

OPTIMALLY STUBBORN

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Optimally Stubborn*

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Abstract

I consider a bargaining game with two types of players – rational and stubborn. Rational players choose demands at each point in time. Stubborn players are restricted to choose from the set of “insistent” strategies that always make the same demand and never accept anything less. However, their initial choice of demand is unrestricted. I characterize the equilibria of this game. I show that while pooling equilibria exist, fully separating equilibria do not. Relative to the case with exogenous behavioral types, strong behavioral predictions emerge: in the limit, players randomize over at most two demands. However, unlike in a world with exogenous types, there is Folk-theorem-like payoff multiplicity.

1 Introduction

This paper endogenizes behavioral types in a bargaining setting. Rational and stubborn types bargain over the division of a pie. The stubborn type can make any demand over the pie but is insistent on the demand once made. What stubborn types arise in equilibrium? Is there a bound on the payoff that the rational type can guarantee himself? Does the rational type mimic the stubborn type, or do types separate when the stubborn type is given some choice over the form of his stubbornness?

I consider a bargaining game with two types of players: rational and stubborn. The game consists of two stages: a demand stage and a concession stage. The game ends when one player concedes, i.e., agrees to his opponent’s demand. Rational players can concede anytime. As in the literature, stubborn players cannot concede to their

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opponent. However, I depart from the literature by giving them some choice over the form of their “stubbornness,” namely, the choice of their initial demand is free.

Even in well-defined environments, real-world agents may not optimize over all possible strategies. Instead, agents may restrict attention to a subset of strategies but optimize within that subset. For instance, some agents may choose not to participate in auctions on platforms such as *eBay* but make the choice of whether to join the platform and which *Buy It Now* prices to accept on the platform optimally. Similarly, agents may restrict attention to vendors in their vicinity but carefully choose from whom to buy within their vicinity. While this paper does not microfound why agents restrict attention to a subset of strategies, it analyzes the implications for behavior and payoffs when such agents are present. In other words, the paper bridges the gap between assuming that a player is either fully behavioral or fully rational in a well-known and tractable environment.

This paper has three main results. First, for a small ex ante probability of stubbornness, equilibria exist where both types randomize over the same set of demands. In contrast, (fully) separating equilibria do not exist. Second, as the ex ante probability of stubbornness vanishes, in any symmetric equilibrium – be it pooling or semiseparating – players randomize over at most two demands. More generally, for any $\epsilon > 0$, when the ex ante probability of stubbornness is sufficiently small, players randomize over offers at most ϵ away from the limit support. Third, despite these stark predictions regarding the structure of the equilibrium support, there is sufficient flexibility in the value of these demands to establish Folk-theorem-like payoff multiplicity for the rational type. Any feasible payoff, and hence delay of any length, can arise in equilibrium when the probability that a player is stubborn is sufficiently small.

Relative to the model with exogenous types [Myerson 1991, Abreu and Gul 2000 (AG)], the behavioral predictions are stronger and the payoff predictions are weaker. In AG, for any (positive) ex ante probability of stubbornness, the rational type may randomize over offers with arbitrary support. Once the stubborn type is given a choice over his initial demand, the behavioral type that allows for payoff predictions in AG regardless of who else is there may not be present. More generally, this paper shows that assuming the presence of any particular behavioral type is not innocuous. If behavioral types are used as a modeling device to sharpen payoff predictions, equilibrium selection or, equivalently, the presence of a particular behavioral type matters.

The reason that separating equilibria fail to exist is that preferences do not satisfy

single-crossing. The intuition for this is as follows: fixing the opponent’s belief and demand, preferences over demands are single-peaked. Players face a tradeoff between the amount received if the opponent concedes and the speed with which the opponent concedes: the higher a player’s demand, the more he obtains when the opponent concedes, but everything else being equal, the higher a player’s demand, the slower the opponent concedes. This tradeoff makes intermediate demands particularly attractive, leading a player’s payoff to be single-peaked in his own demand. However, this single-peakedness looks different for the two types. In particular, single-peakedness is more pronounced for the rational type. When demands are compatible, the two types receive the same payoff. When facing an incompatible demand, the rational type is able to concede while the stubborn type cannot. A rational player is willing to wait to concede as long as he is uncertain about the opponent’s type. Yet, once the player is certain he is facing a stubborn opponent, he strictly prefers to concede. However, a stubborn type cannot do so. This war of attrition takes longer the higher the demands are. As a result, the time at which the rational type has a strict preference for conceding is “far into the future,” and hence, the stubborn type’s cost of not being able to concede is low when demands are high. Intermediate demands, on the other hand, mean that demands are incompatible with a short war of attrition, allowing the rational type to leverage his flexibility. Hence, indifference curves (with the dimensions being a player’s own demand and the opponent’s belief) cross twice – a violation of single-crossing.

In contrast, in the model with exogenous types, the type of equilibrium that exists follows from the distribution of stubborn types that is assumed. In other words, if stubborn types exclusively make low demands, then the rational type does not benefit from mimicking such types, and hence, types separate in equilibrium. If, on the other hand, the stubborn types make demands that are sufficiently high, the rational type mimics every one of those demands, and hence, types pool in equilibrium.

The intuition for why as the ex ante probability of stubbornness vanishes, players randomize over at most two demands is as follows. It starts with two observations already true in AG: in the limit, higher offers immediately concede to lower offers with probability 1. Moreover, despite the rational type being willing to randomize over a large number of offers, the offer closest to (but weakly larger than) $1/2$ is made with probability 1 when there are more than two demands.¹ Given this, the stubborn type

¹To be precise, the lowest offer above $1/2$ is made with probability 1 in the limit unless there are two demands with the lower demand being less than $1/2$ and the higher demand being incompatible

is not indifferent over multiple offers: fixing the demand that the stubborn type faces, the cost of being stubborn is lower the higher the demand. In other words, given the structure imposed by the rational type in terms of offers made and concession behavior, the stubborn type prefers higher demands over lower demands.

Finally, what grants flexibility in the value of the two demands – and hence, what gives rise to Folk-theorem-like payoff multiplicity – is that the probability that a demand is believed to come from a stubborn type can vary across demands. This means that players can be compensated for making otherwise “unattractive” demands by being more likely to be believed to be stubborn; similarly, players can be deterred from making otherwise “attractive” out-of-equilibrium demands by being believed to be rational. In this way, any payoff in $(0, 1/2]$ can be generated in symmetric equilibria. In the model with exogenous types, provided that the “right” stubborn type is present, the rational type can guarantee himself the Rubinstein payoff, i.e., $1/2$ in the symmetric case. More generally, for any payoff in $(0, 1/2]$, one can find a distribution that gives rise to that payoff in equilibrium.

My model builds on the framework by Myerson (1991) and AG. They consider a bargaining environment where there is a small probability of a player being behavioral. Behavioral types in AG have no choice over their actions, and the distribution of behavioral types is exogenously given. The results in this paper stand in contrast with those when the stubborn type cannot choose its initial demand freely (as in AG). First, in AG, for a fixed ex ante probability of stubbornness, the rational type may randomize over offers with arbitrary support. In my model, this is not possible: fix a set of demands (where the set has more than three elements). Then, there exists a threshold ex ante probability of stubbornness below which no equilibrium with this support exists. Second, in AG, when the probability of a player being stubborn is small, there is no delay (and hence, inefficiency), assuming that the “right” stubborn type is present. The right behavioral type is the type that makes a demand proportional to a player’s patience. If the right type is present, a rational player receives a payoff proportional to his patience. My paper shows that the right stubborn type may not be present when given a choice over his initial demand, which implies that the rational type cannot guarantee himself a strictly positive payoff. This Folk-theorem-like payoff multiplicity survives refinements such as D1 when applied to the demand game, taking the expected payoffs in the war of attrition as given.

with the lower demand.

More generally, this paper is related to the literature on reputation (Fudenberg and Levine, 1989 and 1992; Kim 2009; Abreu, Pearce and Stacchetti, 2015; Fanning, 2016 and 2018) and bargaining (Nash, 1953; Abreu and Pearce, 2015). The most closely related papers are Kambe (1999), Abreu and Sethi (2003), Wolitzky (2012) and Atakan and Ekmekci (2014), who all build on AG. Abreu and Sethi (2003) endogenize behavioral types using an evolutionary stability approach. In contrast, in my model, a stubborn type selects his initial demand to maximize his payoff. They show that if a behavioral type is present in an evolutionary equilibrium, the complementary demand must also be present – this is not true in my model. Similar to my model, they find that inefficient delays may occur in equilibrium. Atakan and Ekmekci (2014) endogenize behavioral types in a two-sided search market. The matching market serves as an endogenous outside option. Unlike in my model, stubborn types cannot choose their initial demand, but they can exit the current trade when they are certain that they face a stubborn player. Given the differences in modeling, it is difficult to compare the results of Atakan and Ekmekci (2014) to mine. In Kambe (1999) and Wolitzky (2012), players do not know at the demand stage whether they are behavioral. Rather, in Kambe (1999), a player becomes “committed” with some exogenous probability after initial demands have been chosen.² In Wolitzky (2012), a player announces a (possibly nonstationary) bargaining strategy and becomes committed to it with some exogenous probability after the announcement. In my model, a player knows his type (behavioral or rational) when choosing his demand. Which modeling choice is more reasonable depends on the situation we have in mind. For instance, assuming that players become committed with some exogenous probability may be better suited to analyzing situations where an agent’s flexibility depends on some exogenous shocks – for example, growth rates in the economy may lead a company to impose ex post constraints on a manager’s flexibility in making decisions. In other cases, the agent himself is well aware of his constraints, but these constraints are not common knowledge – a buyer knows his budget constraint, but the seller does not. Similarly, how much a buyer values a good may be private information of the buyer. Unlike in my model, the lower bound on the payoff of a rational player in Wolitzky (2012) is nonzero. Kambe (1999) shows that as the probability of a player being stubborn goes to 0, in any equilibrium, a rational type

²In an extension, Kambe (1999) also considers the case in which a player knows his type when choosing his demand. He focuses on one-demand equilibria, and the results are similar to mine in this special case.

can guarantee himself a payoff proportional to his patience. Why is it “good” for the rational type not to know his type when choosing the initial demand? In Kambe (1999), any demand is equally likely believed to come from a stubborn type since players do not know their type when choosing the demand. As a result, players cannot be incentivized to make more “extreme” demands either by being believed to be stubborn with higher probability if making such a high demand or by the “threat” of being believed to be rational if deviating from this demand. This makes “intermediate” demands particularly attractive – which in turn leads to fast agreement.

The structure of this paper is as follows. I first describe the model in Section 2. Section 3 analyzes the benchmark case with an exogenous distribution of stubborn types as in AG and discusses the preferences of the rational type. Section 4 discusses the necessary conditions for equilibrium existence with endogenous stubborn types. The main results, which focus on pooling equilibria, are presented in Section 5. Section 6 discusses (semi-)separating equilibria. In Section 7, I discuss the robustness of the results. Section 8 concludes the paper.

2 Model

The model and the notation (mostly) follow AG. Time is continuous, and the horizon is infinite. Two players decide on how to split a unit surplus. At time 0, players i and j simultaneously announce demands, α_i and α_j , with $\alpha_i, \alpha_j \in [0, 1]$. If $\alpha_i + \alpha_j \leq 1$, the demands are said to be *compatible*. In this case, the game ends. If $\alpha_i + \alpha_j > 1$, the demands are *incompatible*. In this case, a concession game starts. The game ends when one player concedes. Concession means agreeing to the opponent’s demand.

Each player i is rational with probability $1 - z$ and stubborn with probability z , where $z \in (0, 1)$. Before the game starts, each player privately learns whether he is stubborn or rational. A rational player $i = 1, 2$ can make any demand $\alpha_i \in [0, 1]$ at time 0 and concede to his opponent at any point in time. Stubborn player i can choose his initial demand $\alpha_i \in [0, 1]$ but cannot concede to his opponent. Note that this is unlike in AG, where a stubborn player cannot choose his initial demand.³

A strategy for a stubborn player, i , σ_i^S , is defined by a probability distribution s_i

³In AG, there are $K + 1$ types of players: one rational type and K stubborn types, where K is an arbitrary finite number. A stubborn player of type α_i in AG always demands α_i , accepts any demand of at least α_i , and rejects all smaller demands. They assume an exogenously given finite set of stubborn types: $C = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$.

on $[0, 1]$. A strategy for a rational player i , σ_i^R , is defined by a probability distribution r_i on $[0, 1]$ and a collection of cumulative distributions $F_{\alpha_i, \alpha_j}^{r, i}$ on $\mathcal{R}_+ \cup \{\infty\}$ for all $\alpha_i + \alpha_j > 1$. $F_{\alpha_i, \alpha_j}^{r, i}(t)$ is the probability of rational player i conceding to player j by time t (inclusive), given α_i, α_j . The probability of player i conceding by time t is given by:

$$F_{\alpha_i, \alpha_j}^i(t) = (1 - \pi_i(\alpha_i)) F_{\alpha_i, \alpha_j}^{r, i}(t),$$

where

$$\pi_i(\alpha_i) = \frac{z s_i(\alpha_i)}{z s_i(\alpha_i) + (1 - z) r_i(\alpha_i)} \quad (1)$$

is the posterior probability that player i is stubborn immediately after it is observed that i demands α_i at time zero given σ_i^R and σ_i^S . Therefore,

$$\lim_{t \rightarrow \infty} F_{\alpha_i, \alpha_j}^i(t) \leq 1 - \pi_i(\alpha_i).$$

Note that $F_{\alpha_i, \alpha_j}^i(0)$ may be positive. It is the probability that i immediately concedes to j .

Player i 's discount rate is $\rho > 0$, for $i = 1, 2$. The continuous-time bargaining problem is denoted $B = \{z, \rho\}$. If $\alpha_i + \alpha_j \leq 1$ at $t = 0$, player i receives α_i and $1 - \alpha_j$ with probability $1/2$. Suppose that $\bar{\alpha} = (\alpha_i, \alpha_j)$ is observed at time 0, with $\alpha_i + \alpha_j > 1$. Then, player i 's expected payoff from conceding at time t , given strategy profile $\bar{\sigma} = (\sigma_i, \sigma_j)$, where $\sigma_i = (\sigma_i^R, \sigma_i^S)$, is:

$$U_i(t, \sigma^j | \bar{\alpha}) = \alpha_i \int_{y < t} e^{-\rho y} dF_{\bar{\alpha}}^j(y) + \frac{\alpha_i + 1 - \alpha_j}{2} (F_{\bar{\alpha}}^j(t) - F_{\bar{\alpha}}^j(t^-)) e^{-\rho t} + (1 - \alpha_j) (1 - F_{\bar{\alpha}}^j(t)) e^{-\rho t}, \quad (2)$$

where $F_{\bar{\alpha}}^j(t^-) = \lim_{y \uparrow t} F_{\bar{\alpha}}^j(y)$. Hence, player i receives the discounted value of his demand α_i if player j concedes to i before i concedes to j . If the players concede simultaneously, then player i receives his own demand and the complement of player j 's demand with equal probability. Player i receives the discounted value of the complement of player j 's demand, $1 - \alpha_j$, if player i concedes first. Player i 's expected payoff from never conceding is:

$$U_i(\infty, \sigma^j | \bar{\alpha}) = \alpha_i \int_{y \in [0, \infty)} e^{-\rho y} dF_{\bar{\alpha}}^j(y). \quad (3)$$

This is a stubborn player's payoff from facing a demand that is incompatible with his own demand. Since F_{α_i, α_j}^i describes the concession behavior of a player, unconditional on his type, a rational player i 's concession behavior is described by:

$$\frac{1}{1 - \pi_i(\alpha_i)} F_{\alpha_i, \alpha_j}^i.$$

Therefore, a rational player i 's expected utility from a mixed action F^i conditional on $\bar{\alpha} = (\alpha_i, \alpha_j)$ being observed at time 0 is:

$$U_i(\bar{\sigma} \mid \bar{\alpha}) = \frac{1}{1 - \pi_i(\alpha_i)} \int_{y \in [0, \infty)} U_i(y, \sigma_j \mid \bar{\alpha}) dF_{\bar{\alpha}}^i(y). \quad (4)$$

A rational player i 's expected utility from the strategy profile $\bar{\sigma}$ is:

$$\begin{aligned} U_i(\bar{\sigma}) = & \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j \leq 1 - \alpha_i} \frac{\alpha_i + 1 - \alpha_j}{2} ((1 - z)r_j(\alpha_j) + zs_j(\alpha_j)) \right) \\ & + \sum_{\alpha_i} r_i(\alpha_i) \left(\sum_{\alpha_j > 1 - \alpha_i} U_i(\bar{\sigma} \mid \alpha_i, \alpha_j) ((1 - z)r_j(\alpha_j) + zs_j(\alpha_j)) \right). \end{aligned} \quad (5)$$

The first term is the payoff a rational player receives from demanding α_i when $\alpha_i + \alpha_j \leq 1$. The second term is the payoff from demanding α_i when facing an incompatible demand. At this stage, it is useful for me to introduce two pieces of notation. I denote the probability of player j facing demand α_i by q_i , i.e.,

$$q_i = (1 - z)r_i(\alpha_i) + zs_i(\alpha_i).$$

Moreover, I denote player i 's strength by $\mu_i(\alpha_i)$, where strength is defined as:

$$\mu_i(\alpha_i) = \pi_i(\alpha_i)^{\frac{1}{1 - \alpha_i}}.$$

As becomes clear in the next section, the strength of a player is key to pinning down the war of attrition.

Leaving aside the technical issues of defining continuous-time strategies that would allow revisions of demands, changing one's demand reveals rationality. As we know from Myerson (1991), revealing rationality when the opponent is stubborn with positive probability leads the rational player to immediately concede (i.e., to further revise the

demand to make it compatible with the opponent's).⁴ Hence, revision is essentially equivalent to concession.

For the analysis in $B = \{z, \rho\}$, I use the solution concept of Perfect Bayesian equilibrium (PBE). As usual, a PBE is a profile of strategies $\sigma^* = (\sigma_1^*, \sigma_2^*)$ and a system of beliefs mapping demands into probabilities that a player is stubborn,

$$\pi_i : [0, 1] \rightarrow [0, 1] \quad \text{for } i = 1, 2,$$

such that the strategy maximizes a player's expected utility (given beliefs), and the beliefs are formed according to Bayes' rule, where possible (see Fudenberg and Tirole, 1991 for a formal definition).⁵ Henceforth, equilibrium refers to PBE.

3 Benchmark and preferences of the rational type

In this section, I recall the unique equilibrium outcome when stubborn players have no choice over their initial demand, as in AG.⁶ This serves as a benchmark for the subsequent analysis. I also discuss the preferences of the rational type in some detail and what they imply for the structure of the equilibrium, both in AG and in my model.

There is an exogenously given set of stubborn types $C = \{\alpha_1, \alpha_2, \dots, \alpha_K\}$, where $\alpha_k < \alpha_{k+1}$ and $\alpha_K < 1$. A stubborn player of type α_i always demands α_i , accepts any offer of at least α_i , and rejects all smaller offers.

I denote the probability that stubborn player i is of type α_k by $s_i(\alpha_k)$. Hence, s_i is a probability distribution on C . The continuous-time bargaining problem is denoted $B^{AG} = \{(C, z, s_i, \rho)_{i=1}^2\}$. Proposition 1 (AG) establishes the existence and uniqueness of the equilibrium outcomes with a given distribution of stubborn types.

Proposition 1 (AG, Proposition 2). *For any bargaining game B^{AG} , a PBE exists. Furthermore, all equilibria yield the same distribution over outcomes.*

⁴AG show that any convergent sequence of equilibrium outcomes within a broad family of discrete-time games must converge to the unique continuous-time equilibrium outcome as the maximum time between consecutive demands goes to 0. The modeling of AG differs from mine in some respects – first, they assume that players make initial demands sequentially, while in my model players make initial demands simultaneously. Second, they allow players to choose their demand in the open interval $(0, 1)$, whereas I allow players to choose any demand in $[0, 1]$. However, given the initial demands, my continuation game is identical to that for which AG's convergence result is established.

⁵Note that here I only specify the initial updating of beliefs. Strictly speaking, an equilibrium should also specify beliefs after arbitrary histories, but given that this is a stopping game, the only “surprise” ends the game.

⁶In AG, players make demands sequentially rather than simultaneously (as in my model).

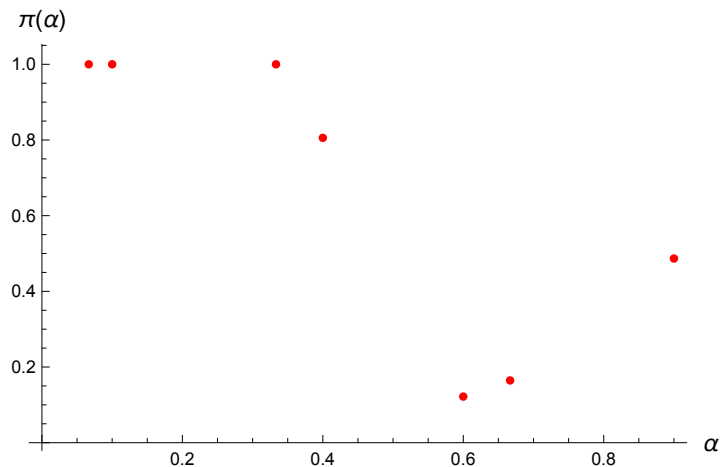


Figure 1: Probability of stubbornness conditional on the demand α , $\pi(\alpha)$, in a PBE with exogenous stubborn types (see body for parameters).

The unique equilibrium outcome in this game can be characterized by the two choices that a rational player makes: (1) when to concede and (2) whom to mimic. In the equilibrium, after the initial choice of demands, (i) at most one player immediately concedes with positive probability; (ii) players concede at a constant rate that makes the opponent indifferent between waiting and conceding; and (iii) there is a finite time, call it T_0 , by which the posterior probability of stubbornness reaches 1 simultaneously for both players and concessions by the rational type stop. Moreover, any demand above some threshold is mimicked with positive probability.

I illustrate the mimicking behavior of the rational type in Figure 1. The figure shows the posterior probability of stubbornness in an equilibrium.⁷ We can see that the lower three demands are not mimicked by the rational type, i.e., $\pi(\alpha|\alpha \leq \frac{1}{3}) = 1$. On the other hand, any demand of $\frac{2}{5}$ or higher is mimicked with positive probability, i.e., $\pi(\alpha|\alpha \geq \frac{2}{5}) < 1$. The U-shaped structure of the posterior probability above the threshold is driven by the concept of strength, as defined and discussed below.

Let me be more precise regarding the rate of concession and the stopping time of the rational type. Player i is indifferent between waiting and conceding if the net cost of waiting is equal to the net benefit of waiting:

$$\rho(1 - \alpha_j) = (\alpha_i - (1 - \alpha_j)) \frac{f_{\alpha_j, \alpha_i}^j(t)}{1 - F_{\alpha_j, \alpha_i}^j(t)},$$

⁷In particular, I choose a PBE with seven stubborn types $C = \{\frac{1}{15}, \frac{1}{10}, \frac{1}{3}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}, \frac{9}{10}\}$, and $z = \frac{1}{3}$.

where $f_{\alpha_j, \alpha_i}^j(t) = dF_{\alpha_j, \alpha_i}^j(t)/dt$. By waiting, a player loses the value of concession over the next instant, which, given a player's impatience, is given by the LHS. The first term on the RHS captures the benefit from being conceded to relative to conceding. The second term on the RHS is the probability with which the opponent concedes in the next instant conditional on not yet having conceded. Therefore, after time 0, player j demanding α_j concedes to player i demanding α_i at a rate

$$\lambda_j^{\alpha_j, \alpha_i} = \frac{\rho(1 - \alpha_j)}{\alpha_i + \alpha_j - 1}.$$

Note that the rate at which player j concedes is decreasing in player i 's demand: the more a player demands, the more he receives conditional on his opponent conceding. Therefore, the rate with which the opponent has to concede to make a player indifferent is lower the higher the player's demand is. Note also that this rate of concession is time-independent. However, only the rational type concedes, which implies that the probability of facing a rational opponent is decreasing over time. Hence, the rational type's rate of concession is increasing over time.

Requirement (iii) pins down the identity of the player who concedes at time 0 and the probability with which this happens.⁸ Let $T_i^{\alpha_i, \alpha_j}$ denote the time at which player i is stubborn with probability 1 conditional on not conceding with positive probability at time 0. Then, the time T_0 is given by:

$$T_0 = \min\{T_1^{\alpha_1, \alpha_2}, T_2^{\alpha_2, \alpha_1}\},$$

where

$$T_i^{\alpha_i, \alpha_j} = -\frac{1}{\lambda_i^{\alpha_i, \alpha_j}} \log \pi_i(\alpha_i)$$

for $i = 1, 2$. Player i is *stronger* than player j if and only if $T_i^{\alpha_i, \alpha_j} < T_j^{\alpha_j, \alpha_i}$. In other words, a player is stronger the sooner is the time at which he is known to be stubborn. Note that

$$T_i^{\alpha_i, \alpha_j} < T_j^{\alpha_j, \alpha_i} \iff \pi_i(\alpha_i)^{\frac{1}{1-\alpha_i}} > \pi_j(\alpha_j)^{\frac{1}{1-\alpha_j}}.$$

For the rest of the paper, I will denote a player's strength by $\mu_i(\alpha_i)$, where

$$\mu_i(\alpha_i) = \pi_i(\alpha_i)^{\frac{1}{1-\alpha_i}}.$$

The weaker player j has to concede with sufficient probability at time zero that conditional on not conceding, and given the concession rates, his probability of stubbornness

⁸For intuition for (iii), see AG page 10.

reaches 1 at the same time as player i . In particular, the probability of immediate concession by player j is given by:

$$F_{\alpha_j, \alpha_i}^j(0) = \max \left\{ 1 - \left(\frac{\mu_j(\alpha_j)}{\mu_i(\alpha_i)} \right)^{1-\alpha_j}, 0 \right\}. \quad (6)$$

The derivation follows AG. The strength of player j relative to player i depends on (i) how likely j is thought to be stubborn conditional on his demand and (ii) how high j 's demand is. Clearly, the more likely a player is thought to be stubborn, the more willing the opponent is to concede. The higher a player's demand, the more willing his opponent is to wait. This is because conditional on giving up, a player obtains less the higher his opponent's demand. Hence, the lower the demand a player makes, the stronger he is because it makes his opponent more willing to concede. Everything else being equal, a player's payoff is increasing in his strength. Consider an incompatible pair of demands. In equilibrium, a weak player is not conceded to with positive probability at time 0. He is indifferent between waiting and conceding and hence must receive what he would receive by conceding immediately. A strong player is conceded to with positive probability at time 0, in which case the player obtains what he demanded, which he strictly prefers over conceding himself. If the strong player is not conceded to at time 0, he also simply receives what he would have received by conceding immediately. The probability with which the opponent concedes to the strong player is strictly increasing in the player's strength. This yields a tradeoff: The more a player demands, the more a player receives conditional on being conceded to immediately. However, the more a player demands, the lower the probability with which the opponent concedes at time 0. This makes intermediate demands particularly attractive for the rational type. The following Lemma is a straightforward consequence of this tradeoff for the rational type. Let $\underline{\alpha}$ denote the lowest demand.

Lemma 1. *Fix any set of demands C . In any symmetric equilibrium with support C , strength $\mu(\alpha)$ is decreasing in $\alpha \in C$, strictly so unless $\alpha \geq 1 - \underline{\alpha}$.*

Proof. See Appendix. □

The key intuition for the proof is that $F_{\alpha_i, \alpha_j}^i(0)$ is increasing in $\mu_j(\alpha_j)$. If $\mu_j(\alpha_j)$ were increasing in α_j , a player would always benefit from increasing his demand α_j . This is inconsistent with a player being indifferent between demands.

Let me return to the U-shaped posterior probability of stubbornness in Figure 1. Suppose that player j demands α_j with probability 1 and is thought to be stubborn with probability $\pi(\alpha_j)$. Then, fixing the probability of player i being stubborn, the preferences of a rational player i are single-peaked in his own demand α_i : he trades off the probability with which his opponent concedes at time 0, with how high his payoff is conditional on his opponent conceding. This implies that in equilibrium, the conditional probability of stubbornness must be single-bottomed in α_i , as Figure 1 shows.

Proposition 2 (AG, Corollary in Section 5). *Let $B_n^{AG} = \{(C, z_n, s_i)_{i=1}^2\}$ be a sequence of continuous-time bargaining games such that $\lim_{n \rightarrow \infty} z_n = 0$. Let ϵ be the mesh of $C \cup \{0, 1\}$.⁹ Then, for n sufficiently large, the equilibrium payoff of player i is at least $\frac{1}{2} - \epsilon$, and hence, the inefficiency due to delay is at most 2ϵ .*

Proposition 2 states that as the probability of a player being stubborn goes to 0, delay and inefficiency disappear provided that the “right” behavioral type is present. By the right type, I mean the type whose presence alone, i.e., regardless of which other behavioral types are present, allows for payoff predictions. The right type in this sense is the type making a demand proportional to a player’s patience. In the symmetric discounting case, the right type is then a type demanding $1/2$, and a rational type can guarantee himself a payoff of $1/2$ in the limit. The loose argument for why the right type is the type demanding $1/2$ is as follows. Since there cannot be any delay, it must mean that with almost probability 1, a player’s demand is immediately accepted. Since this is true for both players, it must mean that a player can mimic type $1/2$.¹⁰ Note that if the right behavioral type is not present, delay and hence inefficiency persist in AG. For instance, if $C = \{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$, then the limit payoff of the rational type is $\frac{2}{5}$.

4 Preferences of the stubborn type

This section addresses the preferences of the stubborn type and what they imply for the demand configurations that can arise in equilibrium. The difference between the rational and the stubborn type is the payoff when facing an incompatible demand coming from a potentially stubborn opponent. Suppose that a rational type faces an incompatible demand. In equilibrium, a rational type is willing to wait until he is sure to face a stubborn opponent. However, once he assigns probability 1 to facing a stubborn

⁹I.e., $\max_k (\alpha_{k+1} - \alpha_k)$, ordering the demands from smallest to largest.

¹⁰More generally, a rational player in AG obtains a payoff proportional to his patience.

opponent, the rational type strictly prefers to concede. However, the stubborn type does not have this “option value of concession” – conditional on facing an incompatible demand from a stubborn opponent, a stubborn player receives a payoff of 0. Hence, when demands are incompatible, the expected payoff to a rational player is strictly higher than the payoff to a stubborn type. Therefore, unless every demand made with positive probability is compatible with every other demand, the equilibrium payoff for a rational player must be strictly higher than the payoff for a stubborn player. Suppose that every demand is compatible with every other demand made with positive probability, i.e., all demands have to lie below $1/2$. This cannot be an equilibrium since a rational player would then strictly prefer to deviate to a demand above $1/2$. Hence, if $|C| > 1$, the payoff a rational player receives is strictly higher than the payoff a stubborn player receives.

Lemma 2. *Fix any set of demands C . In any symmetric equilibrium with support C , the following holds:*

1. *The lowest demand in C , which is played with positive probability by both types, is incompatible with the highest demand in C .*
2. *The set (or, equivalently, number) of compatible equilibrium demands is strictly decreasing in the demand the stubborn type makes with positive probability; i.e., if $\alpha < \alpha'$, with $\alpha, \alpha' \in \text{supp } s$, then there exists $\alpha'' \in C$ such that $\alpha + \alpha'' \leq 1 < \alpha' + \alpha''$.*

Proof. See Appendix. □

While a formal proof can be found in the Appendix, a heuristic argument for the first part of Lemma 2 is as follows: Suppose that the lowest demand were compatible with the highest demand. Then, the payoff from making the lowest demand would be the same for a rational and a stubborn player. However, if it is the same for the lowest demand, it must be the same for every other demand with positive probability. By the above argument, this cannot be.

The heuristic argument for the second part of Lemma 2 is as follows. Fix a set of demands C . Suppose that player j makes all demands in C with positive probability, and suppose further that the rational type of player i is indifferent over all demands in

C . If the stubborn type of player i is indifferent, then the difference in expected payoff between a stubborn and a rational type must be identical for each demand i makes.

Conditional on facing a compatible demand from player j , the stubborn type does not pay a cost for being stubborn. Conditional on facing a certain incompatible demand, call it α_j , the stubborn type's cost for being stubborn is higher the lower his own demand is. To see this, suppose that the opponent, say player j , demands α_K . Let me compare player i demanding α_1 versus demanding α_2 (although this generalizes to any other two consecutive demands when facing any other higher demand).

In equilibrium, player i is willing to wait until time \bar{T}_i to concede (i.e., the time at which player j is known to be stubborn), where of course $\bar{T}_1 < \bar{T}_2$. Recall that (1) the probability of immediate concession by the opponent is decreasing in the demand (i.e., α_K is more likely to concede immediately to α_1 than to α_2); (2) the rate of concession by α_K is higher for α_1 than for α_2 ; and (3) conditional on j not conceding at $t = 0$, the expected payoff for the rational type of player i at any point before \bar{T}_i is $1 - \alpha_K$ regardless of whether i demands α_1 or α_2 . Now suppose that player i does wait until \bar{T}_i to concede (despite j expecting him to concede at the appropriate rate). Then, the probability with which player j concedes (ever) is $1 - \mu_K^{1-\alpha_K}$ – i.e., regardless of the demand i makes. Hence, conditional on j not conceding immediately, the probability with which j concedes (ever) is higher when player i demanded α_2 than when player i demanded α_1 . The stubborn type only pays a cost for being stubborn conditional on his opponent not conceding immediately. Hence, conditional on facing a certain demand, the stubborn type's cost for being stubborn is lower the higher the demand. Therefore, for the expected cost to the stubborn type to be identical for α_1 and α_2 , the opponent must make a demand with positive probability, which imposes a cost of being stubborn on the player when demanding α_2 but not when demanding α_1 . In other words, there must exist a demand that is compatible with α_1 but not with α_2 .

Therefore, for the stubborn type to be indifferent over any two demands in C , the set (or, equivalently, number) of compatible equilibrium demands is strictly decreasing in the demand the stubborn type makes with positive probability.

5 Existence of pooling equilibria

This section establishes the existence of symmetric pooling equilibria and Folk-theorem-like payoff multiplicity. By symmetric equilibria, I mean equilibria where the set of demands over which player i randomizes is identical to the set of demands over which

player j randomizes.¹¹ By pooling equilibria, I mean equilibria where the set of demands over which a player randomizes is identical for the stubborn and the rational type. Since preferences do not satisfy single-crossing, pooling equilibria where players randomize over multiple demands exists and fully separating equilibria (see Section 6) do not. However, as the probability of stubbornness vanishes, an equilibrium in which both types of players assign positive probability to every equilibrium demand must involve one or two demands in its support. Endogenizing the choice of the stubborn type imposes severe restrictions on the number of demands that can be made. Crucially, despite these stark predictions regarding the structure of the equilibrium support, there is sufficient flexibility in the demands themselves to establish Folk-theorem-like payoff multiplicity. Any feasible payoff can arise as an equilibrium payoff when the probability that a player is stubborn is sufficiently small. This implies that even in the limit, delay can be arbitrarily long. In other words, the stubborn type may not find it optimal to choose the “right” demand (in the sense of AG) – the “right” behavioral type needed to derive payoff predictions may not be present. When the right type is not present, even in the limit, delay does not disappear. Throughout, I order demands from lowest to highest, denoting the lowest demand by α_1 and the highest demand by α_K .¹²

5.1 Existence with one demand

Proposition 3 below establishes that equilibria where players make only one demand exist. In such equilibria, there is either an infinitely long delay or immediate agreement but nothing in between.

Proposition 3. *Equilibria where players make only one demand, α , exist. In any such equilibrium, there is either*

- (i) *an infinitely long delay, and $\alpha = 1$, or*
- (ii) *immediate agreement, and $\alpha = \frac{1}{2}$.*

Proof. Suppose that players choose a demand $\alpha < 1/2$. Then, both types of players have an incentive to deviate to $1 - \alpha$. Suppose instead that players choose a demand $1 > \alpha > 1/2$. The expected payoff for a rational player in this candidate equilibrium is

¹¹Given the symmetry assumption, I will simplify notation in the remainder of the paper. In particular, $r_i(\alpha_k) = r_k$, $s_i(\alpha_k) = s_k$, $\mu_i(\alpha_k) = \mu_k$ etc.

¹²This applies in particular to supports.

$1 - \alpha$. The expected payoff for a stubborn player from demanding α is $(1 - \alpha)(1 - z^{\frac{\alpha}{1-\alpha}})$. However, a stubborn player could receive $1 - \alpha$ by demanding $1 - \alpha$. If players demand $1/2$, then $\alpha = 1 - \alpha$, and hence, there is no such deviation. Suppose that $\alpha = 1/2$. Then, if any deviation is believed to come from a rational type, neither player type wants to deviate. If players demand $\alpha = 1$, then similarly, there is no such deviation. \square

Hence, symmetric equilibria with one demand allow for strong predictions in terms of payoffs and behavior. Independent of the probability of stubbornness, there is either no inefficiency or complete surplus dissipation due to the infinitely long delay. Note that in any such equilibrium, the rational type and the stubborn type receive the same payoff. Subsection 5.2 shows that this does not generalize to equilibria with more than one demand. The reader might wonder whether it is reasonable to require players to assign probability 1 to any deviation coming from the rational type – I defer the discussion of off-equilibrium-path beliefs to Section 7, where refinements are introduced.

5.2 Existence with two demands

This subsection establishes that equilibria where the two types of players are mixing over two demands, α_1 and α_2 , exist. After stating the result formally, I provide intuition by discussing the preferences of the two types.

Proposition 4. (a) *Fix a sequence $z^n \rightarrow 0$. Fix a corresponding convergent sequence of equilibria $(\alpha_1^n, \alpha_2^n, r^n, s^n)$. Then, there exist $a_1 \in (0, 1/2]$ and $a_2 \in (1 - a_1, 1]$ such that*

$$\lim_{n \rightarrow \infty} \alpha_1^n = a_1, \quad \lim_{n \rightarrow \infty} \alpha_2^n = a_2. \quad (7)$$

Moreover, along any such sequence,

$$\lim_{n \rightarrow \infty} \begin{pmatrix} r_1^n \\ r_2^n \end{pmatrix} = \begin{cases} \frac{2(a_1 + a_2 - 1)}{2a_2 - 1}, \\ \frac{1 - 2a_1}{2a_2 - 1}, \end{cases} \quad \text{and} \quad \lim_{n \rightarrow \infty} \begin{pmatrix} s_1^n \\ s_2^n \end{pmatrix} = \begin{cases} \frac{1 - a_2}{2 - a_1 - a_2}, \\ \frac{1 - a_1}{2 - a_1 - a_2}. \end{cases} \quad (8)$$

(b) *For any $a_1 \in (0, 1/2]$ and $a_2 \in (1 - a_1, 1]$, there exists a sequence $z^n \rightarrow 0$ and a corresponding convergent sequence of equilibria $(\alpha_1^n, \alpha_2^n, r^n, s^n)$ satisfying (7) and (8).*

Proof. See Appendix. \square

Proposition 4 states that a stubborn player and a rational player can be indifferent between the same two demands, despite the distinct preferences in the reduced game,

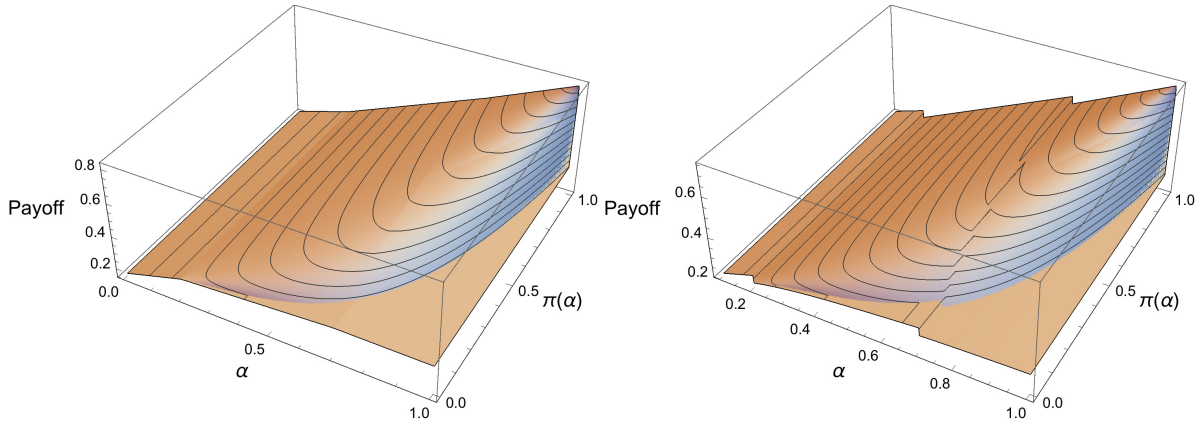


Figure 2: 3D-Payoff profile for a rational type (left) and a stubborn type (right) for a fixed set of demands and posterior probabilities of the opponent.

given their strategic differences. Note that even in the limit, as the probability of stubbornness goes to 0, both types assign strictly positive probability to both demands. This may be surprising to the reader – it implies that unlike in standard models of signaling, preferences in my (reduced-form) model do not satisfy the single-crossing property. Fixing the opponent’s belief and demand, preferences over demands are single-peaked: players face a tradeoff between the amount they receive if the opponent concedes and the speed with which the opponent concedes. Conditional on the opponent conceding, a player receives more the higher his own demand is. However, everything else being equal, the higher a player’s demand, the slower the opponent concedes. This tradeoff makes intermediate demands particularly attractive, leading a player’s payoff to be single-peaked in his own demand. However, the tradeoff is not identical for the stubborn and rational types, and in particular, single-peakedness is more pronounced for the rational type. When demands are compatible, the two types receive the same payoff. When demands are incompatible, the rational type is able to concede while the stubborn type cannot. During the war of attrition, a rational type is willing to wait as long as he is uncertain about the opponent’s type. However, once the player assigns probability 1 to facing a stubborn opponent, he strictly prefers to concede. However, a stubborn type cannot do so. This war of attrition takes longer the higher the demands are: loosely speaking, the higher his demand, the more willing a player is to wait until he is conceded to. As a result, the time at which the rational type has a strict preference for conceding is “far into the future.” Discounting then implies that the stubborn type’s

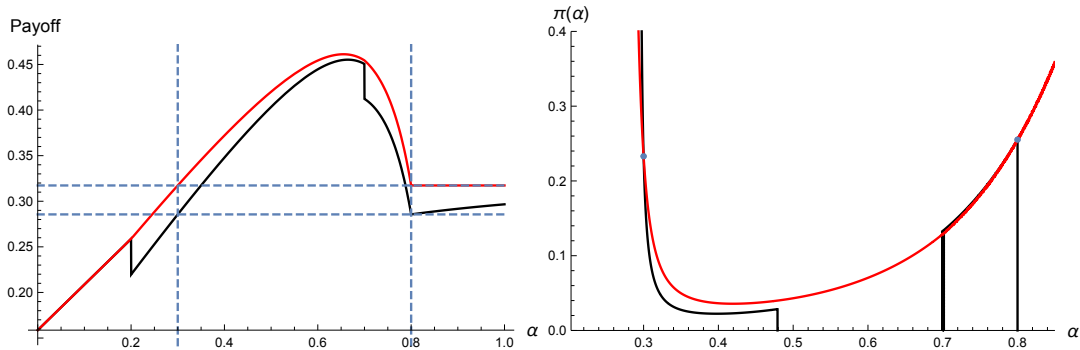


Figure 3: Cross-sections of the 3D-payoff profile (left) and indifference correspondences (right) for rational (red) and stubborn (black) type (see body for parameters).

cost of not being able to concede is low when demands are high. Intermediate demands result in a short war of attrition, allowing the rational type to leverage his flexibility. Hence, when we look at indifference curves (with the dimensions being a player’s own demand and the opponent’s belief), they cross twice – a violation of single-crossing that explains why pooling equilibria, where types randomize over demands, exist and separating equilibria do not (see Section 6).

To see this with the help of an example, fix a pair of demands, say $3/10$ and $8/10$, over which the opponent j randomizes. Moreover, fix an associated probability of stubbornness for each of these demands.¹³ Figure 2 shows the 3D-payoff profile of a rational and stubborn player i as a function of his own demand α and $\pi(\alpha)$. By payoff, I mean the lottery over equilibrium payoffs in the concession game when demands and the associated probabilities of stubbornness are drawn according to this distribution. Therefore, Figure 2 shows the equilibrium payoff of a rational (stubborn) player i when the opponent j mixes over $3/10$ and $8/10$, and I take α_i and $\pi(\alpha_i)$ as given (not necessarily optimal). First, for both types of players, the payoff is increasing in the probability of being thought to be stubborn, $\pi(\alpha)$. This is not surprising: the higher the probability that a player is thought to be stubborn, the more likely an opponent is to concede immediately at time 0. Second, the rational type’s payoff is single-peaked in α . In other words, there is a unique best reply for a rational player to a given demand of the opponent. Third, the stubborn type’s payoff is “nearly” single-peaked in

¹³In particular, I use the equilibrium probabilities of stubbornness conditional on the demands $3/10$ and $8/10$.

α – but there are two discontinuities at the points where a player’s demand becomes incompatible with a demand his opponent makes with positive probability. Hence, there may be no unique best reply for a stubborn player to a given demand of the opponent – fixing the opponent’s demand and probability of stubbornness, a stubborn player’s payoff has three local peaks as we vary α_i (either a demand is compatible with the opponent’s, it is incompatible and stronger, or it is incompatible and weaker).

Consider an equilibrium with $z = 1/100$, $\alpha_1 = 3/10$, and $\alpha_2 = 8/10$. The right panel of Figure 3 shows the indifference correspondences of a rational and stubborn type in this equilibrium (rational type in red, stubborn type in black); in other words, this is a (horizontal) cross-section of Figure 2. We can see that the indifference correspondences cross at $3/10$ and $8/10$. The left panel of Figure 3 shows a (vertical) cross-section of the 3D-payoff profile of player i as a function of α_i and π_i . In particular, I take the cross-section through $(3/10, \pi(3/10))$ and $(8/10, \pi(8/10))$, where $\pi(3/10)$ and $\pi(8/10)$ are the equilibrium probabilities of stubbornness. We can see that there is a discontinuity in the payoff of the stubborn type at $\alpha_i = 2/10$ and $\alpha_i = 7/10$, as α_i becomes incompatible with $8/10$ and $3/10$, respectively. To illustrate, let us consider the range of demands that are compatible if the opponent demands $3/10$ (i.e., demands between 0 and $7/10$). The difference in payoff between the two types is greatest for intermediate demands: demands that are “just incompatible” if the opponent demands $8/10$. When demanding between 0 and $2/10$, the two types receive the same payoff: the probability of facing an incompatible demand is 0. When the probability of facing an incompatible demand is nonzero, the payoffs of the two types differ: the rational type has the option value of concession – if his opponent is known to be stubborn, the rational type can (and strictly prefers to) concede, while the stubborn type cannot. In other words, conditional on facing an incompatible demand from a stubborn type, the stubborn type’s payoff is 0. Conditional on making a demand that is incompatible if the opponent demands $8/10$ but compatible if the opponent demands $3/10$, the difference between the two types is decreasing in the demand. Making a higher demand does not change the probability of facing an incompatible demand, but it does change the length of the war of attrition. The higher the demand is, the longer the war of attrition in the case of incompatible demands. An intermediate demand leads to a short war of attrition, allowing the rational type to leverage his flexibility while the stubborn type cannot. In other words, making a demand that is “just incompatible” with a demand that the opponent makes with positive probability is costly to the stubborn type: the probability of facing an

incompatible demand has increased, but the war of attrition is expected to be short, and hence, the time at which the stubborn type would like to concede but cannot is in the near future.

The content of Proposition 4 is that for any α_1 and α_2 , where $\alpha_1 \in (0, 1/2]$, and $\alpha_2 \in (1 - \alpha_1, 1]$, such a cross-cut as in Figure 3 which makes both types indifferent exists. The difference in the preferences of the two types imposes significant structure on the demand configurations that can occur in equilibrium. In the case of three demands, the lower two demands need to be compatible. Hence, as $z \rightarrow 0$, it becomes “more difficult” to make players indifferent between different demands. This is what I turn to in the next subsection.

5.3 Existence with three or more demands

The proposition below states that players can be made indifferent over more than two demands. However, the demand configurations over which players can be indifferent have a very specific structure. Define

$$(s_1, s_2, s_3) = \begin{cases} \left(\frac{1-a_3}{2-a_1-a_3}, 0, \frac{1-a_1}{2-a_1-a_3} \right) & \text{if } a_1 > 1 - \frac{a_3}{4} - \sqrt{a_3(8 - 7a_3)}, \\ (0, 1, 0) & \text{otherwise.} \end{cases}$$

Proposition 5. (a) *Fix a sequence $z^n \rightarrow 0$ and a corresponding sequence of equilibria whose support $\{\alpha_1^n, \alpha_2^n, \alpha_3^n\}$. Then, there exist $a_1 \in (0, 1/2]$ and $a_3 \in (1 - a_1, 1]$ such that*

$$\lim_{n \rightarrow \infty} (\alpha_1^n, \alpha_2^n, \alpha_3^n) = (a_1, 1 - a_1, a_3). \quad (9)$$

Moreover, along any such sequence,

$$\lim_{n \rightarrow \infty} r^n = (0, 1, 0), \quad \lim_{n \rightarrow \infty} s^n = (s_1, s_2, s_3). \quad (10)$$

(b) *For any $a_1 \in (0, 1/2]$ and $a_3 \in (1 - a_1, 1]$, there exists a sequence $z^n \rightarrow 0$ and a corresponding convergent sequence of equilibria $(\alpha_1^n, \alpha_2^n, \alpha_3^n, r^n, s^n)$ satisfying (9) and (10).*

Proof. See Online Appendix. □

Note that Proposition 5 states that in the limit, players face a demand of $1 - \alpha_1$ with probability 1.

Proposition 6. Fix a sequence $z^n \rightarrow 0$ and a corresponding sequence of equilibria whose support $\{\alpha_1^n, \dots, \alpha_K^n\}$ converge with $K > 3$. Then, there exist $a_1 \in (0, 1/2]$ and $a_K \in (1 - a_1, 1]$ such that

$$\lim_{n \rightarrow \infty} (\alpha_1^n, \dots, \alpha_{k-1}^n, \alpha_k, \dots, \alpha_K^n) = \left(\underbrace{a_1, \dots, a_1}_{\lceil K/2 \rceil - 1 \text{ terms}}, \underbrace{1 - a_1, \dots, 1 - a_1, a_K}_{K - \lceil K/2 \rceil + 1 \text{ terms}} \right), \quad (11)$$

where $k = \lceil K/2 \rceil$. Moreover, along any such sequence,

$$\lim_{n \rightarrow \infty} r^n = \left(\underbrace{0, \dots, 0}_{K-2 \text{ terms}}, 1, 0 \right).$$

Proof. See Online Appendix. □

The intuition for Propositions 5 and 6 is as follows. It starts with two observations already true in AG: as the probability of stubbornness vanishes, higher offers immediately concede to lower offers with probability 1. In addition, despite the rational type being willing to randomize over a large number of offers, the offer closest to (but weakly larger than) $1/2$ is made with probability 1 (when there are more than two demands). Given this, the stubborn type is not indifferent over multiple offers: fixing the demand that the stubborn type faces, the cost of being stubborn is lower the higher the demand. In other words, given the structure imposed by the rational type in terms of offers made and concession behavior, the stubborn type strictly prefers higher demands over lower demands.

Heuristically, the proof proceeds as follows. The difference in the stubborn type's cost between the lowest two demands (referred to as diff-in-diff in what follows), α_1 and α_2 , is given by:

$$\begin{aligned} \Delta_{1,2}^s = & q_{K-1} (1 - \alpha_{K-1}) \mu_2^{\alpha_2} \left(\frac{\mu_{K-1}}{\mu_2} \right)^{1 - \alpha_{K-1}} \\ & - q_K (1 - \alpha_K) \left(\mu_1^{\alpha_1} \left(\frac{\mu_K}{\mu_1} \right)^{1 - \alpha_K} - \mu_2^{\alpha_2} \left(\frac{\mu_K}{\mu_2} \right)^{1 - \alpha_K} \right), \end{aligned} \quad (12)$$

where recall that q_i simply denotes the probability that player j faces demand α_i . Suppose that the opponent, say player j , demands α_K . Let me compare player i demanding α_1 versus demanding α_2 (although this generalizes to any other two consecutive demands when facing any other higher demand). Note that conditional on demanding α_1 , a stubborn type pays a cost for being stubborn iff his opponent demands α_K .

Conditional on demanding α_2 , a stubborn type pays a cost for being stubborn if his opponent demands either α_K or α_{K-1} .

In equilibrium, a (rational) player i is willing to wait until time $T_i^{\alpha_i, \alpha_K}$ to concede (i.e., the time at which player j is known to be stubborn), where of course $T_i^{\alpha_1, \alpha_K} < T_i^{\alpha_2, \alpha_K}$. Recall that (1) the probability of immediate concession by the opponent is decreasing in the demand (i.e., α_K is more likely to concede immediately to α_1 than to α_2); (2) the rate of concession by α_K is higher for α_1 than for α_2 ; and (3) conditional on j not conceding at $t = 0$, the expected payoff for i at any point before $T_i^{\alpha_i, \alpha_K}$ is $1 - \alpha_K$ regardless of whether i demands α_1 or α_2 . Now suppose that player i waits until $T_i^{\alpha_i, \alpha_K}$ to concede (despite j expecting him to concede at the appropriate rate). Then, the probability with which player j concedes (ever) is $1 - \mu_K^{1-\alpha_K}$ – i.e., regardless of the demand i makes. Hence, conditional on j not conceding immediately, the probability with which j concedes (ever) is higher when player i demanded α_2 than when player i demanded α_1 . The stubborn type only pays a cost for being stubborn conditional on his opponent not conceding immediately. Hence, conditional on facing a certain demand, the stubborn type’s cost for being stubborn is lower the higher the demand. In fact, since $\mu_1^{\alpha_1}$ goes to 0 infinitely slower than $\mu_2^{\alpha_2}$, conditional on facing α_K , the stubborn type’s cost for being stubborn is infinitely smaller (in the limit) when demanding α_2 than when demanding α_1 (cf. the last term in (12)).

To offset this difference, we need to make a player facing a demand of α_K a (near) zero-probability event, or we need to make the stubborn type’s cost of being stubborn when demanding α_2 and facing a demand of α_{K-1} “similarly” large as the stubborn type’s cost of being stubborn when demanding α_1 and facing a demand of α_K . Hence, we require the stubborn type’s cost for being stubborn when demanding α_2 to be infinitely larger (in the limit) when facing a demand of α_{K-1} than when facing a demand of α_K . This cost is infinitely larger iff $\mu_K/\mu_{K-1} \rightarrow 0$. Hence, in short, for $\Delta_{1,2}^s = 0$ either (1) $q_K \rightarrow 0$ or (2) $\mu_K/\mu_{K-1} \rightarrow 0$.¹⁴

More generally, conditional on facing demand α_{K-k+1} , the stubborn type’s cost for being stubborn is infinitely smaller when demanding α_{k+1} than when demanding α_k . Therefore, the stubborn type’s diff-in-diff for α_2 and α_3 means that we would want (i) $\mu_{K-1}/\mu_{K-2} \rightarrow 0$ or $q_{K-1} \rightarrow 0$ and (ii) $\mu_K/\mu_{K-1} \rightarrow 0$ or $q_K \rightarrow 0$.

However, if (1) $q_K \rightarrow 0$, or (2) $\mu_K/\mu_{K-1} \rightarrow 0$, for the rational type to be indifferent

¹⁴When $K = 2$, we have $\mu_2/\mu_1 \rightarrow 0$; when $K = 3$, we have $q_K \rightarrow 0$.

between α_1 and α_2 requires:

$$q_i \rightarrow 0, \text{ for any } i \neq K - 1, \quad (13)$$

$$\mu_{K-1}/\mu_2 \rightarrow 1, \text{ and} \quad (14)$$

$$\alpha_1 + \alpha_{K-1} \rightarrow 1. \quad (15)$$

In words, it requires that the second-highest demand be played with probability 1 and that the second-highest demand not concede with positive probability to the second-lowest demand. Recall that the lowest demand, α_1 , is compatible with all but the highest demand α_K and that the second-lowest demand, α_2 , is compatible with all but the highest and second-highest demand. Hence, the payoff from demanding α_2 is strictly higher than that from demanding α_1 , conditional on facing any demand $\alpha_j < \alpha_{K-1}$, and the payoff is at best identical conditional on facing α_{K-1} . Hence, if either the highest demand is not being played, or the highest demand immediately concedes w.p. 1 to both α_1 and α_2 in the limit, the rational type has no strict incentive to demand α_1 over α_2 .¹⁵ For the rational type to then be indifferent between α_1 and α_2 , it must be that every term in the payoff difference for the rational type has to be 0, which in particular requires (13)–(15) to be satisfied. However, if (13)–(15) are satisfied, $\Delta_{2,3}^s = 0$ cannot be satisfied for $K > 3$: the stubborn type’s diff-in-diff for α_2 and α_3 (if $\alpha_3 \leq 1/2$) is given by:

$$\begin{aligned} \Delta_{2,3}^s = & \underbrace{q_{K-2}}_{\rightarrow 0} (1 - \alpha_{K-2}) \underbrace{\mu_3^{\alpha_3} \left(\frac{\mu_{K-2}}{\mu_3} \right)^{1-\alpha_{K-2}}}_{\rightarrow 1} \\ & - \underbrace{q_{K-1}}_{\rightarrow 1} (1 - \alpha_{K-1}) \left(\underbrace{\mu_2^{\alpha_2} \left(\frac{\mu_{K-1}}{\mu_2} \right)^{1-\alpha_{K-1}}}_{\rightarrow 1} - \underbrace{\mu_3^{\alpha_3} \left(\frac{\mu_{K-1}}{\mu_3} \right)^{1-\alpha_{K-1}}}_{\rightarrow 1} \right) \\ & - \underbrace{q_K}_{\rightarrow 0} (1 - \alpha_K) \mu_K^{1-\alpha_K} (\mu_2^{\alpha_2+\alpha_{K-1}} - \mu_3^{\alpha_3+\alpha_{K-1}}). \end{aligned} \quad (16)$$

The key fact is that the second condition, $\mu_{K-1}/\mu_2 \rightarrow 1$, implies that the second-highest demand does not concede with positive probability to any other demand (in the limit). This implies that the stubborn type’s cost of being stubborn when demanding α_3 and facing a demand of α_{K-2} cannot be made “similarly” large as the stubborn type’s cost of being stubborn when demanding α_2 and facing a demand of α_{K-1} . More

¹⁵Recall that $\mu_K/\mu_{K-1} \rightarrow 0$ implies that $\mu_K/\mu_i \rightarrow 0$ for any $i < K$.

precisely, note that the second condition, $\mu_{K-1}/\mu_2 \rightarrow 1$, implies $\mu_{K-2}/\mu_3 \rightarrow 1$ and $\mu_{K-1}/\mu_3 \rightarrow 1$. Hence, $\mu_{K-1}^{1-\alpha_{K-1}} \mu_2^{\alpha_2+\alpha_{K-1}-1}$ is infinitely larger than $\mu_3^{\alpha_3} \left(\frac{\mu_{K-2}}{\mu_3}\right)^{1-\alpha_{K-2}}$. The first condition states that $q_{K-1} \rightarrow 1$ and $q_{K-2} \rightarrow 0$, and hence, the second term in (5) is infinitely larger than any other term in (5). Hence, $\Delta_{2,3}^s = 0$ cannot be satisfied. In particular, the stubborn type would prefer α_3 .¹⁶ To summarize, the key aspects are as follows: (1) conditional on facing a certain demand, call it α_k , the stubborn type's cost of being stubborn is infinitely larger for lower demands. (2) To equalize the stubborn type's cost of being stubborn across the demands he makes, either facing α_k happens with 0 probability (in the limit) or α_{k-1} is infinitely stronger than α_k (in the limit). (3) However, if this is true, then the highest demand concedes immediately w.p. 1 to both the lowest and the second-lowest demand (in the limit). However, then, a rational type would not want to play the lowest demand.

Proposition 6 delivers strong predictions in terms of the size of the support. However, it is restricting attention to equilibria with finitely many demands. A natural question is whether strategies involving a continuum of demands can be part of an equilibrium. Proposition 7 establishes that there exists no equilibrium where players randomize over intervals (of demands).

Proposition 7. *Fix any nonempty, open interval I . Then, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, $I \not\subseteq \text{supp } r$.*

Proof. See Online Appendix. □

The proposition above states the rational type cannot be made indifferent over demands in any interval or in fact over any number of intervals (for $z < \bar{z}$). The proof has essentially two parts: I show first that if the support of the rational type includes an interval, there must be an atom at the lowest demand. Otherwise, the rational type strictly benefits from not making the lowest demand (for any $z > 0$). The second part of the proof is then a straightforward modification of the proof of Proposition 6. It

¹⁶If $\alpha_3 > 1/2$, then

$$\begin{aligned} \Delta_{2,3}^s = & q_{K-2} (1 - \alpha_{K-2}) \mu_{K-2}^{\alpha_3} \\ & - q_{K-1} (1 - \alpha_{K-1}) \mu_{K-1}^{1-\alpha_{K-1}} \left(\mu_2^{\alpha_2+\alpha_{K-1}-1} - \mu_3^{\alpha_3+\alpha_{K-1}-1} \right) \\ & - q_K (1 - \alpha_K) \mu_K^{1-\alpha_K} \left(\mu_2^{\alpha_2+\alpha_K-1} - \mu_3^{\alpha_3+\alpha_K-1} \right). \end{aligned} \quad (17)$$

The same reasoning carries through here as with $\alpha_3 \leq 1/2$ and $K = 4$. When $K = 3$, $\Delta_{2,3}^s = 0$ can be satisfied; when $K = 3$, $\mu_{K-1}/\mu_2 \rightarrow 1$ is true trivially – since $K - 1 = 2$.

shows that there must be no atom at the lowest demand (for $z < \bar{z}$). Hence, strategies involving a continuum of demands cannot be part of an equilibrium.

5.4 Inefficiency and payoffs in the limit

Proposition 4 (in Section 5.2) states that the necessary conditions for equilibrium existence (in Sections 3 and 4) are sufficient if players assign a strictly positive probability to two demands, α_1 and α_2 , only. For such an equilibrium to exist, the lower demand α_1 must be (weakly) less than $1/2$, and α_1 and α_2 must sum to strictly more than 1. In such an equilibrium, when the probability of stubbornness is small, the higher demand α_2 immediately concedes to the lower demand α_1 with probability close to 1. When both players choose the higher demand, they engage in a war of attrition with an expected payoff of $1 - \alpha_2$. Therefore, even in the limit, delay (and, hence, inefficiency) may not disappear. In the limit, the equilibrium payoff for the rational type, v_r , is given by:

$$\lim_{z \rightarrow 0} v_r = \frac{2(a_1 + a_2 - 1)}{2a_2 - 1}(1 - a_1) + \frac{1 - 2a_1}{2a_2 - 1}(1 - a_2) = \frac{1}{2} - \frac{\left(\frac{1}{2} - a_1\right)^2}{a_2 - \frac{1}{2}}. \quad (18)$$

The level of inefficiency is measured by the distance between $1/2$ and the lower demand α_1 and between α_2 and 1, as (18) shows. It is clear that when α_1 is close to 0 (and hence, α_2 close to 1), a rational player's expected equilibrium payoff is close to 0. If, on the other hand, α_1 is close to $1/2$, a rational player's expected payoff is close to $1/2$ (when players are equally patient). By adjusting α_1 and α_2 , one can generate in this fashion any payoff between 0 and $1/2$. Corollary 1 formalizes this insight. Note that when fixing α_1 , a higher α_2 *increases* the limit equilibrium payoff. This may sound surprising at first, given that the symmetric equilibrium with the highest payoff is the one where both types demand $1/2$ with probability 1. Conditional on facing a demand of α_2 , a rational type receives $1 - \alpha_2 < 1/2$ from demanding α_2 . Hence, conditional on facing a demand of α_2 , the rational payoff is higher the lower α_2 is. However, there is another effect that dominates: the probability that the rational type demands α_2 is decreasing in α_2 as seen from the first line in equation 18.

Corollary 1. *Fix any $v \in (0, 1/2]$. Then, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, a symmetric equilibrium exists such that the equilibrium payoff for a rational player is v .*

Proof. This follows immediately from Proposition 4. Fix any equilibrium characterized in Proposition 4. Denote the payoff of a rational player in this equilibrium by v_r . Then,

the limit of this payoff is given by equation 18. Fix any $\epsilon > 0$, and set $a_1 < \epsilon$. Then, $a_2 > 1 - \epsilon$. The result immediately follows. \square

Hence, unlike with an exogenously given distribution of stubborn types, there is a Folk-theorem-like payoff multiplicity when stubborn types can freely choose their initial demand. This is induced by the delay to agreement. For delay to disappear in the limit with exogenous stubborn types, AG requires the “right” stubborn type to be present. In the symmetric discounting case, this would be the type demanding $1/2$. Corollary 1 shows that when the stubborn type is given choice over his initial demand, the right stubborn type may not be present. When he is not, delay (and, hence, inefficiency) does not disappear even when the probability of a player being stubborn is infinitely small. It is natural to ask whether I can derive stronger predictions regarding payoffs (and inefficiency) when using refinements.

6 Existence of (semi-)separating equilibria

In this section, I characterize (semi-)separating equilibria, i.e., equilibria where demands are made with positive probability by some type but not the other. I first show that under a mild assumption on the payoff of the rational type, there exists no equilibrium with a separating demand by the rational type. This immediately implies that there exists no fully separating equilibrium. I then show that there is at most one separating demand by the stubborn type and that such equilibria have a particularly simple form.

6.1 Separating demands by the rational type

Throughout, any separating demand by the rational type is denoted β (or if there are several, β, β' , etc.). The remaining demands are denoted α_1 through α_K (ordered in an increasing manner) as before.

Lemma 3. *If separating demands by the rational type exist, the lowest separating demand must be higher than the highest pooling demand.*

Sketch of proof. Suppose that there was a pooling demand that was higher than some separating demand by the rational type. A rational player would receive a strictly higher payoff from making the highest pooling demand regardless of the demand faced: when facing a compatible demand, the higher demand (i.e., the pooling) would receive a higher payoff. When facing an incompatible demand, the opponent would never concede to the rational demand at time 0; however, the rational demand would immediately

concede to the highest pooling demand. Hence, there cannot be a pooling demand that is higher than a rational demand. \square

Assumption 1 (A1). *Suppose there are two demands, β and β' , which are made exclusively by the rational type:*

$$\beta, \beta' \in \text{supp } r \setminus \text{supp } s.$$

Then, conditional on facing a demand β , a player demanding β' receives as payoff the limit of the (unique) equilibrium payoff in the game, where each player is believed to be stubborn with probability z .

Note that (A1) implies that the probability of immediate concession from β' to β (where both β and β' are made exclusively by the rational type) is given by:

$$\begin{aligned} F_{\beta, \beta'}(0) &= \lim_{z \rightarrow 0} \max \left\{ 1 - z^{\frac{\beta' - \beta}{1 - \beta}}, 0 \right\} \\ &= \begin{cases} 1 & \text{if } \beta < \beta', \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (19)$$

In words, (A1) is equivalent to assuming that the higher rational demand will concede to the lower rational demand with probability 1 at time 0. Denote the payoff from demanding β conditional on facing a demand β' by $v_{\beta}^r |_{\beta'}$. Suppose that $\beta < \beta'$; then, the conditional payoffs are as follows:

$$\begin{aligned} v_{\beta}^r |_{\beta'} &= \lim_{z \rightarrow 0} \left(\beta \left(1 - z^{\frac{\beta' - \beta}{1 - \beta}} \right) + (1 - \beta) z^{\frac{\beta' - \beta}{1 - \beta}} \right) \\ &= \beta. \end{aligned} \quad (20)$$

$$v_{\beta'}^r |_{\beta} = 1 - \beta. \quad (21)$$

Proposition 8. *Under (A1), there exists no symmetric equilibrium with a separating demand by the rational type, i.e., in every symmetric equilibrium, $\text{supp } r \setminus \text{supp } s = \emptyset$.*

Proof. See Online Appendix. \square

Proposition 8 implies that fully separating equilibria do not exist. This stands in contrast to the model with exogenous types: if stubborn types in AG exclusively make low demands, types separate in equilibrium. The proof of Proposition 8 is in two steps:

I first show that there is at most one separating demand by the rational type. The key intuition is as follows. Consider the two highest separating demands, say β and β' with $\beta' > \beta$. Conditional on facing any demand other than β' , the payoffs from demanding β and β' are identical – either demand would concede with probability 1 to any lower demand. However, conditional on facing β' , by (A1), a player receives β by demanding β and $1 - \beta'$ by demanding β' . The second step is then to show that for any candidate separating demand of the rational type, there exists a profitable deviation.

6.2 Separating demands by the stubborn type

The following propositions show that for z sufficiently small, any semiseparating equilibrium has the following support:

$$\text{supp } s = \{\alpha, 1 - \alpha\}; \quad \text{supp } r = \alpha,$$

where $\alpha > \frac{1}{2}$. Moreover, for any $\alpha > \frac{1}{2}$, such a semiseparating equilibrium exists.

Proposition 9. *Fix demands $\{\alpha_1, \dots, \alpha_K\}$, with some $\alpha_i \in \text{supp } s \setminus \text{supp } r$ and $K > 2$. Then, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, there exists no equilibrium with support $\{\alpha_1, \dots, \alpha_K\}$.*

Proof. See Online Appendix. □

Note first that it follows immediately from Lemma 1 that any separating demand by the stubborn type is lower than the lowest pooling demand – otherwise, the rational type would strictly prefer to demand the separating demand of the stubborn type.¹⁷ The proof of Proposition 9 is in three steps. I first show that the lowest separating demand must be incompatible with the highest demand for z small enough. In short, if the highest demand were compatible with the lowest separating demand, the stubborn type would be better off by deviating to a higher demand. The next step is to show that there can be at most one separating demand: note that any separating demand will be conceded to immediately (conditional on facing a rational opponent). Hence, when the probability of facing a rational opponent goes to 1, there is no incentive to make the lower separating demand. The last step is to prove that for a sufficiently small probability of stubbornness, there exists no equilibrium with a separating demand by the stubborn type when there are more than two demands. This proof is a natural extension of the proof of the nonexistence of pooling equilibria with $K > 3$ demands.

¹⁷Any separating offer by the stubborn type α_i has $\mu_i = 1$. Hence, any rational player immediately concedes to α_i .

Proposition 10. *Fix any $\{\alpha_1, \alpha_2\}$. Then, there exists a symmetric equilibrium with $\text{supp } r = \{\alpha_2\}$ and $\text{supp } s = \{\alpha_1, \alpha_2\}$ if and only if $\alpha_1 + \alpha_2 = 1$.*

Proof. See Online Appendix. □

7 Robustness

In this section, I discuss the robustness of the results to modeling choices. First, I briefly discuss asymmetric equilibria. Then, I return to the question of off-equilibrium-path beliefs when I discuss refinements. Finally, I cover sequential move bargaining and briefly discuss other asymmetries in model parameters.

7.1 Other equilibria

Throughout, I have focused on symmetric equilibria. It is clear that asymmetric equilibria exist. For instance, there exists an equilibrium with player i demanding α and player j demanding $1 - \alpha$ for any $\alpha \in [0, 1]$. While the differences in the two types of preferences over demands also impose some structure on the demand configurations that can arise in asymmetric equilibria, it remains an open question whether the strong predictions in terms of the size of the equilibrium support are robust to considering asymmetric equilibria.

7.2 Refinements

Recall that thus far, I have simply assigned probability 1 to any deviation coming from a rational type, and this deterred deviations.

7.2.1 Divinity

Loosely speaking, refinement D1 attaches probability 1 to the type with the strongest incentive to deviate to a given demand. More formally, denote the set of types by $\Theta = \{R, S\}$, where R stands for rational and S for stubborn. Let $u_i^*(\theta)$ be the equilibrium payoff of type $\theta \in \{R, S\}$. Define $D(\theta, S, d)$ to be the set of mixed-strategy best responses (MBRs) F_2 to demand d and beliefs concentrated on S that make type θ strictly prefer d to his equilibrium strategy,

$$D(\theta, S, d) = \cup_{\mu: \mu(S|d)=1} \{F_2 \in MBR(\mu, d) \text{ s.t. } u_1^*(\theta) < u_1(d, F_1, \theta)\},$$

and let $D^0(\theta, S, d)$ be the set of mixed best responses that make type θ exactly indifferent. A type θ is deleted for demand d under criterion D1 if there is a θ' such

that

$$\{D(\theta, \Theta, d) \cup D^0(\theta, \Theta, d)\} \subset D(\theta', \Theta, d).$$

In other words, if the set of best responses (and associated beliefs about a player being stubborn conditional on d) for which a rational player benefits from deviating to d is strictly smaller than the set of best responses for which a stubborn player benefits from deviating to d , then D1 assigns probability 0 to the deviation coming from a rational player. Note that D1 is not defined for dynamic games beyond signaling games. However, first, note that, given the realized demands and associated beliefs, I can compute the expected payoff from the continuation game. Hence, I can associate to my game a corresponding game that ends once demands are chosen. This is the game to which I apply D1.

Proposition 11. 1. *Every one-demand equilibrium satisfies D1.*

2. *A pooling equilibrium with support $\{\alpha_1, \dots, \alpha_K\}$ and $K \geq 2$ satisfies D1 iff $\alpha_2 = 1 - \alpha_1$ (and hence, $K = 3$).*

3. *Every semiseparating equilibrium with $\text{supp } r = \{\alpha_2\}$ and $\text{supp } s = \{\alpha_1, \alpha_2\}$ satisfies D1.*

Proof. See Online Appendix. □

Note that Proposition 11 implies that Folk-theorem-like payoff multiplicity survives D1: first, the payoffs $1/2$ and 0 can be generated by one-demand equilibria. Second, for any payoff $v \in (0, 1/2)$, there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, a symmetric pooling equilibrium with $\{\alpha_1, 1 - \alpha_1, \alpha_3\}$ exists that yields the rational type a payoff of v .¹⁸

7.2.2 Perturbations à la Nash (1953)

To address the problem of equilibrium selection, Nash (1953) suggested introducing some uncertainty over whether a pair of demands are compatible (in the context of the Nash demand game). Let the probability that a pair of demands $\alpha = (\alpha_i, \alpha_j)$ is compatible be given by a function:

$$p : \mathbb{R}^2 \rightarrow [0, 1],$$

¹⁸Alternatively, note that there exists $\bar{z} > 0$ such that for any $z < \bar{z}$, a symmetric semiseparating equilibrium exists such that the equilibrium payoff for a rational player is v . This simply comes from the fact that in semiseparating equilibria, the rational player faces a demand of $\alpha > 1/2$ with probability 1 in the limit, and hence, his payoff in the limit is $1 - \alpha$.

i.e., let $p(\alpha)$ denote the probability that α_i and α_j turn out to be compatible.

To apply this selection method to my game, there are a number of modeling choices that need to be made: What is the function $p(\alpha)$? What is the size of the surplus (for computing payoffs)? What is the size of the surplus if players revise their demands? Moreover, there are important differences from Nash's demand game: in Nash's demand game, the game ends after initial demands have been made; here, players can revise their demands, and hence, the threat of incompatibility does not loom as large.

Using Nash's method for equilibrium selection is particularly attractive when equilibrium demands are exactly compatible and there is a discontinuity in payoffs at the point where demands become incompatible. We have seen that in my model, the payoff of the stubborn type is discontinuous in the player's own demand at the points where it becomes incompatible with a demand his opponent makes with positive probability.¹⁹ One may therefore imagine that introducing such trembles may eliminate multiplicity. However, in any mixed-strategy pooling equilibrium, players randomize over demands that are incompatible. More precisely, the lowest and highest demand over which a player randomizes are incompatible.

7.3 Sequential move bargaining

Suppose that players make demands sequentially, rather than simultaneously; i.e., first, player 1 makes demand α_1 , then player 2 makes demand α_2 . If $\alpha_1 + \alpha_2 > 1$, a concession game starts as before. In this case, the symmetric pooling equilibria in pure strategies (i.e., one-demand equilibria) in the simultaneous move game remain equilibria in the sequential move game.²⁰ However, not surprisingly, the symmetric pooling equilibria in mixed strategies (in the simultaneous move game) are not robust to this change in the bargaining protocol. To see this, consider a simple example with two demands. In particular, suppose that player 1 randomizes over demands $1/3$ and $3/4$. If player 1 demands $1/3$, then player 2 is strictly better off by demanding at least $2/3$: if demanding $1/3$, player 2 would receive $1/2$; if demanding $2/3$, player 2 would receive $2/3$. Hence, the two players cannot be made indifferent over the two demands. By a similar argument, symmetric semiseparating equilibria do not exist.

Of course, restricting attention to symmetric equilibria is somewhat unnatural when

¹⁹Note that, regardless of the opponent's belief, the rational type in this paper always prefers to make a demand that is at least exactly compatible.

²⁰In fact, there exists a symmetric pooling equilibrium in pure strategies iff $\alpha = \{1/2, 1\}$ (as in the simultaneous move game).

moves are sequential. If we allow for asymmetric equilibria, Folk-theorem-like payoff multiplicity arises (as in the simultaneous move game). In particular, fix any $\alpha_1 \in [0, 1]$. Then, there exists an equilibrium where player 1 demands α_1 and player 2 demands $\alpha_2 = 1 - \alpha_1$ in the sequential move game. Player i 's equilibrium payoff in such an equilibrium is simply given by α_i . Hence, there is a Folk-theorem-like payoff multiplicity in the sequential move game. A heuristic argument for the existence of such equilibria is as follows. Provided that player 1 places sufficiently high probability on player 2 being rational conditional on seeing an out-of-equilibrium demand, the rational type receives no more than $1 - \alpha_1$ by demanding more than $1 - \alpha_1$. The stubborn type will receive a payoff strictly less than $1 - \alpha_1$ if he demands more than $1 - \alpha_1$. Moreover, regardless of player 1's belief, player 2 does not want to make a demand less than $1 - \alpha_1$. Hence, player 2 has no incentive to deviate to an out-of-equilibrium demand. By an analogous argument, player 1 has no incentive to deviate.

Note that any such asymmetric pooling equilibrium in pure strategies satisfies refinements such as D1. In short, (i) regardless of the belief of player j , a rational player i is willing to make a demand higher than $\alpha_i = 1 - \alpha_j$; (ii) regardless of player j 's belief, neither type of player i would be willing to make a demand less than $\alpha_i = 1 - \alpha_j$. Hence, there exists no belief of player j (and associated best response) that makes the stubborn type of player i willing to deviate from his equilibrium demand while the rational type of player i is not.

7.4 Model parameters

Allowing players do differ in (i) their ex ante probability of stubbornness or (ii) their patience does not affect the set of equilibria. It does, however, affect players' payoffs. Everything else being equal, an increase in the ex ante probability of stubbornness of a player or similarly in his patience increases the player's payoff. This is analogous to the reasoning and the results in AG.

8 Conclusion

This paper shows that the predictions of the reputation literature are sensitive to the specification of exogenous stubborn types. Once the stubborn type is given a choice over his initial demand, delay (and, hence, inefficiency) may not disappear even when the probability of stubbornness vanishes. Unlike in the literature, I am able to derive strong behavioral predictions in terms of the demand configurations that can occur in

equilibrium.

Within the framework of this paper, a natural extension would be to broaden the set of strategies available to the stubborn type. For instance, it may be natural to introduce an exit option for the stubborn type when known to be facing a stubborn opponent. This may help to better understand the tradeoff between the predictions of the reputation literature and the flexibility given to behavioral types. However, such an extension is unlikely to overturn the results regarding equilibrium payoff multiplicity: further increasing the flexibility of the stubborn type only brings the game closer to a complete-information bargaining model with rational players only.

While this paper focuses on endogenizing behavioral types in a bargaining setting, the idea of endogenizing behavioral types applies more broadly. For instance, some agents may restrict attention to stationary strategies in a repeated game. Whatever drives their preference for this restriction does not mean that they do not choose optimally within the set of stationary strategies. There is a middle ground between rational and behavioral agents, and this paper is a first attempt to explore this territory in a well-known and tractable environment.

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Appendix

Necessary conditions for equilibrium existence

For the proofs of Lemma 1 and 2 that follows it is useful to introduce some notation. Define for $i = 1, 2$, $\mathcal{W}(\alpha_i) = \{\alpha_j | \mu_i(\alpha_i) \leq \mu_j(\alpha_j), \alpha_i + \alpha_j > 1\}$, and $\mathcal{S}(\alpha_i) = \{\alpha_j | \mu_i(\alpha_i) > \mu_j(\alpha_j), \alpha_i + \alpha_j > 1\}$. In a candidate pooling equilibrium, the payoff to the rational type of player 2 from demanding α_2 is:

$$\begin{aligned}
 v_2^r(\alpha_2) &= \int_{\underline{\alpha}}^{1-\alpha_2} \frac{1 - \alpha_i + \alpha_2}{2} dG(\alpha_i) \\
 &\quad + \int_{1-\alpha_2}^{\bar{\alpha}} \left(\alpha_2 - (\alpha_i + \alpha_2 - 1) \min \left\{ \left(\frac{\mu(\alpha_i)}{\mu(\alpha_2)} \right)^{1-\alpha_i}, 1 \right\} \right) dG(\alpha_i),
 \end{aligned} \tag{22}$$

where $\bar{\alpha}$ denotes the highest demand made by player 1 wpp; $\underline{\alpha}$ denotes the lowest demand made by player 1 wpp; and $G(\alpha_i)$ is the cdf over offers by player 1.

Similarly, I can write the payoff of a stubborn player 2 demanding α_2 in a candidate pooling equilibrium as:

$$v_2^s(\alpha_2) = v_2^r(\alpha_2) - \int_{1-\alpha_2}^{\bar{\alpha}} (1 - \alpha_i) \mu_i^{\alpha_2} \max \left\{ 1, \left(\frac{\mu_2}{\mu_i} \right)^{\alpha_i + \alpha_2 - 1} \right\} dG(\alpha_i). \quad (23)$$

Equivalently, for the rational and stubborn type of player 1. Using (22),(23), given $z > 0$, an equilibrium with support \mathcal{C} requires $\forall \alpha, \alpha' \in \mathcal{C}$, and $j = 1, 2$,

$$v_j^r(\alpha) - v_j^r(\alpha') = 0, \quad (24)$$

$$v_j^s(\alpha) - v_j^s(\alpha') = 0, \quad (25)$$

$$G(\bar{\alpha}) = 1, \text{ and} \quad (26)$$

$$\int_{\mathcal{C}} \mu_j(\alpha_i)^{1-\alpha_i} g_j(\alpha_i) d\alpha_i = z, \quad (27)$$

with $g_j(\alpha_i), \mu_j(\alpha_i) \in [0, 1]$.

Proof of Lemma 1. Note first that by definition of strength, any separating offer by the rational type has strength 0 and any separating offer by the stubborn type has strength 1. By straightforward reasoning, the highest separating offer by the stubborn type must be below the lowest pooling offer (otherwise, the rational type would have an incentive to deviate to the separating offer by the stubborn type); and the lowest separating offer by the rational type must be above the highest pooling offer (otherwise, the rational type would have an incentive to deviate to the highest pooling offer – he is more likely conceded to, and receives more conditional on being conceded to). As a result, I will focus on pure pooling equilibria from now on.

Suppose the strength of player 2, $\mu_2(\alpha_2)$, was not decreasing in α_2 .

Case 1: Suppose there exist α'_2 and α''_2 with $\alpha'_2 < \alpha''_2$, such that $\mathcal{W}(\alpha'_2) = \mathcal{W}(\alpha''_2)$, and $\mathcal{S}(\alpha'_2) = \mathcal{S}(\alpha''_2)$.

- (a) Suppose that $\exists \alpha_i \in \mathcal{S}(\alpha'_2)$. Recall that if $\alpha_1 \in \mathcal{S}(\alpha_2)$, then $\alpha_2 + \alpha_1 > 1$, and $\frac{\mu_1(\alpha_1)}{\mu_2(\alpha_2)} < 1$. This implies that (i) fixing strength, the rational player's payoff $v_2^r(\alpha_2)$, as defined in (22), is increasing in α_2 ; and (ii) fixing the offer α_2 , $v_2^r(\alpha_2)$ is increasing in the strength, $\mu_2(\alpha_2)$. Evaluating (22) at $\alpha_2 = \alpha'_2$ and $\alpha_2 = \alpha''_2$, it then follows that $\mu(\alpha'_2) \leq \mu_2(\alpha''_2) \Rightarrow v_2^r(\alpha'_2) < v_2^r(\alpha''_2)$. Hence, for $v_2^r(\alpha'_2) = v_2^r(\alpha''_2)$, it is necessary that $\mu(\alpha'_2) > \mu_2(\alpha''_2)$.

(b) Suppose that $\mathcal{S}(\alpha'_2) = \emptyset$. Note that if $\exists \alpha_i \leq 1 - \alpha'_2$, then

$$\int_{\underline{\alpha}}^{1-\alpha'_2} \frac{1 - \alpha_i + \alpha'_2}{2} dG(\alpha_i) < \int_{\underline{\alpha}}^{1-\alpha''_2} \frac{1 - \alpha_i + \alpha''_2}{2} dG(\alpha_i).$$

Hence, for $v_2^r(\alpha''_2) = v_2^r(\alpha'_2)$, it is necessary that $\alpha_i \in \mathcal{W}(\alpha'_2) = \mathcal{W}(\alpha''_2)$, $\forall \alpha_i$.

Case 2: Suppose there exist α'_2 and α''_2 with $\alpha'_2 < \alpha''_2$, such that $\mathcal{W}(\alpha'_2) \neq \mathcal{W}(\alpha''_2)$, or $\mathcal{S}(\alpha'_2) \neq \mathcal{S}(\alpha''_2)$, or both.

(a) Suppose first that (i) $\exists \alpha_i$ such that $\alpha_i < 1 - \alpha'_2$, and $\alpha_i \in \mathcal{W}(\alpha''_2)$; and (ii) that $\forall \alpha_j \neq \alpha_i, \alpha_j \in \mathcal{W}(\alpha'_2) \iff \alpha_j \in \mathcal{W}(\alpha''_2)$, and $\alpha_j \in \mathcal{S}(\alpha'_2) \iff \alpha_j \in \mathcal{S}(\alpha''_2)$. Then, evaluating (22) at $\alpha_2 = \alpha'_2$ and $\alpha_2 = \alpha''_2$, it is clear that $v_2^r(\alpha''_2) > v_2^r(\alpha'_2)$. Hence, if $\exists \alpha_i$ such that $\alpha_i < 1 - \alpha'_2$, and $\alpha_i \in \mathcal{W}(\alpha''_2)$, then there must exist $\alpha_j \in \mathcal{S}(\alpha'_2) \setminus \mathcal{S}(\alpha''_2)$. But this implies $\mu_2(\alpha'_2) > \mu_2(\alpha''_2)$.

(b) Suppose that $\exists \alpha_i$ such that $\alpha_i = 1 - \alpha'_2$, and $\alpha_i \in \mathcal{W}(\alpha''_2)$, and that $\forall \alpha_j \neq \alpha_i, \alpha_j \in \mathcal{W}(\alpha'_2) \iff \alpha_j \in \mathcal{W}(\alpha''_2)$, and $\alpha_j \in \mathcal{S}(\alpha'_2) \iff \alpha_j \in \mathcal{S}(\alpha''_2)$. By Case 2(a), it must be that $\alpha_i = \underline{\alpha}$ (otherwise, there must exist $\alpha_j \in \mathcal{S}(\alpha'_2) \setminus \mathcal{S}(\alpha''_2)$). However, since $\alpha'_2 < \alpha''_2$, for $v_2^r(\alpha'_2) = v_2^r(\alpha''_2)$, it is necessary that $\mathcal{S}(\alpha'_2) = \mathcal{S}(\alpha''_2) = \emptyset$.

(c) Suppose finally that (i) $\exists \alpha_i \in \mathcal{S}(\alpha''_2) \setminus \mathcal{S}(\alpha'_2)$, and (ii) that $\forall \alpha_j \neq \alpha_i, \alpha_j \in \mathcal{W}(\alpha'_2) \iff \alpha_j \in \mathcal{W}(\alpha''_2)$, and $\alpha_j \in \mathcal{S}(\alpha'_2) \iff \alpha_j \in \mathcal{S}(\alpha''_2)$. Then, evaluating (22) at $\alpha_2 = \alpha'_2$ and $\alpha_2 = \alpha''_2$, it is clear that $v_2^r(\alpha''_2) > v_2^r(\alpha'_2)$. Hence, $\mathcal{S}(\alpha''_2) \setminus \mathcal{S}(\alpha'_2) = \emptyset$. This implies that if there exists $\alpha_i < 1 - \alpha'_2$, then $\mu_2(\alpha'_2) > \mu_2(\alpha''_2)$. If there does not exist $\alpha_i < 1 - \alpha'_2$, then see Case 2 (b).

Hence, either (i) $\alpha_i \in \mathcal{W}(\alpha'_2) = \mathcal{W}(\alpha''_2)$, $\forall \alpha_i$; or (ii) $\underline{\alpha} = 1 - \alpha'_2$, $\underline{\alpha} \in \mathcal{W}(\alpha''_2)$ and $\alpha_i \in \mathcal{W}(\alpha'_2) = \mathcal{W}(\alpha''_2) \setminus \underline{\alpha}$, $\forall \alpha_i > \underline{\alpha}$; or (iii) $\mu_2(\alpha'_2) > \mu_2(\alpha''_2)$. Note that (i) implies that $\alpha'_2 > 1 - \underline{\alpha}$, and that $\mu(\alpha'_2) = \mu(\alpha''_2)$. Hence, $\mu_2(\alpha_2)$ is strictly decreasing in α_2 unless $\alpha_2 \geq 1 - \underline{\alpha}$. \square

NB: If the stubborn type chooses his initial demand, then as we will see in Lemma 3, (i) cannot be. Moreover, for (ii) it must be that $\alpha'_2 = 1 - \underline{\alpha}$ and $\alpha''_2 = \bar{\alpha}$ by Lemma 2, $\underline{\alpha} + \bar{\alpha} > 1$). This then implies $\mu_2(\bar{\alpha}) = \mu_2(1 - \underline{\alpha})$. Hence, $\mu_2(\alpha_2)$ is strictly decreasing in α_2 unless $\alpha_2 = 1 - \underline{\alpha}$, in which case $\mu_2(1 - \underline{\alpha}) = \mu_2(\bar{\alpha})$.

Proof of Lemma 2, Part 1. Suppose not; i.e., suppose that there exists an offer which is compatible with every offer made by the opponent wpp. Then the payoff to a rational and stubborn type from making this offer is identical. This would then have to be true for every other offer made wpp. When facing an incompatible demand, the rational type has the option value of concession, while the stubborn type does not. Hence, there could be no offer higher than $1/2$. But if there is no offer higher than $1/2$, then both types would want to demand at least $1/2$. Hence, there would not be multiple offers being made wpp. Therefore, every offer must be incompatible with at least one offer made by the opponent wpp. \square

Proof of Lemma 2, Part 2. The proof is divided into two parts: I first focus on pooling equilibria; then I turn to semiseparating equilibria.

Pooling equilibria: Consider a candidate pooling equilibrium. Suppose the set of compatible demands is constant between α and α' , with $\alpha < \alpha'$. Suppose further that (24) is satisfied for all $\alpha, \alpha' \in \mathcal{C}$. Then for $j = 2$, I can write (25) as

$$0 = - \int_{1-\alpha}^{\bar{\alpha}} (1 - \alpha_i) \cdot \left(\mu_1(\alpha_i)^\alpha \max \left\{ 1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha - 1} \right\} - \mu_1(\alpha_i)^{\alpha'} \max \left\{ 1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha' - 1} \right\} \right) dG(\alpha_i). \quad (28)$$

(a) Suppose $\mathcal{S}(\alpha) = \mathcal{S}(\alpha') = \emptyset$. Then $\forall \alpha_i > 1 - \alpha$,

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha - 1} \right\} = 1.$$

Similarly, for α' , since $\mu_2(\alpha) \geq \mu_2(\alpha')$. Clearly, $\mu_1(\alpha_i)^\alpha > \mu_1(\alpha_i)^{\alpha'}$. Therefore, if $\mathcal{S}(\alpha) = \mathcal{S}(\alpha') = \emptyset$, the RHS of (28) is strictly negative, and hence, (28) cannot be satisfied.

(b) Suppose $\mathcal{W}(\alpha) = \mathcal{W}(\alpha') = \emptyset$. Then $\forall \alpha_i > 1 - \alpha$,

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha' - 1} \right\} = \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)} \right)^{\alpha_i + \alpha' - 1}, \quad (29)$$

and similarly for α . Hence, using (29), I can simplify the RHS of (28) to:

$$- \int_{1-\alpha}^{\bar{\alpha}} (1 - \alpha_i) \cdot \mu_1(\alpha_i)^{1-\alpha_i} \left(\mu_2(\alpha)^{\alpha+\alpha_i-1} - \mu_2(\alpha')^{\alpha'+\alpha_i-1} \right) dG(\alpha_i). \quad (30)$$

By Lemma 1, $\mu_2(\alpha) \geq \mu_2(\alpha')$. Moreover, note that $0 < \alpha + \alpha_i - 1 < \alpha' + \alpha_i - 1 < 1$. Hence, (30) is strictly negative, and thus, (28) cannot be satisfied.

(c) Suppose $\exists \alpha_i \in \mathcal{S}(\alpha) \setminus \mathcal{S}(\alpha')$. Then

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)} \right)^{\alpha_i+\alpha-1} \right\} = \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)} \right)^{\alpha_i+\alpha-1} \quad \text{and}$$

$$\max \left\{ 1, \left(\frac{\mu_2(\alpha')}{\mu_1(\alpha_i)} \right)^{\alpha_i+\alpha'-1} \right\} = 1.$$

But

$$-g(\alpha_i)(1 - \alpha_i) \cdot \left(\mu_1(\alpha_i)^\alpha \left(\frac{\mu_2(\alpha)}{\mu_1(\alpha_i)} \right)^{\alpha_i+\alpha-1} - \mu_1(\alpha_i)^{\alpha'} \right) < 0 \quad (31)$$

since $\mu_2(\alpha) > \mu_1(\alpha_i)$ and $\alpha < \alpha_i \leq \alpha'$. Hence, (28) cannot be satisfied.

(d) By Lemma 1, $\mathcal{S}(\alpha') \setminus \mathcal{S}(\alpha) = \emptyset$.

Therefore, if for some $\alpha, \alpha' \in \mathcal{C}$, $\nexists \alpha'' \in \mathcal{C}$ such that $\alpha \leq 1 - \alpha'' < \alpha'$, then (28) cannot be satisfied. Hence, for all $\alpha, \alpha' \in \mathcal{C}$, there exists $\alpha'' \in \mathcal{C}$ such that $\alpha \leq 1 - \alpha'' < \alpha'$.

Semiseparating equilibria: Consider a candidate semiseparating equilibrium with K offers, with multiple separating offers. If the set of compatible offers is non-decreasing between any two separating offers, then the stubborn type strictly prefers the higher offer: any separating offer will be conceded to immediately by the rational type, regardless of its value (if less than 1). Hence, the higher offer yields a strictly higher payoff regardless of the offer made by the opponent.

Consider a candidate semiseparating equilibrium with support \mathcal{C} , with one separating offer, α_1 , by the stubborn type. Denote the lowest pooling offer by α_2 , and the highest offer by $\bar{\alpha}$. Suppose $\nexists \alpha_i$ such that $1 - \alpha_2 < \alpha_i \leq 1 - \alpha_1$. Then conditional on facing $\bar{\alpha}$, the difference in payoffs between the rational and stubborn type of player i for α_1 and α_2 is:

$$\begin{aligned} \Delta_{1,2}^r \Big|_{\alpha_j=\bar{\alpha}} - \Delta_{1,2}^s \Big|_{\alpha_j=\bar{\alpha}} &= (1 - \bar{\alpha}) \bar{\mu}^{1-\bar{\alpha}} - (1 - \alpha_K) \left(\frac{\bar{\mu}}{\mu_2} \right)^{1-\bar{\alpha}} \mu_2^{\alpha_2}, \\ &= (1 - \bar{\alpha}) \bar{\mu}^{1-\bar{\alpha}} (1 - \mu_2^{\alpha_2+\bar{\alpha}-1}) > 0. \end{aligned}$$

Hence, the rational type has an incentive to deviate to the separating offer α_1 . \square

Existence of pooling equilibria

Before proving the existence of pooling equilibria, it is helpful to state and prove the following supplementary lemma:

Lemma 4. *There exists a pooling equilibrium with support $\{\alpha_1, \dots, \alpha_K\}$ only if the offers α_1 through α_K along with probabilities q_1 through to q_K , and positive numbers μ_1 through to μ_K solve (36)–(39).*

Proof. Fix $z > 0$, and an equilibrium, specifying $\{\alpha_1, \dots, \alpha_K\}$, $\mu_1, \dots, \mu_K > 0$, and $q_1, \dots, q_K > 0$. For any $k \leq K$, define

$$v_k^r = \sum_{\substack{i \text{ s.t.} \\ \alpha_i \leq 1 - \alpha_k}} q_i \left(\frac{\alpha_k + 1 - \alpha_i}{2} \right) + \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1 - \alpha_k}} q_i \left(\alpha_k \min \left\{ 0, 1 - \left(\frac{\mu_i}{\mu_k} \right)^{1 - \alpha_i} \right\} + (1 - \alpha_i) \min \left\{ 1, \left(\frac{\mu_i}{\mu_k} \right)^{1 - \alpha_i} \right\} \right), \quad (32)$$

$$v_k^s = v_k^r - \sum_{\substack{i \text{ s.t.} \\ \alpha_i > 1 - \alpha_k}} q_i (1 - \alpha_i) \max \left\{ \mu_i^{\alpha_k}, \left(\frac{\mu_i}{\mu_k} \right)^{1 - \alpha_i} \mu_k^{\alpha_k} \right\}. \quad (33)$$

For a detailed derivation of these payoffs see the supplementary material on my website. For any $k, k' \leq K$, define

$$\Delta_{k,k'}^r = v_k^r - v_{k'}^r, \quad (34)$$

$$\Delta_{k,k'}^s = v_k^s - v_{k'}^s. \quad (35)$$

Given z and $\{\alpha_1, \dots, \alpha_K\}$, define the following system in (q_i, μ_i) , $i = 1, \dots, K$:

$$\Delta_{k,k+1}^r = 0, \quad \forall k < K, \quad (36)$$

$$\Delta_{k,k+1}^r - \Delta_{k,k+1}^s = 0, \quad \forall k < K \quad (37)$$

$$\sum_{i=1}^K q_i \mu_i^{1 - \alpha_i} = z, \quad \text{and} \quad (38)$$

$$\sum_{i=1}^K q_i = 1. \quad (39)$$

Note that there are $2K$ equations (and as many variables). For a candidate equilibrium with support $\{\alpha_1, \dots, \alpha_K\}$, both types need to be indifferent over all demands α_1 through to α_K , with probabilities $q_i > 0$, given an ex ante probability of a player being stubborn, z . Equation (36) shows the difference in payoff for a rational type between making a demand of α_k and making a demand of α_{k+1} , conditional on the opponent mixing over the offers α_1 through to α_K . Hence, equation (37) ensures indifference of the rational type between any two offers, α_k and α_{k+1} . In the same manner, equation (37) ensures indifference of the stubborn type between any two offers, simplified using the indifference of the rational type. Equation (39) ensures that the probabilities of being faced with a given offer add up to 1; and equation (38) ensures that the conditional probabilities of stubbornness, $\mu_i^{1-\alpha_i}$, are consistent with the ex ante probability of a player being stubborn, z .

Fix K demands (satisfying Lemmas 1 and 2). Suppose that for all $\bar{z} > 0$, there exists $z < \bar{z}$, such that there exist $q_i > 0$, and $\mu_i > 0$ for $i = 1, 2, \dots, K$ such that (z, α, q, μ) satisfies (36) to (39). Then there exists a sequence $(z^n, \alpha^n, q^n, \mu^n)_{n \in \mathbb{N}}$, with $\lim_{n \rightarrow \infty} z^n \rightarrow 0$, solving (36)–(39), such that it is *not* the case that $\alpha_i^n - \alpha_{i+1}^n \rightarrow 0$ for all i , $i + 1 \leq \lceil K/2 \rceil - 1$ and all $i, i + 1 \geq \lceil K/2 \rceil$ with $i + 1 < K$. Recall, that $\alpha^n, q^n, \mu^n \in [0, 1]$. Hence, without loss, assume that α^n, q^n and μ^n converge. By continuity, $(z = 0, \lim_{z \rightarrow 0} \alpha, \lim_{z \rightarrow 0} q, \lim_{z \rightarrow 0} \mu)$ also solves (36)–(39). In the following, I drop the subscript n ; limits are indicated explicitly by $\lim_{z \rightarrow 0}$ throughout.

In other words, if the system has a solution for small enough z , then for at least one $i \notin \{\lceil K/2 \rceil - 1, K\}$, $\alpha_i \neq \alpha_{i+1}$. \square

Proof of Proposition 4. When $K = 2$, I can write (36) and (37) for $k = 1$ as:

$$q_1 \left(\alpha_1 - \frac{1}{2} \right) + q_2 (\alpha_1 + \alpha_2 - 1) \left(1 - \left(\frac{\mu_2}{\mu_1} \right)^{1-\alpha_2} \right) = 0, \text{ and} \quad (40)$$

$$q_1 (1 - \alpha_1) \mu_1^{\alpha_2} - q_2 (1 - \alpha_2) \mu_2^{1-\alpha_2} (\mu_1^{\alpha_1 + \alpha_2 - 1} - \mu_2^{2\alpha_2 - 1}) = 0. \quad (41)$$

The proof has the following steps. First, in any sequence of equilibria, $\mu_i \rightarrow 0$ for $i = 1, 2$ (Claim 2). Second, an equilibrium with support $\{\alpha_1, \alpha_2\}$ exists in the limit (Claim 3). Finally, I show that the system (36)–(39) can be solved locally around $z = 0$ when $K = 2$, with $q_i \in (0, 1)$, and $\mu_i \in (0, 1)$ for $i = 1, 2$ (Claim 4).

Claim 2. For (36)–(39) to be satisfied when $K = 2$, $\lim_{z \rightarrow 0} \mu_i = 0$ for $i = 1, 2$.

Proof. By (38) and (39), either $\lim_{z \rightarrow 0} q_i = 0$ or $\lim_{z \rightarrow 0} \mu_i = 0$ for $i = 1, 2$. Moreover, if $\lim_{z \rightarrow 0} q_i = 0$, then $\lim_{z \rightarrow 0} \mu_j = 0$. Recall that by Lemma 1, $\mu_2 < \mu_1, \forall z > 0$. Hence, by (38), it follows that $\lim_{z \rightarrow 0} \mu_2 = 0$. If $\lim_{z \rightarrow 0} \mu_2 = 0$, then (36) can only be satisfied if $\lim_{z \rightarrow 0} \mu_1 = 0$: if $\lim_{z \rightarrow 0} q_1 = 0$, then it must be that $\lim_{z \rightarrow 0} l_{2,1} = 1$, and hence, $\lim_{z \rightarrow 0} \mu_1 = 0$. Therefore, $\lim_{z \rightarrow 0} \mu_i = 0$ for $i = 1, 2$. \square

NB. Recall that by Lemma 1, in order for (36) to be satisfied it must be that $\mu_{k+1} \leq \mu_k, \forall k, \forall z > 0$. Hence, all ratios $\frac{\mu_i}{\mu_k}$ and $\frac{\mu_i}{\mu_{k+1}}$ in (36) and (37) are bounded above by 1. Hence, without loss, assume that these ratios converge. Call the ratios $l_{i,k}$ and $l_{i,k+1}$.

Claim 3. *The system (36)–(39) has a solution in the limit when $K = 2$, with*

$$\lim_{z \rightarrow 0} r_1 = \lim_{z \rightarrow 0} q_1 = \frac{2(\alpha_1 + \alpha_2 - 1)}{2\alpha_2 - 1}, \text{ and} \quad (42)$$

$$\lim_{z \rightarrow 0} s_1 = \frac{1 - \alpha_2}{2 - \alpha_1 - \alpha_2}. \quad (43)$$

Proof. I first reduce the system (38)–(41) to two equations (where recall that equations (40) and (41) are simply (36) and (37) for $K = 2$). Then I use Taylor approximations to derive (42) and (43). Using (39), I can replace q_2 by $1 - q_1$ in (40). I can then solve (40) for q_1 as a function of μ_1 and μ_2 only:

$$q_1 = \frac{2(\alpha_1 + \alpha_2 - 1)(1 - l_{2,1}^{1+\alpha_2})}{(2\alpha_2 - 1) - 2(\alpha_1 + \alpha_2 - 1)l_{2,1}^{1+\alpha_2}}. \quad (44)$$

I can then replace q_2 and q_1 (using (44)) in (41) and (38). I can write the stubborn type's indifference, (41), as:

$$\frac{(1 - 2\alpha_1)(1 - \alpha_2)(\mu_2^{1-\alpha_2}\mu_1^{\alpha_1-1} - l_{2,1}^{\alpha_2}) + 2(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)(l_{2,1}^{1-\alpha_2} - 1)}{\mu_1^{-\alpha_2}(2(\alpha_1 + \alpha_2 - 1)l_{2,1}^{1-\alpha_2} - (2\alpha_2 - 1))} = 0. \quad (45)$$

I can then show that

$$\lim_{z \rightarrow 0} \frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}} = \frac{2(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)}{(1 - 2\alpha_1)(1 - \alpha_2)}. \quad (46)$$

More precisely,

$$\mu_1 = \left(\frac{(1 - 2\alpha_1)(1 - \alpha_2)}{2(1 - \alpha_1)(\alpha_1 + \alpha_2 - 1)} \right)^{\frac{1}{1-\alpha_1}} \mu_2^{\frac{1-\alpha_2}{1-\alpha_1}} + \mathcal{O} \left(\mu_2^{\frac{1-\alpha_2}{1-\alpha_1}(1+\alpha_2-\alpha_1)} \right). \quad (47)$$

To derive (46) and (47), note that for (45) to be satisfied either

$$\lim_{z \rightarrow 0} l_{2,1} = K, \quad \text{or} \quad \lim_{z \rightarrow 0} \frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}} = K,$$

where K is some positive constant. If $\lim_{z \rightarrow 0} l_{2,1} = K$, then $\lim_{z \rightarrow 0} \frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}} \rightarrow \infty$, and hence, (45) cannot be satisfied. If $\lim_{z \rightarrow 0} \frac{\mu_2^{1-\alpha_2}}{\mu_1^{1-\alpha_1}} = K$, then $\lim_{z \rightarrow 0} l_{2,1} = 0$. Hence, we can solve (45) for K :

$$K = \frac{2(1-\alpha_1)(\alpha_1 + \alpha_2 - 1)}{(1-2\alpha_1)(1-\alpha_2)}, \quad (48)$$

and (46) follows. Using Taylor approximation, I can then derive (47). Using (47), I can rewrite (38) and (44) as

$$q_1 = \frac{2(\alpha_1 + \alpha_2 - 1)}{2\alpha_2 - 1} - k_1 \mu_2^{\frac{(1+\alpha_2)(1-\alpha_1)-(1-\alpha_2)^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{\frac{2(2\alpha_2-\alpha_1-\alpha_2^2)}{1-\alpha_1}}\right), \quad (49)$$

$$z = \frac{(1-2\alpha_1)(2-\alpha_1-\alpha_2)}{(1-\alpha_1)(2\alpha_2-1)} \mu_2^{1-\alpha_2} + \mathcal{O}\left(\mu_2^{\frac{1-2\alpha_1+\alpha_2(2-\alpha_2)}{1-\alpha_1}}\right), \quad (50)$$

where

$$k_1 = \left(\frac{2(\alpha_1 + \alpha_2 - 1)}{2\alpha_2 - 1}\right)^2 \left(\frac{1-2\alpha_1}{2(\alpha_1 + \alpha_2 - 1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} \left(\frac{1-\alpha_1}{1-\alpha_2}\right)^{\frac{1-\alpha_2}{1-\alpha_1}}.$$

To derive (49), note that I can write $l_{2,1}^{1-\alpha_2}$ as

$$l_{2,1}^{1-\alpha_2} = \left(\frac{(1-2\alpha_1)(1-\alpha_2)}{2(1-\alpha_1)(\alpha_1 + \alpha_2 - 1)}\right)^{-\frac{1-\alpha_2}{1-\alpha_1}} \mu_2^{\frac{(1+\alpha_2)(1-\alpha_1)-(1-\alpha_2)^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{\frac{2(2\alpha_2-\alpha_1-\alpha_2^2)}{1-\alpha_1}}\right).$$

Using (50), and recalling that $s_1 = \frac{\mu_1^{1-\alpha_1} q_1}{z}$, I can now write s_1 as a function of μ_2 only:

$$s_1 = \frac{1-\alpha_2}{2-\alpha_1-\alpha_2} - k_2 \mu_2^{\frac{(1+\alpha_2)(1-\alpha_1)-(1-\alpha_2)^2}{1-\alpha_1}} + \mathcal{O}\left(\mu_2^{\frac{1-2\alpha_1+2\alpha_2-\alpha_2^2}{1-\alpha_1}-1+\alpha_2}\right), \quad (51)$$

where

$$k_2 = \left(\frac{(1-2\alpha_1)(1-\alpha_2)}{2(\alpha_1 + \alpha_2 - 1)(1-\alpha_1)}\right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} \left(\frac{2(\alpha_1 + \alpha_2 - 1)(1-\alpha_1)}{(2\alpha_2 - 1)(2-\alpha_1-\alpha_2)}\right).$$

Hence,

$$\lim_{z \rightarrow 0} r_1 = \frac{2(\alpha_1 + \alpha_2 - 1)}{2\alpha_2 - 1}, \quad \text{and} \quad (52)$$

$$\lim_{z \rightarrow 0} s_1 = \frac{1-\alpha_2}{2-\alpha_1-\alpha_2}. \quad (53)$$

□

Claim 4. *The system (36)–(39) can be solved locally around $z = 0$ when $K = 2$, with $s_1 \in (0, 1)$, $r_1 \in (0, 1)$.*

Proof. As before, I replace q_2 by $1 - q_1$ in equations (38), (40) and (41) (using (39)). In analogue to before, I then solve (40) for q_1 as a function of μ_1 and μ_2 only:

$$q_1 = \frac{2(\alpha_1 + \alpha_2 - 1) \left(1 - \left(\frac{\mu_2}{\mu_1} \right)^{1+\alpha_2} \right)}{(2\alpha_2 - 1) - 2(\alpha_1 + \alpha_2 - 1) \left(\frac{\mu_2}{\mu_1} \right)^{1+\alpha_2}}. \quad (54)$$

Using this, I can then use (38) to solve for μ_2 as a function of z and μ_1 :

$$\mu_2 = \mu_1 \left(\frac{2(\alpha_1 + \alpha_2 - 1) \mu_1^{1-\alpha_1} - (2\alpha_2 - 1)z +}{2(\alpha_1 + \alpha_2 - 1) (\mu_1^{1-\alpha_1} - z) - (1 - 2\alpha_1) \mu_1^{1-\alpha_2}} \right)^{\frac{1}{1-\alpha_2}}. \quad (55)$$

Hence, I can express (41) as a function of μ_1 and z only. Let me introduce two auxiliary variables, p and u , where

$$p = z^{\frac{\alpha_1 - \alpha_2(1-\alpha_1) + 2\alpha_2^2}{(1-\alpha_1)(1-\alpha_2)}}, \quad \text{and} \quad (56)$$

$$u = \mu_1^{1-\alpha_1} z^{-1} - \frac{(1-\alpha_2)(2\alpha_2-1)}{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}. \quad (57)$$

Using the Implicit Function Theorem one can derive:

$$\left. \frac{dp}{du} \right|_{(p,u)=(0,0)} = \frac{(2-\alpha_1-\alpha_2)}{1-\alpha_1} \left(\frac{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}{(1-\alpha_2)(2\alpha_2-1)} \right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}} > 0. \quad (58)$$

I can rewrite (41) as a function of p and u , using (56) and (57). Denote this new function $\Delta_{p,u}^s$. Taking derivatives w.r.t. p and u , evaluating these derivatives at $p = u = 0$, and rearranging, I get (58), which is clearly finite and positive:

$$\begin{aligned} \left. \frac{dp}{du} \right|_{(p,u)=(0,0)} &= - \left. \frac{\partial \Delta_{p,u}^s / \partial u}{\partial \Delta_{p,u}^s / \partial p} \right|_{(p,u)=(0,0)} \\ &= \frac{(2-\alpha_1-\alpha_2)}{1-\alpha_1} \left(\frac{2(2-\alpha_1-\alpha_2)(\alpha_1+\alpha_2-1)}{(1-\alpha_2)(2\alpha_2-1)} \right)^{\frac{\alpha_2-\alpha_1}{1-\alpha_1}}. \end{aligned} \quad (59)$$

Hence, the system (36)–(39) can be solved locally around $z = 0$ when $K = 2$, with $r_1 \in (0, 1)$, and $s_1 \in (0, 1)$. □

□