Mathematical modeling of the space tug transfers between the Lagrange points of the Earth-Moon system

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Abstract. The paper outlines the mathematical modeling of the L1-L2 and L2-L1 missions using electric propulsion. The variation problem of the low thrust spacecraft transfer optimization, with total flight time as the optimization criterion is considered. The locally optimal control programs were obtained by using the Fedorenko method to estimate the derivatives, the gradient method to optimize the control laws and the Runge-Kutta method for the numerical integration of the differential equation system. As the result of optimization, optimal control programs and corresponding trajectories were determined for certain values of acceleration and jet stream velocity of the propulsion system.

Keywords: mathematical modeling, motion simulation, spacecraft, low thrust engine, ballistic optimization, Lagrange point, Earth-Moon system.

1. Introduction

Nowadays the spacefaring nations are developing the missions to achieve the libration points of the Earth-Moon system, especially L1 and L2. The optimal interplanetary trajectories and the trajectories of flights to the Moon pass near the libration point L1 of the Earth-Moon system, as shown in works [1-2]. Moreover, the usage of the Lagrange points will help to decrease the fuel expenses for orbit maintaining, to start from the Earth at any moment without choosing the date of start, to monitor the solar wind and to avoid the radiation from the Earth. One of the main problems of such missions is to determine the optimal control structure of the spacecraft transfers.

2. Mathematical Model

Let us formulate the general statement of the optimization problem. The following parameters are considered:

$\boldsymbol{x}(t) = (\boldsymbol{r}(t),\ \boldsymbol{V}(t),\ m_f(t), \boldsymbol{r}_E(t), \boldsymbol{r}_M(t),\ \boldsymbol{r}_S(t))^T \in \boldsymbol{X}$ is a system state vector corresponding to boundary conditions, defined by the purpose of the transfer and possible restrictions, where $\boldsymbol{X}$ is set of admissible state area;

$\boldsymbol{u}(t) = (\delta(t),\ \boldsymbol{e}(t))^T \in \boldsymbol{U}$ is a vector of control functions, where $\boldsymbol{U}$ is set of admissible control parameters;

$\boldsymbol{p} = (a_0, j_{sp})^T \in \boldsymbol{P}$ is the vector of optimized design parameters. It is limited by set of admissible area of the design parameters $\boldsymbol{P}$.

Here t is the current time, $\boldsymbol{r}(t)$ is a radius vector of the SC, $\boldsymbol{V}(t)$ is a vector of the SC velocity, $m_f(t)$ is a expended fuel mass, $\boldsymbol{r}_E(t)$, $\boldsymbol{r}_M(t)$, $\boldsymbol{r}_S(t)$ are the radius-vectors of the Earth, the Moon and the Sun, $\delta(t)$ is the function of thrust switching, $\boldsymbol{e}(t)$ is the thrusting direction unit vector, a_0 is the nominal acceleration of the SC in the initial orbit, j_{sp} is the specific impulse of propulsion system (PS).

The boundary conditions of the flight are shown in table 1.

Table 1. Boundary conditions of the flights in the Earth-Moon system.

	Finishing time	Radius-vector	Velocity vector	Radius-vector of the Earth	Radius-vector of the Moon	Radius-vector of the Sun	Set of admissible state area
L1	(t_i)	$\mathbf{r}_{L1}(t_1)$	$\mathbf{V}_{L1}(t_1)$	$\mathbf{r}_E(t_1)$	$\mathbf{r}_M(t_1)$	$\mathbf{r}_S(t_1)$	$\mathbf{X}_{L1}$
L2		$\mathbf{r}_{L2}(t_2)$	$\mathbf{V}_{L2}(t_2)$	$\mathbf{r}_E(t_2)$	$\mathbf{r}_M(t_2)$	$\mathbf{r}_S(t_2)$	$\mathbf{X}_{L2}$

Optimizing these space transfers with low thrust we need to determine the vectors $\boldsymbol{u}_{opt}(t)$ and $\boldsymbol{p}_{opt}$ (vectors of optimal control functions and optimal design parameters correspondingly) that provide the minimum duration of flight T to perform the mission purposes according to table 1.

$$T = \min_{\boldsymbol{u}\in\mathbf{U},\, \boldsymbol{p}\in\mathbf{P}} T \big| m = unfixed, \boldsymbol{x} \in \mathbf{X} \qquad (1)$$

The transfers areconsidered in the terms of the BCI frame with the corresponding motion equations [3]. The following assumptions are made: the eccentricity of the Moon and the Earth orbits around barycenter is neglected; the eccentricity of the gravitational fields of the Earth, the Moon and the Sun is neglected, the SC sometimes moves in the Earth and the Moon shadow.

3. Methods

In this work we use the Fedorenko successful linearization method [1] that accepts the limitation on composed functions that have Freshe derivatives. The method is based on making the variation optimal control problem the iteration problem of linear programming. The functional to optimize was chosen as a sum of the fuel used during the transfer and the components accounting for the conditions of the spacecraft final orbit insertion:

$$I = m(t_k^{[n]}) + (r^{[n]}(t_k^{[n]}) - r_k)^2 + (\varphi^{[n]}(t_k^{[n]}) - \omega_M t_k^{[n]})^2 + v_r^{[n]}(t_k^{[n]}))^2 + (v_\varphi^{[n]}(t_k^{[n]}) - v_{\varphi k})^2 \to min, \qquad (2)$$

where r_k is a radius of the final orbit, ω_M – Moon rotation rate around the barycenter of the system, $v_{\varphi k}$ – angular rotation rate on the final orbit, n – the number of the final segment of the transfer.

To solve the settled problem we should find the following variables:

$$\frac{\partial I^*}{\partial T_0^i}, \frac{\partial I^*}{\partial T_k^i}, \frac{\partial I^*}{\partial \lambda_1^i}, \frac{\partial I^*}{\partial a_0}, \frac{\partial I^*}{\partial c_0} \qquad (3)$$

The exact solution of the problem was established with the use of the Pontryagin maximum method and the numerical integration. The analysis of that solution shows, that the transfer trajectory has three general segments of work (figure 1). Each of the general steps of the transfer control program was divided into several segments of work to provide better accuracy.

The thrust is directed at the angle λ_1^i to the radius-vector of the SC. Thus, $\boldsymbol{u}$is a piecewise continuous function which is determined by the following parameters: $\lambda_1^i,\ \partial T_0^i, \partial T_k^i$ (each λ_1^i ∂T_0^i and ∂T_k^i is relevant to the corresponding segment of the trajectory).

So, according to notations of the Fedorenko method we have:

$$u^{[i]} = \{\lambda_1^i\},\ p^{[i]} = \{T_0^i\},\ q = \{a_0, c_0\}, \qquad (4)$$

where a_0 – spacecraft acceleration, c_0 – exhaust velocity. As can be seen form (2) the functional does not include the integral component, it consists only of the terminal component and depends on the following state vector:

$$x^{[i]}(t^{[i]}) = \{r, \varphi, v_r, v_\varphi, m\}^T = \{x_1, x_2, x_3, x_4, x_5\}^T . \qquad (5)$$

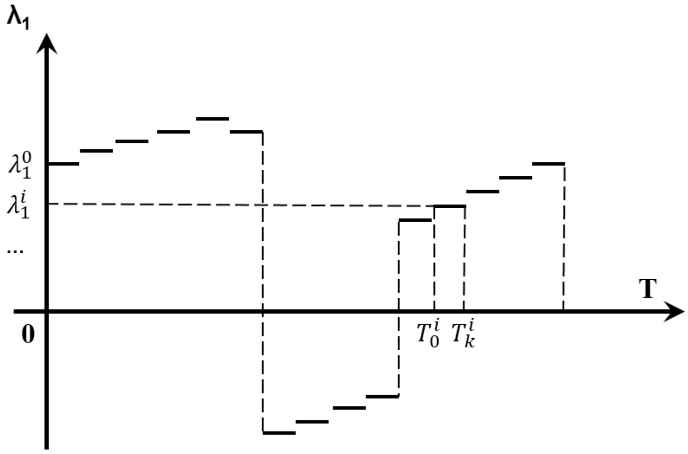

Figure 1. Transfer general stepwise control structure.

Afterhavingthemotionequations [3] withtheboundaryconditions (table 1) integratedwe get the state vector components relevant to the final transfer segment determined.

Let us derive the Hamiltonian for the costate vector $\psi_r, \psi_\varphi, \psi_{v_r}, \psi_{v_\varphi}, \psi_m$:

$$H = \frac{dr}{dr}\psi_r + \frac{d\varphi}{dt}\psi_\varphi + \frac{dv_r}{dt}\psi_{v_r} + \frac{dv_\varphi}{dt}\psi_{v_\varphi} + \frac{dm}{dt}\psi_m . \qquad (6)$$

Then we plug the right parts of the motion equations into (6):

$$H = v_r\psi_r + \frac{v_\varphi}{r}\psi_\varphi + \left(\frac{v_\varphi^2}{r} - \frac{1}{r^2} + \frac{a_0}{1-m}\delta\, Cos\lambda\right)\psi_{v_r} + \left(-\frac{v_r v_\varphi}{r} + \frac{a_0}{1-m}\delta\, Sin\lambda\right)\psi_{v_\varphi} + \beta\psi_m \qquad (7)$$

Here we can estimate the costate functions derivatives by deriving the Hamiltonian with respect to the corresponding state coordinates:

$$\begin{cases} \dfrac{d\psi_r}{dt} = -\dfrac{\partial H}{\partial r} = -\left(\dfrac{v_\varphi}{r^2}\psi_\varphi - \dfrac{v_\varphi^2}{r^2}\psi_{v_r} + \dfrac{2}{r^3}\psi_{v_r} + \dfrac{v_r v_\varphi}{r^2}\psi_{v_\varphi}\right) \\ \dfrac{d\psi_\varphi}{dt} = -\dfrac{\partial H}{\partial \varphi} = 0 \\ \dfrac{d\psi_{v_r}}{dt} = -\dfrac{\partial H}{\partial v_r} = -\left(\psi_r - \dfrac{v_\varphi}{r}\psi_{v_\varphi}\right) \\ \dfrac{d\psi_{v_\varphi}}{dt} = -\dfrac{\partial H}{\partial v_\varphi} = -\left(\dfrac{\psi_\varphi}{r} + 2\dfrac{v_\varphi}{r}\psi_{v_r} - \dfrac{v_r}{r}\psi_{v_\varphi}\right) \\ \dfrac{d\psi_m}{dt} = -\dfrac{\partial H}{\partial m} = -\left(\dfrac{a_0}{(1-m)^2}\delta\, Cos\lambda\, \psi_{v_r} + \dfrac{a_0}{(1-m)^2}\delta\, Sin\lambda\, \psi_{v_\varphi}\right) \end{cases} \qquad (8)$$

Thescalarfunctionμ determines the finishing time of the each segment. In this case each of thefiveμ functions is a null vector, because there is no discontinuous jumps of the state coordinates on the transfer segments boundaries.

To determine the needed derivatives we need to find the following variables:

$$a^{[n]} = -\frac{1}{\mu_x^{[n]} f_k^{[n]} + \mu_t^{[n]}} = -1 \qquad (9)$$

$$d = a^{[n]}\left(F_t + F_x f_k^{[n]}\right) = -1\left(\beta + 2(r^{[n]}(t_k^{[n]}) - r_{L2})\right)\Big|_T \cdot v_r\Big|_T + 2\left(\varphi^{[n]}(t_k^{[n]}) - \omega_M t_k^{[n]}\right)\Big|_T + 2\left(v_\varphi^{[n]}(t_k^{[n]}) - v_{\varphi L2}\right)\Big|_T \cdot\left(-\frac{v_r v_\varphi}{r} + \frac{a_0}{1-m}\delta\, Sin\lambda\right)\Big|_T) \qquad (10)$$

Then the motion equations right parts for the state coordinates should be found:

$$\begin{cases} f_t(T_k) = v_r(T_k) \\ f_\varphi(T_k) = \dfrac{v_\varphi}{r}(T_k) \\ f_{v_r}(T_k) = \left(\dfrac{v_\varphi^2}{r} - \dfrac{1}{r^2} + \dfrac{a_0}{1-m}\delta\, Cos\lambda\right)(T_k). \\ f_{v_\varphi}(T_k) = \left(-\dfrac{v_r v_\varphi}{r} + \dfrac{a_0}{1-m}\delta\, Sin\lambda\right)(T_k) \\ f_m(T_k) = \beta \end{cases} \tag{11}$$

Nowthecostatefunctionsvaluesinthefinalpointscouldbefound:

$$\psi^{[n]}|_T = F_x + d \cdot \mu_x^{[n]}, \tag{12}$$

$$\begin{pmatrix} \psi_r \\ \psi_\varphi \\ \psi_{v_r} \\ \psi_{v_\varphi} \\ \psi_m \end{pmatrix} = \begin{pmatrix} 2(r^{[n]}(t_k^{[n]}) - r_{L2}) \\ 2(\varphi^{[n]}(t_k^{[n]}) - \omega_M t_k^{[n]}) \\ 2 v_r^{[n]}(t_k^{[n]}) \\ 2(v_\varphi^{[n]}(t_k^{[n]}) - v_{\varphi L2}) \\ 1 \end{pmatrix}. \tag{13}$$

Let us find $\dfrac{\partial J}{\partial \lambda_1^i}$.

Firstly, let us settle $\omega^i(s)$ (analytical derivatives of the functional with respect to $u^{[i]}$):

$$\omega^i(s) = \psi_i^T(s) f_u^i(s), \quad i = 1, ..., K, \tag{14}$$

where the ψ_i costate functions are the solution of the equations:

$$\frac{d\psi_i}{dt^i} = -(f_x^i)^T \psi_i, \quad i = 1, ..., K. \tag{15}$$

So, $\omega^i(s)$ would be as follows:

$$\omega = -\psi_{v_r} \frac{a_0}{1-m}\delta\, Sin\lambda_1^i + \psi_{v_\varphi} \frac{a_0}{1-m}\delta\, Cos\lambda_1^i. \tag{16}$$

$$\omega_{\lambda_1^i} = \int_{t_0^i}^{t_k^i} \omega^i(s)\, ds, \tag{17}$$

Now letusfind $\dfrac{\partial J}{\partial \overline{q}} = \left(\dfrac{\partial J}{\partial a_0}; \quad \dfrac{\partial J}{\partial c_0}\right)$.

$$\frac{\partial \overline{f}}{\partial \overline{q}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{Cos\lambda_1^i \cdot \delta}{1-m} & 0 \\ 0 & 0 & \dfrac{Sin\lambda_1^i \cdot \delta}{1-m} & 0 \\ 0 & 0 & \dfrac{\delta}{c_0} & -\dfrac{a_0 \cdot \delta}{c_0^2} \\ 0 & 0 & \dfrac{\partial \overline{f}}{\partial a_0} & \dfrac{\partial \overline{f}}{\partial c_0} \end{pmatrix}. \tag{18}$$

$$\frac{\partial J}{\partial \overline{q}} = D = \sum_{i=1}^{n} \left(\int_{t_0^i}^{t_k^i} (\psi^{[i]})^T \cdot \frac{\partial \overline{f}^{[i]}}{\partial \overline{q}} ds \right) + \frac{\partial F}{\partial \overline{q}} + d \frac{\partial \mu^{[i]}}{\partial \overline{q}}; \tag{19}$$

$$\frac{\partial J}{\partial \overline{q}} = \left(\int_{t_0}^{t_k} (\psi_{v_r} \cdot \frac{Cos\lambda \cdot \delta}{1-m} + \psi_{v_\varphi} \cdot \frac{Sin\lambda \cdot \delta}{1-m} + \psi_m \frac{\delta}{c_0}) ds; \quad \int_{t_0}^{t_k} (-\psi_m \cdot \frac{a_0 \delta}{c_0^2}) ds \right); \qquad (20)$$

$$\frac{\partial J}{\partial a_0} = \int_{t_0}^{t_k} (\psi_{v_r} \cdot \frac{Cos \cdot \delta}{1-m} + \psi_{v_\varphi} \cdot \frac{Sin\lambda \cdot \delta}{1-m} + \psi_m \cdot \frac{\delta}{c_0})\, ds; \qquad (21)$$

$$\frac{\partial J}{\partial c_0} = \int_{t_0}^{t_k} (-\psi_m \cdot \frac{a_0 \cdot \delta}{c_0^2})\, ds. \qquad (22)$$

Now letusfind $\frac{\partial J}{\partial \overline{T}} = \left(\frac{\partial J}{\partial T_0^i} \; \frac{\partial J}{\partial T_k^i} \right) \cdot \begin{pmatrix} \partial T_0^i \\ \partial T_k^i \end{pmatrix}$.

As all the initial boundary conditions are constants, we have:

$$\Pi^0 \partial p^{[0]} = 0\,. \qquad (23)$$

$$\Pi^{[i]} = \left(\psi^{[i+1]}(t_0^{[i+1]})\right)^T \cdot [\frac{\partial \overline{\varphi}^{[i]}}{\partial p^{[i]}} + \overline{c}^{[i]} \frac{\partial \mu^{[i]}}{\partial p^{[i]}} - f^{[i+1]}(t_0^{[i+1]}) \cdot \left(\frac{\partial \tau^{[i]}}{\partial p^{[i]}} + b^{[i]} \frac{\partial \mu^{[i]}}{\partial p^{[i]}} \right)] +$$
$$+ \int_{t_0^{[i]}}^{t_k^{[i]}} (\psi^{[i]}(s))^T \cdot \frac{\partial \overline{f}^{[i]}}{\partial p^{[i]}}(s)\, ds = (\psi^{[i+1]}(t_k^{[i+1]}))^T \cdot [\, f^{[i]}(t_k^{[i]}) - f^{[i+1]}(t_0^{[i+1]})]; \qquad (24)$$

Thus, we found the analytical expressions for all the functional derivatives (3), therefore, the problem is solved.

4. Results

The optimal control programs for the L1-L2 and L2-L1 transfers and corresponding trajectorieswere obtained. The time duration of the L1-L2 transfer was equal to 5.9 days and the duration of the L2-L1 transfer was equal to 6.7 days.

The results of the flight simulation and optimization showed that the general trajectory consists of alternating passive and active segments, but in particular cases one of the segments can disappear, particularly, the first passive segment of the L1-L2 transfer. The obtained trajectories and control programs are in good agreement with the ones derived in [3, 4]. Clearly, for ballistic optimization of mission, it is necessary to balance between fuel consumption and mission duration.

5. Conclusion

The usage of the Fedorenko successful linearization method along with the gradient method in the three-body task framework allows us to obtain optimal steering and the motion trajectories for spacecraft with low-thrust engines. The approach used in the paper is applicable only for the estimative computations, however, it can be improved by adding the gravitational correction caused by the Earth and the Moon oblateness. The above-mentioned conditions will be taken into account in the future works. Findings may be used to design the required -ballistic parameters of the future lunar missions.

6. References

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