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NECESSARY AND SUFFICIENT CONDITIONS FOR DETERMINACY OF ASYMPTOTICALLY STATIONARY EQUILIBRIA IN OLG MODELS

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Abstract. We propose a criterion for determining whether a local policy analysis can be made in a given equilibrium in an overlapping generations model. The criterion can be applied to models with infinite past and future as well as those with a truncated past. The equilibrium is not necessarily a steady state; for example, demographic and type composition of the population or individuals' endowments can change over time. However, asymptotically, the equilibrium should be stationary. The two limiting stationary paths at either end of the time line do not have to be the same. If they are, conditions for local uniqueness are far more stringent for an economy with a truncated past as compared to its counterpart with an infinite past.

In addition, we illustrate our main result using a text-book model with a single physical good, a two-period life-cycle and a single type of consumer. In this model we show how to calculate a response to a policy change using the implicit function theorem.

J.E.L. codes: C02, C62, D50.

1. Introduction

1.1. **Our basic question.** A researcher constructs an overlapping generations (OLG) model and solves for an equilibrium. Her next task is to perform a policy analysis using the implicit function theorem, e.g., to find out how a pension reform will change the allocation of resources across generations. Before undertaking the assessment she has to verify that the equilibrium is determinate, for otherwise the reaction of equilibrium to such a change can be either non-existent or ambiguous. According to Shannon (2008), an equilibrium is called *determinate* if it is "locally unique and local comparative statics can be precisely described." If indeterminate, an equilibrium might be "infinitely sensitive to arbitrarily small changes in parameters." [*ibid.*] Examples of such parameters include policies or regulations. In case of an OLG model, one can analyse policy changes that are effective over a period of time and that target particular groups of individuals. In other words, a policy can be a function of time and individual characteristics. Hence, if the equilibrium in an OLG model is determinate, one can conduct a policy evaluation, examining

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the full effect of such policies on all individuals born at any time. Is there an easy way to assess determinacy of equilibria?

The foundations for the analysis of competitive equilibria in finite economies are well-known [5]. If the model has a finite number of goods and individuals, policy analysis is almost always valid. Even if the original equilibrium fails to be determinate, a small perturbation of parameters around their original values can restore determinacy. These results have been extended, but they still require either a finite number of agents, [25], or a compact set of traded commodities [4]. Overlapping generations models, pioneered by Samuelson (1958), do not fall in either class: both sets, of agents and of dated goods, are infinite and neither of them is compact. Moreover, familiar cases of the OLG models are associated with generic indeterminacy, cf. [10] for the overview. Unfortunately, both the concept of genericity and that of indeterminacy have been interpreted in a variety of ways in the literature, especially when an equilibrium variable is a solution to a dynamic system.

For practical purposes though, a researcher needs to know whether policy evaluation is valid in a given equilibrium for a given model. This is the question we address here.

1.2. **Answering the question.** To provide a direct answer, we develop a criterion for equilibrium determinacy. In order to make the analysis more transparent, we consider a discrete-time endowment model, cf. section 2. There are $L \geq 1$ physical goods traded in every period by heterogenous consumers with different life-spans. Equilibrium price path, as usual, is a solution for the system of market clearing equations, or a zero of the aggregate excess demand. We call an equilibrium stationary if the derivative of the aggregate demand evaluated at this equilibrium is time-invariant, meaning that any of its partial derivatives at time t with respect to price at time k depends only on the difference, $t-k$. A steady state in a stationary economy satisfies this condition. We require the baseline equilibrium to be stationary only asymptotically, i.e., when $t \to +\infty$ in the truncated case and $t \to \pm\infty$ for the economy with infinite past.

We view an equilibrium price (growth path) as a point in the Banach space of uniformly bounded sequences and apply the implicit function theorem (IFT) for that space, cf. also [3], rather than linearizing equilibrium difference equations using an IFT for every time period $t \in \mathbb{Z}(\mathbb{Z}_+)$, as in, e.g., [14].

It is the applicability of the IFT that gives us the indicator of determinacy of equilibria. To assess invertibility of the derivative of equilibrium equation system with respect to prices (evaluated at an equilibrium point), we build upon contributions in spectral analysis, [15], [11]. Our main statement is in section 3 and its proof is in the appendix.

1.3. **A related puzzle in the literature.** As a by-product of our analysis, we address a discrepancy in the OLG literature about indeterminacy of equilibria.

There are several known cases of generic indeterminacy. One includes steady states in a stationary endowment economy with several physical goods and discrete time starting at zero [14], [13]. Indeterminacy in that case is understood as multiplicity of equilibria converging to some steady state as a result of a small policy change at time zero. These results still hold in an OLG economy with some production possibilities [21]. Using a similar concept of indeterminacy and a more general notion of stationarity (in the sense of Liapunov) for the base-line equilibrium, Geanakoplos and Brown (1985) show that when the model has both infinite past and future and the agents perfectly foresee the future policy change, equilibria are generically determinate, which is not the case when the change comes as a surprise at time zero. Demichelis and Polemarchakis (2007) show that in an economy with logarithmic instantaneous utility and a single physical good the equilibrium set is substantially reduced when the time runs from $-\infty$ instead of 0. This also holds when transactions become more frequent, as the discrete model converges to its limit in continuous time. In the limit, the steady states are isolated and increasing equilibria are unique up to a time-shift.

Generic determinacy, understood as applicability of the IFT in the appropriate Banach spaces, can be acheived if the time-line is extended to include an infinite past. Burke (1990) demonstrates this for the nearly stationary equilibria in the economy described in [14]. In a model with Cobb-Douglas production, constant-elastiticy-of-substitution instantaneous utility and arbitrary life-cycle productivity, balanced growth equilibria (BGE) are determinate for almost all parameters [20], if time is a whole real line. In an extended version of this model, BGE are generically determinate in the space of real transfers [12].

From this brief sketch of the literature it is already clear that neither switching from discrete to continuous time nor introducing production possibilities in the model is sufficient to avoid generic indeterminacy. It may be that the culprit is "the beginning of times," or truncation of the past used in the standard OLG models. This was our initial conjecture.

1.4. **Resolving the puzzle.** We find that conditions needed to validate comparative statics in a model with truncated past are, indeed, more restrictive than those for the model with infinite past, but only in the case in which the two asymptotic conditions coincide, e.g., when the baseline equilibrium converges at $t \to \pm\infty$ to the same stationary price growth path and the parameters tend to the same stationary values towards both ends of the time-line. However, such a conclusion does not extend to *asymmetric* baseline equilibria that converge to two different stationary paths, one for each direction in time. In this case, determinacy imposes non-trivial restrictions on parameters, which might be extreme, as we later illustrate.

In section 4 we demonstrate how to calculate equilibrium response to a policy change for a textbook example of an OLG economy with Cobb-Douglas life-time utility using the complete characterization of equilibria offered in [6]. If the economy is eternal and the policy is fully foreseen by the agents, steady state equilibria are determinate, apart from a knife-edge case, in which the amount of money needed to sustain the golden rule equilibrium is zero. In other cases, we illustrate the response of the economy to a small addition of consumption goods to the young in a single period. If the young save in the golden rule equilibrium, the response of prices precedes the change in policy, so perfect foresight is crucial there. In contrast, if the beginning of times is set, or if the policy is perceived as a surprise by the agents, the golden rule equilibrium is indeterminate in case of positive savings by the young. This supports our initial conjecture and illustrates the implications of the main result for symmetric equilibria. However, infinite past is not a perfect remedy against indeterminacy. Increasing equilibria that converge to two different steady states at either end of the time-line are not determinate for *any* choice of parameters in that example, as follows from a simple calculation based on the main result.

Some caveats and applicability of our results to other dynamic systems are discussed in section 5.

2. The model

2.1. **Economy with an infinite past, $\mathscr{E}$.** Time is indexed by the set of integers, $t \in \mathbb{Z}$. An individual of type $\theta \in \Theta$ lives for $1 < \tau^\theta < \infty$ periods, where Θ is a finite set of individual types. Individuals of different types can be born at any time, and their lifespans can overlap. Let $0 \leq N^{\theta}_{t,s} < \infty$ be the number of people of type θ who are of age $s \in \{0, \dots, \tau^{\theta} - 1\}$ at time $t \in \mathbb{Z}$. We assume that at any point in time t the economy is non-empty.

Assumption 1. For any $t \in \mathbb{Z}$, there is at least one type $\theta \in \Theta$, such that $N^{\theta}_{t,s} > 0$ for some $s \in \{0, \dots, \tau^{\theta} - 1\}$.

There are $L \geq 1$ perishable physical goods traded at any period $t \in \mathbb{Z}$. An individual born in period $x \in \mathbb{Z}$ is entitled to a stream of consumption goods $(\omega^{\theta}_{x+s,s} \in \mathbb{R}^L_+)_{s \in \{0,\dots,\tau^{\theta}-1\}}$ during his life-time, and has nothing beyond: $\omega^{\theta}_{x+s,s} = 0$ for all $s < 0$ and $s > \tau^{\theta} - 1$. Resources are limited, both over the life-span of an individual and at any point in time for the economy as a whole. In addition, aggregate resources are bounded away from zero.

Assumption 2. Individual resources: $0 < \sum_{s \in \mathbb{Z}} \omega^{\theta}_{x+s,s} < \infty$, $\forall x \in \mathbb{Z}, \forall \theta \in \Theta$.
Aggregate resources: $0 < \underline{\omega} \leq \sum_{\theta \in \Theta} \sum_{s \in \mathbb{Z}} N^{\theta}_{t,s} \omega^{\theta}_{t,s} < \infty$, $\forall t \in \mathbb{Z}$.

Budget set can be defined in the usual way to include all the consumption streams that can be purchased by selling the endowment at prevailing market prices, $p_t \in \mathbb{R}^L_{++}$. We work with Arrow-Debreu prices, i.e., $p_{t,l}$ is the price of a promise to deliver good l at time t in terms of money of some fixed period, say, 0. Implicit is the assumption that individuals have access to a storage of value across periods, cf. [22], even though the goods are perishable.

Assume that individuals do not derive utility from consuming goods beyond their life-span, as is common in the OLG models. Then, as is well known, [10], [9], individual demand will depend on the prices during one's life-time only.

Let $p^{x,\tau} = (p_x, p_{x+1}, \dots, p_{x+\tau-1}) \in (\mathbb{R}^L_+)^{\tau}$ be the sequence of prices of L goods for τ periods following and including period x. Assume that the solution to the consumer's problem is unique and denote demand of an individual of type $\theta \in \Theta$, who is of age $s \in \{0, \dots, \tau^{\theta} - 1\}$ at time $t \in \mathbb{Z}$ by $c^{\theta}_{t,s}$. It is a function of strictly positive prices during his life-time:

$$c^{\theta}_{t,s} \colon (\mathbb{R}^L_{++})^{\tau^{\theta}} \to (\mathbb{R}^L_+)^{\tau^{\theta}},\ p^{t-s,\tau^{\theta}} \mapsto c^{\theta}_{t,s}(p^{t-s,\tau^{\theta}}) \tag{1}$$

Note that the demand is also a function of parameters, which is implicit in the notation we adopted. First, it depends on income, or the value of the endowment to which he is entitled over the course of his life-time: $\sum_{k=0}^{\tau^{\theta}-1} p_{x+k} \omega^{\theta}_{x+k,k}$. The changes in endowments could be associated with changes in an external policy. Second, individual demand depends on preferences. Both factors can vary $c^{\theta}_{t,s}$ and its dependence on prices.

The rest of the assumptions will be imposed directly on individual demand, cf., e.g. [13]. They follow from the standard set of assumptions on preferences and consumer optimization.[1]

Assumption 3. Homogeneity: $c^{\theta}_{t,s}(p^{t-s,\tau^{\theta}}) = c^{\theta}_{t,s}(\zeta p^{t-s,\tau^{\theta}})$ $\forall \zeta > 0$, $\forall t \in \mathbb{Z}$, $\forall \theta \in \Theta$, $\forall s \in \{0, \dots, \tau^{\theta} - 1\}$.
Differentiability: $c^{\theta}_{t,s}(p^{t-s,\tau^{\theta}})$ is continuously differentiable on $(\mathbb{R}^L_{++})^{\tau^{\theta}}$.

[1] Since there are no infinitely-lived agents in this economy, it is without loss of generality to assume that individual preferences are defined over a finite-dimensional subspace of consumer bundles, a subset of $\mathbb{R}^{\tau^{\theta} L}_+$ for any given type $\theta \in \Theta$: the demand should depend on a finite collection of prices during an individual's life-time in either case. Thus, we avoid the known difficulties of imposing too many implicit restrictions on the choice of the underlying space for prices and quantities by assuming that the demand is smooth, [1], [18].

It is common to impose more structure on demand, such as Walras law, for example, $\sum_{k=0}^{\tau^\theta-1} p_{x+k}\big(c^\theta_{x+k,k}(p^{x,\tau^\theta}) - \omega^\theta_{x+k,k}\big) = 0$, $\forall x \in \mathbb{Z}$, $\forall \theta \in \Theta$, as well as an intertemporal irreducibility assumption in order to be able to rely on the existence of equilibria results in [2], for example. However, even that might not suffice in our case since we focus on a subclass of equilibria described below. This includes one of the most widely used classes of equilibria, the steady states, cf., e.g., [10]. For non-stationary equilibria, a direct computation might be feasible, cf. [6]. We establish existence of equilibria in the neighbourhood of a baseline (provided it exists) using the implicit function theorem.

By homogeneity, demand of any individual born at x will not change if we normalize the prices faced by the individual over his life-time (p^{x,τ^θ}). For that, we divide every element of the price vector $p^{x,\tau^\theta} \in (\mathbb{R}^L_{++})^{\tau^\theta}$ by the price of the first physical good that he encounters when born, $p_{x,1} > 0$. This is equivalent to re-defining individual demand $c^{\tau^\theta}_{t,s}$ to be a function of relative prices, $(q_{t-s}, \dots, q_{t-s+\tau^\theta-1})$, where $q_k \in \mathbb{R}^L_{++}$ for $k = t-s, \dots, t-s+\tau^\theta-1$,[2] and

$$q_{t,l} \overset{\text{def}}{=} \begin{cases} \frac{p_{t,l}}{p_{t,l-1}} & , \text{ if } l > 1 \\ \frac{p_{t,1}}{p_{t-1,L}} & , \text{ if } l = 1 \end{cases}$$

Following the earlier convention, let $q^{x,\tau} \overset{\text{def}}{=} (q_x, q_{x+1}, \dots, q_{x+\tau-1}) \in (\mathbb{R}^L_+)^\tau$ be the sequence of relative prices of L goods for τ periods following and including period x.

2.2. **Equilibria.** Denote the excess demand of individual of type θ who is of age s at time t by $\xi^\theta_{t,s}$ defined on $(\mathbb{R}^L_{++})^{\tau^\theta}$:

$$\xi^\theta_{t,s}(q^{t-s,\tau^\theta}) \overset{\text{def}}{=} c^\theta_{t,s}(q^{t-s,\tau^\theta}) - \omega^\theta_{t,s}$$

Definition 1. An equilibrium price $q = (\dots, q_0, q_1, \dots, q_t, \dots)$ is a strictly positive solution of the equilibrium system of equations:

$$F_t(q) = \sum_{\theta\in\Theta} \sum_{s=0}^{\tau^\theta-1} N^\theta_{t,s} \xi^\theta_{t,s}(q^{t-s,\tau^\theta}) = 0, \ \forall t \in \mathbb{Z} \tag{2}$$

We exclude diverging sequences of relative prices by assuming that the equilibrium price is uniformly bounded over time,[3] $q \in \ell^L_\infty(\mathbb{Z})$. The set solutions to the system (2) and our ability to perform comparative statics depends on the properties of the derivative of F, which maps the set of infinite bounded sequences indexed by integers into itself, $DF\colon \ell^L_\infty(\mathbb{Z}) \to \ell^L_\infty(\mathbb{Z})$.

$$DF_t(q) = \sum_{\theta\in\Theta} \sum_{s=0}^{\tau^\theta-1} N^\theta_{t,s} \sum_{j=0}^{\tau^\theta-1} D_{j+1}\xi^\theta_{t,s}(q_{t-s}, \dots, q_{t-s+j}, \dots, q_{t-s+\tau^\theta-1}), \tag{3}$$

where $D_j\xi^\theta_{t,s}(\cdot)$ is the derivative with respect to j-th argument of $\xi^\theta_{t,s}$.

It will be convenient to assess the derivative of the aggregate demand with respect to each of the prices q_k for $k \in \mathbb{Z}$. For that we change variable in (3): $k = t-s+j$,

[2] By construction, the derivative of each one of the L components of this demand vector is zero with respect to the first relative price, $q_{t-s,1}$.

[3] This is the reason we chose to work with ratios of prices, i.e., to normalize prices as above: the rate of growth of prices is typically bounded by a combination of preference parameters and a bound on resources, at least in the presence of production, cf. [19], while a uniform bound on nominal values is hard to justify. Here the lower bound on aggregate resources ($\underline{\omega}$) should be sufficient, using the standard argument, to assure that prices remain bounded. The downside of this normalization is that we can only work with strictly positive prices.

so that $t - s \leq k \leq t - s + \tau^\theta - 1$ and then switch the order of the summation which can be done since all the sums are finite. Then using $[v]_+ = \max\{0, v\}$ the derivative can be written as follows:

$$(4) \quad (DF_t(q))(\delta q) = \sum_{k=t-\bar\tau+1}^{t+\bar\tau-1} \gamma_{t,k}\delta q_k, \text{ where}$$

$$(5) \quad \gamma_{t,k} \overset{\text{def}}{=} \begin{cases} \sum_{\theta\in\Theta}\sum_{s=[t-k]_+}^{\tau^\theta-1-[k-t]_+} N^\theta_{t,s} D_{k-t+s+1}\xi^\theta_{t,s}(q^{t-s,\tau^\theta}), & \text{if } |k-t| \leq \bar\tau - 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that each $\gamma_{t,k} \in \mathbb{R}^{L\times L}$ is an $L \times L$ matrix of real numbers. This collection of matrices reflects the parameters defining the economy and the policies, cf. equation (5). It involves derivatives of individual demand, hence individual preferences, endowments as well as policies that translate into aggregate changes through demographic and type composition of the population, $N^\theta_{t,s}$. In case $N^\theta_{t,s}$ and the derivatives of $\xi^\theta_{t,s}$ evaluated at some equilibrium are independent of t, the economy and the equilibrium are referred to as stationary (steady states).

Steady states in stationary economies have a convenient property that the coefficients $\gamma_{t,k}$ depend only on the difference, $t - k$, so, DF can be viewed as an "infinite matrix" with equal entries along the "diagonals", i.e., equal if the difference between the column and the row index is the same. In other words, DF is given by a convolution, cf. [12]. Our analysis incorporates also non-stationary equilibria that have this property asymptotically.

Definition 2. An equilibrium price q in economy $\mathscr{E}$ is *asymptotically stationary* (ASE) if there is a collection of matrices $\gamma^+_s, \gamma^-_s \in \mathbb{R}^{L\times L}$ such that

$$(6) \qquad \gamma_{t,t-s} \xrightarrow[t\to+\infty]{} \gamma^+_s, \ \gamma_{t,t-s} \xrightarrow[t\to-\infty]{} \gamma^-_s \quad \forall s \in \mathbb{Z}$$

where $\gamma_{t,k}$ are as defined in (5).

If, in addition, $\gamma^+_s = \gamma^-_s$ for all $s \in \mathbb{Z}$ then the equilibrium is *symmetric*.

A symmetric equilibrium is *stationary* if $\gamma_{t,t-s} = \gamma^+_s$ for all $t, s \in \mathbb{Z}$.

Note that $\gamma_{t,t-s} = 0 = \gamma^\pm_s$ for $|s| > \bar\tau$ and for all $t \in \mathbb{Z}$ by equation (5). Further, there is no restriction on the rate of convergence. Note also that the two asymptotes, γ^+_s, γ^-_s are not necessarily the same if the equilibrium is not symmetric.

2.3. **Economy with truncated past $\mathscr{E}_+$ and its equilibria.** One can either assume that the time "starts" at some fixed $t = 0$ or that the policy change is a surprise (announced in period 0), so none of the equilibrium variables could react before $t = 0$. Hence, the time here is indexed by positive integers, $\mathbb{Z}_+$, and not by $\mathbb{Z}$, as in $\mathscr{E}$. The prices are normalized in the same way, only the price of the first good in the initial period is set to unity, $q_{0,1} = 1$. Otherwise, we impose the same assumptions on $\mathscr{E}_+$ as we did on $\mathscr{E}$.

The only difference in the definition of the derivative of the equilibrium equations is the lower bound on summation assuring that $\delta q_k = 0$ for all $k < 0$, while $\gamma_{t,k}$ are defined as before, in equation (5).

$$(7) \qquad DF^+ : \ell^L_\infty(\mathbb{Z}_+) \to \ell^L_\infty(\mathbb{Z}_+)$$

$$(8) \qquad (DF^+_t(q))(\delta q) = \sum_{k=\max\{t-\bar\tau+1,0\}}^{t+\bar\tau-1} \gamma_{t,k}\delta q_k$$

An equilibrium price q in economy $\mathscr{E}_+$ with truncated past is *ASE* if there is a collection of matrices $\gamma^+_s \in \mathbb{R}^{L\times L}$, such that $\gamma_{t,t-s} \xrightarrow[t\to+\infty]{} \gamma^+_s$ $\forall s \in \mathbb{Z}$, where $\gamma_{t,k}$ are as defined in (5). Stationary equilibrium for $\mathscr{E}_+$ is as defined in definition 2.

2.4. **Policies.** In either economy, policies are exogenous to the equilibrium system and we associate them with the changes that they induce on the aggregate excess demand. It is convenient to think about a policy being a function of time and, possibly, individual characteristics, e.g., a pension reform or a tax policy. To incorporate these examples, we assume that policies, π, belong to a Banach manifold, B_π. In addition, in order to make local comparative statics a meaningful exercise, we assume that at any equilibrium of interest the excess demand F is differentiable with respect to policies.

Therefore, strictly speaking, we should write F as a function of two arguments: the endogenously determined prices, $q \in \ell_\infty^L$, and of exogenous parameters, $\pi \in B_\pi$. In this case, the derivative with respect to prices, as in equation (3) is $\frac{\partial F}{\partial q}$, and, similarly, the derivative with respect to policies is denoted by $\frac{\partial F}{\partial \pi}$.[4]

3. The result

Evaluating a response of an equilibrium variable (δq) to a policy change ($\delta\pi$) can be approximated using an implicit function theorem (IFT). For convenience, we provide a formulation of the IFT that suits our case.

3.1. The implicit function theorem.

Theorem 1 ([23], Thm. 3.8.5.)**.** *Let $\mathscr{E}, \mathscr{F}, \mathscr{G}$ be three Banach spaces, F be a continuously differentiable map from an open set $O \subset \mathscr{E} \times \mathscr{F}$ into $\mathscr{G}$, $F\colon (q,\pi) \mapsto F(q,\pi)$. Let (q^0,π^0) be a point in O, $F(q^0,\pi^0) = 0$.*

If $\frac{\partial F}{\partial q}(q^0,\pi^0)$ is invertible in the space of linear operators from $\mathscr{E}$ to $\mathscr{G}$, then there exist opens sets $A \subset \mathscr{E}$ and $B \subset \mathscr{F}$, $A \times B \subset O$ such that for every $\pi \in B$, there is a unique solution (in q) of the equation $F(q,\pi) = 0$ which belongs to A and there is a continuously differentiable function $\phi\colon B \to A$ such that $F(\phi(\pi),\pi) = 0$. Its derivative is given by

$$\phi'(\pi^0) = -\Big(\frac{\partial F}{\partial q}(q^0,\pi^0)\Big)^{-1} \circ \Big(\frac{\partial F}{\partial \pi}(q^0,\pi^0)\Big) \tag{9}$$

Thus the key assumption to verify is the invertibility of $\frac{\partial F}{\partial q}(q^0,\pi^0)$. To simplify notation in what follows we will be using DF to denote $\frac{\partial F}{\partial q}$ when no confusion arises.

3.2. Formulation of the main result.

Consider the derivative of aggregate excess demand (equilibrium equation) DF as defined in (4) for economy $\mathscr{E}$ evaluated at an ASE with asymptotes $(\gamma_s^+,\gamma_s^-)_{s\in\mathbb{Z}}$. Similarly, DF^+ as in (7) for economy $\mathscr{E}_+$ is evaluated at an ASE with $(\gamma_s^+)_{s\in\mathbb{Z}}$.

Let i denote the imaginary unit ($\sqrt{-1}$), $\mathsf{det}(A)$ denote the determinant of matrix A, I denote the identity matrix. Let

$$\Phi^+(\lambda) \stackrel{\text{def}}{=} \sum_{s=-\infty}^{\infty} e^{\mathrm{i}\lambda s}\gamma_s^+, \quad \Phi^-(\lambda) \stackrel{\text{def}}{=} \sum_{s=-\infty}^{\infty} e^{\mathrm{i}\lambda s}\gamma_s^-, \quad 0 \le \lambda \le 2\pi \tag{10}$$

$$\Gamma^\pm \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid \mathsf{det}[\Phi^\pm(\lambda) - zI] = 0 \text{ for some } 0 \le \lambda \le 2\pi\} \tag{11}$$

$$\Gamma \stackrel{\text{def}}{=} \Gamma^- \cup \Gamma^+ \tag{12}$$

Note that $\Phi^\pm(\lambda) \in \mathbb{C}^{L\times L}$ are $L \times L$ matrices with complex entries that are well-defined. Indeed, the sums in the equation (10) have only a finite number of non-zero elements because by definition of ASE, $\gamma_s^\pm = 0$ for all $s\colon |s| \ge \bar{\tau}$.

[4]In this case $\delta\pi$, being in a tangent space to B_π, belongs to a Banach space.

Finally, if $0 \notin \Gamma^{\pm}$, let $f_{\pm}\colon (0, 2\pi) \to \mathbb{C}$ be defined as $f_{\pm}(\lambda) = \mathsf{det}(\Phi^{\pm}(\lambda))$, and

$$n^{\pm} \overset{\text{def}}{=} \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{f'_{\pm}(\lambda)}{f_{\pm}(\lambda)} d\lambda \tag{13}$$

$$n \overset{\text{def}}{=} n^- - n^+ \tag{14}$$

Note that n is an integer.[5]

Let us start with a simple case. Assume that there is only one physical good traded at each period, $L = 1$, and that the baseline equilibrium is stationary e.g., a steady state in a stationary economy. This implies that $\gamma_{t,t-s}$ is a real number and for all $s \in \mathbb{Z}$, $\gamma_{t,t-s} = \gamma_s^+ = \gamma_s^-$. In this case $\Phi^+ = \Phi^-$ maps a real interval $[0, 2\pi]$ into $\mathbb{C}$. $\Gamma^+ = \Gamma^-$ is its range. It is a closed curve in the complex plane, since $\Phi^+(0) = \Phi^+(2\pi)$. Hence, for the stationary economy with infinite past $0 \notin \Gamma^+ \Rightarrow n^- = n^+$, thus $n = 0$. For this case we can formulate a tight indicator for equilibrium determinacy. The proof is in the appendix.

Proposition 1. *Assume $L = 1$.*

- *(i) In an economy with infinite past, $\mathscr{E}$, DF at a stationary equilibrium with $(\gamma_s^+)_{s\in\mathbb{Z}}$ is invertible if and only if $0 \notin \Gamma^+$.*
- *(ii) In an economy with truncated past, $\mathscr{E}^+$, DF at a stationary equilibrium with $(\gamma_s^+)_{s\in\mathbb{Z}}$ is invertible if and only if $0 \notin \Gamma^+$ and $n^+ = 0$.*

This statement supports our initial conjecture: determinacy is "easier to get" in an economy with infinite past. Truncating the past imposes an additional condition, $n^+ = 0$, for determinacy. As we illustrate in section 4.5, it can restrict the parameters of the model in a non-trivial way. In case of non-stationary equilibria, this feature is preserved only for those that are symmetric, as our main result asserts.

Theorem 2.

- *(i) Consider an asymptotically stationary equilibrium q of economy $\mathscr{E}$ with $(\gamma_s^+, \gamma_s^-)_{s\in\mathbb{Z}}$. Then DF evaluated at q is invertible or can be made invertible by an arbitrarily small perturbation[6] if and only if $0 \notin \Gamma$ and $n = 0$.*
- *(ii) Consider an asymptotically stationary equilibrium q of economy $\mathscr{E}^+$ with $(\gamma_s^+)_{s\in\mathbb{Z}}$. Then DF^+ evaluated at q is invertible or can be made invertible by an arbitrarily small perturbation if and only if $0 \notin \Gamma^+$ and $n^+ = 0$.*
- *(iii) If the equilibrium q of economy $\mathscr{E}$ is asymptotically stationary and* symmetric *then $0 \notin \Gamma \Rightarrow n = 0$.*

Remark 1. If $n \neq 0$ ($n^+ \neq 0$) then DF (DF^+) is not invertible for all sufficiently small of its perturbations.

The proof is based on the work of Gokhberg and Krein(1958), cf. the appendix. The basic building block is the Toeplitz operator T that maps the Banach space of complex bounded infinite sequences indexed by positive integers, $\ell_\infty(\mathbb{Z}_+)^L$, into itself.

$$(T(x))_t = \sum_{k=0}^{+\infty} \gamma_{t-k} x_k, \text{ where } \gamma_s \in \mathbb{R}^{L\times L}, \quad \sum_{s=-\infty}^{\infty} |\gamma_s| < \infty \tag{15}$$

By [11], T has the following property. If $0 \notin \Gamma^+$, then both the dimension of the null space of T ($x \neq 0\colon Tx = 0$) and the dimension of its co-range ($\ell_\infty^L(\mathbb{Z}_+)/\operatorname{\mathsf{Ran}} T$)

[5]This feature of the indicator is very convenient for a numerical calculation of the integral: the approximation can be rather crude. In the examples we consider, the direct analytic calculation is standard.

[6]A small perturbation of a linear operator is equivalent to adding to it another operator which is small in the operator norm.

are finite and their difference, index of the operator T, equals $-n^+$. Our DF^+ for the economy with the truncated past, $\mathscr{E}^+$, can be decomposed into a stationary component, an operator like T, and a non-stationary part and it inherits the above mentioned property of T. A mapping between complex and real numbers translates the results to our case. An economy with infinite past can be viewed as a concatenation of two truncated economies.

Part (iii) of the theorem implies that economies with infinite past are more amenable to comparative statics, as invertibility of DF (up to an arbitrarily small perturbation) relies then on a single condition, $0 \notin \Gamma$. The condition $0 \notin \Gamma$ is non-generic in terms of parameters of the model, which is illustrated in the example that follows and is shown for a continuous-time economy with production and a single physical consumption good traded at any $t \in \mathbb{R}$, cf. [20], [12].

For an asymptotically stationary equilibria in $\mathscr{E}$, conditions $0 \notin \Gamma$ and $n = 0$ are necessary but not sufficient for invertibility of DF. In order to assure the invertibility, one can, for example, verify that the null space of DF is empty, i.e., that $DF(x) = 0 \Rightarrow x = 0$. Alternatively, one could slightly perturb DF. In order to find the desired direction of the perturbation, cf. [26], in general, more information is needed about the model. One potential difficulty is that changing parameters of the model affects the derivative of excess demand for any fixed price and, in addition, can change the equilibrium price at which the derivative is to be evaluated. One possible way to fully control the change in the asymptotic derivative coefficients γ_s is to focus on perturbations of the parameters that keep the equilibrium intact, cf. [12].

Next we use an example economy to illustrate the main result and to demonstrate how to perform comparative statics.

4. Comparative statics in a two-period-life-cycle economy

The main task here is to assess the difference between the (stationary) baseline equilibrium and the new equilibrium path in a "perturbed" economy, whenever it is well-defined. The approximation is based on the implicit function theorem.

4.1. The economy with infinite past and future, $\mathscr{E}^2$, and its equilibria.

Consider an OLG economy as in [10]. There is a single commodity, $L = 1$ and a single type of individual, so superscript θ will be omitted. Assume that $N_{t,s}$ is a constant. An individual is alive for two periods: $\tau = 2$.

An individual born at time $x \in \mathbb{Z}$ is maximizing the life-time utility defined over the infinite consumption streams indexed by time and age of the individual,

$$U_x(\hat{c}) = a \ln \hat{c}_{x,0} + (1-a) \ln \hat{c}_{x+1,1}, \text{ where } a \in (0,1), \hat{c} \in \mathbb{R}_+^{\mathbb{Z}\times\mathbb{Z}} \tag{16}$$

subject to the budget constraint, $\sum_{s\in\mathbb{Z}} p_{x+s}\hat{c}_{x+s,s} \leq \sum_{s\in\mathbb{Z}} p_{x+s}\omega_{x+s,s}$, where $\hat{c}_{t,s} \geq 0$ denotes amount consumed by an individual at time t if he is of age $s \in \mathbb{Z}$ at that time. His endowment is $\omega_{x,0} \geq 0$ when he is young and $\omega_{x+1,1} \geq 0$, when he is old and zero for any other age $s \notin \{0, 1\}$. Therefore, individual demand is zero beyond his life-time, $c_{t,s} = 0, \forall s \in \mathbb{Z}\backslash\{0, 1\}$, and during his life-time it is

$$c_{t,0}(q_t) = a(\omega_{t,0} + q_t\omega_{t+1,1}),\ c_{t+1,1}(q_t) = (1-a)(\frac{\omega_{t,0}}{q_t} + \omega_{t+1,1}),\ q_t = \frac{p_{t+1}}{p_t} \tag{17}$$

By definition 1, an equilibrium price $q \in \ell_\infty(\mathbb{Z})$ has to be a solution to the following system of equations for a given profile of endowments to the young and old, $\omega \in \ell^2_\infty(\mathbb{Z})$:

$$F(q,\omega) = 0, \quad F\colon (\ell_\infty(\mathbb{Z}) \times \ell^2_\infty(\mathbb{Z})) \to \ell_\infty(\mathbb{Z}) \tag{18}$$

$$F_t(q,\omega) = (a-1)\omega_{t,0} + aq_t\omega_{t+1,1} + \frac{(1-a)\omega_{t-1,0}}{q_{t-1}} - a\omega_{t,1} \tag{19}$$

If the endowments are constant over time, so that the young get $\omega_{t,0} = \omega^\alpha$ and the old get $\omega_{t,1} = \omega^\beta$, then, as is well-known, the system of equilibrium equations $F(q, \omega) = 0$ admits two constant solutions in q: the golden rule equilibrium (GRE) and the *balanced* equilibrium (BE), cf. [8].

GRE: $q_t = 1$ with $c_{t,0} = a(\omega^\alpha + \omega^\beta), c_{t,1} = (1-a)(\omega^\alpha + \omega^\beta)$.

BE: $q_t = \kappa \stackrel{\text{def}}{=} \frac{(1-a)\omega^\alpha}{a\omega^\beta}$ with $c_{t,0} = \omega^\alpha, c_{t,1} = \omega^\beta$.

Note that while in the BE the amount of net assets is zero since agents' optimal consumption equals their endowment, in GRE the net assets (money) are typically not zero: $M = \omega^\alpha - a(\omega^\alpha + \omega^\beta) = (\kappa - 1)a\omega^\beta$. The economy is in perpetual debt in the GRE if $\kappa < 1$.

In addition, there are non-constant equilibria. If $\kappa = 1$, all equilibria collapse to a single one. Demichelis and Polemarchakis (2007) derive explicit solutions for all equilibria in this model, which are illustrated in figure 1. In addition to BE and GRE there are also increasing equilibria, indexed by the parameter $v \in \mathbb{R}_{++}$: $q_t = \frac{1+v\kappa^{t+1}}{1+v\kappa^t}$. The graph depicts the case of $\kappa < 1$. The asymptotic behavior of the increasing equilibria is similar in the case of $\kappa > 1$ with the corresponding upper asymptote being κ and the lower one being unity.

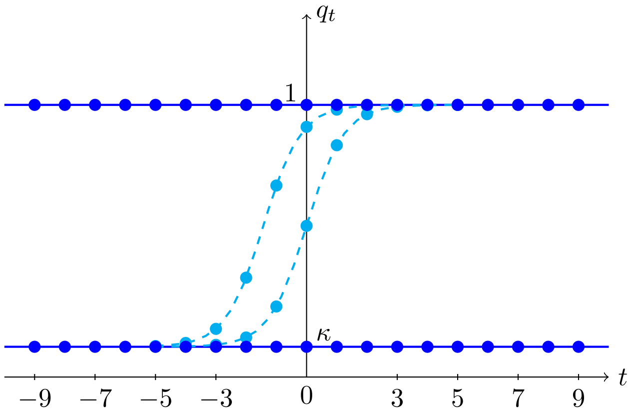

FIGURE 1. The two stationary equilibria, $q_t = 1$ and $q_t = \kappa = 0.2$. There are also two examples of increasing equilibria, $q_t = \frac{1+v\kappa^{t+1}}{1+v\kappa^t}$ for $v = 1, 0.1$, see [6].

It is evident that the two constant equilibria are locally unique in $\ell_\infty(\mathbb{Z})$. Indeed, although there is a continuum of increasing equilibria, which converge to the upper asymptote as $t \to +\infty$ and to the lower asymptote as $t \to -\infty$, none of them remains in an ϵ-neighbourhood of either equilibria for any $0 < \epsilon < \frac{|1-\kappa|}{2}$.

The derivative of F with respect to prices evaluated at an arbitrary strictly positive price vector q is

$$\left(\frac{\partial F_t}{\partial q}(q)\right)(\delta q) = \gamma_{t,t}\delta q_t + \gamma_{t,t-1}\delta q_{t-1} \quad \text{where} \tag{20}$$

$$\gamma_{t,t} = a\omega_{t+1,1}, \quad \gamma_{t,t-1} = -\frac{(1-a)\omega_{t-1,0}}{q_{t-1}^2} \tag{21}$$

As one would expect, both GRE and BE are stationary according to definition 2: all components of the derivative in equation (20) are independent of time: $\forall t \in \mathbb{Z}$,

$$\begin{array}{lll} \text{in GRE:} & \gamma_{t,t} = \gamma_0^\pm = a\omega^\beta, & \gamma_{t,t-1} = \gamma_1^\pm = -(1-a)\omega^\alpha \\ \text{in BE:} & \gamma_{t,t} = \gamma_0^\pm = a\omega^\beta, & \gamma_{t,t-1} = \gamma_1^\pm = -\frac{(1-a)\omega^\alpha}{\kappa^2} \end{array} \tag{22}$$

The increasing equilibria are asymptotically stationary by the same definition. Indeed, the coefficient $\gamma_{t,t} = \gamma_0 = a\omega^\beta$ is independent of time. Since the price q_t converges to κ or 1 at $\pm\infty$, so $\gamma_{t,t-1} = -\frac{(1-a)\omega^\alpha}{q_{t-1}^2}$ converges to either $-(1-a)\omega^\alpha$ or to $-\frac{(1-a)\omega^\alpha}{\kappa^2}$, both are independent of t. In sum, for increasing equilibria,

$$\begin{array}{llll} \text{if } \kappa < 1 & \gamma_0^+ = \gamma_0^- = a\omega^\beta, & \gamma_1^- = -\frac{(1-a)\omega^\alpha}{\kappa^2}, & \gamma_1^+ = -(1-a)\omega^\alpha \\ \text{if } \kappa > 1 & \gamma_0^+ = \gamma_0^- = a\omega^\beta, & \gamma_1^- = -(1-a)\omega^\alpha, & \gamma_1^+ = -\frac{(1-a)\omega^\alpha}{\kappa^2}. \end{array} \tag{23}$$

4.2. **An example of a policy change.** Assume that a policy change targets only the endowments of the young generation, $\delta\pi_t = \delta\omega_{t,0} \in \ell_\infty(\mathbb{Z})$. Then, being evaluated at some strictly positive $q \in \ell_\infty(\mathbb{Z})$, the derivative of F can be written as follows:

$$\Big(\frac{\partial F_t}{\partial \omega_{\cdot,0}}(q)\Big)(\delta\omega_{\cdot,0}) = (1-a)\left(\frac{\delta\omega_{t-1,0}}{q_{t-1}} - \delta\omega_{t,0}\right), \quad \forall t \in \mathbb{Z} \tag{24}$$

At any equilibrium q, for which $\left[\frac{\partial F}{\partial q}(q)\right]^{-1}$ exists, the equilibrium response of prices, by the IFT, is

$$\frac{\partial q}{\partial \omega_{\cdot,0}}(\delta\omega_{\cdot,0}) = -\Big[\frac{\partial F}{\partial q}(q)\Big]^{-1} \circ \Big(\frac{\partial F}{\partial \omega_{\cdot,0}}(q)\Big)(\delta\omega_{\cdot,0}) \tag{25}$$

To simplify, we consider a change in the endowment of the young at some fixed $t = 0$ from $\omega_{0,0} = \omega^\alpha$ to $\omega^\alpha + \varepsilon$, $\varepsilon > 0$, so that $\delta\omega_{t,0} = \varepsilon$ if $t = 0$, whilst otherwise $\delta\omega_{t,0} = 0$.

4.3. **Reaction of the golden rule equilibrium,** $q_t = 1$**.** At this point we need to assure that the derivative $\frac{\partial F}{\partial q}(q = 1)$, cf. equation (20), is an invertible map. We compare the results of the direct computation with the implications of theorem 2 for this case.

Notation 4.1. (i) I denotes the identity operator on ℓ_∞;
(ii) $\$_h$ denotes the shift of a sequence forward by $h \in \mathbb{Z}$: $\big(\$_h(v)\big)(t) = v_{t-h}$;
(iii) $\mathbb{1}_C$ is an indicator function that returns 1 if condition C holds and zero otherwise.

Using this notation we can re-write $\frac{\partial F}{\partial q}(1)$, the derivative of the equilibrium equation with respect to prices, cf. equation (20), evaluated at a GRE ($q_t = 1$), as a map from $\ell_\infty(\mathbb{Z})$ to $\ell_\infty(\mathbb{Z})$,

$$\Big(\frac{\partial F}{\partial q}(1)\Big)(\delta q) = a\omega^\beta(I - \kappa\$_1) \circ \delta q = -a\omega^\beta\kappa\$_1(I - \kappa^{-1}\$_{-1}) \circ \delta q \tag{26}$$

It follows then that its inverse exists if and only if $\kappa \neq 1$.

The same conclusion could be reached directly from theorem 2.(iii). As follows from computation of the derivative in the previous section (20), in a GRE $\gamma_0^\pm = a\omega^\beta$, $\gamma_1^\pm = -(1-a)\omega^\alpha$, so in this case,

$$\Phi^+(\lambda) = \Phi^-(\lambda) = \gamma_0^\pm + \gamma_1^\pm e^{\lambda \mathrm{i}} = \gamma_0^\pm(1 - \kappa e^{\lambda \mathrm{i}}) \tag{27}$$

Therefore, $0 \in \Gamma = \{z \in \mathbb{C} \colon z = \gamma_0^\pm(1 - \kappa e^{\mathrm{i}\lambda}) \mid 0 \le \lambda \le 2\pi\}$ if and only if $\kappa = 1$. In other cases we can calculate the inverse.

If $\kappa < 1$ then using the first equality in equation (26), the inverse can be calculated directly:

$$\Big[\frac{\partial F}{\partial q}(1)\Big]^{-1} = \frac{1}{a\omega^\beta}[I - \kappa\$_1]^{-1} = \frac{1}{a\omega^\beta}\sum_{n=0}^{\infty}\kappa^n\$_n \tag{28}$$

Using the same notation, and equation 24, the derivative of F with respect to the policy is

$$\frac{\partial F}{\partial \omega_{\cdot,0}}(1) = (1-a)(\$_1 - I) \tag{29}$$

Using the implicit function theorem, cf. eq. (25), and our specification of the change in policy, $\delta\omega_{\cdot,0} = \mathbb{1}_{t=0}\varepsilon$, we get

$$\frac{\partial q}{\partial \omega_{\cdot,0}}(\delta\omega_{\cdot,0}) = \frac{1}{a\omega^{\beta}}\Big[\sum_{n=0}^{\infty}\kappa^n \$_n (1-a)(I-\$_1)\Big](\mathbb{1}_{t=0}\varepsilon) \tag{30}$$

$$= \frac{\varepsilon\kappa}{\omega^{\alpha}}\Big(\sum_{n=0}^{\infty}\kappa^n \mathbb{1}_{t=n} - \sum_{n=1}^{\infty}\kappa^{n-1}\mathbb{1}_{t=n}\Big) \tag{31}$$

If $\kappa > 1$ then using the second equality in equation (24), the inverse of the derivative can be calculated directly as well,

$$\Big[\frac{\partial F}{\partial q}(1)\Big]^{-1} = -\frac{1}{a\omega^{\beta}\kappa}\$_{-1}(I-\kappa^{-1}\$_{-1})^{-1} = -\frac{1}{a\omega^{\beta}\kappa}\$_{-1}\sum_{n=0}^{\infty}\kappa^{-n}\$_{-n} \tag{32}$$

Similarly, by the implicit function theorem, we get

$$\frac{\partial q}{\partial \omega_{\cdot,0}}(\delta\omega_{\cdot,0}) = \frac{1}{a\omega^{\beta}\kappa}\$_{-1}\Big[\sum_{n=0}^{\infty}\kappa^{-n}\$_{-n}(1-a)(\$_1 - I)\Big](\mathbb{1}_{t=0}\varepsilon) \tag{33}$$

$$= \frac{\varepsilon}{\omega^{\alpha}}\Big[\sum_{v=-\infty}^{0}\kappa^{v}\mathbb{1}_{t=v} - \sum_{v=-\infty}^{-1}\kappa^{v+1}\mathbb{1}_{t=v}\Big] \tag{34}$$

TABLE 1. Summary of eq. (30), (33). The variation δq_t of the equilibrium price ratio in the economy $\mathscr{E}^2$ due to a change in endowment ($=\varepsilon$) of the young at time $t=0$.

	$\kappa < 1$	$\kappa > 1$
$t \leq -1$	0	$\frac{\varepsilon}{\omega^{\alpha}}\kappa^t(1-\kappa)$
$t = 0$	$\frac{\varepsilon}{\omega^{\alpha}}\kappa$	$\frac{\varepsilon}{\omega^{\alpha}}$
$t \geq 1$	$\frac{\varepsilon}{\omega^{\alpha}}\kappa^t(\kappa-1)$	0

The comparative statics is illustrated in figure 2.

Let us now describe the resulting changes in consumption allocations. First, for any changes in endowments and prices, and for any $t \in \mathbb{Z}$, the equilibrium reaction of the quantities demanded by young and old generations are, respectively,

$$\delta c_{t,0} = a(\delta\omega_{t,0} + q_t\delta\omega_{t+1,1}) + a\omega^{\beta}\delta q_t \tag{35}$$

$$\delta c_{t,1} = (1-a)(\delta\omega_{t-1,0}/q_{t-1} + \delta\omega_{t,1}) - (1-a)\omega^{\alpha}\frac{\delta q_{t-1}}{q_{t-1}^2} \tag{36}$$

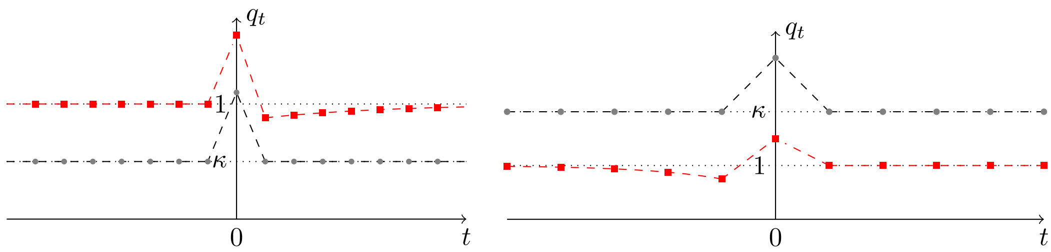

FIGURE 2. In both graphs $\omega^\alpha = 1$. The squares trace the perturbed equilibrium around a GRE according to table 1 $\varepsilon = 0.3$, $\kappa = 0.8$ on the left and $\varepsilon = 0.5$, $\kappa = 2$ on the right. The circles depict the perturbation around the BE according to eq. (51).

4.3.1. $\kappa < 1$. According to table 1 in the new equilibrium, no changes in demand should occur before $t = 0$: policy directly affects only the young at time 0 and prices do not respond beforehand. Relative prices q_t for all $t < 0$ remain unity, so in particular, $q_{t-1} = 1$, hence the consumption by the old at time zero does not change. Thus the first affected generation is the young one at time 0. Evaluating their response using equation (35), we conclude that the quantity demanded increased:

$$\delta c_{0,0} = a\varepsilon + a\omega^\beta \frac{\varepsilon}{\omega^\alpha}\kappa = \varepsilon \tag{37}$$

This equals the increase in supply at that time.

Since $\kappa < 1$, the amount of net assets ($M = (\kappa - 1)a\omega^\beta$) in this equilibrium is negative, so young consume more than their endowment in the unperturbed equilibrium. The change in the relative prices δq_0 is exactly enough to motivate them to consume the extra endowment, thus staying with the same debt.

Starting from time $t = 1$, the endowments are stationary again and the ratio of prices converges back to 1.

This, however is not the end of the story, since the generation born at 0 also consumes in the next period, $t = 1$, and that consumption is subject to two effects. The first one is the direct income effect due to the policy change, while the second one is the indirect response of the equilibrium relative price:

$$\delta c_{1,1} = (1-a)(\delta\omega_{0,0}/q_0 + \delta\omega_{1,1}) - (1-a)\omega_{0,0}\frac{\delta q_0}{q_0^2} \tag{38}$$

$$\delta c_{1,1} = (1-a)\varepsilon - (1-a)\omega^\alpha \frac{\varepsilon}{\omega^\alpha}\kappa = (1-a)\varepsilon(1-\kappa) \tag{39}$$

Thus the old in period 1 who got an increase in endowment in the previous period are enjoying extra consumption as well. To accommodate for that the prices discourage the young at this time to consume as much as they would have consumed without the policy change, i.e., at the baseline, the GRE. Indeed, the price change δq_1 affects the young born at $t = 1$: by equation (35)

$$\delta c_{1,0} = a\omega^\beta \delta q_1 = a\omega^\beta \frac{\varepsilon}{\omega^\alpha}\kappa(\kappa - 1) = (1-a)\varepsilon(\kappa - 1) \tag{40}$$

Let us look at $t \geq 2$. Here we are left with price effects only for both generations:

$$\delta c_{t,0} = a\omega^\beta \delta q_t = a\omega^\beta \frac{\varepsilon}{\omega^\alpha}\kappa^t(\kappa - 1) = (1-a)\varepsilon\kappa^{t-1}(\kappa - 1) < 0 \tag{41}$$

$$\delta c_{t,1} = -(1-a)\omega^\alpha \delta q_{t-1} = (1-a)\varepsilon\kappa^{t-1}(1-\kappa) > 0 \tag{42}$$

So, the price effects exactly cancel each other for any time $t \geq 2$: the young consume less and the old consume more as compared to the golden rule equilibrium. These changes decrease with time $t \to +\infty$, as $\kappa < 1$.

4.3.2. $\kappa > 1$. In this case the young are saving, as $\kappa > 1$ indicates that the net assets, $M = (\kappa - 1)a\omega^\beta$, are positive.

$$\delta c_{0,0} = a\varepsilon + a\omega^\beta \frac{\varepsilon}{\omega^\alpha} = \frac{\varepsilon}{\kappa}(a\kappa + 1 - a) \tag{43}$$

$$\delta c_{0,1} = -(1-a)\omega^\alpha \frac{\varepsilon}{\omega^\alpha}\kappa^{-1}(1-\kappa) = \frac{\varepsilon}{\kappa}(\kappa - 1)(1-a) \tag{44}$$

Thus the young and the old at time $t = 0$ are sharing the increase in the endowment of the young.

Next period, at $t = 1$, the consumption by the old is subject to two negating effects, the increase in real income and the price change at time 0, which cancel each other:

$$\delta c_{1,1} = (1-a)\varepsilon - (1-a)\omega^\alpha \frac{\varepsilon}{\omega^\alpha} = 0 \tag{45}$$

Looking back in time, $t \leq -1$, we have only the price effects, which create some re-distribution from the young to the old:

$$\delta c_{t,0} = a\omega^\beta \frac{\varepsilon}{\omega^\alpha}\kappa^t(1-\kappa) = (1-a)\varepsilon\kappa^{t-1}(1-\kappa) < 0 \tag{46}$$

$$\delta c_{t,1} = (1-a)\varepsilon\kappa^{t-1}(\kappa - 1) > 0 \tag{47}$$

Again, the changes grow smaller and the allocations converge to the golden rule equilibrium as $t \to -\infty$.

4.4. **Reaction of the balanced equilibrium,** $q_t = \kappa$**.** Following the same algorithm we can calculate the reaction of another stationary equilibrium to the same perturbation of endowments of the young at some fixed time 0.

As in case of the GRE, the inverse of $\frac{\partial F}{\partial q}(\kappa)$ exists as long as $\kappa \neq 1$: $\gamma_0 = a\omega^\beta$, $\gamma_1 = -\frac{(1-a)\omega^\alpha}{\kappa^2}$, so in this case,

$$\Phi^\pm(\lambda) = \gamma_0 + \gamma_1 e^{\lambda \mathrm{i}} = \gamma_0(1 - \frac{1}{\kappa}e^{\lambda \mathrm{i}}) \tag{48}$$

Therefore, $0 \in \Gamma = \{z \in \mathbb{C} \colon z = \gamma_0(1 - \frac{1}{\kappa}e^{\mathrm{i}\lambda}) \mid 0 \leq \lambda \leq 2\pi\}$ if and only if $\kappa = 1$.

By the implicit function theorem, (25), if $\kappa \neq 1$,

$$\frac{\partial q}{\partial \omega_{\cdot,0}}(\delta\omega_{\cdot,0}) = \left[a\omega^\beta(I - \kappa^{-1}\$_1)\right]^{-1} \circ (1-a)(I - \kappa^{-1}\$_1)(\delta\omega_{0,0}) \tag{49}$$

$$= \frac{\kappa}{\omega^\alpha}\left[(I - \kappa^{-1}\$_1)\right]^{-1} \circ (I - \kappa^{-1}\$_1)(\varepsilon\mathbb{1}_{t=0}) = \frac{\kappa}{\omega^\alpha}\varepsilon\mathbb{1}_{t=0} \tag{50}$$

Therefore,

$$\delta q_t = \begin{cases} \varepsilon\frac{\kappa}{\omega^\alpha} & t = 0 \\ 0, & t \neq 0 \end{cases} \tag{51}$$

It is easy to check using equation (35) that in this case all the increase in endowment is consumed by the young at time $t = 0$, cf. also (37). Using equation (36) one can verify that the income effect for the old at time $t = 1$ is cancelled out by the effect of an increase in the relative price $\delta q_{t-1} = \delta q_0$.

Hence, in the balanced equilibrium the one-time increase in the endowment of the young is not traded and hence has no long-lasting effects, which is in contrast to the golden-rule equilibrium. Thus such perturbation does not require money to be supported: the perturbed equilibrium is still balanced.

The same feature is preserved if we perturb the consumption by the old at time 0, $\delta\omega_{t,1} = \mathbb{1}_{t=0}\varepsilon$.

$$\frac{\partial q}{\partial \omega_{\cdot,1}}(\delta\omega_{\cdot,1}) = -\big[a\omega^\beta(I - \kappa^{-1}\$_1)\big]^{-1} \circ a(\$_{-1}\kappa - I)(\delta\omega_{t,1}) \tag{52}$$

$$= -\frac{1}{\omega^\beta}\big[(I - \kappa^{-1}\$_1)\big]^{-1} \circ \$_{-1}\kappa \circ (I - \$_1\kappa^{-1})(\varepsilon\mathbb{1}_{t=0}) \tag{53}$$

$$= -\frac{\kappa}{\omega^\beta}\varepsilon\mathbb{1}_{t=-1} \tag{54}$$

The response at this equilibrium is, again, in a single period. This time, only the relative price $q_{-1} = \frac{p_0}{p_{-1}}$ faced by the old generation changes: $\delta q_{-1} = -\frac{\kappa}{\omega^\beta}\varepsilon$. The old who get the endowment at $t = 0$ consume it in full:

$$\delta c_{t,1} = (1-a)(\delta\omega_{t-1,0}/q_{t-1} + \delta\omega_{t,1}) - (1-a)\omega^\alpha \frac{\delta q_{t-1}}{q_{t-1}^2}|_{t=0} \tag{55}$$

$$= (1-a)\varepsilon + (1-a)\omega^\alpha \frac{\frac{\kappa}{\omega^\beta}\varepsilon}{\kappa^2} = (1-a)\varepsilon + (1-a)\omega^\alpha \frac{\varepsilon}{\kappa\omega^\beta} = \varepsilon \tag{56}$$

Beforehand, at $t = -1$, the income and price effects cancel each other for the generation who are born then and are expecting to get a transfer next period:

$$\delta c_{t,0} = a(\delta\omega_{t,0} + q_t\delta\omega_{t+1,1}) + a\omega^\beta\delta q_t|_{t=-1} \tag{57}$$

$$= a\kappa\varepsilon - a\omega^\beta\frac{\kappa}{\omega^\beta}\varepsilon = 0 \tag{58}$$

4.5. **Economy $\mathscr{E}^2_+$ with infinite future and no past.** Truncating the past in the overlapping generations model is equivalent to requiring all consumption allocations in the previous model to be zero before time $t = 0$. At time $t = 0$ a new generation is born, co-existing with the so-called initial old. We want to be precise here about the "initial conditions".

Let us start with the generations born at or after period 0. As is well-known in the literature [10], to enable any generation to consume a bundle $(\hat{c}_{x,0}, \hat{c}_{x+1,1})$ that satisfies the constraint $p_x\hat{c}_{x,0} + p_{x+1}\hat{c}_{x+1,1} \le p_x\omega_{x,0} + p_{x+1}\omega_{x+1,1}$ $\forall x \in \mathbb{Z}$, one has to introduce a store of value between periods, so that the same constraint can be re-written as two period-by-period constraints with $q_x = \frac{p_{x+1}}{p_x}$:

$$\hat{c}_{x,0} + m_x \le \omega_{x,0} \tag{59}$$

$$q_x\hat{c}_{x+1,1} \le q_x\omega_{x+1,1} + m_x \tag{60}$$

One way to model the store of value is in terms of contracts denominated in real terms (m_x). Another way is to use fiat money, $M = p_x(\omega_{x,0} - \hat{c}_{x,0}) = p_{x+1}(\hat{c}_{x+1,1} - \omega_{x+1,1})$. Then $m_x = \frac{M}{p_x}$. The amount of money is the aggregate nominal net debt and it is constant in any competitive equilibrium in an OLG model, which follows from the market clearing conditions, cf. [14], [19].

Optimal consumption by the young who were born at $t \ge 0$ is the same as in the model with the infinite past, $c_{t,0} = a(\omega_{t,0} + q_t\omega_{t+1,1})$. The optimal consumption by the old at any time $t \ge 1$ is unchanged as well: $c_{t,1} = (1-a)(\omega_{t-1,0}/q_{t-1} + \omega_{t,1})$. Therefore, market clearing equations for any $t \ge 1$ remain the same as in the model with the infinite past, $\mathscr{E}^2$.

In contrast to the previous model, there is an additional element here: the summary of the truncated past of the economy, or the "initial conditions". Their effect depends on the interpretation of the store of value.

4.5.1. *Real balances interpretation.* First, assume that the initial old hold a claim m_0 to real goods or a contract that entitles them to receive or requires them to deliver the consumption good. The contracts have to be respected and can not be renegotiated. Then, the consumption by the initial old is fully determined:

$$c_{0,1} = \omega_{0,1} + m_0 \tag{61}$$

The market clearing at time zero then identifies the price ratio, given the parameters of the system. Equilibrium exists only if the claim of the old, m_0, does not exceed $(1-a)\omega_{0,0}$:

$$a(\omega_{0,0} + q_0\omega_{1,1}) + \omega_{0,1} + m_0 = \omega_{0,0} + \omega_{0,1} \Rightarrow \tag{62}$$

$$q_0 = \frac{(1-a)\omega_{0,0} - m_0}{a\omega_{1,1}} \tag{63}$$

Note that m_0 can be negative, in which case the old are returning the loan they took one period before.

To sum up, the equilibrium conditions in this case are as follows:

$$F(q,\omega) = 0, \quad F\colon (\ell_\infty(\mathbb{Z}_+) \times \ell^2_\infty(\mathbb{Z}_+)) \to \ell_\infty(\mathbb{Z}_+) \tag{64}$$

$$F_t(q,\omega) = \begin{cases} (a-1)\omega_{t,0} + aq_t\omega_{t+1,1} + \frac{(1-a)\omega_{t-1,0}}{q_{t-1}} - a\omega_{t,1}, & t \geq 1 \\ (a-1)\omega_{0,0} + aq_0\omega_{1,1} + m_0, & t = 0 \end{cases} \tag{65}$$

Existence of stationary equilibria for a stationary economy with $\omega_{t,0} = \omega^\alpha$, $\omega_{t,1} = \omega^\beta$ now depends on the initial condition, m_0. If $m_0 = 0$, there exists a balanced equilibrium with $q_t = \kappa$. If $m_0 = a\omega^\beta(\kappa - 1)$, then there is a GRE.

As before, we can write the derivatives of F as linear operators, but this time mapping the set $\ell_\infty(\mathbb{Z}_+)$ of bounded sequences on a *half-line* to itself. Otherwise, the form of the derivative $\frac{\partial F}{\partial q}$ evaluated at a stationary equilibrium, is the same. Indeed, in this case, by definition of the shift operator,

$$\$_1(a_0, a_1, \ldots, a_n \ldots) = (0, a_0, a_1, \ldots, a_n \ldots) \quad \forall a \in \ell_\infty(\mathbb{Z}_+)$$

Thus, $\frac{\partial F}{\partial q} = DF^+$ is a map from $\ell_\infty(\mathbb{Z}_+)$ to itself

$$DF^+(\delta q) = a\omega^\beta(I - v\$_1) \circ \delta q, \quad \text{where} \quad v = \begin{cases} \kappa, & q_t = 1; \\ \frac{1}{\kappa}, & q_t = \kappa \end{cases}$$

Note that for any $v < 1$ the inverse of $(I - v\$_1)$ is $\sum_{n=1}^\infty v^n \$_n$. If $v > 1$ this inverse does not exist in $\ell_\infty(\mathbb{Z}_+)$.

It is easy to check that whenever this inverse exists, the response to the change in the endowment of the young at time zero, $\delta\omega_{0,0}$ is the same as in the model with infinite past.[7]

The conditions for invertibility of DF^+ can be easily derived from theorem 2 as well. Note that by definition of Φ^+ (equation (10)) and our calculation of the derivative coefficients at the GRE, equation (22),

$$\Phi^+(\lambda) = \gamma_0^+ + \gamma_1^+ e^{\lambda \mathrm{i}} = \gamma_0^+(1 - \kappa e^{\lambda \mathrm{i}}) \tag{66}$$

It can be verified by a direct computation that $n^+ = 0$ if $\kappa < 1$ and $n^+ = 1$ if $\kappa > 1$, cf. eq. (13). We illustrate it in figure 3.

To summarize, the two conditions, $0 \notin \Gamma^+$ and $n^+ = 0$, imply $\kappa < 1$. Since the equilibrium is stationary, by proposition 1, $\kappa < 1$ is necessary and sufficient for invertibility of DF^+.

[7]As for the perturbation of the endowment of the initial old, $\omega_{0,1}$, the reaction is trivial: the old just consume the extra endowment and no prices change. It can be easily confirmed by examining the market clearing condition (62).

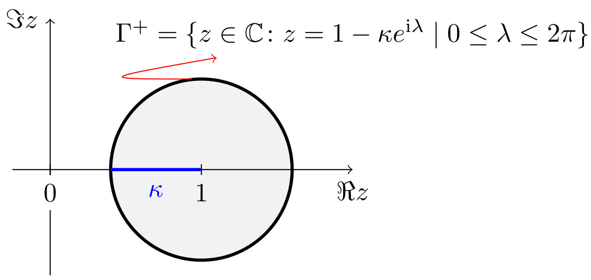

FIGURE 3. $0 \notin \Gamma^+$ assures that the curve Φ^+, or the circle Γ^+ does not pass through zero. $n^+ = 0$ if zero is outside the circle Γ^+ and $n^+ = 1$ if it is inside the circle. Assuming $a\omega^\beta = 1$ and $\kappa < 1$, we get $n^+ = 0$, as in the graph below. If $\kappa > 1$, zero is in the disc so then $n^+ \neq 0$ and DF^+ is not invertible.

This example illustrates another point raised in the discussion. In the model with infinite past, $\mathscr{E}^2$, comparative statics is impossible only for $\kappa = 1$, while with truncated past the condition is harder to satisfy and it is non-generic in parameters.

Next, we present the results for the second interpretation of the store of value in the economy with the truncated past, which has been more widely accepted in the literature, cf. [14], [10].

4.5.2. *Fiat money interpretation.* Assume that the only store of value across periods is fiat money.[8] Now the economy is "shocked": there is a sudden transfer of ε units of good (from an external source) to the young generation in period 0. Whatever price p_0 was believed to prevail before the shock, these "expectations" might be altered thereafter. In this case the consumption by the initial old depends on the new price of the physical good in period 0:

$$c_{0,1} = \omega_{0,1} + \frac{M_0}{p_0} \tag{67}$$

The market clearing at time zero then involves two variables, p_0 and $q_0(p_0) = \frac{p_1}{p_0}$

$$(a-1)\omega_{0,0} + aq_0\omega_{1,1} + \frac{M_0}{p_0} = 0 \tag{68}$$

To incorporate this into the system of difference equations, we can re-define the equilibrium variable: let $\tilde{q}_0 = p_0$ and let $\tilde{q}_t = q_{t-1}$ for $t \ge 1$. Then

$$F(\tilde{q}, \omega) = 0, \quad F\colon (\ell_\infty(\mathbb{Z}_+) \times \ell^2_\infty(\mathbb{Z}_+)) \to \ell_\infty(\mathbb{Z}_+) \tag{69}$$

$$F_t(\tilde{q}, \omega) = \begin{cases} (a-1)\omega_{t,0} + a\tilde{q}_{t+1}\omega_{t+1,1} + \frac{(1-a)\omega_{t-1,0}}{\tilde{q}_t} - a\omega_{t,1}, & t \ge 1 \\ (a-1)\omega_{0,0} + a\omega_{1,1}\tilde{q}_1(\tilde{q}_0) + \frac{M_0}{\tilde{q}_0}, & t = 0 \end{cases} \tag{70}$$

Evaluated at stationary endowments and at a stationary equilibrium $\tilde{q}_t = Q \in \{1, \kappa\}$, $t \ge 1$ the derivative DF^+ is

$$DF^+_t(\tilde{q}, \omega) = \begin{cases} a\omega^\beta \delta\tilde{q}_{t+1} - \frac{(1-a)\omega^\alpha}{Q^2}\delta\tilde{q}_t, & t \ge 1 \\ a\omega^\beta \delta\tilde{q}_1 - (aQ\omega^\beta \frac{1}{\tilde{q}_0} + \frac{M_0}{\tilde{q}_0^2})\delta\tilde{q}_0, & t = 0 \end{cases} \tag{71}$$

So, the equilibrium is asymptotically stationary with $\gamma^+_{-1} = a\omega^\beta$ and $\gamma^+_0 = -\frac{(1-a)\omega^\alpha}{Q^2}$. Then, if $Q = 1$, $\Phi^+(\lambda) = \gamma^+_0 + \gamma^+_{-1}e^{-\lambda\mathrm{i}} = -\gamma_0(1 - \frac{1}{\kappa}e^{-\lambda\mathrm{i}})$. Here, again, $0 \notin \Gamma^+ \iff$

[8]The amount of fiat held by the initial old does not pin down a particular equilibrium in economy $\mathscr{E}$.

$\kappa \neq 1$. Also, $n^+ = 0 \iff \kappa > 1$. Note that the equilibrium is only asymptotically stationary and so, to assure invertibility we need the null space of DF^+ to be empty, i.e., $\{x \neq 0 \colon DF^+(x) = 0\} = \emptyset$. For that it is sufficient to establish that any non-trivial solution to $DF^+(x) = 0$ does not belong to ℓ_∞, being an unbounded sequence. By equation (71) $DF^+(x) = 0$ implies that for any $t \geq 1$, $x_{t+1} = \kappa x_t$. In the first equation, for any choice of x_0, $x_1 = x_0\left(\frac{1}{\tilde{q}_0} + \frac{M_0}{a\omega^\beta \tilde{q}_0^2}\right) = \frac{x_0}{p_0}(1 + \frac{M_0}{a\omega^\beta p_0})$. Clearly, if $\kappa > 1$, the null-space of DF^+ is empty in $\ell_\infty(\mathbb{Z}_+)$. Hence $0 \notin \Gamma^+$ and $n^+ = 0$ are sufficient for invertibility of DF^+ in this case. Both conditions imply $\kappa > 1$.

At the other stationary equilibrium, $Q = \kappa$, the coefficients are $\gamma_{-1}^+ = a\omega^\beta$ and $\gamma_0^+ = -\frac{(1-a)\omega^\alpha}{\kappa^2}$, so $\Phi^+(\lambda) = -\gamma_0^+(1 - \kappa e^{-\lambda \mathrm{i}})$. Here $0 \notin \Gamma^+$ and $n^+ = 0$ are equivalent to $\kappa < 1$. The null space of DF^+ consists of x such that x_0 is the same function of x_1 as above, but for $t \geq 1$, $a\omega^\beta x_{t+1} = \frac{(1-a)\omega^\alpha}{\kappa^2} x_t \Rightarrow x_{t+1} = \frac{1}{\kappa} x_t$. Hence it is empty in ℓ_∞ iff $\kappa < 1$.

In sum, as is evident from table 2 below, the conclusions about determinacy are sensitive to the specification of the store of value, cf. section 5 for discussion.

TABLE 2. Determinacy of stationary equilibria in the example economy with truncated past as a function of parameters and the store of value at the time of the "shock". ('D' stands for the indication of determinacy of equilibrium, while 'N' means that the equilibrium is not determinate.)

Store of value	Equilibria			
	GRE		BE	
	$\kappa < 1$	$\kappa > 1$	$\kappa < 1$	$\kappa > 1$
Real balances	D	N	N	D
Fiat	N	D	D	N

4.6. **Related result by Kehoe and Levine (1985).** The approach taken in [14] is based on linearizing the equilibrium difference equation for every t using an implicit function theorem. We use economy $\mathscr{E}_+^2$ to illustrate and to compare their approach to ours. The system of equilibrium difference equations in [14] is approximated at a stationary equilibrium as follows:

$$\begin{pmatrix} p_t \\ p_{t+1} \end{pmatrix} = G \begin{pmatrix} p_{t-1} \\ p_t \end{pmatrix} \quad \forall t \in \mathbb{Z}_+ \tag{72}$$

Using calculations in [14] and the specification of our example economy, at the golden rule equilibrium $p_t = 1$, $G = \begin{pmatrix} 0 & 1 \\ -\kappa & \kappa + 1 \end{pmatrix}$. The eigenvalues of the matrix are 1 and κ. Thus, if $\kappa < 1$, there is one eigenvalue of matrix G that is less than unity, which is higher than the number of physical goods in each period (1) minus 1, hence this corresponds to case (iii) in [14, p.445]:

> "In this case there is a continuum of locally stable paths. The steady state is indeterminate. Comparative statics is impossible and perfect foresight implausible."

As for the other stationary equilibrium, in which prices grow at rate β, so that $\frac{p_{t+1}}{p_t} = \beta$, the corresponding matrix G is $\begin{pmatrix} 0 & 1 \\ -\beta\kappa & \beta\kappa + 1 \end{pmatrix}$. In our notation, the ratio of prices in this equilibrium is κ, so $\beta = \kappa$, implying that the eigenvalues of G are 1 and κ^2. Hence, again, if $\kappa < 1$ indeterminacy should be present according to [14].

Most likely, the discrepancy in the results (cf. table 2) stems from the differences in our basic definitions. The notion of determinacy in [14] is based on an asymptotic convergence of the perturbed path to *some* steady state, i.e., some form of stability. In contrast, we require the perturbed path to remain in a neighbourhood of the baseline which is not necessarily stationary. We view an equilibrium variable as a full path and not just its component at some point in time t. The path is an element of a metric space, for which the IFT is formulated. In our set up the metric is the lowest upper bound of the distance between the corresponding components of the two infinite sequences, hence the perturbed path might not converge to the baseline.

4.7. **Indeterminacy in $\mathscr{E}^2$.** Surprisingly, none of the increasing equilibria, cf. section 4.1, in the example economy with infinite past is determinate, no matter what the parameters are. As in the symmetric case, $0 \in \Gamma \iff \kappa = 1$, so in this case the equilibrium is indeterminate.

Assume $\kappa < 1$. Then at $t \to +\infty$ the equilibrium price path converges to 1, i.e., to the GRE. Hence, following calculations in section 4.5.1, $\Phi^+(\lambda) = \gamma_0^+ + \gamma_1^+ e^{\lambda i} = \gamma_0^+(1 - \kappa e^{\lambda i})$, and hence $n^+ = 0$. At $t \to -\infty$ the equilibrium price path converges to κ, so $\Phi^-(\lambda) = \gamma_0^- + \gamma_1^- e^{\lambda i} = \gamma_0^-(1 - \frac{1}{\kappa} e^{\lambda i})$, and therefore, following the argument in section 4.5.1 applied to $\frac{1}{\kappa}$, $n^- = 1$. Hence, by theorem 2.(i), $n = n^- - n^+ \neq 0$, implying DF is not invertible.

If $\kappa > 1$, the same argument applies with the definitions of Φ^+ and Φ^- switched. We depict Γ^+ and Γ^- on the complex plane in figure 4.

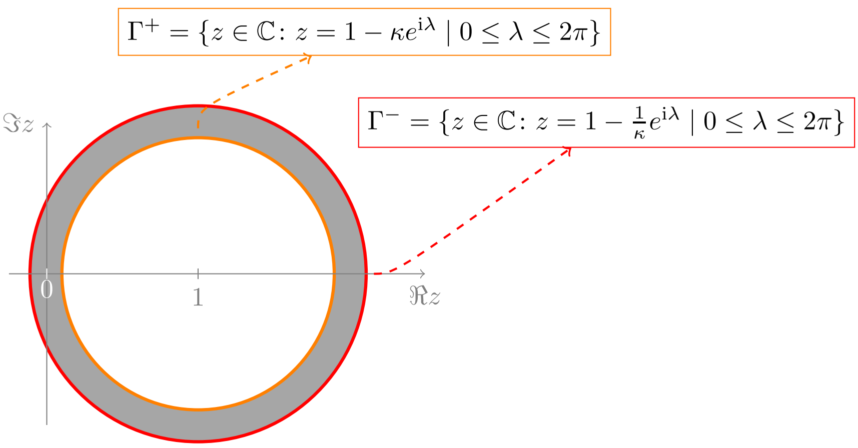

FIGURE 4. 0 is always inside either Γ^+ or Γ^-, but not both, so either $n^- = 1$ (if $\kappa < 1$) or $n^+ = 1$ (if $\kappa > 1$). Hence $n = n^- - n^+$ is never zero, thus DF is not invertible.

Indeterminacy of these equilibria is not surprising. Indeed, each of them is associated with some real number v: $q_t = \frac{1+v\kappa^{t+1}}{1+v\kappa^t}$, cf. [6] and fig. 1. Hence any equilibrium with some v has another one with $v + \epsilon$ in its arbitrarily small ℓ_∞-neighbourhood for small enough $\epsilon > 0$. Thus none is locally unique.

4.8. **Nominal indeterminacy.** The choice of equilibrium variable has implications for determinacy. If one considers two equilibria with different non-normalized prices as "different", then indeterminacy is present for all parameter values even in a steady state of an eternal economy. We use economy $\mathscr{E}^2$ to illustrate.

Following [6], the equilibrium conditions can be written in terms of prices, p:

$$F_t(p) = p_t - \sum_{k=-\infty}^{\infty} g_{t-k} p_k = 0, \ \forall t \in \mathbb{Z} \tag{73}$$

$$g(s) = \begin{cases} \frac{a\omega^\beta}{\omega^\alpha+\omega^\beta}, & s=-1; \\ \frac{(1-a)\omega^\beta + a\omega^\alpha}{\omega^\alpha+\omega^\beta}, & s=0; \\ \frac{(1-a)\omega^\alpha}{\omega^\alpha+\omega^\beta}, & s=1; \\ 0, & \text{otherwise} \end{cases} \tag{74}$$

F, as before, can be defined on $\ell_\infty(\mathbb{Z})$, for example. In our notation, evaluated at any equilibrium, the derivative of F has the following coefficients:

$$\gamma_{-1}^{\pm} = -\frac{a\omega^\beta}{\omega^\alpha+\omega^\beta}, \quad \gamma_0^{\pm} = 1 - \frac{(1-a)\omega^\beta + a\omega^\alpha}{\omega^\alpha+\omega^\beta}, \quad \gamma_1^{\pm} = -\frac{(1-a)\omega^\alpha}{\omega^\alpha+\omega^\beta} \tag{75}$$

This implies that $\Phi^+(0) = \Phi^-(0) = 0$, and hence $0 \in \Gamma$, implying by theorem 2 that the derivative is not invertible. This is true for any parameters $\omega^\alpha, \omega^\beta, a$ for which coefficients $\gamma_s^{\pm}$ are well defined.

This example suggests, in particular, that formulating equilibrium equations in terms of real variables, cf. [8], e.g., quantities (allocations), has the advantage of avoiding nominal indeterminacy, cf. [14] for the discussion.

5. Discussion and Conclusions

For an OLG economy with a single physical good we have formulated necessary and sufficient conditions for determinacy of stationary equilibria for OLG economies with truncated or infinite past. With multiple physical goods and *asymptotic* stationarity of equilibria, the conditions for the determinacy remain necessary. They are sufficient up to a small perturbation of the derivative of the equilibrium equations with respect to (relative) prices. Asymptotically stationary equilibria include price paths that converge to two different steady states at either end of the time line. The criterion for equilibrium determinacy in the economy with an infinite past are less stringent if the equilibrium path converges to the same stationary equilibrium in both directions, i.e., when $t \to \pm\infty$.

The criterion is formulated in terms of the asymptotic behavior of the equilibrium equation system and thus is independent of the non-stationary components of the system.

One advantage of our approach is that it corresponds to the most direct generalization of the common definition of comparative statics: the appropriate IFT implies that in a neighborhood of any determinate equilibrium there is a unique equilibrium for each combination of its perturbed parameters, cf. also [20], [12]. In contrast, if the IFT is applied a countable number of times (for each period $t \in \mathbb{Z}_+$), assuring that such a neighborhood does not collapse to a single point requires more assumptions and a separate proof. Further, given our choice of the metric space for the equilibrium, the price growth sequence of the perturbed equilibria can remain in a neighbourhood of the baseline equilibrium without converging to a stationary path, which is required, e.g., in [14] and [9]. Thus our test for equilibrium determinacy does not rule out policy reforms that yield persistent changes in an equilibrium path.

The same approach can be used more broadly. The crucial property of equations characterizing an equilibrium in an OLG model is limited forward and backward dependence. In other words, an equation at time t contains a finite number of elements of the infinite sequence determined in equilibrium. Indeed, the market clearing at any t is a function of $\bar{\tau} - 1$ prices before and $\bar{\tau}$ prices following and including t, for some finite length of the longest individual life-span $\bar{\tau} > 1$. This implies that the sum of absolute values of the asymptotic coefficients $\sum_{s=-\infty}^{\infty}|\gamma_s|$ is finite, cf. (15). Thus, the theorem can be applied to any dynamic system with limited recall and foresight whose derivative is reduced to a linear map like DF (on ℓ_∞), cf. equation (4), with an arbitrary interpretation of coefficients $(\gamma_{t,k})_{t,k\in\mathbb{Z}_+}$ that satisfy asymptotic stationarity. The explicit form of coefficients $\gamma_{t,k}$ in our case, cf. equation (5), was used only to demonstrate that tastes, demographic composition, longevity and endowments can affect both the conditions for determinacy and comparative statics.

Illustrating our theorem for a simple economy $\mathscr{E}_+^2$, we see that a particular interpretation for the storage of value in an economy with truncated past can lead to different, in fact opposing conditions for invertibility, cf. table 2. There is too much freedom in interpreting what happens at the beginning of times. This might indicate that the equilibrium in our *deterministic* model is inconsistent with *surprise policies*, by definition. In a competitive equilibrium, the price path is known to all the agents and all the allocations are chosen accordingly. Any policy variation is fully foreseen and factored into the prices. To tackle this question one might need to adopt an explicit way of accounting for surprises in the presence of perfect foresight and rational expectations, cf., e.g. [16].

Appendix A. Proof of the main result

Proof of theorem 2. (i): Recall that DF is a linear operator, defined on the set of infinite sequences of real L-dimensional vectors. First, we establish several auxiliary results for operators that act on sequences of *complex* L-dimensional vectors.

A.1. **Definitions and helpful results.**

A.1.1. *Fredholm operators and their properties.*

Definition 3. (i) Let U, V be $\mathbb{K}$-vector spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and $T\colon U \to V$ be a linear map.

Then $\mathsf{Ker}\,T = \{u \in U \mid Tu = 0\} \subset U$, $\mathsf{Ran}\,T = \{Tu \mid u \in U\} \subset V$ and $\mathsf{Coker}\,T = V/\,\mathsf{Ran}\,T$.

(ii) T is *Fredholm* if $\mathsf{Ker}\,T$ and $\mathsf{Coker}\,T$ are finite dimensional. Index T is then

$$\mathsf{index}\,T = \mathsf{dim}\,\mathsf{Ker}\,T - \mathsf{dim}\,\mathsf{Coker}\,T.$$

(iii) An essential spectrum of a bounded linear operator T mapping a complex Banach space into itself is the set $\{z \in \mathbb{C}\colon T - zI \text{ is not Fredholm}\}$, where I is the identity operator.

Next theorem is a summary of the known properties of Fredholm operators that will be useful in our proof.

Theorem 3. *Assume that U, V are $\mathbb{K}$-Banach spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\mathcal{L}(U,V)$ is a set of bounded linear maps, $T \in \mathcal{L}(U,V)$ is a Fredholm map.*

(i) Then there exists $\varepsilon > 0$ such that for every $B \in \mathcal{L}(U,V)$, with $\|B\| < \varepsilon$, $T+B$ is Fredholm and $\mathsf{index}\,T = \mathsf{index}(T+B)$. In particular, if $\mathsf{index}\,T \neq 0$ then $T + B$ is not invertible whenever $\|B\| < \varepsilon$.

(ii) If $B \in \mathcal{L}(U,V)$ is compact then $T + B$ is Fredholm and

$$\mathsf{index}\,T = \mathsf{index}(T + B)$$

(iii) If $\mathsf{index}\,T = 0$ then there exists $B \in \mathcal{L}(U,V)$ such that $T + \varepsilon B$ is invertible for every sufficiently small $\varepsilon > 0$.

A.1.2. *Complexification.* For a real vector space V we form its complexification $V_{\mathbb{C}}$ as follows. Consider first a real vector space $V \oplus V$. With the multiplication by complex numbers given by

$$(x+iy)(u \oplus v) = (xu - yv) \oplus (xv + yu)$$

it becomes a complex vector space which we denote $V_{\mathbb{C}}$. We denote an element $u \oplus v \in V \oplus V = V_{\mathbb{C}}$ by $u + iv$.

Let U be another real vector space and T be $\mathbb{R}$-linear map $U \to V$. Denote by $T_{\mathbb{C}}$ $\mathbb{C}$-linear map $U_{\mathbb{C}} \to V_{\mathbb{C}}$ given by

$$T_{\mathbb{C}}(u + iv) = Tu + iTv.$$

Theorem 4. *(i) T is Fredholm if and only if $T_{\mathbb{C}}$ is Fredholm and $\operatorname{index} T = \operatorname{index} T_{\mathbb{C}}$ (where we use real and complex dimensions to define index for T and $T_{\mathbb{C}}$ respectively).*

(ii) T is invertible if and only if $T_{\mathbb{C}}$ is invertible.

Proof. It follows from the definitions that $\operatorname{Ker} T_{\mathbb{C}} = \operatorname{Ker} T + i \operatorname{Ker} T$, $\operatorname{Ran} T_{\mathbb{C}} = \operatorname{Ran} T + i \operatorname{Ran} T$. Thus, $\dim_{\mathbb{C}} \operatorname{Ker} T_{\mathbb{C}} = \dim_{\mathbb{R}} \operatorname{Ker} T$, $\dim_{\mathbb{C}} \operatorname{Coker} T_{\mathbb{C}} = \dim_{\mathbb{R}} \operatorname{Coker} T$. Both statements then follow. Indeed, T is Fredholm when $\dim_{\mathbb{R}} \operatorname{Ker} T$, $\dim_{\mathbb{R}} \operatorname{Coker} T < \infty$. But this is equivalent to $\dim_{\mathbb{C}} \operatorname{Ker} T_{\mathbb{C}}$, $\dim_{\mathbb{C}} \operatorname{Coker} T_{\mathbb{C}} < \infty$, i.e. $T_{\mathbb{C}}$ being Fredholm. Then

$$\operatorname{index} T = \dim_{\mathbb{R}} \operatorname{Ker} T - \dim_{\mathbb{R}} \operatorname{Coker} T = \dim_{\mathbb{C}} \operatorname{Ker} T_{\mathbb{C}} - \dim_{\mathbb{C}} \operatorname{Coker} T_{\mathbb{C}} = \operatorname{index} T_{\mathbb{C}}$$

Invertibility of T is equivalent to $\dim_{\mathbb{R}} \operatorname{Ker} T = \dim_{\mathbb{R}} \operatorname{Coker} T = 0$. But this is equivalent to $\dim_{\mathbb{C}} \operatorname{Ker} T_{\mathbb{C}} = \dim_{\mathbb{C}} \operatorname{Coker} T_{\mathbb{C}} = 0$, which is equivalent to invertibility of $T_{\mathbb{C}}$. ∎

A.2. **Decomposition of the derivative operator.** Asymptotic stationarity implies an operator (like DF) can be decomposed into a stationary and non-stationary components. The latter is compact.

Let $1 \le p \le \infty$. Let $\mathcal{B} = l_p(\mathbb{Z})^L$ be the Banach space of p-summable sequences of *complex* L-vectors.

Proposition 2. *Consider an operator A defined on $\mathcal{B}$ given by $(Ax)_t = \sum_{k=-\infty}^{\infty} \alpha_{t,k} x_k$, where $\alpha_{t,k} \in \mathbb{C}^{L\times L}$ are $L \times L$ matrices such that (i) there is a number $N\colon |t-k| > N \Rightarrow \alpha_{t,k} = 0$; (ii) $\lim_{t\to\pm\infty} \alpha_{t,s-t} = 0$. Then A is compact.*

Proof. By definition of a compact operator it is sufficient to find a sequence of finite-rank operators that converge to A. To construct such a sequence consider an $2m \times 2m$ matrix G^m such that $(G^m)_t = (\alpha_{t,k})_{k=[t-m]_+}^{2t-[m-t]_+}$. Then $G^m \xrightarrow[m\to\infty]{} A$ in the operator norm, by the two assumptions. ∎

Proposition 3. *Consider operator $T_{\mathbb{C}}\colon \mathcal{B} \to \mathcal{B}$ given by $(T_{\mathbb{C}}x)_t = \sum_{k=-\infty}^{\infty} \gamma_{t,k} x_k$, where $\gamma_{t,k} \in \mathbb{C}^{L\times L}$ such that (i) there is a number $N\colon |t-k| > N \Rightarrow \gamma_{t,k} = 0$; (ii) $\lim_{t\to\pm\infty} \gamma_{t,s-t} = \gamma_s^{\pm}$.*

Let $\mathcal{B}^{\pm} = l_p(\mathbb{Z}_{\pm})^L$, $\mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \ge 0\}$, $\mathbb{Z}_- = \{n \in \mathbb{Z} \mid n < 0\}$.

Define T^+ and T^- acting on $\mathcal{B}^+$ and $\mathcal{B}^-$ respectively using coefficients $\gamma_s^{\pm}$ from the definition of $T_{\mathbb{C}}$:

$$(T^+x)_t = \sum_{k=0}^{\infty} \gamma_{t-k}^{+} x_k,\ n \in \mathbb{Z}_+ \tag{76}$$

$$(T^-x)_t = \sum_{k=-\infty}^{-1} \gamma_{t-k}^{-} x_k,\ n \in \mathbb{Z}_- \tag{77}$$

Then

(i) Essential spectrum of $T_{\mathbb{C}}$ *is the union of essential spectra of* T^+ *and* T^-.
(ii) $T_{\mathbb{C}}$ *is Fredholm if and only if both* T^+ *and* T^- *are, and in this case*

$$\mathsf{index}\, T_{\mathbb{C}} = \mathsf{index}\, T^+ + \mathsf{index}\, T^-.$$

Proof. The space $\mathcal{B}$ decomposes as a direct sum $\mathcal{B} = \mathcal{B}^+ \oplus \mathcal{B}^-$. It follows from proposition 2 that $T_{\mathbb{C}} - (T^+ \oplus T^-)$ is compact. The conclusions now follow by the properties of essential spectra and Fredholm operators, cf. [17]. ■

A.3. **Properties of the stationary components.** Operators T^+ and T^- above are Toeplitz operators and determination of their spectra is given in the work of Gokhberg and Krein.

Theorem 5 (Gokhberg and Krein (1958))**.** *For operators* T^+ *and* T^- *from proposition 3 define*

$$\Phi^+(\lambda) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} e^{\mathrm{i}\lambda k}\gamma_k^+, \quad \Phi^-(\lambda) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} e^{\mathrm{i}\lambda k}\gamma_k^- \tag{78}$$

$$\Gamma^\pm = \{z \in \mathbb{C} \mid \mathsf{det}[\Phi^\pm(\lambda) - zI] = 0 \text{ for some } 0 \le \lambda \le 2\pi\} \tag{79}$$

Then the essential spectrum of $T^\pm$ *is* $\Gamma^\pm$.

Assume that $0 \notin \Gamma^\pm$. *Let* $n^\pm$ *be the winding number of the curve* $\mathsf{det}\,\Phi^\pm(\lambda)$, $0 \le \lambda \le 2\pi$, *around* 0*:*

$$n^\pm \stackrel{\text{def}}{=} \frac{1}{2\pi \mathrm{i}} \int_0^{2\pi} \frac{(\mathsf{det}(\Phi^\pm(\lambda)))'}{\mathsf{det}(\Phi^\pm(\lambda))} d\lambda$$

Then

$$\mathsf{index}\, T^+ = -n^+$$

$$\mathsf{index}\, T^- = n^-$$

Combining proposition 3 and theorem 5 we obtain

Theorem 6. *The operator* $T_{\mathbb{C}}$ *defined in proposition 3 is Fredholm if and only if* $0 \notin \Gamma = \Gamma^+ \cup \Gamma^-$. *In this case*

$$\mathsf{index}\, T_{\mathbb{C}} = -n^+ + n^-$$

A.4. **Recovering invertibility properties of** DF**.** Consider now operator T defined on the space of *real* uniformly bounded vector sequences, $\ell_\infty^L(\mathbb{Z})$.

$$(T_t)(x) = \sum_{k=-\infty}^{+\infty} \gamma_{t,k} x_k, \text{ where } \gamma_{t,k} \in \mathbb{R}^{L\times L}, \tag{80}$$

$$\gamma_{t,k} = 0, \text{ if } |k - t| \le \bar{\tau} - 1 \tag{81}$$

$$\gamma_{t,s-t} \xrightarrow[t\to+\infty]{} \gamma_s^+, \quad \gamma_{t,s-t} \xrightarrow[t\to-\infty]{} \gamma_s^-, \tag{82}$$

Note that $T = DF$ defined in the text, cf. eq. (4), corresponding to an asymptotically stationary equilibrium. Combining the properties of Fredholm operators, theorem 3, the index of the linear operator with asymmetric stationary components, theorem 6 and the translation of these results for real bounded sequences, theorem 4, we get the following corollary which yields part (i) of theorem 2.

Corollary 1. *If* $0 \notin \Gamma$ *and* $n^+ \ne n^-$ *then* T *is not invertible, as well as all of its either sufficiently small perturbations or compact perturbations. If* $0 \notin \Gamma$ *and* $n^+ = n^-$, *then* T *is either invertible or can be made invertible by an arbitrarily small finite rank perturbation.*

Part (ii) of theorem 2 is similar and follows the same steps (using theorem 5 and proposition 3).

Part (iii) of theorem 2 follows by theorems 6, 4, definition of $n = n^- - n^+$ and the equality $n^+ = n^-$ in case of symmetry, $\gamma_s^- = \gamma_s^+$. ■

Proof of proposition 1. Follow the same steps as the proof of theorem 2, but omit the use of theorem 3 and proposition 2 and use a simpler version of theorem 5 (for $L = 1$ only), which gives the necessary and sufficient conditions for invertibility of the Toeplitz operator in this case, cf. e.g., [7], [15]. ■

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