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On some problems and solutions in frame theory

S.Ya. Novikov[1], **M.E. Fedina**[1]

[1]Samara National Research University, Moskovskoe Shosse 34A, Samara, Russia, 443086

Abstract. We define frames for a finite dimensional Hilbert space $\mathbb{H}^M$ as the complete systems in $\mathbb{H}^M$. The basic frame families are classified such as tight and Parseval frames,equal norm frames and equiangular frames. The statements of some problems that have already become famous in the theory of frames are given. Considerable progress has been made in addressing some of them in recent years.

1. Introduction

A *finite frame* is a spanning set for a finite dimensional Hilbert space $\mathbb{H}^M$ that generalizes the notion of a basis by relaxing the need for linear independence. In other words, a family of vectors $\Phi = \{\varphi_n\}_{n=1}^N$ is a *frame* for a real or complex $\mathbb{H}^M$ if there are constants $0 < A \leq B < \infty$ such that for all $x \in \mathbb{H}^M$,

$$A\|x\|^2 \leq \sum_{n=1}^{N} |\langle x, \varphi_n \rangle|^2 \leq B\|x\|^2.$$

In a finite-dimensional space the concept of a frame is equivalent to the completeness of the system, i. e. $\text{span}\{\varphi_n\}_{n=1}^N = \mathbb{H}^M$.

In practice, frames are chiefly used in two ways. The synthesis operator is defined by

$$V^* : h \in \mathbb{H}^N \to \sum_{n=1}^{N} h_n \varphi_n \in \mathbb{H}^M.$$

As such the $M \times N$ matrix representation V^* of the synthesis operator has the frame elements $\{\varphi_n\}_{n=1}^N$ as columns.

The *analysis operator* of the frame is the map $V : \mathbb{H}^M \to \mathbb{H}^N$ given by $(Vx)_n = \langle x, \varphi_n \rangle$, $n = 1, \ldots, N$.

Frames are used to redundantly decompose signals $y = Vx$, before synthesizing the corresponding frame coefficients $z = V^*y = V^*Vx$, and so the frame operator $V^*V : \mathbb{H}^M \to \mathbb{H}^M$ is often analyzed to determine how well this process preserves information about the original signal x.

In particular, if the frame bounds are equal, the frame operator has the form $V^*V = AI_M$, and so signal reconstruction is rather painless: $x = \frac{1}{A}V^*Vx$; in this case the frame is called *tight*.

Oftentimes, it is additionally desirable for the frame elements to have equal or unit norms, in these cases the frames are equal- norm or unit norm respectively.

Moreover, the worst-case coherence between unit norm frame elements $\mu := \max_{n \neq n'} |\langle \varphi_n, \varphi_{n'} \rangle|$ satisfies

$$\mu^2 \geq \frac{N-M}{M(N-1)},$$

and equality is achieved precisely when the frame is tight with $|\langle \varphi_n, \varphi_{n'} \rangle| = \mu$ for all distinct pairs $n, n' \in \{1, \ldots, N\}$. In this case the frame is called an *equiangular* tight frame.

2. The Paulsen problem

The frame $\Phi = \{\varphi_n\}_{n=1}^N$ is called *tight* if the equality $A = B$ is possible in the definition.

In the case $A = B = 1$ it's called *Parseval frame.*

The frame Φ is called *equal norm frame* if there exists $\alpha > 0$ such that $\|\varphi_n\| = \alpha,\ n = 1, \ldots, N$.

The *analysis operator* of the frame is the map $V : \mathbb{H}^M \to \mathbb{H}^N$ given by $(Vx)_n = \langle x, \varphi_n \rangle,\ n = 1, \ldots, N$.

Its adjoint, V^*, is the *synthesis operator* : $h \in \mathbb{H}^N \to \sum\limits_{n=1}^N h_n \varphi_n$.

The *frame operator* is the positive, self adjoint invertible operator $S = V^*V$ on $\mathbb{H}^M$.

The *Gramian* is the operator $G = VV^*$ on $\mathbb{H}^N$.

The unit norm frame Φ is called *equiangular frame* if there exists $\beta \geq 0$ such that $|\langle \varphi_{n'}, \varphi_{n''} \rangle| = \beta$ for all $n' \neq n''$.

The frame Φ is called *equiangular tight frame (ETF)* if it is is an equiangular and tight simultaneously.

If Φ is Parseval frame with analysis operator V, then V is an isometry, since

$$\|Vx\|_2^2 = \sum_{n=1}^N |\langle x, \varphi_n \rangle|^2 = \|x\|^2,\ x \in \mathbb{H}^M.$$

Conversely, if an $N \times M$ matrix V is an isometry, then it is the analysis operator of the Parseval frame.

Parseval frames (and only such frames) satisfy the reconstruction identity

$$x = \sum_{n=1}^N \langle x, \varphi_n \rangle \varphi_n,$$

or $x = V^*Vx,\ S = V^*V = I_{\mathbb{H}^M}$.

The following theorem was the first in the Frame theory, and maybe one of the most important.

Theorem 1. (M.A.Naimark[1]) If $\Phi = \{\varphi_n\}_{n=1}^N$ is Parseval frame for $\mathbb{H}^M$, then there exists an N-dimensional Hilbert space $\mathfrak{H}^N$, and an orthonormal basis $\{b_n\}_{n=1}^N \subset \mathfrak{H}^N$ such that $\mathbb{H}^M$ is a linear subspace of $\mathfrak{H}^N$ and $\varphi_n = P_{\mathbb{H}^M} b_n$ for all n, where $P_{\mathbb{H}^M}$ denotes the orthogonal projection of $\mathfrak{H}^N$ onto $\mathbb{H}^M$.

The converse statement is also true.

Theorem 2. If $\{b_n\}_{n=1}^N$ is an orthonormal basis for $\mathbb{H}^N$, and $\mathbb{H}^M \subset \mathbb{H}^N$ is any M-dimensional linear subspace, then $\Phi = \{P_{\mathbb{H}^M} b_n\}_{n=1}^N$ is Parseval frame for $\mathbb{H}^M$, where $P_{\mathbb{H}^M}$ denotes the orthogonal projection of $\mathbb{H}^N$ onto $\mathbb{H}^M$.

Proof. Denote by $\varphi_n = P_{\mathbb{H}^M} b_n, n = 1, \ldots, N$. We have for $x \in \mathbb{H}^M$

$$\|x\|^2 = \|P_{\mathbb{H}^M} x\|^2 = \sum_{n=1}^N |\langle P_{\mathbb{H}^M} x, b_n \rangle|^2 =$$

$$= \sum_{n=1}^{N} |\langle x, P_{\mathbb{H}^M} b_n \rangle|^2 = \sum_{n=1}^{N} |\langle x, \varphi_n \rangle|^2,$$

i.e. Φ is Parseval frame.

□

The theorems 1 and 2 are generalized in [6] to frames of a general form, which are projections of Riesz bases. Projections of orthogonal systems (generally speaking, incomplete) are considered in detail in [9].

Parseval frame $\{P_{\mathbb{H}^M} b_n\}_{n=1}^N$ from the theorem 2 is *not* equal-norm frame. In fact, the first equal-norm Parseval frame was built by A.I.Maltsev in [8] (of course not using such terms). The article [8] did not fall into the field of view of specialists in the Frame theory; its results will undoubtedly find interesting applications.

Almost all known equal-norm Parseval frame designs are based on a discrete Fourier transform matrix. The elements of this matrix are formed by complex numbers, however, the correct choice of columns and rows of the matrix and simple arithmetic operations with them lead to matrices of synthesis operators for real frames with given properties.

Theorem 3. Equal norm Parseval Frame $\Phi = \{\varphi_n\}_{n=1}^N$ exists in $\mathbb{R}^M$ for any $N \geq M$.

Proof. Define the following orthogonal $N \times N$ matrices separately for even and odd number N. For $N = 2k+1$ we have

$$\sqrt{\frac{2}{N}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \dots & \dots & \dots & \frac{1}{\sqrt{2}} \\ 1 & \cos\frac{2\pi}{N} & \cos\frac{4\pi}{N} & \dots & \cos\frac{2\pi(N-1)}{N} \\ 0 & \sin\frac{2\pi}{N} & \sin\frac{4\pi}{N} & \dots & \sin\frac{2\pi(N-1)}{N} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\frac{2\pi k}{N} & \cos\frac{4\pi k}{N} & \dots & \cos\frac{2\pi k(N-1)}{N} \\ 0 & \sin\frac{2\pi k}{N} & \sin\frac{4\pi k}{N} & \dots & \sin\frac{2\pi k(N-1)}{N} \end{pmatrix}.$$

If M is odd, we delete the last $N - M$ rows and obtain the matrix

$$\begin{pmatrix} \varphi_1| & \varphi_2| & \dots| & \dots| & \varphi_N \end{pmatrix}$$

of the synthesis operator (remind that its columns are frame vectors) for the equal-norm Parseval Frame in $\mathbb{R}^M$.

If M is even, we delete the first $N - M$ rows and obtain the similar matrix for the equal-norm Parseval Frame in $\mathbb{R}^M$.

For $N = 2k$ we define a little different orthogonal matrix

$$\sqrt{\frac{2}{N}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \dots & \dots & \dots & \frac{1}{\sqrt{2}} \\ 1 & \cos\frac{2\pi}{N} & \cos\frac{4\pi}{N} & \dots & \cos\frac{2\pi(N-1)}{N} \\ 0 & \sin\frac{2\pi}{N} & \sin\frac{4\pi}{N} & \dots & \sin\frac{2\pi(N-1)}{N} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \cos\frac{2\pi(k-1)}{N} & \cos\frac{4\pi(k-1)}{N} & \dots & \cos\frac{2\pi(k-1)(N-1)}{N} \\ 0 & \sin\frac{2\pi(k-1)}{N} & \sin\frac{4\pi(k-1)}{N} & \dots & \sin\frac{2\pi(k-1)(N-1)}{N} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \dots & \dots & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

If M is odd, we delete the last $N - M$ rows and obtain the equal-norm Parseval Frame in $\mathbb{R}^M$.

If M is even, we delete the first and $N - M - 1$ last rows and again we obtain the equal-norm Parseval Frame in $\mathbb{R}^M$.

□

Any Frame has a Parseval frame as a natural satellite. Indeed, if $\{e_n\}_{n=1}^N$ is an orthonormal basis for $\mathbb{H}^N$, then

$$Sx = V^*Vx = V^*\left(\sum_{n=1}^{N}\langle x,\varphi_n\rangle e_n\right) =$$

$$= \sum_{n=1}^{N}\langle x,\varphi_n\rangle V^*e_n = \sum_{n=1}^{N}\langle x,\varphi_n\rangle\varphi_n.$$

Hence,

$$\langle Sx,x\rangle = \sum_{n=1}^{N}|\langle x,\varphi_n\rangle|^2.$$

Moreover, $\{\varphi_n\}_{n=1}^N$ is a frame with bounds $A, B > 0$ if and only if $AI_M \le S \le BI_M$.

Also we have that

$$x = SS^{-1}x = \sum_{n=1}^{N}\langle S^{-1}x,\varphi_n\rangle\varphi_n =$$

$$= \sum_{n=1}^{N}\langle x,S^{-1}\varphi_n\rangle\varphi_n = \sum_{n=1}^{N}\langle x,S^{-1/2}\varphi_n\rangle S^{-1/2}\varphi_n,$$

i.e $S^{-1/2}\{\varphi_n\}_{n=1}^N$ is Parseval frame.

Now we define the distance between frames and after that we'll show following [4] that this frame is the nearest Parseval frame to the frame Φ.

The ℓ^2-*distance* between two frames $\Phi = \{\varphi_n\}_{n=1}^N$ and $\Phi' = \{\varphi_n'\}_{n=1}^N$ in $\mathbb{H}^M$ is defined by

$$\operatorname{dist}(\Phi,\Phi') := \sqrt{\sum_{n=1}^{N}\|\varphi_n - \varphi_n'\|^2}.$$

Theorem 4. If $\Phi = \{\varphi_n\}_{n=1}^N$ is a frame for $\mathbb{H}^M$ with the frame operator S, then $\{S^{-1/2}\varphi_n\}_{n=1}^N$ minimizes the ℓ^2-distance between Φ and all possible choices of Parseval frames.

Proof. Let $\{e_m\}_{m=1}^M$ be an orthonormal eigenvector basis for $\mathbb{H}^M$ with respect to S and respective eigenvalues $\{\lambda_m\}_{m=1}^M$. Then we have

$$\sum_{n=1}^{N}\|\varphi_n - S^{-1/2}\varphi_n\|^2 = \sum_{n=1}^{N}\left\|\sum_{m=1}^{M}\langle\varphi_n,e_m\rangle e_m - \frac{1}{\sqrt{\lambda_m}}\langle\varphi_n,e_m\rangle e_m\right\|^2 =$$

$$= \sum_{n=1}^{N}\sum_{m=1}^{M}|\langle\varphi_n,e_m\rangle|^2\left|1-\frac{1}{\sqrt{\lambda_m}}\right|^2 = \sum_{m=1}^{M}\left|1-\frac{1}{\sqrt{\lambda_m}}\right|^2\sum_{n=1}^{N}|\langle\varphi_n,e_m\rangle|^2 =$$

$$= \sum_{m=1}^{M}\left|1-\frac{1}{\sqrt{\lambda_m}}\right|^2\lambda_m = \sum_{m=1}^{M}\left(\lambda_m - 2\sqrt{\lambda_m}+1\right).$$

Now let $\{g_n\}_{n=1}^N$ be an arbitrary Parseval frame for $\mathbb{H}^M$. Using again the basis of eigenvectors and its eigenvalues we obtain

$$\sum_{n=1}^{N}\|\varphi_n - g_n\|^2 = \sum_{n=1}^{N}\left\|\sum_{m=1}^{M}\langle\varphi_n,e_m\rangle e_m - \langle g_n,e_m\rangle e_m\right\|^2 =$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{M} |\langle \varphi_n, e_m \rangle - \langle g_n, e_m \rangle e_m|^2 =$$

$$= \sum_{m=1}^{M} \sum_{n=1}^{N} \left(|\langle \varphi_n, e_m \rangle|^2 + |\langle g_n, e_m \rangle|^2 - 2Re\left[\langle \varphi_n, e_m \rangle \overline{\langle g_n, e_m \rangle} \right] \right) =$$

$$= \sum_{m=1}^{M} \left(\sum_{n=1}^{N} |\langle \varphi_n, e_m \rangle|^2 + \sum_{n=1}^{N} |\langle g_n, e_m \rangle|^2 - 2Re\left[\sum_{n=1}^{N} \langle \varphi_n, e_m \rangle \overline{\langle g_n, e_m \rangle} \right] \right) =$$

$$= \sum_{m=1}^{M} \left(\lambda_m + 1 - 2Re\left[\sum_{n=1}^{N} \langle \varphi_n, e_m \rangle \overline{\langle g_n, e_m \rangle} \right] \right).$$

Now we estimate the last term

$$\sum_{m=1}^{M} Re\left[\sum_{n=1}^{N} \langle \varphi_n, e_m \rangle \overline{\langle g_n, e_m \rangle} \right] \le$$

$$\le \sum_{m=1}^{M} \sum_{n=1}^{N} |\langle \varphi_n, e_m \rangle| \, |\langle g_n, e_m \rangle| \le$$

$$\le \sum_{m=1}^{M} \sqrt{\sum_{n=1}^{N} |\langle \varphi_n, e_m \rangle|^2} \sqrt{\sum_{n=1}^{N} |\langle g_n, e_m \rangle|^2} = \sum_{m=1}^{M} \sqrt{\lambda_m}.$$

Now we see that

$$\sum_{n=1}^{N} \|\varphi_n - g_n\|^2 \ge \sum_{m=1}^{M} \left(\lambda_m - 2\sqrt{\lambda_m} + 1 \right) =$$

$$= \sum_{n=1}^{N} \|\varphi_n - S^{-\frac{1}{2}} \varphi_n\|^2.$$

Since $\left\{ S^{-\frac{1}{2}} \varphi_n \right\}_{n=1}^{N}$ is a Parseval frame for $\mathbb{H}^M$, the theorem follows.

□

The frame $\{S^{-1/2}\varphi_n\}_{n=1}^{N}$ is not an equal-norm frame.

If $\Phi = \{\varphi_n\}_{n=1}^{N}$ is an equal norm Parseval frame for $\mathbb{R}^M$, then $S = Id_M$ and $\|\varphi_n\|^2 = M/N, \ n = 1, \ldots, N$.

We say that Φ is an ϵ-*nearly equal norm Parseval frame* if

$$(1-\epsilon) Id_M \le S \le (1+\epsilon) Id_M,$$

and

$$(1-\epsilon)\frac{M}{N} \le \|\varphi_n\|^2 \le (1+\epsilon)\frac{M}{N}, \ n = 1, \ldots, N.$$

Let ENPF be the set of all equal norm Parseval frames. The **Paulsen problem** is one of the most known and attractive problems in frame theory.

Paulsen problem. For every ϵ-nearly equal norm Parseval frame Φ, is

$$\inf_{\Psi \in \mathrm{ENPF}} \mathrm{dist}^2(\Phi, \Psi)$$

bounded by a fixed polynomial in ϵ and M?

Early results gave bounds on the squared distance that were polynomial in ϵ, M and N [2, 3]. The first bound that was polynomial in ϵ and M was obtained in [7]. They proved that squared distance is at most $O(\epsilon M^{13/2})$.

A much simpler way to a better bound was found in [5].

Theorem 5. For any ϵ-nearly equal norm Parseval frame Φ there is $\Psi \in$ ENPF such that

$$\mathrm{dist}^2(\Phi, \Psi) \leq 20\epsilon M^2.$$

Cahill and Casazza [2] gave a family of examples of ϵ-nearly equal norm Parseval frames where the

$$\mathrm{dist}^2(\Phi, \Psi) \geq c\epsilon M.$$

It is an interesting open question to close the gap.

3. Acknowledgements

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4. References

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