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Recommended Citation

G.S. Staples, D. Wylie. Clifford algebra decompositions of conformal orthogonal group elements, Clifford Analysis, Clifford Algebras and Their Applications, 4 (2015), 223-240

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Clifford algebra decompositions of conformal orthogonal group elements

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Abstract

Beginning with a finite-dimensional vector space V equipped with a nondegenerate quadratic form Q, we consider the decompositions of elements of the conformal orthogonal group $\mathrm{CO}_Q(V)$, defined as the direct product of the orthogonal group $O_Q(V)$ with dilations. Utilizing the correspondence between conformal orthogonal group elements and "decomposable" elements of the associated Clifford algebra, $\mathcal{C}\ell_Q(V)$, a decomposition algorithm is developed. Preliminary results on complexity reductions that can be realized passing from additive to multiplicative representations of invertible elements are also presented with examples. The approach here is based on group actions in the conformal orthogonal group. Algorithms are implemented in *Mathematica* using the **CliffMath** package.
Keywords: Clifford algebras; Lipschitz group; representation theory; decomposition; complexity; conformal transformations
AMS Subj. Class. 15A66; 15A75; 68W30

1 Introduction

Beginning with a finite-dimensional vector space V equipped with a nondegenerate quadratic form Q, we consider the decompositions of particular elements of the Clifford Lipschitz group Γ in the Clifford algebra $\mathcal{C}\ell_Q(V)$. These elements represent the conformal orthogonal group $\mathrm{CO}_Q(V)$, defined as the direct product of the orthogonal group $O_Q(V)$ with dilations.

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In Euclidean Clifford algebras, it is well known that elements $\mathfrak{u} \in \Gamma$ satisfying $\mathfrak{u}\tilde{\mathfrak{u}} = \alpha \in \mathbb{R}$ represent scaled orthogonal transformations on V; i.e., $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\overline{\mathfrak{u}}$ is a conformal orthogonal transformation on V. When $\mathfrak{u}\tilde{\mathfrak{u}} = \pm 1$, one sees that the mapping $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\overline{\mathfrak{u}}$ is an element of the orthogonal group $\mathrm{O}(n)$. More precisely, such an element $\mathfrak{u}$ is an element of the *Pin group*. The geometric significance of these mappings is detailed in a number of works, including (but not limited to) [2] and [6].

When an invertible element $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ can be written as an ordered Clifford product of anisotropic vectors from V, such a multiplicative representation $\mathfrak{u} = \prod_{i=1}^{k} \mathbf{v}_i$ is called a *decomposition* of $\mathfrak{u}$. The goal of the current paper is to consider decompositions of Clifford group elements, with an eye toward efficient symbolic computation. While the theoretical underpinnings have been understood and studied in various forms for decades, the advent of newer computing technologies and algorithms have shed a new light on these concepts.

The basic problem considered here is not new. To wit, versor factorization algorithms can be found in the work of Christian Perwass [7], and efficient blade factorization algorithms are found in the works of Dorst and Fontijne [3], [4].

More recently, the general problem of factorization in Clifford algebras of arbitrary signature was considered by Helmstetter [5]. The Lipschitz monoid (or Lipschitz semi-group) is the multiplicative monoid generated in $\mathcal{C}\ell_Q(V)$ over a field $\Bbbk$ by all scalars in $\Bbbk$, all vectors in V, and all $1 + \mathbf{xy}$ where $\mathbf{x}$ and $\mathbf{y}$ are vectors that span a totally isotropic plane. The elements of this monoid are called the Lipschitzian elements. Given a Lipschitzian element a in a Clifford algebra $\mathcal{C}\ell_Q(V)$ over a field $\Bbbk$ containing at least three scalars, Helmstetter showed that, if a is not in the subalgebra generated by a totally isotropic subspace of V, then it is a product of linearly independent vectors of V.

The current work is an extension of work begun in Wylie's master's thesis [10], where only Euclidean Clifford algebras were considered. Decomposition algorithms have been extended to Clifford algebras of arbitrary signature and implemented in *Mathematica*.

When the quadratic form Q is definite, the decomposable elements of $\mathcal{C}\ell_Q(V)$ are precisely the elements of the Clifford Lipschitz group. When Q is indefinite, we pass to a proper subset of Γ. In particular, an element $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ is said to be *decomposable* if there exists a collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of linearly independent anisotropic vectors such that $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$ *and* if the "top form" (i.e., grade-k part) of $\mathfrak{u}$ is invertible.

In Section 3, the basic theory underlying Clifford algebra decomposition of conformal orthogonal group elements is laid out. Decomposition algorithms are presented as pseudocode. For motivation, the Euclidean case is considered first, and a geometric algorithm called VersorFactor is presented for decomposing a transformation into elementary rotations and reflections, combined with scaling.

Passing to indefinite quadratic forms, a more general algorithm, CliffordDecomp (Algorithm 2), is developed for decomposition of elements of the Clifford Lipschitz group which satisfy the decomposability criteria mentioned previously. When Algorithm 2 is applied to an invertible blade, the output is an orthogonal collection of vectors.

A faster algorithm for decomposing blades is the FastBladeFactor algorithm (Algorithm 3). This algorithm is essentially the same as Fontijne's blade factorization[3, 4], except that Clifford multiplication and grade projections now take the place of geometric contractions. The combinatorial approach to writing and implementing the algorithm makes symbolic computations very efficient; geometric contractions have been avoided by using differences of sets, and computation of the ∞-norm of a blade to identify a starting point for the factorization has been eliminated by choosing the first term of the sum, as determined by a canonical ordering of multiindices. Unlike the CliffordDecomp algorithm, FastBladeFactor does *not* return an orthogonal collection of blades, but simply a set whose exterior product is equal to the input blade.

FastBladeFactor offers two significant advantages over CliffordDecomp: it works on null (noninvertible) blades, and it runs much more quickly than CliffordDecomp. Where CliffordDecomp computes the image of a probing vector under the mapping $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$, using the full additive representation of the blade $\mathfrak{u}$, FastBladeFactor makes use of a single basis blade chosen from that additive representation. Relative differences in processing times are illustrated in Section 4.

Experimental results were obtained using *Mathematica* 10 with the **CliffMath** [1] package on a MacBook Pro equipped with 2.4 GHz Intel Core i7 processor and 8GB of 1333 MHz DDR3 memory. Numerous trials were processed to compare the complexity between blade and versor factorizations, comparisons with changing grade and fixed dimension, and changing dimension with fixed grade.

We note that the comparisons here illustrate the relative differences in

[1] Available through `http://www.siue.edu/~sstaple`.

complexity of decomposing elements of different types. It is beyond the scope of the current paper to perform comparisons among algorithms devised by other authors because such comparisons are heavily implementation-dependent.

The paper concludes in Section 5 with a preliminary discussion of reductions in the complexity of representations and further avenues of research. The combinatorial set-theoretic approach to Clifford algebra computations using *Mathematica* and the implementations contained herein are original with the current authors.

For convenience, Table 1 details the various notation and font distinctions used throughout the paper.

Notation	Meaning
$\mathcal{C}\ell_Q(V)$	Clifford algebra of quadratic form Q of V.
$\mathcal{C}\ell_n$	Euclidean Clifford algebra of $\mathbb{R}^n$.
$\mathbb{R}^*$	Invertible real numbers, $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$
$\mathbf{v}_i$	Vector: lowercase, bold, single index.
$\mathbf{v}_I$	Multi-index notation for basis blades.
	$\mathbf{v}_I := \prod_{\ell \in I} \mathbf{v}_\ell = \mathbf{v}_{I_1} \wedge \cdots \wedge \mathbf{v}_{I_{\vert I \vert}}$.
v_I, v_i	Scalar coefficients in canonical expansions.
$\mathfrak{w}$	product of linearly independent invertible vectors;
	$\mathfrak{w} = \mathbf{w}_1 \cdots \mathbf{w}_k$
$\sharp\mathfrak{w}$	Grade of element $\mathfrak{w}$; i.e., $\mathfrak{w} := \mathbf{w}_1 \cdots \mathbf{w}_{\sharp\mathfrak{u}}$.
$\langle u \rangle_\ell$	Grade-ℓ part of $u \in \mathcal{C}\ell_Q(V)$
π_ℓ	Canonical grade-ℓ projection operator: $\pi_\ell(u) := \langle u \rangle_\ell$.
$v \lrcorner u$, $v \llcorner u$	Geometric left and right contraction, respectively.
$v \wedge u$	Exterior product.
$\mathbf{x} \vert \mathfrak{w}$	$\mathbf{x}$ "divides" $\mathfrak{w}$; i.e., $\mathfrak{w} = \pm \mathbf{x}\mathfrak{v}$ for decomposable $\mathfrak{v}$, invertible $\mathbf{x}$.
$\tilde{u}$	Reversion of $u \in \mathcal{C}\ell_Q(V)$: $\tilde{u} = \sum_{k=0}^{\dim V} (-1)^{\frac{n(n-1)}{2}} \langle u \rangle_k$
$\hat{u}$	Grade involution: $\hat{u} = \sum_{k=0}^{\dim V} (-1)^k \langle u \rangle_k$
$\overline{u}$	Clifford conjugate: $\overline{u} = \sum_{k=0}^{\dim V} (-1)^{\frac{n(n+1)}{2}} \langle u \rangle_k$
$\varphi_{\mathfrak{w}}$	Blade conjugation operator on V: $\mathbf{x} \mapsto \mathfrak{w}\mathbf{x}\widehat{\mathfrak{w}^{-1}}$

Table 1: Summary of Notation

2 Preliminaries

Let V be an n-dimensional vector space over $\mathbb{R}$ equipped with a nondegenerate quadratic form Q. Associate with Q the symmetric bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_Q = \frac{1}{2} \left[Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y}) \right].$$

The *exterior product* on V satisfies the canonical anti commutation relation (CAR) $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$. Geometrically, the exterior product of two vectors represents an oriented parallelogram generated by the two vectors. By associative extension, the exterior product of k linearly independent vectors represents an oriented k-volume. It follows immediately from the CAR that the exterior product of linearly *dependent* vectors is zero.

The *Clifford algebra* $\mathcal{C}\ell_Q(V)$ is the real algebra obtained from associative linear extension of the Clifford vector product

$$\mathbf{x}\,\mathbf{y} := \langle \mathbf{x}, \mathbf{y} \rangle_Q + \mathbf{x} \wedge \mathbf{y}, \ \forall \mathbf{x}, \mathbf{y} \in V. \tag{2.1}$$

Given a nondegenerate quadratic form Q, the mapping $\| \cdot \|_Q : V \to \mathbb{R}$ defined by

$$\|\mathbf{x}\|_Q = |\langle \mathbf{x}, \mathbf{x} \rangle_Q|^{1/2}, \ \ (\mathbf{x} \in V)$$

is readily seen to be a seminorm, referred to henceforth as the Q*-seminorm* on V.

A vector $\mathbf{x}$ is said to be *anisotropic* if $\|\mathbf{x}\|_Q \neq 0$. A set S of Q-orthogonal vectors is said to be Q*-orthonormal* if $\|\mathbf{x}\|_Q = 1$ for all $\mathbf{x} \in S$.

Note that since Q is nondegenerate, all vectors of a Q-orthogonal basis for V must be anisotropic. Given a collection of Q-orthogonal vectors $\{\mathbf{x}_i\}$, a Q-orthonormal basis $\{\mathbf{u}_i : 1 \le i \le n\}$ for V is obtained by defining

$$\mathbf{u}_i := \frac{\mathbf{x}_i}{\|\mathbf{x}_i\|_Q},$$

for each $i = 1, \ldots, n$. In particular, for each $i = 1, \ldots, n$,

$$\mathbf{u}_i{}^2 = \langle \mathbf{u}_i, \mathbf{u}_i \rangle_Q = \frac{\langle \mathbf{x}_i, \mathbf{x}_i \rangle_Q}{|\langle \mathbf{x}_i, \mathbf{x}_i \rangle_Q|} = \pm 1.$$

These vectors then generate the Clifford algebra $\mathcal{C}\ell_Q(V)$.

Generally speaking, the exterior product of k linearly independent vectors is called a k*-blade* or *blade of grade* k. When the vectors are Q-orthogonal, one sees from (2.1) that the Clifford product coincides with the exterior product.

Given an arbitrary Q-orthogonal basis $\{\mathbf{e}_i : 1 \leq i \leq n\}$ for V, multi-index notation for canonical basis blades is adopted in the following manner. Denote the n-set $\{1, \ldots, n\}$ by $[n]$, and denote the associated *power set* by $2^{[n]}$. The ordered product of basis vectors (i.e., algebra generators) is then conveniently denoted by

$$\prod_{i \in I} \mathbf{e}_i = \mathbf{e}_I,$$

for any subset $I \subseteq [n]$, also denoted $I \in 2^{[n]}$.

These products of generators are referred to as basis *blades* for the algebra. The *grade* of a basis blade is defined to be the cardinality of its multi-index. An arbitrary element $u \in \mathcal{C}\ell_Q(V)$ has a canonical basis blade decomposition of the form

$$u = \sum_{I \subseteq [n]} u_I \, \mathbf{e}_I,$$

where $u_I \in \mathbb{R}$ for each multi-index I. The *grade-k part* of $u \in \mathcal{C}\ell_Q(V)$ is then naturally defined by $\langle u \rangle_k := \sum_{|I|=k} u_I \mathbf{e}_I$. It is now evident that $\mathcal{C}\ell_Q(V)$ has a canonical vector space decomposition of the form

$$\mathcal{C}\ell_Q(V) = \bigoplus_{k=0}^{n} \langle \mathcal{C}\ell_Q(V) \rangle_k.$$

Example 2.1. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ denote an orthonormal basis for the two-dimensional Euclidean space $\mathbb{R}^2$. The associated quadratic form is $Q(x, y) = x^2 + y^2$, and a general element of the Clifford algebra $\mathcal{C}\ell_Q(\mathbb{R}^2)$ is of the form

$$a_0 + a_1 \, \mathbf{e}_1 + a_2 \, \mathbf{e}_2 + a_{\{1,2\}} \, \mathbf{e}_{\{1,2\}},$$

where $a_I \in \mathbb{R}$ for each multi index $I \in 2^{[2]}$.

An arbitrary element $u \in \mathcal{C}\ell_Q(V)$ is said to be *homogeneous of grade k* if $\langle u \rangle_k \neq 0$ and $\langle u \rangle_\ell = 0$ for all $\ell \neq k$. As the degree of a polynomial refers to the maximal exponent appearing in terms of the polynomial, an arbitrary multivector $u \in \mathcal{C}\ell_Q(V)$ is said to be *heterogeneous of grade k* if $\langle u \rangle_k \neq 0$ and $\langle u \rangle_\ell = 0$ for $\ell > k$.

It is not difficult to see that $\mathcal{C}\ell_Q(V)$ contains the following two subspaces: $\mathcal{C}\ell_Q(V)^+ = \text{span}\{\mathbf{e}_I : |I| \equiv 0 \pmod 2\}$, called the *even subalgebra* of $\mathcal{C}\ell_Q(V)$, and $\mathcal{C}\ell_Q(V)^- := \text{span}\{\mathbf{e}_I : |I| \equiv 1 \pmod 2\}$, which is a subspace, but not a subalgebra.

The *reversion* on $\mathcal{C}\ell_Q(V)$ is defined on arbitrary blade $\mathfrak{u} = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_{\sharp\mathfrak{u}}$ by

$$\tilde{\mathfrak{u}} := \mathbf{u}_{\sharp u} \wedge \cdots \wedge \mathbf{u}_1 = (-1)^{\sharp\mathfrak{u}(\sharp\mathfrak{u}-1)/2}\mathfrak{u},$$

and is extended linearly to all of $\mathcal{C}\ell_Q(V)$. Similarly, the *grade involution* is defined by linear extension of $\hat{\mathfrak{u}} := (-1)^{\sharp\mathfrak{u}}\mathfrak{u}$, and *Clifford conjugation* is defined as the composition of reversion and grade involution. Specifically, Clifford conjugation acts on an arbitrary blade $\mathfrak{u}$ according to $\overline{\mathfrak{u}} := (-1)^{\sharp\mathfrak{u}(\sharp\mathfrak{u}+1)/2}\mathfrak{u}$.

By utilizing reversion, the inner product $\langle\cdot,\cdot\rangle_Q$ is seen to extend to the full algebra $\mathcal{C}\ell_Q(V)$ by bilinear linear extension of

$$\langle\mathfrak{b}_1, \mathfrak{b}_2\rangle_Q := \langle\mathfrak{b}_1\widetilde{\mathfrak{b}_2}\rangle_0$$

for arbitrary basis blades $\mathfrak{b}_1, \mathfrak{b}_2$.

Given the Clifford product, the *left contraction* operator is now conveniently defined for vector $\mathbf{x}$ and arbitrary multivector $v \in \mathcal{C}\ell_Q(V)$ by linear extension of

$$\mathbf{x}v = \mathbf{x} \lrcorner v + \mathbf{x} \wedge v.$$

A similar definition holds for the *right contraction*, i.e., $u\mathbf{x} := u \llcorner \mathbf{x} + u \wedge \mathbf{x}$. The left and right contraction operators then extend associatively to blades and linearly to arbitrary elements $u, v \in \mathcal{C}\ell_Q(V)$. Moreover, left and right contractions are dual to the exterior product and satisfy the following:

$$\langle u \lrcorner v, w\rangle_Q = \langle v, \tilde{u} \wedge w\rangle_Q\,,$$
$$\langle u \llcorner v, w\rangle_Q = \langle u, w \wedge \tilde{v}\rangle_Q\,.$$

2.1 Motivation: The problem in the Euclidean case

When Q positive definite, $V \cong \mathbb{R}^n$ with the standard (Euclidean) inner product. The associated Clifford algebra is denoted $\mathcal{C}\ell_n$ for simplicity. Suppose $u \in \mathcal{C}\ell_n$ is written in terms of a generating set of orthonormal vectors $\{\mathbf{e}_i : 1 \leq i \leq n\}$ for $\mathbb{R}^n$; i.e., $u = \sum_{I\in 2^{[n]}} u_I\mathbf{e}_I$, where, $[n] = \{1, \ldots, n\}$ denotes the n-set, and $2^{[n]}$ is the corresponding power set.

Let $\mathcal{C}\ell_n{}^*$ denote the multiplicative group of invertible Clifford elements. In particular,

$$\mathcal{C}\ell_n{}^* = \{u \in \mathcal{C}\ell_n : u\tilde{u} \in \mathbb{R}^*\}.$$

The inverse of $u \in \mathcal{C}\ell_n$ is then seen to be $u^{-1} = \dfrac{\tilde{u}}{u\tilde{u}}$.

Definition 2.2. An element $\mathfrak{u} \in \mathcal{C}\ell_n$ is said to be *decomposable* if $\mathfrak{u} = \mathbf{v}_1 \cdots \mathbf{v}_k$ for some linearly independent collection of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ in $\mathcal{C}\ell_n$. Equivalently, $\mathfrak{u}$ is decomposable if and only if it satisfies the following conditions:

1. $\mathfrak{u} \in \mathcal{C}\ell_n^+ \cup \mathcal{C}\ell_n^-$;
2. For all $\mathbf{x} \in V$, $\mathfrak{u}\mathbf{x}\overline{\mathfrak{u}} \in V$.

In fact, the decomposable elements of $\mathcal{C}\ell_n$ are precisely the elements of the Clifford Lipschitz group, Γ_n.

The *pin group* $\mathrm{Pin}(n) = \{u \in \mathcal{C}\ell_n^+ \cup \mathcal{C}\ell_n^- : u\tilde{u} = \pm 1\}$ is a double covering of $O(n)$. The *spin group* $\mathrm{Spin}(n) = \{u \in \mathcal{C}\ell_n^+ \cup \mathcal{C}\ell_n^- : u\tilde{u} = 1\}$ is a double covering of $SO(n)$. One quickly sees that decomposable elements $\mathfrak{u} \in \mathcal{C}\ell_n^+ \cup \mathcal{C}\ell_n^-$ satisfying $\mathfrak{u}\tilde{\mathfrak{u}} = \alpha \neq 0$ provide a double covering of the conformal orthogonal group $\mathrm{CO}(n)$.

For convenience, let $\sharp\mathfrak{u}$ denote the maximum grade among nonzero terms in the canonical basis blade expansion of $\mathfrak{u}$. the additive representation of $\mathfrak{u}$ with respect to any basis $\{\mathbf{e}_i : 1 \leq i \leq n\}$ of V is then of the form $\mathfrak{u} = \sum_{\substack{I \subseteq [n] \\ (|I| - \sharp\mathfrak{u}) \equiv 0 \pmod 2}} u_I \mathbf{e}_I$. When $k = \sharp\mathfrak{u}$, $\mathfrak{u}$ will also be referred to as a *decomposable k-element* of $\mathcal{C}\ell_n$. A problem providing motivation now is to *efficiently* represent such an element, which consists of as many as $\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{n}{k - 2\ell}$ nonzero terms.

As a consequence of the definition of a decomposable element, there exists a constant $\alpha \in \mathbb{R}$ and a linearly independent collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of unit vectors in $\mathbb{R}^n$ such that

$$\alpha \mathbf{w}_1 \cdots \mathbf{w}_k = \mathfrak{u}.$$

In the context of geometric algebra, any element constructed as the product of a number of non-null vectors is commonly referred to as a *versor*. The element $\mathfrak{u}$ described above is correctly regarded as a k-versor.

Given a unit vector $\mathbf{u}$ and an arbitrary vector $\mathbf{x} \in \mathbb{R}^n$, it is well-known and easily verified that computing the geometric product $-\mathbf{uxu}$ yields a vector $\mathbf{x}'$ obtained by reflection of $\mathbf{x}$ through the hyperplane orthogonal to $\mathbf{u}$.

By considering compositions of reflections, one similarly easily verifies that given a second unit vector $\mathbf{v}$, the geometric product $\mathbf{uvxvu}$ gives a vector $\mathbf{x}'$ obtained by rotating $\mathbf{x}$ in the $\mathbf{uv}$-plane by twice the angle measured from $\mathbf{v}$ to $\mathbf{u}$.

When $\mathfrak{u}$ is a product of vectors in $\mathcal{C}\ell_n$, the mapping $\varphi_{\mathfrak{u}} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\varphi_{\mathfrak{u}}(\mathbf{x}) = \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$$

is an orthogonal transformation on $\mathbb{R}^n$. More generally, $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\overline{\mathfrak{u}}$, where $\overline{\mathfrak{u}}$ denotes the Clifford conjugate of $\mathfrak{u}$, is a conformal orthogonal mapping on $\mathbb{R}^n$.

Utilizing these basic facts allows one to develop and implement an efficient algorithm for factoring versors and blades in $\mathcal{C}\ell_n$. The same algorithm works equally well in the negative-definite Clifford algebra $\mathcal{C}\ell_{0,n}$.

2.2 Versor decomposition in definite signatures

When $A \in \mathrm{SO}(n)$ acts as plane rotation in $\mathbb{R}^n$, there exists a two-versor $\mathfrak{b} \in \mathcal{C}\ell_n$ such that

$$A\mathbf{x} = \mathfrak{b}\mathbf{x}\mathfrak{b}^{-1}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Beginning with such a versor, written explicitly in terms of a fixed basis in $\mathcal{C}\ell_n$, one task of interest is to obtain a factorization $\mathfrak{b} = \mathbf{b}_1\mathbf{b}_2$, where $\mathbf{b}_1, \mathbf{b}_2$ are unit vectors of $\mathbb{R}^n$. An intuitive geometric approach to accomplish this is to first apply a "probing vector." The normalized component of this vector lying in the plane of rotation represents one factor, $\mathbf{b}_1$, of the versor. This factor is rotated to its image, $\mathbf{u}$, by the action of the versor. Halfway between the probing vector's projection and the projection's image lies the second factor, $\mathbf{b}_2 = (\mathbf{b}_1 + \mathbf{u})/\|\mathbf{b}_1 + \mathbf{u}\|$, of the versor (see Figure 1). A nice description of the ideas behind this process can be found in the work of Aragón-Gonzales, Aragón, *et al.* [1].

By normalizing $\mathbf{b}_1$ and $\mathbf{u}$, one guarantees that the angle between $\mathbf{b}_1$ and $\mathbf{b}_2$ is $\theta/2$, where θ is the angle measured from $\mathbf{b}_1$ to $\mathbf{u}$. For arbitrary $\mathbf{x} \in \mathbb{R}^n$, it follows that $\mathbf{b}_2\mathbf{b}_1\mathbf{x}\mathbf{b}_1\mathbf{b}_2$ is rotation of $\mathbf{x}$ by angle θ in the $\mathbf{b}_1\mathbf{b}_2$-plane.

A natural extension of this geometric approach allows one to iteratively factor blades and versors in Clifford algebras of definite signature. Consider now a $2k$-versor $\mathfrak{b}$ such that $\varphi_{\mathfrak{b}} : \mathbf{x} \mapsto \mathfrak{b}\mathbf{x}\mathfrak{b}^{-1}$ represents the composition of k plane rotations of $\mathbf{x}$ in $\mathbb{R}^n$. An important assumption is that the linear operator $\varphi_{\mathfrak{b}}$ does *not* have -1 as an eigenvalue [2].

[2]The requirement that -1 is not an eigenvalue of $\varphi_{\mathfrak{b}}$ ensures that $\mathbf{b}_\ell' := (\mathbf{b}_\ell + \mathbf{u})/\|\mathbf{b}_\ell + \mathbf{u}\|$ is well-defined.

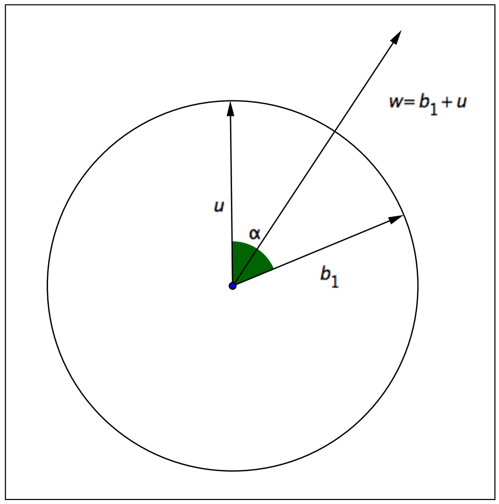

Figure 1: Applying a probing vector to factor a two-versor.

When $\mathfrak{b}$ is a versor of odd grade, one vector can be "factored out" before reverting to the iterated rotor factorization. Moreover, the group action can be generalized from $O(n)$ to $CO(n)$ by considering arbitrary scalar multiples of rotors and versors. An implementation of this approach is seen in Algorithm 1.

Example 2.3. Consider $\mathfrak{b} = 4 + 8\mathbf{e}_{\{1,2\}} + 6\mathbf{e}_{\{1,3\}} - 6\mathbf{e}_{\{2,3\}} \in \mathcal{C}\ell_3$. The action of $\mathbf{x} \mapsto \mathfrak{b}\mathbf{x}\overline{\mathfrak{b}}$ is the composition of a plane rotation and dilation by factor $\mathfrak{b}\tilde{\mathfrak{b}} = 152$ in $\mathbb{R}^3$. Letting $\mathbf{p} = \mathbf{e}_1$ serve as a "probing vector," we compute $\mathbf{p}' = \mathfrak{b}\mathbf{p}\widehat{\mathfrak{b}^{-1}}$ and obtain $\mathbf{p}' = -\frac{6}{19}\mathbf{e}_1 + \frac{1}{19}\mathbf{e}_2 - \frac{18}{19}\mathbf{e}_3$. Letting $\mathbf{b}_1 = (\mathbf{p}-\mathbf{p}')/\|\mathbf{p}-\mathbf{p}'\|$, we obtain the normalized projection $\mathbf{b}_1$ of $\mathbf{p}$ into the plane of rotation. In particular,

$$\mathbf{b}_1 = -\frac{5}{\sqrt{38}}\mathbf{e}_1 + \frac{1}{5\sqrt{38}}\mathbf{e}_2 - \frac{9\sqrt{2}}{5\sqrt{19}}\mathbf{e}_3.$$

Computing $\mathbf{u} = \mathfrak{b}\mathbf{b}_1\widehat{\mathfrak{b}^{-1}}$, we obtain

$$\mathbf{u} = \frac{275\mathbf{e}_1 + 293\mathbf{e}_2 + 426\mathbf{e}_3}{95\sqrt{38}}.$$

Computing the unit vector $\mathbf{b}_2$, which lies halfway between $\mathbf{b}_1$ and its image, we obtain

$$\mathbf{b}_2 = (\mathbf{b}_1 + \mathbf{u})/\|\mathbf{b}_1 + \mathbf{u}\| = -\frac{50}{95}\mathbf{e}_1 + \frac{78}{95}\mathbf{e}_2 + \frac{21}{95}\mathbf{e}_3,$$

The rotation induced by $\mathfrak{b}$ now corresponds to the composition of two reflections across the orthogonal complements of $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively. Note that $\mathbf{b}_2$ is the normalization of $\mathbf{w}$ in Figure 1. The factorization of $\mathfrak{b}$ is then given by

$$\mathfrak{b} = \sqrt{152}\,\mathbf{b}_2\mathbf{b}_1 = 4 + 8\mathbf{e}_{\{1,2\}} + 6\mathbf{e}_{\{1,3\}} - 6\mathbf{e}_{\{2,3\}}.$$

3 Decomposable elements of $\mathrm{CO}_Q(V)$

To maintain generality in the theoretical background, let Q denote a nondegenerate quadratic form, and let V be an n-dimensional real vector space with inner product $\langle , \rangle_Q$ induced by Q. The Clifford algebra of this space is then denoted by $\mathcal{C}\ell_Q(V)$. The conformal orthogonal group $\mathrm{CO}_Q(V)$ is then direct product of dilations and Q-orthogonal linear transformations of V.

The concept of a blade is commonplace in Clifford algebras, where it refers to the Clifford product of a collection of pairwise-orthogonal vectors. In such cases, the exterior product coincides with the Clifford (geometric) product.

For a positive integer k, a *blade of grade* k, or k*-blade*, is a homogeneous multivector $\mathfrak{u}$ of grade k that can be written in the form $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$ for some Q*-orthogonal* collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\} \subset V$.

A nonzero element $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ is said to be *invertible* if $\mathfrak{u}\tilde{\mathfrak{u}}$ is a nonzero scalar. In this case, $\mathfrak{u}^{-1} = \dfrac{\tilde{\mathfrak{u}}}{\mathfrak{u}\tilde{\mathfrak{u}}}$.

Due to complications arising from the use of indefinite quadratic forms, we tighten our definition of decomposable elements for the general case. As a result, the decomposable elements of $\mathcal{C}\ell_Q(V)$ are no longer in one-to-one correspondence with elements of the Clifford Lipschitz group.

Definition 3.1. An invertible element $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$ of grade k is said to be *decomposable* if there exists a linearly independent collection $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ of anisotropic vectors in V such that $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$ and $\langle \mathfrak{u} \rangle_k$ is invertible [3]. In this case, $\mathfrak{u}$ is referred to as a *decomposable* k-element.

As a consequence of this definition, any decomposable element $\mathfrak{u}$ is either even or odd; i.e., $\mathfrak{u} \in \mathcal{C}\ell_Q(V)^+ \cup \mathcal{C}\ell_Q(V)^-$. Further, invertibility is guaranteed by $\mathfrak{u}\tilde{\mathfrak{u}} \in \mathbb{R}^*$. The next definition lends meaning to the notion of whether a vector can be said to "divide" a blade or decomposable element.

[3] Requiring invertibility of the top form makes the *decomposable* elements of $\mathcal{C}\ell_Q(V)$ a proper subset of the Lipschitz group of $\mathcal{C}\ell_Q(V)$.

input : Additive representation of $\mathfrak{u}$, an invertible k-versor, expanded w.r.t. generators $\{\mathbf{e}_i : 1 \leq i \leq n\}$.
output: Vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that $\mathfrak{u} = \alpha \mathbf{b}_k \cdots \mathbf{b}_1$.

```
ℓ ← 1;
u' ← u;
while ♯u' > 1 do
    Choose a random unit vector x ∈ ℝ^n and compute its image under
    the action of u'. ;
    Let x ∈ ℝ^n such that x⌟u = 0 and ‖x‖ = 1;
    x' ← u' x \widehat{u'^{-1}};
    b_ℓ ← (x − u' x \widehat{u'^{-1}}) / ‖x − u' x \widehat{u'^{-1}}‖;
    If u' is of odd grade, factor out a vector (reflection). Otherwise,
    factor out a 2-versor (plane rotation).;
    if ♯u' ≡ 1 (mod 2) then
        u' ← u' b_ℓ;
        ℓ ← ℓ + 1;
    else
        z ← u' b_ℓ u'^{-1};
        if ⟨b_ℓ, z⟩ ≠ −1;
        then
            b_{ℓ+1} ← (b_ℓ + z) / ‖b_ℓ + z‖;
            w ← b_{ℓ+1} b_ℓ;
            ℓ ← ℓ + 2;
        else
            w ← b_ℓ;
            ℓ ← ℓ + 1;
        end
        Compute lower-grade versor;
        u' ← u' w^{-1};
    end
end
return {b_1, ..., b_{ℓ−1}, u'};
```

Algorithm 1: VersorFactor: Factor Versors in Definite Signatures

Definition 3.2. Let $\mathfrak{u}$ be a decomposable element in $\mathcal{C}\ell_Q(V)$. An anisotropic vector $\mathbf{w} \in V$ is said to *divide* $\mathfrak{u}$ if and only if there exists a decomposable element $\mathfrak{u}' \in \mathcal{C}\ell_Q(V)$ of grade $\sharp\mathfrak{u} - 1$ such that $\mathfrak{u} = \pm\mathbf{w}\mathfrak{u}'$. In this case, one writes $\mathbf{w}\big|\mathfrak{u}$.

A basic result inherent to the decomposition algorithms is the following.

Lemma 3.3. *If* $\mathfrak{u}$ *is a decomposable* k*-element, then the grade-*k *part of* $\mathfrak{u}$ *is a* k*-blade and any anisotropic vector* $\mathbf{v}$ *dividing this blade also divides* $\mathfrak{u}$*.*

Proof. If $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$ is a decomposable k-element, then the grade-k part $\langle\mathfrak{u}\rangle_k$ represents an oriented k-volume in V. Any factorization of this blade thereby spans a k-dimensional subspace of V, and by decomposability there exists an anisotropic basis β for this subspace. Any vector $\mathbf{v} \in \beta$ divides the blade $\langle\mathfrak{u}\rangle_k$. Writing $\mathbf{v}$ as a linear combination of the (unknown) vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ then gives

$$\begin{aligned}
\mathbf{v}^{-1}\mathfrak{u} &= \frac{1}{\mathbf{v}^2}(a_1\mathbf{w}_1 + \cdots + a_k\mathbf{w}_k)\mathbf{w}_1 \cdots \mathbf{w}_k \\
&= \frac{1}{\mathbf{v}^2}\sum_{j=1}^{k} a_j \mathbf{w}_1 \cdots \check{\mathbf{w}}_j \cdots \mathbf{w}_k,
\end{aligned}$$

where $\check{\mathbf{w}}_j$ indicates the omission of $\mathbf{w}_j$ from the product. Letting $\mathfrak{u}' = \mathbf{v}^{-1}\mathfrak{u}$, associativity guarantees that $\mathfrak{u} = \mathbf{v}\mathfrak{u}'$ where $\mathfrak{u}'$ is a $(k-1)$-element. Decomposability of $\mathfrak{u}'$ depends on its invertibility; i.e. $\mathfrak{u}'\widetilde{\mathfrak{u}'} \in \mathbb{R}^*$ is required. This is verified by computation:

$$\mathfrak{u}'\widetilde{\mathfrak{u}'} = (\mathbf{v}^{-1}\mathfrak{u})(\widetilde{\mathbf{v}^{-1}\mathfrak{u}}) = \mathbf{v}^{-1}(\mathfrak{u}\tilde{\mathfrak{u}})\mathbf{v}^{-1} = \frac{\mathfrak{u}\tilde{\mathfrak{u}}}{\mathbf{v}^2} \in \mathbb{R}^*.$$

□

The theoretical basis for an essential tool used in the decomposition algorithms is provided by the following proposition.

Theorem 3.4. *Given a decomposable* k*-element* $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k \in \mathcal{C}\ell_Q(V)$*, let* $n = \dim V$ *and define* $\varphi_{\mathfrak{u}} \in \mathrm{O}_Q(V)$ *by*

$$\varphi_{\mathfrak{u}}(\mathbf{v}) = \mathfrak{u}\mathbf{v}\widehat{\mathfrak{u}^{-1}}.$$

Then $\varphi_{\mathfrak{u}}$ *has an eigenspace* $\mathcal{E}$ *of dimension* $n-k$ *with corresponding eigenvalue* 1*.*

Proof. If $\mathbf{v}$ is in the orthogonal complement of $\mathfrak{u}$, one sees immediately that

$$\begin{aligned}
\mathfrak{u}\mathbf{v}\widehat{\mathfrak{u}^{-1}} &= \frac{1}{\mathfrak{u}\tilde{\mathfrak{u}}}(\mathbf{w}_1\cdots\mathbf{w}_k)\mathbf{v}(\widehat{\mathbf{w}_k\cdots\mathbf{w}}_1) \\
&= \frac{(-1)^k}{\mathfrak{u}\tilde{\mathfrak{u}}}(\mathbf{w}_1\cdots\mathbf{w}_k)\mathbf{v}(\mathbf{w}_k\cdots\mathbf{w}_1) \\
&= \mathbf{v}.
\end{aligned}$$

Hence, $\dim\mathcal{E} \geq n-k$.

On the other hand, since $\langle\mathfrak{u}\rangle_k = \mathbf{w}_1\wedge\cdots\wedge\mathbf{w}_k$, which is invertible by our definition of decomposability, there exists an anisotropic orthogonal collection $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ such that $\mathbf{v}_1\cdots\mathbf{v}_k = \langle\mathfrak{u}\rangle_k$. Setting $\mathfrak{w} = \mathbf{v}_1\cdots\mathbf{v}_k$, it follows that

$$\begin{aligned}
\mathfrak{w}\mathbf{v}_k\widehat{\mathfrak{w}^{-1}} &= \frac{1}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_k)\mathbf{v}_k(\widehat{\mathbf{v}_k\cdots\mathbf{v}}_1) \\
&= \frac{(-1)^k}{\mathfrak{w}\tilde{\mathfrak{w}}}{\mathbf{v}_k}^2(\mathbf{v}_1\cdots\mathbf{v}_k)(\mathbf{v}_{k-1}\cdots\mathbf{v}_1) \\
&= \frac{(-1)^k}{\mathfrak{w}\tilde{\mathfrak{w}}}{\mathbf{v}_k}^2(-1)^{\frac{k(k-1)}{2}+\frac{(k-1)(k-2)}{2}}(\mathbf{v}_k\cdots\mathbf{v}_1)(\mathbf{v}_1\cdots\mathbf{v}_{k-1}) \\
&= \frac{(-1)^k}{\mathfrak{w}\tilde{\mathfrak{w}}}{\mathbf{v}_k}^2(-1)^{(k-1)}(\mathbf{v}_k\cdots\mathbf{v}_1)(\mathbf{v}_1\cdots\mathbf{v}_{k-1}) \\
&= -\frac{{\mathbf{v}_k}^2}{\mathfrak{w}\tilde{\mathfrak{w}}}\mathbf{v}_k(\mathbf{v}_{k-1}\cdots\mathbf{v}_1)(\mathbf{v}_1\cdots\mathbf{v}_{k-1}) \\
&= -\mathbf{v}_k.
\end{aligned}$$

The corresponding result is similarly obtained for $\mathbf{v}_1$. For $1 < j < k$, one can consider

$$
\begin{aligned}
\mathfrak{w}\mathbf{v}_j\widehat{\mathfrak{w}^{-1}} &= \frac{1}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_k)\mathbf{v}_j(\widehat{\mathbf{v}_k\cdots\mathbf{v}_1}) \\
&= \frac{(-1)^k}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_{j-1}\mathbf{v}_j\cdots\mathbf{v}_k)\mathbf{v}_j(\mathbf{v}_k\cdots\mathbf{v}_j\mathbf{v}_{j-1}\cdots\mathbf{v}_1) \\
&= \frac{(-1)^k}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_{j-1})(\mathbf{v}_k\cdots\mathbf{v}_j)\mathbf{v}_j(\mathbf{v}_j\cdots\mathbf{v}_k)(\mathbf{v}_{j-1}\cdots\mathbf{v}_1) \\
&= \frac{(-1)^k{\mathbf{v}_j}^2}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_{j-1})(\mathbf{v}_k\cdots\mathbf{v}_{j+1}\mathbf{v}_j)(\mathbf{v}_{j+1}\cdots\mathbf{v}_k)(\mathbf{v}_{j-1}\cdots\mathbf{v}_1) \\
&= \frac{(-1)^{k+(k-j)^2}{\mathbf{v}_j}^2}{\mathfrak{w}\tilde{\mathfrak{w}}}(\mathbf{v}_1\cdots\mathbf{v}_{j-1})(\mathbf{v}_j\cdots\mathbf{v}_k)(\mathbf{v}_k\cdots\mathbf{v}_{j+1})(\mathbf{v}_{j-1}\cdots\mathbf{v}_1) \\
&= \left(\frac{(-1)^{k+(k-j)^2}}{\mathfrak{w}\tilde{\mathfrak{w}}}\prod_{\ell=j}^{k}{\mathbf{v}_\ell}^2\right)(\mathbf{v}_1\cdots\mathbf{v}_{j-1})\mathbf{v}_j(\mathbf{v}_{j-1}\cdots\mathbf{v}_1) \\
&= \left(\frac{(-1)^{k+(k-j)^2+(j-1)^2}}{\mathfrak{w}\tilde{\mathfrak{w}}}\prod_{\ell=j}^{k}{\mathbf{v}_\ell}^2\right)\mathbf{v}_j(\mathbf{v}_{j-1}\cdots\mathbf{v}_1)(\mathbf{v}_1\cdots\mathbf{v}_{j-1}) \\
&= (-1)^{k(k+1)+2(j^2-kj-j)+1}\mathbf{v}_j \\
&= -\mathbf{v}_j.
\end{aligned}
$$

It follows that $\text{span}\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$ is an eigenspace of the transformation $\mathbf{x}\mapsto\mathfrak{w}\mathbf{x}\widehat{\mathfrak{w}^{-1}}$ corresponding to eigenvalue -1. Letting $\mathbf{v}\in\text{span}\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$, it is not difficult to see that writing $\mathfrak{u}=\mathfrak{w}+\mathfrak{u}'$ implies

$$\mathfrak{u}\mathbf{v}\widehat{\mathfrak{u}^{-1}} = \frac{1}{\mathfrak{u}\tilde{\mathfrak{u}}}\left(\mathfrak{w}\mathbf{v}\overline{\mathfrak{w}} + \mathfrak{u}'\mathbf{v}\overline{\mathfrak{w}} + \mathfrak{w}\mathbf{v}\overline{\mathfrak{u}'} + \mathfrak{u}'\mathbf{v}\overline{\mathfrak{u}'}\right).$$

Observe that $\mathbf{v}\overline{\mathfrak{w}}$ and $\mathfrak{w}\mathbf{v}$ are blades of grade $k-1$ orthogonal to $\mathbf{v}$, while the highest grade terms of $\mathfrak{u}'$ are of grade $k-2$. Consequently, the "cross terms" contribute no components parallel to $\mathbf{v}$. In other words, a little algebra shows that $\mathfrak{u}\mathbf{v}\widehat{\mathfrak{u}^{-1}}=\mathbf{v}$ implies

$$\mathfrak{w}\mathbf{v}\overline{\mathfrak{w}} + \mathfrak{u}'\mathbf{v}\overline{\mathfrak{u}'} = (\mathfrak{u}\tilde{\mathfrak{u}})\mathbf{v}. \tag{3.1}$$

Note that writing $\mathfrak{u}=\mathfrak{w}+\mathfrak{u}'$ gives

$$\mathfrak{u}\tilde{\mathfrak{u}} = \mathfrak{w}\tilde{\mathfrak{w}} + \mathfrak{u}'\widetilde{\mathfrak{u}'}.$$

Further, if $\mathbf{v}$ divides $\mathfrak{w}$, one sees that $\mathfrak{u}'\mathbf{v}\overline{\mathfrak{u}'}=\lambda\mathbf{v}$ implies $\lambda=\mathfrak{u}'\widetilde{\mathfrak{u}'}$. Finally, a little algebra applied to (3.1) yields

$$
\begin{aligned}
\mathfrak{u}'\mathbf{v}\overline{\mathfrak{u}'} &= (\mathfrak{w}\tilde{\mathfrak{w}} + \mathfrak{u}\tilde{\mathfrak{u}})\mathbf{v} \\
&= (2\mathfrak{w}\tilde{\mathfrak{w}} + \mathfrak{u}'\widetilde{\mathfrak{u}'})\mathbf{v}.
\end{aligned}
$$

This implies $\mathfrak{u}'\widetilde{\mathfrak{u}'} = 2\mathfrak{w}\tilde{\mathfrak{w}} + \mathfrak{u}'\widetilde{\mathfrak{u}'}$. Since $\mathfrak{w}$ is anisotropic, this is a contradiction. It follows that $\mathbf{v} \in V_\mathfrak{u}$ implies $\varphi_\mathfrak{u}(\mathbf{v}) \neq \mathbf{v}$, so that $\dim \mathcal{E} \leq n - k$. □

Corollary 3.5. *Let $\mathbf{x} \in V$ be arbitrary. Then $\mathbf{x} - \varphi_\mathfrak{u}(\mathbf{x}) \in V_\mathfrak{u}$. In other words, the operator $\pi_\mathfrak{u} := \mathbb{I} - \varphi_\mathfrak{u}$ is a projection into the subspace determined by $\mathfrak{u}$.*

Proof. Write $V = V_\mathfrak{u} \oplus V_\mathfrak{u}'$, where $V_\mathfrak{u}'$ is the orthogonal complement of $V_\mathfrak{u}$ in V. Then, letting $\mathbf{x} = \mathbf{w} + \mathbf{w}' \in V$ be arbitrary,

$$\begin{aligned}
\pi_\mathfrak{u}(\mathbf{x}) &= \frac{1}{2}\left(\varphi_\mathfrak{u}(\mathbf{x}) - \mathbf{x}\right) \\
&= \frac{1}{2}\left(\mathfrak{u}(\mathbf{w} + \mathbf{w}')\widehat{\mathfrak{u}^{-1}} - (\mathbf{w} + \mathbf{w}')\right) \\
&= \frac{1}{2}\left(\mathfrak{u}\mathbf{w}\widehat{\mathfrak{u}^{-1}} - \mathbf{w} + \mathbf{w}' - \mathbf{w}'\right) \\
&= \frac{1}{2}\left(\mathfrak{u}\mathbf{w}\widehat{\mathfrak{u}^{-1}} - \mathbf{w}\right) \in V_\mathfrak{u}.
\end{aligned}$$

□

Given $\mathfrak{u} = \mathbf{w}_1 \cdots \mathbf{w}_k$, it will be convenient to refer to $V_\mathfrak{u} = \mathrm{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ as the $\mathfrak{u}$-*subspace* of V. As seen in Theorem 3.4, when the orthogonal complement of the $\mathfrak{u}$-subspace is nontrivial, any unit vector of $V_{\mathfrak{u}^\star}$ is an eigenvector of $\varphi_\mathfrak{u}$ having eigenvalue 1. This observation allows one to define a $\mathfrak{u}$-subspace projection by

$$\pi_\mathfrak{u}(\mathbf{x}) := \frac{1}{2}\left(\mathbf{x} - \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}\right).$$

It is clear that the null space of $\pi_\mathfrak{u}$ is $V_{\mathfrak{u}^\star}$, so that the range is $\mathfrak{u}$.

Now that all tools are in hand, it is possible to formalize a decomposition algorithm for decomposable elements of $\mathcal{C}\ell_Q(V)$. Algorithm 2 makes use of the projection operator defined in Corollary 3.5 to obtain component vectors of decomposable elements. When the algorithm is applied to a blade, the result is an orthogonal collection of vectors whose product is the blade.

3.1 Blade Factorization

Algorithm 3 provides an efficient method for blade decomposition. Unlike the approach of Algorithm 2, it makes use of a *single term* of the canonical expansion to obtain each vector of the decomposition, as opposed to computing the full blade conjugation. That is, subspace projections are computed using a single basis blade from the expansion in place of the expansion itself. As a result, FastBladeFactor requires less time (see Figure 4) and also

input : $\mathfrak{b}$, a decomposable k-element.
output: $\{\mathbf{b}_k, \ldots, \mathbf{b}_1\}$ such that $\mathfrak{b} = \mathbf{b}_k \cdots \mathbf{b}_1$.
;

$\ell \leftarrow 1$;
$\mathfrak{u} \leftarrow \mathfrak{b}/\|\mathfrak{b}\|$;

while $\sharp\mathfrak{u} > 1$ **do**

  Choose random anisotropic vector $\mathbf{x} \in V$ *such that* $\mathbf{x} \lrcorner \mathfrak{u} \neq 0$ *and compute its image under the action of* $\varphi_{\mathfrak{u}}$. ;

  Let $\mathbf{x} \in V$ such that $\mathbf{x} \lrcorner \mathfrak{u} \neq 0$ and $\mathbf{x}^2 \neq 0$;
  $\mathbf{x}' \leftarrow \mathfrak{u}\mathbf{x}\widehat{\mathfrak{u}^{-1}}$;

  if $(\mathbf{x} - \mathbf{x}')^2 \neq 0$ **then**
    $\mathbf{b}_{\sharp\mathfrak{u}} \leftarrow (\mathbf{x} - \mathbf{x}')/\|\mathbf{x} - \mathbf{x}'\|$;

    if $\mathfrak{u}\,{\mathbf{b}_{\sharp\mathfrak{u}}}^{-1}$ *is decomposable* **then**
      $\mathfrak{u} \leftarrow \mathfrak{u}\,{\mathbf{b}_{\sharp\mathfrak{u}}}^{-1}$;
    end
  end
end
return $\{\mathbf{b}_k, \ldots, \mathbf{b}_2, \|\mathfrak{b}\|\mathfrak{u}\}$;

Algorithm 2: CliffordDecomp

yields decompositions of null (i.e., noninvertible) blades. Unlike Algorithm 2, FastBladeFactor does not return a Q-orthogonal decomposition but simply a collection whose exterior product is the blade.

The combinatorial approach developed here effectively computes geometric contractions using differences of multi indices. Further, because multi indices are well ordered by $\mathbf{f}_I \preceq \mathbf{f}_J \Leftrightarrow \sum_{i\in I} 2^{i-1} \leq \sum_{j\in J} 2^{j-1}$, the following function is well defined:

$$\text{FirstTerm}\left(\sum_I \alpha_I \mathbf{f}_I\right) := \min_{\{\mathbf{f}_X : \alpha_X \neq 0\}} \alpha_X \mathbf{f}_X.$$

The FirstTerm procedure thereby provides the means for choosing the term that drives the blade's decomposition.

Remark 3.6. Like Algorithm 3, Fontijne's algorithm [4] also utilizes a single term from the blade's canonical expansion to compute the blade's constituent vectors. The initial term selected in Fontijne's algorithm is a term whose scalar coefficient is equal in magnitude to the ∞-norm of the blade. This term is then used in conjunction with geometric contractions to extract the blade's constituent vectors.

input : Blade $\mathfrak{b} \in \mathcal{C}\ell_Q(V)$ of grade k expressed as a sum $\sum_I \alpha_I \mathbf{f}_I$.
output: Scalar α and set of vectors $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ such that $\mathfrak{b} = \alpha \mathbf{b}_k \wedge \cdots \wedge \mathbf{b}_1$.

$\alpha_M \mathbf{f}_{M=\{m_1,\ldots,m_k\}} \leftarrow$FirstTerm($\mathfrak{b}$);
for $\ell \leftarrow 1$ **to** k **do**
  $\mathfrak{u} \leftarrow \mathbf{f}_{M\setminus\{m_\ell\}}$;
  $\mathbf{b}_\ell \leftarrow \langle \mathfrak{b}\mathfrak{u}^{-1} \rangle_1$;
end
return $\{\alpha_M, \mathbf{b}_1, \ldots, \mathbf{b}_k\}$;

Algorithm 3: FastBladeFactor

4 Decomposition examples with *Mathematica*

All results appearing here were obtained using *Mathematica* 10 with the **CliffMath** package running on a MacBook Pro with 2.4 GHz Intel Core i7 processor and 8 GB of 1333 MHz DDR3 memory. The interested (*Mathematica* adept) reader can find code and examples online by clicking the "Research" link at `http://www.siue.edu/~sstaple`.

Example 4.1. To compare the algorithms involving general element decomposition, blade decomposition, and fast blade factoring, consider the randomly-generated grade-5 element of $\mathcal{C}\ell_8$ seen in Figure 2. First, the element is decomposed using Algorithm 2.

The grade-5 part of the element is a 5-blade. Applying Algorithm 2 to this blade results in the factorization seen in Figure 3. Applying Algorithm 3 to the blade results in the non-orthogonal factors seen in Figure 3 in 1/18 of the time. All decompositions were subsequently verified to reproduce the original elements.

Example 4.2. In Figure 4, runtimes are compared for decomposition of grade-4 blades and general elements in $\mathcal{C}\ell_6$ and $\mathcal{C}\ell_7$. In each case, five hundred elements of grade 4 were randomly-generated.

```
6784 e{1} + 6678 e{2} + 10 984 e{3} + 7576 e{4} - 6205 e{5} - 102 e{6} - 5149 e{7} + 1202 e{8} - 676 e{1,2,3} - 1752 e{1,2,4} +
  3229 e{1,2,5} - 2116 e{1,2,6} - 1015 e{1,2,7} + 2368 e{1,2,8} - 7568 e{1,3,4} + 6862 e{1,3,5} + 1036 e{1,3,6} +
  9822 e{1,3,7} - 4980 e{1,3,8} - 2208 e{1,4,5} - 4776 e{1,4,6} - 5872 e{1,4,7} + 1608 e{1,4,8} + 3779 e{1,5,6} +
  5302 e{1,5,7} - 4159 e{1,5,8} - 11 535 e{1,6,7} + 11 564 e{1,6,8} + 5533 e{1,7,8} - 5368 e{2,3,4} + 2145 e{2,3,5} +
  4456 e{2,3,6} + 11 825 e{2,3,7} - 8856 e{2,3,8} - 4177 e{2,4,5} - 2312 e{2,4,6} - 3317 e{2,4,7} - 1372 e{2,4,8} +
  1736 e{2,5,6} + 1840 e{2,5,7} - 1356 e{2,5,8} - 9764 e{2,6,7} + 11 044 e{2,6,8} + 7064 e{2,7,8} - 4316 e{3,4,5} -
  8776 e{3,4,6} - 14 732 e{3,4,7} + 6824 e{3,4,8} + 6963 e{3,5,6} + 12 360 e{3,5,7} - 10 073 e{3,5,8} - 19 315 e{3,6,7} +
  18 832 e{3,6,8} + 6919 e{3,7,8} - 115 e{4,5,6} + 2226 e{4,5,7} - 3565 e{4,5,8} - 9345 e{4,6,7} + 12 092 e{4,6,8} +
  6359 e{4,7,8} + 7762 e{5,6,7} - 9970 e{5,6,8} - 7278 e{5,7,8} + 6650 e{6,7,8} - 1610 e{1,2,3,4,5} + 3104 e{1,2,3,4,6} +
  4254 e{1,2,3,4,7} - 4088 e{1,2,3,4,8} - 3010 e{1,2,3,5,6} - 6230 e{1,2,3,5,7} + 5180 e{1,2,3,5,8} + 4058 e{1,2,3,6,7} -
  2344 e{1,2,3,6,8} + 2132 e{1,2,3,7,8} + 1986 e{1,2,4,5,6} + 1756 e{1,2,4,5,7} - 462 e{1,2,4,5,8} + 1862 e{1,2,4,6,7} -
  4152 e{1,2,4,6,8} - 3238 e{1,2,4,7,8} - 4402 e{1,2,5,6,7} + 5526 e{1,2,5,6,8} + 3862 e{1,2,5,7,8} - 4022 e{1,2,6,7,8} +
  278 e{1,3,4,5,6} - 3172 e{1,3,4,5,7} + 4634 e{1,3,4,5,8} + 6850 e{1,3,4,6,7} - 9640 e{1,3,4,6,8} - 4190 e{1,3,4,7,8} -
  7006 e{1,3,5,6,7} + 9558 e{1,3,5,6,8} + 7726 e{1,3,5,7,8} - 7430 e{1,3,6,7,8} + 4216 e{1,4,5,6,7} - 5796 e{1,4,5,6,8} -
  4144 e{1,4,5,7,8} + 3380 e{1,4,6,7,8} + 1116 e{1,5,6,7,8} - 3440 e{2,3,4,5,6} - 7810 e{2,3,4,5,7} + 7070 e{2,3,4,5,8} +
  5968 e{2,3,4,6,7} - 4896 e{2,3,4,6,8} + 1150 e{2,3,4,7,8} - 1290 e{2,3,5,6,7} + 2150 e{2,3,5,6,8} + 2230 e{2,3,5,7,8} -
  1894 e{2,3,6,7,8} + 5882 e{2,4,5,6,7} - 7734 e{2,4,5,6,8} - 5470 e{2,4,5,7,8} + 5046 e{2,4,6,7,8} + 776 e{2,5,6,7,8} +
  8126 e{3,4,5,6,7} - 11 122 e{3,4,5,6,8} - 8550 e{3,4,5,7,8} + 7730 e{3,4,6,7,8} + 908 e{3,5,6,7,8} + 748 e{4,5,6,7,8}
```

```
In[20]:= Timing[Bf = CliffordDecomp[B]]
Out[20]= {8.438983,
  {0.0630904 e{1} + 0.264971 e{2} - 0.390084 e{3} + 0.147402 e{4} - 0.188183 e{5} + 0.748969 e{6} + 0.136306 e{7} +
    0.370096 e{8}, 0.171923 e{1} - 0.380025 e{2} - 0.579994 e{3} + 0.143699 e{4} + 0.554634 e{5} - 0.320797 e{6} +
    0.193767 e{7} - 0.144584 e{8}, 0.423971 e{1} + 0.232783 e{2} + 0.30033 e{3} + 0.431616 e{4} -
    0.282986 e{5} + 0.31425 e{6} + 0.348015 e{7} - 0.435456 e{8}, 0.653833 e{1} + 0.148705 e{2} +
    0.184215 e{3} + 0.519516 e{4} + 0.304638 e{5} - 0.363147 e{6} - 0.137426 e{7} + 0.0546965 e{8},
   31 906.1 e{1} + 10 450.9 e{2} - 7684.13 e{3} - 3218.17 e{4} + 42 554.1 e{5} + 3429.73 e{6} + 42 580. e{7} - 14 537.7 e{8}}}
```

Figure 2: Decomposable grade-5 element in $\mathcal{C}\ell_8$ and its decomposition.

```
In[24]:= Timing[Blf = CliffordDecomp[Bl]]
Out[24]= {2.055691, {0.436433 e{1} - 0.0787571 e{2} - 0.161208 e{3} +
    0.830435 e{4} - 0.225138 e{5} - 0.000738118 e{6} - 0.170457 e{7} - 0.0892718 e{8},
   0.180493 e{1} + 0.321771 e{2} + 0.856856 e{3} - 0.219418 e{5} - 0.192729 e{6} - 0.113192 e{7} - 0.177712 e{8},
   0.486322 e{1} + 0.110234 e{2} + 0.752037 e{5} - 0.391758 e{6} + 0.154873 e{7} + 0.0912149 e{8},
   0.246575 e{1} + 0.546527 e{2} + 0.621213 e{6} + 0.119359 e{7} + 0.490262 e{8},
   -5505.05 e{1} - 11 762.3 e{6} - 26 704.6 e{7} + 24 174.3 e{8}}}
```

```
In[26]:= Timing[Blff = FastBladeFactor[Bl]]
Out[26]= {0.107119, {e{1} + (344 e{6})/161 + (781 e{7})/161 - (101 e{8})/23, -e{2} - (139 e{6})/805 + (1586 e{7})/805 - (331 e{8})/115,
   e{3} - (993 e{6})/805 - (878 e{7})/805 + (33 e{8})/115, -e{4} + (43 e{6})/23 + (89 e{7})/23 - (74 e{8})/23, e{5} - (1552 e{6})/805 - (2127 e{7})/805 + (292 e{8})/115, -1610}}
```

Figure 3: Decomposition and fast decomposition of the 5-blade (grade-5 part) in Figure 2.

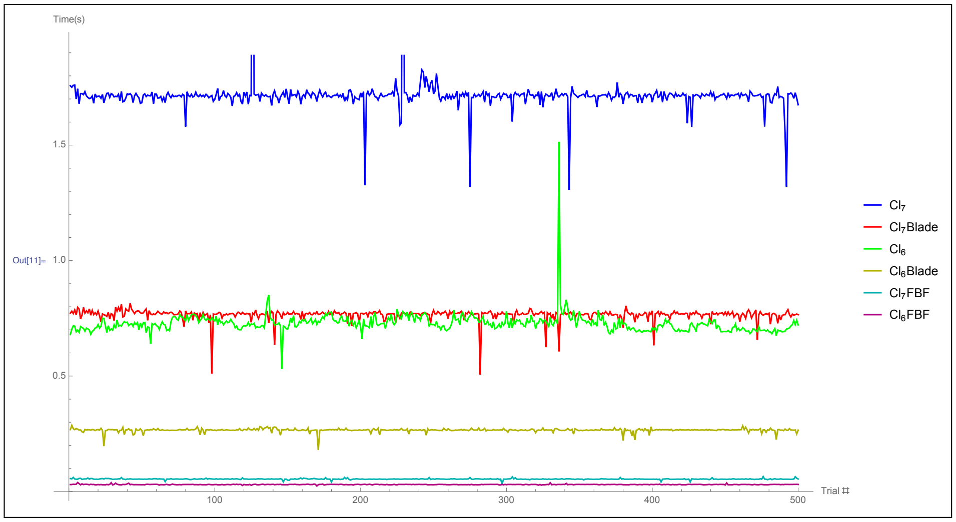

Figure 4: Computation time required for decomposition of grade-4 elements and 4-blades in $\mathcal{C}\ell_6$ and $\mathcal{C}\ell_7$.

5 Complexity of representations and avenues for further research

Consider the geometric product of a set of k linearly independent anisotropic vectors in the Clifford algebra $\mathcal{C}\ell_Q(V)$, where $k \leq n = \dim V$. Fixing an orthonormal basis $\{\mathbf{e}_i\}$ of V as generators of the algebra, one immediately notices that the additive representation of the product may contain up to $\sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{n}{k-2\ell}$ nonzero terms. As seen in Figure 4, blade factorization is significantly more efficient than general decomposition of Clifford elements. Fast blade factorization is faster still.

This leads to the following question: If one wishes to represent decomposable elements as a sum of (multiplicative representations of) blades, how many blades must be factored? The next preliminary result leads to an upper bound.

Lemma 5.1. *Let $\mathfrak{b}$ be an invertible k-blade. Let $\mathbf{w}$ be an anisotropic vector, $\mathbf{w} = \mathbf{w}_{\mathfrak{b}'} + \mathbf{w}_{\mathfrak{b}}$ where $\mathbf{w}_{\mathfrak{b}'}$ is orthogonal to $\mathfrak{b}$ and $\mathbf{w}_{\mathfrak{b}}$ divides $\mathfrak{b}$. Then $\mathfrak{b}\mathbf{w}$ can be written as a sum of a $(k+1)$-blade, $\mathfrak{b} \wedge \mathbf{w}_{\mathfrak{b}'}$, and a $(k-1)$-blade, $\mathfrak{b} \llcorner \mathbf{w}_{\mathfrak{b}}$.*

Proof. Let $\mathfrak{b} = \mathbf{v}_1 \cdots \mathbf{v}_k$ where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are orthogonal invertible vectors and let $\mathbf{w}$ be an anisotropic vector of the form $\mathbf{w} = \mathbf{w}_{\mathfrak{b}'} + \mathbf{w}_{\mathfrak{b}}$ where $\mathbf{w}_{\mathfrak{b}'}$ is orthogonal to $\mathfrak{b}$ and $\mathbf{w}_{\mathfrak{b}}$ divides $\mathfrak{b}$. Then

$$\begin{aligned}\mathfrak{b}\mathbf{w} &= \mathfrak{b}(\mathbf{w}_{\mathfrak{b}'} + \mathbf{w}_{\mathfrak{b}}) \\ &= (\mathbf{v}_1 \cdots \mathbf{v}_k)\mathbf{w}_{\mathfrak{b}'} + (\mathbf{v}_1 \cdots \mathbf{v}_k)\mathbf{w}_{\mathfrak{b}}.\end{aligned}$$

Since $\mathbf{w}_{\mathfrak{b}'}$ is orthogonal to $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$,

$$(\mathbf{v}_1 \cdots \mathbf{v}_k)\mathbf{w}_{\mathfrak{b}'} = (\mathbf{v}_1 \cdots \mathbf{v}_k) \wedge \mathbf{w}_{\mathfrak{b}'}$$

which is a $k+1$-blade. Now, since $\mathbf{w}_{\mathfrak{b}}$ divides $\mathfrak{b}$, $\mathbf{w}_{\mathfrak{b}}$ exists in the space spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$. Performing Gram-Schmidt orthonormalization on $\mathbf{v}_1, \ldots, \mathbf{v}_k$ with $\mathbf{w}_1 = \mathbf{w}_{\mathfrak{b}}$ gives

$$\mathbf{v}_1 \cdots \mathbf{v}_k = \alpha \mathbf{w}_{\mathfrak{b}} \mathbf{w}_2 \cdots \mathbf{w}_k$$

for some $\alpha \in \mathbb{R}$, so that

$$\begin{aligned}(\mathbf{v}_1 \cdots \mathbf{v}_k)\mathbf{w}_{\mathfrak{b}} &= (\mathbf{v}_1 \cdots \mathbf{v}_k) \llcorner \mathbf{w}_{\mathfrak{b}} \\ &= (\alpha \mathbf{w}_{\mathfrak{b}} \mathbf{w}_2 \cdots \mathbf{w}_k) \llcorner \mathbf{w}_{\mathfrak{b}} \\ &= \alpha(-1)^{k-1}(\mathbf{w}_{\mathfrak{b}} \llcorner \mathbf{w}_{\mathfrak{b}})\mathbf{w}_2\mathbf{w}_3 \cdots \mathbf{w}_k \\ &= \alpha(-1)^{k-1}\|\mathbf{w}_{\mathfrak{b}}\|^2 \mathbf{w}_2\mathbf{w}_3 \cdots \mathbf{w}_k\end{aligned}$$

which is a $(k-1)$-blade. □

Observe that the two blades obtained in Lemma 5.1 are not necessarily invertible. It is true nonetheless that the product of a blade yields *at most* two invertible blades. Iterated application of Lemma 5.1 thereby leads immediately to an upper bound on the number of blades required to express the grade j part of a decomposable element.

Lemma 5.2. *If $\mathfrak{v} \in \mathcal{C}\ell_Q(V)$ is a decomposable k-element for $k \leq \dim V$, then $c_{k,j}$, as defined below, gives an upper bound on the number of blades required to express $\langle \mathfrak{v} \rangle_j$ as a sum of blades. This upper bound satisfies the following recurrence:*

$$c_{k,j} = \begin{cases} \frac{(-1)^{k-j}+1}{2} & \text{if } j = 0 \text{ or } 1 \\ 1 & \text{if } j = k \\ c_{k-1,j-1} + c_{k-1,j+1} & \text{if } 1 < j < k \\ 0 & \text{if } j > k \end{cases}$$

Proof. Values of $c_{k,j}$ for $1 \leq k \leq 10$ are shown in Table 2. For readability, entries with value zero have been left blank. The first line of $c_{k,j}$ and the first two columns of the table follow from the summing of scalars and the summing of vectors. The grade k part of a decomposable k-element is a k-blade, giving the second line of $c_{k,j}$ which corresponds to the diagonal line of 1's in the table where $j = k$.

As shown previously, the product of a j-blade and a vector yields, *at most*, a $(j-1)$-blade and a $(j+1)$-blade. Distributing shows that the product of a vector and the sum of c j-blades yields at most c $(j-1)$-blades and c $(j+1)$-blades. So the grade j part of any k-element comes from the product of a vector with the grade $(j-1)$ and $(j+1)$ parts of a $(k-1)$-element. This gives the third line of $c_{k,j}$ and the pattern seen in the table in which each entry below the line $j = k$ is the sum of the two entries above it diagonally. □

$k \backslash j$	0	1	2	3	4	5	6	7	8	9	10	T_k
1		1										1
2	1		1									2
3		1		1								2
4	1		2		1							4
5		1		3		1						5
6	1		4		4		1					10
7		1		8		5		1				15
8	1		9		13		6		1			30
9		1		22		19		7		1		50
10	1		23		41		26		8		1	100

Table 2: Values of $c_{k,j}$

An upper bound on the number of blades required to represent a product of k vectors is $T_k = \sum_{i=0}^{k} c_{k,i}$, shown on the table in the column at the far right. Values of T_k $(k \leq \dim V)$ satisfy the following recurrence:

$$T_k = \begin{cases} 2T_{k-1} & \text{if } k \text{ is even} \\ 2T_{k-1} - (c_{k-1,2} + 1) & \text{if } k \text{ is odd.} \end{cases}$$

Letting $n = \dim V$, an upper bound on the number of scalars required to describe these blades is $c_{k,0} + n \sum_{i=1}^{k} i c_{k,i}$. This can be compared to the maximum number of terms in the expanded form of a product of k vectors, $\sum_{\substack{0 \le i \le k \\ i \equiv k \pmod 2}} \binom{n}{i}$.

In light of these considerations, an open problem is how to develop a strategy for writing a homogeneous element as a sum of blades. More to the point, one desires a method for recognizing a *minimal* collection of blades that make up a general homogeneous element.

A very nice special case occurs when a homogeneous element $w \in \mathcal{C}\ell_Q(V)$ can be written as a sum of pairwise-orthogonal blades [4]. In this case, the fast blade factor algorithm can be used to "intelligently" pick apart the element, one blade at a time. Each pass of the algorithm factors one blade of the sum, resulting in a sort of "division algorithm" for elements of this type. Algorithm 4 makes this idea more formal.

input : $w \in \mathcal{C}\ell_Q(V)$, a sum of m pairwise-orthogonal blades.
output: Sets of the form $F_i = \{\alpha_i, \mathbf{v}_{ij} : 1 \le i \le m, 1 \le j \le k_i\}$ such that $w = \sum_{i=1}^{m} \alpha_i(\mathbf{v}_{i1} \wedge \cdots \wedge \mathbf{v}_{ik_i})$.

```
while w ≠ 0 do
    {α_i, v_i1, ..., v_ik_i} ← FastBladeFactor[w];
    f ← α_i v_i1 ∧ ··· ∧ v_ik_i;
    w ← w − f;
end
return {{α_i, v_i1, ..., v_ik_i} : 1 ≤ i ≤ m}
```

Algorithm 4: OrthoSumDecomp

A few more avenues of exploration are listed below.

- Factorization of permutations. Permutation matrices are 0-1 symmetric matrices and hence represent orthogonal linear transformations. Multiplicative Clifford algebra representations exist naturally. The study of particular classes of permutations via Clifford algebraic methods might be interesting.
- Graph theory. Adjacency matrices of undirected graphs are 0-1 symmetric matrices. Under suitable conditions, a graph's adjacency matrix can be decomposed into a sum of orthogonal matrices. The relation-

[4]Two blades $\mathfrak{u}$, $\mathfrak{v}$ are considered to be *orthogonal* if their associated subspaces have trivial intersection.

ship between such graphs and their Clifford algebra representations could be an interesting area of study.

- Multiplicative representations of Kravchuk transforms. Kravchuk polynomials and Kravchuk matrices are naturally related to Clifford algebras [9]. Symbolic computations can be implemented using Clifford factorizations of Kravchuk matrices.

References

[1] G. Aragón-González, J. L. Aragón, M. A. Rodríguez-Andrade, L. Verde-Star, Reflections, Rotations, and Pythagorean Numbers, *Advances in Applied Clifford Algebras*, **19** (2008), 1–14.

[2] C. Chevalley, *The Construction and Study of Certain Important Algebras*, Math. Soc. of Japan, Tokyo, 1955.

[3] L. Dorst, D. Fontijne, *Efficient Algorithms for Factorization and Join of Blades*, Geometric Algebra Computing, E. Bayro-Corrochano, G. Scheuermann, Eds., Springer, London, 2010, pp. 457–476.

[4] D. Fontijne, *Efficient Implementation of Geometric Algebra*, Ph.D. thesis, University of Amsterdam, 2007.

[5] J. Helmstetter, Factorization of Lipschitzian elements, *Advances in Applied Clifford Algebras*, **24** (2014), 675–712.

[6] P. Lounesto, E. Latvamaa, Conformal Transformations and Clifford Algebras, *Proc. Am. Math. Soc.*, **79** (1980), 533–538.

[7] C. Perwass, *Geometric Algebra with Applications in Engineering*, Springer-Verlag, Berlin, 2009.

[8] G.S. Staples, **CliffMath2014** for *Mathematica*. Available for download at `http://www.siue.edu/~sstaple/index_files/research.htm`.

[9] G.S. Staples, Kravchuk polynomials and induced/reduced operators on Clifford algebras, *Complex Analysis and Operator Theory*, **9** (2015), 445–478.
`http://dx.doi.org/10.1007/s11785-014-0377-z`

[10] D. Wylie, Factoring blades and versors in Euclidean Clifford algebras, *M.S. Thesis*, Southern Illinois University Edwardsville, 2014.