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The Application of Linear Programming to Team Decision Problems

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1. Introduction

In a team decision problem there are two or more decision variables, and these different decisions can be made to depend upon different aspects of the environment, or information variables. The choice of optimal rules for selecting information variables and for making decisions is the central problem of the economic theory of teams. In the previous paper [1], Marschak gave an introduction to the main concepts of this theory. In the present paper I shall show how the technique of linear programming can be used to solve a typical class of team decision problems.

The "character" of a decision problem is determined by the form of the function that is to be maximized, which I shall call, with Marschak, the payoff function. Much of the available data about business leads naturally to the formulation of decision problems in terms of what might be called convex polyhedral payoff functions, i.e., the space of decision variables can be divided into regions, whose boundaries are linear, such that within each region the payoff is a linear function of the decision variables. As is well known, such a problem is amenable to linear programming, and as I have shown in another paper [2], the introduction of probabilistic uncertainty, and of the further complications of a team situation, does not destroy the linear character of a programming problem, although it may result in a substantial increase in the "size" of the problem.

In this paper I will illustrate these ideas by means of an example; a general formulation has been given in the paper just referred to, but the reader will probably have no trouble in providing such a generalization himself. The example to be used is about as simple as it can be without sacrificing any of the three features that I want to illustrate, namely, (1) uncertainty, (2) different decision variables can be made to depend upon different information variables, and (3) a nondegenerate convex polyhedral payoff function. (Therefore, the reader should not expect too much in the way of realism!)

2. An Example

Consider a "firm" with two activities, which I will label "production" and "promotion," and suppose that the levels of expenditure on these two activities must be chosen for one period to come. Let $\underline{a}$ denote the amount of money alloted to production, and let $\underline{xa}$ be the resulting quantity produced (there is only one commodity concerned), where x can be interpreted as the "productivity" of the production activity. Similarly, let $\underline{b}$ denote the amount of money alloted to promotion, and let $\underline{yb}$ be the resulting demand generated. The quantity actually sold will therefore be the smaller of the two quantities, $\underline{xa}$ and $\underline{yb}$; if both the product and the demand generated are "perishable," and if the units are chosen so that the price of the commodity is 1, then the profit resulting from the pair of expenditures (a,b) is

$$\min(xa, yb) - (a + b) .$$

If the business were at all profitable, then the firm would of course expand its scale of operation indefinitely, were it not for the fact that its immediate supply of capital is limited. This limit is not absolute, but there is a substantial cost attached to obtaining more capital than is immediately available. Letting $\underline{k}$ denote the capital limit, and $\underline{f}$ denote the cost per dollar of additional capital, the firm's profit, as a function of the decision variables a and b , is given by the payoff function

$$u(a,b;x,y) = \min (xa,yb) - (a+b) - f \max (0,a+b-k) \quad (1)$$

The function u just defined is indeed convex and polyhedral, for any reasonable given values of x and y , and its contours are shown in the accompanying figure (page 3a).

It is easy to see that, for given x and y , the function u attains its maximum when

$$\begin{aligned} ax &= by , \\ a + b &= k , \end{aligned} \quad (2)$$

that is to say, when

$$\begin{aligned} a &= \frac{yk}{x+y} , \\ b &= \frac{xk}{x+y} , \end{aligned} \quad (3)$$

and the maximum value of u is

$$\max_{a,b} u = \frac{xyk}{x+y} . \quad (4)$$

Suppose now that the firm is uncertain about the actual values of the "productivity" parameters, x and y , that will prevail during the period in question, and that these values can be predicted accurately only at some substantial cost. Two extreme alternatives suggest themselves. The firm could pay the cost and obtain the relevant information, and then make the appropriate decisions, as given by equation (3). This alternative will be called the case of full information. On the other hand, the firm could rely only upon its

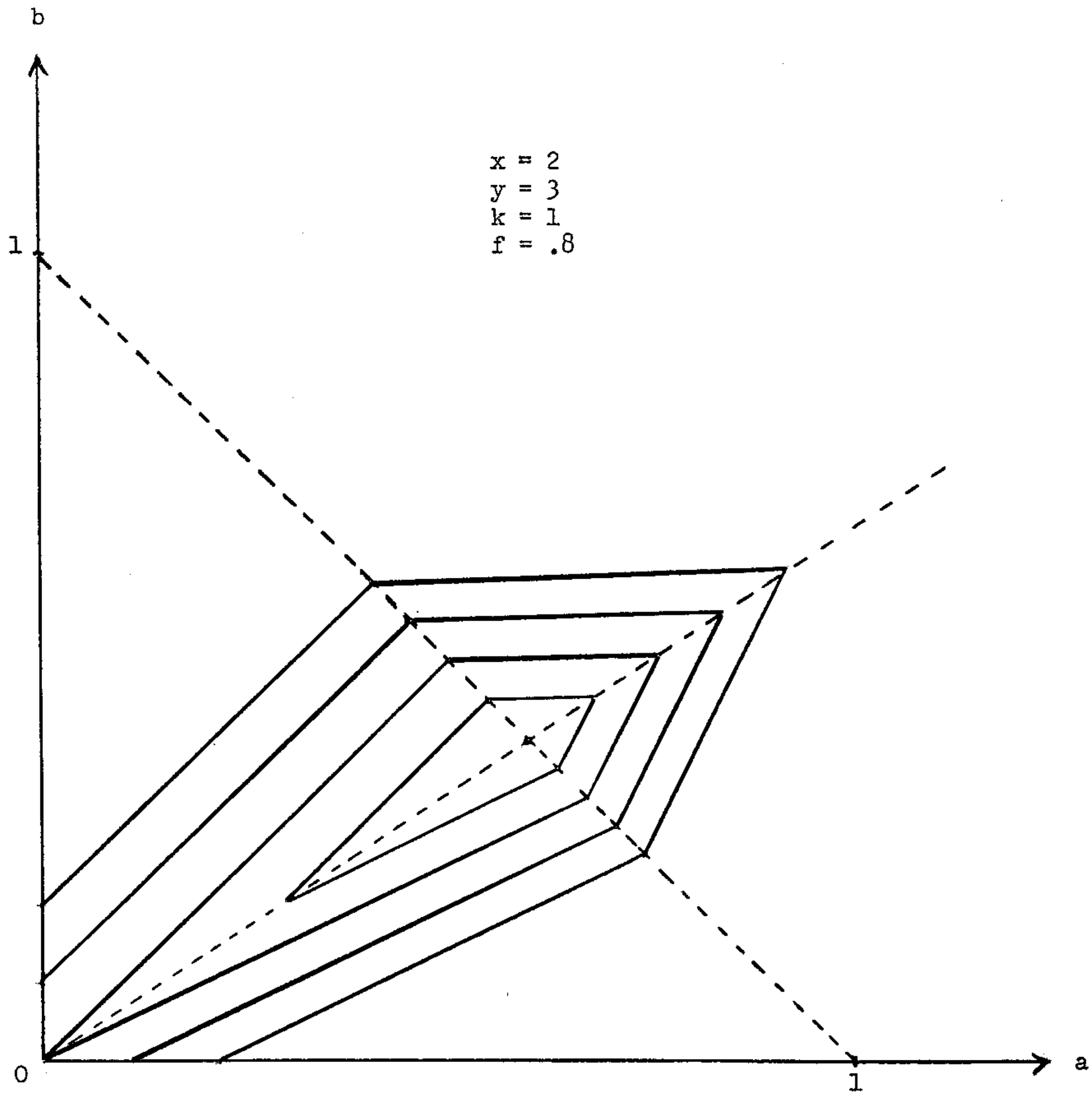

Figure 1. Iso-profit curves for u(a,b;x,y).

knowledge of the probability distribution of x and y , which is assumed to be known, and choose that pair of expenditures that maximizes the expected, or average, profit. This will be called the case of routine operation. Each of these two alternatives involves a different structure of information. Which alternative is the better depends upon which one results in the higher expected profit, net of the cost of information.

A third, intermediate, alternative is suggested under the circumstance described by Marschak in the previous paper as "cospecialization of action and information." In this case, it costs less for the person in charge of production to get the needed information about the parameter x than it does for the person in charge of promotion to do so, and the reverse holds for the parameter y . If, in addition, communication between these two persons is costly, it may be desirable to have the decision about the variable a made by the production manager only in the light of knowledge about x , and the decision about the variable b made by the promotion manager only in the light of knowledge about y , all however according to a decision rule agreed upon in advance. This last alternative will be called the case of decentralization.

The possibility of costly communication may seem far-fetched in the context of this simple example; however, if instead of a and b one thinks of two fairly complicated sequences of decision, with each person (or department) getting new information all the time, then it might indeed be costly to achieve a complete exchange of information between the two.

As a primary step towards solving the over-all problem of choosing both a best information structure and best decision rules, one must, at least in principle, solve the various "sub-optimizing" problems of choosing the best decision rules for given information structures, and this is the type of problem that will be considered in detail in the rest of this paper. Before doing so (in the next section), it may be helpful to look at the results for some given numerical values of the parameters.

Suppose that x and y are statistically independent, and can each take on one of two values, with equal probability, the values being given in the following table.

x \ y	2.8	3.6
3.0	1/4	1/4
3.4	1/4	1/4

Table 1

Joint Probability Distribution of x and y

Suppose, furthermore, that the amount of free capital (k) equals 1,000 dollars, and that the cost of additional capital (f) is 1.7 dollars per dollar. (It is clear from equations (2) and (3) that the value of k merely determines the scale of operation, and does not influence the relative expenditures.) The maximum possible expected profit for each of the three alternative information structures described above is given in Table 2.

	Routine	Decentralized	Full Information
Maximum Expected Profit	498	541	592

Table 2. Maximum Expected Profit

In the "routine" case, the decision rules are, in a sense, degenerate; a best single value of a and a best single value of b are to be chosen. In the "decentralized case," however, a pair of values (a_1, a_2) and a pair of values (b_1, b_2) are to be chosen, where a_1 denotes the expenditure that will be made by the production manager if he learns that x will have the value 3.0, a_2 is the expenditure corresponding to $x = 3.4$, etc. In the "full information" case, there are four values a_{ij} and four values b_{ij} to be chosen, where a_{ij} denotes the expenditure that will be made on production corresponding to the pair of parameter values (x_i, y_j), etc. Table 3 shows the best decision rules for each of the three information structures.

Parameter Values	Best Decision Rules		
	Routine	Decentralized	Full Information
x = 3.0	a = 483	512*	483
y = 2.8	b = 517	548*	517
3.0	483	512	452
3.6	517	426	548
3.4	483	452	545
2.8	517	548	455
3.4	483	452	514
3.6	517	426	486

* Total expenditure exceeds immediate supply of capital.

Table 3. Best Decision Rules

An interesting feature of the "decentralized" case (for this numerical example) is that, under the best decision rules, the capital limit is actually exceeded by a small amount whenever a and b both take on their largest values, an event that occurs with probability 1/4 . This is so in spite of the relatively high value assumed for the cost of additional capital (equivalent to an interest rate of 170 per cent!). This illustrates an important point about the role of joint constraints in a team decision problem. It often happens that a constraint in a programming problem does not represent a condition that is impossible to violate, but rather a condition that appears to the programmer to be obviously uneconomical to violate. However, if different decisions are based upon different information variables, as in the "decentralized case," a certain degree of lack of complete coordination will typically be introduced, and to require that a given constraint never be violated may turn out to be too harsh a restriction (that is to say, uneconomical) when the true cost of a violation is actually weighed against the possible advantages of flexibility.

Even in this simple example there are many other conceivable information structures besides the three already mentioned. Some of these will be mentioned in Section 4. In the next section I will take up the problem of computing best decision rules for any given information structure.

3. Computing the Optimal Decision Rules

The procedure for computing the optimal decision rules for a given information structure is an immediate application of the following lemma. The lemma is stated in terms of two decision variables and two random parameters, in order to make more transparent its relation to the example of the previous section, but the generalization to any number of decision variables and random parameters is obvious.

Lemma. Suppose that

$$u(a,b;x,y) = \min_n f_n(a,b;x,y) \ , \qquad (n = 1, \ldots, N) \tag{5}$$

where, for every n, x, and y, f_n is linear in a and b. Suppose also that x and y take on only a finite number of values, with probabilities $p(x,y)$. Furthermore, let

$r = R(x,y)$ be the information on which action a is based,

$s = S(x,y)$ be the information on which action b is based,

A denote any function of r (a decision rule for a),

B denote any function of s (a decision rule for b),

Z denote any function of x and y.

Then the following two maximization problems are equivalent, in the sense that the maximum values are the same, and $(\hat{A},\hat{B})$ is a solution of Problem I if and only if there is a $\hat{Z}$ such that $(\hat{Z},\hat{A},\hat{B})$ is a solution of Problem II.

Problem I. Choose A and B so as to maximize

$$Eu(A[R(x,y)],B[S(x,y)];x,y) \ , \tag{6}$$

subject to $A(r), B(s)$ nonnegative.

Problem II. Choose Z, A and B so as to maximize

$$EZ(x,y) , \tag{7}$$

subject to $Z(x,y)$, $A(r)$, $B(s)$ nonnegative, and to the further constraints that

$$Z(x,y) \leq f_n(A[R(x,y)], B[S(x,y)]; x,y) \tag{8}$$

for every n, x and y.

(Note: the symbol E denotes mathematical expectation with respect to the random parameters x and y.)

Since $EZ(x,y) = \sum_{x,y} p(x,y)Z(x,y)$ is a linear function of the "variables" $Z(x,y)$, and since the constraints (8) are linear in $Z(x,y)$, $A(r)$ and $B(s)$, Problem II is a standard linear programming problem.

Returning to the example of the previous section, let the function u be given by equation (1); then it is easy to see that u can be expressed in the form (5) by taking

$$\begin{aligned} f_1 &= (x-1)a - b , \\ f_2 &= (x-1-f)a - (1+f)b + fc , \\ f_3 &= -a + (y-1)b , \\ f_4 &= -(1+f)a + (y-1-f)b + fc . \end{aligned} \tag{9}$$

Consider the decentralization example; there one has the information structure

$$R(x,y) = x, \quad S(x,y) = y . \tag{10}$$

Suppose, furthermore, that x and y can each take on one of two values, as in the numerical example; then A will take on one of two values, say a_1 and a_2, according as x equals x_1 or x_2; and likewise for B. $Z(x,y)$, however, will take on one of four values, say z_{ij}, corresponding to the four pairs (x_1,y_j). In this case Problem II takes the form:

> Choose z_{ij}, a_i, b_j, so as to maximize $\sum_{ij} p_{ij} z_{ij}$, subject to z_{ij}, a_i, b_j nonnegative, and the set of linear constraints presented in matrix ("detached coefficient") form in Table 4.

z_{11}	z_{12}	z_{21}	z_{22}	a_1	a_2	b_1	b_2	$\leqq$
E	0	0	0	$-G_1$	0	$-H_1$	0	F
0	E	0	0	$-G_1$	0	0	$-H_2$	F
0	0	E	0	0	$-G_2$	$-H_1$	0	F
0	0	0	E	0	$-G_2$	0	$-H_2$	F

where

$$E = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad G_i = \begin{bmatrix} x_i - 1 \\ x_i - 1 - f \\ -1 \\ -1-f \end{bmatrix}, \quad H_j = \begin{bmatrix} -1 \\ -1-f \\ y_j - 1 \\ y_j - 1 - f \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ fc \\ 0 \\ fc \end{bmatrix}.$$

Table 4. Constraint Matrix

The fortunate pattern of 1's and 0's in the left half of the constraint matrix of Table 4 is characteristic of a linear problem derived from one with a polyhedral profit function; from a computational point of view, the addition of the variables z_{ij} does not represent a significant increase in the number of variables. The scattering of 0's throughout the right half of the constraint matrix is typical of a team decision problem.

More generally, for the decentralized case in this example, if x can take on I values, and y can take on J values, then for Problem II there will be $IJ + I + J$ variables, and $4IJ$ constraints, but from the point of view of computation time, the number of variables is effectively only on the order of $I + J$.

Even so, the number of possible values for the random parameters must not be allowed to get too large, for feasible computation along these lines; this is especially so for problems with more decision variables.

4. Interpretation of Sequential Decision Problems as Team Problems.

Thus far in this paper the different decision variables in a team decision problem have been interpreted as the decisions of different persons. Another class of problems with the same formal structure arises from sequential decision problems for even a single "person" (e.g., inventory and production scheduling problems In this case the different decision variables correspond to decisions taken at different points of time. Thus, if there is a decision to be made in each of two successive time periods, and information about the parameter values also tends to become known sequentially, then, using the notation of the lemma of Section 3, the following information structures are likely to be relevant:

$$(11)\quad \begin{cases} R(x,y) = \text{constant} \\ S(x,y) = x \end{cases}$$

$$(12)\quad \begin{cases} R(x,y) = x \\ S(x,y) = (x,y) \end{cases}$$

The technique of Section 3 applies just as well, of course, to these information structures as it did to the ones considered there. However, the special "triangular" character of the information structures that arise in single-person sequential problems often leads to computational simplifications that do not apply to team problems in general. On the other hand, it is clear that sequential or "dynamic" elements can be incorporated into a team decision problem, without altering the basic mathematical framework.

References

[1] J. Marschak, "Theory of teams: Introduction," Cowles Foundation Discussion Paper No. 31, May 28, 1957 (dittoed)

[2] R. Radner, "The linear team: An example of linear programming under uncertainty," Proceedings of the Second symposium in Linear Programming, Washington: National Bureau of Standards, 1955, pp. 381-396.