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By

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Abstract

We revisit the revenue comparison of standard auction formats, including first-price, second-price, and English auctions. We rank auctions according to their revenue guarantees, i.e., the greatest lower bound of revenue across all informational environments, where we hold fixed the distribution of bidders' values. We conclude that if we restrict attention to the symmetric affiliated models of Milgrom and Weber (1982) and monotonic pure-strategy equilibria, first-price, second-price, and English auctions all have the same revenue guarantee, which is equal to that of the first-price auction as characterized by Bergemann, Brooks, and Morris (2017a). If we consider all equilibria or if we allow more general models of information, then first-price auctions have a greater revenue guarantee than all other auctions considered.

KEYWORDS: Revenue guarantee, common values, affiliated values, revenue equivalence, revenue ranking, first-price auction, second-price auction, English auction.

JEL CLASSIFICATION: C72, D44, D82, D83.

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1 Introduction

In auction theory, the revenue equivalence theorem is a central result that helps us understand the relationship between the choice of auction format and the resulting revenue. In an environment with *independent private values*, it states the surprising result that many standard auction formats, including first-price, second-price, and English auctions, all deliver the same expected revenue (Myerson, 1981). By contrast, in an environment with *affiliated values*, there is a revenue ranking theorem that establishes that the first-price auction achieves less revenue than the second-price auction which in turn generates less revenue than the English auction (Milgrom and Weber, 1982, hereafter MW). Against this background, we provide a new revenue ranking theorem for environments with common or interdependent values, one that reverses the received ranking when values are affiliated. Subsequently, we also establish a new revenue equivalence theorem under favorable equilibrium selection.

For a given auction format, say the first-price auction, the resulting auction outcome is conventionally analyzed for a fixed distribution of the values of the bidders *and* for a fixed information structure that generates the signals that the bidder have before submitting their bids. Revenue and welfare in any given auction can be strongly affected by the specific form of information, e.g., Fang and Morris (2006) and Bergemann, Brooks, and Morris (2017a, hereafter BBM). This presents a challenge for comparative auction theory, since it may be difficult to pin down the correct model of information, either through introspection or measurement. Given such, we propose to rank auctions by a criterion that is less sensitive to misspecification of the informational environment. In particular, we evaluate an auction according to its *revenue guarantee*: the greatest lower bound on the auction's revenue that holds across all information structures. Importantly, this guarantee is computed while holding fixed the payoff environment—that is, the distribution over the bidders' values.

We establish a revenue guarantee ranking for the auctions studied in the affiliated value model of MW, namely the first-price auction, the second-price auction and the English auction. Our main results are exposited for the case of pure common values, with an extension to interdependent values in Section 5. A first step to obtain a revenue guarantee ranking is to establish the revenue guarantee of the first-price auction. Here we appeal to an earlier result in BBM that establishes that the lowest revenue in the first-price auction arises in an information structure that we refer to as a maximum signal model. This information structure which supports the lowest revenue is one in which the bidders receive identical and independent signals, and the value of the object equals the maximum of all the signals.

We then ask what is the equilibrium revenue in the maximum signal model of the second-price and the English auction. Here, we obtain the first surprising result. In this specific common value model, there is an equilibrium in which bidders behave as if each bidder's value is equal to their individual signal rather than the common value given by the maximum signal. Thus, the bidders act as if they are in an independent private value environment, and all but the bidder with highest signal bid as if their value is lower than their true value. Given this *strategic equivalence* with the independent private value model, we can appeal to the standard revenue equivalence result to conclude that all three auctions generate the same revenue in the maximum signal information structure. As a result, the revenue guarantee of the second-price auction or the English auction can be *at most equal* to the revenue guarantee of the first-price auction. Strategic equivalence, and hence the revenue guarantee ranking, can be extended to any "standard" auction that admits an equilibrium in monotonic pure-strategies in the independent private-value model.

Note that the revenue guarantee for the first-price auction in BBM is valid across all *equilibria*, as well as all information structures. But given that the second-price and English auction have other, less revenue-favorable equilibria, the revenue guarantee of the first-price auction must be strictly higher than those of second-price and English auctions. Thus, Theorem 1 reverses the revenue ranking established in MW.

At the same time, second-price and English auctions have compelling equilibria in monotonic pure strategies when the information structure admits a strong ordering on signals, e.g., when values are affiliated. We may ask, what is the revenue guarantee ranking if we restrict attention to symmetric affiliated values and monotonic equilibria? This approach is similar to the revenue equivalence theorem with symmetric independent private values which establishes the equivalence result in well-behaved informational environments and under favorable equilibrium selection.

Theorem 2 shows that in the affiliated common-value model, first-price, second-price and English auction are *revenue guarantee equivalent*. This result is established by showing that the critical maximum signal model is itself affiliated, so that the weak ranking of Theorem 1 is preserved. At the same time, it is a result of MW that the first-price auction generates *weakly lower* revenue than the other auction formats when restricting attention to favorable equilibria that excludes bidding ring like equilibria for the second-price and English auction. We therefore conclude that all of these mechanisms must have exactly the same revenue guarantee in affiliated environments and under monotonic equilibria. The maximum signal model thus has a remarkable property. If we take as a measure of the winner's curse the difference between the expected value of the object and the expected equilibrium revenue, then the maximum signal model maximizes the winner's curse uniformly across all three auction formats.[1]

Thus, Theorem 1 and Theorem 2 offer a different perspective of the revenue ranking result in the affiliated value model. If we are concerned with the robustness of the revenue comparison across all informational environments, we find that the English auction, and the second-price auction lose their advantage, as stated in the revenue guarantee equivalence theorem. Moreover, if we are at the same time concerned with the equilibrium selection, and seek to offer a revenue guarantee that is valid across all information structures and all equilibria then we find that the first-price auction offers better guarantees than either the second-price or the English auction.

In light of our results, a natural question is: what is the mechanism with the greatest revenue guarantee? This question is answered by Bergemann, Brooks, and Morris (2016) when there are two bidders and binary common values and by Brooks and Du (2018) for general common value models. The revenue-guarantee maximizing auctions look quite different from the standard auctions considered here, and necessarily involve randomized allocations to optimally hedge ambiguity about the information structure. We view these results as complementary to our revenue guarantee rankings: revenue guarantees are one of many criteria that could be used in selecting an auction format, and while the standard auctions considered here do not achieve optimal revenue guarantee, they have other desirable attributes aside from revenue guarantees.

Our analysis shares the interest in performance guarantees that is at the core of much recent work on auction theory in theoretical computer science, see e.g. Roughgarden et al. (2017). The majority of these results obtains guarantees through approximation algorithms. By contrast, the central revenue guarantee that emerged from the first-price auction here arises as an exact equilibrium of a critical information structure, namely the maximum signal model.

We establish Theorem 1 and 2 for common values with affiliated signals. Towards the end we discuss extensions of these results to more general settings. We consider interdependent rather than common values. We argue that the revenue guarantee ranking extends immediately to more general interdependent value environments. Thus, the earlier restriction to common values is done for simplicity of exposition rather than logical necessity. Extending revenue guarantee equivalence is more subtle, but there is a sharp sense in which this result

[1] Bulow and Klemperer (2002) were the first to study the maximum signal model in the context of second-price auctions. They showed that bidding one's signal is an equilibrium and that the resulting revenue is less than what the seller would obtain with a posted price. Bergemann, Brooks, and Morris (2017b) characterize the optimal auction in the maximum signal model. They show that the posted price is optimal when the good must be sold, but otherwise the optimal mechanism has a different form.

would also extend to more general environments. Finally, we give extensions to rankings of other mechanisms, including those with reserve prices.

2 Model

There are N bidders for a single unit of a good. The set of bidder indices is $\mathcal{N} = \{1, \ldots, N\}$. Bidders' values $(v_1, \ldots, v_N)$ are jointly distributed according to a measure $\pi(dv_1, \ldots, dv_N)$.

For our main results, we will consider environments where values are common. We will say that the environment is *common-values* if $v_1 = \cdots = v_N$ with probability one. In this case, we denote by $H(v)$ the cumulative distribution of the bidders' common value, and let $[\underline{v}, \overline{v}]$ denote the convex hull of its support. We assume that H is non-atomic.

An *information structure* consists of measurable sets of signals S_i, a joint probability measure $\mu(ds_1, \ldots, ds_N)$ on signal profiles in $S = S_1 \times \cdots \times S_N$, and a measurable interim expected value function

$$w : S \to \mathbb{R}^N,$$

where $w(s)$ is interpreted as the interim expectation of the value profile conditional on the signals. We say that w is *consistent with the prior* π (or simply *consistent*) if $v \sim \pi$ is a mean-preserving spread of $w(s)$ where $s \sim \mu$, meaning that there is a random variable ϵ that is correlated with s such that $\mathbb{E}[\epsilon|s] = 0$ and $w(s) + \epsilon$ is distributed according to π. A representative information structure is denoted $\mathcal{I}$.

An information structure is *symmetric* if $S_1 = \cdots = S_N$, π is exchangeable, and w is symmetric, in the sense that for all permutations $\xi : \mathcal{N} \to \mathcal{N}$, we have

$$w_{\xi(i)}\left(s_{\xi(1)}, \ldots, s_{\xi(N)}\right) = w_i(s_1, \ldots, s_N).$$

An information structure has *private values* if w_i is constant in s_{-i}. An information structure is *independent* if the s_i are independent random variables.

A *mechanism* consists of measurable sets of messages M_i for each player, $M = \times_{i=1}^N M_i$, allocations $q : M \to [0,1]^N$ with $\sum_{i=1}^N q_i(m) \leq 1$ for all m, and transfers to the seller $t : m \to \mathbb{R}_+^N$. A representative mechanism is denoted $\mathcal{M}$. A pair of an information structure $\mathcal{I}$ and mechanism $\mathcal{M}$ comprise a Bayesian game. A *Bayes Nash equilibrium* of that game is a profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_N)$, where $\sigma_i : S_i \to \Delta(M_i)$ and each player's strategy maximizes their ex ante welfare:

$$U_i(\sigma_i, \sigma_{-i}; \mathcal{M}, \mathcal{I}) = \int_{s\in S}\int_{m\in M} (w(s)\, q_i(m) - t_i(m))(\sigma_i, \sigma_{-i})(dm|s)\, \mu(ds).$$

A strategy profile induces revenue

$$R(\sigma; \mathcal{M}, \mathcal{I}) = \int_{s\in S}\int_{m\in M} \sum_{i=1}^{N} t_i(m)(\sigma_i, \sigma_{-i})(dm|s)\, \mu(ds).$$

R is a *revenue guarantee* of the mechanism $\mathcal{M}$ if for all $\mathcal{I}$ and for all Bayes Nash equilibria σ of $(\mathcal{M}, \mathcal{I})$, $R(\mathcal{M}, \mathcal{I}, \sigma) \geq R$. R is *the revenue guarantee of* $\mathcal{M}$ if it is a revenue guarantee, and if there is no higher guarantee.

3 Revenue Guarantee Ranking

We establish a revenue guarantee ranking across a number of classic auction formats, including the first-price, the second-price and the English auction. We begin the analysis by establishing a revenue guarantee for the first-price auction.

3.1 Revenue Guarantee of First-Price Auction

The determination of the revenue guarantee of the first price auction will use some insights and formalism established recently in Bergemann et al. (2017a). For a real-vector $x \in \mathbb{R}^{\mathrm{N}}$, we let $x^{(k)}$ denote the k-th highest element of the vector. Thus, $x^{(1)}$ is the first-order statistic, $x^{(2)}$ is the second-order statistic, etc.

The first-price auction $\mathcal{M}^{FPA}$ is defined as follows: $M_i = \mathbb{R}_+$,

$$q_i^{FPA}(m) = \begin{cases} \frac{1}{|\arg\max_j m_j|} & \text{if } i \in \arg\max_j m_j, \\ 0 & \text{otherwise;} \end{cases}$$

and

$$t_i^{FPA}(m) = q_i^{FPA}(m)\, m^{(1)}.$$

A specific information structure is given the maximum signal model. For a given distribution $H(v)$ of the common value v, the distribution $G(s_i)$ of the individual signal s_i is chosen to satisfy

$$G(x) = (H(x))^{1/N}.$$

Thus, we can interpret the common value v to be determined as the maximum of the N independent and identical signals:

$$v(s_1, \dots, s_N) = \max\{s_1, \dots, s_N\}. \quad (1)$$

Theorem 1 in BBM establishes that the revenue guarantee of the first-price auction is given by

$$R^{FPA} = \mathbb{E}_{(s_1,\dots,s_N)\overset{iid}{\sim}G}\left[s^{(2)}\right] \quad (2)$$

where $G(x) = (H(x))^{1/N}$. In other words, the revenue guarantee in the first price auction, R^{FPA} is the expected second-highest of N draws from the cumulative distribution G. This level of revenue is attained in a Bayes Nash equilibrium on the information structure in which bidders receive signals s_i that are independent draws from G, and $w(s) = s^{(1)}$, i.e., the maximum signal. We call this the *maximum signal* information structure, and denote it by $\mathcal{I}^*$. There is a monotonic pure-strategy equilibrium of the first-price auction on this information structure in which a type s_i bids

$$\beta^{FPA}(s_i) = \mathbb{E}_{s_{-i}\overset{iid}{\sim}G}\left[s_{-i}^{(1)} | s_{-i}^{(1)} \leq s_i\right]. \quad (3)$$

By this, we mean that the strategy $\sigma(\cdot|s_i)$ puts probability one on $\beta^{FPA}(s_i)$. We hereafter adopt this notation for pure strategies.

Proposition 1 (Bergemann, Brooks, and Morris, 2017a)**.**
The revenue guarantee of the first-price auction is R^{FPA}. Moreover, the strategies β^{FPA} are an equilibrium of $\left(\mathcal{M}^{FPA}, \mathcal{I}^*\right)$ and $R^{FPA} = R\left(\beta^{FPA}; \mathcal{M}^{FPA}, \mathcal{I}^*\right)$.

The first step in the proof of this result establishes that R^{FPA} is a lower bound on revenue of $\mathcal{M}^{FPA}$ in any equilibrium in any information structure. The second step shows that β^{FPA} is an equilibrium in which revenue is R^{FPA}, so that the lower bound is attained. It is this second step that is the most relevant for the new results in our paper.

The information structure $\mathcal{I}^*$ is strategically very similar to another information structure, which we denote by $\mathcal{I}^{IPV}$: as before signals are independent draws from G, but now

$$w_i(s) = s_i.$$

In other words, $\mathcal{I}^{IPV}$ is the *independent private values* information structure in which the individual values are distributed by G but highest value among the N values has the same distribution H as the value in maximum signal model. We note that in the independent private value model $\mathcal{I}^{IPV}$ derived from $\mathcal{I}^*$, all the bidders except for the bidder with the highest signal, have a strictly lower value for the object than in the corresponding common value model.

It is a standard result that there is a monotonic pure-strategy equilibrium of $\left(\mathcal{M}^{FPA}, \mathcal{I}^{IPV}\right)$ in which a bidder with value s_i bids the expected highest of the others' signals, conditional on others' signals being less than s_i, i.e., $\beta^{FPA}(s_i)$. We can use this to show that β^{FPA} is also an equilibrium of $\left(\mathcal{M}^{FPA}, \mathcal{I}^{*}\right)$. First, consider a deviation in which a type s_i bids $\beta^{FPA}(s_i')$ for some $s_i' \leq s_i$. Then the bidder only wins when the highest of the others' signals is less than s_i, in which case the highest signal, and hence the value, is just s_i. The deviator's surplus is therefore the same as what it would be in $\mathcal{I}^{IPV}$, which we know is less than or equal to the equilibrium surplus. On the other hand, by deviating to a higher bid, the deviator's surplus would be

$$\begin{aligned}\int_{x=\underline{v}}^{s_i'} \left(\max\{x, s_i\} - \beta^{FPA}(s_i')\right) d\left(G(x)^{N-1}\right) &= \int_{x=\underline{v}}^{s_i'} \left(\max\{x, s_i\} - x\right) d\left(G(x)^{N-1}\right) \\ &= \int_{x=\underline{v}}^{s_i} (s_i - x)\, d\left(G(x)^{N-1}\right),\end{aligned}$$

which is independent of s_i'. Finally, it is clear that bidding above $\beta^{FPA}(\overline{v})$ is not attractive.

From the revenue equivalence theorem (Myerson, 1981), we know that revenue in this equilibrium must be equal to that of the second-price auction, which is the expected second-highest value, thus giving us the formula (2).

3.2 Revenue Ranking

Our primary interest is to compare the revenue guarantee of the first-price auction to that of other mechanisms. We will refer to a mechanism as *standard* if (i) messages are one-dimensional bids and (ii) the high bidder is allocated the good, as in the first-price auction. We say a mechanism is *private-value efficient* if there is a monotonic pure-strategy equilibrium when values are symmetric, independent, and private. First-price auctions, second-price auctions, all-pay auctions, and the war-of-attrition are all examples of standard private-value efficient auctions.

Our first main result is a ranking of revenue guarantees of standard and private-value efficient mechanisms.

Theorem 1 (Revenue Guarantee Ranking).
If $\mathcal{M}$ is a standard and private-value-efficient mechanism, then R^{FPA} is greater than any revenue guarantee of $\mathcal{M}$.

To prove the theorem, we first establish the following result:

Proposition 2 (Strategic Equivalence)**.**
Suppose that $\mathcal{M}$ is a standard mechanism and β is a symmetric monotonic pure-strategy equilibrium of $\left(\mathcal{M},\mathcal{I}^{IPV}\right)$. Then β is also an equilibrium of $(\mathcal{M},\mathcal{I}^{*})$.

Proof of Proposition 2. When others use the strategy β, bidding $\beta(s_i)$ must result in the bidder winning when $s_i \geq \max_{j\neq i} s_j$ and making an interim payment $T(s_i)$. This "direct" allocation is precisely the one that is induced by the first-price auction. Moreover, from the revenue equivalence theorem, we know that the interim expected payment must be the same as that induced by the first-price auction, $T^{FPA}(s_i)$, up to a constant that depends on i but not on s_i:

$$T(s_i) = T^{FPA}(s_i) + c_i.$$

Thus, s_i profits from a deviation to $\beta(s_i)$ to $\beta(s_i')$ in the game $(\mathcal{M},\mathcal{I}^{*})$ if and only if s_i profits from a deviation from $\beta^{FPA}(s_i)$ to $\beta^{FPA}(s_i')$ in the game $\left(\mathcal{M}^{FPA},\mathcal{I}^{*}\right)$. Since the latter deviation is unprofitable, the former must be as well. Finally, it cannot be that there is any type that wants to deviate to a message that is not sent in equilibrium. The fact that there are no atoms implies that for any message, there is an equilibrium message which induces the same allocation. If any of the out-of-equilibrium messages were a profitable deviation, they would have to have a lower expected transfer than the equilibrium message, which contradicts the hypothesis that β is an equilibrium of $\left(\mathcal{M},\mathcal{I}^{IPV}\right)$. □

We now complete the proof of Theorem 1.

Proof of Theorem 1. To prove the result, we will simply exhibit an information structure and equilibrium in which revenue is equal to R^{FPA}. The information structure is $\mathcal{I}^{*}$. From the private-value efficiency hypothesis, we know that $\left(\mathcal{M},\mathcal{I}^{IPV}\right)$ must have an equilibrium in symmetric monotonic pure-strategies, which we denote by β. From Lemma 2, β is also an equilibrium of the game $(\mathcal{M},\mathcal{I}^{*})$. This implies the result, since the revenue-equivalence theorem implies that

$$R(\beta;\mathcal{M},\mathcal{I}^{*}) = R\left(\beta;\mathcal{M},\mathcal{I}^{IPV}\right) \leq R\left(\beta^{FPA};\mathcal{M}^{FPA},\mathcal{I}^{IPV}\right) = R^{FPA},$$

where the inequality follows from the optimality of the first-price auction with symmetric and independent private values. □

This theorem immediately demonstrates the maxmin optimality of the first-price auction among standard and private-value efficient mechanisms.

Corollary 1 (Optimality of the First-Price Auction).
The first-price auction maximizes the revenue guarantee among standard mechanisms that are private-value efficient.

In particular, the first-price auction has a greater revenue guarantee than second-price auctions, English auctions, all-pay auctions, the war of attrition, and all combinations of these mechanisms. While Theorem 1 and Corollary 1 only show a weak ranking, in the case of second-price and English auctions, the ranking is clearly strict, since these mechanisms have "bidding ring" equilibria in which one bidder makes a high bid and the others effectively do not participate in the auction.[2]

4 Revenue Guarantee Equivalence

The notion of a revenue guarantee in Section 2 requires that the revenue bound holds across all equilibria. We could therefore have quite easily concluded that the second-price and English auctions would have lower revenue guarantees than the first-price auction, without the use of Theorem 1, since the former mechanisms have "bidding ring" equilibria in which one bidder bids a large amount and the others bid zero. We might find the revenue ranking unappealing if it depended on the unfavorable selection of such equilibria, especially since the second-price and English auctions are known to have very appealing equilibria in well-behaved environments, such as the affiliated values setting studied by MW. Our next result shows that even if we restrict attention to affiliated values information structures and if we select the monotonic pure-strategy equilibrium, the first-price auction still performs weakly better than the second-price and English auctions. In fact, they all perform equally well.

We now proceed formally. An information structure has *affiliated signals* if (i) $S_i = \mathbb{R}$ for all i, (ii) π is absolutely continuous with respect to Lebesgue measure and has a density $f(s)$, and (iii) the density f is affiliated in the sense of MW, i.e., f is log supermodular. An *affiliated values information structure* is an information structure with affiliated signals and also satisfies (iv) $w_i(s)$ is weakly increasing in each coordinate.

A second-price auction $\mathcal{M}^{SPA}$ has an allocation rule $q^{SPA} = q^{FPA}$ that is the same as that of the first-price auction, but the payment is the second-highest bid, i.e.,

$$t_i^{SPA}(m) = q_i^{SPA}(m)\, m^{(2)}.$$

[2]In personal communication, Ziwei Wang has given an example of an information structure and equilibrium in which revenue from the all-pay auction is strictly lower than R^{FPA}.

As MW show, when values are affiliated, this mechanism has a monotonic pure-strategy equilibrium in which a type s_i bids

$$\beta^{SPA}(s_i) = \mathbb{E}_\mu\left[w(s') \,|s_i' = s_i, s_{-i}^{(1)} = s_i\right].$$

We say that R is an *affiliated values revenue guarantee for the second-price auction* if for any affiliated values information structure $\mathcal{I}$, $R\left(\beta^{SPA}; \mathcal{M}^{SPA}, \mathcal{I}\right) \geq R$. As before, R is *the* revenue guarantee if it is a revenue guarantee and it is greater than any other revenue guarantee.

The English auction $\mathcal{M}^{EA}$ has messages that are actually collections of mappings $m_i^I : \mathbb{R}_+^I \to \mathbb{R}_+$ for all $I \subseteq \mathcal{N} \setminus \{i\}$ that say, as a function of the drop-out prices of bidders in I, at which price bidder i should drop out of the auction. Our convention is that $m_i^\emptyset$ is a constant. A profile of messages induces an outcome wherein the first bidder to drop is the one with the smallest $m_i^\emptyset$, which is the price at which that bidder drops out, and the second bidder drops out at price $m_j^i\left(m_i^\emptyset\right)$, etc. The auction ends when only one bidder remains, and the remaining bidder gets the good (breaking ties equally if more than one bidder drops out simultaneously to end the auction), and pays the price at which the penultimate bidder dropped out. For a more formal description of the English auction, see MW. They show that there is an equilibrium of this game in which, conditional on the first K bidders dropping out at prices $y^{N-1} \leq \cdots \leq y^{N-1-K}$, a bidder with signal $s_i \geq y^{N-1-K}$ drops out at price

$$\beta^{EA}\left(s_i, y^{N-1}, \ldots, y^{N-1+K}\right) = \mathbb{E}_\mu\left[w(s') \,|s_i' = s_i, s_{-i}^{(k)} = y^k \;\forall k \geq N-1+K, s_{-i}^{(k)} = s_i \;\forall k \leq N-K\right].$$

We say that R is an *affiliated values revenue guarantee for the English auction* if for any affiliated values information structure $\mathcal{I}$, $R\left(\beta^{EA}; \mathcal{M}^{EA}, \mathcal{I}\right) \geq R$. The revenue guarantee is the best possible such guarantee.

Theorem 2 (Revenue Guarantee Equivalence)**.**
The second-price auction and the English auction have the same affiliated values revenue guarantee as the first-price auction given by R^{FPA}.

Proof of Theorem 2. The proof proceeds by two short steps.

Step 1: MW show that for any affiliated values information structure, there is an equilibrium of the first-price auction in which revenue is lower than both $R\left(\beta^{SPA}; \mathcal{M}^{SPA}, \mathcal{I}\right)$ and $R\left(\beta^{EA}; \mathcal{M}^{EA}, \mathcal{I}\right)$. This proves that revenue in the second-price and English auctions must be at least the revenue guarantee of the first-price auction, R^{FPA}.

Step 2: It is easy to verify that the information structure $\mathcal{I}^*$ is symmetric and has affiliated values. Moreover, the equilibria of the second-price and English auctions reduces to bidding your signal and dropping out when the price reaches your signal, respectively. Both of these equilibria induce revenue equal to the expected second-highest signal, which is R^{FPA}. □

In a sense, Theorem 2 shows that the information structure $\mathcal{I}^*$ has the strongest winner's curse of any affiliated values environment with the given distribution over the common value. By winner's curse, we mean the adverse selection from winning the good under a mechanism and equilibrium in which the high-signal bidder is allocated the good. It is well known that the presence of a winner's curse induces the bidders to shade their bids, so that they bid based on their pivotal value on the marginal event that they win. This updating is particularly severe in the maximum signal information structure $\mathcal{I}^*$. To wit, here learning that one has the highest signal means that the value is *exactly* equal to one's own signal, whereas at the moment when the bidder only knows his own signal, it is only a *lower bound* on the true value of the object. Thus, at interim stage, the signal of each bidder is the *greatest lower bound* for the value, and at the ex-post stage, the signal of the winning bidder is *least upper bound.*

If we measure the degree of adverse selection in terms of the difference between expected value and expected revenue, that difference is the largest under the monotonic equilibrium of $\mathcal{I}^*$, regardless of which of the standard auctions we use to measure the effect.

Thus, we find that when we restrict attention to well-behaved (symmetric and affiliated value) environments, second-price and English auctions do no better than the first-price auction in the worst case. At the same time, if we relax these hypotheses (symmetry, affiliated values, favorable equilibrium selection), the worst-case for the first-price auction must remain the same, while for these other mechanisms it can only decrease.

5 Extensions

5.1 Revenue Guarantee Rankings with Interdependent Values

The analysis of the first-price auction in BBM goes well beyond the common value case. In that paper, we characterize the revenue guarantee of the first-price auction as long as the joint distribution of values π is exchangeable, thus including interdependent as well as private values. We could similarly extend the robust revenue ranking of Theorem 1 to cover this more general environment with minimal conceptual innovation, although some additional notation is required.

In general, the critical worst-case information structure is defined as follows. For any realized vector of values $v \in \mathbb{R}^N$ among the N bidders, let

$$\alpha(v) = \frac{1}{N-1}\left(\sum_{i \in \mathcal{N}} v_i - v^{(1)}\right)$$

denote the average of the $N-1$ lowest values. Let H denote the distribution of $\alpha(v)$, where v is distributed according to π. We continue to maintain the hypothesis that H has no atoms. In the critical information structure, the bidders receive as before independent one-dimensional signals

$$s_i \sim G(x) = (H(x))^{1/N}.$$

The values can then be written in terms of the signals as follows:

$$w_i(s) = \begin{cases} s_i, & \text{if } s_i \notin H(s); \\ \mathbb{E}_\pi\left[v^{(1)} | \alpha(v) = s_i\right], & \text{otherwise.} \end{cases}$$

Thus, the highest signal is equal to the average of the $N-1$ lowest values, and the high-value bidder gets the highest signal. We continue to denote this information structure by $\mathcal{I}^*$.

The first-price auction continues to have an equilibrium on this information structure which is described by (3), which attains the generalized revenue guarantee, still given by (2) (although with the redefined G and H). This is shown in BBM. Moreover, by exactly the same steps as in the proof of Theorem 1, we could show that any standard and private-value efficient mechanism $\mathcal{M}$ must have an equilibrium with the same expected revenue, so that any revenue guarantee of $\mathcal{M}$ must be weakly less than R^{FPA}. The only step in the proof that changes is that when we evaluate a downward deviation, the deviator's value is even higher than it would be in the "as if" independent private value model $\mathcal{I}^{IPV}$. Thus, downward deviations are even less attractive than before. The argument for upward deviations is unchanged, and in fact bidders are indifferent to all upward deviations.

5.2 Revenue Guarantee Equivalence with Interdependent Values

Generalizing Theorem 2 is more subtle. The interdependent values version of $\mathcal{I}^*$ is not affiliated. To see why, consider a simple case in which there are two bidders and v_i are independent draws from the cumulative distribution F. In that case, under $\mathcal{I}^*$, the bidders receive independent signals, and the highest signal is equal to the smallest value:

$\max_{i=1,2} s_i = \min_{i=1,2} v_i$. For example, if we take F to be uniform on $[0,1]$, then

$$w_i(s) = \begin{cases} s_i, & \text{if } s_i < s_j; \\ \frac{1+s_i}{2}, & \text{otherwise.} \end{cases}$$

Thus, there is a downward jump in the value function $w_i(s)$ as a function of s_j when $s_j = s_i$.

This discontinuity means that there are multiple monotonic pure strategy equilibria. In particular, bidding s_i is an equilibrium, but so is bidding $(1+s_i)/2$. In either case, the equilibrium winner will be the bidder with the higher value, and the winner always pays a price less than their value, so that downward deviations are not attractive. On the other hand, increasing one's bid generally leads to a downward jump in the value on the marginal event when one wins, so upward deviations are not attractive either. Similarly, there are multiple monotonic pure-strategy equilibria of the first-price auction, with the one described by (3) being the lowest. In the uniform example, the lowest equilibrium of the FPA is to bid

$$\underline{\beta}(s_i) = \frac{1}{G(s_i)} \int_{x=0}^{s_i} x dG(x),$$

but it is easily verified that the following monotonic strategy is also an equilibrium:

$$\overline{\beta}(s_i) = \frac{1}{G(s_i)} \int_{x=0}^{s_i} \frac{1+x}{2} dG(x),$$

in which revenue is strictly higher. In effect, when there is a gap between the highest and second-highest values, there are different equilibria corresponding to different ways of "selecting" which value in the gap is treated as the value in the pivotal event where the bidders tie.

So, in order to generalize Theorem 2 beyond common values, we have to both expand the range of information structures that we consider, and also to decide which of the symmetric and monotonic pure-strategy equilibrium the bidders should play. This can be done as follows. We will say that an information structure is one of *generalized affiliated values* if the signals are affiliated and if the value function can be written as

$$w_i(s) = \hat{w}_i(s) + \mathbb{I}_{s_i \geq \max_{j \neq i} s_j} \tilde{w}_i(s)$$

where] $\hat{w}_i$ is monotonic and $\tilde{w}_i$ is non-negative. It is easily verified that when information is symmetric and generalized affiliated, there are monotonic pure-strategy equilibria of first-price, second-price, and English auctions, where bidders act "as if" the value function were $\hat{w}_i$. We refer to this as the *minimal monotonic equilibrium*. Moreover, the linkage principle of MW applies to these equilibria, so that revenue in this equilibrium is greater under the English auction than it is under the second-price auction than it is under the first-price auction. We define the *generalized affiliated values revenue guarantee* to be minimum revenue in the minimal monotonic equilibrium across all generalized affiliated values information structures with the given prior as a mean-preserving spread. The linkage principle implies that the generalized affiliated values revenue guarantee for the first-price auction is weakly lower than that of second-price and English auctions. Finally, $\mathcal{I}^*$ is a generalized affiliated values information structure, and in this information structure, the minimal monotonic equilibria of these auctions are all revenue equivalent. Thus, we conclude that first-price, second-price, and English auctions all have the same generalized affiliated values revenue guarantee.

5.3 Auctions with Reserve Prices

It is well-known that adding a minimum bid can raise revenue in private value environments. This occurs when there are bidder types that have relatively low gains from trade compared to their information rent. We can extend our results to the case where there is a reserve price. Specifically, consider the first-price auction with reserve r, denoted $\mathcal{M}^{FPA}(r)$:

$$q_i^{FPA}(m) = \begin{cases} \frac{1}{|W(m)|} & \text{if } i \in W(m); \\ 0 & \text{otherwise,} \end{cases}$$

where

$$W(m) = \left\{ i \in \mathcal{N} | m_i = m^{(1)}, m_i \geq r \right\}$$

is the set of high bidders whose bids exceed the reserve, and

$$t_i^{FPA}(m) = q_i^{FPA}(m)\, m^{(1)}.$$

Note that this is a different mechanism from the no-reserve first-price auction considered above, and hence it has a distinct revenue-minimizing information structure. BBM show that it has the following structure: Let x_i be independent draws from $G(x) = (H(x))^{1/N}$. Bidder i's signal is

$$s_i = \begin{cases} x_i & \text{if } x_i \geq \hat{v}; \\ r & \text{otherwise,} \end{cases}$$

where $\hat{v}$ solves

$$\int_{v=0}^{\hat{v}} vH(dv) = r.$$

The value function is again simply

$$w(s) = s^{(1)}.$$

We denote this information structure by $\mathcal{I}^*(r)$. We can think of this information structure as being derived from $\mathcal{I}^*$, where signals below $\hat{v}$ are pooled together as a single signal r. The cutoff $\hat{v}$ is chosen so that the expected value is r conditional on the highest signal being r.

As before, there is a monotonic pure-strategy equilibrium of the first-price auction:

$$\beta^{FPA}(s;r) = \begin{cases} 0 & \text{if } s_i = r; \\ \mathbb{E}_{s_{-i}}\left[\max\left\{r, s_{-i}^{(1)}\right\} | s_{-i}^{(1)} \leq s_i\right] & \text{otherwise.} \end{cases}$$

BBM show that this information structure and equilibrium achieve the revenue guarantee of $\mathcal{M}^{FPA}(r)$, which is

$$R^{FPA}(r) = \mathbb{E}_{(x_1,\ldots,x_N)\overset{iid}{\sim}G}\left[\max\left\{r, x^{(2)}\right\}\mathbb{I}_{x^{(1)}\geq\hat{v}}\right].$$

Again, there is a strategic equivalence result that says that the same strategies would be an equilibrium even if bidders treated their signals as private values. Let us denote this information structure by $\mathcal{I}^{IPV}(r)$. Moreover, the revenue equivalence theorem says that revenue on $\mathcal{I}^{IPV}(r)$ is the same as what would obtain with a second-price auction with a reserve price of r, thus yielding the formula for $R^{FPA}(r)$.

We could extend Theorem 1 to reserve price auctions as follows. Suppose there is another mechanism $\mathcal{M}$, that results in the same allocation in $\mathcal{I}^{IPV}(r)$. This could be a second-price or English auction with reserve r, or it could be an all-pay auction, albeit with a different reserve price. In order to have an apples-to-apples comparison, we hold the screening level fixed, so that the allocation is conditionally efficient when the highest value is greater than r, but the seller keeps the good when the highest value is weakly less than r. The fact that these mechanisms are revenue equivalent to the first-price auction on $\mathcal{I}^{IPV}(r)$, and the strategic equivalence of the induced direct mechanism between $\mathcal{I}^{IPV}(r)$ and $\mathcal{I}^*(r)$, means that there is an equilibrium and information structure in which $\mathcal{M}$ has revenue equal to $R^{FPA}(r)$, so the revenue guarantee for $\mathcal{M}$ is weakly below $R^{FPA}(r)$.

Theorem 2 can be extended as well. The type space $\mathcal{I}^*(r)$ is still one of affiliated values, and the revenue ranking of MW in affiliated environments extends to first-price, second-price, and English auctions with a common reserve price (*ibid*, Section 7, pp. 1111-1113). Moreover, the equilibrium $\beta^{FPA}(\cdot;r)$ coincides with the one described by MW. Thus, we conclude that first-price, second-price, and English auctions with reserve price r are revenue guarantee equivalent, with a guarantee of $R^{FPA}(r)$.

5.4 Revenue Guarantee Equivalence with Other Mechanisms

Theorem 1 is quite general and covers all standard and private-value-efficient mechanisms. Theorem 2, on the other hand, is specific to first-price, second-price, and English auctions. Theorem 2 would extend to cover a mechanism $\mathcal{M}$ if that mechanism generates weakly more revenue than the first-price auction when values are symmetric and affiliated. While MW prove this revenue ranking for second-price and English auctions, their proof technique can be adapted to cover more mechanisms. In particular, the critical feature of the second-price auction that yields the revenue ranking is that (i) only the winner pays and (ii) the winner's payment is increasing in other bidder's reports (*ibid*, Theorem 15, p. 1109). Any mechanism that satisfies the same conditions and has an equilibrium in monotonic pure-strategies must generate weakly more revenue than the first-price auction. Thus, for example, Theorem 2 would extend to cover convex combinations of first-price and second-price auctions, where the winner pays a weighted average of the highest and second-highest bids, provided a monotonic equilibrium exists. Lizzeri and Persico (2000) proved existence of a monotonic equilibrium when there are two bidders.

In addition, we expect that the characterization of affiliated revenue guarantees can be extended beyond monotonic winning payments. Krishna and Morgan (1997) give conditions under which the all-pay auction and the war-of-attrition always generate more revenue than the first-price auction. Theorem 2 will extend to these mechanisms as well, as long as $\mathcal{I}^*$ satisfies their additional conditions, which boils down to a hazard rate condition on the distribution G. The takeaway is that Theorem 2 can extend well-beyond second-price and English auctions.

5.5 Releasing More Information

MW famously gave conditions under which releasing public information about the value will raise revenue from first-price, second-price, and English auctions. Analogous results hold for revenue guarantees. First, suppose the seller has access to a signal that can be publicly revealed to the bidders. We claim that for any mechanism, revealing the signal must raise the revenue guarantee. Why? Since all information structures are allowed, it is always possible that the bidders' already have access to this signal. Revealing the signal may, however, rule out some information structures, e.g., no information, so that the revenue guarantee will weakly increase. At the same time, our revenue guarantee ranking will continue to hold ex post for each realized public signal, so that the ranking continues to hold ex ante as well.

Similarly, revenue guarantee equivalence would continue to hold if the seller releases public information that is affiliated with the value, as long as we restrict attention to information structures that are jointly affiliated with the value and the public signal. Again, conditional on the public signal, the information structure is still affiliated. At the same time, it is possible that conditional on the public signal, the bidders' get independent signals and the maximum signal is equal to the value. Thus, revenue guarantee equivalence will hold ex post conditional on each realization of the public signal.

The bottom line is that releasing public information always helps the seller, but it is also preserves the dominance of the first-price auction in terms of revenue guarantees.

6 Conclusion

We presented a novel version of the revenue equivalence and revenue ranking theorems. We compared the auction format in terms of a revenue guarantee across all information environments rather than in terms of the revenue from a specific information environment. The revenue guarantee identified the greatest lower bound across all information structures (and all equilibria). This analysis yields a powerful new argument in favor of first-price auctions as achieving a greater revenue guarantee than other standard mechanisms, such as second-price and English auctions.

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