A fast one dimensional total variation regularization algorithm

A. Makovetskii[a], S. Voronin[a], V. Kober[a]

[a] *Chelyabinsk State University, 454001, 129 Bratiev Kashirinykh street, Chelyabinsk, Russia*

Abstract

Denoising has numerous applications in communications, control, machine learning, and many other fields of engineering and science. A common way to solve the problem utilizes the total variation (TV) regularization. Many efficient numerical algorithms have been developed for solving the TV regularization problem. Condat described a fast direct algorithm to compute the processed 1D signal. In this paper, we propose a variant of the Condat's algorithm based on the direct 1D TV regularization problem. The usage of the Condat algorithm with the taut string approach leads to a clear geometric description of the extremal function.

Keywords: image restoration; total variation; denoising; exact solutions

1. Introduction

One of the most known techniques for denosing of noisy signals and images was proposed by Rudin, Osher, and Fatemi [1]. This is a total variation (TV) regularization problem. Let $J(u)$ be the following functional in the functional space L_2:

$$J(u) = \| u - u_0 \|_{L_2}^2 + \lambda\, TV(u), \tag{1}$$

where $\| u - u_0 \|_{L_2}^2$ is called a fidelity term and $\lambda TV(u)$ is called a regularization term. Here u_0 is an observed signal that is distorted by additive noise n,

$$u_0 = v + n. \tag{2}$$

Consider the following variational problem:

$$u_* = \arg\min_{u \in BV(\Omega)} J(u). \tag{3}$$

where u_* is an extremal function for $J(u)$. Numerical results have shown that TV regularization is quite useful in image restoration [2-4]. Here we consider a one dimensional TV (1D TV) regularization problem. In [5,6] Strong and Chan considered the behavior of explicit solutions of the 1D TV problem when the parameter λ in Eq. (1) is sufficiently small. The exact solutions of one dimensional TV regularization problem and of two dimensional radial symmetric TV regularization problem were considered in [7-10]. Recently, Condat [11,12] proposed explicit solutions of the 1D TV problem as well as a direct fast algorithm for the case of discrete functions. The algorithm is very fast and has complexity of $O(n)$ for typical discrete functions. In contrast, the proposed approach to finding exact solutions has a clear geometrical meaning.

In this paper, we propose a variant of the Condat's approach based on the direct 1D TV regularization problem. The usage of the Condat approach with the taut string method [12] leads to a clear geometric description of the extremal function.

2. Formulation of 1D TV regularization as a discrete problem

Let u_0 be a discrete function $u_0 = \{u_0^1, \dots, u_0^n\}$. For the function u_0 the problem (1) takes following form:

$$J(u) = \sum_{i=1}^{n} (u^i - u_0^i)^2 + \lambda \sum_{i=1}^{n-1} | u^{i+1} - u^i |. \tag{4}$$

The functional $J(u)$ is convex. Thus for the extremal (minimum) function u_* the subgradient $\nabla J(u)$ satisfies the condition:

$$\mathbf{0} \in \nabla J(u_*). \tag{5}$$

Remark. The subgradient $\nabla f(x)$ of the function $f(x) = |x|$:

$$\nabla f(x) = \begin{cases} 1, if\ x > 0 \\ -1, x < 0 \\ [-1; 1], x = 0 \end{cases}. \tag{6}$$

2.1. Computation of the subgradient

Consider subgradient $\nabla J(u)$:

$$\nabla J(u) = \sum_{i=1}^{n} \nabla (u^i - u_0^i)^2 + \lambda \sum_{i=1}^{n-1} \nabla | u^{i+1} - u^i |. \tag{7}$$

$$\sum_{i=1}^{n} \nabla (u^i - u_0^i)^2 = (u^1 - u_0^1,\ u^2 - u_0^2,\ \dots, u^{1n-1} - u_0^{n-1},\ u^n - u_0^n). \tag{8}$$

By analogy with (6) the subgradients $\nabla |u^{i+1} - u^i|$, $i = 1, \dots, n-1$, can be written as:

$$\nabla|u^2-u^1| = \begin{cases} (-1,1,0,0,0,\dots,0,0), if\ u^2 > u^1 \\ (1,-1,0,0,0,\dots,0,0), if\ u^2 < u^1 \\ \{(\delta^1,-\delta^1,0,0,0,\dots,0,0) | \delta^1 \in [-1;1]\}, if\ u^2 = u^1 \end{cases}, \tag{9}$$

$$\nabla|u^3-u^2| = \begin{cases} (0,-1,1,0,0,\dots,0,0), if\ u^3 > u^2 \\ (0,1,-1,0,0,\dots,0,0), if\ u^3 < u^2 \\ \{(0,\delta^2,-\delta^2,0,0,\dots,0,0) | \delta^2 \in [-1;1]\}, if\ u^3 = u^2 \end{cases}, \tag{10}$$

...

$$\nabla|u^{n-1}-u^{n-2}| = \begin{cases} (0,0,0,0,0,\dots,-1,1,0), if\ u^{n-1} > u^{n-2} \\ (0,0,0,0,0,\dots,1,-1,0), if\ u^{n-1} < u^{n-2} \\ \{(0,0,0,0,0,\dots,\delta^{n-2},-\delta^{n-2},0) | \delta^{n-2} \in [-1;1]\}, if\ u^{n-1} = u^{n-2} \end{cases}, \tag{11}$$

$$\nabla|u^{n}-u^{n-1}| = \begin{cases} (0,0,0,0,0,\dots 0,-1,1), if\ u^{n} > u^{n-1} \\ (0,0,0,0,0,\dots 0,1,-1), if\ u^{n} < u^{n-1} \\ \{(0,0,0,0,0,\dots,0,\delta^{n-1},-\delta^{n-1}) | \delta^{n-1} \in [-1;1]\}, if\ u^{n} = u^{n-1} \end{cases}, \tag{12}$$

$$\sum_{i=1}^{n-1} \nabla| u^{i+1}-u^i| = \{(\delta^1,\delta^2-\delta^1,\delta^3-\delta^2,\delta^4-\delta^3,\dots,\delta^{n-1}-\delta^{n-2},-\delta^{n-1}) \mid \delta^i=-1, if\ u^{i+1} > u^i,\ \delta^i = 1, if\ u^{i+1} < u^i,$$
$$\delta^i \in [-1;1], if\ u^{i+1} = u^i, i=1,\dots,n-1\}. \tag{13}$$

From expressions (8) and (13) we get the following parameterization of the subradient:

$$\begin{cases} (\nabla J(u))^1 = (u^1-u_0^1) + \lambda\delta^1 \\ (\nabla J(u))^2 = (u^2-u_0^2) + \lambda\delta^2 - \lambda\delta^1 \\ (\nabla J(u))^3 = (u^3-u_0^3) + \lambda\delta^3 - \lambda\delta^2 \\ \dots \\ (\nabla J(u))^{n-1} = (u^{n-1}-u_0^{n-1}) + \lambda\delta^{n-1} - \lambda\delta^{n-2} \\ (\nabla J(u))^n = (u^n-u_0^n) + \lambda\delta^{n-1} \end{cases}. \tag{14}$$

where

$$\delta^i = \begin{cases} -1, if\ u^{i+1} > u^i \\ 1, if\ u^{i+1} < u^i \\ \in [-1;1], if\ u^{i+1} = u^i \end{cases}. \tag{15}$$

Since $(\nabla J(u_*))^i = 0, i = 1,\dots,n-1$ for some values of the parameters δ^i satisfying (15) we get:

$$\begin{cases} u_*^1 = u_0^1 - \lambda\delta^1 \\ u_*^2 = u_0^2 - \lambda\delta^2 + \lambda\delta^1 \\ u_*^3 = u_0^3 - \lambda\delta^3 + \lambda\delta^2 \\ \dots \\ u_*^{n-1} = u_0^{n-1} - \lambda\delta^{n-1} + \lambda\delta^{n-2} \\ u_*^n = u_0^n + \lambda\delta^{n-1} \end{cases}. \tag{16}$$

Consider the sequence of the cumulative sums:

$$\begin{cases} u_*^1 = u_0^1 - \lambda\delta^1 \\ u_*^2 + u_*^1 = u_0^2 + u_0^1 - \lambda\delta^2 \\ u_*^3 + u_*^2 + u_*^1 = u_0^3 + u_0^2 + u_0^1 - \lambda\delta^3 \\ \dots \\ u_*^{n-1} + \dots + u_*^1 = u_0^{n-1} + \dots + u_0^1 - \lambda\delta^{n-1} \\ u_*^n + \dots + u_*^1 = u_0^n + \dots + u_0^1 \end{cases}. \tag{17}$$

Consider such variables $U^1,\dots,U^n$ and $U_0^1,\dots,U_0^n$, that:

$$\begin{cases} U^1 = u_*^1, U_0^1 = u_0^1 \\ U^2 = u_*^2 + u_*^1, U_0^2 = u_0^2 + u_0^1 \\ \dots \\ U^{n-1} = u_*^{n-1} + \dots + u_*^1, U_0^{n-1} = u_0^{n-1} + \dots + u_0^1 \\ U^n = u_*^n + \dots + u_*^1, U_0^n = u_0^n + \dots + u_0^1 \end{cases}. \tag{18}$$

So the solution of the problem (3) is reduced to the solution of the problem:

$$\begin{cases} U^1 = U_0^1 - \lambda\delta^1 \\ U^2 = U_0^2 - \lambda\delta^2 \\ U^3 = U_0^3 - \lambda\delta^3 \\ \dots \\ U^{n-1} = U_0^{n-1} - \lambda\delta^{n-1} \\ U^n = U_0^n \end{cases}, \qquad (19)$$

with given discrete function U_0 and unknown discrete functions U and δ satisfying to the conditions (15).

Consider additional variables $U^0 = U_0^0 = 0$. Note that then for any $i = 1, \dots, n-1$ the condition $u^{i+1} > u^i$ is equivalent to the condition $U^{i+1} - 2U^i + U^{i-1} > 0$, the condition $u^{i+1} < u^i$ is equivalent to the condition $U^{i+1} - 2U^i + U^{i-1} < 0$, the condition $u^{i+1} = u^i$ is equivalent to the condition $U^{i+1} - 2U^i + U^{i-1} = 0$.

Then the set of equations (19) can be rewritten taking into account additional variables:

$$\begin{cases} U^0 = U_0^0 = 0 \\ U^1 = U_0^1 - \lambda\delta^1 \\ U^2 = U_0^2 - \lambda\delta^2 \\ U^3 = U_0^3 - \lambda\delta^3 \\ \dots \\ U^{n-1} = U_0^{n-1} - \lambda\delta^{n-1} \\ U^n = U_0^n \end{cases}, \qquad (20)$$

where

$$\delta^i = \begin{cases} -1, if\ U^{i+1} - 2U^i + U^{i-1} > 0 \\ 1, if\ U^{i+1} - 2U^i + U^{i-1} < 0 \\ \in [-1; 1], if\ U^{i+1} - 2U^i + U^{i-1} = 0 \end{cases}. \qquad (21)$$

2.2. Construction the „tube''

The values $U_0^0, U_0^1, \dots, U_0^n$ of the discrete function U_0 defines a piecewise linear curve, which is an axial line of the tube. The values $U_0^0, U_0^1 + \lambda, \dots, U_0^{n-1} + \lambda, U_0^n$ form the upper piecewise linear border of the tube, the values $U_0^0, U_0^1 - \lambda, \dots, U_0^{n-1} - \lambda, U_0^n$ form the bottom piecewise linear border of the tube. Figure 1 shows an example of a tube.

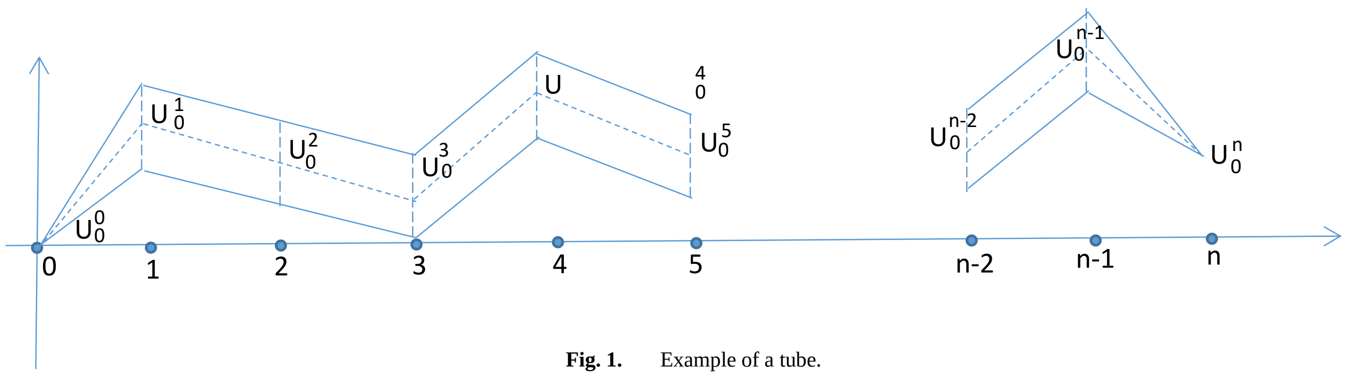

Fig. 1. Example of a tube.

2.3. Description of the extremal function U

Since $\delta^i, i = 1, \dots, n-1$, take values in the segment $[-1; 1]$, a piecewise linear curve defined by the values $U^1, \dots, U^n$ of a discrete function U (i.e. solution of the problem (20)) entirely belongs to the tube.

If the second discrete derivative equals zero, $U^{i+1} - 2U^i + U^{i-1} = 0$ then the piecewise linear curve defined by the values $U^1, \dots, U^n$ of a discrete function U in the neighborhood of the point i is a straight line.

If the second discrete derivative is positive, $U^{i+1} - 2U^i + U^{i-1} > 0$ then from (21) we see that $\delta^i = -1$ and (20) shows us that $U^i = U_0^i + \lambda$, i.e. U^i belongs to the upper border of the tube.

If the second discrete derivative is negative, $U^{i+1} - 2U^i + U^{i-1} < 0$ then from (21) we see that $\delta^i = 1$ and (20) shows us that $U^i = U_0^i - \lambda$, i.e. U^i belongs to the lower border of the tube.

It means that a piecewise linear curve defined by the values $U^0, \dots, U^n$ of a discrete function U exactly coincides with so called „taut string" connecting the endpoints of the tube.

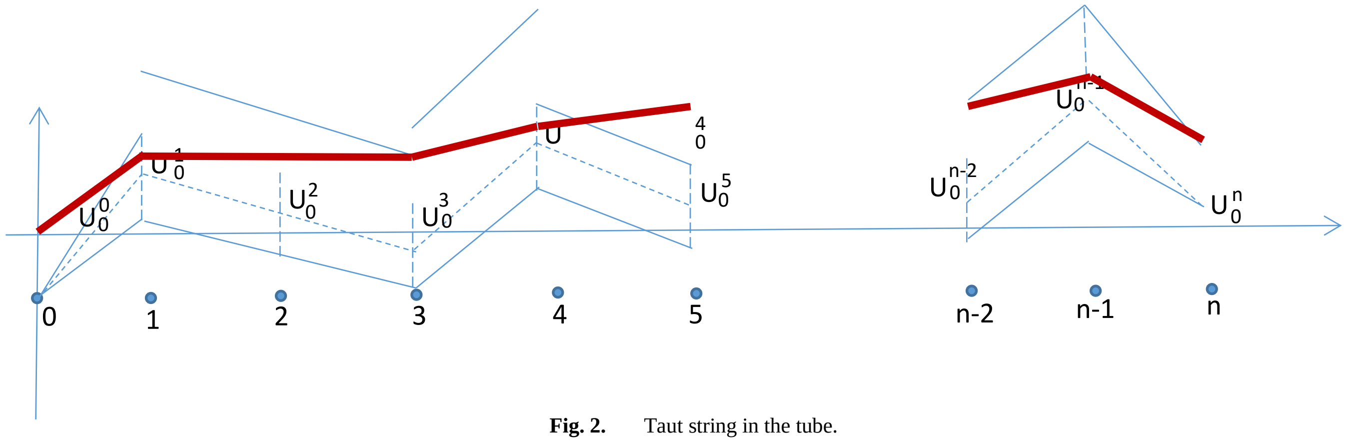

Fig. 2. Taut string in the tube.

Conclusion

In this paper, we propose a variant of the Condat's approach based on the direct 1D TV regularization problem. The usage of the Condat approach with the taut string method leads to a clear geometric description of the extremal function.

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