View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Yale University

Yale University

EliScholar – A Digital Platform for Scholarly Publishing at Yale

Cowles Foundation Discussion Papers

Cowles Foundation

6-1-1993

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money: Part I

Martin Shubik

Shuntian Yao

Follow this and additional works at: https://elischolar.library.yale.edu/cowles-discussion-paper-series

Part of the Economics Commons

Recommended Citation

Shubik, Martin and Yao, Shuntian, "The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money: Part I" (1993). *Cowles Foundation Discussion Papers*. 1289.
https://elischolar.library.yale.edu/cowles-discussion-paper-series/1289

This Discussion Paper is brought to you for free and open access by the Cowles Foundation at EliScholar – A Digital Platform for Scholarly Publishing at Yale. It has been accepted for inclusion in Cowles Foundation Discussion Papers by an authorized administrator of EliScholar – A Digital Platform for Scholarly Publishing at Yale. For more information, please contact elischolar@yale.edu.

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1046

Note: Cowles Foundation Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. Requests for single copies of a Paper will be filled by the Cowles Foundation within the limits of the supply. References in publications to Discussion Papers (other than mere acknowledgment by a writer that he has access to such unpublished material) should be cleared with the author to protect the tentative character of these papers.

THE MONEY RATE OF INTEREST AND THE INFLUENCE OF ASSETS IN A MULTISTAGE ECONOMY WITH GOLD OR PAPER MONEY
PART I

Martin Shubik and Shuntian Yao

June 1993

The Money Rate of Interest and the Influence of Assets in a Multistage Economy with Gold or Paper Money

Part I

Martin Shubik and Shuntian Yao

1. Introduction

1.1 General context

In several previous papers the meaning was studied of there being enough money to support trustless trade in a market involving one period of exchange. Dubey and Shubik (1978), Dubey and Shapley (1978, 1992) and Shubik (1991) have explored the two aspects of using a commodity money in a one period exchange economy. There must be "enough money" and it must be distributed appropriately among all traders.

A commodity money is differentiated in trade from other commodities by its use as a means of payment. In one period trade, in essence, the use of money is for transactions or intratemporal trade. As long as the commodity satisfies the double coincidence of wants, i.e. it is desired as an item of value by all traders; then if its marginal value relative to any other commodity is bounded from below:

$$(1) \qquad \frac{1}{p_j} = \frac{(\partial \varphi_i / \partial x^i_{m+1})}{\partial \varphi_i / \partial x^i_j} > \Delta$$

for all endowments and for all j, then if there is enough gold or commodity money all trade will be on an immediate value for value basis. Thus we may interpret the one period requirement that all pay in cash as a condition for anonymous trustless trade.

In an economy involving only one period, consumable goods are not distinguished from storable consumables (such as tobacco or tins of food) or from consumer durables such as houses, consumer appliances, automobiles, clothes or jewelry. When two or more periods are to be considered the distinctions between assets and their services, between stocks and flows may need to be made.

In a paper studying two period trade, Dubey, Geanakoplos and Shubik (1992) were able to establish that generically, in an economy involving more than one time period and more than one commodity other than gold, if gold or any other durable is used as money, even if it is in sufficient supply, it cannot be well distributed. This means that it is not possible to achieve efficient trade without introducing loan markets.

The formal statement of the theorem is as follows:

THEOREM 1 (DGS): *Let money be gold-like. Then, for generic utility functions $u \in U$, any equilibrium is inefficient.*

A simple interpretation of an economy which uses a commodity money for all trades of goods, but also has loan markets is that there has been a division of labor between trade and the assessment of trust. All trade involving physical commodities is carried out for cash. But there are special financial loan markets where cash can be loaned or borrowed in exchange for IOU notes. These markets involve both knowledge and force. Credit evaluation takes place, assets are often put up as security and bankruptcy laws are used to limit the possibilities for obtaining strategic advantage by defaulting on debt. In this paper we do not go into the detail of credit assessment, but merely make explicit a simple bankruptcy rule which describes both the punishment of the debtors and the nature of the losses of the creditors.

An excuse for this simplification at this point is that without the presence of exogenous uncertainty we expect to be able to examine equilibria where efficient trade is achieved without active bankruptcy. If there were exogenous uncertainty present we would expect to find active bankruptcy and the details of what happens would become more important to the examination of the solution.

A modern economy uses paper not gold for most or all transactions. Furthermore it has large volumes of credit much of which is secured not merely by gold but by other durable assets. In actuality, especially when paper is to be issued against other assets, there is an intermediary such as a bank which issues one form of standard paper (such as a banknote or check) in exchange for the security of assets such as gold, jewelry, machinery, automobiles, buildings and inventories.

The actual process of contract drawing, evaluating, confirming the existence and title to assets involves institutional detail which differentiates mortgage markets from consumer durable financing and from many other specialized loan markets and procedures. We avoid institutional detail here, but point out the fundamental difference between using one or more assets to secure loans. When a loan is denominated in gold to be paid back in gold a futures price, the gold rate of interest is fixed. When a loan is backed by a collection of assets their relative prices may change from period to period and hence no simple one dimensional measure that is invariant is available. This does not mean that it is impossible to have 100% asset backed loans. It does mean that for this to be the case in all eventualities we require some form of bounding over the domain of disequilibrium prices. This is consistent with lending practices in many instances. Thus when a banker will only grant a 50% loan on the market value of some asset this takes into account that in a possible disequilibrium he will still be covered with a 50% fall in price and that this may be regarded as a highly low probability event.

1.2 Categories of goods

There are many different taxonomical schemes we could consider in categorizing all goods noting features such as moveability, durability, fungibility and so forth. Here we concentrate only on three divisions. They are (1) perishable consumables; (2) storable consumables and (3) durables. For the purposes at hand here a perishable consumable yields its value on consumption. As a first order approximation we can consider that consumption requires only a point of time. A perishable is either consumed in the current period or its value is lost. A perfect storable consumable can be saved from period to period at no cost, but provides its services when consumed. A perfect durable may be regarded as providing a constant stream of services without depreciation[1]. Gold and land are possibly the two closest approximations to this type of durable.

---

[1]We could also consider a storable durable. For example one could have bought a 1933 Cadillac, put it up on blocks and started to use it "as good as new" in 1990.

If a durable is used as a money, it is reasonable to expect that while it is being used in transactions, its value as a consumer durable is lost. Thus we should note this loss. As a first order approximation we might regard this loss as minor, hence if markets meet near the end of the period we can use the convention that the original owner of gold during any period is credited with the consumer services of that period for all of the gold he holds at the start. This is equivalent to the idea that any durable sold is sold *ex dividend*. If we place the markets near the start of the period we can use the convention that the seller receives the consumption value of the gold and the buyer the consumption value of the asset. Basically every asset during a single period of time is split into its present flow and its future stock value. Thus there are two spot markets for it, a market for its service and a market for the asset without its current service.

In our first models as we consider only a finite horizon we must make the assumption that although gold might last for ever, its value is zero after the economy has ended.

1.3 Why a three period model

The two period model of an economy clearly provides the first step in any attempt to investigate multiperiod models of economic activity. But there are several phenomena which cannot be captured until a three stage model is considered. In particular it could be that the need for money varies from period to period, but with only two periods this need can only fall or rise. With three periods all of the problems in the need to vary the money supply arise.

Another reason considering the three period model is that long and short term loans can be distinguished and the need for them is made clear. Furthermore if exogenous uncertainty is considered then a term structure of the interest rates emerges. We do not deal with this phenomenon in this paper; but regard this as a preliminary exploration which must be done prior to considering this extra complication.

When the phenomenon under consideration can be examined with a two period model it is easier to do so than to expand to three periods. In sections 3 and 4 most of the observations can be gleaned from two period models.

2. The Three Period Model with Perishable Commodities and Gold

2.1 The basic structure

We consider a three period economy with n types of traders, where all trade of goods is carried out in spot markets in exchange for gold. There are m perishables available each period and they are exchanged directly for gold. There are six loan markets to be considered. These are composed of three intratemporal markets, two one period loan markets and one two period loan market. These are illustrated in Figure 1. If we require that we have a completely defined playable game it is necessary to be specific about all sequencing of moves. Here, as is illustrated in Figure 1, we note that we assume that the three loan markets in period 1 are all simultaneous. In period 2 the new intratemporal loan market and the one period loan market are simultaneous, as are the repayments of the intratemporal loan and the one period loan from the previous period. However, in case of default, the seniority of the payback of the loans must be specified. We assume that the shorter term loans are senior, i.e. they are paid first.

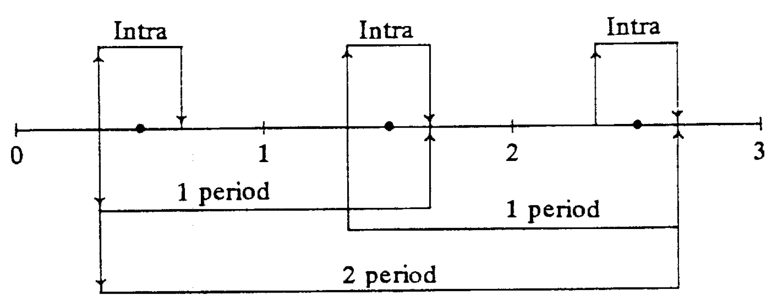

Figure 1
Loan and Goods Markets

A problem concerning the roll over of loans must be settled. All loans except the intratemporal loans overlap with other loans. Thus it is possible to borrow a second time in the one period loan market without having paid back the first loan. This opens up the possibilities for strategic behavior where individuals borrow more and more to pay off previous debts. This can be avoided by specifying an upper bound to the total amount of IOU notes an individual can have outstanding.

It would appear that if an individual can, in essence, roll over a sequence of one period loans there is no need for longer period loan markets. In terms of actual equilibria, if we include these markets in an economy without exogenous uncertainty, all that we will obtain is a multiplicity of solutions where short term loan roll overs are the equivalent of longer term loans. In actuality there are many transactions features such as the cost of contracting and investigation which differentiate among these. In particular, even at a high level of abstraction, as soon as uncertainty is considered, if there is a random variable influencing each period then borrowing twice in the one period markets is far different from borrowing once in a two period market.

The following notation is used:

$u^i_{0,t}$ = percentage of gold offered in the intratemporal loan market during period t

$u^i_{1,t}$ = percentage of gold offered in the one period loan market at t

$u^i_{2,t}$ = percentage of gold offered in the two period loan market at t

and

$v^i_{0,t}$ = percentage of IOUs offered in the intratemporal loan market during period t

$v^i_{1,t}$ = percentage of IOUs offered in the one period loan market at t

$v^i_{2,t}$ = percentage of IOUs offered in the two period loan market at t

In the goods market

$b^i_{j,t}$ = percentage of gold bid for the j^{th} commodity during period t

$q^i_{j,t}$ = amount of commodity j offered in the goods market at t

and for $j = 1, \ldots, m$

(2) $$p_{j,t} = \frac{\sum_{i=1}^{n} b^i_{j,t} A^i_t}{\sum_{i=1}^{n} q^i_t}$$

As all spot prices are quoted in terms of the money, the spot price of money in terms of itself each period is one.

We assume that there is an upper bound on the amount of IOU notes (denominated in money) that an individual is permitted to have outstanding at any period.

V^i = the upper bound on i's IOU notes.

The endowment of gold of individual i at the start of the economy is given by A^i and each period the new endowment of perishables owned by individual i is given by:

(3) $$(a^i_{1t}, a^i_{2t}, \ldots, a^i_{mt}) \quad \text{for} \quad t = 1, \ldots, 3 \ .$$

x^i_{jt} = consumption of i of j in period t for $j = 1, \ldots, m+1$.

The utility function of an individual i is generally described by:

(4) $$U^i = \varphi^i(x^i_{1,1}, \ldots, x^i_{m+1,3}) + \mu \sum \min[0, d^i_{h,t}] \quad \text{where} \quad h + t \leq 3 \ .$$

The μ is the bankruptcy parameter and the $d_{h,t}$ is the default of a loan of length h at time t.

We define three stages of the amount of money each period. The first is the money at the start, the second after financing and the third after trade and the settlement of debts due. These are described as:

$$A^i = \text{Gold before the loan market} = A^i_1$$

(5)
$$\tilde{A}^i_1 = \left(1 - \sum_{h=0}^{2} u^i_{h,1}\right)A^i + \sum_{k=0}^{2} \frac{v^i_{k,1}V^i}{(1 + \rho_{k,1})}$$

$$\dot{A}^i_1 = \max\left[\tilde{A}^i_1\left(1 - \sum_{j=1}^{m} b^i_{j,1}\right) + \sum_{j=1}^{m} p_{j,1}q^i_{j,1} + \frac{u^i_{0,1}A^iR_{0,1}}{\int_I u^v_{0,1}A^v dv} - v^i_{0,1}V^i \ , \ 0\right] = A^i_2$$

where

(6)
$$R^i_{0,1} = \int_I r^i_{0,1} di \ , \quad \text{where}$$

$$r^i_{0,1} = \min\left[v^i_{0,1}V^i, \left[\tilde{A}^i_1\left(1 - \sum_{j=1}^{m} b^i_{jt}\right) + \sum_{j=1}^{m} p_{j,1}q^i_{j,1} + u^i_{0,1}(1 + \rho_{0,1})\right]\right] .$$

For period 2 we have A^i_2 = gold at the start of period 2, and

$$\tilde{A}^i_2 = \left(1 - \sum_{h=0}^{1} u^i_{h,2}\right)A^i_2 + \sum_{h=0}^{1} \frac{v^i_{h,2}V^i}{1 + \rho_{h,2}}$$

$$r^i_{0,2} = \min\left[v^i_{0,2}V^i, \tilde{A}^i_2\left(1 - \sum b^i_{j,2}\right) + \sum p_{j,2}q^i_{j,2}\right]$$

(7) $$R_{0,2} = \int_I r^i_{0,2}di$$

$$r^i_{1,1} = \min\left[v^i_{1,1}V^i, \tilde{A}^i_2\left(1 - \sum b^i_{j,2}\right) + \sum p_{j,2}q^i_{j,2} - v^i_{0,2}V^i\right]$$

$$R_{1,1} = \int_I r^i_{1,1}di$$

From A^i_3, the starting amount of gold in period 3 the updating is:

(8) $$A^i_3 = \max\left[\left(\tilde{A}^i_2\left(1 - \sum_{j=1}^{m} b^i_{j,t}\right) + \sum_{j=1}^{m} p_{j,t}q^i_{j,t}\right) + \frac{u^i_{0,2}A^i_2R_{0,2}}{\int_I u^v_{0,2}A^v_2 dv} + \frac{u^i_{1,1}A^i_1R_{1,1}}{\int_I u^v_{1,1}A^v_1 dv} - v^i_{0,2}V^i - v^i_{1,1}V^i),\ 0\right]$$

$$A^i_3 = (1 - u^i_{0,1})A^i_3 + \frac{v^i_{0,3}V^i}{1 + \rho_{0,3}}$$

$$r^i_{0,3} = \min\left[v^i_{0,3}V^i, \tilde{A}^i_3\left(1 - \sum_{j=1}^{m} b^i_{j,3}\right) + \sum_{j=1}^{m} p_{j,3}q^i_{j,3}\right]$$

$$R_{0,3} = \int_I r_{0,3}di$$

(9) $$r^i_{1,2} = \min\left[v^i_{1,2}V^i, \tilde{A}^i_3\left(1 - \sum_{j=1}^{m} b^i_{j,3}\right) + \sum_{j=1}^{m} p_{j,3}q^i_{j,3} - v^i_{o,3}V^i\right]$$

$$R_{1,2} = \int_I r^i_{1,2}di$$

$$r^i_{2,1} = \min\left[v^i_{2,1}V^i, \tilde{A}^i_3\left(1 - \sum_{j=1}^{m} b^i_{j,3}\right) + \sum_{j=1}^{m} p_{j,3}q^i_{j,3} - v^i_{0,3}V^i - v^i_{1,2}V^i\right]$$

$$R_{2,1} = \int_I r^i_{2,1}di$$

and the final holding of gold (which equals consumption in the last period) is given by:

(10) $$A^i_f = \max\left[\tilde{A}^i_3\left(1 - \sum_{j=1}^{m} b^i_{j,3}\right) + \sum_{j=1}^{m} p_{j,3} + \sum_{s=0}^{2} \frac{u^i_{2-s,s+1}A^i_{s+1}R_{2-s,s+1}}{\int_I u^v_{2-s,s+1}A^v_{s+1}dv} - \sum_{s=0}^{2} v^i_{2-s,s+1}V^i,\ 0\right]$$

2.2 Low information strategies

We have purposely begun with an extremely implausible game to take advantage of its simpler mathematical structure in order to establish the existence of equilibrium with little labor. We may then use this to confirm existence for a far more plausible game. In particular we consider a game in which all players are in the dark, i.e. obtain no information during the course of the game. This being the situation each must submit a full feasible plan to the referee at the start. We can do this by having individuals bid a percentage of the various items they might hold.

When we do this a problem arises with how we are to handle credit markets. This can be taken care of by regarding the amount of IOU notes (denominated in the money) that an individual is permitted to issue is bounded. The bounding of the amount of debt an individual can create, in essence, gives a physical meaning to the supply of IOU notes; if we were to play an experimental game, it would be as though we gave all individuals a supply of say, green chips, called IOU notes. Once an individual runs out of them he cannot enter into any credit market until he has redeemed some of them.

A strategy by a trader in this game is of the form

$$(11)\quad \begin{aligned}&\Big(u^i_{2,1},\ v^i_{2,1},\ u^i_{1,1},\ v^i_{1,1},\ u^i_{0,1},\ v_{0,1},\ b^i_{j,1},\ q^i_{j,1},\ x^i_{j,1};\ u^i_{1,2},\ v^i_{1,2},\\ &\quad u^i_{0,2},\ v^i_{0,2},\ b^i_{j,2},\ q^i_{J,2},\ x^i_{j,2};\ u^i_{0,3},\ v^i_{0,3},\ b_{j,3},\ q^i_{j,3}\Big)\end{aligned}$$

where the $u^i_{2,1}$, $v^i_{2,1}$, ... are as described in 2.1. We denote this game by G.

THEOREM: *Assume that the utility function φ^i for any individual i is continuous, increasing in each variable and strictly concave. Then G has at least one type-symmetric NE.*

PROOF: Obviously the low information strategy set Σ^i for every individual i is convex and compact. Assume a low information strategy profile σ is given. Consider the situation where no one can vary his strategy in σ except individual i. Consider the best responses of i to all others playing in σ. By the continuity of φ^i and the compactness of Σ^i, there exists some $\overline{\sigma}^i \in \Sigma^i$ which maximizes the utility of i. Consider the subset of i's best responses

$$(12) \qquad B^i = \{\bar{\sigma}^i \in \Sigma^i : u^i_{h,s} \cdot v^i_{h,s} = 0 \text{ and } h^i_{j,s} \cdot q^i_{j,s} = 0\} .$$

By the strict concavity of φ^i, every $\bar{\sigma}^i$ in B^i must lead to the same $x^i = (x^i_{1,1}, \ldots, x^i_{m+1,3})$. But then any two different elements in B^i can be different only possibly in the intraperiod loans $u^i_{0,s}$ (or $v^i_{0,s}$) in case the interest rate for it is zero. But obviously in this case a convex combination of these two best responses is still a best response for i. Therefore in any situation, B^i is convex. On the other hand, the closedness of B^i follows from the continuity of φ^i.

Note that for any two individuals of the same type, the above mentioned sets of best responses to σ, for them are the same. Thus we have an induced mapping

$$(13) \qquad \Psi : \mathop{\mathrm{X}}_{\alpha=1}^{n} \Sigma^\alpha \to 2^{\mathop{\mathrm{X}}_{\alpha=1}^{n} \Sigma^\alpha}$$

defined by

$$(14) \qquad \sigma \xrightarrow{\Psi} B = \mathop{\mathrm{X}}_{\alpha=1}^{n} B^\alpha$$

where the superscript α runs over the set of n different types. The upper-semi continuity of Ψ follows from the continuity of the v^i. By Kakutani's Fixed Point Theorem, there exists some $\mathring{\sigma}$ such that

$$\mathring{\sigma} \in \Psi(\mathring{\sigma})$$

which is a type-symmetric NE as required. Q.E.D.

2.3 The high information game

An important property of games with complete information is that any pure strategy equilibrium point of the low information game is also a pure strategy in the game with more refined information (see Dubey and Shubik, 1981). A further property is that for a game with a continuum of agents and no atomic strategic players, all equilibria are perfect.

The more realistic game has the traders know their own results and market prices after each move. Strategies are no longer percentages but actual quantities. But from the results in Section 2.2 we know that this new game will also have the NEs of the game in 2.2 remain as NEs, thus the proof is an immediate consequence of the prior proof.

2.4 Enough money

A natural question to explore is the relationship between the amount of money needed[2] to be able to attain some CE as an NE for a one period economy and the amount needed for a T period economy. Before we give the precise notation it is necessary to make the question concerning the comparison more precise. Fortunately this can be done directly by considering an Arrow-Debreu treatment of a T period economy regarding all trades as taking place in the first period by a lavish use of futures markets. We then compare the money requirements for the related game where only spot markets are used for goods, but loans of money are possible. Even if we limit our concern to situations in which the same vector of perishable goods is introduced into the economy each period, if we maintain total generality of the utility functions, we can only obtain relatively weak bounds on the relationship between the money required for one period trade with full futures and T period trade with only spot goods markets. But by adding somewhat more structure to the description of individual preferences closer bound can be considered. For this reason in Sections 2.5 to 2.7 several specific restrictions on utility are considered. In Section 2.8 we consider the growth of the size of the economy.

Suppose that we consider a particular equilibrium point to a T period trading economy with n types of traders trading in m goods each period. At the equilibrium point a set of prices will exist. These prices are the futures prices for the consumption of goods or the use of services for a single period.

A game theoretic interpretation of the general equilibrium model is that all futures markets

[2]There is a deep unanswered question that we note, but do not treat in this paper. As the amount of a monetary commodity is increased when the money is not linear, separable, it is likely that the set of CEs may change as the amount of money changes. As the amount of money is increased from very little, at some point, it will permit one out of the set of CEs to be attainable as an NE. As we increase the money supply we expect that at some point we will reach a game at which all CEs of the new game will be attainable as NEs, thus it is more accurate to discuss the lower and upper bounds of enough money than just "enough money." This is clearly illustrated when the money is linear separable. The question is: can we characterize in an informative manner the relationship between the upper and lower bounds.

exist, i.e. it is possible for individuals at time 1 to buy a promise for the future delivery of a consumable or a period of service of a durable at any point in the future. This being the case, then all individuals can achieve optimal trade by trading only in the first period. If we require that an individual buying a future must pay the future price at this time, the question concerning the existence of enough money is as though we were examining a one period strategic market game. An item left out of this formulation is the motivation of the seller of a future to deliver, if he has been paid in advance. In actuality there are failure-to-deliver rules and they tend to be enforced, hence it is reasonable to assume that delivery will be forthcoming if the seller has the good.

Suppose that we select one of the mT commodities or services as the money. In particular suppose that the first component in each one of the T blocks of prices of m goods and services each period represents the price of the service rendered by a unit of gold during that period. We can represent the presence of an asset gold by assuming that there are A units of the services of gold available each period. A commodity money is, in general, a durable or a storable consumable and not a service. Thus if we were to normalize prices selecting the price of the money as one, we would not fix the price of the services of gold in the first period as one, but would fix the price of the asset at one. But if relative prices at the CE are given by:

$$(p_{1,1}, \ldots, p_{m,T}) ,$$

then the price of the asset gold is given by:

$$(16) \qquad \hat{p}_1 = \sum_{t=1}^{T} p_{1,t} .$$

If gold is the money we normalize the first m prices by dividing by the value of the asset. We normalize the subsequent sets of m prices by normalizing for the depreciated asset. Thus during the t^{th} period we divide prices by:

$$(17) \qquad \hat{p}_t = \sum_{s=t}^{T} p_{1,s} .$$

The condition of enough money for the one period game is

$$(18)\qquad \sum_{t=1}^{T}\sum_{j=2}^{m}\sum_{i=1}^{n}\max[(x_{j,t}^{i}-a_{j,t}^{i}),\,0]p_{j,t}\le A\ .$$

The conditions for enough money for trade in spot markets may be written

$$(19)\qquad \sum_{j=2}^{m}\sum_{n=1}^{m}\max[(x_{j,t}^{i}-a_{j,t}^{i}),\,0]p_{j,t}^{*}\le A\quad\text{for}\quad t=1,\,\ldots,\,T$$

where the p^* represent the normalized prices. It follows immediately that if there is enough gold for trade in futures markets then there is, in general, more than enough gold for trade in the spot markets as all of the constraints on spot trading are less binding than on the futures trading.

We note that the ratio between the (futures) price of the asset gold at period t and period $t+1$ can be interpreted as the rate of interest for gold, or its rental price which is the marginal utility of its service value for one period if there is enough gold. If there is not enough gold at some period then the rate of interest reflects both the marginal utility and the shadow price of the capacity constraint on trade.

2.5 Time separable utility functions

Without placing some special structure on the utility functions one can only make a relatively weak ordinal comparison between the requirements of enough money in different markets. If we consider that the utility functions are separable in time, with or without a discount factor and gold enters the utility function of all traders either generally or as a linear separable term we can be far more specific. The three cases are as follows. We consider utility functions of the following forms:

$$\phi^i = \sum_{t=1}^{T} \beta^{t-1} \varphi_t^i(x_{1,t}^i, \ldots, x_{m+1,t}^i)$$

$$\text{(20)} \qquad \phi^i = \sum_{t=1}^{T} \beta^{t-1} [\varphi(x_{1,t}^i, \ldots, x_{m,t}^i) + x_{m+1,t}^i]$$

$$\phi^i = \sum_{t=1}^{T} \beta^{t-1} [\varphi(x_{1,t}^i, \ldots x_{m,t}^i)] + \sum_{t=1}^{T} x_{m+1,t}^i .$$

We examine that situation in which there is the same endowment of perishable resources each period and the amount of gold is fixed.

Suppose that we consider the first functional form above. If the functional forms are different from period to period, consider the one period services of gold as the money and calculate for each period as though it were a separate economy, the amount of gold services sufficient for enough money in that period. The upper bound on the requirements of the asset gold will be the largest of the amounts required for any single period economy. This amount is independent of the value of β.

Suppose that $\beta = 1$ (the second and third forms are equivalent) then for a finite T, the asset gold is depreciating by its one period marginal use value, but as in the spot markets, it is always the numeraire, the one period money rate of interest at time t (for $t = 1, 2, \ldots, T-1$) is given by $\rho = 1/(T-t)$, for $T \geq 2$. As the horizon becomes longer the initial interest rate tends to zero. As the distribution of consumption from period to period is unchanged, but all that is happening is that spot prices are increasing, we may regard this rate of interest as an inflation rate due to the depreciation of the value of money.

When $\beta < 1$ both the finite and the infinite horizon problem are well defined and, for the third form, the rate of interest at time t is given by: $\rho = \{1/(T-t)\} + \{(1-\beta)/\beta\}$, for $T \geq 2$. In the infinite horizon economy the inflation rate drops to zero, spot prices are constant and the interest rate reflects the marginal consumption utility of the money for a single period.

With the linear separable gold without discount, the amount of money required to have the CE attainable as an NE (i.e. for the NE to be interior) is enough to finance the last period, when the service value and asset value of gold coincide. Thus until the last period much of the gold is not needed for transactions.

A simple example shows that if the economy is financed by tea which enters as a linear separable term, if $\beta < 1$ efficient exchange cannot be achieved, as the CE would require that all is consumed at the first period. Suppose that the utility functions and endowments are:

$$(21) \qquad U^1 = \sum_{t=1}^{T} \beta^{t-1}[\sqrt{x_t} + z_t] \quad \text{and} \quad U^2 = \sum_{t=1}^{T} \beta^{t-1}[\sqrt{y} + z_t]$$

endowments are [0, 1] and [1, 0] per period of the perishables and each has one unit of tea. For $T = 2$ 1/2 of the tea is saved for money in period 2 and 1/2 will be consumed in period 1. As T becomes large the trade in the economy with the storable consumable approaches efficiency as the loss of utility by keeping enough tea in circulation as money is small relative to the many trades for which it is used.

2.6 Life cycle utilities

The idea of an infinite horizon utility function with a "natural" time discount does not appear to be too appealing unless one reinterprets it as indicating a probability of death. It is well known that in an overlapping generations model even if a "natural discount" is included, in the stationary state its influence disappears in a standing wave. The life cycle utility function appears to offer a more interesting alternative, although better still would be one which combines both the life cycle and the uncertainty of living longer. Suppose that individuals live for T periods. What are the most general set of utility functions we can construct which enable us to consider a T period economy which can be naturally generalized to a stationary overlapping generations economy? We suggest the following form for an individual of type k and age t at period 1:

$$(22) \qquad \phi^k = \varphi^k(f_t^k(x_{1,1}^k, \ldots, x_{m+1,1}^k), \ldots, f_{(t+j)\mathrm{mod}(T+1)}^k(x_{1,j+1}^k, \ldots, x_{m+1,j+1}^k))$$

where $j = 0, 1, \ldots, T-1$.

In essence, each of the K types is broken down into T further types characterized by age.

When we consider the full OLG model in Part II we may abandon the fiction of gluing together a $T-t$ year old with a t year old to make a synthetic T year old (for $t = 0, ..., T-1$). Here we examine the simplest example of these type of utility functions where we consider T types of traders distinguished from each other only by a shift in discount parameters β_t on T separable terms, with a linear separable gold. The utility function for an individual of age t is:

$$(23) \qquad \phi^t = \sum_{j=0}^{T-1} \beta_{(t+j)\mathrm{mod}(T)}[\varphi(x^t_{1,j+1}, ..., x^t_{m,j+1}) + x^t_{m+1,j+1}] .$$

It is straightforward to observe that the β_t have no influence on the interest rate. If all goods are perishables except for gold which is money, then the only interest rate is that caused by the depreciation of the asset gold.

2.7 A growth in inputs without population growth

A key concern in any theory of a money rate of interest is how does the supply of money and the money rate of interest tie in with the possibility for real growth in the economy. We consider an extremely simple example where we have gold as a linearly separable money as in the second example in Equation (16) above.

We consider an infinite horizon economy with a common discount factor $\beta < 1$ and a growth by a factor $(1+\alpha)$ in the input of perishable commodities each period. Before considering the solution we observe that an extra natural condition on the utility functions must be added if growth is to be taken into account. It seems reasonable to assume that an upper bound on the growth of utility as all inputs are increased equally is certainly $h(1)$, i.e.

$$(24) \qquad \varphi(kx_1, ..., kx_m) \le k\varphi(x_1, ..., x_m) .$$

We show by means of a simple example that if money is of the form of a linear separable gold, then with growth, beyond some point, there will not be enough money. As growth continues efficient trade requires that the money supply grow at a rate basically determined by the specifics of the economy and in general *not* in a linear manner.

Assume that there are two types of individuals and two perishable consumer goods which each grow at a rate of $\alpha > 0$ per period. For simplicity, the endowment for Type 1 at period t

is $(\overset{X}{e^{\alpha t}}, \overset{Y}{0})$, for Type 2 is $(\overset{X}{0}, \overset{Y}{e^{\alpha t}})$. Each type possesses an amount of gold at period 1, say M. Assume the utility functions for the two types are

$$(25)\qquad \begin{aligned} \varphi^1 &= \sqrt{y^1} + g^1 \\ \varphi^2 &= \sqrt{x^2} + g^2 \end{aligned}$$

(Note that we have $\sqrt{ky^1} < k\sqrt{y_1}$ for $k > 1$.)

Now obviously the CE of the system is

Type 1: $[(0, e^{\alpha t}), M]$ for $t = 1, 2, \ldots$

Type 2: $[e^{\alpha t}, 0), M]$.

Note that at period t,

$$(26)\qquad \begin{aligned} \frac{\partial \varphi^1}{\partial y^1} &= \frac{1}{2}\left(y^2\right)^{-\frac{1}{2}} = \frac{1}{2}\left(e^{\alpha t}\right)^{\frac{1}{2}} = \frac{1}{2}e^{-\frac{1}{2}\alpha t} \\ \frac{\partial \varphi^2}{\partial x^2} &= \frac{1}{2}e^{-\frac{1}{2}\alpha t} \end{aligned}$$

Thus if we normalize gold prices to be 1 always, then the price vector at period t is

$$(27)\qquad \left(\overset{X}{\frac{1}{2}e^{-\frac{1}{2}\alpha t}}, \overset{Y}{\frac{1}{2}e^{-\frac{1}{2}\alpha t}}, \overset{g}{1}\right)$$

It is easy to see that the price of both commodities with respect to gold tend to zero exponentially.

But(!) the transaction costs, on the other hand, at period t, is, for any individual,

$$(28)\qquad (e^{\alpha t})\left(\frac{1}{2}e^{-\frac{1}{2}\alpha t}\right) = \frac{1}{2}e^{\frac{1}{2}\alpha t} \to +\infty .$$

It tends to $+\infty$ as $t \to \infty$. Thus any amount M is not sufficient when t is sufficiently large. the only "solution" if everything including money grows in quantity, but gold does not, then the utility of gold cannot be linearly separable if there is enough money.

If the endowments continue to grow while the amount of gold is fixed, trade is limited and the shadow price of the capacity constraint will become positive and will continue to grow.

Enough money

If money is not separable it is possible that there will always be enough to accommodate growth. This requires that the elasticity of demand for the perishables is more than one as the supply grows, i.e. in terms of gold as the money the decrease in price per unit for all trade more than offsets the increase in the amount of money require to buy the additional units for sale.

In Part II we consider the additional possibility of population growth.

3. The Three Period Model with Perishable Commodities and Tea

In Section 2 we considered an economy with a consumer durable as money. We could consider instead a storable consumable, such as bricks of tea, cans of sardines or bundles of tobacco as the money. Dubey, Geanakoplos and Shubik (1992) were able to show that one could use a storable consumable as a money and the money could be in sufficient supply and well distributed.

THEOREM 2 (DGS): *Suppose the money commodity $\ell+1$ is tobacco-like, and agent α likes $\ell+1$, for all α in the set of agents. There exists an M^* such that, if the minimum monetary endowment of any agent is larger than M^* any equilibrium is efficient.*

3.1 Enough money: Gold or tea?

Suppose that we have two economies with trading in perishables which are identical in all aspects except that one uses gold as money and the other uses tea. The similarity is such that all individuals in both instances have the similar utility functions, where in one, the services of gold enter in the same way as the consumption of tea enters into the other. In order to match endowments, if A units of gold are distributed initially in the economy which uses gold, then AT units of tea must be distributed at the start of the economy which uses tea. They both have loan markets for money.

THEOREM: *For an economy with time separable utility functions of the form:*

$$U^i = \sum_{t=1}^{T} \varphi_i(x^i_{1,t}, \ldots, x^i_{m+1,t})$$

where the $m+1^{st}$ *component in each period may be interpreted as the services of either gold, or tea consumed. The economy with tea requires MT units of money compared with the analogous economy with M units of gold.*

The proof is simple. The economy with gold will require the most gold at the last period, as the value of the asset depreciates. Suppose at the T^{th} period M units are required. But at the last period the same number of units of tea will be required. But if the models are directly comparable this requires MT unit of tea at the start.

A simple example illustrates this lower bound. We compare two economies where the first is run using gold as money and the second using tea.

(29)
$$\varphi^1 = y^1_1 + y^1_2 + 5 \ln g^1_1 + 5 \ln g^1_2$$
$$\varphi^2 = x^2_1 + x^2_2 + 5 \ln g^2_1 + 5 \ln g^2_2$$

(30)

	x_1	y_1	x_2	y_2	g_1	g_2
Initial	<10,	0;	5,	0,	5,	0>
	<0,	10;	0,	5,	5,	0>

the prices in terms of <u>service of gold are</u>

$$p_{11} = 1 \ , \ p_{21} = 1 \ , \ p_{31} = 2 \ , \ p_{12} = 1 \ , \ p_{22} = 1 \ , \ p_{32} = 1$$

the prices in terms of the asset gold are:

(31)
$$p_{11} = 1/2 \ , \ p_{21} = 1/2 \ , \ p_{31} = 1 \ , \ p_{12} = 1 \ , \ p_{22} = 1 \ , \ p_{32} = 1$$

The CE allocation is

<0, 10; 0, 5, 5, 5>

<10, 0; 5, 0, 5, 5>

We see here that money is exactly enough.

$$y_1 + y_2 + g_1 + g_2 = 10 + 5 + 2 \times 5 = 25$$

$$\max y_1 + y_2 + 5 \ln g_1 + 5 \ln(25 - y_1 - y_2 - g_1)$$

$$1 = \frac{5}{25 - y_1 - y_2 - g_1} \tag{32}$$

$$\frac{5}{g_1} = \frac{5}{25 - y_1 - y_2 - g_1}$$

$$\rightarrow g_1 = 5 \ , \ y_1 = 10 \ , \ y_2 = 5$$

We now turn to the economy with tea as money:

$$\varphi^1 = y_1^1 + y_2^1 + 5 \ln t_1^1 + 5 \ln t_2^1 \tag{33}$$

$$\varphi^2 = x_1^2 + x_2^2 + 5 \ln t_1^2 + 5 \ln t_2^2$$

	x_1	y_1	x_2	y_2	t_1	t_2
Initial	<10,	0;	5,	0,	10,	0>
	<0,	10;	0,	5,	10,	0>

CE

<0, 10; 0, 5, 5, 5>

<10, 0; 5, 0, 5, 5>

The spot prices are: $p_{11} = 1$, $p_{21} = 1$, $p_{31} = 1$, $p_{12} = p_{22} = p_{32} = 1$

There is just enough tea for money in period 1.

$$y_1 + y_2 + t_1 + t_2 = 10 + 5 + 10 + 0 = 25$$

$$\max y_1 + y_2 + 5 \ln t_1 + 5 \ln(25 - y_1 - y_2 - t_1)$$

$$1 = \frac{5}{25 - y_1 - y_2 - t_1} \tag{34}$$

$$\frac{5}{t_1} = \frac{5}{25 - y_1 - y_2 - t_1}$$

$$\rightarrow t_1 = 5 \ , \ y_1 = 10 \ , \ y_2 = 5$$

3.2 A digression on transactions costs and float loss

In the models in Section 2 we sidestepped the possibility that there might be a float loss from gold tied up in trade. We credited the final owner during any period with all of the value from the consumption use of the gold. Suppose instead we were to acknowledge some finite float, or transactions loss of consumption use of the gold while it is being used in transactions. Formally we can do this by considering that a small amount of time Δt is lost in carrying out a goods for gold transaction and it is lost to all. Thus we modify our payoffs to reflect this loss of utility to the system. An individual who receives k units of gold derives, in that period, only $(1-\Delta t)k$ units of service from it. Thus there has to be at least a small positive rate of interest for a transactions or intratemporal loan.

As can be seen from Figure 1, intraperiod loans do not overlap with each other, thus it is not possible to construct longer term loans out of them by taking out new financing from a later loan to pay back an outstanding loan. In building a formal game theoretic model where individuals may take out intraperiod loans and interperiod loans we may imagine two "credit lines," one consisting of say, "grey chips" and another of "green chips," the first representing the amount of short term or transactions, bridging or intratemporal debt that an individual can have outstanding and the second being the amount of intertemporal debt of any length that the individual can incur. Operationally the bidding of a grey chip in the intratemporal market between gold and grey chips means that the individual has issued a promise to pay one unit of gold in order to redeem his grey chip or short term IOU note.

At equilibrium, as all debts will be paid back, the introduction of these two sets of chips appear to be much ado about nothing. But they enable us to well define a fully dynamic structure and they serve to bound the strategy sets of all individual bidding for loanable funds. Furthermore a certain amount of reflection concerning the actual functioning of loan markets indicates this is a reasonable way to model borrowing as any individual borrower when viewed from the point of view of the lender has a plausible credit limit. If someone whose is earning \$50,000 a year and who has assets valued optimistically at current market of \$200,000 offers to assume a debt load of \$10,000,000 as contrasted with say \$100,000 it is unlikely that he will be able to borrow much, if any more, in the first instance than in the second. An attempt to model a realistic game both from the point of view of approximating reality (whatever that is!) and from the point of view of developing a plausible theory, point to the proposition that bounding the amount of debt an individual is permitted to assume, is both realistic and mathematically sound, as it bounds his strategy set. When the model is viewed as a playable game, we have a modeling choice, we could introduce simply one set of chips for all IOU notes to be issued; if a player issues all of these chips to others he can borrow no further. However, as a matter of highlighting the distinction between transactions loans and intertemporal or longer term borrowing, a case can be made for distinguishing among two types of chips.

3.3 Transactions and the number of markets with durables

When gold is money and there is an asset for sale which produces services, as noted in 1.2 we can split the asset into two pieces. In particular, we can consider that in each period there is a spot market for this period's services of the asset and a spot market for the asset stripped of its services for the period. But even limiting ourselves to the single period, if trade takes place near the start of the period, we could consider three spot markets associated with each asset. They are a market for the service; a market for the asset with its services for the period attached and a market for the asset without the services attached. This extra distinction at first glance may appear to be nit picking, but in some instances it may make a difference in the financing. Suppose only a market for the service and the asset with the service exist and they meet simultaneously. If an individual who owns a unit of the asset wishes to consume a unit of the service, but also wishes to sell the asset he must sell the asset with the service and simultaneously buy the service. If a "stripped asset" market existed he would keep the service and sell the stripped asset and this would require no financing whatsoever.

For specificity and simplicity here we assume only two markets per durable; they are the service market and the asset market with the current service attached, unless we specify otherwise.

If we were to consider futures markets we could contemplate selling assets with various lengths of income streams stripped from them. We do not consider these at this time.

4. The Three Period Model with Storables and Durables

For the further models we consider either gold or paper as the money.

4.1 The game with storables and gold

In the models considered in Section 2 the consumer commodities were all perishables. We modify the models of that section by replacing the perishables by consumer storables. The only difference is that formerly no inventories could be carried forward, now we consider this enlargement of strategic choice and ask whether it influences the amount of money needed and the needs for borrowing.

As in 2.1 we assume that there are n types of individuals trading for T periods, but they now are trading m storable consumable goods where each period there is a new endowment denoted by:

$$a_t^i = (a_{1,t}^i, \ldots, a_{m,t}^i)$$

where A^i is the initial endowment of money for i and

$$\varphi^i(x_{1,1}^i, \ldots, x_{m,1}^i, x_{m+1,1}^i, \ldots, x_{m+1,3}^i)$$

is the utility function for i.

A strategy is of the form:

$$(35) \quad \begin{aligned} \Big(&u_{2,1}^i, v_{2,1}^i, u_{1,1}^i, v_{1,1}^i, u_{0,1}^i, v_{0,1}^i, b_{j,1}^i, q_{j,1}^i, x_{j,1}; u_{1,2}^i, v_{1,2}^i, \\ &u_{0,2}^i, v_{0,2}^i, b_{j,2}^i, q_{j,2}^i, x_{j,2}^i; u_{0,3}^i, v_{0,3}^i, v_{0,3}^i, b_{j,3}^i, q_{j,3}^i\Big) \end{aligned}$$

As before we assume that all bids are given in percentages, thus price is defined as:

$$(36) \qquad p_{j,t} = \frac{\sum_{i=1}^{n} b_{j,t} \bar{A}_t^i}{\sum_{i=1}^{n} q_{j,t}^i Q_{j,t}^i}$$

where

$$\tilde{A}^i_1 = \left(1 - \sum_{h=0}^{2} u^i_{h,1}\right)A^i + \sum_{h=0}^{2} \frac{u^i_{h,1}V^i}{1 + \rho_{h,1}}$$

(37)
$$A^i_2 = \max\left[\tilde{A}^i_1\left(1 - \sum_{j=1}^{m} b^i_{j,1}\right) + \sum_{j=1}^{m} p_{j,1}q^i_{j,1}Q^i_{j,1} + \frac{v^i_{0,1}A^i R_{0,1}}{\int_I u^{\nu}_{0,1}A^{\nu}d\nu} - v^i_{0,1}V^i,\ 0\right]$$

$$Q^i_{j,1} = a^i_{j,1}$$

$$Q^i_{j,2} + Q^i_{j,1}(1-q^i_{j,1}) + \frac{b^i_{j,1}}{p_{j,1}} - x^i_{j,1}$$

the remaining equations are similar to those for the perishable consumer goods model.

The equilibrium existence proofs for this and the following model are, in essence, the same as that for the model with perishables. But the interesting question is not the existence of equilibrium but difference in flexibility between the economy with perishables and storables. Without the need to develop all of the formal mathematics we can see from the two period model that on a comparative basis the economy with the storables has fewer constraints on an optimization that is otherwise the same. For the models to be strictly comparable they must have the same utility functions and resource base, with the only difference being that for good j the initial supply of the storable will be $(a_{j1} + a_{j2})$, whereas for the economy with perishables the supply in the first period is a_{j1}, and the supply in the second period is a_{j2}.

A simple example illustrates the advantage of the extra flexibility derived from the goods being storable and also shows that the "enough money conditions" with perishables and storables are such that trade with storables may requires the same amount if storage costs are zero

(38)
$$\varphi^1 = \min[x^1_1, y^1_1] + 10\min[x^1_2, y^1_2] + g^1_1 + g^1_2$$
$$\varphi^2 = 10\min[x^2_1, y^2_1] + \min[x^2_2, y^2_2] + g^2_1 + g^2_2$$

We consider two sets of endowments. The first instance is where both types of trader have no gold whatsoever. Traders of type 1 have (4, 0, 0; 0, 0, 0) and type 2 have (0, 4, 0; 0, 0, 0); in words, the first have 4 units of the first commodity as an endowment in the first period and the second have 4 units of the second commodity. If there is a clearinghouse for intratemporal trade, then if the commodities are perishables, the CE for this model has a final distribution of (2, 2, 0; 0, 0, 0) for each trader and a utility for type 1 of 2 and for type 2 of 20. If the commodities are storable the solution remains the same if payments in gold are required as there is no gold or a way to extend intertemporal credit. Suppose that gold were also required for intratemporal settlement. Then neither the economy with perishables nor the economy with durables could achieve the CE.

We now consider a different set of endowments. Traders of type 1 have (4, 0, 20; 0, 0, 0), as their endowment and traders of type 2 have (0, 4, 20; 0, 0, 0). A CE for the economy with perishables has a final has a final distribution of (2, 2, 20; 0, 0, 20) for type 1 and (2, 2, 20; 0, 0, 20) for type 2. The third and sixth entries in the endowments indicate the new endowments of the asset gold. The third and sixth entries in the final consumptions indicate the use of the services of gold. A CE for the economy with storables has as its final distribution (0, 0, 20; 2, 2, 20) for type 1 and (2, 2, 20; 0, 0, 20) for type 2. This is also an NE which is achieved by the traders trading goods for gold in the first period and then those of type 1 store them until the second period.

4.2 The game with durables and gold

Our next comparison involves an economy in which all goods are durable, and for simplicity they all last for the full T time periods. Furthermore all rental markets exist. Thus one can buy the asset or rent it instead for any number of periods. We model this game without any loan markets because we can show that loan markets are not necessary without exogenous uncertainty in order to achieve optimality if there is enough money.

If the money is sufficient but not originally well distributed then an intratemporal loan market is required but no intertemporal laons are needed.

Intuitively this should be more or less obvious when we note that in order to make the model strictly comparable with the models with perishables or storable consumables we must introduce all the assets as well as money in the first period. But this means that all individuals have all their lifetimes' wealth immediately available, and there are $2m$ spotmakets, one for the asset and other for its immediate service. The availability of all asset and rental markets provides the flexibility to sell or rent precisely enough to satisfy wants for each subsequent period.

The formal model is as follows:

(39)

endowments are: $(a^i_{1,1}, \ldots, a^i_{m,1}, A^i; 0, \ldots, 0)$

The utility functions as before are: $\varphi^i(x^i_{1,1}, \ldots, x^i_{m+1,T})$

A strategy is of the form:

$$(b^i_{j,1,1}, b^i_{j,1,2}, b^i_{j,1,3}; q^i_{j,1,1}, q^i_{j,1,2}, q^i_{j,1,3}; b^i_{j,2,1}, b^i_{j,2,2}; q^i_{j,2,1}, q^i_{j,2,2}; b^i_{j,3,1}, q^i_{j,3,1})$$

The triple subscripts indicate the good, the time period and the length of time for which the asset is leased. In a more complicated model one might wish to distinguish the possibility that one rents an asset for its full life as contrasted with buying it. In this model, at this level of abstraction rental for for all remaining periods of life is equavlent to purchase.

We must rule out the possibility that an individual can sell a rented asset.

The prices are for various rental lengths, or sale:

$$p_{j,t,\tau} = \frac{\sum_{i=1}^{n} b^i_{j,t,\tau} A^i}{\sum_{i=1}^{n} q^i_{j,t,\tau} Q_{j,t}} \tag{40}$$

where

$$A^i_1 = A^i , \quad Q^i_{j,1} = a^i_{j,1}$$

$$A^i_{t+1} = A^i_t\left(1 - \sum_{j=1}^{m} \sum_{\tau \le 4-t} b^i_{j,t,\tau}\right) + \sum_{j=1}^{m} \sum_{T \le 4-t} a_{j,t,\tau} Q^i_{j,1} p_{j,t,\tau} (39) \tag{41}$$

$$Q^i_{j,t+1} = Q^i_{j,t}\left(1 - \sum_{t \le 4-t} q^i_{j,t,\tau}\right) + b_{j,t,4-t}\frac{A^i_t}{p_{j,t,4-t}} .$$

The constraints are given by:

$$(42)\qquad \sum_{j=1}^{m} \sum_{\tau \le 4-t} b^i_{j,t,\tau} = 1 \quad \text{and} \quad \sum_{\tau \le 4-t} q^i_{j,t,\tau} \le 1 \ .$$

The levels of consumption are given by

$$x^i_{j,t} = Q^i_{j,t+1} + \sum_{\tau \le 4-t} b_{j,t,\tau} \frac{A^i_t}{p_{j,t,\tau}} + (1 - |t-2|) b_{j,t-1,2} \frac{A^i_{t-1}}{p_{j,t-1}} \ .$$

The proof of the existence of efficient NE follows in the same manner as the previous existence proof.

The intuition behind the role of the assets is straightforward. As they are all stores of value, if there are both asset and rental markets considerable flexibility is provided for the avoidance of risk trades which only involve only exchanges of perceived value for value.

A major theme developed here is that a role of the exchange of money and other assets is to minimize the need of trust in trade. In essence, assets, including a commodity money serve as hostages which minimize the need for borrowing and trust.

5 Paper Money with Demonetized Gold

5.1 From gold to paper

In all of the models above, a commodity served as a money. However since the 1700s the trend has been towards paper currencies, although the Chinese had experimented with paper currency far earlier. When we attempt to introduce a paper currency which intrinsically has no consumption value into a finite horizon game we run into the well known problem in supporting its value at the end of the game.

Viewing the model as an experimental game a way to handle this problem is to construct a game in which the agents bid to borrow a fixed amount of money. At the end of the game they are required to return it to the referee or suffer a default penalty. This well defines a physical process and motivates the individuals to keep money to return at the end of the game. A far more satisfactory model would have an infinite horizon and overlapping generations. In this manner paper money could be treated as an asset in positive net supply, like gold, with no other offsetting claim. This is done in Part II.

The formal model

There are n types of individual and m perishable consumer goods. Paper money is introduced by the referee who auctions off an amount M which is to be returned (with any accrued) interest at the end of the game, or a penalty (measured in units of money) is assessed, where the size of the penalty is related to the size of the default.

Each individual of Type i is endowed with a vector a^{it} of consumer goods at each period t. At the beginning of the first period, every individual bids for the paper money using his personal IOU notes. The issue of IOU notes by individual i is bounded by some amount V^i denominated in money. More formally:

$$(43) \qquad v^i \in [0, \bar{V}^i] \quad \text{let} \quad v = \int v^i di \ .$$

Then i receives an amount of paper money $A^i = v_i M/v$. A default penalty of size μ min[0, d^i] is levied against anyone who fails to pay back, where d^i is the amount of his debt (negative). There are two one-period loan markets and three intraperiod loan markets. The mechanism is essentially the same as the model in 2.1 but with gold replaced by paper. A strategy by an individual i is:

$$(44) \qquad \begin{pmatrix} v^i;\ u^i_{11},\ v^i_{11},\ u^i_{01},\ v^i_{01},\ b^i_{j1},\ q^i_{j1},\ q^i_{j1};\ u^i_{12},\ v^i_{12}; \\ u^i_{02},\ v^i_{02},\ b^i_{j2},\ q^i_{j2};\ u^i_{03},\ v^i_{03},\ b^i_{j3},\ q^i_{j3} \end{pmatrix}$$

where all the entries are in percentages, except v^i and the q^{ii}_{jt}.

Let $A^i_1 = v^i M/v$ as the above A^i. Let $1 + \rho_{ki} = (\int v^i_{k1}(V^i - v^i)di)/(\int u^i_{k1} A^i_1 di)$

$$(45) \qquad \tilde{A}^i_1 = \left(1 - \sum_{k=0}^{1} u^i_{k,1}\right) A^i_1 + \sum_{k=0}^{1} \frac{v^i_{k,1}(V^i - v^i)}{1 + \rho_{ki}} \ .$$

Then the prices at period 1 can be calculated by

$$(46) \qquad p_{j1} = \frac{\sum_i b^i_{j1} \tilde{A}^i_1}{\sum_i q^i_{j1}} \ .$$

Now consider the return of the first intraperiod loan

$$(47)\qquad r^i_{0,1} = \min\left\{v^i_{01}(V^i - v^i),\ \tilde{A}^i_1\left(1 - \sum_j b^i_{j1}\right) + \sum_j p_{ji} q^i_{ji}\right\}$$

$$(48)\qquad R_{0,1} = \int r^i_{0,1}\, di\ .$$

Then we have

$$(49)\qquad A^i_2 = \max\left\{\tilde{A}^i_1\left(1 - \sum_{j=1}^{m} b^i_{j1}\right) + \sum_{j=1}^{m} p_{j,1} q^i_{j1} + \frac{u^i_{0,1} A^i_1 R_{0,1}}{\int u^v_{0,1} A^v_1 dv} - v^i_{0,1}(V^i - v^i),\ 0\right\}.$$

In a similar way as in the first model, we can find A^i_f and all the x^i_{jt}. thus the utility of i is

$$(50)\qquad \varphi^i = \sum \varphi^i_t(x^i_t) + \mu \sum_{\substack{k+t \le 3 \\ k \le 1}} \min\{0, d^i_{kt}\} + \mu \min\{0, d^i\}\ .$$

The existence of equilibrium is obvious. We note two simple results:

PROPOSITION 1: *Any CE of the original exchange economy can be achieved with any small amount of paper M.*

PROOF: Let (x^i_t) be a CE allocation with prices p_{jt} (j = 1, ..., m; t = 1, 2, 3). Without loss of generality, we assume that

$$\frac{\partial \varphi^i}{\partial x^i_{jt}} \Big/ \frac{\partial \varphi^i}{\partial x^i_{j't'}} = \frac{p_{jt}}{p_{j't'}}\ .$$

Let $P = \sum_i \sum_{j,t} p_{jt}\{0,\ x^i_{jt} - a^i_{jt}\}$. P is the total transactions money value. Now consider that all $v^i = M/n \Rightarrow \rho = 0$. Then each individual at the beginning of period 1 will receive M/n units of paper. Then consider his following bid and offer

$$q^i_{jt} = \max\{0,\ a^i_{jt} - x^i_{jt}\}$$

$$(51)\qquad b^i_{jt} = \max\left\{0,\ \frac{p_{jt}(x^i_{jt} - a^i_{jt})}{P}\frac{M}{n}\right\}\ .$$

It is easy to check that we *realize* the price vector. It is also obvious that everyone get the CE outcome.

This observation on the possibility for using any amount of money is completely well known and basically common knowledge and in straight general equilibrium form appears in many books and papers (for example Debreu, 1959). Thus a natural question to ask is what is the "value added" by using a strictly game theoretic formulation and requiring that the economy should be described as a game which can be played. We suggest that in meeting these criteria two requirements appear which enable the analysis to extend further into the understanding of the financial system than a general equilibrium formulation. In particular it is necessary to be explicit about default and reorganization, no matter how simplistically they are modeled they are a logical necessity of the game of strategy. Furthermore one needs to be precise in defining what is meant by money, credit, IOU notes, spot markets, futures markets and making these distinctions operational. In doing so mathematical problems concerning the boundedness of strategy sets appear and one is led into considering bounds on how much debt an individual is permitted to incur. But even casual observation of credit markets suggests that such bounds rather than being mere mathematical conveniences have economic meaning. In particular we observe that the relationship between the money supply and the default rules are critical in determining the leeway in the money supply and how closely paper can simulate a durable or a storable consumable money.

The price level with paper money

In the theorem above it appears that any arbitrary positive amount of money is always sufficient to support efficient trade. But we did not make explicit the relationship between the amount of money and the default penalty which must directly tie in the money supply with an individual's preferences as the penalty is related to a default denominated in money.

In most of our discussion we have described the default penalty at time t as a separable term of the variety μ min[Debt$_t$, 0], where debt is the amount due but unpaid at time t[3]. The

[3]At a more detailed level of precision we should distinguish insolvency from bankruptcy at this point. But for the purpose at hand we do not do so.

number μ gives us a simple form for the penalty which has an immediate economic interpretation as Hick's marginal utility of income (with a minus sign attached). Until one introduces exogenous uncertainty and expects to find an equilibrium involving active bankruptcy this functional form is as general as one needs, as on the margin of bankruptcy the value of an extra unit of money will be less than or equal to μ.

In the one period models of strategic market games involving borrowing and lending it has been shown (see Shubik and Wilson, 1977 and Shubik, 1993) that the possibilities for strategic bankruptcy at various levels of prices depend on the relationship between the amount of money and the default parameter, or the pair $[M, \mu]$. A way of looking at the Arrow-Debreu model is that both the default penalty and the money supply are unboundedly large. The result is that the price level is defined on the open interval $(0, \infty)$. When individuals can monetize their own debt by bidding IOU notes denominated in a monetary unit with a default penalty denominated in a monetary unit then the price level is defined on the half open interval $[p^*, \infty)$. When a finite amount M of paper money is issued and must be returned[4] then the price level is defined on the closed interval $[p_*, p^*]$ where $p^* > p_*$ in general, but it can be shown[5] that there will exist a default penalty such that the price level i completely specified. Any pair $(kM, \mu/k)$ picked, as noted, for $k > 0$ will completely fix the price level. Intuitively, the M provides an upper bound on the price level and the μ provides a lower bound on the price level that can be sustained without strategic default. The lower the price level, the more valuable a marginal unit of money becomes, hence the more an individual may be motivated to default. The game with a linearly separable gold as money and no credit markets has a unique price level.

We may extend this analysis to the multistage game.

---

[4]Assuming that velocity of circulation of money is bounded.

[5]Essentially by normalizing prices setting trade at a selected CE equal to M and the fixing the bankruptcy penalty to equal the largest of the Lagrangian multipliers at that CE. (See Shubik and Wilson, 1977.)

THEOREM: *A finite multiperiod exchange strategic market game with time separable utilities will have an efficient NE with a uniquely defined price level if for any amount of fiat money M the maximal transactions need is at the T th period and the* μ *is selected so that the price level is uniquely defined at that period.*

Suppose we consider a specific CE of the related exchange economy. We calculate the spot market requirements each period. Suppose that the last period requirements are the most. We may now consider the last period as a one period game. If we select M and μ in such a manner that the price level is unique, then if M units of outside money are auctioned at the start, the outside rate of interest will be zero and as money can be carried from period to period all prices must line up with the prices ruling at the last period hence they will be unique.

If the period of maximal need occurs before the last period there may a bounded zone of price levels each characterized by a positive inside money market rate of interest which offsets some of the potential deflation in the price system. Two simple examples clarify this remark. We consider two economies each with an amount of fiat, M; each of which obtain a new stationary supply of perishables each period. The economies differ only in the nature of the default penalties. The utility functions are of the form:

$$\phi^i = \sum_{t=1}^{T} \beta^{t-1}\varphi(x_t^i) + \mu \min[0, \text{debt}] \quad \text{or, alternatively}$$

(52)

$$\phi^i = \sum_{t=1}^{T} \beta^{t-1}[\varphi(x_t^i) + \mu \min[0, \text{debt}]] \ .$$

In the first utility function the bankruptcy penalty is not influenced by the diminishing utility brought about by β^t; in the second instance, it is influenced. If the penalty is selected so that in both instances the price level in the first period is unique, then if in both instances the first period price level is normalized to 1 then, for the first utility function any price level ranging from β^t to 1 can be supported. They will be matched by an inside money market rate of interest ranging between 0 and $1+\rho = 1/\beta$. The game with the second default penalty has a unique price level and a zero rate of interest in an inside money or loan market.

We may consider permitting $T \to \infty$ and the same results hold for both models. In Part II we consider the overlapping generations model and argue that in a stationary state with a fixed supply of fiat the price level is unique.

5.2 Expectations and the end of the game

One of the key paradoxes to be faced in dealing with the role of paper money in an economy is how does it get into the system and what happens to it "at the end of time"? A way to deal with this problem is to construct a game in which the referee does the original financing of the players by auctioning off, or alternatively handing out either paper money or gold (or bricks of tea) at the start of the game and requiring that each trader return to the referee his initial supply of "outside" money at the end of the game, or otherwise suffer a penalty. This model is easy to define for the finite horizon. But what happens in the infinite horizon where it appears that the money never has to be returned.

We may reasonably argue that we can study the finite game defined for T periods and then investigate what happens when T becomes large. Hopefully the limiting behavior of the solutions obtained should approach behavior at the limit. By approaching the limit via a finite game where individuals are required to pay back the money introduced at the start we require that "the books balance at infinity." But there is also an interpretation of the conditions for the payback at the end of the game. They are in essence, the expectations of the traders.

When we consider the strategic market game as an experimental game and consider briefing the players we tell them what to expect at the end of the game. In essence we "price default." We could also post prices at which the referee is willing to buy left over resources such as land or reproducible durable goods or storables. In such an instance the players know for certain what prices will be at the end of T time periods. In actuality these expectations are not specified by the referee but are constructed by all from experience and the information at hand.

A gold standard, or replacing gold by paper

If an economy switches from using gold (which it has in sufficient supply) to using paper backed by gold, at this level of abstraction the same results are obtained as though it were using gold. In fact the results are somewhat better (as can be seen from the bank of Amsterdam, see Kindelberger, 1984) as the wear and tear and other losses are minimized.

Keeping the gold but demonetizing it

If there were enough gold then if it is an infinite durable and if paper money is issued together with a bankruptcy rule which uniquely fixes the price level at the point of maximal monetary need then except for a factor in price between gold and paper the economic results would be the same as using gold as currency.

5.3 Varying the money supply with paper or gold

Whenever the gold rate of interest equals its consumption service value or when the paper money rate of interest is zero there is no need to vary the money supply as long as there is sufficient money to begin with. This is illustrated in Figure 2. In essence there will be enough money in the society if it can satisfy, efficiently, the largest trading requirement for any period without causing a change efficient trade.[6]

Figure 2 shows a simple diagram for the period by period money requirements of a three period economy with different (perishable inputs).

Consider two types of traders, each trading for three periods with utility functions of the form:

(53) $$(x_1y_2)^{1/4} + (x_2y_2)^{1/4} + (x_3y_3)^{1/4} + \mu \min[0, d_3]$$

where d_3 = debt at end of trade.

[6]If, for example, there is not enough money in some period to support efficient trade and the traders hold storable consumables, they will ship forward inventories which would have otherwise been exchanged and consumed.

Let the endowments of traders of type 1 be:

(8, 8; 4, 4; 2, 2) , and that of type 2

(0, 0; 28, 28; 0, 0)

The CE is (4, 4; 16, 16; 1, 1) each

(54) with prices $\left(\frac{1}{8}; \frac{1}{16}; \frac{1}{4}\right)$

$$\text{Wealth } 1 = 16\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) + 4\left(\frac{1}{4}\right) = 3\frac{1}{3}$$

$$2 = 56\left(\frac{1}{16}\right) = 3\frac{1}{2}$$

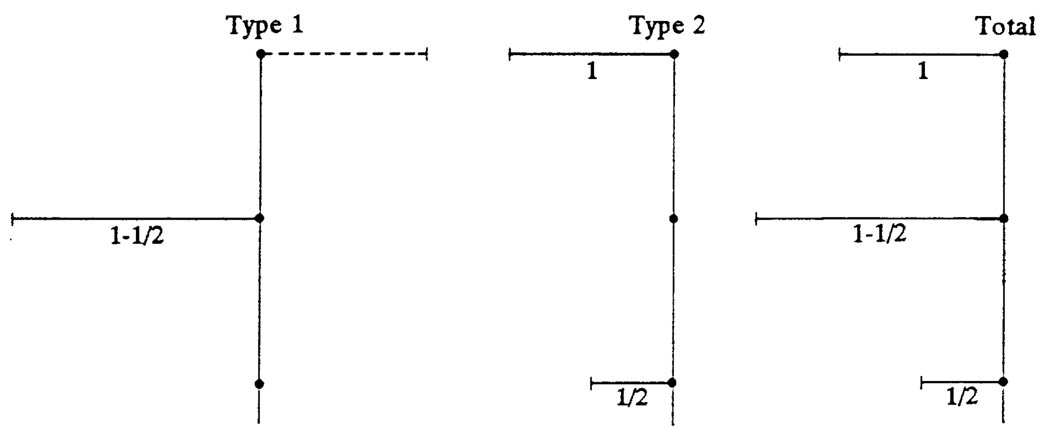

Figure 2
Cash requirements

Put in $M = 3/2$, $\mu = 1/4$. Outside ρ^* must be $\rho^* = 0$ otherwise someone would be going bankrupt. Could there be inside lending such that $\rho > 0$.

Suppose $\mu = 1/2$ then in periods 1, 2 and 3 *if* bankruptcy is to be avoided, then prices can be anywhere from

(55) $$\frac{1}{8(1+\gamma)}, \frac{1}{16(1+\gamma)}, \frac{1}{4(1+\gamma)}, \quad 0 \le \gamma \le 1 .$$

The amount of money needed for trade will vary from 3/4 to 3/2. We now claim that there exists a solution which has an inside rate of interest of $\rho_1{}^* = 1$ in the first period and $\rho_2{}^* = 0$, in the second period and an outside rate of interest of $\rho = 0$ when $\mu = 1/2$ and $M = 3/2$. Suppose traders of type i bid for all of the outside money. The distribution is (3/2, 0). If the prices are as in equation 55 with $\rho = 1$ then traders of type 2 will borrow 1/2 in period 1 and repay 1 in period 2. They then spend 1/2 in period 3. Figure 3 shows the cash flows with borrowing.

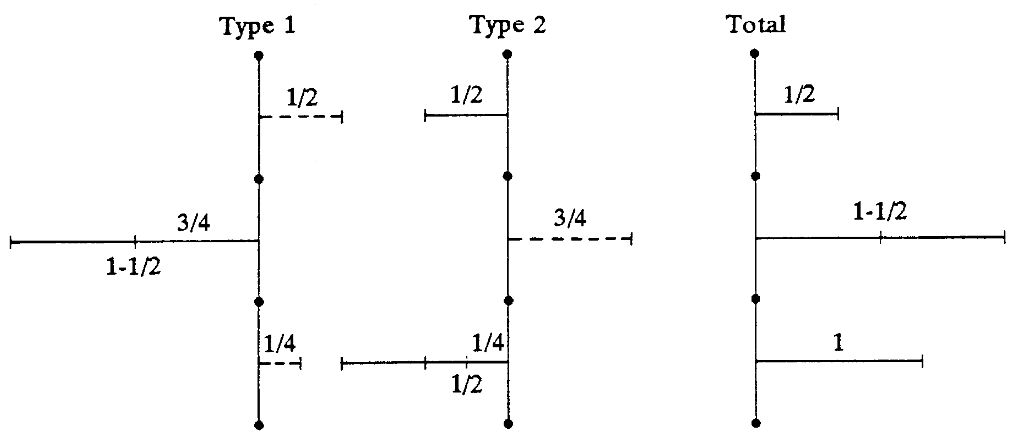

Figure 3

Cash flow with borrowing

Three periods are sufficient to illustrate the six patterns of monetary needs in the three periods. If H is highest, L is lowest and M the middle we have HLM, HML, LMH, LHM, MLH, MHL. Generically the system will hit a maximal requirement once and a minimal requirement once. Thus most of the time fiat money will be lying slack or unemployed and if the economy uses a linear separable gold as money most of the time much of it will be used for consumption only. This perfectly consistent with a zero interest rate for fiat or a rate equal to the marginal use of its service for gold, but this does not match our experience with economies we know. It is reasonable to ask why.

In essence, the money rate of interest as we experience it is rarely zero. In the friction-free, uncertainty-free stationary equilibrium world all of the conditions which conspire to give a rate of interest of zero to money have been selected. When there is active bankruptcy the money rate will reflect this; when there are transactions costs or when there is growth the money rate will be positive. If this were so, we would need to vary the money supply as money would not be hoarded or held as a slack. But there would need to be a counterparty to take up the slack when money is in oversupply and possibly to supply it if it is not in sufficient supply. A simple mechanism which enables the money supply to follow the contour of needs from period to period can be constructed by introducing the government as a strategic dummy every period and handing out at the start of the game an appropriate distribution of interest bearing government bonds to both the government and the traders. It will always be feasible to find an appropriate initial issue of bonds which will enable the traders to trade in and out of bonds leaving no "idle balances" with the net transfer of payments between the government and the traders being zero. All that one needs to do is to select the size of the money and bond issue in such a manner that the interest paid on borrowin exactly offsets the interest paid on lending at whatever the rate of interest happens to be. By continuity we can always adjust the amounts appropriately.

The problem with the observation above involves the difference between a highly simplified theory and the complexity of practise. An allseeing government with perfect foresight could make the calculations to put together the appropriate default penalty, quantity of money issued and quantity of bonds available in a manner that prices are determined. But, in fact, the prediction and computation requirements are such that it is unlikely that one can expect anything more than a crude approximation of the correct relationship among the quantities.

5.4 Labor, credit and assets

Even at the level of simplified theorizing presented here, a few crude aggregate statistics may serve to sweeten the intuition. The theme here is that money payment is trustless trade and that in a mass society where much trade is anonymous there are great gains in efficiency to be had by having a division of labor which concentrates credit evaluation and security guarantees in a special financial evaluation sector.

The old banking rule of "character, competence and collateral" has three components, two of which go in one direction and the third goes in the opposite direction. The investigation of character and competence requires deep knowledge of the individual and the context of his operations. But if an individual has collateral of value which can be easily claimed by the creditor, then the investigation of character and competence need not be as intensive as it would be if he had no assets. Thus, the expense of credit granting and the need for detailed information varies inversely with the availability of assets which can serve to guarantee the loan.

In a modern democratic society, the most valuable economic asset of the society is its human capital; but as slavery or indentured servitude are no longer condoned, the asset value of human capital as a backing for a loan is diminished to the amount of salary that can be garnished.

Using the 1992 *Statistical Abstract of the United States* (SAUS) and some "back of the envelope calculations we come up with some crude estimates or "guesstimates" about the availability of assets to back loans in the United as well as the money and credit structure.

(1) LAND is the nonreproducible, nondepreciating asset: The total land area of the United States is 3,536,000 square miles. A recent evaluation (1991) for the average price of agricultural land was $682 per acre. There is a considerable amount of desert and other unused land. In contrast the are central urban areas with some land worth over $40,000,000 per acre. If we evaluated all of the USA at the agricultural land level we obtain 3,536,000 x 640 x 682 = $1.543E12, with the federal government holding 29.2%

(2) REPRODUCIBLE CAPITAL: The estimate of the total value of reproducible capital in the USA fiven by SAUS (Table 735) is $1.490E13, of which 2E12 are consumer durables; 10.1E12 are other privately held assets and 2.5E12 are assets held by the government.

(3) HUMAN CAPITAL: Extrapolating Kendrick's 1965 Cobb-Douglas estimate for the U.S. economy his evaluation of human capital was approximately twice that of physical capital. Here this would give $2.98E13. A different (and cruder) estimate could be had by considering the size of the working population (123,500,000), the average wage ($23,600) and a working life of say, 45 years, and attributing 20% of expenditures as nonproductive inputs we obtain $2.63E13 or an individual human capital worth of $212,000 per member of the workforce, or averaging over the whole population $104,000 per capita.

Table 1
Capital Assets (Billions)

		Private	Government
Land	1,543	1,089	454
Reproducible cap.	14,900	12,400	2,500
Human cap.	29,800	29,800	0
Total	46,243	43,289	2,954

Credit market debt (in billions) in 1991 was as follows:

Government	3,569
Corporate	2,194
Individual	5,377

The corporate debt can be broken down into:

Bonds	1,174	Corp. Net worth 4,207 (1988, Table 834)
Mortgages	118	
Bank Loans	536	
Open Market Paper	99	
Other	170	

The individual debt can be broken down into:

Mortgages	3,887	Housing value 5,080 (Table 1222)
Cons Credit	793	Cons Durables 2,000
Policy Loans	71	
Bank Loans	196	
Other	296	
Tax exempt	95	

We note that there is $7,600 billion of nongovernmental debt against which there is around $13,600 billion of privately owned assets not counting human capital. Thus at the macro level there is little need for nonsecured lending. Even although there appears to be an asset coverage of nearly 2:1 this presumes a competitive market for liquidation; but bankruptcy phrases such as "10 cents in the dollar" warn us that assets may be dissipated, may be specialized with only a thin market and may be administratively costly to liquidate. The failure rate among firms in the U.S. has recently been running between 3/4-1% and in 1990 the liabilities of bankrupt firms were around $64 billion (Table 845)

In what may be now termed as "old fashioned" banking, character and competence counted, possibly even more than collateral in the making of new enterprise loans. With current laws, even though some garnishing of salary is feasible, it is unlikely that as much as 5% of human capital per se is available as security. The assessment of honesty, character and competence may still be worth more than the security of being able to garnish.

These few crude figures suggest that the asset stucture of the United States is such that more or less secured lending is feasible and to be expected. In routine loans postulating a more or less steady state this permits a considerable reduction in the need for careful investigation and evaluation of the individual. Unfortunately this is not so in the most dynamic part of the economy when new and untried enterprises must be financed.

Government debt is considerably different from private debt as is indicated from the observation that debt is around $3,600 billion while total assets are at most $2,950 billion. The security is a mix of force, custom, law and belief in the full faith and credit of the central government.

In 1990 the GDP of the United States was $5,514 billion. The amounts of money or near monies employed to finance this production were (in $billions, Table 802):

Currency	267
Trav. check	8
Demand dep.	289
other chble dep.	333
Total M1	898
M2	3439
M3	4989

In Section 3 we considered economies using gold or tea as the currencies. There is of the order of 100,000 troy tons of mined gold in the world gold in the world. The USA monetary reserves in 1991 were reported at $11.1 billion (Table 1318) with gold evaluated at $42.22 per ounce. Adjusting to recent market price of around $360 we obtain $95.1 billion and adjusting for the approximation that government may only hold a little more than 40% of the gold available in the U.S. we obtain around $228 billion or somewhat less than the amount of currency in the United States.

An upper bound on per capita tea consumption in the USA is 2 pounds, this evaluated at \$5 per pound gives a value of around \$2.5 billion. In a small country it might still be feasible to consider gold as a currency, the possibilities for tea are not promising.

5.5 Monetary policy, expectations and theory

Clearly the Philosopher's Stone of the macroeconomic advisor is a dynamic theory which can be applied. This essay into multistage exchange economies is far removed from such practicalities. Rather it addresses a simpler, but nevertheless operationally interesting set of questions. If the world was as simple as that portrayed in the games specified here what would the referee (government) have to know in order to devise its monetary policy in a way to facilitate optimal exchange. Furthermore what weapons are available to it?

Can we build a playable experimental game to cover the models described above? The answer is yes. Expectations are supplied by the referee specifying the salvage prices to be paid for left over assets at the end of play. The supply of bonds, initial issue of money and default parameter can be calculated so that there is a unique price level specified at a noncooperative equilibrium. But this sort of experimental control over a game is what an actual government does not have over the economy it may try to guide.

References

Debreu, G., 1959. *Theory of Value*. New York: Wiley.

Dubey, P., J. Geanakoplos, and M. Shubik, 1992. Is gold an efficient store of value? CFDP 1031, Yale University.

Dubey, P., and L. S. Shapley, 1977. Noncooperative exchange with a continuum of traders, Rand p-5964, Santa Monica, CA. also Cowles Foundation Discussion Paper No. 447, Yale University, New Haven, CT.

Dubey, P., and M. Shubik, 1986. Information, conditions, communications and general equilibrium. *Math of O.R.6* 2: 186-189.

Dubey, P., and M. Shubik, 1978. The noncooperative equilibria of a closed trading economy with market supply and bidding strategies. *Journal of Economic Theory* 17(1): 1-20.

Kindleberger, C. P., 1984. *A Financial History of Western Europe*. London: Allen and Unwin.

Shubik, M. 1990. The transactions trust demand for money. *Journal of Economics* 52: 211-232.

Shubik, M., and C. Wilson, 1977. The optimal bankruptcy rule in a trading economy using fiat money, *Zeitschrift fur Nationalokonomie* 37: 337-354.