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COWLES FOUNDATION DISCUSSION PAPER No. 16

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A Note on the Modigliani-Hohn Production Smoothing Model*

Alan S. Manne

September 10, 1956

* Research undertaken by the Cowles Foundation for Research in Economics under Contract Nonr-609(01) with the Office of Naval Research.)

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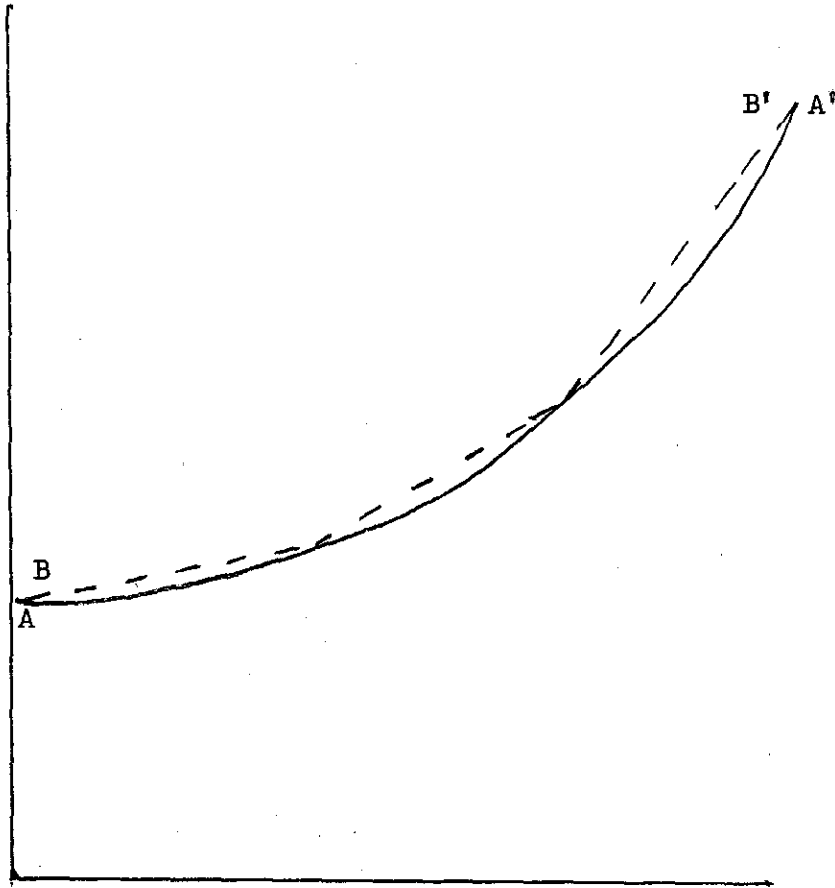
1. Introduction

Modigliani and Hohn [5] have formulated a production planning and inventory control model that appears relevant to an important class of non-stochastic decision-making problems. In their own words:

We consider the problem of scheduling the production of a given commodity x over T equal, successive periods of time in such a way as
 (1) to meet initially known requirements s_1, s_2, \dots, s_T in these periods while
 (2) incurring the lowest possible cost. [5, p. 46]

It is the purpose of this note to suggest: (1) that the Modigliani-Hohn problem may be formulated in terms of linear programming as well as through the calculus model originally employed; (2) that since the linear programming version is equivalent to a "transportation problem" [3], even large systems may be solved readily by hand; and (3) that the linear programming version is especially well-suited to trace out the cost implications of stabilizing the work force at alternative levels. Like the "caterer problem" [4] and the "warehousing problem" [1], this linear programming model represents another instance in which every basis is pure triangular and contains no elements but zero, +1, and -1.

Total
production
costs
(\$ / period)



Production rate
(output / period)

$$(2.3) \quad x_{jt} + y_{jt} = a_{jt} \quad \left\{ \begin{array}{l} j = 1, 2, \dots, J \\ t = 1, 2, \dots, T \end{array} \right.$$

And condition (2.4) indicates the identity between the inventory carried over from the end of one period and that on hand at the beginning of the next: *

$$(2.4) \quad h_t = H_{t+1} \quad (t = 1, 2, \dots, T-1)$$

Finally, there are the usual non-negativity restrictions on all unknowns:

$$(2.5) \quad \begin{array}{l} x_{jt} \geq 0 \\ y_{jt} \geq 0 \\ h_t \geq 0 \end{array} \quad \left\{ \begin{array}{l} j = 1, 2, \dots, J \\ t = 1, 2, \dots, T \end{array} \right. \quad (t = 1, 2, \dots, T-1)$$

3. Equivalence to the transportation model

To show that the problem just defined is equivalent to a transportation model, it suffices to illustrate the array for $J=T=3$. (See Table 1.) In this instance, 22 variables altogether are present: $x_{11}, x_{21}, \dots, x_{33}; y_{11}, y_{21}, \dots, y_{33}; h_1, h_2, H_2$, and H_3 . The remaining cells of the 4×11 array are empty. There are 14 restrictions on the row and column totals of Table 1: the three row totals corresponding to equation group (2.2), the first nine column totals corresponding to equation group (2.3), and the last two column totals to group (2.4). No restriction is placed upon the sum of the row containing the "slack variables", y_{jt} .

* The "costless" variables H_t , along with the identity conditions (2.4), are really superfluous. They are introduced here only to facilitate setting up the linear programming problem in the format of the "transportation" problem.

period t	1			2			3			Inventory carried over		Requirements
	1	2	3	1	2	3	1	2	3	period 1 to 2	period 2 to 3	
shift j	c_1	c_2	c_3	c_1	c_2	c_3	c_1	c_2	c_3	$-\alpha$	α	$s_1 - H_1$
	x_{11}	x_{21}	x_{31}	x_{12}	x_{22}	x_{32}	x_{13}	x_{23}	x_{33}	$-h_1$	h_2	s_2
	y_{11}	y_{21}	y_{31}	y_{12}	y_{22}	y_{32}	y_{13}	y_{23}	y_{33}	$+H_2$	$-h_2$	$s_3 + h_3$
upper limits on output, a_{jt}	a_{11}	a_{21}	a_{31}	a_{12}	a_{22}	a_{32}	a_{13}	a_{23}	a_{33}	0	0	

General form of LP problem:

$$\text{Minimize } \sum_l c_l z_l$$

$$\text{subject to } z_l \geq 0$$

$$\sum_l b_{ql} z_l = d_q$$

Transportation form

$$\text{Minimize } \sum_{hi} c_{hi} z_{hi}$$

$$\text{subject to } z_{hi} \geq 0$$

$$\sum_i z_{hi} = d_h$$

$$\sum_h z_{hi} = e_i$$

$$\left. \begin{array}{l} \sum_i z_{hi} = d_h \\ \sum_h z_{hi} = e_i \end{array} \right\} \text{where } \sum_h d_h = \sum_i e_i$$

The above problem can be put in ^{quasi-}transportation form by specifying

c_{hi} as inserted above in period, z_{hi} as printed above (of which two, $-h_1$ & $-h_2$ are nonpositive!)

$$[e_i] = [a_{11}, \dots, a_{33}, 0, 0]$$

$$[d_h] = [s_1 - H_1, H_2, s_2 - h_2, d_4]$$

$$d_4 = \sum_i e_i - \sum_{h=1}^3 d_h$$

What are the initial characteristics?

What method? 8.

In the case of an array (such as the one shown in Table 1, Dantzig has proved the following theorem: At each iteration, it will be possible to find either a row or a column containing exactly one basis variable. Solving for this variable and then working with the reduced array formed by deleting the corresponding row or column, it is proved by induction that every basis must be strictly triangular. Furthermore, since the only coefficients entering into the equations of the initial linear programming problem are zero, ^{and as} +1, and -1, the only arithmetic operations required for a solution consist of addition and subtraction.

[Dantzig, 3, p. 365].

4. An illustrative example

Suppose that the constants and the cost coefficients for a 3-shift, 3 time period problem are as given in Table 2. For these data, Table 3 then contains the minimum cost solution as computed by the "transportation" method. Values for the dual variables u_t , the "shadow prices" associated with each of the sales requirement equations, are also entered in Table 3. In this, as in any optimal solution to this class of linear programming problems, the following conditions are necessary, and may be employed to

facilitate computations: *

* Note that the conditions of (4.1) correspond to the Modigliani-Hohn calculus inequalities (3.10), [5, p.50].

~~Problems for a hypothetical 3-shift, 3 time period~~
smoothing problem.

<u>Constants</u> (units of output)	<u>Cost coefficients</u> (\$10 ³ /unit of output)
$s_1 - H_1 = 80$	$c_1 = 10$
$s_2 = 160$	$c_2 = 15$
$s_3 + h_3 = 240$	$c_3 = 20$
$a_{jt} = 100$ (all j, t)	$\alpha = 4$

Table 3. Optimal solution to hypothetical smoothing problem

time shift j period t	x_{jt}			H_t	h_t	Requirements	u_t (\$10 ³)
	1	2	3				
1	100	0	0	--	20	80	11
2	100	80	0	20	40	160	15
3	100	100	0	40	--	240	19
a_{jt}	100	100	100				
c_j (\$10 ³)	10	15	20				
Inventory + labor costs = $\alpha \sum_{t=1}^{T-1} h_t$ + $\sum_{j=1}^J c_j \sum_{t=1}^T x_{jt}$ = \$5,940,000.							

$$(4.1) \quad u_{t+1} \leq u_t + \alpha \quad (t = 1, 2, \dots, T-1)$$

$$(4.2) \quad (u_t + \alpha - u_{t+1}) (h_t) = 0 \quad (t = 1, 2, \dots, T-1) *$$

(4.3) If, for any given shift j and period t ,

$$x_{jt} > 0, \text{ then } x_{j-1, t} = a_{j-1, t} \quad (j = 2, 3, \dots, J; t = 1, 2, \dots, T)$$

$$\forall x_{jt} < a_{jt} \text{ then } x_{j+1, t} = 0 \quad (j = 1, \dots, J-1)$$

(4.4) If, for any given shift j and period t ,

$$a_{jt} > x_{jt} > 0, \text{ then } u_t = c_j. \text{ If } x_{jt} = a_{jt}, \text{ then } u_t \geq c_j.$$

$$\text{And if } x_{jt} = 0, \text{ then } u_t \leq c_j. \quad (\text{all } j, t)$$

(4.5) The "costless" variables H_2, H_3, \dots, H_T will all be contained in an optimal basis - although some of them may take on the value of zero.

5. Work stabilization policies

The model with which we have been dealing includes no explicit costs for altering the rate of production or the size of the work force from one period to the next. Indeed, in any concrete application, it may be extremely difficult for a firm to place dollar cost estimates on the loss of employee and community good will that results from deliberate fluctuations in employment. As an alternative to making arbitrary estimates of these "rate of change" costs, it will frequently be desirable to calculate several alternative inventory and production plans - each corresponding to a different assumption as to the "work force commitment", i.e., the level below which employment will not be reduced during any of the time periods covered by the production plan.

* In the terminology of Modigliani and Hohn, if $u_t + \alpha > u_{t+1}$, period t and $t+1$ belong to two different "intervals". "Small" changes in $s_{t+1}, s_{t+2}, \dots, s_T + h_T$ could not alter the optimal solution for periods 1, 2, ..., t .

In the case of our hypothetical example, the optimum solution shown in Table 3 already provides a minimum of one-shift employment during each of the three periods. The bar chart of Figure 2 compares the cost of this plan with those of two further linear programming solutions - one that assumes a work force commitment of 1.6 shifts and one of 2.0. * Interestingly enough, the total of inventory plus labor costs increases very little from a 1.0 to a 1.6 shift commitment - from \$5.94 millions to 6.34. In going from the 1.6 to the 2.0 shift level, however, some of the employees must remain idle, and a substantial cost increase occurs. The total goes up to \$7.66 millions. When presented with an explicit choice of this sort, few managements would hesitate to choose the 1.6 shift policy - despite their initial reluctance to place a dollar value on the worth of stabilizing employment.

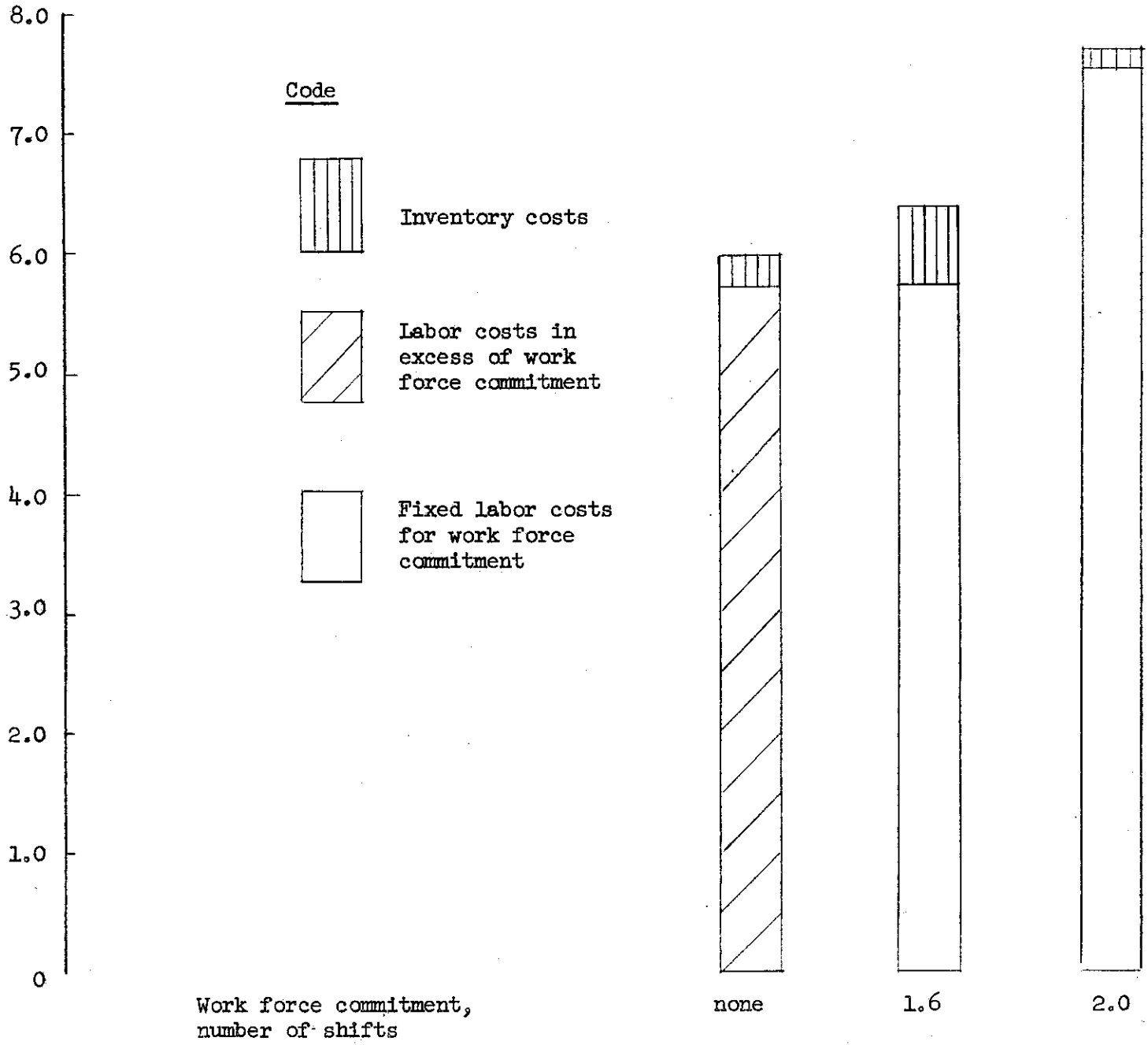
In order to obtain a linear programming solution for the cases of 1.6 and 2.0 shifts, only a few changes need be made in the constants and cost coefficients shown in Tables 2 and 3:

Work force commitment, number of shifts	none (Tables 2 and 3)	1.6	2.0
a_{1t}	100	160	0
a_{2t}	100	40	200
c_1	10	0	0
c_2	15	15	0

* Note that 1.6 shifts yield 480 units of output during the three periods - exactly the cumulated production requirements.

Three-period
costs, millions
of dollars

Figure 2



These alterations consist simply of recognizing that the effect of a work force stabilization policy is to make the marginal labor costs zero for output rates that fall short of using up the fixed labor commitment, but that the marginal costs increase discontinuously at production rates in excess of this level. Here it is apparent that a linear programming idealization approaches economic reality much more closely than a calculus model which assumes that the marginal cost curve is a continuous one. Both from the viewpoint of problem formulation and of numerical computations, the linear programming version seems to be as useable as its calculus counterpart.

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