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Multiplicative Bidding and Convergence to Equilibrium

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by

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Abstract

General equilibrium strategies may be relatively difficult to determine. While multiplicative strategies may be much simpler to calculate, they are not in general in equilibrium. An example shows that such strategies may indeed be quite far from equilibrium. However, if the example is iterated using Bayesian decision analysis, the strategies quickly converge to being very nearly in equilibrium

Introduction

The survey of auctions and bidding by Engelbrecht-Wiggans (1978) reveals that many bidding models assume that individuals will bid a multiple of an unbiased estimate of an object's true value. Relatively few attempts have been made to justify such an assumption; some work has been reported showing that equilibrium strategies may in certain cases be represented by a function well approximated by a multiplicative function. Such work however does not resolve whether or not multiplicative strategies are close to equilibrium in terms of expected profit.

This paper briefly presents a model of auctions as games with incomplete information. The mathematics of determining general equilibrium and

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multiplicative strategies is outlined for auctions with symmetric bidders. Finally, an example is considered to indicate the degree to which multiplicative strategies fail to be in equilibrium. In this example, however, if players participate in a series of similar auctions, always using a Bayesian analysis of the previous auction to determine their bidding strategy, then the strategies will be relatively too close to equilibrium after a very few iterations.

Model

Auctions may be modelled as games with incomplete information as defined by Harsanyi (1967a, 1967b, 1967c). In such models, "nature" chooses the true value of the object, the reservation price of the object, and the number of players from a probability distribution known to all players. Although nature's choice for the random variables are not revealed to the players, the players obtain some insight into those choices by observing an information random variable. The joint distribution of the players' information random variables depends on the outcome chosen by nature; the conditional density functions are known to all players. Using the knowledge about the distributions, a player must specify a bidding strategy; a bidding strategy is a function from possible outcomes of the information random variable to possible bids. Thus, a player's bidding strategy specifies his bid as soon as he observes the outcome of his information random variable.

We will consider a somewhat simplified version of this model. The true value of the object z and the number of players n will be the outcomes of the independent random variables Z and N with probability densities $h(z)$ and $p(n)$ respectively. In this discussion, N will either be degenerate or Poisson distributed with mean u . The reservation price C is degenerate (i.e., fixed at some known value c).

The information random variables will be independent identically distributed random variables with density $f(x)$ and distribution function $F(x|z)$. The information random variables will be assumed to be "monotonic" in the true value; in particular $\Pr(Z \geq z | X_i = x_i^*) \geq \Pr(Z \geq z | X_i = x_i)$ whenever $x_i^* \geq x_i$. Finally, our goal is to find a differentiable (almost everywhere) monotonically increasing function $b(x)$ such that all players using this as their bidding strategy results in a Nash (1950) equilibrium.

From any one player's viewpoint, the probability of having the winning bid when all players are using the same bidding strategy $b(x)$ is equivalent to the probability that the value he observes for his information random variable exceeds each of the values observed by the remaining players. In particular, under a symmetric set of equilibrium strategies, the probability $X_i = x_i$ results in a winning bid is precisely the probability that $Y \leq x_i$ where Y is defined to be the maximum of all X_j with $j \neq i$. Thus, define the probability density $g(y)$ and the probability distribution $G(y)$; $G(y)$ is given by $F(x)^{n-1}$ if the number of players is fixed at n , or by $\exp(-u(1 - F(x))$ if the number of competing players (not counting player i) is Poisson with mean u .

General Equilibrium

The expected profits to a player using the bidding strategy $B(x)$ when the remaining players are using the strategy $b(x)$ is given by

$$E(\$|x) = \frac{\int_z (z - b(x))G(B^{-1}(b(x))|z)\mathfrak{f}(x|z)h(z)dz}{\int_z \mathfrak{f}(x|z)h(z)d\theta} \qquad (I)$$

$$E(\$) = \int_x \int_z (z - b(x)) G(B^{-1}(b(x)) | z) f(x|z) h(z) dz dx \qquad \text{(II)}$$

Differentiating (I), setting it equal to zero, and evaluating it at $B(x) = b(x)$ results in the following equilibrium condition on $b(x)$:

$$b(x) \int_z g(x|z) f(x|z) h(z) dz + b'(x) \int_z G(x|z) f(x|z) h(z) dz$$

$$= \int_z z g(x|z) f(x|z) h(z) d \qquad \text{(III)}$$

Equation (III) is a first order linear differential equation; it is thus possible to write an explicit symbolic expression for the function $b(x)$. The required initial condition may be obtained by finding the x^* which satisfies

$$\int_z (z-c) f(x^*|z) G(x^*|z) h(z) dz = 0 \qquad \text{(IV)}$$

and setting $b(x^*) = R$, the reservation price. Unfortunately, in practical situations, evaluating the double and triple integrals in the symbolic expression for $b(x)$ is very difficult. Although the differential equation can be solved analytically for very simple distributions $f(x)$ and $h(z)$, the author has yet to find "reasonable" $f(x)$ and $h(z)$ for which $b(x)$ can be obtained analytically (as opposed to numerically).

Multiplicative Strategies

One alternative to the general equilibrium approach is to find strategies which are in equilibrium within some restricted class of possible strategies. A common restriction is to consider only multiplicative strategies; a player's bid $b(x) = b \cdot x$. Such an assumption appears quite often, explicitly or implicitly, in oil lease bidding models.

Along with multiplicative strategies, a common assumption is that the information random variable is simply a random multiple of the true value. In such cases, $\mathfrak{f}(x|z) = \frac{1}{z}f\left(\frac{x}{z}\right)$, and $G(x|z) = G\left(\frac{x}{z}\right)$. In oil lease auctions, f is typically assumed to be the log-normal density.

Using the restriction to multiplicative strategies and assuming that information is multiplicative, equation (II) may be differentiated and evaluated at $B = b$ to obtain the equilibrium bid fraction b , whenever $\int_z zh(z)dz < \infty$.

$$b = \frac{\int_w wg(w)f(w)dw}{\int_w w^2 g(w)f(w)dw + \int_w wG(w)f(w)dw} \tag{V}$$

Notice that this equilibrium fraction b is independent of the function h(z) ; no assumptions need be made about the distribution of nature's choices of the true value (other than finite expected value) in order to determine the equilibrium bid fraction. Furthermore, the integrals involved in calculating this fraction are relatively simple.

The equilibrium bid fraction b is plotted as a function of the mean number u of Poisson distributed competitors, in Figure 1, for the case when the logarithm of x/z is assumed to be normally distributed with mean zero and variance s^2 . The larger the variance s^2 , the larger the variance in the error, and the more conservatively players should bid. For very small mean numbers of competitors, there is a significant chance that no competing bids will be submitted, and the equilibrium bid fraction will be small to take advantage of these possible bargains. Conversely, as the mean number becomes large, the "bidders' curse" has an increasing influence, and again players bid conservatively. Although Capen, Clapp and Campbell (1971) determine optimal bid fraction for an individual in a decision theoretic model, their results have the same qualitative features.

Unfortunately, there is no assurance that the strategies $b(x) = b\cdot x$ are in equilibrium if players are not restricted to using multiplicative strategies. Rothkopf (1969, 1971, 1977a) shows, under conditions equivalent to assuming that $h(z)$ is a diffuse uniform distribution on $[0,\infty)$, that multiplicative strategies are still in equilibrium if players are not restricted. Winkler and Brooks (1977) prove a corresponding result for additive strategies for the case of additive errors and a diffuse distribution on z. For other $h(z)$, this is not in general true. The difficulty in general arises that for non-diffuse $h(z)$, there is at least a vague sense of scale, and players should perceive large information outcomes as possibly arising out of large errors rather than large true values and vice versa; a posterior Bayes estimate of the true value will in general not be proportional to the information outcome, but will rather shrink large information outcomes down and increase small information outcomes up.

Even though multiplicative strategies are in general not in equilibrium, the question remains whether they are close enough to be reasonable approximations. Rothkopf (1977b) attacks this question by calculating equilibrium linear strategies $(b(x) = a + b\cdot x)$ and shows that if the variance of the error is small compared to the variance in nature's choices for the true value, then the multiplicative term strongly dominates the additive factor; slightly less precisely, but perhaps more intuitively, as $h(z)$ becomes relatively diffuse compared to the spread in the error, multiplicative strategies are close to equilibrium linear strategies. This approach does not, however, examine how much multiplicative strategies and equilibrium strategies differ in the expected profit realized by a player. A subsequent example shows that this difference may be substantial.

An alternative to multiplicative strategies is to use multiples of Bayes estimates, where the Bayes estimate is given by

$$E(z|x) = \frac{\int_z z f(x|z)h(z)dz}{\int_z f(x|z)h(z)dz} \qquad \text{(VI)}$$

Unfortunately, the bid fractions will now depend on $h(z)$; Figure 2 plots the equilibrium Bayes multiplier for the case when the logarithm of z is distributed normally with mean zero and variance t^2 specified in the graph and the logarithm of the error x/z is distributed normal with mean zero and variance s^2 equal to one. While Bayes multipliers may be closer to being in equilibrium than multiplicative strategies, they are also not in general in equilibrium.

Bayesian Iteration

Since multiplicative strategies are not in equilibrium, one might ask what would happen if players using multiplicative strategies were allowed to use more general bidding strategies. In particular, how would players bid in subsequent similar auctions. if they each performed a Bayesian analysis on the previous auction; in subsequent auctions, an individual uses the strategy $B(x)$ which maximizes equation (I) for each x , where $b(x)$ denotes the strategy used by all players in the previous auction. If this process is repeated yet a third, and fourth, etc., time, would the bidding strategies eventually converge to an equilibrium?

In order to investigate these questions, consider an example in which the logarithm of the true value (in \$1000's) is normally distributed with mean 8.5 and variance 2.5; the logarithm of the multiplicative error is normally distributed with mean zero and variance one; and the number of competitors is Poisson distributed with mean $u = 5$. These values and assumptions were chosen to provide a simple example of a nature similar to a typical offshore oil lease sale. (Although we will compare the distribution of bids under equilibrium bidding with the distribution reported by the U.S. Geological Survey (1978) for OCS Sale #40, this is only to indicate that numbers in the example are of a magnitude encountered in actual auctions. The choice of distributions and parameters is in no way intended as an endorsement of the validity of such choices for models of oil lease sales.)

In Figure 3, the bidding equilibrium multiplicative bidding strategy is plotted as a function of the posterior Bayes estimate. The optimal (Bayesian) response of an individual player to such bidding strategies, for this example, happens to be indistinguishable (within the accuracy of our numerical routines) from the equilibrium Bayes strategy. Notice, that Table 1 indicates that if all other players are using the multiplicative strategy, any individual may increase his expected profit by about one half by switching from the multiplicative strategy to his optimal (Bayesian) response. Since, one unit in the table corresponds to approximately a million dollars, this implies that if palyers in off shore oil auctions indeed bid multiplicatively, any individual could gain on the order of a half million dollars per lease by using his optimal response; multiplicative strategies are clearly not very near to equilibrium!

If all players bid multiples of Bayes estimates, then some improvement is still possible for any individual who deviates. Table 1, however, indicates that this potential improvement is less than that obtained by going from multiples of information outcomes to multiples of Bayes estimates. Figure 3 also plots the optimal response to strategies which are a multiple of Bayes estimates.

This process may be iterated. One can assume that each player uses the optimal response to the strategies used in the previous auction. The relative expected profits and the different bidding strategies are given in Table 1 and Figure 3. Notice that after only a very few iterations, such a procedure converges on a strategy which is as close to equilibrium as we can distinguish using our (relatively crude) numerical methods. Finally, in Figure 4, the distribution of bids under the above limiting distribution is compared to the distribution observed in OCS Sale #40; while the match is far from perfect, the correspondence is close enough to suggest that multiplicative strategies may indeed be hundreds of thousands of dollars from being in equilibrium.

Conclusion

In this paper, auctions are modelled as games with incomplete information; in particular, we consider symmetric games with multiplicative errors. When strategies are restricted to be multiples of the information observed, the equilibrium strategies are relatively simple to calculate compared to the general (unrestricted) equilibrium strategies. Equilibrium multiplicative strategies are independent of the distribution of true value for an object, but are unfortunately not necessarily very close to being in equilibrium.

As an alternative, one might consider bidding strategies which are multiples of posterior Bayes estimates. Such estimates are more difficult to calculate, but appear to be closer to equilibrium, at least in our particular example. Questions remains as to what conditions on the distribution of true values and information result in Bayes estimates being closer to equilibrium, and how close in general are Bayes multipliers to the optimal response to multiplicative strategies.

An example is presented showing that multiplicative strategies, as well and multiples of Bayes estimates, are not very close to equilibrium. However, under repeated application of a Bayesian analysis, either of these strategies quickly converges to a strategy which is very close to equilibrium. For what conditions on the model does such convergence occur? If, in general, strategies converge rapidly to equilibrium, then this technique may be used to calculate equilibrium strategies. Such convergence would also assure that even less sophisticated minded bidders would use equilibrium strategies after a relatively small number of "learning experiences."

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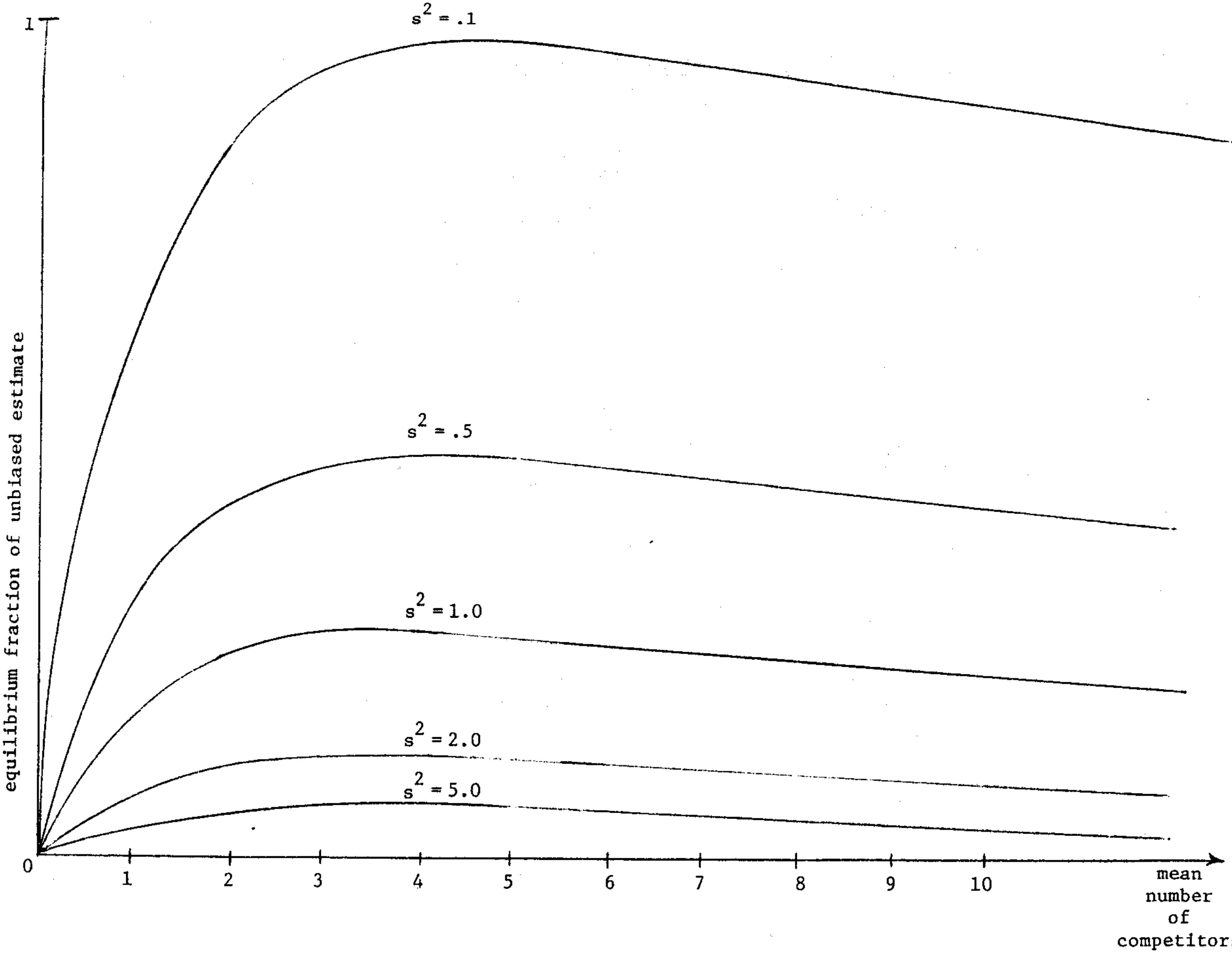
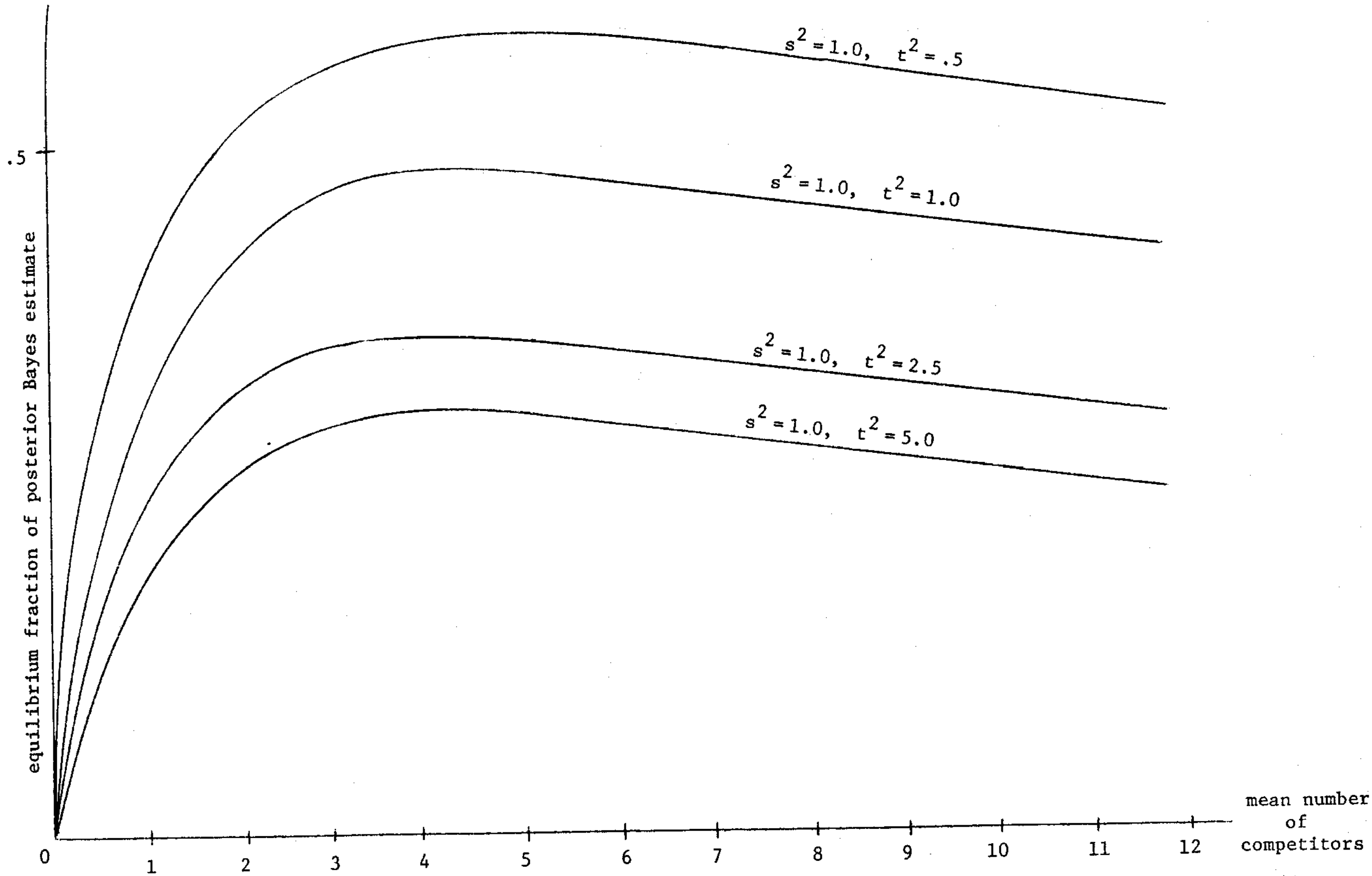
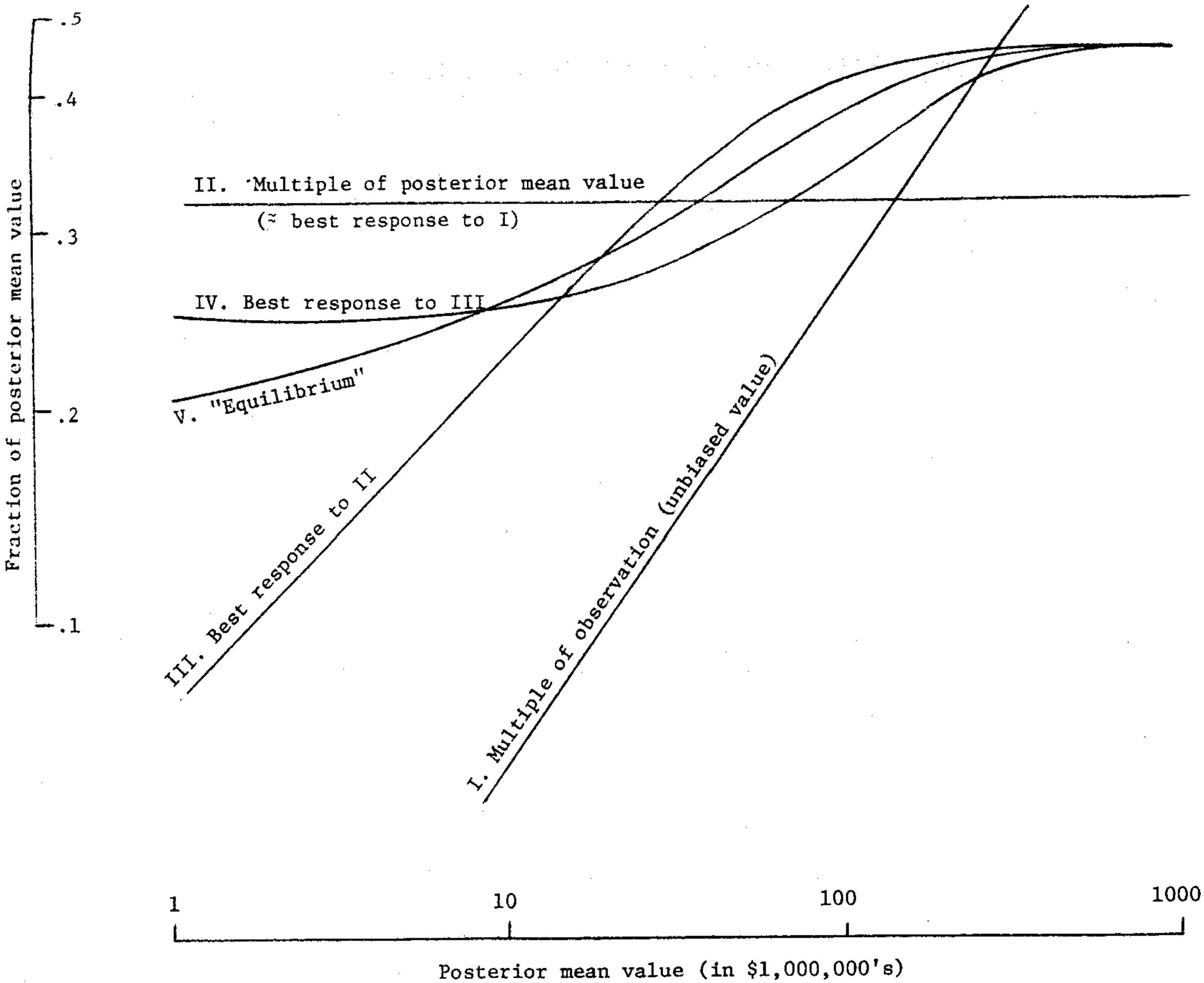

equilibrium fraction of unbiased estimate
1
0
s2 = .1
s2 = .5
s2 = 1.0
s2 = 2.0
s2 = 5.0
1
2
3
4
5
6
7
8
9
10
mean number of competitors

FIGURE 1



.5
equilibrium fraction of posterior Bayes estimate
s² = 1.0, t² = .5
s² = 1.0, t² = 1.0
s² = 1.0, t² = 2.5
s² = 1.0, t² = 5.0
0
1
2
3
4
5
6
7
8
9
10
11
12
mean number of competitors

FIGURE 2



.5
.4
.3
.2
.1
Fraction of posterior mean value
II. Multiple of posterior mean value
(≈ best response to I)
IV. Best response to III
V. "Equilibrium"
III. Best response to II
I. Multiple of observation (unbiased value)
1
10
100
1000
Posterior mean value (in $1,000,000's)

FIGURE 3

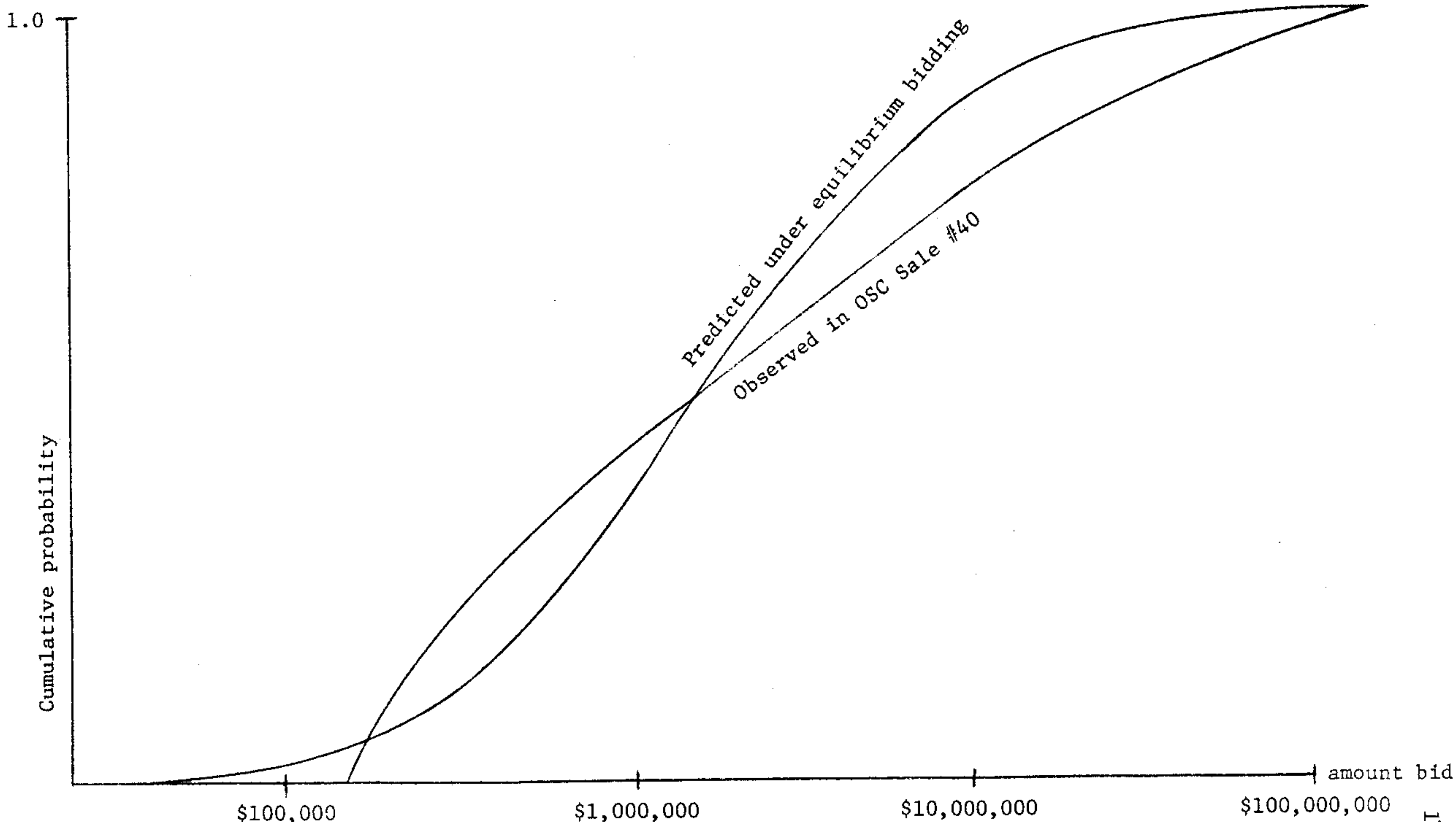

1.0
Cumulative probability
Predicted under equilibrium bidding
Observed in OSC Sale #40
$100,000
$1,000,000
$10,000,000
$100,000,000
amount bid

FIGURE 4

TABLE 1

Bidding Strategy of Remaining Players	Bidding Strategy I Multiplicative	II Multiple of Bayes Estimate	III Best Response to II	IV Best Response to III	V "Equilibrium"
I	1.32	1.76	1.62	1.76	1.72
II	.95	1.14	1.22	1.19	1.20
III	.66	.81	.82	.86	.85
V	.76	.96	.95	.99	1.00

Relative Expected Profit as a Function of Bidding Strategy Versus Strategy Used by Remaining Bidders (Each unit is slightly over $1,000,000)