\circ$  (c)  $\sin 72.4^\circ$  (d)  $\csc \frac{3\pi}{8}$  (e)  $\sec \frac{\pi}{10}$  (f)  $\cot \frac{\pi}{14}$

**Practice 11.** (a)  $20^\circ$  (b)  $85.4^\circ$  (c)  $40.4^\circ$

**Practice 13.**  $\sin \theta = \frac{2}{5}$ ,  $\csc \theta = \frac{5}{2}$ ,  $\tan \theta = \frac{2\sqrt{21}}{21}$ ,  $\cot \theta = \frac{\sqrt{21}}{2}$ ,  $\sec \theta = \frac{5\sqrt{21}}{21}$

**Practice 15.**  $67.6^\circ$

**Practice 17.** 139 feet

**Practice 19.** (a)  $\frac{\sqrt{13}}{7}$  (b)  $\csc \theta = \frac{7}{6}$ ,  $\sec \theta = \frac{7\sqrt{13}}{13}$ ,  $\tan \theta = \frac{6\sqrt{13}}{13}$ ,  $\cot \theta = \frac{\sqrt{13}}{6}$  (d)  $59^\circ$

**Practice 21.** 112 feet

**Practice 23.** Divide both sides of the equation by  $\sin^2 \theta$

**UNIT 3: Trigonometric Functions of Any Angle**

**Practice 25.** (a)  $\sin \theta = -\frac{2\sqrt{2}}{3}$ ,  $\csc \theta = -\frac{3\sqrt{2}}{4}$ ,  $\tan \theta = -2\sqrt{2}$ ,  $\cot \theta = -\frac{\sqrt{2}}{4}$ ,  $\sec \theta = 3$

(b)  $\csc \theta = \frac{17}{8}$ ,  $\cos \theta = \frac{15}{17}$ ,  $\sec \theta = \frac{17}{15}$ ,  $\tan \theta = \frac{8}{15}$ ,  $\cot \theta = \frac{15}{8}$

(c)  $\sin \theta = \frac{2\sqrt{5}}{5}$ ,  $\csc \theta = \frac{\sqrt{5}}{2}$ ,  $\cos \theta = -\frac{\sqrt{5}}{5}$ ,  $\sec \theta = -\sqrt{5}$ ,  $\tan \theta = -2$ ,  $\cot \theta = -\frac{1}{2}$

**Practice 27.** (a)  $-\tan 80^\circ$  (b)  $+\cot 15^\circ$  (c)  $+\sin 34^\circ$  (d)  $+\tan \frac{\pi}{5}$  (e)  $-\cos \frac{\pi}{5}$  (f)  $-\csc \frac{2\pi}{9}$

**Practice 29.** (a) (i) only (b) (i), (ii) and (iii) (c) (i) and (iii)

**UNIT 4: Graphs of the Sine Curve**

**Practice 31.** Period =  $2\pi$  Amplitude =  $\frac{3}{4}$  Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, 0)$ ,  $(\frac{\pi}{2}, -\frac{3}{4})$ ,  $(\pi, 0)$ ,  $(\frac{3\pi}{2}, \frac{3}{4})$ ,  $(2\pi, 0)$

**Practice 33.** Period =  $\frac{2\pi}{3}$  Amplitude = 1 Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, 0)$ ,  $(\frac{\pi}{6}, 1)$ ,  $(\frac{\pi}{3}, 0)$ ,  $(\frac{\pi}{2}, -1)$ ,  $(\frac{2\pi}{3}, 0)$

**Practice 35.** Period =  $\frac{\pi}{2}$  Amplitude =  $\frac{1}{5}$  Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, 0)$ ,  $(\frac{\pi}{8}, -\frac{1}{5})$ ,  $(\frac{\pi}{4}, 0)$ ,  $(\frac{3\pi}{8}, \frac{1}{5})$ ,  $(\frac{\pi}{2}, 0)$

**UNIT 5: Graphs of the Cosine Curve**

**Practice 37.** Period =  $2\pi$       Amplitude =  $\frac{1}{4}$       Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, -\frac{1}{4})$ ,  $(\frac{\pi}{2}, 0)$ ,  $(\pi, \frac{1}{4})$ ,  $(\frac{3\pi}{2}, 0)$ ,  $(2\pi, -\frac{1}{4})$

**Practice 39.** Period =  $8\pi$       Amplitude = 1      Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, 1)$ ,  $(2\pi, 0)$ ,  $(4\pi, -1)$ ,  $(6\pi, 0)$ ,  $(8\pi, 1)$

**Practice 41.** Period =  $\frac{\pi}{2}$       Amplitude = 3      Center-Line =  $x$ -axis

Coordinates of the 5 points:  $(0, 3)$ ,  $(\frac{\pi}{8}, 0)$ ,  $(\frac{\pi}{4}, -3)$ ,  $(\frac{3\pi}{8}, 0)$ ,  $(\frac{\pi}{2}, 3)$