Entanglement in two-atom Jaynes-Cummings model with Kerr nonlinearity

E.K. Bashkirov[1], **M.M. Evseev**[1], **A.E. Gorchakova-Zakatova**[1], **A. Othman**[1]

[1]Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443086

Abstract. The dynamics of quantum system consisting of two superconducting circuits interacting with one-mode quantum electromagnetic field of microwave coplanar cavity is studied. The influence of the Kerr medium on qubit-qubit entanglement is examined. We showed that Kerr medium can greatly enhance the amount of qubit-qubit entanglement.

1. Introduction

Coupling distant qubits is an important goal for quantum information and for its potential applications. This kind of coupling needs the study of interaction between qubits and photons, which has been widely studied in a cavity quantum electrodynamics (CQED) [1]. Resent years a lot of schemes are proposed for generating, controlling and protecting the atomic entanglement in CQED systems such as trapped and cooled ions or neutral atoms, superconducting circuits, spins in solids, quantum dots interacting with quantum fields of cavities [2]. The theoretical investigations of such schemes are based on a Jaynes-Cummings model (JCM) and its generalizations [3]. It is well known that JCM is the simplest physical model that describes the interaction of a natural or artificial two-level atom (qubit) with a single-mode cavity field, and was used to understand a wide variety of phenomena in quantum optics and condensed matter systems. During last decades the numerous generalizations of the JCM have been investigated (see references in [4]). Particularly, Buzek [5] studied the dynamics of atom interacting with field of one-mode cavity with a Kerr-like medium in a framework of nonlinear JCM [5]. A material whose refractive index depends on the intensity of the light field is called a Kerr medium. A light beam travelling through such a material acquires a phase shift $\phi = \chi\tau I$, where χ is the Kerr constant, τ is the interaction time of the light field with the material, and I is the intensity of the beam. The Kerr effect is a widely used phenomenon in nonlinear quantum optics, and has been successfully used to generate quadrature and amplitude squeezed states, parametrically convert frequencies, and create ultra-fast pulses [3]. In practice, Kerr nonlinearities in atomic systems χ are, however, often small in comparison to photon loss rate κ, making the observation of these non-classical states of light difficult. As an alternative approach, strong photon-photon interaction can readily be realized in superconducting quantum circuits, with χ/κ 30 demonstrated experimentally [6]. Note, that in the field of quantum optics with microwave circuits, the direct analogue of the Kerr effect is naturally created by the nonlinear inductance of the Josephson junctions. Later the experimental and theoretical investigations of superconducting circuits dynamics in coplanar cavities with Kerr media was investigated in a lot of papers (see Ref. in [6]-[11]).

Resent years we studied the dynamics of atom-atom entanglement for different generalizations of two-atom JCM. We explored the influence of dipole-dipole interaction, atomic coherence, detuning, Stark shift and non-degenerate and Raman two-photon transitions on atomic entanglement for superconducting qubits interacting with microwave fields of coplanar cavities [12]-[17]. It is of interest to expand our researches to more complicated CQED systems with strong photon-photon interaction. In this paper, we study the influence of Kerr nonlinearity on atomic entanglement for system consisting of two identical superconducting qubits interacting with one-mode microwave cavity field.

2. Model and its exact solution

We consider two identical superconducting circuits (qubits) resonantly interacting with cavity mode of 1D coplanar resonator. We suppose that qubit-field coupling constants are equal. The suppose also that there is an additional Kerr medium in the cavity. In a frame rotating with the twice field frequency, the interaction Hamiltonian for the system under rotating wave approximation can be written as

$$H = \sum_{i=1}^{2} \hbar\gamma(\sigma_i^+ a + a^+ \sigma_i^-) + +\chi a^{+2} a^2, \tag{1}$$

where $(1/2)\sigma_i^z$ is the inversion operator for the ith qubit ($i = 1, 2$), $\sigma_i^+ = |+\rangle_{ii}\langle -|$, and $\sigma_i^- = |-\rangle_{ii}\langle +|$ are the transition operators between the excited $|+\rangle_i$ and the ground $|-\rangle_i$ states in the ith qubit, a^+ and a are the creation and the annihilation operators of photons of the cavity mode, γ is the coupling constant between qubits and the cavity mode, χ is the dispersive part of the third- order nonlinearity of Kerr medium in the cavity. The initial qubits state is assumed to be separable such as

$$|\Psi(0)\rangle_A = |+, -\rangle \tag{2}$$

or

$$|\Psi(0)\rangle_A = |+, +\rangle \tag{3}.$$

The initial cavity mode state is assumed to be Fock one-mode state $|\Psi(0)\rangle_F = |n\rangle$, where $n = 1, 2, ...$

To obtain the exact dynamics of the model under consideration we use the so-called "dressed" states or eigenfunctions of interaction Hamiltonian (1). Suppose that the excitation number of the qubits-field system is n ($n \geq 0$). The evolution of the system is confined in the subspace $|-, -, n+2\rangle$, $|+, -, n+1\rangle$, $|-, +, n+1\rangle$, $|+, +, n\rangle$. Thus, the eigenfunctions of the Hamiltonian (1) can be written as

$$|\Psi_{in}\rangle = w_{in}(X_{i1n}|-, -, n+2\rangle + X_{i2n}|+, -, n+1\rangle +$$

$$+X_{i3n}|-, +, n+1\rangle + X_{i4n}|+, +, n\rangle) \quad (i = 1, 2, 3, 4), \tag{4}$$

where

$$w_{in} = 1/\sqrt{|X_{i1n}|^2 + |X_{i2n}|^2 + |X_{i3n}|^2 + |X_{i4n}|^2}$$

and

$$X_{11,n} = 0, \quad X_{12,n} = -1, \quad X_{13,n} = 1, \quad X_{14,n} = 0,$$

$$X_{i1,n} = -\frac{2n\tilde{\chi} - n^2\tilde{\chi} - n^3\tilde{\chi} + 2\varepsilon_{in} + n\varepsilon_{in}}{\sqrt{1+n}\sqrt{2+n}(2\tilde{\chi} + 3n\tilde{\chi} + n^2\tilde{\chi} - \varepsilon_{in})},$$

$$X_{i2,n} = \frac{\sqrt{1+n}\left(2\tilde{\chi} + 3n\tilde{\chi} + n^2\tilde{\chi} - \varepsilon_{in}\right)}{2+n} +$$

$$+\frac{2+n-(n(1+n)\tilde{\chi}-\varepsilon_{in})(1+n)(2+n)\tilde{\chi}-\varepsilon_{in})(2\tilde{\chi}+3n\tilde{\chi}+5n^2\tilde{\chi}+2n^3\tilde{\chi}-3\varepsilon_{in}-2n\varepsilon_{in})}{(2+n)(\sqrt{1+n}(2+n)-\sqrt{1+n}(2+n-(n(1+n)\tilde{\chi}-\varepsilon_{in})(1+n)(2+n)\tilde{\chi}-\varepsilon_{in}))},$$

$$X_{i3,n}=-\frac{2\tilde{\chi}+3n\tilde{\chi}+5n^2\tilde{\chi}+2n^3\tilde{\chi}-3\varepsilon_{in}-2n\varepsilon_{in}}{\sqrt{1+n}(2+n)-\sqrt{1+n}(2+n-(n(1+n)\tilde{\chi}-\varepsilon_{in})((1+n)(2+n)\tilde{\chi}-\varepsilon_{in}))},$$

$$X_{i4,n}=1,$$

where $\tilde{\chi}=\chi/\hbar\gamma$. The corresponding scaled eigenvalues are

$$\varepsilon_{1n}=E_{1n}/\hbar\gamma=\left(n+n^2\right)\tilde{\chi},$$

$$\varepsilon_{2n}=E_{2n}/\hbar\gamma=\frac{1}{3}D_n-\frac{2^{1/3}A_n}{3(B_n+C_n)^{1/3}}+\frac{(B_n+C_n)^{1/3}}{3\,2^{1/3}},$$

$$\varepsilon_{3n}=E_{3n}/\hbar\gamma=\frac{1}{3}D_n+\frac{\left(1+i\sqrt{3}\right)A_n}{3\,2^{2/3}(B_n+C_n)^{1/3}}-\frac{\left(1-i\sqrt{3}\right)(B_n+C_n)^{1/3}}{6\,2^{1/3}},$$

$$\varepsilon_{4n}=E_{4n}/\hbar\gamma=\frac{1}{3}D_n+\frac{\left(1-i\sqrt{3}\right)A_n}{3\,2^{2/3}(B_n+C_n)^{1/3}}-\frac{\left(1+i\sqrt{3}\right)(B_n+C_n)^{1/3}}{6\,2^{1/3}},$$

where

$$A_n=-18-12n-4\tilde{\chi}^2-12n\tilde{\chi}^2-12n^2\tilde{\chi},\quad B_n=72n\tilde{\chi}+16\tilde{\chi}^3+72n\tilde{\chi}^3+72n^2\tilde{\chi}^3,$$

$$C_n=\sqrt{4A_n^3+B_n^2},\quad D_n=2\tilde{\chi}+3n\tilde{\chi}+3n^2\tilde{\chi}.$$

Assume that the system is initially prepared in the state $|+,-,n+1\rangle$ $(n\geq 0)$. In this case, at time instant t, the whole system will evolve to

$$|\Psi(t)\rangle=C_{12,n}|-,-,n+2\rangle+C_{22,n}|+,-,n+1\rangle+C_{32,n}|-,+,n+1\rangle+C_{42,n}|+,+,n\rangle,\qquad(5)$$

where

$$C_{i2,n}=e^{-\imath E_{1n}t/\hbar}\,w_{1n}\,Y_{2in}\,X_{1in}+e^{-\imath E_{2n}t/\hbar}\,w_{2n}\,Y_{2in}X_{2in}+\qquad(6)$$
$$+e^{-\imath E_{3n}t/\hbar}\,w_{3n}\,Y_{2in}\,X_{3in}+e^{-\imath E_{4n}t/\hbar}\,w_{4n}\,Y_{2in}\,X_{4in}\quad(i=1,2,3,4)$$

and

$$Y_{ijn}=w_{jn}X^{*}_{jin}.$$

Assume that the system is initially prepared in the state $|-,+,n+1\rangle$ $(n\geq 0)$. The time-dependent wave function can be presented as

$$|\Psi(t)\rangle=C_{13,n}|-,-,n+2\rangle+C_{23,n}|+,-,n+1\rangle+C_{33,n}|-,+,n+1\rangle+C_{43,n}|+,+,n\rangle,\qquad(7)$$

where the coefficients $C_{i3,n}$ can be obtained from (6) by replacing Y_{2in} with Y_{3in} $(i=1,2,3,4)$.

For initial states $|+,+,n\rangle$ and $|-,-,n+2\rangle$ $(n\geq 0)$ the time-dependent wave functions are

$$|\Psi(t)\rangle=C_{11,n}|-,-,n+2\rangle+C_{21,n}|+,-,n+1\rangle+C_{31,n}|-,+,n+1\rangle+C_{41,n}|+,+,n\rangle.\qquad(8)$$

$$|\Psi(t)\rangle=C_{14,n}|-,-,n+2\rangle+C_{24,n}|+,-,n+1\rangle+C_{34,n}|-,+,n+1\rangle+C_{44,n}|+,+,n\rangle.\qquad(9)$$

The coefficients $C_{i1,n}$ $(C_{i4,n})$ can be obtain from (6) by replacing Y_{2in} with Y_{1in} (Y_{4in}) $(i=1,2,3,4)$.

Using expressions (4)–(9) one can obtain the density operator for the whole system $\rho(t)$. Taking a trace over the field variables $\rho_A(t)=Tr_F\rho(t)$ one can also obtain the reduced atomic density operator.

3. Negativity calculation

For two-qubit system described by the density operator $\rho_A(t)$, a measure of entanglement or negativity can be defined in terms of the negative eigenvalues μ_i^- of partial transpose of a reduced atomic density matrix ($\rho_A^{T_1}$)

$$\varepsilon = -2\sum \mu_i^- .$$

For separable initial atomic states (2) or (3) the reduced atomic density matrix is

$$\rho_A(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{23}(t)^* & \rho_{33}(t) & 0 \\ 0 & 0 & 0 & \rho_{44}(t) \end{pmatrix}. \tag{10}$$

The partial transpose of a reduced atomic density matrix (10) has the form

$$\rho_A^{T_1}(t) = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & \rho_{23}(t)^* \\ 0 & \rho_{22}(t) & 0 & 0 \\ 0 & 0 & \rho_{33}(t) & 0 \\ \rho_{23}(t) & 0 & 0 & \rho_{44}(t) \end{pmatrix}. \tag{11}$$

The elements of matrix (11) for initial atomic state $|+,+\rangle$ are

$$\rho_{11} = |C_{41,n}(,t)|^2, \quad \rho_{22} = |C_{21,n}(t)|^2,$$
$$\rho_{33} = |C_{31,n}(t)|^2, \quad \rho_{44} = |C_{11,n}(,t)|^2,$$
$$\rho_{23} = C_{21,n}(t)C_{31,n}(t)^*.$$

For initial atomic state $|+,-\rangle$ these take the form

$$\rho_{11} = |C_{42,n-1}(,t)|^2, \quad \rho_{22} = |C_{22,n-1}(t)|^2,$$
$$\rho_{33} = |C_{32,n-1}(t)|^2, \quad \rho_{44} = C_{12,n-1}(t)|^2,$$
$$\rho_{23} = C_{22,n-1}(t)C_{32,n-1}(t)^*.$$

Matrix (11) has only one eigenvalue, which may take a negative value. As a result we have

$$\varepsilon(t) = \sqrt{(\rho_{11}(t) - \rho_{44}(t))^2 + 4|\rho_{23}(t)|^2} - \rho_{11}(t) - \rho_{44}(t). \tag{12}$$

The results of numerical calculations of negativity (12) are shown in Figs. 1-2.

4. Results and discussions

The negativity for a separable initial atomic state $|+,-\rangle$ is plotted in Fig. 1 as a function of a scaled time γt for fixed values of photon number $n = 1$ and different values of Kerr nonlinearity coefficient $\tilde{\chi} = 0$ (solid), $\tilde{\chi} = 0.3$ (dashed) and $\tilde{\chi} = 0.5$ (dotted). One can see from Fig. 1 that with the increase of Kerr non-linearity coefficient, the maximum value of negativity increases. Therefore, the Kerr medium can use to control the amount of entanglement between qubits. The negativity for a separable initial atomic state $|+,+\rangle$ is plotted in Fig. 2 as a function of a scaled time gt for fixed values of photon number $n = 1$ and different values of Kerr non-linearity coefficient $\tilde{\chi} = 0.3$ (solid) and $\tilde{\chi} = 0.5$ (dashed) . One can easily see from Fig. 2 that as the Kerr non-linearity coefficient increases, larger entanglement is obtainable. Note that for zero Kerr non-linearity coefficient the entanglement between atoms doesn't arise during the evolution. This results is in accordance with previous investigations [14]-[17].

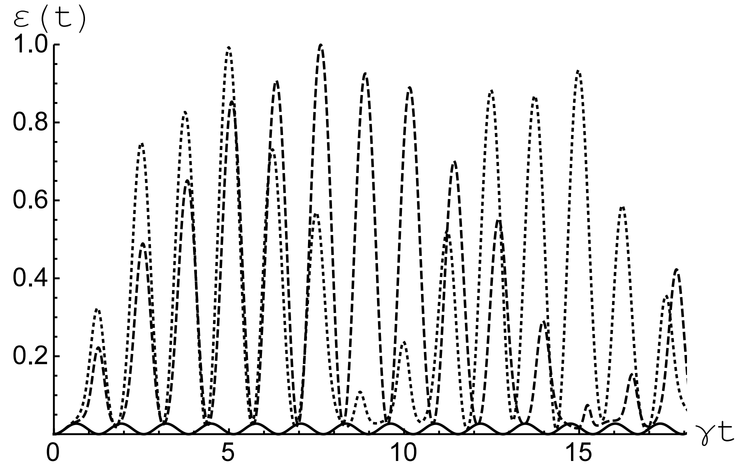

Figure 1. The negativity as a function of a scaled time γt for initial separable atomic state (2) and $\tilde{\chi} = 0$ (solid), $\tilde{\chi} = 0.3$ (dashed) and $\tilde{\chi} = 0.5$ (dotted) The cavity photon number $n = 1$.

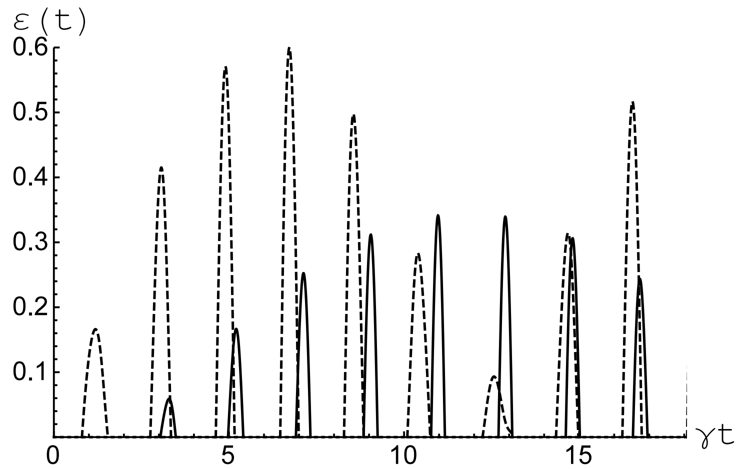

Figure 2. The negativity as a function of a scaled time γt for initial separable atomic state (3) and $\tilde{\chi} = 0.3$ (dashed), $\tilde{\chi} = 0,5$ (dotted) The cavity photon number $n = 2$.

5. Conclusion

We studied dynamics of two superconducting qubits interacting with cavity field in the framework of two-atom JCM with the Kerr medium, and examined the influence of the Kerr medium on the atom-atom entanglement. The atomic entanglement behavior for separable initial atomic states and cavity Fock state was a subject of n our investigation. We derived that for initial atomic state $|+,-\rangle$ the Kerr nonlinearity enhances the amount of entanglement. We also found that for the initial state $|+,+\rangle$ the Kerr nonlinearity causes entanglement of the atoms, while for a model without nonlinearity, this initial state does not cause atomic entanglement during system evolution. These results may be useful for quantum information processing based on the entanglement.

6. References

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