6-1-1993

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money: Part II

Martin Shubik

Shuntian Yao

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125 Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1050

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money Part II

by

Martin Shubik and Shuntian Yao

June 1993

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money

Part II

Martin Shubik and Shuntian Yao

Abstract

We consider the relationship between the length of life of individuals and the assets they own and their influence on trustless trade. In particular in some structures a role for government or an outside bank may be called for to support an equilibrium. An example of an OLG model with production illustrates the need for expanding the fiat money supply if population growth is greater than zero.

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money
Part II

Martin Shubik and Shuntian Yao

1 Introduction

In a previous paper we have explored some of the implications of using a durable commodity money such as gold, a storable consumable, such as bricks of tea or tobacco or fiat in economies with assets of various lives. We explored models which were portrayed as either exchange economies with a finite number of time periods or otherwise infinite horizon economies with "immortal" individuals. The overlapping generations model is clearly far more satisfactory as a representation of long term economic activity. In this paper we extend the observations concerning the role of different types of money and durable assets to two sets of models. They are the sequential generations model and then the overlapping generations model. We consider the sequential generation model of trade as way to both help to justify the treatment of a finite model with a price paid for long term assets which might be left over at the end of the life of a generation and to serve as a simple introduction to overlapping generations.

2 Sequential Generations and the Government Connection

Figure 1 shows the model of sequential generations where all individuals live until age T, they are then replaced by a new (but temporally separated) generation. There is however an infinitely lived strategic dummy called the government or the referee with a given strategy announced in advance.

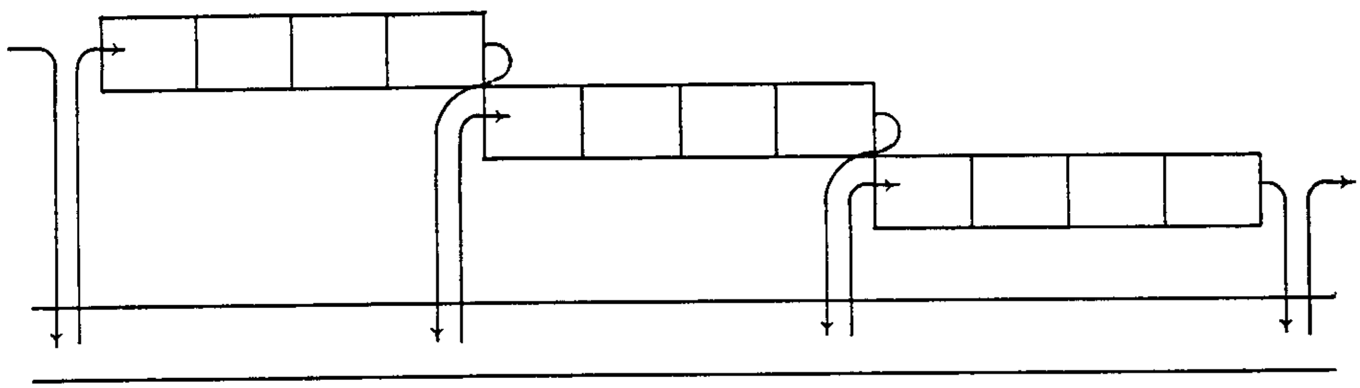


Figure 1

Sequential generations

2.1 Gold and perishables

We see immediately that if all goods are perishables and gold is money we have the same solution as in Part I, except that now, at least from the viewpoint of the government (or the experimenter) the game continues beyond T and a way of interpreting what happens to the gold at the end of T is that it reverts to the government. But if it reverts to the government we may now include as a well-defined playable game, the game where gold is used as money and the individuals begin with no gold but are able to bid for a supply from the referee, but are required to return (possibly with interest) any amount they have borrowed, or otherwise they suffer a default penalty.

Furthermore if there are durables present which may be left over we can consider that the government, as part of its strategy posts prices at which it will buy the remaining durables.

In this section we confine our remarks to perishable goods and gold.

There are two ways in which we can introduce gold. In the first, the government offers M units for sale at the start of every T periods and the traders all bid using their IOU notes for their share of this amount. The second way is to have the government specify an interest rate at which gold can be borrowed.[1] In either case, at the end of T periods we must specify how

[1]There are several modeling problems when the rate of interest is greater than zero. For the books to balance the government must buy assets, but with perishables, unless there is a lag in production the only consistent interest rate will be zero.

accounts are settled before the next game begins. One way to do this is to consider that a default penalty is imposed on any trader who fails to pay back what he owes at the end. The problems in the interpretation of this penalty when an individual must settle debts at the end of his life are noted in Section 2.4, but for now we merely consider the penalty as part of the formal rules of the game.

We might also wish to consider the possibility that some of the individuals may start with a positive amount of gold, but this can never lead to a stationary sequence of games unless we give out the initial stake of gold to each new generation.

Model 1: Endogenous interest

Suppose that the government puts up M units of gold for auction. We assume, basically for ease in calculation and exposition that the gold is linear separable in the utility functions which are of the following form:

$$(1) \qquad \phi^i = \varphi^i(x^i_{1,1}, \ldots, x^i_{m,T}) + \sum_{t=1}^{T} x^i_{m+1,t} + \mu \min[0, \text{debt}^i] \ .$$

Model 2: Exogenous interest

Instead of having the government auction off M units of gold we may consider a game where the government holds a large amount of gold and announces a rate of interest at which it will lend the gold In order to fully define the game we must bound the IOU notes which can be offered by the individuals and have a rationing rule if the amount of money demanded exceeds the supply held by the government. Rationing can be avoided by making sure that:

$$(2) \qquad M > \frac{\int_I V^i di}{(1 + \rho)}$$

where V^i = the upper bound on IOU notes which can be bid.

2.2 Gold and durables

(1) All goods are durable with life of T or less

Suppose that all goods are consumer durables with life T or less. When the generations do not overlap we obtain some somewhat pathological cases when assets are introduced, but for the sake of completeness we note them. We need to consider two cases. The first is where all endowments are received early enough that there is nothing (except the money) left over at the end. The second is where there is some array of durables left over. If an efficient stationary state is to be achieved, then the government must be an active buyer and seller of assets which overlap the generations.

Case 1: No leftovers

If a generation lasts for T periods, but the life of any durable is less than T and divides into T, no assets will be left over. The solution is similar to that given in Part I, for a finite economy, modified by the bidding for gold.

Case 2: Left over capital stock

A way of handling the possibility of there being capital stock left over without any inheritance is to have the government "make a market." This is illustrated in Figure 2 for the instance where $T = 3$ and $L = 2$.

We note that if durables of all ages up to T exist, then if a stationary state is to exist, if a good of life k enters the economy once every k years only it can only do so cyclically where the periodicity is determined by the multiplication of the relative primes of the length of life of the durables. Thus for $T = 20$, with durables of all ages less than or equal to 20, if there is a stationary economy its periodicity is once every $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 = 9{,}699{,}690$ periods. If the length of life is 70 years and durables of every age are to be considered then the periodicity for a stationary state becomes around once every 7.86×10^{24} years. With overlapping generations we can however obtain a stationary state if the same profile of goods enters each period.

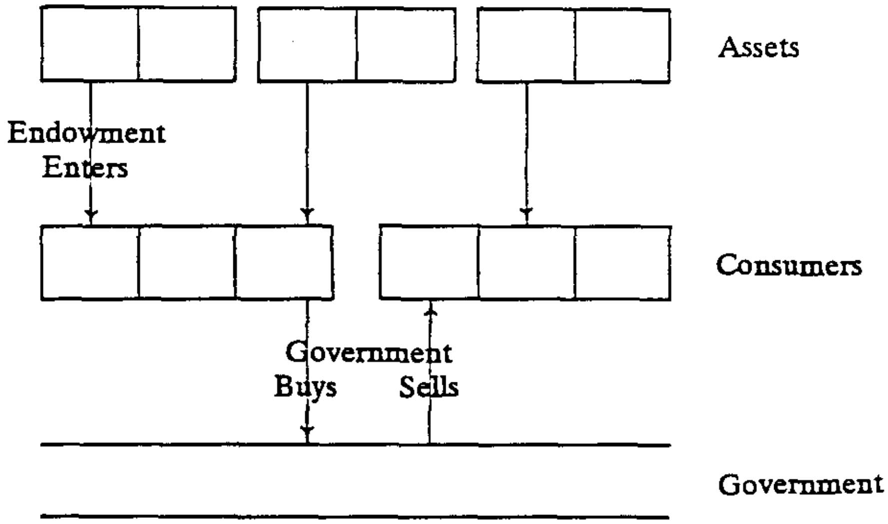


Figure 2

Consumer life 3, asset life 2

(2) All goods are durables and have lives of T or more

The situation with durables whose lives are longer than that of an individual poses no new difficulties if a government market at the end of every generation and the start of every new generation is introduced. This is illustrated in Figure 3 for the instance where $T = 3$ and $L = 4$.

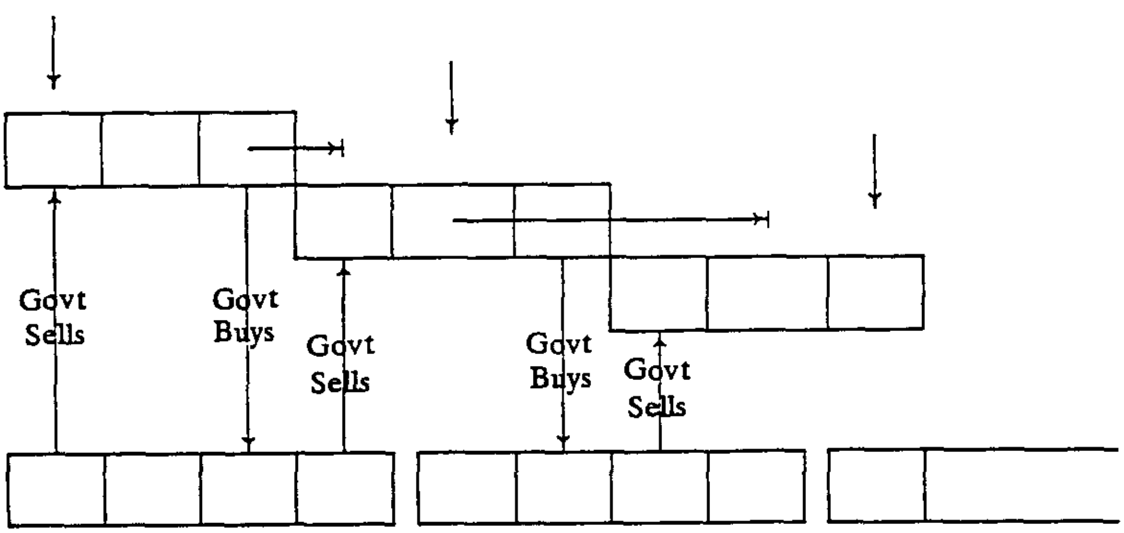


Figure 3

Consumer life 3, asset life 4

2.3 The default penalty and credit limits

If we regard our models as experimental games, then there is no difficulty in interpreting default penalties which come at the end of the game where the player settles up with the referee who is clearly in a position to adjudicate the score of the player. If however the model is meant to be a parable reflecting actual economic life, then there are some weakness to be considered. In general creditors are well aware that one cannot collect debts from the dead who leave no estates, thus there is a tendency not to lend to those who may achieve such a state, or to require security to avoid such a possibility.

Another common tendency among creditors is to place bounds on the amount of debt individuals can generate. At first glance the introduction of such a bound into a mathematical model appears to violate the desire for generality often striven for in mathematical economics. But there is generality and spurious generality. As the implicit assumptions of general equilibrium theory have infinitely harsh bankruptcy penalties there is no need to introduce lending limits as any defaulter is mathematically shot on sight. If we make the weaker assumption that default laws of a lesser harshness exist the danger appears that a scoundrel seeing a loophole will try to borrow without constraint. But in actuality unbounded borrowing does not exist and the introduction on the limit of the IOU notes an individual can issue, has a benefit both in the mathematical analysis and in the design of experimental games. In particular from the viewpoint of the mathematics it provides a condition to help maintain compactness on a strategy set. From the viewpoint of the experimental gaming, it turns the mysteries of what is meant by bank credit or by personal credit into simple physical objects with well defined quantities in a playable game. In the game, the limitations on the debtor's promises can easily be represented by a quantity of colored Poker chips which can be traded for a money, which is a physical good or a different form of Poker chip.

The bound on the issue of IOU notes is, in essence, an indication of the length of the individual's line of credit. In defining the credit game the rules must specify not merely how a would-be debtor can bid his IOU notes, but whether creditors can trade the IOU notes of others which they hold. In the modern world of finance banks may sell their loan portfolios.

2.4 Tea as money

Suppose that the government puts up M units of tea for auction. We assume general utility functions modified by the bankruptcy penalty:

$$\phi^i = \varphi^i(x^i_{1,1}, \ldots, x^i_{m+1,T}) + \mu \min[0, \text{debt}^i] \tag{3}$$

where μ is the default penalty parameter. A strategy by i is of the form:

$$\begin{pmatrix} v^i_{2,1}, u^i_{1,1}, v^i_{1,1}, u^i_{0,1}, v_{0,1}, b^i_{j,1}, q^i_{j,1}, x^i_{j,1}; u^i_{1,2}, v^i_{1,2}, \\ u^i_{0,2}, v^i_{0,2}, b^i_{j,2}, q^i_{j,2}, x^i_{j,2}; u^i_{0,3}, v^i_{0,3}, b^i_{j,3}, q^i_{j,3} \end{pmatrix} \tag{4}$$

where the first entry is the bid (in IOU notes) for tea. We require the weak condition that

$$\frac{\partial\varphi}{\partial x_{m+1,t}} / \frac{\partial\varphi}{\partial x_{j,t}} \geq \Delta \quad \text{for all } j \text{ and } t \ , \tag{5}$$

when evaluated for any amount of tea for all other consumptions.[2] In words, a unit of tea is always moderately valuable. We now consider a CE of the economy without tea. Calculate the Lagrangian multipliers at some CE for the normalization $p{\cdot}a = W$, consider a redefined utility function of the form

$$\phi^i = \varphi^i(x^i_{1,1}, \ldots, x^i_{m,T}, 0) + \max[\lambda^1, \ldots, \lambda^n]x^i_g + \mu \min[0, \text{debt}^i]$$

we then calculate the amount of the newly introduced "shadow gold" needed to serve as enough money; say this is M^*. We now set the bankruptcy penalty $\mu \geq \Delta$ and now auction off M^* units

---

[2]Note: the general requirement is

$$\frac{\partial\varphi}{\partial x_{m+1,t}} / \frac{\partial\varphi}{\partial x_{j,t}} \geq \Delta > 0 \ ,$$

the equality

$$\text{"}\ \frac{1}{p_j} = \frac{\dfrac{\partial\varphi}{\partial x_{m+1,t}}}{\dfrac{\partial\varphi}{\partial x_{j,t}}}\ \text{"}$$

holds only at CEs.

of tea. Tea is clearly a store of value in any quantity, but as it must be returned and the default penalty is sufficiently high, if its marginal utility at zero consumption is low enough it is borrowed for transactions but never consumed.

2.5 Fiat as money

Suppose that the government puts up M units of fiat for auction. Here fiat money does not appear in the utility functions, except in default. The function is of the following form:

$$\phi^i = \varphi^i(x^i_{1,1}, \ldots, x^i_{m,T}) + \mu \min[0, \text{debt}^i]$$

where μ is the default penalty parameter. A strategy by i is of exactly the same form as in 2.1:

$$\Big(v^i_{2,1}, u^i_{1,1}, v^i_{1,1}, u^i_{0,1}, v_{0,1}, b^i_{j,1}, q^i_{j,1}, x^i_{j,1}; u^i_{1,2}, v^i_{1,2}, u^i_{0,2}, v^i_{0,2}, b^i_{j,2}, q^i_{j,2}, x^i_{j,2}; u^i_{0,3}, v^i_{0,3}, b^i_{j,3}, q^i_{j,3}\Big)$$

where the first entry is the bid (in IOU notes) for fiat.

The amount of fiat offered by the referee is always sufficient, provided that the bankruptcy penalty is high enough. The rate of interest which emerges is $\rho = 0$.

2.6 Fiat, production, interest and population growth

In all of our models without production, unless there is an artifact such as a natural discount factor or there is a wedge in the price system caused by transactions costs, or a commodity money is in short supply and a positive shadow price for the value of gold appears, we tend to end up with a zero rate of interest. Matters change when production is considered. The adding of production to a fully defined OLG strategic market game even at its simplest with only two generations is somewhat complex, but fortunately an extremely simple sequential generations model serves to indicate many of the features involving the emergence of a positive rate of interest with production and growth.

Staying with the simplest possible model we consider a sequential economy where each individual lives for one period and is the replaced by another. As land lasts for ever and there is no trade with other individuals and no inheritance, if we are to look for the market or economic solution we will have to introduce government into the scheme as the agent which buys from the old and sells to the newborn.

We ask can we find an economy with a stationary set of prices and consumptions from generation to generation with a zero rate of interest and can we find one with a growth in population and a positive rate of interest. Furthermore how is the value of the money supported and how does it come into and leave the economy. Given an exogenous population growth, in what sense is there an optimal interest rate?, furthermore what influence does the interest rate have?

Two simple examples illustrate the possibility for government to control economic growth via the interest rate. In the first example we consider an economy with only labor and food. In the second example there is land, labor and food.

We assume that an individual of any generation has a utility function of the form:

$$\phi = \sqrt{\hat{x}-x} + \sqrt{B-y} + \sqrt{\hat{z}-z} + \mu \min[0, \text{debt}] \tag{9}$$

where the $\hat{x}$ and $\hat{z}$ represent purchases of land and food from the government and the B represents a given amount of labor owned by the individual.

Each individual has a production process for the production of more food with a one period time lag. It is of the form:

$$\hat{x}_{t+1} = (x_t \cdot y_t \cdot z_t)^{1/3}$$

(10) or, if no land

$$\hat{x}_{t+1} = (x_t . y_t)^{1/2}$$

CASE 1: No land is needed in production or consumption; $p = 1$ = price of food, $\rho \geq 0$ = the interest rate, a strategy for an individual is: (m, x, y) where m = amount of money borrowed; x = food used for production and y = labor used for production. Suppose that total labor = 2 when m is chosen, he can buy m units of food. Thus:

$$\max \sqrt{m-x} + \sqrt{2-y} - \mu\{0, \text{debt}\} \tag{11}$$

where debt = $(xy)^{1/2} \cdot 1 - (1+\rho)m$ when μ is large enough then at equilibrium

(12) $(xy)^{1/2} = (1+\rho)m$.

From which

(13) $$x = \frac{m^2(1+\rho)^2}{y} .$$

Thus (1) *becomes*

(14) $$\underline{\max}\ u = \sqrt{m - \frac{m^2(1+\rho)^2}{y}} + \sqrt{2-y} .$$

Therefore

(15) $$\frac{du}{dm} = \frac{1}{2}\left[m - \frac{m^2(1+\rho)^2}{y}\right]^{-1/2}\left[1 - \frac{2(1+\rho)^2 m}{y}\right] = 0$$

(16) $$\frac{du}{dy} = \frac{1}{2}\left[m - \frac{m^2(1+\rho)^2}{y}\right]^{-1/2} \cdot \frac{m^2(1+\rho)^2}{y^2} - \frac{1}{2}(2-y)^{-1/2} = 0 .$$

From (15), $\boxed{y = 2m(1+\rho)^2 .}$

Substituting into (16),

$$\left(\frac{1}{2}m\right)^{-1/2} \cdot \frac{1}{4(1+\rho)^2} = \left[2 - 2m(1+\rho)^2\right]^{-1/2}$$

$$\frac{1}{2}m \cdot 16(1+\rho)^4 = 2 - 2m(1+\rho)^2$$

(17) $$[4(1+\rho)^4 + (1+\rho)^2]m = 1$$

$$m = \frac{1}{(1+\rho)^2[1 + 4(1+\rho)^2]}$$

$$y = \frac{2}{1 + 4(1+\rho)^2} ; \quad x = \frac{1}{2(1+\rho)^2[1 + 4(1+\rho)^2]} .$$

Thus what he borrows is $1/\{(1+\rho)^2[1 + 4(1+\rho)^2]\}$; he consumes half of the food; he spends $2/\{1 + 4(1+\rho)^2\}$ of labor for production. When $\rho \to \infty$, $m \to 0$, he approaches to just consuming the leisure as it becomes too expensive to service a loan.

We see the possibility for the existence of a wide range of interest rates controlling growth.

A reasonable further question to consider is how would the interest rate be influenced by the presence of land, an infinite durable which might enter into production or both production and consumption. Our second example considers this. Now land is needed in production. We may select $p_f = p_l = 1$, since the prices of food and the use of land are the same, and their functions in production are symmetric. We assume that $\rho \geq 0$. A strategy for an agent is $(m; x, y, z)$ where m = money borrowed; x = money for food put in production and y = money for land. , thus $x = y$. z = labor utilized

Thus

$$(18)\qquad \begin{aligned} &\max \sqrt{m-2x} + \sqrt{2-z} + \mu\{0, \text{debt}\} \\ &\text{debt} = \left(x^2 z\right)^{1/3} - (1+\rho)m \end{aligned}$$

when μ is large enough, at equilibrium, debt = 0

$$\left(x^2 z\right)^{1/3} = (1+\rho)m$$

$$(19)\qquad \therefore z = \frac{m^3(1+\rho)^3}{x^2}$$

$$\therefore \varphi = \sqrt{m-2x} + \sqrt{2 - \frac{m^3(1+\rho)^3}{x^2}} .$$

$$(20)\qquad \frac{\partial\varphi}{\partial x} = \frac{1}{2}(m-2x)^{-1/2}(-2) + \frac{1}{2}\left[2 - \frac{m^3(1+\rho)^3}{x^2}\right]^{-1/2} \cdot \frac{2m^3(1+\rho)^3}{x^3} = 0$$

$$\rightarrow \left[2 - \frac{m^3(1+\rho)^3)}{x^2}\right]^{-1/2} \cdot \frac{m^3(1+\rho)^3}{x^3} = (m-2x)^{-1/2}$$

$$(21)\qquad \frac{\partial\varphi}{\partial m} = \frac{1}{2}(m-2x)^{-1/2} + \frac{1}{2}\left[2 - \frac{m^3(1+\rho)^3}{x^2}\right]^{-1/2}\left(-\frac{m^3(1+\rho)^3}{x^2}\right) = 0$$

$$\rightarrow \left[2 - \frac{m^3(1+\rho)^3)}{x^2}\right]^{-1/2}\left(\frac{m^3(1+\rho)^3}{x^2}\right) = (m-2x)^{-1/2}$$

$$(20) \div (21) \rightarrow \frac{m}{3x} = 1$$

$$\boxed{x = \frac{m}{3}} \quad \boxed{y = \frac{m}{3}}$$

$$\rightarrow \left[2 - 9m(1+\rho)^3\right]^{-1/2} \cdot 27(1+\rho)^3 = \left(\frac{m}{3}\right)^{-1/2}$$

$$\left[2 - 9m(1+\rho)^3\right]^{-1/2} \cdot \left[\frac{1}{2y^2(1+\rho)^6}\right]^{-1/2} = \left(\frac{m}{3}\right)^{-1/2}$$

$$\frac{[2 - 9m(1+\rho)^3]}{27^2(1+\rho)^6} = \frac{m}{3}$$

$$2 - 9m(1+\rho)^3 = 243(1+\rho)^6 m$$

$$m = \frac{2}{243(1+\rho)^6 + 9(1+\rho)^3}$$

$$y = x = \frac{2}{729(1+\rho)^6 + 27(1+\rho)^3}$$

$$\boxed{z = 9m(1+\rho)^3 = \frac{2(1+\rho)^3}{27(1+\rho)^6 + (1+\rho)^3}}$$

(22) $$z = \frac{2}{27(1+\rho)^3 + 1} \, .$$

The rate of interest and optimal growth?

In the analysis above the rate of interest was a control variable of the government. No question of optimality was invoked. If we wish to venture into noting the possibility for some form of intergenerational optimality and equity we can consider the meaning of an "optimal interest rate."

Suppose that each period both the population and all inputs were increases by some factor say, $(1+\alpha)$ then it is immediate that the optimal money supply becomes $(1+\alpha)M$. This very special case with scaler growth in all aspects of goods and population is consistent with Friedman's optimal growth of the money supply.

3 Overlapping Generations and the Infinite Horizon

3.1 Gold or tea money with perishable goods

Gold:

We now extend our observations to an overlapping generations economy with n different types of agents, each of whom lives for T periods. The most general type of utility function which preserves the possibility for full stationarity each period is indicated below:

Consider the following OLG model. There are m commodities, 1, ..., m, available at each period t = ... -2, -1, 0, 1, 2, ... At any period t, there are k kinds of individuals born, each kind consisting of a continuum of measure 1. Every individual lives for T periods. Every individual of kind k born at period t receives a vector of the m commodities at the beginning of periods t, ..., $t+T-1$:

$$(23) \qquad a^k(t) = a^{k1},\ a^k(t+1) = a^{k2},\ ...,\ a^k(t+T-1) = a^{kT}$$

where $a^{kt} = (a_1^{kt}, ..., a_m^{kt})$, and a_j^{kt} is the amount of commodity j he receives at the beginning of the t^{th} period of his life. Note here we use the variable inside the ( ) for *period timing*, and the second superscript for *age*. In particular, from (23) we observe that all individuals of the same kind receive the same endowment when they are of the *same age*.

Assume that the utility function of each individual of kind k born at period t is

$$(24) \qquad \varphi^k(x^k(t),\ x_{m+1}^k(t);\ ...;\ x^k(t+T-1),\ x_{m+1}^k(t+T-1))$$

where $x^k(s)$ is the perishable commodity vector he consumes at period s, $x_{m+1}^k(s)$ the gold service he receives at period s, ($s = t$, ..., $t+T-1$). Also note that (24) can be rewritten in a form with age superscripts as follows

$$(24') \qquad \varphi^k(x^{k1},\ x_{m+1}^{k1};\ ...;\ x^{kT},\ x_{m+1}^{kt})$$

where, x^{kt}, say, is the vector of perishables he consumes when he is in his t^{th} period of his life.

Now we see that we have a stationary population over different periods. Our interests are in looking for the stationary NE of the system when some sufficient amount of gold used as money is distributed in the system over the older generations.

Now let us consider the following optimization problem for a new born individual of kind k. For simplicity of argument, we assume that $T = 3$. Assume that three stationary prices are given

(25) $$p^1 = p^2 = p^3 = p$$

where the price of the asset gold has been normalized to 1. Note that in each period there is an amount G of the asset gold available in the market. We first consider the simplest problem: markets occur only at the end of each period, all the trades must be financed by the asset gold. Other more complicated payment timing situations can be discussed in a similar way.

For the p^t given in (25), we can calculate an optimal final holding for an individual at the end of each period. Let it be

(26) $$\begin{aligned} &(x^{k1}, y^{k1}_{m+1}), (k^{k2}, y^{k2}_{m+1}), (x^{k3}, y^{k3}_{m+1}) \\ &\text{S.T. } p(x^{k1} + x^{k2} + x^{k3}) \le p(a^{k1} + a^{k2} + a^{k3}) \ . \end{aligned}$$

Obviously we must always have $y^{k\,3}_{m+1} = 0$. Note that we should have the following constraints (excess demands ≤ 0):

(27) $$\begin{aligned} &\sum_k (y^{k1}_{m+1} + y^{k2}_{m+1}) \le G \ ; \\ &\sum_k x^{kt} \le \sum_k a^{kt} \ . \end{aligned}$$

Just as in the proof of the existence of CE for a one period exchange economy, we can calculate the excess demands for every commodity including gold. From the Competitive Equilibrium Existence Theorem we know that there must exist some $\hat{p}$ such that the utility of every individual i^{k1} is maximized by the following final holding pattern:

(28) $$\begin{aligned} &(\hat{x}^{k1}, \hat{y}^{k1}_{m+1}), (\hat{x}^{k2}, \hat{y}^{k2}_{m+1}), (\hat{x}^{k3}, \hat{y}^{k3}_{m+1}) \ . \\ &\qquad \hat{y}^{k,3}_{m+1} = 0 \end{aligned}$$

At the same time, the following constraints are all satisfied:

$$(29)\qquad \begin{aligned}&\sum_k (\hat{x}^{k1} + \hat{x}^{k2} + \hat{x}^{k3}) = \sum_k (a^{k1} + a^{k2} + a^{k3})\\ &\sum_k (\hat{y}^{k1}_{m+1} + \hat{y}^{k2}_{m+1}) = G \ .\end{aligned}$$

It is easy to see that the corresponding $\hat{p}$ is a stationary price vector for the OLG, and the final holding pattern by (28) is the stationary CE allocation.

Note also that if G is sufficiently large, and if loan markets are available at each period, this stationary CE allocation can always be achieved as an efficient NE. In fact, in addition to the lending or borrowing in intraperiod loan markets, the strategy an individual of kind k will play can be described as the following bid-and-offer at age t:

bidding an amount for good j :

$$b^{kt}_j = \hat{p}_j \cdot \max\{0, \hat{x}^{kt}_j - a^{kt}_j\}$$

offering an amount of good j :

$$q^{kt}_j = \max\{0, a^{kt}_j - \hat{x}^{kt}_j\} \ .$$

It is not difficult to check that this leads to $\hat{p}$ and the CE allocation.

EXAMPLE 1: For simplicity we look at an example with period-separable and gold linear separable utility functions. Assume $k = 1$, $T = 3$. And the utility function of a newborn is

$$(30)\qquad \varphi^1 = \sqrt{x^1} + y^1 + \sqrt{x^2} + y^2 + \sqrt{x^3} + y^3$$

where x^t is the amount of perishable good he consumes at age t; y^t the consumption of the services of gold at age t. Assume that the endowment for a newborn is given by

$$\left(2\frac{4}{81}, 0; 2\frac{4}{25}, 0; 0, 0\right)$$

Let the strategy for him be: sell 2 units of perishable at period 1, sell 2 units again at period 2, bid 4 units of gold for the perishable at period 3. Then at every period, we have a stationary price vector $\hat{p} = (1, 1)$.

To check that it gives an efficient NE, note that for a newborn, the optimization problem is

(31) $$\max \sqrt{2\frac{4}{81} - u} + \left(\sqrt{2\frac{4}{25} - v} + u\right) + (u + v + \sqrt{u+v}) \; .$$

Calculating the partial derivates:

$$\frac{\partial}{\partial u} : -\frac{1}{2}\left(2\frac{4}{81} - u\right)^{-1/2} + 1 + 1 + \frac{1}{2}(u+v)^{-1/2} = 0$$

$$\frac{\partial}{\partial v} : -\frac{1}{2}\left(2\frac{4}{25} - v\right)^{-1/2} + 1 + \frac{1}{2}(u+v)^{-1/2} = 0 \; .$$

It is not difficult to check that

$$(u, v) = (2, 2)$$

is the optimal solution.

Tea:

For an OLG model with tea as money, we must raise some questions before we go into the construction of the model.

The first observation is that the tea model is different from the gold model in the feature that the gold inside the system must be passed to each generation. Its services are consumed, but the asset gold lasts forever; but in a model with tea as currency, if tea is consumed, it cannot be reserved for later use. The asset gold is never consumed at an efficient equilibrium even though the oldest find no future value for gold. But in the tea model this mechanism does not work, since tea can be drunk instantly. Thus for an "inside-tea" model, unless we place some strong constraint on the marginal utility of tea, in general an efficient SNE may or may not exist.

Of course we can consider an "outside-tea" model. Every new generation is given some tea, first used for transactions and then drunk and more tea is created for each generation. When more tea is introduced and eventually drunk, we do have existence results. We do not go into detail, but illustrate the process with an example.

We recall that in the gold model, because of the "wealth" limitation of the newborn, when the asset life L is greater than the trader's life T, we in general have no efficient active SNE. But for the tea case, since the newborn are endowed with tea this limitation does not apply. Thus we can still achieve an efficient SNE.

EXAMPLE 2: We consider one durable good with $L = 4$. One type of individual with life $T = 3$, and each has a utility function

$$\varphi^i = \sqrt{x_1^i} + \sqrt{x_2^i} + \sqrt{x_3^i} + \frac{\sqrt{3}}{4}\left(\sqrt{t_1^i} + \sqrt{t_2^i} + \sqrt{t_3^i}\right) \tag{32}$$

where x_t^i is the asset service he receives at period t, and t_t^i is the tea he consumes at period t. Assume that each newborn is endowed with 1 unit of the new asset when he is born; and receives in each period of his life 2/3 units of tea.

It is not difficult to check that the following gives a stationary CE of the system:

$$p = (1; 1)$$

(where the *service price* of the asset is always 1).

Young	Middle-aged	Old
$(1, 0, \frac{1}{3}, 0; 0)$	$(0, 1, 0, \frac{1}{3}; 0)$	$(0, 0, \frac{2}{3}, \frac{2}{3}; 2)$
	↑	↑
	$\frac{2}{3}$ - service	service only from middle aged

It is easy to check that all the markets are cleared. Every individual receives 4/3 of a unit of service of the durable at each period, and he does not drink the tea until the end of the last period.

This CE can be achieved by the following strategies:

	Young	Mid-Aged	Old
Period 1 begins	$(1, 0, 0, 0; \frac{2}{3})$	$(0, 1, 0, \frac{1}{3}; \frac{2}{3})$	$(0, 0, 1, \frac{2}{3}, \frac{2}{3})$
	buy $\frac{1}{3}$ units of 3-year old asset	buy $\frac{2}{3}$ units of 3-year old asset without service	sell $\frac{1}{3}$ units of 3 year old asset; sell $\frac{2}{3}$ units of 3-year old asset without service
Period 1 ends	$(1, 0, \frac{1}{3}, 0; 0)$	$(0, 1, 0, \frac{1}{3}; 0)$	$(0, 0, \frac{2}{3}, \frac{2}{3}; 2)$
	|	|	
Period 2 begins	$(0, 1, 0, \frac{1}{3}; \frac{2}{3})$	$(0, 0, 1, \frac{2}{3}; 0)$	

By an argument similar to that given in Section 2, if the marginal utilities are right we can show that an "almost efficient" SNE can be constructed where tea is used for currency and is never drunk, thus there is a small finite loss of utility from the undrunk tea. but the loss relative to the infinite horizon is arbitrarily small.

3.2 Gold and durables

We have considered an overlapping generations economy with gold as an infinitely lived durable and all other commodities are perishables. A new set of features appear when the goods are durables. In all instances gold, which is the money, has an infinite life without depreciation.

Consider the situation with durable commodities. We still assume that T is the length of life for each individual. Let L be the length of the life of the durable assets. We assume that each newborn is endowed with a vector of new asset a^k, but only once in his life. When $L \le T$, we have a situation similar to the prior analysis and the SNE existence theorem remains. But when $L > T$, we cannot always have the existence of an SNE. This can be illustrated with a simple example.

EXAMPLE 1: Let $T = 2$, $L = 4$. Let there be one kind of asset in the system. Thus at each period, there are the assets of 1, 2, 3 or 4 years old, and the amount of asset at each age is the same. According to our assumption, the old generation owns all the older assets (2, 3 and 4 years old). In order to achieve an efficient stationary equilibrium, the asset of 2 and 3 years old should be sold to the first generation. But even without selling the last period services, they are worth at least 2 units of "one period" services to the newborn. On the other hand, the old cannot buy more than one unit of services from the young. Thus if there were an SNE there must be some gold flowing from the young to the old. But then this amount of gold must be wasted by the old who die without having the opportunity to sell it.

We now consider the conditions under which an SNE will exist. In order to state our general results, we first given some definitions.

DEFINITION 1: An efficient stationary allocation is said to be asset-holding stationary, if in this allocation, at each period every individual has total ownership of that class of all the assets he holds.

To explain this concept, let us examine the following example.

EXAMPLE 2: There is one kind of individual, i.e. $k = 1$ and $T = 3, L = 2$. One unit of the new asset is given to the newborn; one unit of gold is owned by the mid-aged; and one unit of the old asset is owned by the old. Markets are available at the end of each period.

	Young		Mid-aged		Old
	(1, 0, 0; 0)		(0, 0, 0; 1)		(0, 1, 0; 0)
sell 1 unit new asset by the end of the period	next period	buy 1 unit of asset at the end of period 1	next period		do nothing
Holding at next period	(0, 0, 0; 1)		(0, 1, 0; 0)		(0, 0, 0; 0)

DEFINITION 2: An efficient stationary outcome is said to be asset-service consuming stationary if the consumption of asset service for the same kind of individuals at their age is stationary.

Note that asset-holding stationary implies asset-service consuming stationary, but not vice versa.

PROPOSITION 1: In any *efficient stationary* equilibrium with *all markets active*, if the price of one unit of service of asset j is p_j, then an asset j which can still last for ℓ periods must have price $\ell \cdot p_j$.

It is trivial when $\ell \leq T$. Now consider $\ell > T$. Take $T = 3$, $\ell = 4$ for example. If the price of this asset is *less than* $4p_j$, no individual would prefer buying the same asset which can last for 3 years at a price $3p_j$, because he had better buy a 4 period lasting asset j in price less than $4p_j$ and sell one unit of service. By an induction process we come to the conclusion.

THEOREM 1: In the OLG model with $L > T$, there can never exist an efficient active stationary asset-holding equilibrium.

PROOF: We use backward induction. Assume that *after trading* at some period, the distribution of assets over different generations is given by

Generations	Assets					
	Aged-1	Aged-2	Aged-3	...	...	Aged-L
Aged-1						
Aged-2						
⋮						
T-1	a^{1T-1}	a^{2T-1}	...	...	$a^{L-1,T-1}$	$a^{L,T-1}$
Aged-T	a^{1T}	a^{2T}	...	...	...	$a^{L,T}$

First since this is an *efficient* asset-holding stationary equilibrium, we must have

$$a^{\ell T} = 0\,,\ \ \ell = 1, ..., L-1\ .$$

Otherwise some resources have to be wasted. Thus the total *asset* wealth for the aged-T generation is only $a^{LT}\cdot p = W_T \le$ total wealth of the aged-L assets, say W_L.

Now look at the total wealth for the aged-($T-1$) generation. After trade, he may have assets of different ages. All those of age $\ell < L-1$ will pass to the next period when he becomes aged-T. These assets are still younger than aged-L, which can be only used in exchange for aged-L assets (through gold), since we are looking at an "*asset-holding*" equilibrium. In fact, all the $a^{1,T-1}$, $a^{2,T-1}$, ..., $a^{L-1,T-1}$ passed to next period contribute to *part* of the a^{LT}. Thus the wealth of those for aged-($T-1$) is not greater than $2a^{LT}\cdot p$. And the total wealth of the ($T-1$) generation is not greater than $2a^{LT}\cdot p + a^{L,T-1}\cdot p \le 2(a^{LT} + a^{L,T-1})\cdot p \le 2W_L$.

By mathematical induction, the total wealth of the aged-($T-$t) generation cannot be greater than tW_L. But then the total wealth owned by all older generations from aged-2 to aged-T cannot be greater than

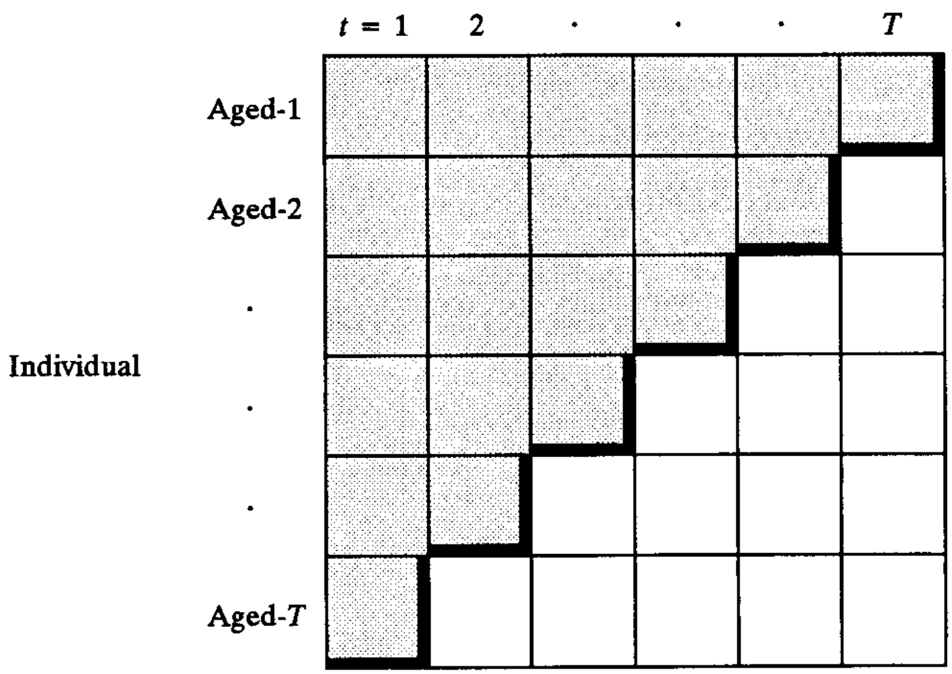
$$(33)\qquad W_L + 2W_L + \cdots + (T-1)W_L = \frac{T(T-1)}{2}W_L .$$

On the other hand, the total asset wealth in the system should be

$$(34)\qquad (L-1)W_L + (L-2)W_L + \cdots + W_L = \frac{L(L-1)}{2}W_L > \frac{T(T-1)}{2}W_L .$$

Thus any "asset-holding" stationary equilibrium can never be sufficient when $L > T$.

THEOREM 2: If $L \geq 2T$, then in the OLG model we described above, there never exists any efficient active stationary asset-consuming equilibrium.



PROOF: Consider some period $t = 1$. The asset endowments for those older generations are: a^k of aged-2; a^k of aged-3, ...; a^k of aged-L. Let p be the price vector for a^k of those aged-L. Then the total initial asset wealth owned by the older generation is, according to Proposition 1:

$$(35)\qquad \sum_k (a^k \cdot (L-1)p + \cdots + a^k \cdot p) = \frac{1}{2}(L-1)L\sum_k a^k \cdot p$$

Among those older generations, the largest total asset services received in the next $(T-1)$ periods must be of generation aged-2. Thus his total asset services should be not less than

$$(36)\qquad \frac{1}{2(T-1)}L(L-1)\sum_k a^k \cdot p .$$

Now the total services for the newborn in his life time should be not less than (35). But his total initial wealth is only

(37) $$\sum a^k \cdot Lp = L\sum_k a^k \cdot p \ .$$

Thus we cannot have an efficient active SNE unless

(38) $$L\sum_k a^k \cdot p > \frac{1}{2(T-1)}L(L-1)\sum_k a^k \cdot p$$

which is equivalent to

(39) $$\frac{L-1}{T-1} < 2\ , \quad \text{or} \quad L < 2T-1\ . \qquad \text{Q.E.D.}$$

It is interesting to note that sometimes when an efficient active SNE does not exist, we can calculate some non-efficient SNE.

In our model with asset life of L and people with life of T, we know that the SNE existence in general is not true when $L > T$. In this more complicated case with different asset lives ℓ_1, ..., ℓ_m, it is reasonable to confine our attention to the situation where

(1) $$\max\{\ell_1, ..., \ell_m\} = L \le T\ .$$

To get some intuition, we first examine a simple example with some characteristic features.

EXAMPLE 1: There is one type of individual (this can be easily extended to two types), $T = 3$. There are three goods, of which the lives are $\ell_1 = 3$, $\ell_2 = 2$, $\ell_3 = 1$. Thus the last good can be regarded as perishable. Every newborn individual receives *only in the first period* 5/32 units of good 1, 5/16 units of good 2, and he receives in every period of his life, 1/4 units of good 3. The utility function of any individual is

(2) $$\varphi^i = \sum_{t=1}^{3}\left(\sqrt{x_t^i} + \sqrt{y_t^i} + \sqrt{z_t^i} + g_t^i\right)$$

where x_t^i, y_t^i, z_t^i and g_t^i are the consumptions of asset services of goods 1 and 2, the consumption of the perishable, and the service of gold in period t.

A feature of this endowment pattern is that if there were no trade or gold in the system. Instead of cyclical with period 6, it is stationary:

$((1,1)^1; 1)$,	$((1,1)^2, 1)$,	$((1,0)^3, 1)$,			
	$((1,1)^1, 1)$,	$((1,1)^2, 1)$,	$((1,0)^3, 1)$,		
		$((1,1)^1, 1)$,	$((1,1)^2, 1)$,	$((1,0)^3, 1)$,	
			$((1,1)^1, 1)$,	$((1,1)^2, 1)$,	$((1,0)^3, 1)$

where $(1, 1)^1 = (5/32, 5/16)$ of aged-1 asset vector;

$(1, 1)^2 = (5/32, 5/16)$ of aged-2 asset vector;

$(1, 0)^3 = (5/32, 0)$ of aged-3 asset vector;

The 3rd $1 = 1/4$ units of perishable.

Therefore it is reasonable to expect to have a stationary CE. In fact, we can check that the following gives an SCE, if there are markets available only at the end of the first two periods.

Price vector (in asset gold): (1, 1, 1; 1) allocation of services and consumption:

Young	Mid-aged	Old
(5/32, 5/16, 1/4; 0)	(1/16, 1/16, 1/4; 11/32)	(1/4, 1/4, 1/4; 0)

In fact, while the marginal utility from the perishable exactly equals to that from gold service, the decision for an individual is to choose u, v, s, t such that

$$\underline{\max} \sqrt{\frac{5}{32}} + \sqrt{\frac{5}{16}} + \sqrt{1/4}$$

$$+ \sqrt{\frac{5}{32} - u} + \sqrt{\frac{5}{16} - v} + \sqrt{\frac{1}{4}} + u + v$$

$$+ \sqrt{\frac{5}{32} + s} + \sqrt{t} + \sqrt{\frac{1}{4}} + u + v = s - t$$

where u = amount of good 1 service sold at end of period 1;

v = amount of good 2 service sold at end of period 2;

s = amount of good 1 service bought at end of period 1;

t = amount of good 1 service bought at end of period 2.

The constraint is

(3) $$\frac{5}{32}\cdot 1 + \frac{5}{16}\cdot 1 + \frac{1}{4}\cdot 1 + \left(\frac{5}{32} - u\right)\cdot 1 + \left(\frac{5}{16} - v\right)\cdot 1 + \frac{1}{4}\cdot 1 + \left(\frac{5}{32} + s\right)\cdot 1$$
$$+ t\cdot 1 + \frac{1}{4}\cdot 1 \le \frac{5}{32}\cdot 3 + \frac{5}{16}\cdot 2 + 3\cdot\frac{1}{4}\cdot 1$$

and

(4) $$u+v-s-t \ge 0\,.$$

It is easy to heck by substitution that

(5) $$u = \frac{3}{32}\,,\quad v = \frac{1}{4}\,,\quad s = \frac{3}{32}\,,\quad t = \frac{1}{4}$$

gives a unique solution to (3) and (4). And (5) leads to the consumption pattern in (*). It is also easy to check that, under the strategies (u, v, s, t), the stationary price-vector (1, 1, 1; 1) is always market clearing for each period. Thus if at some beginning there is an amount 11/32 of gold in the hands of the middle-aged generation, the CE can also be achieved as an SNE.

In general, in an OLG economy with assets of life lengths ℓ_1, ℓ_2, ..., ℓ_m, where $\max_j\{\ell_j\}$ $= L \le T$, if there are all kinds of markets available, the existence of an efficient SNE still remains true. The proof is basically the same as the previous one.

3.3 Bequests and inheritance

In this paper we have purposefully made the counterfactual assumptions that individuals do not identify with their offspring and transfer resources to them. Biologically, this clearly flies in the face of the facts on Nature and nurture. But in order to explore the possibilities of apparently cooperative or mutually beneficial behavior in a world with mass anonymous purely individualistic selfish behavior this exercise cuts out other variables and enables us to see where "rules of the game" must appear to guide the process. These rules may be interpreted as the elemental aspects of government, society and law required for the loveless anonymous mass economy.

3.4 The necessity of taxation for optimality

In the models considered here we have considered the device of having various individuals owe the government for the funds they have borrowed to purchase capital stock or to finance transactions. Neither of these parables appear to fit closely to the facts. In general it is the government who owes the people for the national debt, not vice-versa. That is so if we leave out taxation. But in the world we inhabit the "take" of a government in the form of various taxes is anywhere from a few percent up to 40-60% of the GNP. Furthermore individuals tend to pay taxes and relatively few die deeply in debt. Thus it appears that government and its policies to provide a binding device which can provide the context for overall efficiency in an environment inhabited by finitely lived selfish individuals

It is relatively easy to construct special cases with OLG where government is necessary if the generations are to be connected in a manner such that they can achieve joint optimality.

4 Overlapping Generations with Production

In our models above the resources are assumed to rain in exogenously from the outside. A way more consistent with human experience is that each individual is provided with endowments of labor and that some resources (such as sunlight and air) are provided exogenously. Otherwise, for the resources such as the durables discussed above are the results of production. In this paper we do not attempt to deal with the complexities of production in full but we do explore some simple examples.

In particular we concentrate on the two period OLG model with tea (i.e. a storable consumable) as money, with labor as the perishable good, land as an infinite durable whose services are used both in production and consumption and food as a produced consumable which can be used for consumption or further production.

There are many variations, even to models at this level of simplicity, hence in an attempt to construct a playable game we indicate the choices made and the possible implications of some of the alternatives, prior to developing the notation.

Main purpose: To construct a playable strategic market game with a commodity money, overlapping generations, production and a default rule to examine the need for government or an outside economic agency to adjust the money supply and to consider the money rate of interest in a stationary or a growth economy.

Agents and their preferences: The agents live for two periods. Their preferences are defined over all consumables and services and the commodity money and can be represented by smooth utility functions. They have no direct preferences for bequests and have a disutility to being punished for default. If an agent dies with an owned good left over, a societal rule is required to specify its disposal.

The birth process is exogenously given and there is no identification between parent and child.

Labor: Each agent at each period is endowed with one unit of a perishable service called labor/leisure. An agent cannot consume more than one unit of leisure, but can hire labor of others when producing. The asset, "human capital" cannot be traded, only its services can be sold.

Food: Food is produced from land, labor and food. An important modeling choice must be made in the description of production. For simplicity we assume that all production is individual (a world with small farmers). The reason for doing this is to avoid the conceptual difficulties introduced by postulating jointly owned firms. An immediate consequence is that if production functions are identical for identical individuals then production will be homogenous of order 1 if the number of individuals and resources grow together.

Production takes time. The simplest model has production take one period. In this instance this implies that the old will not produce, as their product would be available after death. Food can be consumed or used in production. It is possible that it can be stored. In the first model, for simplicity we assume that it cannot be stored.

Land: Land is an infinite nondepreciating durable which provides a stream of services. It exists in finite (but large quantities) and is owned by the agents. Its services can be traded, or the asset can be traded.

Money: Money is first assumed fiat and then to be a storable consumable, such as bricks of tea which are assumed to not depreciate at all and hence can be stored indefinitely. The government is assumed to have a large store of money of money which it is prepared lend. We must also consider the possibility of starting individuals with initial endowments of unborrowed money.

Government: The government is a strategic dummy. In the first model we consider the government trading only in money. But there are neither purely logical nor historical reasons to rule out the possibility that it also trades in land or other goods.

The meaning of being a strategic dummy is that its strategy is given in advance to all other players hence there is no need to consider its preference or choice structure or details over the domain of its strategy set.

The extensive form of the game, for two periods is such that: first the individuals finance, then trade resources, then consume, after which the young may produce. Production is not delivered until the next period, after the refinancing. Trade then takes place, individuals consume and then the next young may produce. Births are at the start of period t and deaths at the end of period $t+1$.

Consider an OLG with production, where each individual lives for 2 periods, and the population grows at the rate of α in every period. An individual is born with 1 units of labor as endowment but nothing else. He can borrow any amount m of paper money from the government to buy food and labor from the old generation, then produce food. When he is old he can sell part of his food and labor to the young generation, and return the debt to the government with some interest. We want to find the stationary efficient equilibrium with α as the interest rate.

Let m be the amount a young individual borrows. Let b be the amount of food he buys to eat under the price p (assume the price of labor is 1). Let x be the food for production. Let y_1 be the amount of labor of his used for production. Let z be the amount of labor bought from the old generation. Let c be the amount of food this individual consumes when he is old, and y_2 the labor he wants to sell when old.

The object of his is to maximize his utility on food and leisure:

(1) $$\underline{\max}\ \sqrt{b} + \sqrt{2 - y_1} + \sqrt{c} + \sqrt{2 - y_2}$$

when the punishment against bankruptcy is harsh enough, at the SNE he should have exactly enough money to pay back the debt and interests. Assume his production function is

(2) $$2\sqrt{x(y_1 + z)}$$

since his spending in period 1 should be exactly

(3) $$m = (b+x)p + z$$

And his income in period 2 is

(4) $$\left[2\sqrt{x(y_1 + x)} - c\right]p + y_2$$

We must have

(5) $$\left[2\sqrt{x(y_1 + z)} - c\right]p + y_2 = (1+\rho)m = (1+\rho)[(b+x)p + z]$$

from which one obtains

(6) $$2 - y_2 = 2 + \left[2\sqrt{x(y_1 + z)} - c\right]p - (1+\rho)[(b+x)p + z]$$

Thus (1) can be rewritten as

(7) $$\max \pi = \varphi(b, x, y_1, c, z, y_2)$$

$$= \sqrt{b} + \sqrt{2-y_1} + \sqrt{c} + \sqrt{2 + \left[2\sqrt{x(y_1+z)} - c\right]p - (1+\rho)[(b+x)p + z]}$$

Let □ be the value of $2 - y_2$ at the stationary state. From (7), we calculate

(8) $$\frac{\partial\pi}{\partial x} = 0 \rightarrow x^{-1/2}(y_1 + z)^{1/2}p = (1+\rho)p\ ;$$

$$\frac{\partial\pi}{\partial z} = 0 \rightarrow x^{1/2}(y_1 + z)^{-1/2}p = (1+\rho)\ .$$

Thus $(y_1 + z) = px$ (at SNE) and $p = (1+\rho)^2$. Note that at SNE, $z = y_2/(1+\rho)$. We then have

(9) $$\left(y_1 + \frac{y_2}{1+\rho}\right) = px \ , \quad p = (1+\rho)^2$$

From (7), we calculate

(10) $$\frac{\partial\pi}{\partial b} = 0 \rightarrow \frac{1}{2}b^{-1/2} = \frac{1}{2}\square^{-1/2}(1+\rho)p$$

which implies

(11) $$b = \frac{\square}{(1+\rho)^2 p^2} = \frac{\square}{(1+\rho)^6}$$

similarly we can get

(12) $$c = \frac{\square}{p^2} = \frac{\square}{(1+\rho)^4}$$

Finally we have

(13) $$\frac{\partial\pi}{\partial y_1} = 0 \Rightarrow \frac{1}{2}(2-y_1)^{-1/2} = \frac{1}{2}\square^{-1/2}x^{1/2}(y_1+z)^{-1/2}p = \frac{1}{2}\square^{-1/2}(1+\rho)$$

which gives

(14) $$y_1 = 2 - \frac{\square}{(1+\rho)^2}$$

By the definition of $\square$ we have

(15) $$y_2 = 2-\square \ , \quad z = \frac{y_2}{1+\rho} = \frac{2-\square}{1+\rho}$$

Now we see

(16) $$y_1 + z = 2 - \frac{\square}{(1+\rho)^2} + \frac{2-\square}{(1+\rho)}$$

$$= (1-\rho)^{-2}\{2(1+\rho)^2 + 2(1+\rho) - \square(1+\rho)\}$$

$$= (1+\rho)^{-2}[2(1+\rho)^2 + 2(1+\rho) - \square(1+\rho) - \square]$$

Hence from (9)

(17) $$x = \frac{y_1 + z}{p} = (1+\rho)^{-4}[2(1+\rho)^2 + 2(1+\rho) - \Box(1+\rho) - \Box]$$

So at SNE

(18) $$2\sqrt{x(y_1+z)} = 2(1+\rho)^{-3}[2(1+\rho)^2 + 2(1+\rho) - \Box(1+\rho) - \Box]$$

Note that amount produced = amount consumed + amount sold

$$\therefore\ 2(1+\rho)^{-3}[2(1+\rho)^2 + 2(1+\rho) - \Box(1+\rho) - \Box]$$

$$= \frac{\Box}{(1+\rho)^4} + \left\{\frac{\Box}{(1+\alpha)^6} + (1+\rho)^{-4}[2(1+\rho)^2 + 2(1+\rho) - \Box(1+\rho) - \Box]\right\}(1+\rho)$$

$$\therefore\ (1+\rho)^{-3}[2(1+\rho)^2 + 2(1+\rho) - \Box(1+\rho) - \Box]$$

$$\frac{\Box}{(1+\rho)^4} + \frac{\Box}{(1+\rho)^5}$$

$$\Rightarrow\ 2(1+\rho)^4 + 2(1+\rho)^3 - \Box(1+\rho)^3 - \Box(1+\rho)^2 = \Box(1+\alpha) + \Box$$

$$\therefore\ \Box[(1+\alpha)^3 + (1+\alpha)^2 + (1+\alpha) + 1] = 2(1+\rho)^3[1 + (1+\rho)]$$

From which for $\alpha = \rho$

(19) $$\Box = \frac{2(1+\rho)^3}{1 + (1+\rho)^2}$$

so we get

(20) $$b = \frac{2}{(1+\rho)^3[1 + (1+\rho)^2]}\ ;\ x = (1+\rho)^{-4}\left[2(1+\rho)^2 + 2(1+\rho) - \frac{2(1+\rho)^4 + 2(1+\rho)}{1 + (1+\rho)^2}\right];$$

$$c = \frac{2}{(1+\rho)[1 + (1+\rho)^2]}\ ;\ y+1 = 2 - \frac{2(1+\rho)}{1 + (1+\rho)^2}\ ;\ y_2 = 2 - \frac{2(1+\rho)^3}{1 + (1+\rho)^3}\ ;$$

$$z = \frac{2 - \dfrac{2(1+\rho)^3}{1 + (1+\rho)^2}}{1+\rho}\ ;\ m = (b + y_1)p + z$$

It is easy to see when $\rho = 0$, $b = c = y_1 = y_2 = 1, x = 2, m = 4$.

We may observe several features from this model. Not only does there exist a stationary equilibrium with a rate of interest of zero, but the rate of interest is a control variable. If it is set at other than zero, as in Section 2.6 a different solution will be obtained.

5 Concluding Remarks

5.1 The influence of uncertainty

As soon as uncertainty is introduced it is feasible that for any reasonable definition of equilibrium there will be active default (see for example, Dubey, Geanakoplos and Shubik, 1988) and the role of the default penalty becomes critical. Furthermore the definition of what is meant by optimal no longer is covered by simple Pareto optimality.

5.2 The time structure of the rate of interest

With the presence of exogenous uncertainty with incomplete markets and with goods which may last for several periods, without even having to invoke subjective expectations the conditions for the emergence of a time structure of the rate of interest are present. In particular time conditions on preferences, endowments and production lengths may all conspire to make it desirable for individuals to borrow for two or more periods, but with exogenous uncertainty the two period rate of interest will no longer be the root of the product of the two one period rates as the default probabilities will differ.

5.3 Money, trust, love and institutions

In carrying out this exercise our prime stress is to indicate some of the implications to be drawn from considering both the genrational aspects of individuals andand assets. We also wishs to show where laws or institutions emerge as logical necessities of well-defining a playable game. We further stress that as empirical work in political-economy in general involves the study of situations involving many confounding factors, the approach suggested here is that playable games which could be used experimentally in a gaming laboratory may provide as good, if not a better source of the testing of the glimmerings of an economic science than appeals to "proofs" by correlations (but usually little causality) in large econometric models replete with dummy variables and the many implicit assumptions which must creep into any political-economic model with many variables.

Reference

Dubey, P., J. Geanakoplos and M. Shubik, 1988. Bankruptcy and efficiency in a general equilibrium model with incomplete markets, Cowles Foundation Discussion Paper No. 879, Yale University.