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World Income Components: Measuring and Exploiting International Risk Sharing Opportunities*

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Abstract

We provide a method for decomposing the variance of world national incomes (present values) into components in such a way as to indicate the most important risk-sharing opportunities among nations of the world. We identify risk-sharing opportunities in terms of eigenvectors of a variance matrix of deviations of the present value of country incomes from their respective shares (adjusted for population and risk aversion) of world income.

The method is applied to data on national incomes of six large countries 1870–1992 (Maddison [1995]): Canada, France, Germany, Italy, United Kingdom and United States. The method reveals that, assuming symmetric risk aversions, the most important risk sharing contract to devise for these countries would be essentially a national income swap between the United States, and together on the other side, France, Germany and Italy, i.e., approximately a US–Europe national income swap. A contract that is essentially a France–Germany swap is the second most important risk-sharing contract.

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World Income Components: Measuring and Exploiting International Risk Sharing Opportunities

In this paper, we propose major international risk-sharing arrangements that are, according to a welfare criterion for the countries we study, the most important risk-sharing arrangements that could be made for these countries. To do this, we develop a method for characterizing the risk structure of the present value of world incomes (wealth). It is useful to discover what the most important risk-sharing arrangements actually are, to provide guidance for future efforts to manage risks.

The importance of, and feasibility of, devising international risk-sharing arrangements in terms of large income aggregates was argued in Shiller [1993b]. Our method of identifying risk sharing arrangements (based on Athanasoulis [1995] and Shiller and Athanasoulis [1995]) is related to principal components analysis. Demange and Laroque [1995] have, independently of us, derived a similar theory but did not apply it to data. We apply the theory to the present value of national incomes (strictly speaking, present values of gross domestic products, GDPs) of the nations of the world. Our method takes account of the relative size and variability of different countries' incomes, as well as the tendency of certain national incomes to move together, to suggest the most important opportunities for risk sharing. Our method differs from standard principal components analysis applied to national incomes in that, among other things, ours relates to a variance decomposition of risk-sharing opportunities, not of national incomes (present values) themselves. Although the variance matrix that underlies the analysis cannot be precisely estimated, we think that enough is known to provide some rough suggestions of the kinds of risk sharing arrangements that are most important, very basic suggestions that might not have seemed natural or obvious without our method. We arrive at our proposals for risk-sharing arrangements using data on annual real *per capita* GDPs for six countries 1870–1992, measured in 1990 Geary–Khamis dollars; see Maddison [1995].

A product of our analysis is a set of world income components, or indices (that is, linear combinations) of national incomes, designed to be used as the basis for settlement of risk management contracts. Although the contracts that exploit the risk sharing opportunities we discover might take many forms, we will refer to the contracts as income component securities, tradable claims on the indices created by contracts that define cash payments based on published index values. The securities would exist with zero net value, for every long there is a short, and the dividends, defined in terms of the published indices, would be paid by the shorts to the longs. We imagine that these contracts would be traded on securities markets just as other securities are traded today. The contracts might also be called income component futures contracts, again cash settled based on published index values, and be traded at futures exchanges. Alternatively, the exploitation of the risk sharing opportunities could take the form of agreements among governments that are settled in terms of the published indices, or even of governments' creating institutions to allow individuals to sell claims on their future income, claims which could be bundled together and securitized to create marketable assets resembling the securities defined here. We will not refer further to such possibilities here; we will persist in this paper in giving our contracts an interpretation as marketable securities cash settled using the published indices.

Some of the securities could be described as insurance policies for certain groups of countries; calling a security an insurance policy is most appropriate when the variation in the index is highly negatively correlated with the income of one country, and the people in that country buy the security to reduce their income risk. Some of the securities could also be described as swaps of certain groups of national incomes for other groups; calling one of our securities a swap is most appropriate when the index gives negative weights to roughly half of the national incomes. We shall see below that even when the optimal securities that we derive can be thought of roughly as insurance policies, they are always swaps since a swapping of risks is always involved with an optimally defined contract. Our analysis does not begin with any preconceived notions whether we want to create insurance policies or swaps, or any other instrument: our analysis goes directly for the most advantageous risk-sharing arrangement.

Our study of risk-sharing opportunities among national incomes (present values) is potentially very important, since national incomes are measures of total economic welfare of the countries, and since there have been historically large variations in real national incomes. Moreover, there is very little effective risk diversification across nations today (see, for example, Obstfeld [1993], Tesar and Werner [1992]). It is obvious that national governments do not make significant risk sharing arrangements with each other; even within the European Union, Sala-i-Martin and Sachs [1992] estimate, a one dollar adverse shock to the national income of one country creates, all things considered, less than a 0.005 dollars reduction of that country's tax payment to the European Community. Our empirical results show that there is substantial risk-sharing that can be exploited within the European Union.

Because there do not now exist any markets for national incomes or for any large income aggregates, and because there is very little income risk sharing among governments, when we set up any such contracts we must consider how they would work pretty much in isolation. National incomes consist primarily of labor incomes, which are not now represented by any securities markets. Existing markets for stocks, bonds, and real estate are markets for claims on the rents of factors of production other than labor or are residual claims, and there is no reason to expect dividends in these markets to correlate well with labor income. There is instead some evidence that they do not correlate well, see Shiller [1993b] and Bottazzi, Pesenti and van Wincoop [1996].

In attempting to define a small number of income component risk sharing arrangements that will allow maximal risk sharing given the number of contracts, we are essentially seeking to define the best first world risk market to set up, as well as the best second and/or third markets. We assume that the number of markets introduced must be kept small, especially at the beginning. By analogy, there are not many stock index futures markets in the world, indeed, from a world perspective, not many aggregate liquid risk management markets at all.

Another reason for confining our attention to only one or a few contracts is that it is useful for us to be able to prescribe in simple terms the most important risk management actions that should be taken by large groups of people. Simple prescriptions are what most people take from existing models. The capital asset pricing model (CAPM) in finance, to which our method is related, is most often used by practitioners not to arrive at complicated definitions of optimal portfolios, but just for the simple prescription that investors should hold the market portfolio of investable assets, and we now have many indexed funds that are designed to allow them to do just this. The problem with this commonly-given prescription is that it is not really the logical consequence of the foundations of the CAPM, since it disregards the correlation of investment returns with innovations in other income, other income which is much larger in the aggregate than income from existing investable assets. We seek here to devise a method to replace this simple prescription associated with the CAPM with a more sensible simple prescription, though any such prescription cannot be taken until the new contracts are created.

We derive (Section 3 below), for securities whose dividends are linear combinations of national incomes, an expression for the prices of the securities in general equilibrium. The linear combinations defining the securities are then chosen so as to maximize social welfare (Section 4); this then defines our income component securities. It turns out that the optimal securities are defined in terms of eigenvectors of a variance matrix of deviations of national incomes (present values) from their respective contract-year shares (weighted in terms of population and risk aversions) of world income. Having made a specification of utility functions, we are able to derive estimates of the welfare increase in dollars generated by the creation of the new contracts.

In Section 5 below, we show some illustrative examples of the theory, and in Section 6 we discuss how to apply our method of defining the income component securities to the data. In Section 7 we present results for six countries and in Section 8 we conduct some bootstrap experiments to gauge the robustness of our results. A discussion follows in Section 9 to interpret these results as suggesting genuine opportunities for important new contracts. We have been unable to find in the literature or popular media any proposals for markets resembling ours.

1 Assumed Utility Function for Individuals[1]

We will assume that each individual in country c maximizes:

$$U_c = E_0 u_c(\tilde{W}_c) \tag{1}$$

where E_0 is the expectation operator conditional on information at time 0 and $u_c(\cdot)$ is the utility function for each individual in country c, continuous everywhere on the real line, $u_c' > 0$ and $u_c'' < 0$, and where $\tilde{W}_c$ is the individual's after-hedging wealth, unknown at time 0. As we will apply this theory to the data below, $\tilde{W}_c$ will be the present value of after-hedging income accruing to a person in country c who is alive at time 0; $\tilde{W}_c$ is not the market value at time 0 of future income, but the present value of actual future income.

[1] A table of symbols and a table of basic relations appear near the end of this paper.

2 Definition of Contracts and Risk Structure

All contracts are made at time 0. To buy one security, the long must transfer value, a claim on a fixed stream of payments whose value is P_n (the price) at time 0, to the short in the contract. In exchange for this, the long receives a claim on an uncertain "dividend" $\tilde{D}_n$ (in our applications, actually the present value of future dividends) from the short in the contract, where $\tilde{D}_n$ is defined in terms of individual incomes in such a way that $E_0\tilde{D}_n$ equals 0. Thus, the price P_n is the known fixed payment for the contract and the dividend $\tilde{D}_n$ is the unknown (random) benefit from being long in the contract.

Let $\tilde{\mathbf{W}}_p$ be the C-element row vector whose cth element is $\tilde{W}_{cp}$, the wealth before hedging. We assume that a C-element column vector $\mathbf{A}_n$ is defined in contract n at time 0, and the contract specifies that $\tilde{D}_n = (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}_n$. The dividend $\tilde{D}_n$ is our nth world income component, a linear combination of demeaned present values of pre-hedged world incomes, and our objective below will be to define this dividend optimally by choosing the best vector $\mathbf{A}_n$.

In our intended application for this theory, it must be recognized that the payments in fulfillment of the claims specified in the contract will take place gradually over time. It will be supposed that P_n is the present value of a stream of future fixed payments each year, $t = 1, 2, ...$ specified in the contract, and $\tilde{\mathbf{D}}_n$ will be the present value of dividend payments in each subsequent year $t = 1, 2, ...$ where the payments are defined by the formula in the contract. We may specify that dividends are paid each year in accordance with that year's income, using the same $\mathbf{A}_n$ vector, and this specification will be consistent with assumptions above about the relation between $\tilde{\mathbf{D}}_n$ and $\tilde{\mathbf{W}}_p$. We define the real time-t dividend $\tilde{D}_{tn} = (\tilde{\mathbf{y}}_t - E_0\tilde{\mathbf{y}}_t)\mathbf{A}_n$ to the long in the contract, where $\tilde{\mathbf{y}}_t$ is the C-element row vector whose cth element is the real income at time t of an individual in country c before hedging. The dividend $\tilde{D}_{tn}$ is not known at time 0 since time-t national incomes are not yet known.

We adopt the convention that $\mathbf{A}_n$ is defined so that the price P_n is nonnegative. If P_n were negative we would multiply $\mathbf{A}_n$ by minus one. This is just a convention defining who is called long and who short. The dividend $\tilde{D}_n$ differs from the usual dividend on a security in that it can be either positive or negative. There is no free disposal in this contract, so there is no problem with negative dividends.

In writing contracts in terms of demeaned incomes (present values), incomes minus their expected values at time 0, we are assuming here that public expected values of future real incomes are objective public knowledge at time 0, so that contracts can be written in terms of these expectations. In practice some proxy for the expectations would have to be used by contract designers. Our use of expectations in the contract definition and omission of a constant term from the formula defining $\tilde{D}_n$ is essentially only a normalization rule for price, so that P_n is the expected payment, conditional on information at time 0 and $\tilde{D}_n$ is the unexpected payment. If contract designers misrepresent public expectations when they design the contract, then the result will only be a change in the market-clearing contract price; the contract price P_n will then not have the precise interpretation we give it.

Let $\tilde{\mathbf{D}}$ denote the N-element row vector whose nth element is $\tilde{D}_n$. Then, $\tilde{\mathbf{D}} = (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}$ where $\mathbf{A}$ is the $C \times N$ matrix whose nth column is $\mathbf{A}_n$. Let $\mathbf{P}$ denote the N-element column vector whose nth element is P_n.

3 Solving the Individual's Problem and Defining Equilibrium

In this section we derive, for exogenously given $\mathbf{A}$, the equilibrium prices and quantities of the securities demanded by all individuals under the assumption that incomes are jointly normally distributed. Let us define $\mathbf{q}_c$ as the N-element column vector whose nth element is the number of securities n purchased by a single person in country c. Our objective, then, is to define a simple aggregate demand function for securities, $\mathbf{q}_c$ as a function of $\mathbf{P}$, and then summing these demands over all agents will allow us to obtain equilibrium prices, since net supply is zero.

Wealth $\tilde{W}_c$ of an individual in country c, after hedging, is:

$$\tilde{W}_c = \tilde{W}_{cp} - P'q_c + (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}\mathbf{q}_c. \tag{2}$$

Substituting this expression into expression (1), and differentiating with respect to $\mathbf{q}_c$, we derive the first order condition:

$$\mathbf{q}_c = E_0[u_c'(\tilde{W}_{cp} - \mathbf{P}'q_c + (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}\mathbf{q}_c)(-\mathbf{P}' + (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}] = 0. \tag{3}$$

We assume that incomes (present values) are normally distributed, and using Stein's lemma,[2] we can rewrite this first order condition in the form:

$$-E_0 u_c' \mathbf{P}' + E_0 u_c'' [\mathbf{\Omega}_c' \mathbf{A} + \mathbf{q}_c' \mathbf{A}' \mathbf{\Omega} \mathbf{A}] = 0 \tag{4}$$

where $\mathbf{\Omega}$ is the $C \times C$ variance matrix of individual incomes (present values) by country, $\mathbf{\Omega} = E_0((\tilde{\mathbf{W}}_p - E_0 \tilde{\mathbf{W}}_p)'(\tilde{\mathbf{W}}_p - E_0 \tilde{\mathbf{W}}_p))$, and $\mathbf{\Omega}_c$ is its cth column. The normalization we choose for $\mathbf{A}$ is $\mathbf{A}'\mathbf{\Omega}\mathbf{A} = I$. Substituting this in the above expression we derive an expression for $\mathbf{q}_c$:

$$\mathbf{q}_c = -\mathbf{P}/\gamma_c - \mathbf{A}'\mathbf{\Omega}_c \tag{5}$$

where $\gamma_c = -E_0 u_c'' / E_0 u_c'$ is the coefficient of absolute risk aversion. In general, γ_c will depend on prices $\mathbf{P}$ as well as the matrix $\mathbf{A}$, although in the special case where the utility function is of the constant absolute risk aversion form, γ_c will be a constant. Defining the $C \times C$ diagonal matrix Γ whose cth diagonal element is γ_c and the C-element column vector ι whose elements all equal one, we can write the matrix $\mathbf{q}$ whose cth column is $\mathbf{q}_c$ as:

$$\mathbf{q} = -\mathbf{P}\iota'\Gamma^{-1} - \mathbf{A}'\mathbf{\Omega} \tag{6}$$

Note that if γ_c is a constant for all c (utilities are constant absolute risk aversion) then demand for contract n is not affected by prices of other contracts; this property of demand is a consequence of the fact that dividends on the securities are constructed to be uncorrelated with each other, and of the mean-variance utility assumption. This demand schedule implies that individuals in country c will purchase more of the security the lower the price and the lower the country's covariance with the security. They will hold a positive quantity of a security at a positive price only if the covariance is sufficiently negative so that the security is providing enough risk reduction to warrant paying the price. Note also that a person in a country whose own income is riskless will hold negative quantities of all securities, that is, be a seller of securities, (since we are normalizing all securities to have nonnegative prices). This means that in terms of these contracts the individuals in this country are strictly in the insurance business of accepting risk in return for an insurance premium.

Market clearing requires that $\mathbf{qPOP}\iota = 0$, where $\mathbf{POP}$ is a $C \times C$ diagonal matrix with the populations of the C countries on the diagonal. It

[2] Stein's lemma states that if x and y are jointly normally distributed and $E(y) = 0$ and $f(\cdot)$ is a continuous function then $E(f(x)y) = E(fN(x))\text{cov}(x, y)$.

follows immediately that market clearing prices are:

$$\mathbf{P} = -\mathbf{A}'\boldsymbol{\Omega}\mathbf{POP}\iota\left(\iota'\Gamma^{-1}\mathbf{POP}\iota\right)^{-1}. \tag{7}$$

If one substitutes (7) into (6), one discovers that:

$$\mathbf{q} = -\mathbf{A}'\boldsymbol{\Omega}\mathbf{M} \tag{8}$$

where $\mathbf{M}$ is defined as $I - \iota(\iota'\Gamma^{-1}\mathbf{POP}\iota)^{-1}\iota'\Gamma^{-1}$. The $C \times C$ matrix $\mathbf{M}$ is idempotent, of rank $C-1$. M is the matrix such that $\tilde{\mathbf{W}}_p\mathbf{M}$ is the C-element row vector whose cth element is the difference between an individual in country c's income (present value) and that person's share in world income (present value), where shares are allocated according to risk aversions. If everyone's coefficient of risk aversion is the same, then the cth element of $\tilde{\mathbf{W}}_p\mathbf{M}$ is the difference of country $c = s$ *per capita* income and world *per capita* income. Thus, in general equilibrium, a country will hold positive quantities of a security only if the deviations of its income from its share of world income covary negatively (as measured by $\boldsymbol{\Omega}$) with the dividends on the security. Prices depend only on covariances with world income, not variances, even though markets are incomplete, a result which corresponds to a property of the CAPM that has been noted by many people and was observed first by Mayers [1972]. Oh [1996] showed that when new markets are created in mean variance economies, prices of existing assets are unchanged, as we observe here.

4 Contract Designer's Problem

Let us now turn to the contract designer's problem, which is to define a small number, N, of optimal securities. We assume that the contract designer wishes to choose $\mathbf{A}$ to maximize, following Nash [1953], the sum of the log of the expected utilities over all individuals, where the individual's utilities are maximized given $\mathbf{A}$ and equilibrium prices $\mathbf{P}$. For generality, we allow this to be a weighted sum, where the log utility for an individual in country c is multiplied by a weight, w_c, before summing so that the contract designer maximizes:

$$S = \sum_{c=1}^{C} w_c POP_c \ln(U_c). \tag{9}$$

Note that, unless we bias our weights in favor of certain countries, this social welfare function is independent of national boundaries. For example, social welfare would be unchanged if we lumped together two identical countries that had the same weight. Our use of nations in our analysis arises essentially because nationalities are indicators of incomes, and because we have data on nations. We could apply this method to less aggregated groups of people.

Our method of defining optimal securities is to substitute equation (2) for $\tilde{W}_c$ into utility, equation (1), and then into (9). One then maximizes social welfare with respect to A subject to the constraint that $\mathbf{A}'\boldsymbol{\Omega}\mathbf{A} = I$. We set up the Lagrangian that represents the constraint that diagonal elements of $A'\Omega A$ equal 1. The Lagrangian is,

$$L = \sum_{c=1}^{C} w_c POP_c \ln E_0 u_c(\tilde{W}_{cp} - \mathbf{P}'\mathbf{q}_c + \tilde{\mathbf{D}}\mathbf{q}_c) - \sum_{n=1}^{N} \sum_{m=1}^{n} (\mathbf{A}'_m \boldsymbol{\Omega} \mathbf{A}_n - e(m,n))\lambda_{mn} \tag{10}$$

where $e(m,n)$ is a function that equals zero unless $m = n$, where it is one, and where the Langrange multipliers λ_{mn} correspond to the constraints implicit in $\mathbf{A}'\boldsymbol{\Omega}\mathbf{A} = I$. First order conditions for a maximum are:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{A}_n} &= \sum_{c=1}^{C} w_c POP_c E_0 \left(u'_c \frac{\partial(\tilde{W}_c - \mathbf{P}'\mathbf{q}_c + \tilde{\mathbf{D}}\mathbf{q}_c)/\partial \mathbf{A}_n}{E_0 u_c} \right) \\ &- \boldsymbol{\Omega}\mathbf{A}_n \lambda_{mn} - \sum_{n=1}^{N} \sum_{m=1}^{n} \boldsymbol{\Omega}\mathbf{A}_n \lambda_{mn} = 0, \; n = 1, ..., N. \end{aligned} \tag{11}$$

$$\frac{\partial L}{\partial \lambda_{nn}} = n\mathbf{A}'_n \boldsymbol{\Omega}\mathbf{A} - 1 = 0, \; n = 1, ..., N. \tag{12}$$

$$\frac{\partial L}{\partial \lambda_{mn}} = m\mathbf{A}'_n \boldsymbol{\Omega}\mathbf{A} = 0, \; m \neq n. \tag{13}$$

where $\mathbf{P}$ depends on $\mathbf{A}_n$ via equation (7), $\mathbf{q}_c$ depends on $\mathbf{A}_n$ via equation (8), and $\tilde{\mathbf{D}}$ depends on $\mathbf{A}_n$ via its definition $\tilde{\mathbf{D}} = (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}$. For most utility functions $u_c(\cdot)$ and assumptions about the stochastic properties of income (present value) before hedging $\tilde{\mathbf{W}}_p$, the solution to these equations will have to be solved numerically.

The above equations are much simpler, however, if we assume a constant absolute risk aversion utility function along with our normality assumption for the distribution of pre-hedged income (present value) $\tilde{\mathbf{W}}_p$. In this case, we need consider explicitly only the diagonal constraints in $\mathbf{A}'\boldsymbol{\Omega}\mathbf{A} = I$. In matrix form, the first order conditions reduce to:

$$\mathbf{M}\boldsymbol{\Gamma}^2\mathbf{wPOPM}'\boldsymbol{\Omega}\mathbf{A} = \mathbf{A}\Lambda \tag{14}$$

where Λ is a diagonal matrix whose nth diagonal element is λ_{nn} (see Appendix A for derivation). Thus, the desired matrix $\mathbf{A}$ is determined as an eigenmatrix of the expression that premultiplies $\mathbf{A}$. To show that the off-diagonal elements of $\mathbf{A}'\mathbf{\Omega}\mathbf{A}$ are then zero, first factor $\mathbf{\Omega}$ (which is positive definite by assumption) into $\mathbf{S}'\mathbf{S}$, and premultiply (14) by $\mathbf{S}$. It follows that $\mathbf{SA}$ is an eigenmatrix of a positive definite symmetric matrix, and so $\mathbf{A}'\mathbf{S}'\mathbf{S}\mathbf{A} = \mathbf{A}'\mathbf{\Omega}\mathbf{A}$ is diagonal.

Since $\mathbf{q} = -\mathbf{A}'\mathbf{\Omega}\mathbf{M}$, we can rewrite (14) in terms of $\mathbf{q}$:

$$\mathbf{M}'\mathbf{\Omega}\mathbf{M}\mathbf{\Gamma}^2\mathbf{w}\mathbf{P}\mathbf{O}\mathbf{P}\mathbf{q} = \mathbf{q}\Lambda. \tag{15}$$

In the special case where $\mathbf{w} = \mathbf{\Gamma}^2\mathbf{P}\mathbf{O}\mathbf{P}^{-1}$, then $\mathbf{q}$ is just the matrix of the N eigenvectors of $\mathbf{M}'\mathbf{\Omega}\mathbf{M}$ corresponding to the highest eigenvalues. $\mathbf{M}'\mathbf{\Omega}\mathbf{M}$ is the variance matrix of deviations of individual incomes (present values by country) minus their share of world incomes (present values), and if all countries have the same populations and risk aversions, then $\mathbf{M}'\mathbf{\Omega}\mathbf{M}$ is just the variance matrix of individual incomes (present values by country) minus average individual income (present values) in the world.

We now show that all of our optimal contracts will be essentially swaps. Postmultiply (14) by Λ^{-1}, one finds that $\mathbf{A}$ equals $\mathbf{M}$ times a product of other matrices. Hence, since $\mathbf{M}$ is idempotent, $\mathbf{MA} = \mathbf{A}$. It follows, since $\iota'\Gamma^{-1}\mathbf{M} = 0$, that $\iota'\Gamma^{-1}\mathbf{A} = 0$. Thus, a sum of the elements in each column of $\mathbf{A}$ weighted by inverses of coefficients of absolute risk aversions equals zero. Since all elements of $\mathbf{\Gamma}$ are nonnegative, we cannot have a situation in which all elements of a column of $\mathbf{A}$ are positive.

We can now produce measures that place a dollar value on the availability of these income component securities. The dollar present discounted value of these markets, F_c, is the solution to

$$E_0 u_c(\tilde{W}_c) = E_0 u_c(\tilde{W}_{cp} + F_c) \tag{16}$$

where F_c is a constant. If utility is of the CARA form, then

$$F_c = \tfrac{1}{2}\gamma_c \mathbf{q}_c' \mathbf{q}_c. \tag{17}$$

See Appendix B for the derivation.

5 Some Illustrative Examples

To clarify what we have done let us consider some very simple examples of our theory. The first example, example A, illustrates how the population of a country affects the optimal contracts when all else is the same across countries. The second example, example B, illustrates the effect of unequal risk on contract design. The third example, example C, is used to illustrate how our theory creates groupings of countries of the world, how it decides which countries should be grouped with each other as having positive weights in the **A** matrix and which should be grouped together as having negative weights.

In our example A, we show the importance of the population of the countries in contract design by using a two country case, case 1, and a three country case, case 2. In these cases we assume all countries have unit coefficient of risk aversion, unit variance and the covariances across countries are zero. All individuals have the same expected *per capita* wealth, the only difference is the population in each country. For case 1 let population in country 2 be ten times larger than population in country 1. Despite the unequal total national wealth in the two countries, the optimal contract is an equal swap of demeaned *per capita* wealth. Each agent in the world would like to trade off some of her wealth for some of the wealth agents in the other country have. Since we normalize the price of the contract to be positive, it must be that the contract covaries negatively with the world and thus country 1 weights positively in the contract while country 2 weights negatively in the contract. This must be the case since both countries have equal but opposite weights in the contract and country 2 is larger. Agents in country 1 will sell the contract, and receive a payment, while agents in country 2 will buy the contract. The agents in country 1 benefit enormously by both reducing their risk and receiving a premium for doing so. The agents in country 2 have much less risk reduction and also pay a premium to achieve the reduction.

In case 2, we add in a third country, country 3. Let her population equal the population of country 2. Then the first best contract is an equal swap of demeaned *per capita* wealth between country 2 and country 3 with country 1 having a weight of virtually zero. The price of this contract is virtually zero and both countries, 2 and 3, benefit equally by this symmetric swap of risk. The small country hardly benefits at all. As the number of countries in the exercise are increased, the less populous countries tend to have less benefit in the first few contracts. The only way less populous countries can benefit in the first few contracts, is if they happen to correlate well with some more populous country.

For example B, we suppose that there are only two countries and that γ_1 and $\gamma_2 = 1$ and the first country has a variance of 1, the second a variance of 3 and the covariances are zero. The populations of the two countries are the same. The optimal contract will be a one-for-one swap of each country's demeaned *per capita* wealth. That the swap is one-for-one may seem strange given the unequal risk of *per capita* wealth across the two countries but one must remember that the contract must be constructed so that each agent in the world can take a position in the contract and hold a share of the world. Thus there must be a one-for-one swap of risk. Given the equal populations across the two countries, since country two is riskier, agents in country 2 will pay for this contract while agents in country 1 receive a payment. In the end all agents hold a share of the world.

In our example C, we move to a four-country case with nonsingular $\mathbf{\Omega}$ and we assume that the world divides naturally, in terms of correlations, into two blocks: within each block the countries are highly correlated with each other, but there is no correlation between blocks. Let us assume that there are four countries with an $\mathbf{\Omega}$ matrix given by expression (18):

$$\mathbf{\Omega} = \begin{bmatrix} 1.0 & 0.9 & 0 & 0 \\ 0.9 & 1.0 & 0 & 0 \\ 0 & 0 & 1.0 & 0.9 \\ 0 & 0 & 0.9 & 1.0 \end{bmatrix}. \tag{18}$$

Supposing that the populations and contract-year incomes of all four countries are the same, and that we give all four countries the same weight, we take $\mathbf{w}$, $\mathbf{\Gamma}$ and $\mathbf{POP}$ all to be identity matrices. Then $\mathbf{M}'\mathbf{\Omega}\mathbf{M}\mathbf{w}\Gamma\mathbf{\Gamma}^2\mathbf{POP}$ equals $\mathbf{M}'\mathbf{\Omega}\mathbf{M}$ which has the form given by expression (19),

$$\mathbf{M}'\mathbf{\Omega}\mathbf{M} = \begin{bmatrix} .525 & .425 & -.475 & -.475 \\ .425 & .525 & -.475 & -.475 \\ -.475 & -.475 & .525 & .425 \\ -.475 & -.475 & .425 & .525 \end{bmatrix}, \tag{19}$$

$$\mathbf{A} = .36 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}. \tag{20}$$

This matrix has one eigenvalue equal to 1.9 and two eigenvalues both equal to 0.1. The vector $\mathbf{A}$, derived as shown above using the eigenvector corresponding to the largest eigenvalue is given by expression (20).

$$\mathbf{q} = .69 \begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix} \tag{21}$$

Thus, except for scaling, the component may be described as just a short position in the first two countries and an equal and opposite long position in the other two. This contract is, as we might expect, a swap between the two blocks of countries. This component is quite different from the first principal component of $\mathbf{\Omega}$. That matrix has two first principal components, both with the same eigenvalue. These components are proportional to the vectors $[1\ 1\ 0\ 0]'$ and $[0\ 0\ 1\ 1]'$; if we created a contract in either of these, then we would not provide any means for the two groups of countries to swap their risks. The vector q is given by expression (21). The first two countries are short the component, the second two are long the component. Note also that in this case, where there is symmetry between the two blocks of countries, the price P is zero.

$$\mathbf{I}+\mathbf{Aq}=\begin{bmatrix} .75 & -.25 & .25 & .25 \\ -.25 & .75 & .25 & .25 \\ .25 & .25 & .75 & -.25 \\ .25 & .25 & -.25 & .75 \end{bmatrix}. \tag{22}$$

It is instructive to look in example C at the $C \times C$ matrix $\mathbf{I}+\mathbf{Aq}$, expression (22), whose cth column gives the after-hedging exposure of an individual in country c to risks faced by individuals in each of the four countries. Not all elements of this matrix equal .25, as would be the case if we had included all three possible contracts and thereby spanned the world risk sharing opportunities, resulting in each country holding one quarter of the world. Since we have only one contract for trading income, it is not possible for each country to hold the world income portfolio, but the holdings shown in expression (22) do nearly as well for risk reduction, given the covariance matrix $\mathbf{\Omega}$ that was assumed. For example, for country 1 the holding of .75 times its own income minus .25 times country two's income is almost as good as the holding of .25 times its own income and .25 times country two's income, given the high correlation between the two.

If we were in example C to create the next two contracts, the contracts whose creation would have the next-highest contribution to social welfare, then each of these contracts would entail a swap between the pairs of countries within each block; again the contract price will be zero. The risk reduction afforded by such swaps is much smaller because the countries are so highly correlated within each pair.

6 Data Analysis

Our data set 1870–1992 covers a period with two world wars and two major depressions (the depressions of the 1890s and 1930s).[3] These events might be viewed as regime changes, not easily amenable to econometric analysis. One approach might be to delete these events from the sample used for data analysis, but then one would be eliminating the events that caused some of the biggest income changes. The objective of risk management is to hedge the big events that make major changes in income levels, and although such big events are inherently infrequent and have "special" causes that are not likely to be repeated exactly, we do not want to delete these from our analysis. The comovements that country incomes show even in reaction to these "special causes" may well be indicators of the tendencies for incomes to move together in response to other "special causes" in the future, since the comovements indicate relatedness and similarity of the countries. We therefore use the entire sample for our analysis.

In our empirical work we assume that each contract signer individually earns his or her share of *per capita* national income in subsequent years from sources other than the contracts we define here. We define the present value of *per capita* income (wealth) $\tilde{W}_{cp}$ as:

$$\tilde{W}_{cp} = \sum_{t=1}^{\infty} \frac{\tilde{y}_{tc}}{(1+\rho)^t}. \tag{23}$$

The unforecastable variability of $\tilde{W}_{cp}$ at time 0 represents the total income uncertainty as of time 0 in country c for all who are around at time 0 to sign contracts. We measure this variability by the conditional variance matrix at time 0 of national income accruing to current populations at time 0.

Estimating the variance matrix $\mathbf{\Omega}$ is not a trivial matter; it is supposed to reflect the *conditional* variance matrix at the time the contract is signed for the present value of future national incomes. To estimate such a variance matrix, we need first to form some representation of the conditional expected value each year for future national income.

[3] Maddison [1995] used the Geary–Khamis approach for producing real GDPs so that international comparisons can be made. It relies on the concept of purchasing power parity of currencies and international average prices of commodities. It allows for multilateral rather than binary comparison. The International Comparison Project for OECd countries, such as those in our sample, provide high quality data for Purchasing Power Parity computations.

There are many models that might be used to provide estimates of $\mathbf{\Omega}$. Estimating time series models, such as autoregressive models, for the national income of each country would help us to separate out which components of national incomes are forecastable and which are not. Estimating spatial models, such as the spatial autoregressive models or other Markov random field models, would allow us to put structure on the matrix $\mathbf{\Omega}$ so that fewer parameters would be estimated allowing for more accurate estimates. Spatial models could use measures of economic distance between countries, measures depending on a number of variables (see our earlier paper [1995]).

For this paper we use a very simple methods to estimate, a method that is intended to be transparent and simple. Our estimate $\hat{\mathbf{\Omega}}$ of the c_1c_2th element of $\mathbf{\Omega}$ is:

$$c_1, c_2 = gdp_{1992,c_1} gdp_{1992,c_2} \text{cov}\left(\sum_{\tau=1}^{30} \frac{gdp_{t+\tau,c_1}}{(1+\rho)^\tau gdp_{t,c_1}}, \sum_{\tau=1}^{30} \frac{gdp_{t+\tau,c_2}}{(1+\rho)^\tau gdp_{t,c_2}}\right) \tag{24}$$

where $gdp_{t,c}$ is *per capita* real GDP in year t of country c. The sample period for the covariance calculations is $t = 1870$ to $t = 1962$ (93 overlapping observations) allowing for 30 years after the start of the interval. We chose for the discount rate ρ a value of 5% a year; with this value $1/(1+\rho)30 = .23$, so the truncation of the present value at 30 years may not present too serious a problem.[4]

Table 1 shows statistics from our data set 1870–1992 on income and population, as well as the estimated variance matrix $\mathbf{\Omega}$ for present values and the corresponding correlation matrix. Almost all correlations are positive, and none is very negative. Correlations show some relation both to geographical distance and to cultural difference. The US and Canada, which share both a border and a language, show the highest correlation of any pair of countries. Both the US and Canada also correlate well with the UK, which is more distant from them but with which they share a language and other cultural ties. Both the US and Canada show low correlation with all remaining countries in the sample, which are more distant than the UK and speak other languages.

We choose γ_c as three divided by an estimate of the expected present value of GDP for country c over the next thirty years. We chose the number 3 as a typical estimate of the coefficient of relative risk aversion in empirical studies. The optimal contracts (the estimated $\mathbf{A}$ matrices) are of course

[4]The contract definitions, A matirix, are similar for ρ evaluated at 3%, 7% and 10%.

unaffected by this choice, though the estimated prices and welfare gains will be affected. The estimate of the expected present value of GDP was formed by discounting hypothetical future values of GDP as follows:

$$E\tilde{W}_{cp} = gdp_{1992,c} \sum_{\tau=1}^{30} \frac{gdp_{t+\tau,c}}{(1+\rho)^{\tau} gdp_{t,c}} \tag{25}$$

where the mean is calculated with sample $t = 1870$ to $t = 1962$ (93 observations). The discount rate used is the same as that used to define our variance matrix.

7 Optimal New Markets Implied by Our Analysis

Our results show that the first, most important, optimal contract (the first column of the **A** matrix, Table 2A column 1) is essentially a swap between the United States, on one side, and France, Germany and Italy on the other. It might be described as a US–Europe swap, where Europe is defined in the traditional sense, and not as the European Union. The UK and Canada enter in the swap on the same side as the US, but with small coefficients. In practice, since it is important to keep contract definitions simple, we would recommend that the coefficients be rounded to simple, easily understood numbers: a swap of US *per capita* income against a simple average of France, Germany and Italy's *per capita* incomes. Such a contract would omit the UK completely, but, in terms of our estimated covariance matrix, one might say that the UK is already halfway between the US and continental Europe, and is thus already effectively "hedged." Our analysis has helped us, as contract designers, avoid the trap of using obvious political boundaries in defining contracts, and we do not make the mistake of creating a US–European Union swap. It is possible, of course, that the Economic and Monetary Union (EMU) will make the UK economy behave in the future more like its counterparts on the continent. Contract designers will have to decide how important the EMU is relative to historical cultural and economic ties in deciding how much this analysis needs to be modified judgmentally.

The price of the first contract is $92.85 (in present value in 1990 dollars, see Table 2B, row 1), and, under our assumption that people make optimal use of these contracts, each person in the United States sells 39.6 contracts (see Table 2C column 1), resulting in value received for each person with a present value of about $3680 (Table 2D, column 1), or about 0.9% of the present value of income (Table 2E, column 1). With our assumed 5%

interest rate, this amounts to about $200 extra sure income for each person in the US per year. These people in the US also see a reduction in the standard deviation of the present value of income of $5250 (Table 2F, column 1), which corresponds to a reduction in the estimated US variance of real income present values of 18.7% (see Table 2G, column 1). Thus, people in the US see both an increase in sure income and a reduction in variance; they do not need to pay for the insurance. Even though people in the US pay less than nothing for a reduction in variance, the welfare gain to Americans has a present value of only $5570 (Table 2H, column 1).[5] France has a welfare gain with a present value of $19,800 *per capita*, and Italy has a welfare gain of $17,200 *per capita* with this contract alone. The greater welfare gain, even though these countries must pay money rather than receive money for the variance reduction, occurs because they experience a greater reduction of the standard deviation of the present value of income. France gets rid of nearly 73.1% of the uncertainty about the present value of income with this contract alone.

The second contract (the second column of the **A** matrix, Table 2A column 2) might be described as essentially a Germany–France swap; their coefficients in the second column of the **A** matrix are roughly equal and opposite, and other countries' coefficients are much smaller. These two countries both gain substantially from this swap; the present value *per capita* consumer surplus from this second contract is $13,100 for Germany and $6,630 for France. The third contract (the third column of the **A** matrix, Table 2A column 3) is essentially an Italy–France swap, again their coefficients are roughly equal and opposite, and other coefficients are much smaller. The present-value consumer surplus for this third contract is $3060 for Italy and $1720 for France. The remaining two contracts, shown here for completeness, may not be worth creating, since in each case no more than one country receives consumer surplus in the thousands of dollars.

8 Robustness of Results

To see how robust our results are for this swap of risk between the US and continental Europe, we conducted a bootstrap experiment. For each coun-

[5]We suspect that empirical work using sample covariance matrices may understate consumer surplus, since distributions of present values of national incomes may be "fat tailed," and hence non-normal. The risk-management benefits of the contracts may be greater than shown by our analysis in that they help to prevent potential disasters bigger than observed in the historical sample.

try, we estimate an AR(1) model of the annual growth rates over our sample, 1870–1992, with a constant. We assemble the C univariate regressions into a C-element vector autoregression (VAR(1)) with a diagonal coefficient matrix. We take the first growth rates, 1870-1871, as given and then draw from the vector residuals with replacement and by recursive substitution construct a 122-element time series of the yearly growth rates from the model. We do this 100,000 times and estimate Ω using equation (24) and then evaluate the first column of the $\mathbf{A}$ matrix 100,000 times.

We normalized the $\mathbf{A}$ vector so that the US weight is always positive. We are interested in whether the coefficients generally have the correct sign. Thus since the US is positive 100% of the time we would like to see France, Germany and Italy to be negative most of the time. We find that France has a negative coefficient about 91% of the time, Germany 52.5% of the time and Italy 92% of the time. Canada has a positive coefficient 53% of the time and the UK 26% of the time.

Thus the swap of risk between the US and continental Europe is fairly robust, with perhaps the exception of Germany. Similar results are obtained if one uses a random walk to estimate the growth rates. While there is still substantial uncertainty in the estimate of the $\mathbf{A}$ matrix, we feel that there is enough accuracy here to provide some guidance for the establishment of new markets.

9 Discussion

In practice, given human psychology and a desire for simplicity, it might possibly be better to create markets for the components of the swaps, rather than the swaps themselves. For example, we might, instead of creating a US–Europe swap, create markets for the US and Europe separately, allowing people to create a swap between them by buying one and shorting the other. However, the reasons for creating such markets for the sides of the swap rather than for the swap itself would be purely psychological: their primary use for risk management would still be to use them to create swaps. Even if we were to design such markets that are not swaps, our analysis would still remain fundamental to the definition of the contracts.

In an earlier version of this paper [1995] we did an analyis of twelve countries around the world, but using only post-World War II data (Summers and Heston [1991]). With such a short sample, the results are less reliable, although the data do cover the world more completely. In the results of that paper we found that a swap between Europe and other countries (the other countries were primarily the US and Japan) was very important, but in that paper this swap was the second contract, not the first. The first contract there was essentially a US–Japan swap. In the post war data, the variance of incomes in Japan was higher than the variance of incomes in Europe.

As a result of this analysis we wish to propose to major exchanges that trading begin in contracts that allow long-term US–Europe income swaps and US–Japan income swaps. These proposals are serious; they can and should be implemented. Other such long-term contracts, as illustrated in the results here or analogous contracts for countries not analyzed here, might also be created.

Appendix A

One can prove that equation (14) holds in two ways. Either one can take exponential utility and substitute into the first order condition, (11), or notice that under the assumptions of joint normality of wealth across countries and CARA utility that expected utiltiy is easily solved. Substitute this into the Lagrangian, (10), and solve. We will show this using the latter. Expected utility assuming CARA is given by

$$\begin{aligned} U_c &= E_0 u_c(\tilde{W}_c) = -\exp[-\gamma_c(\tilde{W}_{cp} - \mathbf{P}'\mathbf{q}_c + \tilde{\mathbf{D}}\mathbf{q}_c)] \\ &= \exp[-\gamma_c(E_0\tilde{W}_{cp} - \mathbf{P}'\mathbf{q}_c) + \tfrac{1}{2}\gamma_c^2(\Omega_{cc} + \mathbf{q}_c'\mathbf{q}_c + 2\mathbf{q}_c'\mathbf{A}'\Omega_c)]. \qquad (1) \end{aligned}$$

Substituting this into the Lagrangian and ignoring the off diagonal constraints we obtain

$$\begin{aligned} L &= \sum_{c=1}^{C} w_c POP_c[\gamma_c(E_0\tilde{W}_{cp} - \mathbf{P}'\mathbf{q}_c) \qquad (2) \\ &\quad - \tfrac{1}{2}\gamma_c^2(\Omega_{cc} + \mathbf{q}_c'\mathbf{q}_c + 2\mathbf{q}_c'\mathbf{A}'\Omega_c)] - \sum_{n=1}^{N}(\mathbf{A}_n'\boldsymbol{\Omega}\mathbf{A}_n - 1)\lambda_{nn}. \end{aligned}$$

Substituting equation (7) and (8) into (A2) we obtain:

$$\begin{aligned} L &= \sum_{n=1}^{N}[\mathbf{A}_n'\boldsymbol{\Omega}\mathbf{M}\boldsymbol{\Gamma}^2\mathbf{wPOPM}'\boldsymbol{\Omega}\mathbf{A}_n] \\ &\quad + \sum_{c=1}^{C}[E_0\tilde{W}_{cp} + \Omega_{cc}] - \sum_{n=1}^{N}(\mathbf{A}_n'\boldsymbol{\Omega}\mathbf{A}_n - 1)\lambda_{nn}. \qquad (3) \end{aligned}$$

Taking the first order conditions and putting them in matrix form gives us the desired result.

Appendix B

To obtain the welfare gain we solve for F_c such that:

$$E_0 u_c(\tilde{W}_c) = E_0 u_c(\tilde{W}_{cp} + F_c). \tag{1}$$

With CARA utility the expression becomes:

Using equations (7) and (8) we rewrite (B2) as:

$$\begin{aligned} & -\exp[-\gamma_c(E_0\tilde{W}_{cp} - \mathbf{P}'\mathbf{q}_c) + \tfrac{1}{2}\gamma_c^2(\Omega_{cc} + \mathbf{q}_c'\mathbf{q}_c + 2\mathbf{q}_c'\mathbf{A}'\Omega_c)] \\ & - \ \exp[-\gamma_c(E_0\tilde{W}_{cp} + F_c) + \tfrac{1}{2}\gamma_c^2(\Omega_{cc})]. \end{aligned} \tag{2}$$

Thus we have

$$F_c = \tfrac{1}{2}\gamma_c \mathbf{q}_c'\mathbf{q}_c. \tag{3}$$

Table 1

Statistics on Income and Population

A. Income, population, and present value of income

Country	Income *per capita*	Population in 000's	Present Value of Income
Canada	18159	28436	487401.71
France	17959	57372	475901.59
Germany	19351	64846	514781.31
Italy	16229	57900	442040.64
United Kingdom	15738	57848	367555.35
United States	21558	255610	551945.41

B. Estimated $\mathbf{\Omega}$ matrix, variance matrix for real *per capita* income present values in 1992, income measured in thousands of 1990 dollars

Country	Canada	France	Germany	Italy	United Kingdom	United States
Canada	2855.13					
France	263.20	8645.09				
Germany	1498.69	4825.24	11325.34			
Italy	1126.66	6505.99	6337.67	7475.87		
United Kingdom	880.28	665.13	1790.30	1171.96	680.78	
United States	2348.70	–984.72	1084.94	–127.00	671.35	2859.37

C. Correlation matrix corresponding to $\mathbf{\Omega}$ in above panel: Correlation matrix for real *per capita* income present values

Country	Canada	France	Germany	Italy	United Kingdom	United States
Canada	1.00					
France	0.05	1.00				
Germany	0.26	0.49	1.00			
Italy	0.24	0.81	0.69	1.00		
United Kingdom	0.63	0.27	0.64	0.52	1.00	
United States	0.82	–0.20	0.19	–0.03	0.48	1.00

Table 2

Characteristics of Optimal Securities and Resulting General Equilibrium

A. A matrix in percent (The nth column gives the percent of *per capita* income of each country that a short in one contract n pays the long.)

Country	1	2	3	4	5
Canada	0.071%	0.071%	–0.335%	0.092%	–3.828%
France	–0.383%	0.785%	1.535%	–0.210%	–0.336%
Germany	–0.274%	–1.127%	0.674%	–0.289%	–0.464%
Italy	–0.406%	0.200%	–2.250%	–0.637%	1.028%
United Kingdom	0.043%	0.009%	–0.181%	4.481%	0.114%
United States	0.814%	0.146%	0.259%	–2.132%	2.585%

B. Price P_n, $n = 1, ..., 5$, in 1990 dollars

Contract	Price
1	92.85
2	121.74
3	51.21
4	180.64
5	102.65

C. q Matrix, giving numbers of contracts an individual in each country purchases

Country	1	2	3	4	5
Canada	–23.30	–8.00	9.98	–2.23	23.24
France	69.06	–39.99	–20.40	0.71	1.00
Germany	52.47	58.40	–9.01	1.22	1.42
Italy	61.99	–8.89	26.14	3.20	–2.47
United Kingdom	–2.93	–0.54	1.71	–20.00	–1.74
United States	–39.60	–2.81	–0.56	3.58	–2.22

D. Total (Present Value) price paid $q_{nc}P_n$ by a person in country c for all contracts n purchased in equilibrium (in 1990 $1000)

Country	1	2	3	4	5	Total
Canada	–2.16	–0.97	0.51	–0.40	2.39	–0.64
France	6.41	–4.87	–1.04	0.13	0.10	0.73
Germany	4.87	7.11	–0.46	0.22	0.15	11.89
Italy	5.76	–1.08	1.34	0.58	–0.25	6.34
United Kingdom	–0.27	–0.07	0.09	–3.61	–0.18	–4.04
United Kingdom	–3.68	–0.34	–0.03	0.65	–0.23	–3.63

E. Total Price (Present Value) paid by a person in country c for all of contract n purchased in equilibrium as percent of present value of *per capita* real income in that country

Country	1	2	3	4	5	Total
Canada	–0.6%	–0.3%	0.1%	–0.1%	0.6%	–0.2%
France	1.8%	–1.3%	–0.3%	0.0%	0.0%	0.2%
Germany	1.2%	1.8%	–0.1%	0.1%	0.0%	3.0%
Italy	1.7%	–0.3%	0.4%	0.2%	–0.1%	1.9%
United Kingdom	–0.1%	–0.0%	0.0%	–1.3%	–0.1%	–1.4%
United Kingdom	–0.9%	–0.1%	–0.0%	0.2%	–0.1%	–0.9%

F. Increase in standard deviation of present value of real income (in 1990 $1000) caused by each contract

Country	1	2	3	4	5	Total
Canada	-0.08	1.63	-2.16	0.88	-11.89	-11.63
France	-44.73	-2.31	-0.89	-0.17	-0.14	-48.23
Germany	-20.99	-28.59	0.18	-0.28	-0.19	-49.86
Italy	-38.01	0.94	-5.88	-0.81	0.29	-43.47
United Kingdom	0.81	0.23	-0.38	4.93	0.58	6.18
United States	-5.25	0.82	0.07	-1.85	0.55	-5.66

G. Percent increase in variance of real income caused by each contract (Elements of Panel E above divided by prehedging variance, in percent)						
Country	1	2	3	4	5	Total
Canada	–0.3%	6.2%	–7.9%	3.3%	–39.6%	–38.3%
France	–73.1%	–4.9%	–1.9%	–0.4%	–0.3%	–80.5%
Germany	–35.6%	–46.5%	0.3%	–0.5%	–0.4%	–82.6%
Italy	–68.6%	2.2%	–13.1%	–1.9%	0.7%	–80.7%
United Kingdom	6.3%	1.8%	–2.9%	41.4%	4.5%	51.1%
United States	–18.7%	3.1%	0.3%	–6.8%	2.1%	–20.1%

H. Consumer surplus, $F_{cN} - F_{c(N-1)}$, added by each contract, measured in 1990 $1000						
Country	1	2	3	4	5	Total
Canada	2.20	0.26	0.40	0.02	2.19	5.07
France	19.76	6.63	1.72	0.00	0.00	28.12
Germany	10.54	13.05	0.31	0.01	0.01	23.92
Italy	17.21	0.35	3.06	0.05	0.03	20.70
United Kingdom	0.05	0.00	0.02	2.12	0.02	2.20
United States	5.57	0.03	0.00	0.05	0.02	5.67

Mathematical Symbols

Note: Matrices and vectors are represented as bold-faced symbols.

A. Latin Symbols

- $\mathbf{A}$ $C \times N$ matrix whose cnth element is the share of country c's income that is included in the dividend
- $\tilde{\mathbf{D}}$ N element row vector of real dividends paid by a short to a long in each contract. By our definitions, $\tilde{\mathbf{D}} = (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}$. $\tilde{\mathbf{D}}$ is unknown at time 0.
- $\tilde{\mathbf{D}}_t$ N-element row vector whose nth element is $\tilde{\mathbf{D}}_{tn}$, the real dividend paid by a short to a long in year t in $(\tilde{\mathbf{y}} - E_0\tilde{\mathbf{y}})\mathbf{A}$. $\tilde{\mathbf{D}}_t$ for $t > 0$ is unknown at time 0.
- N Number of contracts, income component securities defined here; i.e., the number of distinct contract types available for use in risk management.
- $\mathbf{P}$ N-element column vector whose nth element is the real price of income component security n according to contract n at time 0, and amount paid at time 0 from the long in one contract to the short. Each element of $\mathbf{P}$ is made to be nonnegative by choosing the sign of the corresponding column of $\mathbf{A}$.
- POP_c Population of country c in the contract year, year 0.
- $\mathbf{POP}$ $C \times C$ matrix whose cth diagonal element is POP_c.
- $\mathbf{q}$ $N \times C$ matrix whose ncth element is the number of the nth income component security demanded by the representative individual in country c.
- S Social welfare, a weighted sum of utilities.
- t Year, contract year is 0, first year following contract is 1.
- $\tilde{W}_c$ Wealth in country c after hedging, unknown at time 0; the present value of after-hedging income.
- $\tilde{W}_{cp}$ Wealth in country c pre-hedging, unknown at time 0; the present value of pre-hedging income.
- $\mathbf{w}$ $C \times C$ diagonal matrix, whose cth diagonal element is the weight given to one individual in country c by the contract designer in the social welfare function used to derive the optimal income component securities.

$\tilde{\mathbf{W}}$ The C-element row vector whose cth element is the present value at time 0 of $\tilde{y}_{tc}$, $t = 1, 2, ...$

$\tilde{y}_{tc}$ Real *per capita* income in year t of country c before hedging. In our data, $\tilde{y}_{tc}$ is *per capita* gross domestic product of country c, measured in 1990 dollars.

B. Greek Symbols

γ_c Risk aversion parameter for country c.

$\mathbf{\Gamma}$ $C \times C$ diagonal matrix whose cth diagonal element equals γ_c.

ι $C \times 1$ vector, all of whose elements equal one.

λ_n A Lagrangian multiplier for the contract designer's problem, corresponding to the constraint that $\mathbf{A}_n' \mathbf{\Omega} \mathbf{A}_n = 1$, where $\mathbf{A}_n$ is the nth column of $\mathbf{A}$; also the nth eigenvalue of a matrix defined in that problem.

Λ $N \times N$ diagonal matrix, whose nth diagonal element is λ_n.

$\boldsymbol{\mu}$ $N \times 1$ vector, whose elements are Lagrangian multipliers for the constraints that the N markets clear.

$\mathbf{\Omega}$ $C \times C$ matrix, the variance matrix, conditional on information at year 0, whose $c_1 c_2$th element is the conditional covariance between countries c_1 and c_2 of present value of a representative individual's real income. In empirical work, *per capital* real income is used for the representative individual's income. $\mathbf{\Omega} = E_0(\mathbf{y} - E_0\mathbf{y})'(\mathbf{y} - E_0\mathbf{y})$.

Summary of Basic Relations

$\tilde{\mathbf{D}} = (\tilde{\mathbf{W}}_p - E_0\tilde{\mathbf{W}}_p)\mathbf{A}$
$\mathbf{P} = -\mathbf{A}'\boldsymbol{\Omega}\mathbf{POP}\iota(\iota'\boldsymbol{\Gamma}^{-1}\mathbf{POP}\iota)^{-1}$
$\mathbf{q} = -(\mathbf{A}'\Omega + \mathbf{P}\iota'\boldsymbol{\Gamma}^{-1})$
$\mathbf{q} = -\mathbf{A}'\boldsymbol{\Omega}\mathbf{M}$
$\mathbf{qPOP}\iota = 0$
$\mathbf{A}'\boldsymbol{\Gamma}^{-1}\iota = 0$
$\mathbf{qA} = -\mathbf{I}$
$\mathbf{A} = -\mathbf{Mw}\boldsymbol{\Gamma}^2\mathbf{POPq}'\Lambda^{-1}$
$\mathbf{A} = \mathbf{MA}$
$\mathbf{q} = \mathbf{qM}$
$\tilde{\mathbf{W}}_p A = (\tilde{\mathbf{W}}_p\mathbf{M})\mathbf{A}$
$\mathbf{M} = \mathbf{I} - \mathbf{POP}\iota(\iota'\boldsymbol{\Gamma}^{-1}\mathbf{POP}\iota)^{-1}\iota'\boldsymbol{\Gamma}^{-1}$
$\mathbf{MPOP}\iota = 0$, $\iota'\boldsymbol{\Gamma}^{-1}\mathbf{M} = 0$
$\mathbf{M}'\boldsymbol{\Gamma}^{-1}\mathbf{POP}^{-1}\mathbf{M} = \boldsymbol{\Gamma}^{-1}\mathbf{M}$
$\tilde{\mathbf{W}}_p\mathbf{M}$ = vector, cth element is the difference between an individual in country c's wealth less her share of world wealth according to populations weighted by risk aversions.
$\mathbf{A}'\boldsymbol{\Omega}\mathbf{Mw}\boldsymbol{\Gamma}^2\mathbf{POPM}'\boldsymbol{\Omega}\mathbf{A} = \mathbf{qw}\boldsymbol{\Gamma}^2\mathbf{POPq}' = \Lambda$
$\mathbf{M}'\boldsymbol{\Omega}\mathbf{Mw}\boldsymbol{\Gamma}^2\mathbf{POPq}' = \mathbf{q}'\Lambda$

Call $\mathbf{e}$ the matrix of first N eigenvectors of $(\mathbf{w}\boldsymbol{\Gamma}^2\mathbf{POP})^{.5}\mathbf{M}'\boldsymbol{\Omega}\mathbf{M}(\mathbf{w}\boldsymbol{\Gamma}^2\mathbf{POP})^{.5}$ so that $\mathbf{e}'\mathbf{e} = \mathbf{I}$, then $\mathbf{q}' = (\mathbf{w}\boldsymbol{\Gamma}^2\mathbf{POP})^{-.5}\mathbf{e}\Lambda^{.5}$, where Λ is diagonal matrix of the eigenvalues.

Adding $k\boldsymbol{\Gamma}^{-1}\iota\iota'\boldsymbol{\Gamma}^{-1}$ to $\boldsymbol{\Omega}$ for any scalar $k > 0$ has no effect on A or q.

If $N = C - 1$: $\mathbf{M}'\boldsymbol{\Omega}\mathbf{MPOP} = \mathbf{q}'$ and $\mathbf{Aq} = -\mathbf{M}$ and $\mathbf{I} + \mathbf{Aq} = \mathbf{POP}\iota(\iota'\boldsymbol{\Gamma}^{-1}\mathbf{POP}\iota)^{-1}\iota'\boldsymbol{\Gamma}^{-1}$

References

Allen, Franklin, and Douglass Gale, *Financial Innovation and Risk Sharing*, MIT Press, Cambridge MA, 1994.

Athanasoulis, Stefano, "Essays in Risk Sharing, Derivatives Design, and Macroeconomic Policy," unpublished Ph.D dissertation, Yale University, 1995.

Athanasoulis, Stefano, and Robert J. Shiller, "The Significance of the Market Portfolio," NBER Technical Working Paper, Cambridge MA, 1997.

Atkeson, Andrew, and Tamim Bayoumi, "Do Private Capital Markets Insure against Risks in a Common Currency Area? Evidence from the United States," unpublished paper, University of Chicago, 1991.

Atkeson, Andrew, and Christopher Phelan, "Reconsidering the Costs of Business Cycles with Incomplete Markets," *NBER Macroeconomics Annual 1994*, MIT Press, Cambridge MA, 1994, pp. 187–206.

Baxter, Marianne, Urban J. Jermann, and Robert G. King, "Nontraded Goods, Nontraded Factors, and International Non-Diversification," xeroxed, University of Virginia, 1994.

Bottazzi, L., P. Pesenti and E. van Wincoop, "Wages, Profits and the International Portfolio Puzzle," *European Economic Review*, 40: 219–254, 1996.

Cole, Harold L., and Maurice Obstfeld, "Commodity Trade and International Risk Sharing: How Much Do Financial Markets Matter?" *Journal of Monetary Economics*, 28: 3–24, 1991.

DeMange, Gabrielle, and Guy LaRoque, "Optimality of Incomplete Markets," *Journal of Economic Theory*, 65: 218–32, 1995

Duffie, Darrell, and Matthew O. Jackson, "Optimal Innovation of Futures Contracts," *The Review of Financial Studies,* 2: 275–96, 1989.

Duffie, Darrell, and Rohit Rahi, "Financial Market Innovation and Security Design: An Introduction," *Journal of Economic Theory,* 65: 1–42, 1995.

Geanakoplos, John, "An Introduction to General Equilibrium with Incomplete Markets," *Journal of Mathematical Economics*, 19: 1–38, 1990.

Maddison, Angus, *Monitoring the World Economy 1820–1992*, OECD, Paris, 1995.

Mayers, David, "Non-Marketable Assets and Capital Market Equilibrium under Uncertainty," in Michael Jensen (ed.), *Studies in the Theory of Capital Markets*, Praeger, New York, 1972, pp. 223–248.

Nash, John F., "The Bargaining Problem," *Econometrica*, 1950.

Obstfeld, Maurice, "Are Industrial Country Consumption risks Globally Diversified?" NBER Working Paper No. 4308, 1993.

Obstfeld, Maurice, "Risk-Taking, Global Diversification, and Growth," *American Economic Review*,1995.

Oh, Gyutaeg, "Some Results in the CAPM with Nontraded Endowments," *Management Science*, 42(2): 286–293, February 1996.

Sala-i-Martin, Xavier, and Jeffrey Sachs, "Fiscal Federalism and Optimum Currency Areas: Evidence for Europe from the United States," in M. B. Canzoneri et al. (eds.), *Establishing a Central Bank: Issues in Europe and Lessons from the U.S.*, Cambridge University Press, Cambridge, 1992.

Shiller, Robert J., *Macro Markets: Creating Institutions for Managing Society's Largest Economic Risks*, Oxford University Press, Oxford, 1993. (a)

Shiller, Robert J., "Measuring Asset Values for Cash Settlement in Derivative Markets: Hedonic Repeated Measures Indices and Perpetual Futures," *Journal of Finance*, July, 1993. (b)

Shiller, Robert J., and Stefano Athanasoulis, "World Income Components: Measuring and Exploiting International Risk Sharing Opportunities," NBER Working Paper No. 5095, 1995.

Summers, Robert, and Alan Heston, "The Penn World Table (Mark 5): An Expanded Set of International Comparisons," *Quarterly Journal of Economics*, 106: 1–41, 1991.

Tesar, Linda and Ingrid Werner, "Home Bias and the Globalization of Securities Markets," NBER Working Paper No. 4218, 1992.