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Transactions Loans, Intertemporal Loans, Variable Velocity, the Rates of Interest and Commodity Money: Part 1. Transactions Loans

Martin Shubik

Shuntian Yao

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

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Martin Shubik and Shintian Yao

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PART I. TRANSACTIONS LOANS

by

Martin Shubik and Shuntian Yao

1. INTRODUCTION

Several models of exchange are presented here to illustrate various conceptual problems in the microeconomic modeling of the velocity of money and the meaning of and cost of liquidity in an exchange economy without exogenous uncertainty.

The celebrated equation linking the amount of money M, its velocity V and the price level P and quantity of goods traded Q has the reassuring specificity of an equation in an elementary physics textbook:

$$MV = PQ. \tag{1}$$

The discussion here is not a direct continuation of the Cambridge-Cambridge debate. But rather the micro economic approach adopted here lays stress on the concept of a playable game with all the physical aspects of process spelled out.

No attempt is made at spurious realism. A series of well defined, but clearly overly simple models are considered in order to try to isolate and examine the complex of factors which go to make up the various aspects of velocity and liquidity.

We purposely consider the use of a commodity money to avoid any mysticism concerning its supply. Furthermore we take as an axiom of our system that all individuals are required to pay for spot transactions in the commodity money.

We believe that in mass anonymous markets, payment in lawful money, be it gold or paper minimizes the need for trust and paperwork. But a defense of the reasons for accepting this axiom is not of concern in this paper.

Several basic phenomena must be all eventually treated simultaneously. They are: (1) the meaning of enough money to achieve efficiency; (2) the meaning of the velocity of money; (3) the relationship between the velocity of money and enough money for efficient trade; (4) the time consumed in transacting and the sequencing of transactions; (5) the way markets clear; (6) the meaning of a transactions loan market; (7) the meaning of an intertemporal loan market; (8) the relationship between the transaction loan and intertemporal loan rates of interest; (9) the relationship between the rates of interest and enough money and last, but by no means least, (10) the influence of the physical properties of the goods and money trade on the analysis.

It appears that virtually all of the phenomena we wish to consider can be illustrated in models with only one or two time periods, hence in Part I of this investigation we limit our concerns to one period models and in Part II the invesdigation is extended to two period models.

Several paradoxical problems arise when we take a single period and subdivide it so that it contains several markets or rounds of transactions. In particular we note that stock and flow aspects of consumption and sequencing aspects of loan markets may influence considerably the analysis we propose.

In keeping with the division of difficulties we first consider single period exchange, summarize our results and extend them to cover single period exchange with many trading sessions. Then, in Part II, two period trade is considered.

For simplicity our remarks are confined to games with a continuum of traders.

2. THE ONE PERIOD EXCHANGE MODEL WITH ONE TRANSACTION

A one period strategic market game modeling exchange is a single simultaneous move game in strategic form where the moves selected by all players determine the market prices and the final holdings of all players. There are many variations of basic exchange which have been constructed (see Shubik, 1990a). But if we regard each of them as a playable game they all include a lag of the form shown in Figure 1.

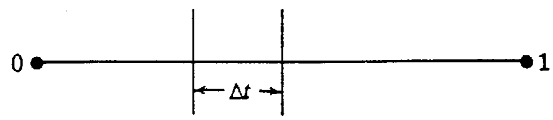

Figure 1
One period, one market trade

There is a single time period [0,1] and within that period there is a single trading session which starts at time t and ends at time $t+\Delta t$. At the start of that period individuals have put up goods and money, prices are formed and the markets are cleared and all individuals obtain goods that they have bought and money from the goods they have sold.[1] The utility function assumed is of the form:

$$U_i(x_1^i, x_2^i, \ldots, x_{m+1}^i) \tag{2}$$

where the $m+1^{st}$ good is the commodity money.

Implicit in this utility function is the assumption that no distinction is made between consumable goods and durable goods. Each individual i begins with a vector of resources $a^i = (a_1^i, a_2^i, \ldots, a_{m+1}^i)$ and ends up with a vector of resources x^i. The individual is then assumed to consume all of x^i. The possibility that individuals lose control over their resources in transit is regarded as not relevant. The distinction between a durable and a consumable is irrelevant. Stocks and flows are not distinguished.

2.1. Enough money

If all traders are all insignificant in size so that no single individual influences prices, and all goods are consumables then the noncooperative equilibria (NEs) of the game will coincide with the competitive equilibria of the related exchange economy provided that there is enough money to facilitate trade.

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[1]There are many variations of the float, or time between payment and receipt which can be modeled. For the purposes at hand we need to be specific and clear about one. Simplicity suggests that we assume that there is the same lag in both the delivery of money and all other goods.

The mathematical meaning of "enough money" is that the NEs are interior. This means that the constraint that the individual cannot spend more of the means of payment than he holds initially, is not binding on any individual's plans.

The condition for the existence of enough money is characterized both by the total amount of money in the system and by its distribution. It is possible that an economy may not have enough money no matter how it is distributed; it is also possible that a redistribution might give rise to interior solutions. These statements can be made precise and are illustrated by means of specific examples.

If there is enough money, but it is badly distributed it can be shown that a loan market will bring about efficiency at an interest rate of $\rho = 0$.

If there is not enough money in the economy, but loans are feasible there may be an active loan market with $\rho > 0$. The price of the loan reflects the shadow price of the capacity constraint caused by the need to pay in money for all transactions.

Depending upon the specifics of trade the need for money in trade will depend on the nature of the strategies employed. For example, if we use the game suggested by Dubey and Shubik (1978) a strategy s^i by a trader of type i is of the form (b^i, q^i) where $\sum b_j^i \le a_{m+1}^i$, all $b_j^i \ge 0$ and $0 \le q_j^i \le a_j^i$. A strategy consists of a vector of money bids, where the sum of the bids cannot exceed cash on hand, and an amount offered of every other commodity, where the amount offered cannot exceed one's initial endowment of the commodity.

We can now specify the three conditions concerning the sufficiency and distribution of money. Associated with the exchange economy E will be a set of competitive equilibria (CEs).

Enough money, well distributed

The game Γ associated with E will have enough money that is well distributed with respect to a CE k if:

$$\sum_{j=1}^{m} p_{jk} \max[0, (x_{jk}^i - a_j^i)] \le a_{m+1}^i$$

for all i, where p_{jk} are the prices at the k^{th} CE and x^i_{jk} is the final holding of good j by individual i at CE k.[2]

Enough money, badly distributed

The game Γ will have enough money that is badly distributed with respect to a CE k if:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_j k \max[0, (x^i_{jk} - a^i_j)] \leq \sum_{i=1}^{n} a^i_{m+1}$$

and for some i

$$\sum_{j=1}^{m} p_j k \max[0, (x^i_{jk} - a^i_j)] > a^i_{m+1}.$$

Not enough money

The game Γ will not have enough money with respect to a CE k if:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p_{jk} \max[0, (x^i_{jk} - a^i_j)] \geq \sum_{i=1}^{n} a^i_{m+1}.$$

When the money and all other goods are consumables and we make the assumption that the period is long enough they can all be consumed after trade this amounts to being able to "use one's money and eat it too". It is used as a store of value in exchange and then can be consumed.

2.2. Durables and consumables: stocks and flows

Even in a one period model with one trading session we can nevertheless make some distinctions between the types of goods. As a first approximation we may imagine that a consumable is approximated by point consumption, i.e. it can be costlessly stored through the period and then consumed at any speed the individual chooses, including in the limit being completely consumed at the point of time t. In contrast at the other extreme is the durable

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[2]In the game price is given by $\int b^i_j / \int q^i_j$. It is also important to note that the inequalities are specified for the instance where the individuals are assumed to either buy or sell; buying *and* selling in the same market would call for more money.

which yields a constant stream of services over which (except to not use them) the user has no control.

In essence the distinction comes down to one of strategic choice. One could imagine drinking a cup of tea as a constant flow over several hours while watching a movie. However the first order approximations have the tea consumed in a few minutes and the movie in an hour or two with a distinct utility loss if a ten minute segment is missed. Figure 2 shows the consumption distinction. Suppose that we use chocolates or cigarettes as money in an experimental game. They can be used in trade and then after accounts have been settled within the period the chocolates can be eaten or the cigarettes smoked. If instead the money were gold jewelry or transistor radios there would be a loss of utility for the period Δt if the money were used in trade. This implies that the distinction between a consumable or durable money is that one incurs no opportunity cost loss when used for money if transactions take time and the other does.

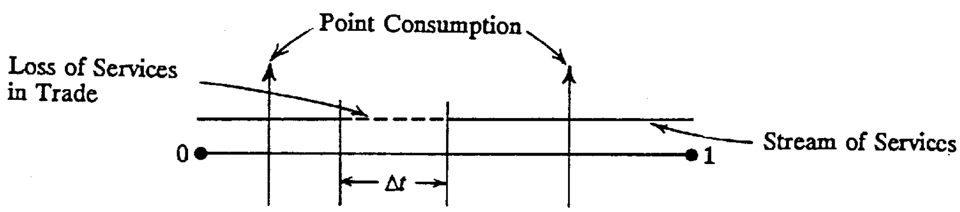

Figure 2
A consumable or durable money

2.3. Efficiency and the rate of interest

CASE 1: Consumable money. The results noted in Shubik (1990b) apply when the money is regarded as a (nondepreciating) consumable. These have been noted above in 2.1.

CASE 2: Durable money. When the money is durable and transactions take time even with enough money trade will not be efficient.[3] When there is not enough money the rate of

[3] It is important to distinguish between the feasible set for exchange in a game with a trading mechanism and the set when trading mechanisms are ignored. Unless the initial distribution is such that no trade is required the cooperative game can only attain the open set of distributions excluding the Pareto set, This can be done by depressing prices.

interest must reflect not only the capacity constraint, but also use foregone and depletion as the stream of remaining services diminish.

2.4. Depletion and enough money

Unless we postulate that a commodity money provides an infinite constant stream of services (like idealized gold in an infinite horizon game, or like fiat earning a constant rate of interest) then as time progresses in a finite horizon game the remaining worth of the durable approaches zero. Thus the purchasing power of the money will depend on the time location of the market. In Figure 1 the market starts at $t = .25$ and lasts to $t = .5$; if instead the market started at $t = .5$ and lasted to $t = .75$ the value of the durable would have diminished.

2.5. Does borrowing lending and repaying take time?

If we introduce a transactions loan market we must be explicit about when the loan starts, when it is to be paid back and whether the activities take time. Figure 3 shows a transactions loan market where the processing of the loan strategically freezes the use of the money so that for a period Δt_1 neither the borrower nor the lender get to use the money while the paperwork and the transfer of ownership takes place. Similarly at the other end there may be a Δt_2 of time lost.

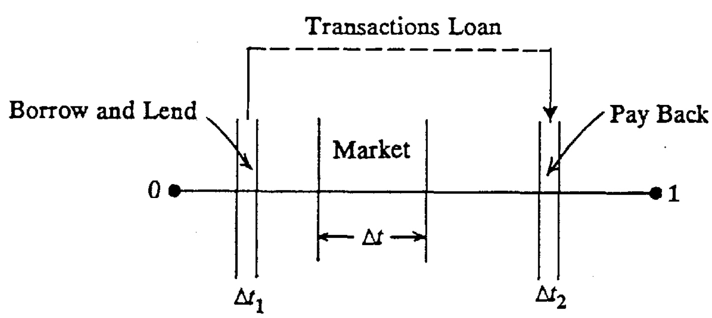

Figure 3
A transactions loan

We might argue that many financial transactions are so much faster than physical transactions that we can approximate them as point transactions.[4]

The simplest model for transactions borrowing is a line of credit at the bank which also receives payments thus we might model the transaction loan as taking place at precisely the start of the market and it is paid back as soon as the market clears.

2.6. What default rules are required with lending?

As soon as any form of lending is introduced some form of penalty rule is required to cover the possibility that the system reaches a state where a borrower cannot meet his commitments. This point has been covered elsewhere (Shubik and Wilson, 1977; Dubey and Shubik, 1978, 1979; Dubey, Geanakoplos and Shubik, 1988). Generally it can be handled by redefining the domain of the utility function so that it is defined on $\mathbf{R}^+_m \times \mathbf{R}$ rather than $\mathbf{R}^+_{m+1}$.

2.7 Does lending help the lenders?

A natural question to ask is does the existence of a loan market help the lenders? Intuitively the answer appears to be that it can go either way. Suppose that one type of trader has no money, but would have a high demand for good A and another type has a great amount of the monetary commodity and would be willing to lend some. But they also have a high utility for A. If they could all agree not to lend they might gain more from the lowered demand for A than they gain from their interest earning combined with a higher demand for A.

This observation be made completely formal by comparing two games which differ only in the presence of a transactions loan market.

EXAMPLE 1: (Lenders improve.) There are two types of traders, each consisting of a continuum of individuals of the same measure. There are three consumer goods, with the third one used as money. The utility functions for the two types of traders are, respectively:

[4] This is not true for initial credit investigations or complex mergers, but holds for the use of a valid credit card or payment by check.

$$u^1 = \ell n(1 + x^1) + 0.1\ \ell n(1 + y^1) + 0.1z^1$$
$$u^2 = 0.1\ \ell n(1 + x^2) + \ell n(1 + y^2) + 0.1z^2\ .$$

The initial endowments for the two types are (0, 9, 0) and (9, 0, 18) respectively.

First we can easily find the CE (unique) allocation ((9, 0, 0), (0, 9, 18)), the associated price vector is (1, 1, 1).

Now we look at the NE allocations when we consider this problem from the point of view of a strategic market game.

If there is no loan market for money, obviously what the traders can do is that the second type of traders buy all the second good from the first type of traders with half of their money. Then the final holdings will be (0, 0, 9) for a Type 1 individual and (9, 9, 9) for a Type 2 individual. The utility from the consumption will be $u^1 = .9$ and $u^2 = 0.1\ \ell n\ 10 + \ell n\ 10 + .9 = 2.0$. The zero trade yields $u^1 = .1$ and $u^2 = 1.9$.

On the other hand, if there is a loan market for money, and if the utility functions are modified as follows:

$$u^1 = \ell n(1 + x^1) + 0.1\ \ell n(1 + y^1) + f(z^1)$$
$$u^2 = 0.1\ \ell n(1 + x^2) + \ell n(1 + y^2) + f(z^2)$$

where f is defined by

$$f(z) = \begin{cases} 0.1z & \text{if } z \geq 0 \\ 100z & \text{if } z < 0 \end{cases}$$

(this implies avoidance of bankruptcy or default). Then the CE allocation ((9,0,0), (0,9,18)) can be achieved as an NE allocation of a strategic market game. The strategies are

(9, 0; 0, 9; 9, 0) for Type 1 traders, the meaning is: borrowing 9 units of money from Type 2 (promising to return the same amount after trading); sending 9 units of the second good for sale; sending 9 units of money for the purchasing of the first good;

(0, 9; 9, 0; 0, 9) for Type 2 individuals, the meaning is: lending 9 units of money to Type 1, sending 9 units of good 1 for sale; sending 9 units of money for the purchasing of good 2.

After trading, Type 1 individuals return 9 units of money to Type 2 so the consumption vectors are (9, 0, 0), (0, 9, 18) respectively. The utilities are $u^1 = \ell n\ 10 = 1$ and $u^2 = \ell n\ 10 + 1.8 = 2.8$ individuals of both types increase their gain.

In this example without lending there is some trade, with lending the CE can be achieved and all are better off.

EXAMPLE 2 (Lenders do worse). There are three types of traders, each consisting of a continuum of individuals. The first and the third type constitute [0, 1], the second [0, 2]. There are three consumer goods, with the third one used as money. The utility functions and initial endowments are, respectively:

$$u^1 = 20\sqrt{x^1} + y^1 + z^1, \ (0,\ 100,\ 550)$$

$$u^2 = x^2 + 20\sqrt{y^2} + z^2, \ (100,\ 0,\ 100)$$

$$u^3 = 200\sqrt{x^3} + y^3 + z^3, \ (0,\ 350,\ 0)$$

Consider this problem as an SMG. There are two different cases:

Without money loan market. An NE is given by the following strategies:

(0, 100; 100, 0) by Type 1

(50, 0; 0, 100) by Type 2

(0, 0; 100; 0) by Type 3

The allocation of goods is

((100, 0, 550), (50, 100, 50) (0, 250, 100))

The utility of a first type individual is then

$$u^1 = 20\sqrt{100} + 550 = 750.$$

Now assume there is a money loan market. The third type of traders will borrow 350 units of money from Type 1 at the beginning. Then the strategies are

(0, 50; 0, 150) for Type 1

(100, 0; 0, 100) for Type 2

(0, 350; 350, 0) for Type 3

Note that the price vector now is $\bar{p} = (2, 1, 1)$, and the allocation after return

⟨(25, 250, 350)(0, 100, 200)(175, 0, 0)⟩

The utility for Type 1 individual is

$$u^1 = 20\sqrt{25} + 350 + 250 = 700.$$

Thus we can see that the existence of loan market harms the lenders. On the other hand this NE with a loan market is also a CE of the exchange economy, the associated price vector is (2, 1, 1).

3. THE ONE PERIOD MODEL WITH MANY TRADING SESSIONS

We may consider the possibility of more than one trading session in a single period. This will give the traders the opportunity to use the same amount of money more than once. Thus the concept of velocity of money now appears as a strategic choice of the players.

Figure 4 shows a one period game with two trading sessions, described by [0,. 25] and [.5, .75]. If the money is consumable then this in essence doubles its amount. But if it is a durable the doubling is obtained at the price of losing its use for twice as long.

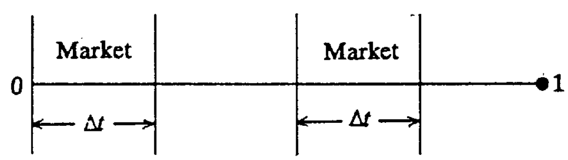

Figure 4
Two trading sessions

3.1. *k* trading sessions with a consumable money and no loans

If all goods including money are consumables, then the more trading periods there are the more trade can be effected with a given amount of money with no utility loss. Thus intuitively a fast enough velocity will guarantee that any amount of money is sufficient. Dubey, Sahi and Shubik (1990) have formalized this intuitive idea and established that any amount of money will be enough provided that its velocity is fast enough.[5] However for any specific exchange economy there will be some finite k^* that will be large enough for enough money.

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[5]This statement must be qualified with the observation that for a commodity money it is possible to select a money that will never be sufficient if its supply is increased, but with an increase in velocity as its the marginal utility is unaffected, eventually there will be enough money.

When we contemplate actual trade there appears to be an upper bound to the velocity of exchange. This bound is now more determined by bounds on the speed of human data processing, reading and verifying than it is on the speed of data processing.

We may now consider a formal model of exchange with k market sessions per period and all commodities are consumables. Suppose that the interval is divided into k trading sessions where each session is of length $\Delta t = 1/k$. This implies that the markets are always open. In actual human affairs there are few 24 hour markets, except for a few stores and international financial markets.[6] We can model k trading sessions in two slightly different symmetric ways. These are shown in Figures 5a and 5b. In Figure 5a there are four trading sessions each of which lasts for a quarter of a period.[7] In Figure 5b there are four trading sessions, each lasting for .125 of a period with an intervening period of rest.[8]

Figure 5
Continuous or frequent trade

We assume that any consumable is sufficiently storable that it lasts without deterioration for the whole period, hence it can be consumed or utilized anywhere. In particular in the model behind Figure 5a there are 4 intervals to be distinguished and in 5b there are 8 intervals. We select the model of Figure 5a. Then generally the number of intervals in which consumption and trade can take place are the same. We can express the utility function of an individual i as being of the form:

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[6]A hospital or power station may be open for servicing, but it is difficult to carry out financial transactions with them at 4.30 a.m.

[7]We need to specify if the intervals are open or closed. In Part II this will turn out to be of some importance, but here it can be finessed.

[8]Anyone who has tried to shop in a small French town at lunch time will recognize the plausibility of this model. Although the history of most old stock markets goes from operations of one or two days a week for a few hours to far more frequent trade.

$$u^i = U_i\left(\sum_{t=1}^{k} x^i_{1,t}, \sum_{t=1}^{k} x^i_{2,t}, \dots, \sum_{t=1}^{k} x^i_{m+1,t}\right) \tag{3}$$

where the $x^i_{j,t}$ are interpreted as consumption during the interval Δt and not the stock. We need a different symbol to indicate the stock of any good. A strategy for an individual i given k trading sessions is a vector of $3km + k$ dimensions of the form:

$$(b^i_{1,1}, q^i_{1,1}, x^i_{1,1}, \dots, b^i_{m,k}, q^i_{m,k}, x^i_{m,k}; x^i_{m+1,1}, \dots, x^i_{m+1,k}) \tag{4}$$

where $z^i_{j,t}$ is the stock of item j at the start of interval t, held by i. We have the following conditions on the moves:

$$\begin{gathered} z^i_{j,1} = a^i_{j,1} \\ z_{j,t} = z^i_{j,t-1} - q^i_{j,t-1} + b^i_{j,t-1}/p_{j,t-1} - x^i_{j,t-1} \\ \text{where } 0 \le q^i_{j,t-1} \le z^i_{j,t-1} \end{gathered} \tag{5}$$

and

$$\begin{gathered} 0 \le b^i_{j,t} \\ \sum_{j=1}^{m} b^i_{j,t} \le z^i_{m+1,t} \\ 0 \le x^i_{j,t} \le z^i_{j,t} \\ 0 \le x^i_{j,t} \le z^i_{j,t} - q^i_{j,t} + b^i_{j,t}/p_{j,t} \end{gathered} \tag{6}$$

Implicit in this formulation is that the sequencing of moves is bid, offer then consume.[9]

This model is a straightforward modification of the Dubey-Shubik (1978) bid offer model.

THEOREM 1. *Assume there are n types of traders with each type consisting of a continuum* [0, 1] *of individuals. The utility function of any trader is* C^1, *strictly concave and increasing in every variable. Then there exist at least one type-symmetric equilibrium.*

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[9]Implicitly assumed in this formulation is that you cannot consume the item you bought in the same period as it is purchased. Somewhat different inequalities based on ending inventories would permit individuals to consume the new purchases immediately at the point of receipt.

PROOF. We do not appeal to the CE existence theorem here, since we want our proof to be valid for the model in the next section where a durable money is used and there is a loss of value during the trade.

Consider first an ε-perturbed game at the beginning as in the usual case. We first look at the so-called percentage-strategies, i.e., strategies of i look like

$$= (\beta^i_{11}, \rho^i_{11}, \kappa^i_{11}, \ldots, \beta^i_{mt}, \rho^i_{mt}, \ldots, \kappa^i_{mt}; \kappa^i_{m+1,1}, \kappa^i_{m+1,2}, \ldots, \kappa^i_{m+1t})$$

where $\Gamma^i_{j\tau}$ is the percentage of money sent for bidding of good j, $\rho^i_{j\tau}$ the percentage of good j sent for sale, and $x^i_{j\tau}$ the percentage of good being consumed after trade, in trade session τ. These figures must satisfy the following constraints:

$$\beta^i_{j\tau} \geq 0, \ \sum_j \beta^i_{j\tau} \leq 1, \ \ 0 \leq x^i_{j\tau} \leq 1 \ .$$

And the relation between a percentage strategy and a quantity strategy is given by

$$z^i_{j1} = a^i_j, \ z^i_{m+11} = a^i_{m+1}$$

$$b^i_{j\tau} = \beta^i_{j\tau} z^i_{m+1,\tau}, \ q^i_{j\tau} = \rho^i_{j\tau} z^i_{j\tau}, \ x^i_{j\tau} = \kappa^i_{j\tau}(z^i_{j\tau} - q^i_{j\tau} + b^i_{j\tau}/p^i_{j\tau})$$

$$x^i_{m+1,\tau} = \kappa^i_{m+1,\tau}\left(z^i_{m+1,\tau} = \sum_j b^i_{j\tau} + \sum_j q^i_{j\tau}/p_{j\tau}\right) \ \ (j = 1, \ldots, m; \ i = 1, \ldots, n) \ .$$

It is obvious that the set of percentage-strategy is compact and convex.

Assume that a type-symmetric percentage strategy profile σ is given. Then k positive price vectors are then determined. Let us consider the best response of an individual of type i. Due to the compactness of the strategy set, there is a strategy σ^{*i} for him to maximize his utility.

We claim that this optimal percentage strategy of i is unique. In fact if there are two different percentage strategies σ^i and $\bar{\sigma}^i$ are all optimal for i, there corresponding quantity strategies s^i and $\bar{s}^i$ are then all optimal. Note that $s^i \neq \bar{s}^i$. Let $\bar{\bar{s}}^i = (s^i + \bar{s}^i)/2$. Due to the linearity in (5) and (6), $\bar{\bar{s}}$ is feasible, and we know from $\bar{\bar{x}}^i_{j\tau} = (x^i_{j\tau} + \bar{x}^i_{j\tau})/2$ $(j = 1, \ldots, m+1)$ and the strict concavity of u^i that $\bar{\bar{s}}^i$ must give a higher utility to i than s^i and $\bar{s}^i$. Let $\bar{\bar{\sigma}}^i$ be the corresponding percentage strategy of $\bar{\bar{s}}^i$. Then $\bar{\bar{\sigma}}^i$ is better than either of σ^i or $\bar{\sigma}^i$. This contradiction shows that the optimal σ^{*i} is unique. What is more, the σ^{*i} must be the same for every individual of type i, and the mapping $\sigma \rightarrow \sigma^{*i}$ is continuous.

The type-symmetric mapping $\sigma \to \sigma^{*i}$ induces a continuous mapping from $\Sigma = \mathop{\mathrm{X}}_{i=1}^{n} \Sigma^i$ into Σ itself, with Σ^i being the percentage-stategy set of a representative of type i. Since Σ is convex and compact, by the Brouwer's Fixed Point Theorem there is a fixed point σ^*, which gives the percentage-strategy Nash equilibrium for the perturbed bame.

A percentage-strategy NE of the original game can be obtained by a limit process. Since in total there is a continuum of traders, this percentage-strategy equilibrium can be directly translated into a quantity-strategy equilibrium. Q.E.D.

3.2. *k* trading sessions with a durable money and no loans

A perfect consumer durable yields value or utility to its owner continuously and requires no maintenance. The model in 3.1 had everything as a storable consumable. As a good approximation the consumable money in an experimental game could be chocolate bars. In this model all goods except money are consumables and money is a consumer durable. Thus all trade (if it involves a finite amount of time) must involve a utility loss. We may assume that the utility functions are of the form:

$$u^i(x_0^i, x_1^i, \ldots, x_m^i)$$

where x_j^i $(j = 1, \ldots, m)$ is a function of the amounts of commodity j *consumed* in each of the k periods, and x_0^i is a function of the amounts of money *held* in each of the k trading periods. Thus:

$$x_j^i = f_j^i(x_{j1}^i, \ldots, x_{jk}^i) \quad (j = 0, 1, \ldots, k)$$

where f_j^i is concave, and increases in every variable. The specific shape poses an empirical question concerning the valuation of availability of durables and the complementarities among them.

We could call the time value of loss of alternative use of money tied up in trade a float loss. But that word is more specifically used to refer to money in the bank clearing system. The loss here (which includes the float) is the overall transactions loss of any alternative use of money while it is tied up in trade.

However if we are accounting for loss of alternative use for the durable money in a more complex models with other durables we should also account for the loss of use of every other durable which yields continuous services.

THEOREM 2. *Under the same assumptions on the set of traders and functions u^i, there exists at least one NE for the game of k trading sections with a consumer durable money.*

PROOF. According to our assumptions on the u^i and the f_j^i, the utility of individual i still depends on the x_{jt}^i in the same manner as in Section 1. So the proof of Theorem 1 also applied here. Q.E.D.

Concentrating on the velocity of money we select a simple example for purposes of illustration with three goods where the commodity utilized as the money is a durable which provides utility or value only when it is not employed in exchange. The other two commodities are also durables with the same delivery loss of time as the money in their trade.

There are two types of traders, all with the same utility function. Each has a utility function of the form:

$$\frac{1}{k}\left(2\sum_{t=1}^{k}\sqrt{x_t y_t} + \alpha\sum_{t=1}^{k} z_t\right)$$

where $t = 1, ..., k$, the x, y and z are the amount of the durables held at time t for use other than trade and the α is a parameter.

Assume that traders of type 1 have an initial endowment of $(2, 0, A)$ and those of type 2 have $(0, 2, A)$. By inspection the competitive equilibrium (CE) solution is $2\sqrt{(1)(1)} + A = 2 + A$ with the price being $(1, 1, 1)$ and the final endowments for traders of each type $(1, 1, A)$.

The amount of money required to finance trade is 1 for each trader if there is only one round of trade.

As the example is so simple it can be solved without developing the considerable notation required for the general case. If $A \geq 1$ and $k > 1$ then all traders trade once in the first period and the payoff to each is $k/(k-1)(2 + A)$. If $A \leq 1$ and $k > 1$ the traders trade using all the money in the earliest periods until they obtain the right proportions of the first two commodities. If they are unable to do so by the k-1st period they stop. There will never be trade in period k because money if spent is worthless in the last period.

We compare two transparent cases with the same data as above. We assume that there are two trading sessions available in the single period.

By inspection when the endowments are (2, 0, 1) and (0, 2, 1) and $\alpha = 1$, efficient proportions can be established in the first round and the payoffs are $2 + 1/2 = 2.5$. This is not as high as in the general equilibrium solution which yields 3. This is because the money yields consumption value for only one half of the period.

When the endowments are (2, 0, 1/2) and (0, 2, 1/2) the individuals will only trade in the first period. The equilibrium yields (1.3106, .6894, .5) and (.6894, 1.3106, .5) with prices of (.725268, .725268, 1), the payoff is 1.90688 instead of the general equilibrium payoff of 2.5. This difference is the foregone use of .5 of a unit of money for half of the period, and the effect of the trading constraint caused by there not being enough money.

If it were feasible to trade for k sessions per period where $k > 3$ then the equilibrium with velocity 2 dominates that with velocity of 1 and as k becomes large the payoff approaches that of the CE as the foregone losses due to trading time approach zero. But this implies that both goods and money can be moved arbitrarily fast through markets.

The parameter α was introduced in the utility function so that a trivially easy sensitivity analysis can be performed to see what might happen if "the worth of money" were to increase or decrease. Setting $\alpha = 2$ or .5 doubles or halves the utility of money. For a fixed amount of money, say .5 these changes tend to decrease or increase velocity. In a multiperiod model a change in the rate of interest has the effect of making money more or less valuable. An increase in the rate of interest should increase the value of money, decrease borrowing and increase velocity.

3.3 k trading sessions with a consumable money and loans

We know that with enough trading periods there will always be enough money and hence a CE can be attained as an NE. If there are not enough trading sessions either the money will be enough but badly distributed or the money will not be enough. In the first instance a loan market should help to achieve efficiency.

Example 1 in 2.7 above shows that for efficiency one round of trade with a loan market is sufficient, otherwise two rounds are needed. The rate of interest is zero.

Example 3 below shows an instance with two rounds of trading where the presence of a loan market can substitute for a different pattern of trading which can achieve the same (CE) outcome without using loans.

EXAMPLE 3. Two types of traders and three consumer goods just as in Example 1, with slightly different utility functions and initial endowments:

$$u^1 = \ln(1 + x^1) + 0.1 \ln(1 + y^1) + 0.1 \ln(1 + z^1)$$
$$u^2 = 0.1 \ln(1 + x^2) + \ln(1 + y^2) + 0.1z^2$$

(0, 9, 0) for Type 1 individuals, $(9, 0, 2(10 - \sqrt{10}))$ for Type 2 individuals.

First the CE can be easily computed: $\langle(9, 0, 0), (0, 9, 2(10 - \sqrt{10}))\rangle$ with price vector $\bar{p} = (1, 1, 1)$.

Considering this problem as an SMG, with modified utility functions similar to those in Example 1, the CE allocation can be achieved as an NE in two different ways:

With money loan market, the strategies at the first trade section are:

$(10 - \sqrt{10}, 0; 0, 10 - \sqrt{10}; 10 - \sqrt{10}, 0)$ for Type 1 traders, meaning: borrowing $10 - \sqrt{10}$ units of money from Type 2, sending $10 - \sqrt{10}$ units of good 2 for sale, sending $10 - \sqrt{10}$ units of money for the purchasing of good 1;

$(0, 10 - \sqrt{10}; 10 - \sqrt{10}, 0; 0, 10 - \sqrt{10})$ for Type 2 traders, meaning: lending $10 - \sqrt{10}$ units of money to Type 1, sending $10 - \sqrt{10}$ units of good 1 for sale, sending $10 - \sqrt{10}$ units of money to bid for good 2.

And the strategies at the second trade section are:

$(0, \sqrt{10} - 1; \sqrt{10} - 1; 0)$ for Type 1 traders;

$(\sqrt{10} - 1, 0; 0, \sqrt{10} - 1)$ for Type 2 traders.

After the trading, Type 1 traders return $10 - \sqrt{10}$ units of money to Type 2, and both consume everything in hand: (9, 0, 0) for Type 1 and $(0, 9, 2(10 - \sqrt{10}))$ for Type 2.

In the case without a money loan, the strategies for the two types are, respectively:

	1st Trade Section	2nd Trade Section
Type 1	(0, 9; 0, 0)	(0, 0; 9, 0)
Type 2	(0, 0; 0, 9)	(9, 0; 0, 0)

It is not difficult to check this really given an NE of the SMG.

EXAMPLE 4: Everything is the same as in Example 3, except that there are slight differences in the utility functions:

$$u^1 = \ln(1 + x^1) + 0.1 \ln(1 + z^1)$$
$$u^2 = \min\{\ln(1 + y^2), \ln 10\} + 0.1 \ln(1 + x^2) + 0.1z^2$$

First the CE is just the same as in Example 3:

$\langle(9, 0, 0), (0, 9, 2(10 - \sqrt{10}))\rangle, \tilde{p} = (1, 1, 1)$

When we consider it as an SMG, there are two different cases:

$(10 - \sqrt{10}, 0; 9, 9; 10 - \sqrt{10}, 0)$ for Type 1 at 1st trade section

$(0, 10 - \sqrt{10}; 10 - \sqrt{10}, 0; 0, 10 - \sqrt{10})$ for Type 2 at 1st trade section

$(0, 0; 10 - \sqrt{10}, 0)$ for Type 1 at 2nd trade section

$(10 - \sqrt{10}, 0; 0, 0)$ for Type 2 at 2nd trade section.

If there is no money loan market, there can be many different NEs of the following form:

	1st Trade Section	2nd Trade Section
Type 1	(0, 9; 0, 0)	$(0, 0; \varepsilon, 0)$
Type 2	$(0, 0; 0, \varepsilon)$	$(\delta(\varepsilon), 0; 0, 0)$

where $\varepsilon > 0$ can be arbitrarily small, and $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. The associated allocation is then

$$\langle(\delta(\varepsilon), 0, 0), (9 - \delta(\varepsilon), 9, 2(10 - \sqrt{10}))\rangle$$

obviously, when ε is small, the lenders really get improvements.

In this example with two trading sessions we provide a somewhat different example from Example 2 above in Section 2.7 where we showed that it was robustly possible for a loan market to make the lenders worse off.

EXAMPLE 5: Two types of traders, each consisting of a continuum [0, 1] of individuals. Three consumer goods, with the third one used as money. The utility functions and the initial endowments are given below:

$$u^1 = x^1 + 101 \ln(1 + y^1) + z^1; \; (200, 0, 0)$$
$$u^2 = 101 \ln(1 + x^2) + y^2 + z^2; \; (0, 200, 2)$$

First, it is easy to see that the only CE is given by

$$\langle(100, 100, 0), (100, 100, 2)\rangle, \bar{p} = (1, 1, 1)$$

Consider this problem as an SMG with the utility functions slightly modified, this CE can be achieved as an NE in two different ways, both need a hundred of trade sections.

With money loan, the strategies look like:

(1, 0; 1, 0; 0, 1), (1, 0; 0, 1), ..., (1, 0; 0, 1) for Type 1

(0, 1; 0, 1; 1, 0), (0, 1; 1, 0), ..., (0, 1; 1, 0) for Type 2

Without money loan, the strategies look like:

(2, 0; 0, 0), (0, 0; 0, 2), ..., (2, 0; 0, 0), (0, 0; 0, 2) for Type 1

(0, 0; 2, 0), (0, 2; 0, 0), ..., (0, 0; 2, 0), (0, 2; 0, 0) for Type 2

In this simple example without loans there is not enough money to achieve efficiency unless there are 100 market sessions. If there are loans efficiency still requires the 100 sessions but the trading pattern is different and the rate of interest is zero. If there are fewer markets the rate of interest is positive reflecting the shortage of money. We consider the one and the two trading session games showing the drop in the rate of interest as velocity increases.

3.4. *k* trading sessions with a durable money and loans

EXAMPLE 6: Two types of traders with each type consisting of a continuum [0, 1] of individuals. The initial endowment is given by

(4, 0, 0) for a type I indiviudal; (0, 4, 3) for a type II individual

where the first component and the second one are the amounts of commodity 1 and commodity 2, respectively, both of them are consumable; and the third component is the amount of a consumer durable money. Assume during the whole period there are n ($n \geq 2$) trade sections.

Assume that each trader consumes the first two goods only at the end of the period, i.e. at $t = 1$. Assume that the utility of keeping a unit of money from time t to t' is $e^{-t} - e^{-t'}$.[10] The utility function for the traders are given by[11]

$$u^1 = x_n^1 + 2\sqrt{2y_n^1} + z_1^1\left(e^0 - e^{-\frac{1}{n}}\right) + z_2^1\left(e^{-\frac{1}{n}} - e^{-\frac{2}{n}}\right) + \cdots + z_n^1\left(e^{-\frac{n-1}{n}} - e^{-1}\right) + z_{n+1}^1 e^{-1} \,;$$

$$u^2 = 2\sqrt{2x_n^2} + y_n^2 + z_1^2\left(e^0 - e^{-\frac{1}{n}}\right) + z_2^2\left(e^{-\frac{1}{n}} - e^{-\frac{2}{n}}\right) + \cdots + z_n^2\left(e^{-\frac{n-1}{n}} - e^{-1}\right) + z_{n+1}^2 e^{-1}$$

where the x_n^i and the y_n^i are the amount of goods 1 and 2, respectively, that an individual i consumes at the end of the period, and z_j^i are the amount of money i keeps in stock in the j^{th} section, ($j = 1, \ldots, n$) and z_{n+1}^i is the amount of money after the end of the period.

It is not difficult to check that the following strategies given an NE of the SMG:

For an individual of type I, borrow an amount $e^{-1/n}$ of money at the beginning, send an amount $e^{-1/n}$ of good 1 for sale at section 1, send an amount $e^{-1/n}$ of money for buying good 2 at section 1; send an amount $e^{-1/n}$ of good 1 for sale at section 2, send an amount 1 of money for buying good 2 at section 2.

For an individual of type II, lend an amount $e^{-1/n}$ of money at the beginning, send an amount $e^{-2/n}$ of good 2 for sale at section 1, send an amount 1 of money for buying good 1 at section 1; send an amount $e^{-2/n}$ of good 2 for sale at section 2, send an amount $e^{-1/n}$ of money for buying good 1 at section 2.

The price vector for the first section is $(e^{1/n}, e^{1/n})$, and the price vector for the second section is $(e^{2/n}, e^{2/n})$. The allocation at the end of the second period is given by $(4 - 2e^{-1/n}, 2e^{-2/n}, e^{1/n})$ for the first type traders, and $(2e^{-1/n}, 4 - 2e^{-2/n}, 3 - e^{1/n})$ for the second type. The first type traders pay back the second type with an amount $e^{1/n}$ of money once the trade ends. Therefore from the beginning of the third section to the end of the period the holdings are $(4 - 2e^{-1/n}, 2e^{-2/n}, 0)$ for the first type traders, and $(2e^{-1/n}, 4 - 2e^{-2/n}, 3)$ for the second type.

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[10]For the durable consumer good we assume that is provides a continuous stream of utility when it is kept in stock from time zero to infinity, and its utility is decreasing so that the total utility it can give is finite. This assumption is necessary for the existence of an NE because of the time lag of receiving payment with respect to the moment it is being paid.

[11]After the last trade section, a trader is assumed to keep his money forever if there is no other period for trade.

Note that in the above example, when n tends to infinity, i.e. when the speed of transaction becomes extremely high, the price vectors all become (1, 1), and the allocation becomes (2, 2, 0) for a type I individual and (2, 2, 3) for a type II one. Thus the CE of the trading economy is obtained.

OBSERVATION: The NEs of SMG with a consumer durable money, with or without a transaction loan market and with k trading periods are generically inefficient, but as $k \to \infty$ they approach the CEs and if there is a transactions loan market, the offer rate of interest approaches $\partial U/\partial x_{m+1}$ and borrowing will cease once the market rate falls below this level.

In a case with two types of traders, an argument similar to that in Shubik and Yao (1989) in the proof of a similar theorem can be adapted here, only some modification is needed.

4. ENOUGH MONEY, VELOCITY AND THE TRANSACTIONS RATE OF INTEREST

In this paper we have attempted to offer a strategic interpretation of the velocity of money. This involved a detailed concerned for the specifics of transacting. Clearly this is a highly special and somewhat contrived set of models, but the main task was to get a well defined even though special, tractable set of models in which all the phenomena appear and can be isolated. The next step is to generalize mechanisms. We suspect that the specific details of how much money, what losses through transactions delays will be highly dependent on technology, custom and institutional detail but all and any well defined process will show the system properties of tradeoffs among velocity, quantity of money and transaction loan interest rates.

Summarizing, we observe that the one period model of exchange using the one period an "an atom" fails to distinguish between stocks and flows. Once one distinguishes consumption from transactions one can naturally make the distinction by subdividing the single period to indicate what part of the time is given over to trading.[12] Once one breaks down the single period into intervals it becomes necessary to make distinctions among the goods traded. As a

[12]This includes the possibility that there may be both trading and consumption taking place in parallel.

first order approximation a consumable good can be considered to be an item with point consumption and a durable as an item which provides a stream of services.

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