6-1-1992

Expectations Driven Nonlinear Business Cycles

Jean-Michel Grandmont

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 1022

EXPECTATIONS DRIVEN NONLINEAR BUSINESS CYCLES

J. M. Grandmont

June 1992

EXPECTATIONS DRIVEN NONLINEAR BUSINESS CYCLES

J.M. Grandmont *

There are two traditional conflicting views about the workings of a market economy. The socalled "Classical" school stresses the virtues of free markets and their intrinsic internal stability. According to that school, persistent, nonexplosive fluctuations should be essentially due to repeated external macroeconomic shocks to the "fundamental" characteristics of the system (technology, tastes, resources). Productivity of capital equipment is assumed to vary randomly across the economy by following an exogenously given stationary stochastic process. Demand for goods and services, supply of labor are by assumption subject to exogenous aggregate random shocks. Government agencies and monetary authorities regularly throw dices when tuning up their policy instruments. Models of the "Classical" vintage typically assume that expectations are self-fulfilling, i.e. every individual's assessment of the future (a probability distribution) is correct at any moment given his information. According to that view, expectations cannot be an independent source of economic fluctuations. Business cycles are forced by aggregate random exogenous shocks to the "fundamentals" and without them, no persistent economic fluctuations would be observed.

There are from time to time aggregate macroeconomic shocks to "fundamentals" that are large enough to generate significant business fluctuations, e.g. the oil shocks of 1973 and 1978, or the 1990-91 Gulf crisis. Policy may be wrongly conceived and ill-timed, and thus may indeed at times generate macroeconomic fluctuations rather than dampening them. Yet it is hard to find a convincing rationale for the exogenous processes of aggregate random shocks to productivity, demand and supply, or policy that are so central in Classical models. There is another view, often associated to socalled "Keynesian" thinking, that starts with the observation that individual prophecies about aggregate outcomes often tend to be self-fulfilling if shared by sufficiently many people, and that suggests this phenomenon as a potential source of significant internal instability of socioeconomic systems in the absence of any outside regulation. Widespread fears of a crash in financial markets increase the likelihood of such a crash occurring. Widespread expectations of a larger aggregate demand generate new investments, hence more employment and more demand. According to that view, aggregate changes of expectations are likely to occur rather frequently in an unpredictable way and are potentially major sources of endogenous business fluctuations (*market psychology, animal spirits*).

Early Keynesian macroeconomic models that were designed along this line more than thirty years ago tended to produce simple periodic outcomes that could hardly claim to account for the dynamic complexity of actual economic time series. These models also appeared to rely upon the implausible assumption that economic units were stubbornly myopic and made systematic forecasting errors along the cycle although the periodic macroeconomic pattern was easily recognizable : if the requirement that expectations were self-fulfilling was imposed on these models, cycles seemed not to occur any more. There has been a new spur of interest in the "Keynesian" approach in the past decade or so. The new class of models makes explicit the intertemporal behavior of individual economic units and shows that given the "fundamental" characteristics of the system, a large multiplicity of such expectations-driven business cycles can occur even when expectations are required to be correct. Complex nonexplosive deterministic or stochastic fluctuations occur because economic units collectively predict - rationally - that these fluctuations will be observed. In fact, one is extremely likely to get such expectations-driven business cycles, in a *nonlinear* framework, whenever the local dynamics near a steady state has a local characteristic root of modulus close to one, a feature that has been claimed to be present in economic time series. Models of this type succeed, under increasingly plausible assumptions, in producing complex trajectories that tend to replicate more and more closely the qualitative dynamic regularities exhibited by actual macroeconomic time series. The first part of the lecture is devoted to a review of the concepts and methods that are employed in the study of such expectations-driven business cycles – they are partly borrowed from the theory of nonlinear dynamical systems (bifurcations, chaos) – and of the macroeconomic patterns they generate.

The studies I just alluded to all analyze the occurrence of persistent, nonexplosive fluctuations generated by volatile forecasts, by *assuming* self-fulfilling expectations. In many cases the assumption is almost self-contradictory since it leads to many possible equilibria. At any rate the informational requirements underlying the postulate are extraordinarily demanding, and one would like to inquire whether or not the dynamics of an economic system are likely to converge to a path with self-fulfilling expectations when the learning processes traders employ during the transition are taken explicitly into account. The second part of the lecture reviews recent studies suggesting that self-fulfilling expectations are then often unstable. These studies tend to show that learning may be an independent source of endogenous expectations-driven business cycles, and moreover that significant forecasting mistakes may never vanish in the corresponding learning dynamics, even in the "long run".

1. SELF-FULFILLING EXPECTATIONS [1]

In order to fix ideas, we shall focus throughout this section on a simple version of the socalled "overlapping generations" model. People are assumed to participate in the market for two periods. They work and save their wages in the form of money when "young" (in the first period of their lifes) and consume when "old" (one would get similar results if people were assumed to work and consume in both periods). Generations overlap so that old members of a generation coexist and trade in any given period with the young members of the following generation. Population is stationary and there are no bequests. On the other hand, one unit of output (a perishable consumption good) can be produced from one unit of output in the same period (productive capital will be considered at the end of the section). Finally, the total money stock is assumed to be constant over time.

The essential feature of such an overlapping generations model, which will enable us to generate expectations-driven endogenous fluctuations, is that it has an important built-in capital market imperfection : the assets of the members of a generation are constrained to be zero at the end of their lifes. Such capital market imperfections are in a sense necessary to ensure that traders do not arbitrage away endogenous fluctuations although they are correctly foreseen (another possibility, not considered here, is to assume a high time preference so that traders do not care arbitraging away the fluctuations ; on this point see Boldrin's surveys (1988, 1991)). The particular "story" we use to illustrate the analysis is chosen for its simplicity but the argument is much more general. For instance, one would get the same *local* results (i.e. near a steady state), if we assumed that people cared about the welfare of their offsprings and allowed bequests, but made the (realistic) assumption that nobody can borrow against the income of their children. Or if we assumed that people are infinitely long-lived but face liquidity (cash-in-advance) constraints expressing the difficulty of workers to borrow against future labor income. In this latter case, the interpretation of the length of a period would be much shorter — say a few months — than in the case of the overlapping generations model (see Woodford (1986)).

Deterministic fluctuations

The "fundamentals" of the simple economic system under consideration are constant. We wish to analyze first how deterministic endogenous fluctuations can nevertheless arise.

Let us assume for simplicity that the aggregate behavior of all members of a generation can be represented by a "representative" houdehold. A representative household born at date t supplies the quantity of labor $0 \leqq \ell_t \leqq \ell^*$ in the current period (where $\ell^* > 0$ is the labor endowment), saves the amount of money $m_t \geqq 0$ and consumes $c_{t+1} \geqq 0$ of the good at date t+1. Tastes are represented by the separable utility function $V_1(\ell^* - \ell_t) + V_2(c_{t+1})$, in which $\ell^* - \ell_t$ can be interpreted as "leisure".

Markets are competitive. In equilibrium, the real wage should accordingly be equal to the marginal productivity of labor, which is 1. So we can identify in each period the money wage $w_t > 0$ with the price of the good p_t. Under this convention, the behavior of a representative household born at t is to maximize his utility function under the budget constraints $p_t\ell_t = m_t = p_{t+1}\ c_{t+1}$, in which p_{t+1} is the price expected to prevail in the future. The first order condition of this problem is $\ell_t\ V_1'(\ell^* - \ell_t) = c_{t+1}\ V_2'(c_{t+1})$, in which $V_1'(\ell^* - \ell_t)$ is the marginal utility of leisure and $V_2'(c_{t+1})$ is the marginal utility of future consumption. If we identify the left hand and right hand sides of this equality with $v_1(\ell_t)$ and $v_2(c_{t+1})$, this yields

$$(1.1) \qquad v_1(\ell_t) = v_2(c_{t+1}).$$

Under the usual assumption that marginal utility is decreasing, the function v_1 is increasing and has therefore an inverse v_1^{-1} . Then (1.1) can be rewritten

$$(1.2) \qquad \ell_t = v_1^{-1}[v_2(c_{t+1})] \equiv \chi(c_{t+1}).$$

The graph of χ is the locus of all optimal pairs $(\ell_t\ ,\ c_{t+1})$ chosen by a trader and represents accordingly his *offer curve.* In equilibrium at t, the young household's labor supply ℓ_t is equal to output y_t , which is itself equal to the old's consumption c_t. Under perfect foresight, expected consumption c_{t+1} is equal to the future actual output level y_{t+1}. Thus a deterministic equilibrium with perfect foresight is characterized by a sequence of outputs $y_t > 0$ that satisfy $y_t = \chi(y_{t+1})$ for all $t \geqq 1$.

The occurrence of cyclic deterministic equilibria will depend therefore upon the shape of the offer curve. Increasing the price of future consumption induces as usual substitution and income effects. The substitution effect results from the fact that future consumption becomes more expensive than current leisure. It should thus generate less future consumption and more leisure, or equivalently a lower current labor supply. The income effect is associated to the loss of purchasing power resulting from a higher price. It leads to a lower consumption of both the good and leisure, hence more labor supplied. Both effects work in the same direction for future consumption but in opposite directions for the current labor supply. The function χ representing the offer curve is thus increasing when the substitution effect dominates, decreasing otherwise. If one looks back at the definition of the offer curve given in (1.1), one sees that the substitution effect dominates whenever the "concavity" of the utility of future consumption, as measured by $-c\ V_2''(c)/V_2'(c)$, is small (less than 1) and that the income effect dominates when this concavity is large.

If the substitution effect dominates everywhere, χ is monotonically increasing also everywhere and there can be no cycle of period $k \geqq 2$. This is a case where the Classical viewpoint prevails : no persistent endogenous deterministic fluctuations can be observed. But assume that there is a strong conflict between substitution and income effects, so that the function χ is single peaked with a maximum occurring at the output level y^*, and displays a big hump : $\chi(y^*)$ is large while the second and third iterates, i.e. $\chi^2(y^*) = \chi(\chi(y^*))$ and $\chi^3(y^*) = \chi(\chi^2(y^*))$, are smaller than y^* (Fig. 1). This is a situation where the Keynesian viewpoint is valid, as it gives rise to multiple persistent nonexplosive endogenous fluctuations. In fact this configuration leads to the existence of a cycle of period 3. Such a cycle can be identified with a fixed point of χ^3 that differs from a stationary state. In the situation described here, one has $\chi^3(y) > y$ for $y > 0$ small while $\chi^3(y^*) < y^*$, and χ^3 must accordingly have a fixed point between 0 and y^*. Then from a theorem by Sarkovskii, there are in fact *infinitely many cycles with perfect foresight* : at least one cycle with period k for every positive integer k (Benhabib and Day (1982), Grandmont (1985a)).

Fig. 1

Stochastic endogenous fluctuations

The foregoing deterministic cycles arise through a mechanism of self-fulfilling prophecies. Although the fundamentals are constant over time, traders collectively predict that prices and quantities will fluctuate and this prophecy turns out to be correct in equilibrium. There is no incentive for an individual household to deviate as he has no impact on the market as whole. There is no reason to stop there and one should also ask whether the same phenomenon can occur when traders predict that *stochastic* fluctuations will occur. The answer is indeed yes. There are also many nonexplosive stochastic equilibria generated by self-fulfilling expectations that vary randomly when the income effect is significant (Azariadis (1981), Azariadis and Guesnerie (1986), Chiappori and Guesnerie (1989), Farmer and Woodford (1987), Grandmont (1985b, 1986, 1989)).

Assume that a representative household believes at date t that the future price p_{t+1} of the good will be random. He seeks to maximize the expected value of his utility function, i.e. $V_1(\ell^* - \ell_t) + E_t V_2(c_{t+1})$, in which future consumption is now random since it has to satisfy the budget

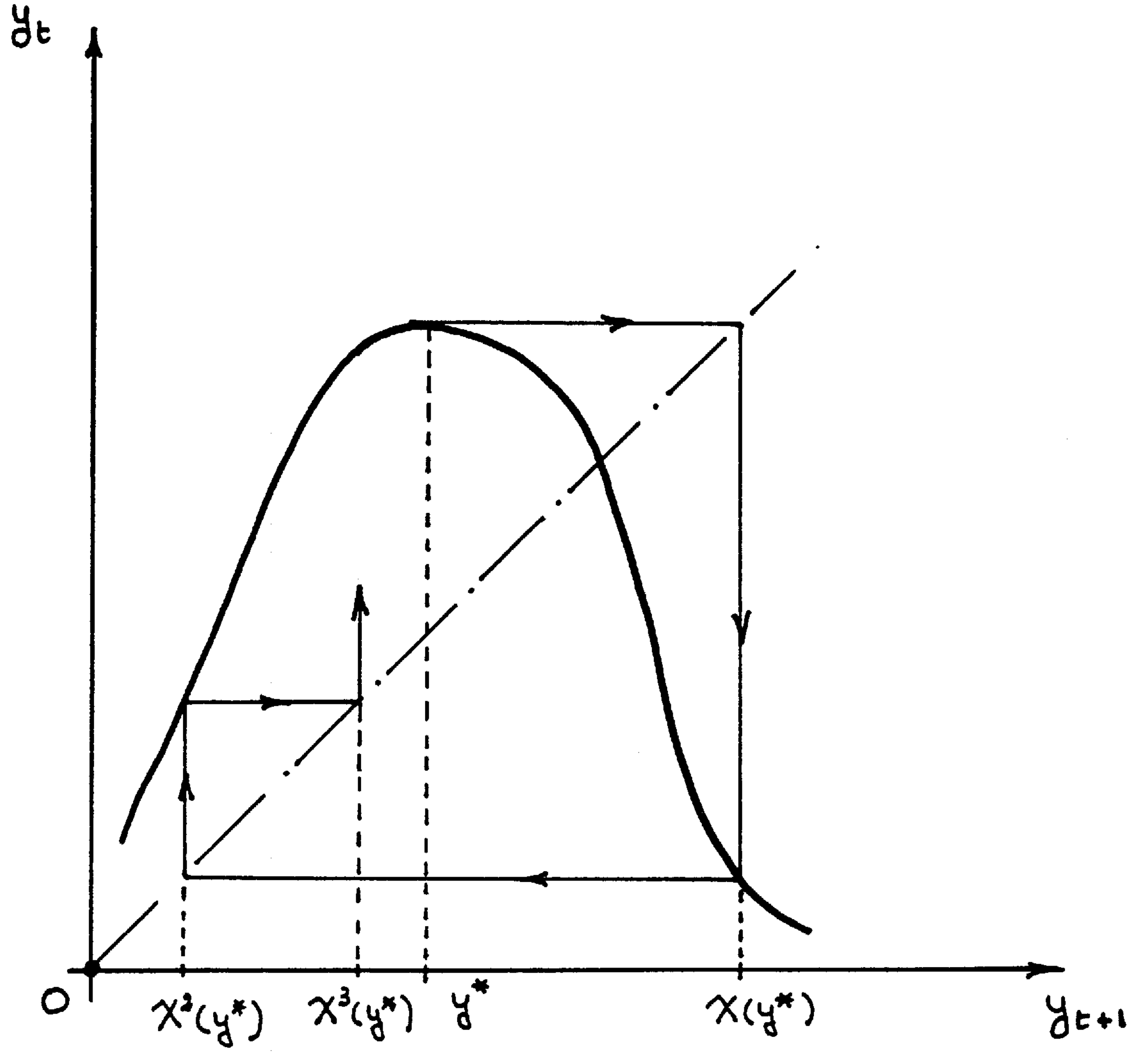


Fig.1

constraints $p_t\ \ell_t = m_t = p_{t+1}\ c_{t+1}$. The first order condition asociated to this decision problem is the analogue of (1.1)

$$(1.3) \qquad v_1(\ell_t) = E_t\ v_2(c_{t+1}).$$

In equilibrium one has here again $y_t = \ell_t = c_t$. Under the assumption of self-fulfilling expectations, expected consumption c_{t+1} is equal to actual future output y_{t+1}. Thus a stochastic equilibrium with self-fulfilling expectations is characterized by a sequence of random outputs $y_t > 0$ such that

$$(1.4) \qquad v_1(y_t) = E_t[v_2(y_{t+1})]$$

for all $t \geqq 1$. In the expression (1.4), the symbol E_t means that people condition their expectations at t on the information available at that date, which can be described by the values taken by an arbitrary exogenous stochastic process of signals s_t, that is

$$E_t = E[\cdot|(s_t, s_{t-1}, \ldots)].$$

The signals s_t are often referred to as *sunspots*, as they describe random variables, the variations of which do not influence at all the "fundamentals" of the system. The terminology is in a sense misleading in as much as it may lend credit to the idea that stochastic endogenous fluctuations are due to people believing in "crazy" theories about the economy. This is wrong. Predictions made everyday by economic experts about the fate of the economy, financial markets and the like are indeed true "sunspots" since they have a negligible direct influence on the "fundamentals". Everyday experience strongly suggests that such predictions are indeed somewhat erratic. The point made here is that if enough people base their expectations on such "sunspots", their variations may be translated into actual fluctuations of equilibrium prices and quantities, even under the constraint that expectations be self-fulfilling.

The general solution of (1.4) is

$$(1.5) \qquad v_2(y_{t+1}) = v_1(y_t) + \varepsilon_{t+1}$$

in which ε_{t+1} is an arbitrary process of stochastic "shocks" satisfying $E_t\ \varepsilon_{t+1} = 0$. It takes indeed the form generally encountered in empirical studies of economic time series, with an important difference in interpretation : *whereas the shocks* ε_{t+1} *are usually interpreted by econometricians finding an equation like* (1.5) *as shocks to the fundamentals, they should be interpreted here as shocks to expectations.*

A geometric characterization of stochastic equilibria is relatively easy in this simple framework. Let [a,b] be the smallest interval containing all y_t with probability 1. Since the expectation operator E_t amounts essentially to taking averages, $v_1(y_t)$ belongs to $v_2([a,b])$. A continuity argument then shows that the minimal invariant interval [a,b] must satisfy

(1.6) $$v_1([a,b]) \subset v_2([a,b]) \quad \text{or} \quad [a,b] \subset \chi([a,b]).$$

One can also show that if the inclusion in (1.6) is strict, i.e. if [a,b] is contained in the interior of its image by χ, it is possible to construct a stochastic equilibrium satisfying (1.4) or (1.5) that stays in the interval at all times by choosing appropriately the disturbances ε_{t+1} (and the sunspot process). In fact there are *infinitely many* of such stochastic equilibria for a *given* invariant interval, see Grandmont (1985b, 1986, 1989) (one can impose stationarity, the existence of an invariant measure, etc ...).

This geometric characterization allows us to reduce the search for stochastic equilibria to the much simpler search for invariant intervals. Then it follows immediately that no stochastic equilibrium sequence of outputs can stay away from autarchy (y = 0) when the substitution effect dominates everywhere (when χ is everywhere increasing), except for the trivial sequence $y_t = \bar{y} > 0$, in which $\bar{y}$ is the unique deterministic monetary stationary state : the only invariant interval [a,b] with a > 0 is indeed $[\bar{y},\bar{y}]$ in that case. The "Classical" viewpoint is thus valid when the substitution effect dominates (Fig. 2.a). By contrast, there are infinitely many invariant intervals satisfying (1.6) when the income effect is strong enough near the stationary state so that $\chi'(\bar{y}) < -1$. It suffices to take $a < \bar{y}$ arbitrary close to $\bar{y}$ and b in $(\chi^{-1}(a), \chi(a))$, see Fig. 2b. The argument shows that such invariant intervals exist arbitrarily near the stationary state, i.e. in *every neighborhood* of $\bar{y}$. This is due to the fact that the (local) inverse χ^{-1} is contracting near the stationary state. When the income effect is significant, the "Keynesian" viewpoint is therefore correct : shifts in expectations can be a significant source of economic fluctuations, independently of any variations of the fundamentals.

Fig. 2.a

Fig. 2.b

Unit roots (local bifurcations)

Many economic time series display a lot of "persistence" (the consequences of a shock tend to persist a long time). In other words, some "eigenvalues" of the system tend to have a modulus close to 1. Some econometricians have gone as far as to claim that quite a few economic time series can be adequately represented as *linear* random walks. The recent literature on nonlinear economic dynamics has shown that in such circumstances, *a very small amount of nonlinearity can dramatically affect the analysis* (Grandmont (1989)).

The proper technique to study what happens when local eigenvalues have a modulus close to 1 is local bifurcation theory. To illustrate the point in the simple framework under consideration, consider a family of economies, indexed by some parameter α, or equivalently of offer curves χ_α, and assume that the income effect is progressively increased so that the slope of the offer curve at the stationary state goes through -1 when the index α goes up ($\chi'_\alpha(\bar{y}_\alpha)$ is greater than -1 when $\alpha < 0$, equal to -1 when $\alpha = 0$ and less than -1 when $\alpha > 0$). The dynamic evolution of deterministic equilibria with perfect foresight near the stationary state is then given by inverting locally the equation $y_t = \chi(y_{t+1})$. Thus they go away from the steady state when $\alpha < 0$ and converge to it when $\alpha > 0$. As for stochastic equilibria, our previous arguments show that one can construct an infinity of them in every neighborhood of the stationary state if and only if $\chi'_\alpha(\bar{y}_\alpha) < -1$, i.e. if and only if $\alpha > 0$.

Were the map χ linear, these statements would be the only ones one could make and they would in fact hold *globally*. But the world is nonlinear and so is the offer curve. The picture one gets then is rather different. The two typical cases are summarized in the local bifurcation diagrams of

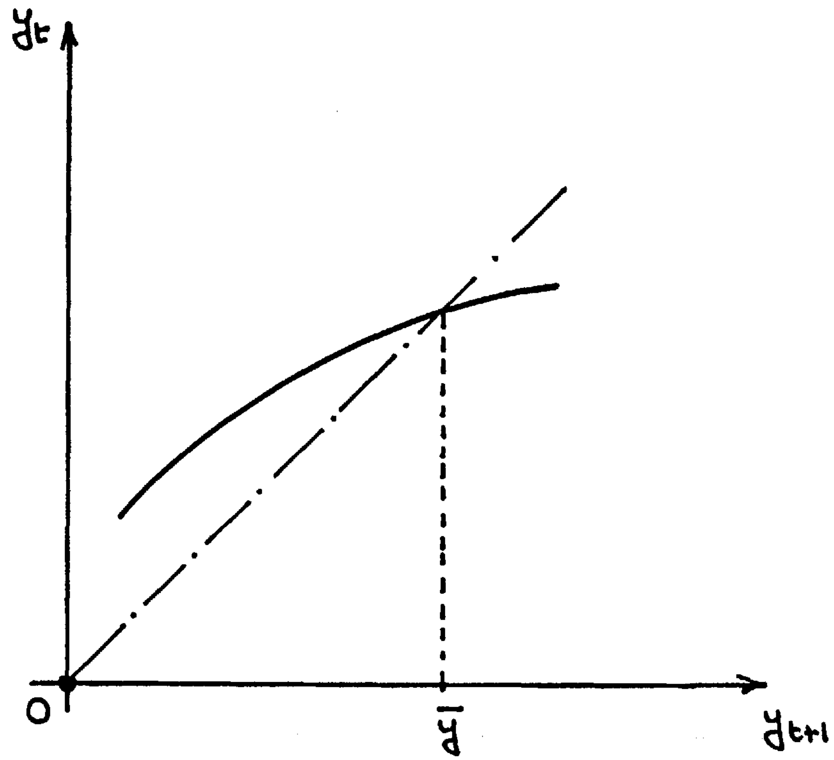


Fig. 2.a

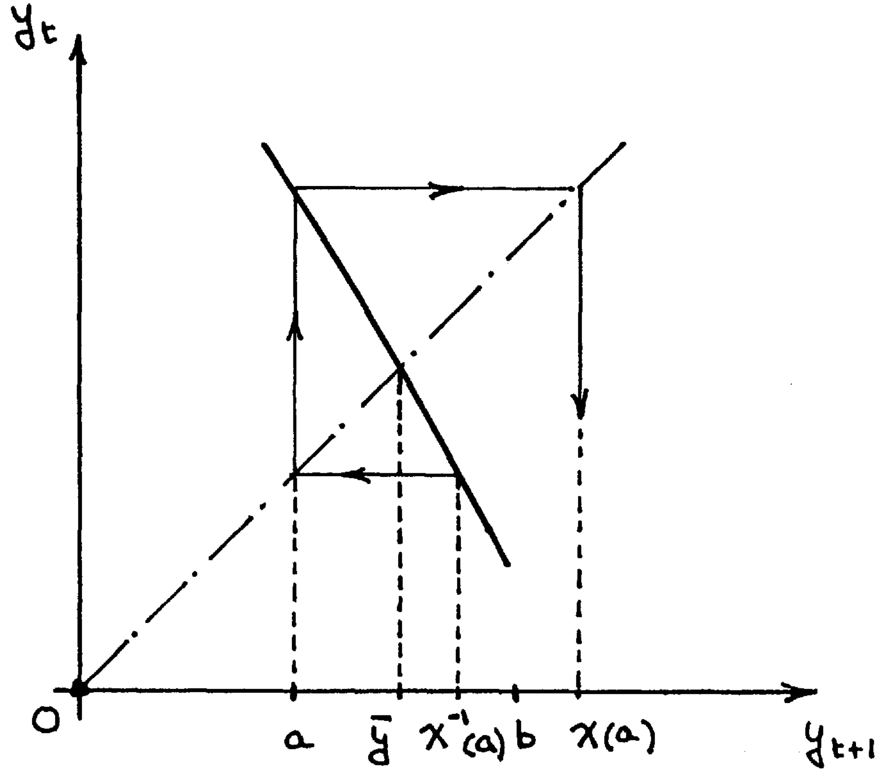
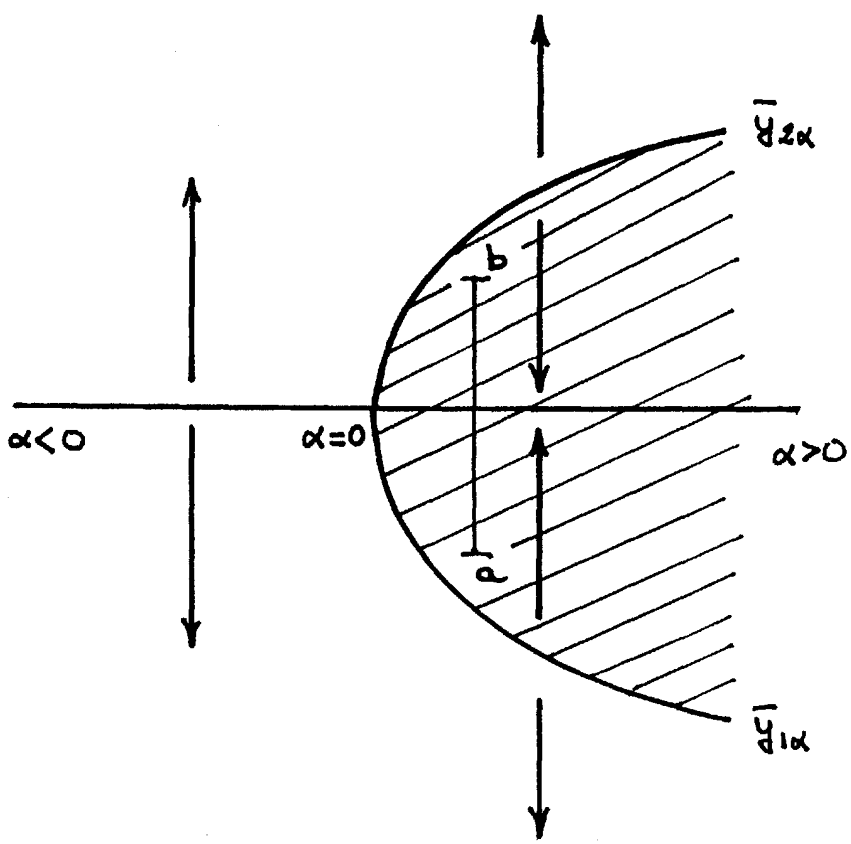


Fig. 2.b

Fig. 3.a, 3.b, in which the indexing parameter α is shown horizontally while the motion of outputs, for a given α, is sketched vertically. The common feature is the presence of a cycle of period 2 $(\bar{y}_{1\alpha}, \bar{y}_{2\alpha})$. In Fig. 3.a, the period 2 cycle appears after the bifurcation, i.e. for $\alpha > 0$, and deterministic equilibria with perfect foresight go away from it, as shown by the arrows. For $\alpha > 0$, it is possible, as we already know, to construct invariant intervals [a,b] and thus stochastic equilibria, but they cannot lie arbitrarily far away from the steady state as would have been predicted in a linear specification. Invariant intervals have here to be in the interior of the interval $(\bar{y}_{1\alpha}, \bar{y}_{2\alpha})$ determined by the cycle, and their union spans that interval. This corresponds to the shaded area in Fig. 3.a.

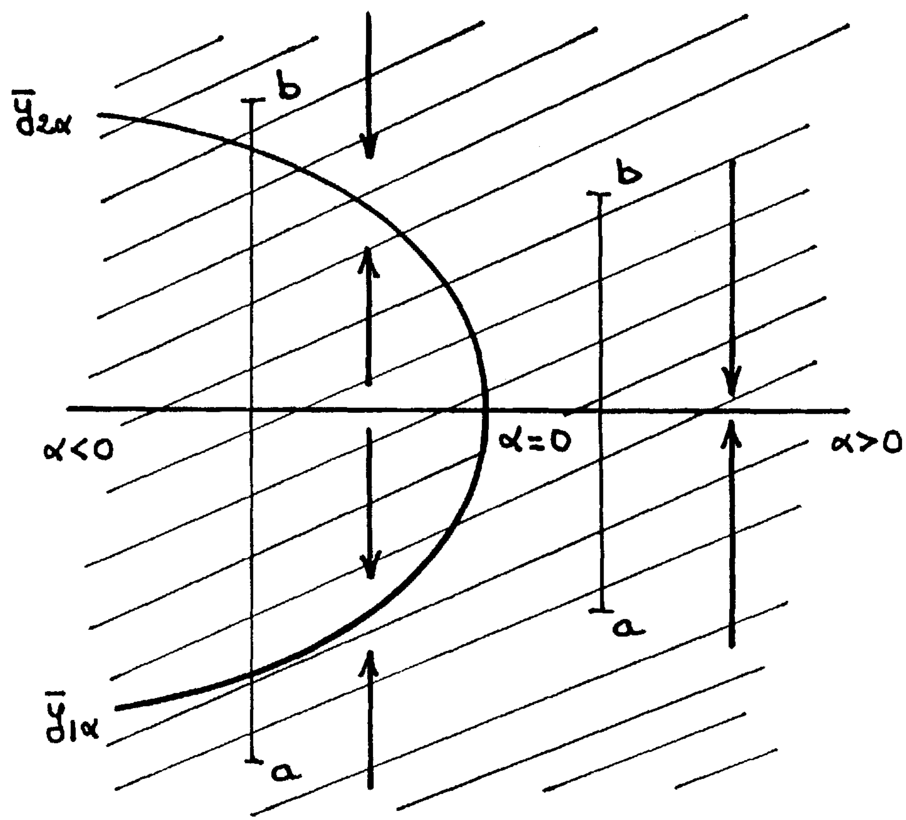
In Fig. 3.b, the cycle appears before the bifurcation, i.e. for $\alpha < 0$, and it is attracting. When $\alpha > 0$, invariant intervals [a,b], thus stochastic equilibria, can be constructed arbitrarily near the steady state and their union spans the whole neighborhood of $\bar{y}_\alpha$ that is represented in the local bifurcation diagram. If $\alpha < 0$, the map χ_α^{-1} is locally expanding and in a linear world we would be led to the conclusion that no nondegenerate invariant interval, hence no stochastic equilibrium, exists in such a case. This is not correct in the nonlinear case. True enough, there is no invariant interval [a,b] with $a \neq b$ included in the open interval $(\bar{y}_{1\alpha}, \bar{y}_{2\alpha})$ determined by the cycle. But infinitely many such invariant intervals can nevertheless be constructed : each of them contains in its interior the period 2 cycle and their union spans the whole neighborhood of the steady state represented in the bifurcation diagram. Nonlinearity, however small, matters a lot in such a configuration ! Moreover, whether Fig. 3.a or 3.b obtains depends upon derivatives of the offer curve up to order 3, hence upon the derivatives of the traders' utility function up to the 4th order, and economic theory provides no information on these. The lesson is clear. In view of the persistence displayed by economic time series, taking into account even small nonlinearities in empirical work may indeed generate quite significantly different results about the dynamics of the economic system.

<u>Fig. 3.a</u> <u>Fig. 3.b</u>



$\bar{y}_{2\alpha}$
b
a
$\alpha<0$
$\alpha=0$
$\alpha>0$
$\bar{y}_{1\alpha}$

Fig. 3.a



$\bar{y}_{2\alpha}$
b
b
$\alpha<0$
$\alpha=0$
$\alpha>0$
a
$\bar{y}_{1\alpha}$
a

Fig. 3.b

Money growth as sunspot

The foregoing analysis has interesting implications concerning the nonneutrality of variations in the money supply. Adding public spending and/or taxes (e.g. lump-sum money transfers) in the present model leads to the usual conclusion that deterministic or random fluctuations of the money stock through these channels do have real effects — after all these are variations of the "fundamentals" ! (Grandmont (1986)). Let us consider now monetary policy, or more specifically, money transfers that are proportional (even out of equilibrium) to the traders' money balances. Macroeconomists sometimes claim that such changes of the money supply cannot have any real effect when traders have complete information and self-fulfilling expectations, as these changes are identical to changes in monetary units. To have real consequences, such changes should be "misperceived" by the private sector (Lucas (1972), Barro (1981)). It is fairly easy to see with the help of the foregoing example that the claim is in general false. The main point is that households can use the rates of growth of the money supply as "sunspots" that influence their expectations and thus real allocations.

Assume that in the above model money transfers are made at each date t at the gross rate $x_t > 0$, proportionally to the old trader's money balance m_t — even out of equilibrium. It is readily verified that the equation describing the dynamic evolution of real equilibrium variables is the same as before : it is still given by (1.4). Now assume that the traders have complete information and condition their expectations on the money growth rates : in (1.4) the conditional expectation operator E_t is in fact $E[\cdot|x_t, x_{t-1}, \ldots)]$. In other words, the money growth rates play the role that "sunspots" s_t played in the previous analysis. There are always nonstationary stochastic equilibria in which variations of the money stock have real effects : their general form is given by (1.5). If we focus on "nonexplosive" stochastic equilibria that stay away from autarcy ($y_t \geqq a > 0$ or prices remain bounded above), there are two cases to consider. In the "Classical" case where the substitution effect dominates (χ is increasing) everywhere, the only possible equilibrium is $y_t = \bar{y}$ at all dates (Fig. 2.a). Changes in the rate of growth of the money supply are then indeed superneutral since they are exactly compensated by changes in the price level, and there is no variation of output. The story is completely different in the "Keynesian" case where there is a strong income effect as in Fig. 2.b or Fig. 3. For there are then many nonexplosive stochastic equilibria near the stationary state where changes in the money growth rate have real effects. These equilibria generally imply arbitrary, positive or negative, correlations between money, inflation, and output and involve various degrees of persistence, i.e. of dependence of current output on past money growth rates. These conclusions hold in particular when the growth rates x_t are i.i.d., satisfy $E(x_t) = 1$ and have continuous density with support $[\alpha,\beta]$ with $0 < \alpha < \beta$, as in Lucas (1972). They are also valid for more complex processes.

Productive investment and endogenous fluctuations

The model considered up to now owed its simplicity to the fact that it was "one-dimensional" : the state of the system at any date could be described by a single real number. Despite its pedagogical usefulness, this simplicity had a price. We had indeed to introduce a lot of nonlinearity in the system, i.e. to make the map χ representing the offer curve decreasing by assuming the presence of a strong income effect, in fact too strong to be empirically plausible. We wish to show now that if we take the (realistic) step of considering productive investment, most of the phenomena we have been talking about still occur (there are many endogenous expectations-driven business cycles) *under the assumption that the substitution effect dominates*, i.e. when the function χ is increasing everywhere.

The households' sector is the same as before. We assume now that output y_t in period t is produced not only from the labor ℓ_t supplied during the same period by the young household, but also from the stock of capital equipment k_{t-1} that is available at outset of the period, these two inputs having to be used in fixed proportions. Specifically, $y_t = \text{Min}\{\ell_t, k_{t-1}/a\}$, where a is the capital-output ratio (or 1/a is the productivity of capital). Current output y_t is then partly directed toward consumption, partly invested. Thus in a market equilibrium at t, one has $y_t = c_t + i_t$, where c_t is consumption of the old and i_t is investment. The capital stock available for production at the outset of period t+1 is then

$$k_t = k_{t-1}(1-\delta) + i_t \ , \qquad (1.7)$$

where $0 < \delta \leq 1$ is the (given) depreciation rate of capital equipment. To simplify matters, it is assumed that workers do not have access to direct or indirect ownership of capital, while the production sector is operated by entrepreneurs seeking to maximize profits. [2]

We first look at deterministic equilibria with perfect foresight in this context. The households' optimal choices are still represented by the aggregate offer curve, i.e. by $v_1(\ell_t) = v_2(c_{t+1})$ or $\ell_t = \chi(c_{t+1})$. We focus on the standard case where the substitution effect dominates everywhere, i.e. $\chi(c)$ is increasing for all $c > 0$. The productive sector yields $y_t = \ell_t = k_{t-1}/a$. Under perfect foresight, expected consumption c_{t+1} is equal to actual future output y_{t+1} minus actual future investment i_{t+1}. But on account of (1.7)

$$(1.8) \qquad i_{t+1} = k_{t+1} - k_t(1-\delta) = a[y_{t+2} - y_{t+1}(1-\delta)].$$

Intertemporal equilibria with perfect foresight are therefore characterized by sequences of outputs $y_t > 0$ such that

$$(1.9) \qquad y_t = \chi[(1 + a(1-\delta))\, y_{t+1} - a y_{t+2}],$$

for all $t \geq 1$.

We shall focus on the occurrence of endogenous expectations-driven business cycles near a stationary solution $y_t = \bar{y} > 0$ of (1.9), i.e. that satisfies $\bar{y} = \chi[(1 - a\delta)\, \bar{y}]$. It is easy to see that such a steady state exists (and is unique) whenever $a\delta < 1$, provided that the slope of the offer curve at the origin is large enough, i.e. $\lim_{c \to 0_+} \chi(c)/c > 1/(1 - a\delta)$. [3]
The simplest technique to get cyclical behavior near the steady state is to generate a local bifurcation. To this effect, one looks at the eigenvalues of the local dynamics obtained by linearizing (1.9) around the stationary solution $\bar{y}$ and one varies the "fundamental" characteristics of the system (or a parameter, say α, indexing these characteristics) so as to get a change of stability. The interesting point is that (1.9) is in effect two-dimensional since it involves two lagged state variables. It is then relatively easy to show that the only case which can occur when the map χ is increasing as here, is that a pair of complex conjugate eigenvalues

crosses the unit circle — one gets then a socalled Hopf bifurcation. As argued earlier, analysis of a configuration of this sort is particularly relevant if one believes in the "unit root" hypothesis in macroeconomics, i.e. that economic time series display eigenvalues with a modulus close to 1.

It is possible to show that such a Hopf bifurcation will indeed occur if one holds the capital-output ratio constant and greater than $1/(1+\delta)$, and if one varies the "concavity" of an old trader's utility for consumption, as measured by the coefficient $- c V_2''(c)/V_2'(c)$, between 0 and 1. These assumptions are quite compatible with those which macroeconomists usually employ about utility functions and with actual data : the capital-output ratio is about 3 in industrialized countries. The steady state $\bar{y}$ is then first stable in the local dynamics determined by (1.9) and becomes unstable when the Hopf bifurcation occurs as the concavity of V_2 is increased. If the system was linear, that would be all one could say. But to be realistic, the offer curve must be nonlinear. Then an invariant closed curve C must appear in the picture near the stationary solution in the state space (y_t, y_{t-1}), on which the motion is periodic (may be with a complicated long period) or quasi-periodic. Figures 4 and 5 are local bifurcation diagrams that describe the two typical cases that are bound to occur, where the indexing bifurcation parameter α is shown horizontally while the motion of the system takes place, for each α, in a plane that is perpendicular to the α-axis. In Fig. 4, the invariant closed curve appears after the bifurcation and it is stable. In Fig. 5, it is present before the bifurcation but it is repulsive. Whether one of two cases arises depends on higher order derivatives of the offer curve.

<u>Fig. 4</u>

<u>Fig. 5</u>

The argument shows how endogenous deterministic fluctuations may occur near the steady state when the local characteristic roots of the system have close to unit modulus. As in the onedimensional case considered earlier, the story does not stop there and one should inquire about the occurrence of stochastic endogenous fluctuations as well. Stochastic equilibria with self-fulfilling expectations are here characterized by

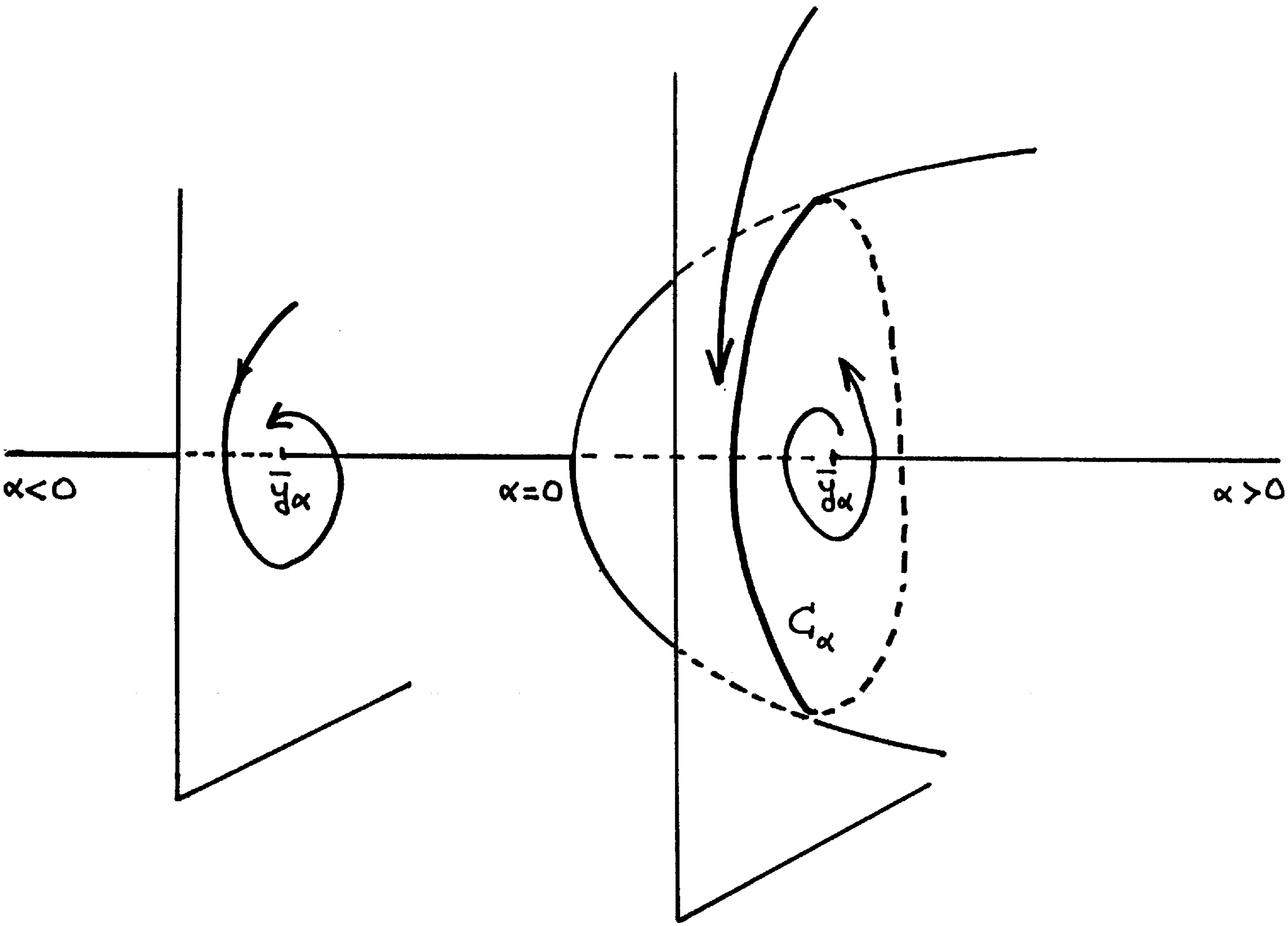


Fig. 4

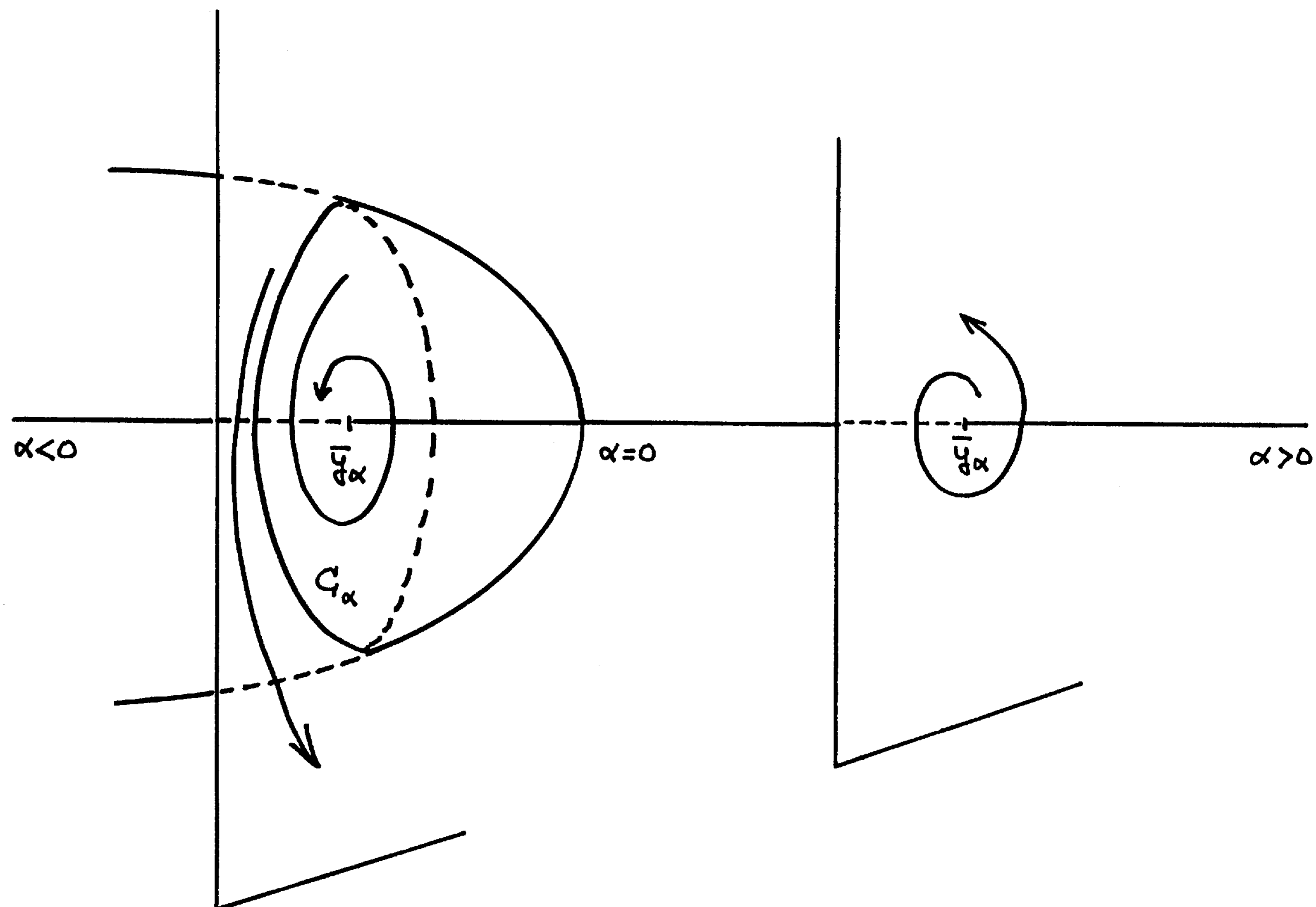


Fig. 5

random sequences of output levels $y_t > 0$ that satisfy for all $t \geqq 1$ a relation analogous to (1.9)

$$(1.10) \qquad v_1(y_t) = E_t \; v_2[(1 + a(1-\delta)) \; y_{t+1} - a \; y_{t+2}],$$

in which E_t means that traders condition their expectations on the observed values $(s_t, s_{t-1}, \ldots)$ of some exogenous stochastic process of "sunspots", the variations of which do not affect the "fundamentals". The general (non necessarily stationary) solutions of (1.10) take the form

$$(1.11) \qquad v_2[(1 + a(1-\delta)) \; y_{t+1} - a \; y_{t+2}] = v_1(y_t) + \varepsilon_{t+1} \; ,$$

in which ε_{t+1} is an arbitrary sequence of random variables satisfying $E_t \; \varepsilon_{t+1} = 0$. Here again, the recursive equation (1.11) resembles analogous relations found in econometric works, with the important difference that the disturbances ε_t should not be interpreted as "shocks" to the fundamentals but rather as "shocks" to expectations [4].

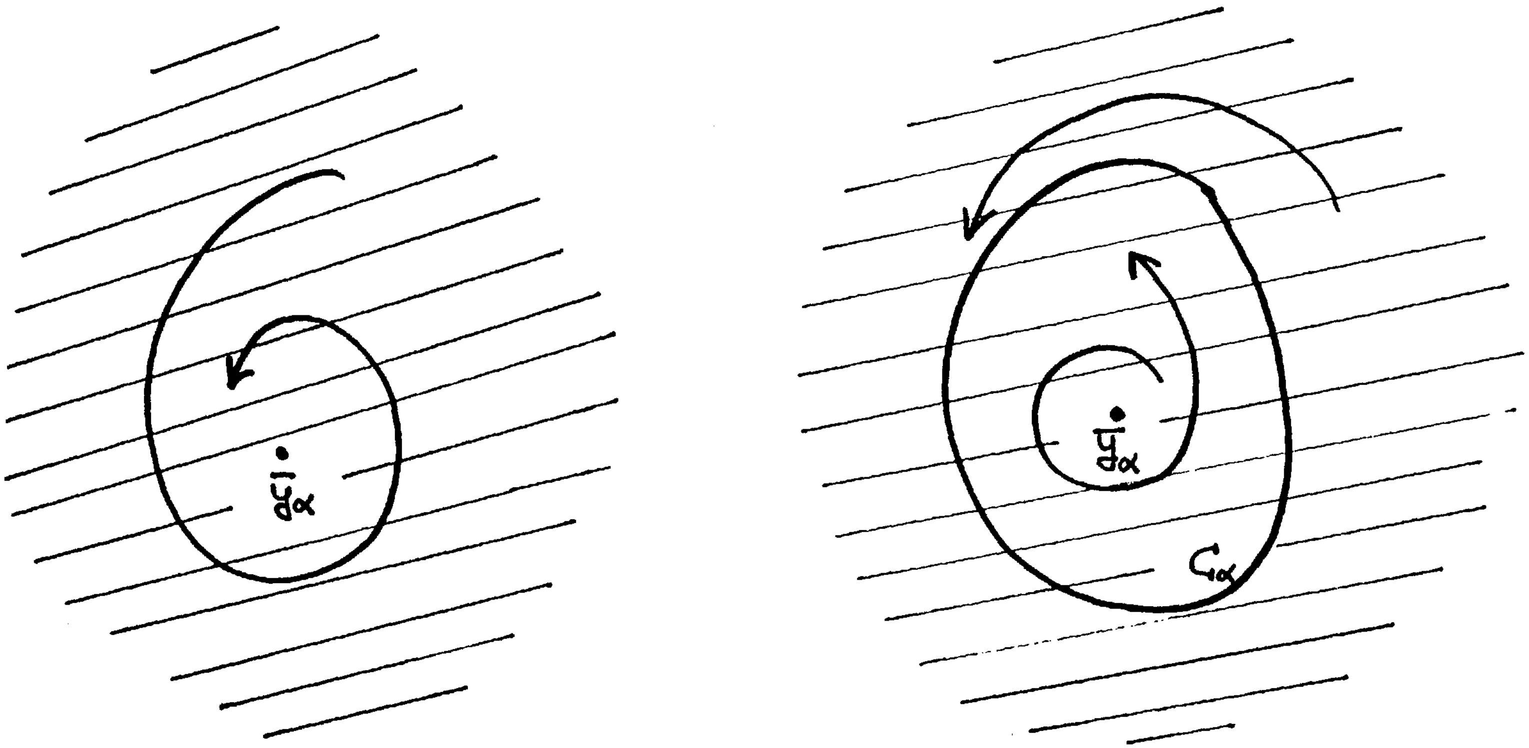
A natural question is whether nonexplosive endogenous stochastic fluctuations of this sort can arise locally, i.e. near the steady state. As in the onedimensinal case studied earlier, one can construct infinitely many such stochastic equilibria in *every* neighborhood of the stationary state if, and in general only if, it is stable in the local deterministic dynamics with perfect foresight defined implicitly by (1.9) — this will be the case if the capital-output ratio is held constant but the concavity of the utility of consumption, i.e. $-c \; V_2''(c)/V_2'(c)$, is small (see Woodford (1986)).

Consideration of what happens for systems displaying a high degree of persistence, or specifically when a local (Hopf) bifurcation occurs, yields also here interesting phenomena that are due to the presence of an invariant closed curve near the steady state, as described in the two local bifurcation diagrams of Fig. 4 and 5. In both cases, the steady state is stable before the bifurcation (when $\alpha < 0$), and therefore nonexplosive stochastic equilibria exist arbitrarily near it. In the case of Fig. 4, the union of the supports of these stochastic equilibria fills the whole neighborhood of the stationary state represented in the local bifurcation diagram, in agreement with what would have been predicted in a linear specification. In the case of Fig. 5, however, each of these supports has to lie in the interior of the region inside the closed curve C_{α} , and their union spans that region. These two cases correspond to the shaded areas in Fig. 6 and 7, for $\alpha < 0$. On the other hand, after the bifurcation (when $\alpha > 0$), there are no nondegenerate stochastic equilibria that would stay in an extremely small neighborhood of the steady state since it is then unstable. Would the world be linear and thus similar in the small and in the large, inexistence would also hold globally. In the nonlinear case described in Fig. 5, inexistence indeed carries over to the whole neighborhood of the stationary state under consideration where $\alpha > 0$. Thus there is no shaded area in Fig. 7 for $\alpha > 0$. The picture is dramatically different in the situation of Fig. 4, due to the presence of the stable invariant closed curve near the steady state. There are no stochastic equilibria, other than the steady state itself, that lie wholly in the interior of the region inside the invariant closed curve. But there are many stochastic equilibria, the support of which *contains* in its interior the invariant closed curve, and the union of these supports fills the whole neighborhood of the steady state represented in the bifurcation diagram. This corresponds to the shaded area in Fig. 6 for $\alpha > 0$. Here is another case where a very small amount of nonlinearity changes significantly the conclusions of the analysis, by comparison with what we would have gotten had we (wrongly) chosen a linear specification [5].

Fig. 6

Fig. 7

I hope this last example will convince the reader that complex endogenous expectations-driven business cycles can arise under quite plausible assumptions and furthermore, that it is important to incorporate nonlinearities when studying such fluctuations, especially when the eigenvalues of the system have a modulus close to 1. The main "non standard" assumption that was responsible for the occurrence of endogenous fluctuations in the above model, apart from the presence of plausible capital markets imperfections, was that capital equipment and labor had to be combined in fixed proportions in production, or more generally that the elasticity of substitution between labor and capital was low. I would



$\alpha < 0$ $\alpha > 0$

Fig. 6

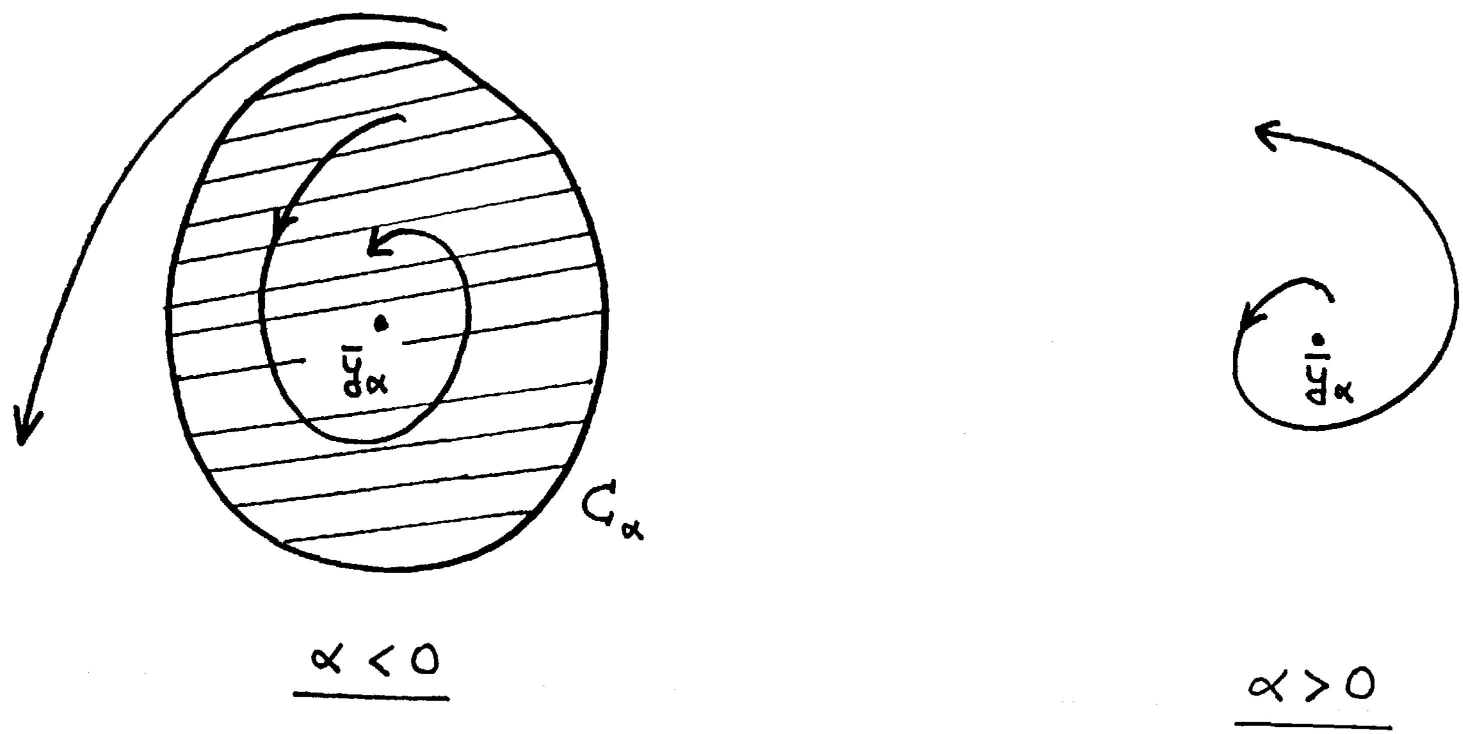


Fig. 7

expect the same sort of results to hold with a more elastic technology if we incorporate in the model the realistic feature that there are significant costs to incur when adjusting the capital stock through investment, or when changing employment, so that the "effective" elasticity of substitution is in fact low. There is apparently still some way to go before models of this type are operational enough to change actual econometric practice. But I believe the studies reviewed here suggest strongly that endogenous business cycles models have become more and more credible alternatives to describe observed fluctuations in our economies.

2. LEARNING : ARE RATIONAL EXPECTATIONS UNSTABLE ?

An essential feature of economic models is that expectations matter. Individual expectations about the future influence current decisions, hence observed market outcomes. On the other hand, observations about market outcomes determine individual expectations. In the absence of firmly established empirical facts about actual individual forecasting behavior, quite a few economic theorists have chosen to impose in their models some kind of consistency between expectations and realized market outcomes, on the ground that arbitrary assumptions about expectations would allow to explain almost everything — and thus nothing. A common modelling strategy is indeed to postulate "rational" expectations, i.e. that an individual's expectations about the future are correct at any moment, given current information. In fact, we have imposed exactly this consistency requirement in the first part of the lecture.

A frequently heard defense of the hypothesis is that it should be the asymptotic outcome of a dynamic process in which individuals learn about the laws of motion of their environment. It is by no means clear, however, that taking into account learning should lead to convergence to a dynamic state where the "rational" expectations hypothesis is satisfied. Economic "reality" is not independent of how individual players conceive it. Changing beliefs about the economic system modify the laws of motion of that system. The second part of this lecture is devoted to a brief review of recent studies suggesting that, indeed, taking into account learning often generates dynamic local instability (Champsaur (1983), Benassy and Blad (1989), Grandmont and Laroque (1991), Grandmont (1990)). Such results seem to agree with (admittedly casual) empirical observations. The economic time series that display most volatility are those for which it appears that expectations are important in shaping current decisions (investment in capital equipment, inventories, durable goods, financial and stock markets). As we shall see shortly, imposing "rational" expectations would lead to the exact opposite and counterfactual conclusion : under the hypothesis the more expectations matter, the more stable (in the absence of exogenous shocks to the "fundamentals") the market should be. By contrast, the studies reviewed below suggest that *local dynamic instability induced by learning is most likely to occur in markets where expectations matter significantly*. They suggest further that in such markets, the dynamics with learning may be highly nonlinear and generate complex trajectories, and moreover that forecasting mistakes may never vanish, even in the "long run".

Smooth foregasting rules

To illustrate the point most simply, we consider a deterministic formulation (no random shocks) in which the state of system in period t is described by a single real number x_t. The state at t is determined by the decisions made by the traders in the past, which we summarize by the immediately preceding state x_{t-1}, and by the traders' forecast about the future x^e_{t+1}, through the temporary equilibrium relation

$$(2.1) \qquad T(x_{t-1}\ ,\ x_t\ ,\ x^e_{t+1}) = 0.$$

We assume that there is a large number of traders, each of whom has a negligible influence on the market as a whole, so strategic considerations are also negligible. In (2.1), x^e_{t+1} should be interpreted as an *average* forecast (each individual's forecast being weighted by its relative influence on the dynamic evolution). The state variable can be viewed, say, as a price and (2.1) as a market clearing condition. The analysis will be local, i.e. near a steady state defined by $T(\bar{x},\bar{x},\bar{x}) = 0$. We shall assume throughout that T is smooth and denote by b_0 , b_1 and a the partial derivatives of T with respect to x_t , x_{t-1} and x^e_{t+1} , evaluated at the stationary state. The parameter a, which measures the local influence of expectations in the market under consideration, is of course assumed to be different from 0, otherwise the issues we wish to analyze would disappear.

The other ingredient of the model is a specification of how forecasts are made. The traders mental processes may be quite sophisticated : they may have "models" of the world depending upon a number of unknown parameters, reestimate these parameters at each date by using past data and use these estimates to forecast the future. It turns out we do not have, for the purpose of the present analysis, to specify in great details the traders' mental processes. In *all* cases, forecasts with depend on, and only on, past data. To simplify matters, we assume that the average forecast is a time-independent function of a finite (but possibly very large) array of past states [6]

(2.2) $$x^e_{t+1} = \psi(x_t, x_{t-1}, \ldots, x_{t-L}).$$

We shall assume throughout that when presented with a long constant sequence of states equal to $\bar{x}$, then $x^e_{t+1} = \bar{x}$. We postulate (in this subsection) that learning is regular enough so that the forecasting rule ψ is smooth, and we shall denote by $c_0, \ldots, c_L$ its partial derivatives with respect to $x_t, \ldots, x_{t-L}$, evaluated again at the steady state.

The dynamics with learning that will be actually observed is defined by putting (2.1) and (2.2) together

(2.3) $$T(x_{t-1}, x_t, \psi(x_t, x_{t-1}, \ldots, x_{t-L})) = 0.$$

Clearly $x_t = \bar{x}$ is a stationary solution of (2.3). If we assume that the partial derivative of (2.3) with respect to x_t at the steady state differs from 0, i.e. $b_0 + ac_0 \neq 0$, the actual dynamics with learning is well defined near the stationary solution $x_t = \bar{x}$. The issue is to analyze its stability.

The usual procedure to evaluate local stability is to linearize the equation near the steady state, look at the corresponding characteristic polynomial and to see whether the resulting eigenvalues are stable (have modulus less than 1) or not. It is intuitively clear that all the information we need to proceed here is in fact embodied in the local behavior of (2.1) and (2.2). Indeed, (2.3) is obtained by "coupling" the dynamical systems (2.1) and (2.2) — in which the forecast x^e_{t+1} would be replaced by the actual state x_{t+1} — in such a way that the variable x_{t+1}

actually disappears. Stability or instability of the actual dynamics with learning will accordingly be a consequence of the interaction of the local eigenvalues of (2.1) and those of (2.2).

The local eigenvalues of (2.1) are the two roots of the characteristic polynomial obtained by replacing x^e_{t+1} by x_{t+1} and linearizing near the steady state

$$(2.4) \qquad Q_T(z) \equiv b_0 + b_1 z + az^2 = 0.$$

The corresponding local eigenvalues λ_1, λ_2 summarize the local behavior of the economic system *under the assumption of perfect foresight*. One remarks that, as announced earlier, the hypothesis leads to the counterfactual conclusion that the more expectations matter (the larger the coefficient a is, given b_0 and b_1), the smaller the modulus of the two perfect foresight roots λ_1 , λ_2 and thus the more stable the local dynamics of the system should be.

The same procedure applied to the forecasting rule yields the polynomial

$$(2.5) \qquad Q_\psi(z) \equiv z^{L+1} - \Sigma_0^L c_j z^{L-j} = 0.$$

Since the characteristic polynomial is obtained by linearizing (2.2), the corresponding L+1 roots $\mu_1,\ldots,\mu_{L+1}$ (the local eigenvalues of the forecasting rule) describe the set of regularities, i.e. the trends and frequencies, that traders are on average able to filter out of current and past deviations Δx_t , $\Delta x_{t-1},\ldots,\Delta x_{t-L}$ from the stationary state. If people extrapolate constant sequences ($\psi(x,x,\ldots,x) \equiv x$ near $\bar{x}$) then $\mu = 1$ is solution of (2.5). If they extrapolate sequences that oscillate between two values ($\psi(x,y,x,y,\ldots) \equiv y$ near $\bar{x}$), then $\mu = 1$ and $\mu = -1$ are solutions of (2.5). If people are able to recognize and willing to extrapolate the specific trend r from past deviations, then $\mu = r$ is a solution of (2.5). More generally, the fact that $\mu = re^{i\theta}$ is a local eigenvalue of the forecasting rule means that people are able to recognize the trend r and the frequency associated to θ in past deviations from the stationary state. Of course, a smooth forecasting rule essentially acts locally as a linear filter and can extract only a finite set of regularities from a finite amount of data. When the memory L is large, and if the traders are relatively sophisticated, one should expect the set of local eigenvalues $\mu_1,\ldots,\mu_{L+1}$ of the forecasting rule to be somewhat spread out in the complex plane. It turns out that this configuration leads to local instability of the actual learning dynamics, especially when expectations matter significantly, i.e. when the coefficient a is large.

Specifically, let $\mu_1^* < \mu_2^*$ be the smallest and largest *real* local eigenvalues of the forecasting rule. Consider the situation where the two perfect foresight roots λ_1, λ_2 are either both complex or where, if they are real, they belong to the open interval (μ_1^*,μ_2^*). Then it is easy to show that the characteristic polynomial associated to the actual learning dynamics (2.3) has a real root ρ that lies outside the interval $[\mu_1^*,\mu_2^*]$. If we make the mild assumption that people are willing to extrapolate long sequences that oscillate between two arbitrary values x and y near the steady state, then as noted earlier, $\mu = 1$ and $\mu = -1$ are local eigenvalues of the forecasting rule. In that case $\mu_1^* \leqq -1$ and $\mu_2^* \geqq 1$, and the actual learning dynamics is bound to be locally unstable. This configuration, and thus local instability, is most likely to occur when expectations matter significantly, i.e. when the coefficient a measuring the local influence of forecasts on the evolution of the system is relatively large, for then the modulus of the two perfect foresight roots λ_1, λ_2 is small.

Discontinuous forecasting rules

A smooth forecasting rule (locally, essentially a linear filter) can only extract a finite set of trends and of frequencies from a finite sequence of past deviations from the steady state. It is not difficult to think of learning rules, e.g. through least squares regressions on such past deviations, that would allow to recognize and extrapolate, say, *any* real trend present in past data. Of course, one is then bound to lose smoothness and even continuity of the associated forecasting rule. Yet one would like to inquire whether the previous instability results carry over to such learning processes. We are going to show that this is indeed the case.

To simplify matters, we set $\bar{x} = 0$ (so x_t stands now for a *deviation* from the steady state) and linearize (2.1)

(2.6) $$b_1 x_{t-1} + b_0 x_t + a x^e_{t+1} = 0$$

with $a \neq 0$ (expectations matter) and $b_0 \neq 0$ so that we can actually solve (2.6) for the current state x_t. As for expectations, we assume that people believe, say, that the law of motion of the system is

(2.7) $$x_n = \beta x_{n-1} + \varepsilon_n \qquad \text{or} \qquad x_n = (\beta + \varepsilon_n)\, x_{n-1}$$

where β is an unknown coefficient and ε_n is white noise. Forecasts are generated as follows. At the outset of period t, traders form an estimate of the unknown coefficient β by looking at past states

(2.8) $$\beta_t = g(x_{t-1},\dots,x_{t-L}),$$

and they formulate a forecast by iterating twice the relation (2.7)

(2.9) $$x^e_{t+1} = \beta^2_t\, x_{t-1}.$$

The relations (2.8) and (2.9) together define a forecasting rule $x^e_{t+1} = \psi(x_{t-1},\dots,x_{t-L})$ exactly as before. One possible interpretation of this learning procedure is that people know where the steady state lies, but try to improve their performances by forecasting growth rates. We may, however, lose continuity if we wish, as here, that people be able to filter a continuum of trends out of past deviations from the steady state. For instance, if people estimate the models (2.7) through least squares, they will get

$$\beta_t = \frac{x_{t-1}\, x_{t-2} + \dots + x_{t-L+1}\, x_{t-L}}{x^2_{t-2} + \dots + x^2_{t-L}},$$

or

$$\beta_t = \frac{1}{L-1}\left[\frac{x_{t-1}}{x_{t-2}} + \dots + \frac{x_{t-L+1}}{x_{t-L}}\right],$$

which are only defined out of the steady state and in fact highly discontinuous there. The nice feature of the above least squares learning schemes is that the estimates β_t are averages of past ratios x_{t-j+1}/x_{t-j}. As a result, the forecasting rule generated by (2.8) and (2.9) has the property that for every real number r,

$$\psi(r^{L-1} x,\dots,rx,x) \equiv r^{L+1} x.$$

People can extract any real trend from past deviations from the stationary state, or in other words, any real number is a "local eigenvalue" of the forecasting rule. The price to pay for this nice feature is the loss of continuity.

The actual learning dynamics is obtained as before by putting together (2.6) with the forecasting rule defined by (2.8) and (2.9), which yields

$$(2.10) \qquad x_t = -b_0^{-1}[b_1 + a\,\beta_t^2]\, x_{t-1} \equiv \Omega(\beta_t)\, x_{t-1},$$

β_t given by (2.8).

The relation (2.8) defining the actual learning dynamics involves a map Ω (introduced in the literature on the subject by Marcet and Sargent (1989)) that has a remarkably simple interpretation. Indeed it describes the link that exists between the *beliefs* people have at the outset of period t about the dynamics of the growth rates x_n/x_{n-1}, as summarized by the estimate β_t, and the *actual* ratio x_t/x_{t-1} that will be observed in that period. It is easy to verify that the fixed points of Ω coincide with the perfect foresight roots λ_1, λ_2 when these are real, and that Ω has no fixed points when λ_1, λ_2 are complex (the equation $\Omega(\beta) = \beta$ is in fact identical to (2.5) with $z = \beta$).

The smallest and largest real "eigenvalues" of the forecasting rule are $-\infty$ and $+\infty$ whenever the estimate (2.8) is an average of past ratios x_{t-j+1}/x_{t-j} , $j = 2,\dots,L$. By analogy with the smooth case discussed earlier, we should expect that the actual learning dynamics is locally unstable in the present case as well, and this for *all* configurations of the two perfect foresight roots λ_1 and λ_2. It can be shown that this conjecture is indeed true under quite general conditions (Grandmont and Laroque (1991)). When the perfect foresight roots λ_1 , λ_2 are complex, local instability occurs for all initial conditions. Suppose now that they are real, with $|\lambda_1| < |\lambda_2|$. Then local instability occurs for an open set of initial conditions, i.e. when the initial ratios x_{t-1}/x_{t-2} , etc ... have all the same sign as λ_2 and a modulus larger than $|\lambda_2|$. Were the forecasting rule smooth, getting local unstability for an open set of initial conditions would imply instability for almost every departure from the steady state. This may not be true here as the forecasting rule is discontinuous. If the map Ω is contracting at the perfect foresight root of smallest modulus, i.e. $|\Omega'(\lambda_1)| < 1$, and if that root is stable, i.e. $|\lambda_1| < 1$, then one will get local stability whenever all initial growth rates x_{t-1}/x_{t-2}, etc ... are close enough to λ_1. Of course if $|\Omega'(\lambda_1)| > 1$, the phenomenon disappears. Be it as it may, the size of the open set for which local instability occurs becomes larger as the coefficient a measuring the relative influence of expectations goes up (the modulus of the two perfect foresight roots goes down). Thus we reach the same qualitative conclusion as in the smooth case : *the more expectations matter, the more probable learning induced local instability becomes, and the more volatile market outcomes should be.*

To illustrate these points, we consider the particular situation where the estimate β_t is x_{t-1}/x_{t-2}. One gets then even sharper results that can be visualized through simple diagrams. In view of (2.10) the actual dynamics with learning is in that case described by the recurrence equation $x_t/x_{t-1} = \Omega(x_{t-1}/x_{t-2})$. The curve representing Ω (a parabola) is pictured in Fig. 8.a,b in the case where the sum of the two perfect foresight roots, i.e. $\lambda_1 + \lambda_2 = -b_o/a$, is positive (the reader will verify that one gets identical results in the opposite case, with the asymptotic branches of the parabola going down). The two roots λ_1, λ_2 are complex in Fig. 8.a, and one gets local instability for all initial conditions. The two perfect foresight roots are real in Fig. 8.b. There local instability occurs whenever the modulus of the initial ratio x_{t-1}/x_{t-2} exceeds $|\lambda_2|$. The ratios x_t/x_{t-1} converge to λ_1 whenever $|x_{t-1}/x_{t-2}| < \lambda_1$, and one gets local stability if $|\lambda_1| < 1$. The size of the region for which one gets instability grows as $|\lambda_2|$ goes down, i.e. when the coefficient a measuring the influence of expectations becomes large.

<u>Fig. 8.a</u>

<u>Fig. 8.b</u>

The foregoing analysis suggests strongly that learning is most likely to generate local instability in markets where expectations matter significantly. Another interesting feature is that although the world may be simple and close to linear (see (2.1) or (2.6)), the mental processes

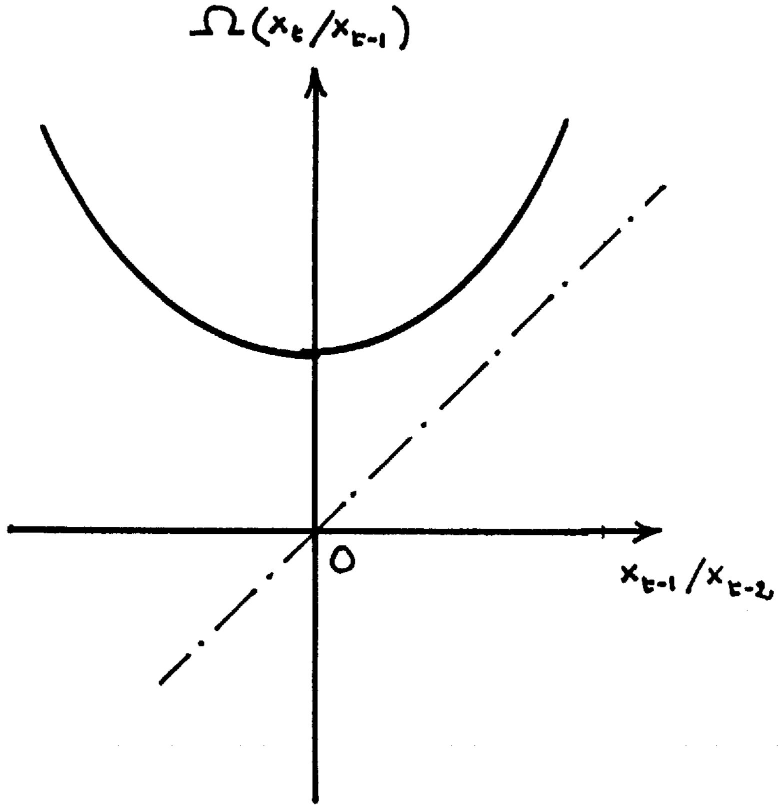


Fig.8.a

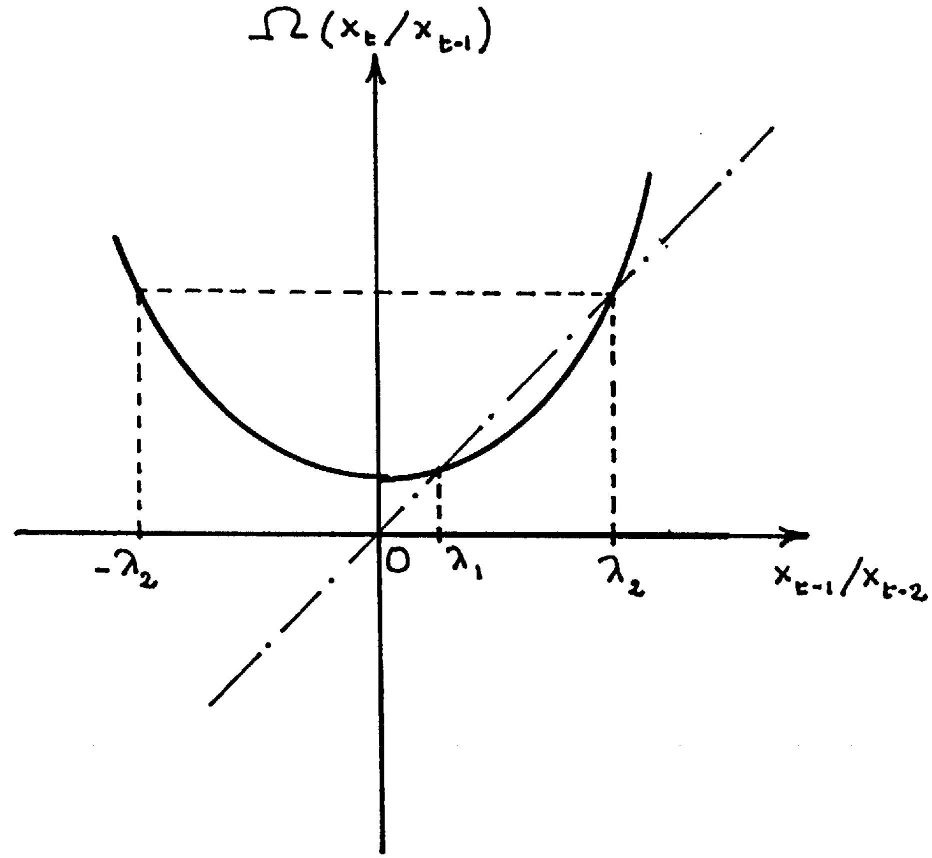


Fig.8.b

employed by economic agents when trying to learn the laws of motion of the system may be highly nonlinear. Then these nonlinearities will also show up in the actual observed dynamics : here, (2.10) involves the map Ω which describes a parabola. That feature suggests that the actual learning dynamics may generate non only local instability but also quite complex, even chaotic, nonexplosive expectations-driven fluctuations. The point is illustrated in Fig. 9, in the simple case where the estimate β_t is equal to x_{t-1}/x_{t-2}. There, the map Ω is expanding at both perfect foresight roots. The actual ratios x_t/x_{t-1} cannot converge to either of them but they may follow chaotic trajectories that are trapped in some invariant interval $[-c,c]$. [7]

Fig. 9

All this opens promising and largely unexplored avenues in business cycles theory. Although the "fundamentals" of the economic system may display only small nonlinearities and may not vary much over time, the traders' learning schemes are presumably highly nonlinear and this may lead to complicated expectations-driven nonexplosive fluctuations along which forecasting mistakes may never vanish, even in the "long run". This cannot, however, be the end of the story. For such a situation to be robust, of course, one should require some degree of consistency between the actual dynamics and private beliefs, so that traders have no incentive to change their views about how the world works. One might envision for instance a situation in which traders attribute their forecasting mistakes to "noise", although the observed dynamics are actually deterministic but chaotic, as in Fig. 9. As I said, this is largely unknown territory and should be the subject of further research.

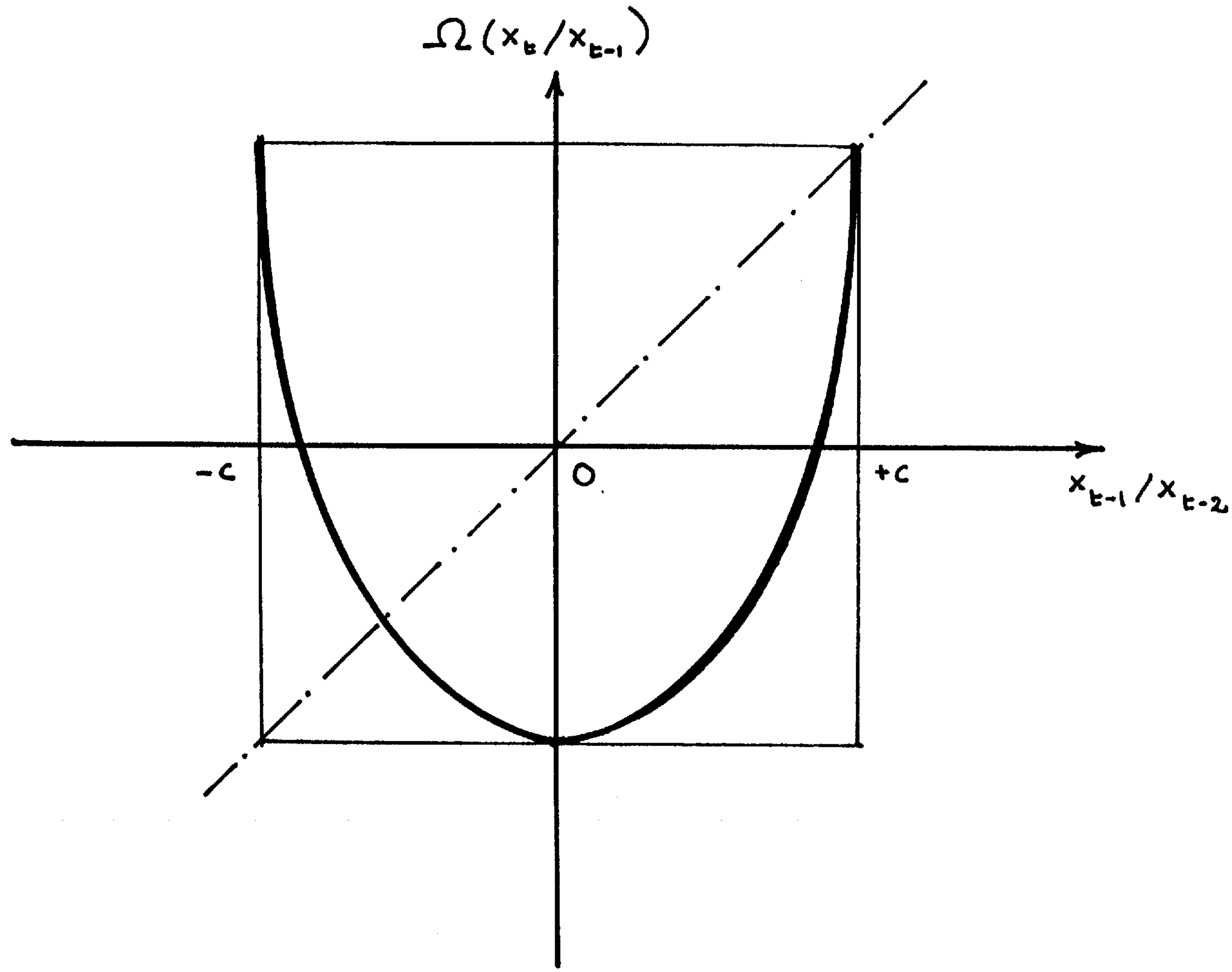


Fig. 9

REFERENCES

ANDERSON, P., K. ARROW and D. PINES (Eds.) (1988), *The economy as an evolving complex system*, vol. V, Santa Fé Institute Studies, Addison Wesley.

AZARIADIS, C. (1981), "Self-Fulfilling Prophecies", *Journal of Economic Theory 25*, 380-396.

AZARIADIS, C. and R. GUESNERIE (1986), "Sunspots and Cycles", *Review of Economic Studies 53*, 725-737.

BARNETT, W., J. GEWEKE and K. SHELL (Eds.) (1989), *Economic Complexity : Chaos, sunspots, bubbles and nonlinearity*, Cambridge University Press.

BARNETT, W., B. CORNET, C. d'ASPREMONT, J. GABSZEWICZ and A. MAS-COLELL (Eds.) (1991), *Equilibrium Theory and Applications*, Cambridge University Press.

BARRO, R.J. (1981), *Money, Expectations and Business Cycles*, Academic Press.

BAUMOL, W. and J. BENHABIB (1989), "Chaos : Significance, Mechanism and Economic Applications", *Journal of Economic Perspectives 3*, 77-105.

BENASSY, J.P. and M. BLAD (1989), "On Learning and Rational Expectations in an Overlapping Generations Model", *Journal of Economic Dynamics and Control 13*, 379-400.

BENHABIB, J. and R. DAY (1982), "A Characterization of Erratic Dynamics in the Overlapping Generations Model", *Journal of Economic Dynamics and Control 4*, 37-55.

BOLDRIN, M. (1988), "Persistent Oscillations and Chaos in Dynamic Economic Models : Notes for a Survey", in ANDERSON, P., K. ARROW and D. PINES (Eds.).

BOLDRIN, M. (1991), "Perfectly Competitive Models of Endogenous Business Fluctuations", *European Economic Review 35*, 300-305.

BOLDRIN, M. and M. WOODFORD (1990), "Equilibrium Models Displaying Endogenous Fluctuations and Chaos", *Journal of Monetary Economics 25*, 189-222.

BROCK, W. (1988), "Nonlinearity and Complex Dynamics in Economics and Finance", in ANDERSON, P., K. ARROW and D. PINES (Eds.).

BROCK, W. and W. DECHERT (1991), "Nonlinear Dynamical Systems : Instability and Chaos in Economics", in HILDENBRAND, W. and H. SONNENSCHEIN (Eds.).

CHAMPSAUR, P. (1983), "On the Stability of Rational Expectations Equilibria", CORE DP 8324, Catholic University of Louvain, Belgium. French version in *Cahiers du Séminaire d'Econométrie 26* (1984), 47-65.

CHIAPPORI, P.A. and R. GUESNERIE (1989), "On Stationary Sunspot Equilibria of Order k", in BARNETT, W., J. GEWEKE and K. SHELL (Eds.).

CHIAPPORI, P.A. and R. GUESNERIE (1991), "Sunspot Equilibria in Sequential Markets Models", in HILDENBRAND, W. and H. SONNENSCHEIN (Eds.).

FARMER, R. (1986), "Deficits and Cycles", *Journal of Economic Theory 40*, 77-88.

FARMER, R. and M. WOODFORD (1987), "Self-Fulfilling Prophecies and the Business Cycle", *Cuadernos Economicos de ICE.*

GRANDMONT, J.M. (1985a), "On Endogenous Competitive Business Cycles", *Econometrica 53*, 995-1045.

GRANDMONT, J.M. (1985b), "Cycles Concurrentiels Endogènes", *Cahiers du Séminaire d'Econométrie.*

GRANDMONT, J.M. (1986), "Stabilizing Competitive Business Cycles", *Journal of Economic Theory 40*, 57-76.

GRANDMONT, J.M. (1989), "Local Bifurcations and Stationary Sunspots", in BARNETT, W., J. GEWEKE and K. SHELL (Eds.).

GRANDMONT, J.M. (1990), "Expectations Formation and Stability of the Economic System", Presidential address, Econometric Society World Congress, Barcelona, forthcoming.

GRANDMONT, J.M. and G. LAROQUE (1991), "Economic Dynamics with Learning : Some Instability Examples", in BARNETT, W., B. CORNET, C. d'ASPREMONT, J. GABSZEWICZ and A. MAS-COLELL (Eds.).

GUESNERIE, R. (1986), "Stationary Sunspot Equilibria in an N Commodity World", *Journal of Economic Theory 40*, 103-127.

HILDENBRAND, W. and H. SONNENSCHEIN (Eds.) (1991), *Handbook of Mathematical Economics*, Vol. IV, North Holland.

HOMMES, C. (1991), "Adaptive Learning and Roads to Chaos", *Economic Letters 36*, 127-132.

HOMMES, C. (1992), *Adaptive Expectations and Nonlinear Cobweb Models : Stable Cycles, Chaos and Bifurcations*, Ph.D. Dissertation, University of Gröningen.

LUCAS, R. E., Jr. (1972), "Expectations and the Neutrality of Money", *Journal of Economic Theory 4*, 103-24.

MARCET and SARGENT (1989), "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models", *Journal of Economic Theory 48*, 337-368.

REICHLIN, P. (1986), "Equilibrium Cycles in an Overlapping Generations Economy with Production", *Journal of Economic Theory 40*, 89-102.

WOODFORD, M. (1986), "Stationary Sunspot Equilibria in a Finance Constrained Economy", *Journal of Economic Theory 40*, 128-137.

FOOTNOTES

* CNRS/CEPREMAP, 142, rue du Chevaleret, 75013 PARIS, and Yale University. The material of this paper is based on lectures given at the Rheinisch-Westfälische Akademie de Wissenchaften in Dusseldorf, in April 1991, and at a workshop on business cycles organized by the Trade Union for Economic Research in Stockholm (FIEF), in August 1991. The paper is being published by Westdeuscher Verlag for the German Academy, and in the proceedings of the Stockholm conference (FIEF studies on business cycles) by Oxford University Press.

[1] The material of this section is intended only to be a simple introduction to the subject. For more extensive (and often more technical) surveys, see Baumol and Benhabib (1989), Boldrin (1988, 1991), Boldrin and Woodford (1990), Brock (1988), Brock and Dechert (1991), Chiappori and Guesnerie (1991).

[2] Models of this type have been considered by Farmer (1986), Reichlin (1986), Woodford (1986). Here again one could assume that workers are in fact infinitely long lived but face liquidity (cash-in-advance) constraints stating that they cannot borrow against future labor income. One would get the same local results as in the analysis that follows (Woodford (1986)).

[3] The constraint that investment should be nonnegative was not explicitly taken into account in the analysis. Stationary investment however is positive, as it is given by $\bar{i} = \delta\bar{k} = a\delta\bar{y} > 0$. Investment will thus be automatically positive for small fluctuations around the steady state.

[4] Output y_{t+2} at date t+2 is only affected by shocks at t+1 since it is a "predetermined" variables, i.e. $y_{t+2} = k_{t+1}/a$.

[5] I would like to warn the reader that the foregoing statements concerning what happens to stochastic endogenous fluctuations along a Hopf bifurcation are in fact conjectures that appear most likely in view of what we know of the one dimensional case (Grandmont (1989)), but for which there are no complete rigorous proofs as yet.

[6] Expectations may also depend on past forecasts, so as to allow people to learn from their past mistakes by comparing past forecasts with actual realizations. The results would be qualitatively the same (Grandmont (1990)).

[7] Examples of cycles and chaos generated by learning have been provided for instance by Hommes (1991, 1992).