Multiparameter Golay m-complementary sequences and transforms

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Abstract. In this paper, we develop the family of Golay-Rudin-Shapiro (GRS) *m*-complementary sequences. It based on a new generalized iteration generating construction with n unitary $(m \times m)$-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n$.

Keywords: generalized, m-complementary, sequences, multiparameter, Fourier-Golay-Rudin-Shapiro transforms, OFDM telecommunication systems.

1. Introduction

Binary ± 1-valued *Golay-Rudin-Shapiro* sequences (2-GRS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by Golay [1,2,3] in 1949-1951, Shapiro [4,5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of “complementary pairs" of finite sequences all of whose entries are ± 1. This was motivated by a highly non-trivial application to infrared spectrometry. Then he gave an explicit construction for binary Golay complementary pairs of length 2^m and later [3] noted that the construction implies the existence of at least $2^m m!/2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 1^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ (Turyn, [7]), but do not exist for any length N having a prime factor congruent to the modulo 4 (Eliahou et al., [8]). In 1951, H. S. Shapiro [4,5] introduced what became known, after 1963, as the “Rudin-Shapiro” polynomial pairs. Shapiro's work was entirely inpure mathematics. Budisin [9,10,11] using the work of Sivaswamy [12] gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it coincides with those given explicitly by Golay. For a survey of results on binary and nonbinary Golay complementary pairs, see Byrnes [13] and Fan, Darnel, [14], respectively. In 1999, Davis and Jedwab [15] gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed-Muller codes.

Discrete classical *Fourier-Golay-Rudin-Shapiro Transforms* (FGRST) in bases of different Golay-Rudin-Shapiro sequences can be used in many signal processing applications: multiresolution by discrete orthogonal wavelet decomposition, digital audition, digital video broadcasting, communication systems (Orthogonal Frequency Division Multiplexing - OFDM, Multi-Code-Division Multiple Access - MCDA), radar, and cryptographic systems.

For building the classical FGRST in bases of classical Golay-Rudin-Shapiro sequences the following actors are used: 1) the Abelian group $\mathbf{Z}_2^n$, 2) 2-point Fourier transform F_2, and 3) the complex field $\mathbf{C}$; i.e., these transforms are associated with the triple $\left(\mathbf{Z}_2^n, \mathsf{F}_2, \mathbf{C}\right)$. In this work, we develop a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-, and Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n, \mathsf{F}_2, \mathbf{C}\right)$, but with $\left(\mathbf{Z}_m^n, \mathbf{U}_m, \mathsf{A}lg\right)$ or with $\left(\mathbf{Z}_m^n, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathsf{A}lg\right)$, where $\mathbf{U}_m$ or $\left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}$ are an single or a set of arbitrary unitary $(m \times m)$-transforms instead of $\mathsf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $\mathsf{A}lg$ is an algebras (Clifford algebras), or finite rings ($\mathbf{Z}_N$) instead of the complex field $\mathbf{C}$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay-Rudin-Shapiro* m-ary sequences) is described. In Section 3 we propose method based on new generalized iteration rule with n unitary $(m \times m)$-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n$.

2. The object of the study. New iteration construction for Golay m-arysequences

2.1. Basic definitions

We begin by describing the original Golay 2- and m-complementary sequences.

Definition 1. Let $\mathrm{com}^0(t) := \left(c_0, c_1, ..., c_{N-1}\right)$ and $\mathrm{com}^1(t) = \left(s_0, s_1, ..., s_{N-1}\right)$, where $c_i, s_i \in \mathbf{B}_2 = \left\{\pm 1\right\}$. The sequences $\mathrm{com}^0(t), \mathrm{com}^1(t)$ are called the *2-complementary ($\left(\pm 1\right)$-valued)* or *Goley complementary pair* over $\left\{\pm 1\right\}$, if $COR^0(\tau) + COR^1(\tau) = N\delta(\tau)$, or $\left(\left|\mathrm{COM}^0(z)\right|^2 + \left|\mathrm{COM}^1(z)\right|^2\right)_{|z|=1} = N$, where $COR^0(\tau), COR^1(\tau)$ are the periodic correlation functions of $\mathrm{com}^0(t), \mathrm{com}^1(t)$, respectively, and $\mathrm{COM}^0(z) = \mathsf{Z}\left\{\mathrm{com}^0(t)\right\}$, $\mathrm{COM}^1(z) = \mathsf{Z}\left\{\mathrm{com}^1(t)\right\}$ are their Z–transforms. Any sequence, which is a member of a Golay complementary pair, is called the *Golay sequence* and its Z–transform $\mathrm{COM}_k(z) = \mathsf{Z}\left\{\mathrm{com}_k(t)\right\}$ is called the *Golay-Shapiro-Rudin polynomial (GSRP).*

Definition 2. A generalization of Golay complementary pair, known as the *Golay m-complementary* m-element set of complex-valued sequences [16],

$$\begin{cases} \mathrm{com}_0(t) := \left(c_0(0), \; c_0(1), \; ..., \; c_0(m-1)\right), \\ \mathrm{com}_1(t) := \left(c_1(0), \; c_1(1), \; ..., \; c_1(m-1)\right), \\ \text{...}, \\ \mathrm{com}_{m-1}(t) := \left(c_{m-1}(0), c_{m-1}(1), ..., c_{m-1}(m-1)\right) \end{cases}$$

is defined by $\sum_{k=0}^{m-1} COR_k(\tau) = m \cdot \delta(\tau)$, or $\sum_{k=0}^{m-1} \left|\mathrm{COM}_1(z)\right|^2 = m$, where $\left\{COR_k(\tau)\right\}_{k=0}^{m-1}$ are the periodic autocorrelation functions of $\left\{\mathrm{com}_k(t)\right\}_{k=1}^{m}$ and $\mathrm{COM}_k(z) = \mathsf{Z}\left\{\mathrm{com}_k(t)\right\}$, $k = 0,1,...,m-1$ are their Z–transforms.

2.2. Golay matrix

We use two symbols $\boldsymbol{\alpha}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ and $\mathbf{t}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ for numeration of Golay sequences and discrete time, respectively. For integer $\boldsymbol{\alpha}_n \in [0, m^{n-1}-1]$ and $\mathbf{t}_n \in [0, m^{n-1}-1]$ we shall use m-ary codes $\vec{\boldsymbol{\alpha}}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$, $\vec{\mathbf{t}}_n = (t_1, t_2, ..., t_n)$ where $\alpha_i, t_i \in \{0, 1, ..., m-1\} = \mathbf{Z}_m$, $i = 1, 2, ..., n$.

Let $\vec{\boldsymbol{\alpha}}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, ..., t_n)$ be m-ary codes, then define

$$\boldsymbol{\alpha}_n = |\vec{\boldsymbol{\alpha}}_n| = |(\alpha_1, \alpha_2, ..., \alpha_n)| = \sum_{i=1}^{n} \alpha_{n-i+1} m^{i-1}, \; \mathbf{t}_n = |\vec{\mathbf{t}}_n| = |(t_1, t_2, ..., t_n)| = \sum_{i=1}^{n} t_{n-i+1} m^{n-i}$$

be integers whose m-ary codes are $\vec{\boldsymbol{\alpha}}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, ..., t_n)$, where α_n, t_1 are less significant bits (LSB) and α_1, t_n are most significant bits (MSB) of $\vec{\boldsymbol{\alpha}}_n = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\vec{\mathbf{t}}_n = (t_1, t_2, ..., t_n)$, respectively. Obviously,

$$\begin{array}{llll}
\vec{\boldsymbol{\alpha}}_1 = (\alpha_1) \in \mathbf{Z}_m, & \boldsymbol{\alpha}_1 = \alpha_1 \in \mathbf{Z}_m, & \vec{\mathbf{t}}_1 = (t_1) \in \mathbf{Z}_m, & \mathbf{t}_1 = t_1 \in \mathbf{Z}_m, \\
\vec{\boldsymbol{\alpha}}_2 = (\vec{\boldsymbol{\alpha}}_1, \alpha_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & (\boldsymbol{\alpha}_1, \alpha_2) \in \mathbf{Z}_m \times \mathbf{Z}_m, & \vec{\mathbf{t}}_2 = (\vec{\mathbf{t}}_1, t_2) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & (\mathbf{t}_1, t_2) \in \mathbf{Z}_m \times \mathbf{Z}_m, \\
\vec{\boldsymbol{\alpha}}_3 = (\vec{\boldsymbol{\alpha}}_2, \alpha_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & (\boldsymbol{\alpha}_2, \alpha_3) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m, & \vec{\mathbf{t}}_3 = (\vec{\mathbf{t}}_2, t_3) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & (\mathbf{t}_2, t_3) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m, \\
\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots, \\
\vec{\boldsymbol{\alpha}}_n = (\vec{\boldsymbol{\alpha}}_{n-1}, \alpha_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & (\boldsymbol{\alpha}_{n-1}, \alpha_n) \in \mathbf{Z}^1_{m^{n-1}} \times \mathbf{Z}_m & \vec{\mathbf{t}}_n = (\vec{\mathbf{t}}_{n-1}, t_n) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & (\mathbf{t}_{n-1}, t_n) \in \mathbf{Z}_{m^{n-1}} \times \mathbf{Z}_m
\end{array}$$

Let $\left\{\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})\right\}$ be m^{n+1}-element set of m-complementary sequences (of length m^{n+1}), where $\boldsymbol{\alpha}_{\mathbf{n+1}}, \mathbf{t}_{\mathbf{n+1}} = 0, 1, 2, ..., m^{n+1}\text{-}1$. They are form rows of a $\left(m^{n+1} \times m^{n+1}\right)$-matrix

$$\mathbf{G}^{[n+1]}_{m^{n+1}} = \left[\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})\right]_{\boldsymbol{\alpha}_{n+1}, \mathbf{t}_{n+1}=0}^{m^{n+1}\text{-}1} = \left[\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}\right]_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}\text{-}1},$$

that is called *the m-Golay matrix*. Here index $[n+1]$ shows that *m*-Golay matrix have been obtained on the $n+1$ iteration step. We are going to group these rows (sequences) into m^n collections consisting of m-element sets of *m*-complementary Golay sequences

$$\mathbf{G}^{[n+1]}_{m^{n+1}} = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1}) = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_n=0}^{m^n-1} \left(\mathop{\Big(}\limits_{\alpha_{n+1}=0}^{m-1} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n, \alpha_{n+1})}(\mathbf{t}_{n+1}) \right) = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_n=0}^{m^n\text{-}1} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1}) \\ \ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,m-1)}(\mathbf{t}_{n+1}) \end{bmatrix}. \quad (1)$$

Let us to select the more fine structure of the m-Golay matrix:

$$\mathbf{G}^{[n+1]}_{m^{n+1}} = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1}) = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n}\text{-}\mathbf{1}} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1}) \\ \ldots\ldots\ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_{n-1}=0}^{m^{n-1}\text{-}1} \left(\mathop{\Big(}\limits_{\alpha_n=0}^{m\text{-}1} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n 1)}(\mathbf{t}_{n+1}) \\ \ldots\ldots\ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n m-1)}(\mathbf{t}_{n+1}) \end{bmatrix} \right) = \mathop{\Big(}\limits_{\boldsymbol{\alpha}_{n-1}=0}^{m^{n-1}\text{-}1} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}) \\ \ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) \\ \ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1}) \\ \hline \mathrm{M} \\ \mathrm{M} \\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-11)}(\mathbf{t}_{n+1}) \\ \ldots \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}(\mathbf{t}_{n+1}) \end{bmatrix}. \quad (2)$$

Example 1. For $n=1$ and $n=2$ we have, respectively,

$$\mathbf{G}^{[1]}_{3^1}=\left[\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)\right]^{2}_{\boldsymbol{\alpha}_1,\mathbf{t}_1=0}=\mathop{\Bigg(}_{\boldsymbol{\alpha}_1=0}^{2}\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)=\begin{bmatrix}\mathrm{com}^{[1]}_{(0)}(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_{(1)}(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_{(2)}(\mathbf{t}_1)\end{bmatrix},$$

$$\mathbf{G}^{[2]}_{3^2}=\left[\mathrm{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)\right]^{8}_{\boldsymbol{\alpha}_2,\mathbf{t}_2=0}=\mathop{\Bigg(}_{\boldsymbol{\alpha}_2=0}^{8}\mathrm{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)=\mathop{\Bigg(}_{\boldsymbol{\alpha}_1=0}^{2}\begin{bmatrix}\mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,2)}(\mathbf{t}_2)\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[2]}_{(0,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(0,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(0,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(2,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(2,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(2,2)}(\mathbf{t}_2)\end{bmatrix}.$$

The matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}$ is constructed by an iteration construction. The initial matrix $\mathbf{G}^{[1]}_{m^1}$ is formed by starting with an arbitrary unitary (orthogonal) $(m\times m)$-matrix

$$\mathbf{U}=\left[U_\alpha(t)\right]:=\mathbf{G}^{[1]}_{m^1}=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_2(\mathbf{t}_1)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[1]}_{m-1}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}U_0(0) & U_0(1) & U_0(2) & \dots & U_0(m-1)\\ U_1(0) & U_1(1) & U_1(2) & \dots & U_1(m-1)\\ U_2(0) & U_2(1) & U_2(2) & \dots & U_2(m-1)\\ \dots & \dots & \dots & \dots & \dots\\ U_{m-1}(0) & U_{m-1}(1) & U_{m-1}(2) & \dots & U_{m-1}(m-1)\end{bmatrix}\in SU(\mathcal{A}lg,m).$$

where $U_\alpha(t)\in\mathcal{A}lg$, $\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)=\left(U_{\alpha_1}(0),\ U_{\alpha_1}(1),\ U_{\alpha_1}(2),\ \dots,\ U_{\alpha_1}(m-1)\right)$, $\boldsymbol{\alpha}_1=0,1,2,\dots,m-1$.

Example 2. The initial matrix $\mathbf{G}^{[1]}_{m^1}$ can be the Fourier transform on Abelian group $\mathbf{Z}_m$

$$\mathbf{G}^{[1]}_{m^1}=\mathcal{F}_m=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_2(\mathbf{t}_1)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[1]}_{m-1}(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & \dots & 1\\ 1 & \varepsilon^{1\cdot1} & \varepsilon^{1\cdot2} & \dots & \varepsilon^{1\cdot(m-1)}\\ 1 & \varepsilon^{2\cdot1} & \varepsilon^{2\cdot2} & \dots & \varepsilon^{2\cdot(m-1)}\\ \dots & \dots & \dots & \dots & \dots\\ 1 & \varepsilon^{(m-1)\cdot1} & \varepsilon^{(m-1)\cdot2} & \dots & \varepsilon^{(m-1)\cdot(m-1)}\end{bmatrix},\qquad(3)$$

where $\mathrm{com}^{[1]}_k(\mathbf{t}_1)=\left(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\dots,\varepsilon^{k\cdot(m-1)}\right)$, $k=0,1,2,\dots,m-1$ are characters of $\mathbf{Z}_m$.

It is easy to check that $\left(\left|\mathrm{COM}_0(z)\right|^2+\left|\mathrm{COM}_1(z)\right|^2+\dots+\left|\mathrm{COM}_{m\text{-}1}(z)\right|^2\right)_{|z|=1}=m.$ Hence, initial sequences in the form of rows of an unitary matrix are the Golay m-complementary sequences.

3. Methods

The matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}$ is constructed by an iteration construction

$$\mathbf{G}^{[1]}_{m^1}(\mathbf{U}_1)\xrightarrow{\mathbf{U}_2}\mathbf{G}^{[2]}_{m^2}(\mathbf{U}_1,\mathbf{U}_2)\xrightarrow{\mathbf{U}_3}\dots\xrightarrow{\mathbf{U}_n}\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n)\xrightarrow{\mathbf{U}_{n+1}}\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n,\mathbf{U}_{n+1}).\qquad(4)$$

where $\mathcal{U}_n:=\left\{\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n\right\}$, $\mathcal{U}_{n+1}:=\left\{\mathbf{U}_1,\mathbf{U}_2,\dots,\mathbf{U}_n,\mathbf{U}_{n+1}\right\}=\left\{\mathcal{U}_n,\mathbf{U}_{n+1}\right\}$. Here

$$\mathbf{U}_s = \left[U_\alpha^s(t)\right]_{\alpha,t=0}^{m-1} = \begin{bmatrix} U_0^s(0) & U_0^s(1) & U_0^s(2) & \ldots & U_0^s(m-1) \\ U_1^s(0) & U_1^s(1) & U_1^s(2) & \ldots & U_1^s(m-1) \\ U_2^s(0) & U_2^s(1) & U_2^s(2) & \ldots & U_2^s(m-1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ U_{m-1}^s(0) & U_{m-1}^s(1) & U_{m-1}^s(2) & \ldots & U_{m-1}^s(m-1) \end{bmatrix} \in SU(\mathcal{A}lg, m)$$

are a sequence of unitary $(m\times m)$-transforms, belonging to the special unitary group $SU(\mathcal{A}lg,m)$ over an algebra $\mathcal{A}lg$, where $s=1,2,\ldots,n+1$ and $U_\alpha^s(t)\in\mathcal{A}lg$. Let us assume that we have m-Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n)=\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ (depending on n previous transforms $\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n$). We need to construct the next m-Golay matrix $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n,\mathbf{U}_{n+1})=\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ using only $\mathbf{G}_{m^n}^{[n]}(\mathbf{U}_1,\mathbf{U}_2,\ldots,\mathbf{U}_n)$ and $\mathbf{U}_{n+1}$ We are going to use for m-Golay matrix $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ the same structure as in (1):

$$\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)=\bigcup_{\boldsymbol{\alpha}_n=0}^{m^n-1}\mathrm{com}_{(\boldsymbol{\alpha}_n)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)=\bigcup_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n-1}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix}. \qquad (5)$$

For constructing $\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1})$ from $\mathbf{G}_{m^n}^{[n]}(\mathcal{U}_n)$ we take each m-complementary set in the following form

$$\left\{\mathbf{com}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\right\}=\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix}$$ and construct shifted versa of their components

$$\left\{{}^{(k)}\mathbf{Tcom}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_{n+1})\right\}=\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n+\mathbf{m}^n\cdot 0)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n+\mathbf{m}^n\cdot 1)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n+\mathbf{m}^n\cdot(m-1))\end{bmatrix}=\begin{bmatrix}\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\\ \ldots\ldots\ldots\ldots\\ \mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}=$$

$$=\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)},\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)},\ldots,\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\right\}\cdot\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n)\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n)\end{bmatrix}=$$

$$=\mathrm{diag}\left\{\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(0\oplus k)},\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}(1\oplus k)},\ldots,\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}((m-1)\oplus k)}\right\}\cdot\left\{\mathbf{com}_{(\boldsymbol{\alpha}_{n-1},\cdot)}^{[n]}(\mathbf{t}_n\mid\mathcal{U}_{n+1})\right\} \qquad (6)$$

where $k=0,1,\ldots,m-1$ and $\mathbf{T}_{\mathbf{t}_n}^{\mathbf{m^n}s}$ is the shift operator on $\mathbf{m^n}s$ discrete positions in time domain $\mathbf{T}_{\mathbf{t}_n}^{m^n s}f(\mathbf{t}_n):=f(\mathbf{t}_n+m^n\mathrm{s})$.

Now we construct the general building blocks for the Golay $(m^{n+1}\times m^{n+1})$-matrix $\mathbf{G}_{m^{n+1}}$ as:

$$\mathbf{U}_{n+1}\cdot\left[{}^{(k)}\mathbf{Tcom}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\mid \mathrm{U}_{n+1})\right]=$$

$$=\mathbf{U}_{n+1}\cdot\operatorname{diag}\left\{\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n},\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n},\ldots,\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\right\}\cdot\left[\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\mid \mathrm{U}_n)\right]=$$

$$={}^{(k)}\mathbf{U}_{n+1}\cdot\left[\mathbf{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\cdot)}(\mathbf{t}_n\mid \mathrm{U}_{n+1})\right] \tag{7}$$

where

$${}^{(k)}\mathbf{U}_{n+1}=\mathbf{U}_{n+1}\cdot\operatorname{diag}\left\{\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n},\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n},\ldots,\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\right\}$$

$$=\begin{bmatrix} U_0^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & U_0^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & U_0^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\\ U_1^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & U_1^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & U_1^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\\ \ldots & \ldots & \ldots & \ldots\\ U_{m-1}^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & U_{m-1}^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & U_{m-1}^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\end{bmatrix}.$$

Using building blocks of $(m^n\times m^n)$-matrix $\mathbf{G}_{m^n}$ we construct the Golay $(m^{n+1}\times m^{n+1})$-matrix $\mathbf{G}_{m^{n+1}}$ according to the following iteration rule:

$$\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}\to
\begin{bmatrix} {}^{(0)}\mathbf{U}_{n+1}\cdot\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}\\ \hline {}^{(1)}\mathbf{U}_{n+1}\cdot\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}\\ \hline \ldots\\ \hline {}^{(m-1)}\mathbf{U}_{n+1}\cdot\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}\end{bmatrix}=
\begin{bmatrix}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\end{bmatrix}\\ \hline \begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\end{bmatrix}\\ \hline \ldots\\ \hline \begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-11)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\end{bmatrix}\end{bmatrix}=\mathbf{G}_{m^{n+1}}(\mathrm{U}_{n+1}),$$

where

$$\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},k,0)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},k,1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\\ \ldots\ldots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},k,m-1)}(\mathbf{t}_{n+1}\mid \mathrm{U}_{n+1})\end{bmatrix}={}^{(k)}\mathbf{U}_{n+1}\cdot\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}=$$

$$=\begin{bmatrix} a_0^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_0^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & a_0^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\\ a_1^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_1^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & a_1^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\\ \ldots & \ldots & \ldots & \ldots\\ a_{m-1}^{n+1}(0)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(0\oplus k)}_{\mathbf{t}_n} & a_{m-1}^{n+1}(1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}(1\oplus k)}_{\mathbf{t}_n} & \ldots & a_{m-1}^{n+1}(m-1)\mathbf{T}^{\mathbf{m}^{\mathbf{n}}((m-1)\oplus k)}_{\mathbf{t}_n}\end{bmatrix}\cdot\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid \mathrm{U}_n)\\ \ldots\ldots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid \mathrm{U}_n)\end{bmatrix}.$$

Hence,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}\,|\,\mathsf{U}_{n+1})=\sum_{\beta_n=0}^{m-1}a^{n+1}_{\alpha_{n+1}}(\beta_n)\mathbf{T}^{\mathbf{m^n}(\beta_n\oplus\alpha_n)}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n\,|\,\mathsf{U}_n).$$

Since $\mathbf{t}_{n+1}=(\mathbf{t}_n,t_{n+1})$, then believing $t_{n+1}=\alpha_n\underset{m}{\oplus}\beta_n$, we obtain:

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}\,|\,\mathsf{U}_{n+1})=\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1}\,|\,\mathsf{U}_{n+1})=$$

$$=\sum_{t_{n+1}=0}^{m-1}a^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n\,|\,\mathsf{U}_n)=$$

$$=\sum_{t_{n+1}=0}^{m-1}a^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n+m^n t_{n+1}\,|\,\mathsf{U}_n). \tag{8}$$

So,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1}\,|\,\mathsf{U}_{n+1})=U^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n\,|\,\mathsf{U}_n). \tag{9}$$

It is finally recurrent relation between m-complementary sequences of $\mathbf{G}^{[n+1]}_{m^{n+1}}[\mathsf{U}_{n+1}]$ and $\mathbf{G}^{[n]}_{m^n}[\mathsf{U}_n]$. From (9) we obtain two expressions for $\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1}\,|\,\mathsf{U}_{n+1})$:

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\prod_{s=1}^{n}U^{s+1}_{\alpha_{s+1}\underset{m}{\oplus}t_{s+2}}(\alpha_s\underset{m}{\oplus}t_{s+1}), \tag{10}$$

where $\alpha_0,t_{n+2}\equiv 0$.

In particular, for all matrices in the form of the Fourier $(m\times m)$-transform $\mathbf{U}^1_m=\mathbf{U}^2_m=...=\mathbf{U}^n_m=[\varepsilon^{\alpha t}_m]$ we have

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1})=\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1})=\varepsilon_m^{\sum_{s=1}^{n}(\alpha_s\underset{m}{\oplus}t_{s+1})\left(\alpha_{s+1}\underset{m}{\oplus}t_{s+2}\right)}, \tag{11}$$

where $\alpha_0,t_{n+2}\equiv 0$.

New sequences in (9) are orthogonal and m-complementary sequences.

4. Conclusion

In this paper, we have shown a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-,or Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}^n_2,\mathsf{F}_2,\mathbf{C})$,but with $(\mathbf{Z}^n_m,\mathbf{U}_m,\mathcal{A}lg)$ or with $(\mathbf{Z}^n_m,\{\mathbf{U}^1_m,\mathbf{U}^2_m,...,\mathbf{U}^n_m\},\mathcal{A}lg)$, where $\mathbf{U}_m$ or $\{\mathbf{U}^1_m,\mathbf{U}^2_m,...,\mathbf{U}^n_m\}$ are an single or a set of arbitrary unitary $(m\times m)$-transforms instead of $\mathsf{F}_2=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, and $\mathcal{A}lg$ is an algebras (Clifford algebras), finite rings ($\mathbf{Z}_N$) or finite Galois fields ($\mathbf{GF}(q)$) instead of the complex field $\mathbf{C}$.

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6. References

[1] Golay, M.J.E. Multislit spectrometry / M.J.E. Golay // J. Optical Society Am. – 1949. – P. 39-43.

[2] Golay, M.J.E. Complementary series / M.J.E. Golay // IRE Trans. Information Theory. – 1961. – Vol. IT-7. – P. 82-87.
[3] Golay, M.J.E. Sieves for low autocorrelation binary sequences / M.J.E. Golay // IEEE Trans. Inform. Theory. – 1977. – IT-23. – P. 43-51.
[4] Shapiro, H.S. Extremal problems for polynomials and power series / H.S. Shapiro // ScM. Thesis, Massachusetts Institute of Technology. – 1951.
[5] Shapiro, H.S. Apower series with small partial sums / H.S. Shapiro // Notices of the AMS. – 958. – Vol. 6(3). – P. 366-378.
[6] Rudin, W. Some theorems on Fourier coefficients / W. Rudin // Proc. Amer. Math. Soc. – 1959. – Vol. 10. – P. 855-859.
[7] Turyn, R.J. Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings / R.J. Turyn // J. Combin. Theory (A). – 1974. – Vol. 16. – P. 313-333.
[8] Eliahou, S. A new restriction on the lengths of Golay complementary sequences / S. Eliahou, M. Kervaire, B. Saffari // J. Combin. Theory (A). – 1990. – Vol. 55. – P. 49-59.
[9] Budisin, S.Z. New complementary pairs of sequences / S.Z. Budisin // Electron. Lett. – 1990. – Vol. 26. – . 881-883.
[10] Budisin, S.Z. Efficient pulse compressor for Golay complementary sequences / S.Z. Budisin // Electron. Lett. – 1991. – Vol. 27. – P. 219-220.
[11] Budisin, S.Z. Designing radar signals using complementary sequences. / S.Z. Budisin, B.M. Popovic, L.M. Indjin // Proc. IEE Conf. RADAR, 1987.
[12] Sivaswamy, R. Multiphase complementary codes / R. Sivaswamy // IEEE Trans. Inform. Theory. – 1978. – Vol. IT-24. – P. 546-552.
[13] Byrnes, J.S. Quadrature mirror filters, low crest factor arrays, functions achieving optimal uncertainty principle bounds, and complete orthonormal sequences — a unified approach / J.S. Byrnes // Applied and Computational Harmonic Analysis. – 1994. – P. 261-264.
[14] Fan, P. Sequence Design for Communications Applications (Communications Systems, Technologies and Applications) / P. Fan, M. Darnell // Taunton. – U.K.: Res. Studies, 1996.
[15] Davis, J.A. Peak-to-Mean Power Control in OFDM, Golay Complementary Sequences, and Reed-Muller Codes / J.A. Davis, J. Jedwab // IEEE Trans. Inform. Theory. – 1999. – Vol. IT-45(7). – P. 2397-2417.
[16] Lei, Z.X. Some properties of generalized Rudin-Shapiro polynomials / Z.X. Lei // Chinese Ann. Math. – 1991. – Vol. 2. – P. 145-153.