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First Price Auctions with General Information Structures: Implications for Bidding and Revenue*

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Abstract

We explore the impact of private information in sealed bid first price auctions. For a given symmetric and arbitrarily correlated prior distribution over values, we characterize the lowest winning bid distribution that can arise across all information structures and equilibria. The information and equilibrium attaining this minimum leave bidders uncertain whether they will win or lose and indifferent between their equilibrium bids and all higher bids. Our results provide lower bounds for bids and revenue with asymmetric distributions over values.

We report further analytic and computational characterizations of revenue and bidder surplus including upper bounds on revenue. Our work has implications for the identification of value distributions from winning bid data and for the informationally robust comparison of alternative bidding mechanisms.

Keywords: First price auction, information structure, Bayes correlated equilibrium, private values, interdependent values, common values, revenue, surplus, welfare bounds, reserve price.

JEL Classification: C72, D44, D82, D83.

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1 Introduction

The first price auction has been the subject of extensive theoretical study for over fifty years. It is still fair to say, however, that its properties are well-understood only in relatively special cases. Under complete information, the first price auction reduces to classic Bertrand competition. Under incomplete information, most work focuses on the case of one dimensional type spaces. For example, when bidders know their private values, it is typically assumed that bidders *only* know their private value and have no additional sources of information. Beyond the known private values case, it is typically assumed that bidders have one-dimensional types that are jointly affiliated and that values are increasing in the profile of types (Milgrom and Weber, 1982). Thus, a strong relationship is assumed between each bidder's belief about his own value and his beliefs about others' values. But in first price auctions, unlike in second price auctions, bidders' beliefs about others' values is of central strategic importance, since in equilibrium others' values are informative about what they will bid. In many situations, it is unnatural to impose strong restrictions on the relationship between the conceptually distinct beliefs about one's own value and about others' values.

In this paper, we derive results about equilibrium behavior in the first price auction that hold across *all* common prior information structures. For a given symmetric prior joint distribution over value profiles, we study what can happen for all (symmetric or asymmetric) information structures specifying bidders' information about their own and others' values. As we will discuss in detail below, our setting thus incorporates all existing models with symmetric distributions of values.[1] For any such value distribution, we identify a lower bound on the distribution of winning bids in the sense of first-order stochastic dominance. In other words, no matter what the true information structure is, the distribution of winning bids must first-order stochastically dominate the bound that we describe. In addition, we construct an equilibrium and information structure in which this lower bound is attained. This minimum winning bid distribution therefore pins down the minimum amount of revenue that can be generated by the auction in expectation. Moreover, the minimum winning bid distribution is attained in an efficient equilibrium. As a result, this equilibrium also attains an upper bound on the expected surplus of the bidders, which is equal to the maximum feasible surplus minus minimum revenue.[2]

[1] We will discuss what happens with asymmetric value distributions in the body of the paper. Some results extend as stated to the asymmetric value distributions, and for the others, we will report weaker analogue results.

[2] Where no confusion results, we will write "revenue" for ex-ante expected revenue, "bidder surplus" for ex-ante expected surplus, etc.

To illustrate our main result, consider the case when there are two bidders with a common value that is uniformly distributed on the interval $[0, 1]$. Since all bidders have the same value, the allocation of the good does not affect the expected total surplus, which is always $1/2$. All that remains to be determined is how that surplus is split between the seller and the bidders, with bidder surplus will be $1/2$ minus revenue. What does existing theory tell us about possible welfare outcomes in this example? If both bidders know nothing about the value, there is Bertrand competition, both bidders will bid the expected value of $1/2$ and so revenue will be $1/2$. If both bidders knew the value, there would again be Bertrand competition, and revenue would remain $1/2$ in expectation. If one bidder knew the value, and the other bidder knew nothing, then Engelbrecht-Wiggans, Milgrom, and Weber (1983)[3] have shown that the informed bidder would bid half the true value and the uninformed bidder would bid uniformly on the interval $[0, 1/2]$. These strategies lead to revenue of $1/3$, while the informed bidder gets surplus of $1/6$ and the uninformed bidder gets zero. Our main result applied to this example shows that no matter what the information structure and equilibrium are, the distribution of the winning bid must first-order stochastically dominate a uniform distribution over $[0, 1/3]$, so that revenue is at least $1/6$. This lower bound is achieved in an information structure and equilibrium where the winning bid is always equal one third of the value. We will study this example in detail in Section 3.

Let us give a brief intuition for how our bounds are obtained. If the distribution of winning bids places too high of a probability on low bids, then some bidder would find that a modest increase in their bid would result in a relatively large increase in the probability of winning, so that such a deviation would be attractive. This suggests that the relevant constraints for pinning down minimum bidding are those associated with deviating to higher bids. Indeed, we show that the minimum winning bid distribution is characterized by bidders being indifferent to *all* upward deviations. In fact, it turns out to be sufficient to look at a relatively small class of such deviations: For some bid b, we say that a bidder *uniformly deviates up to b* if, whenever they would have bid less than b in equilibrium, they switch to bidding b. It is clearly necessary for equilibrium that the bidders should not want to uniformly deviate upward. Moreover, it turns out that it is possible to evaluate the merits of a uniform upward deviation using just the distribution of winning bids, and not using any information about losing bids. This motivates a relaxed program in which we minimize revenue over distributions of winning bids, subject to only the uniform upward incentive constraints. The solution to this relaxed program gives us a lower bound on the distribution

[3]Engelbrecht-Wiggans, Milgrom, and Weber (1983) give a simple and natural specification of information in a pure common value environment under which the bidders can obtain positive rents. We will periodically compare our results with those of Engelbrecht-Wiggans et al. (1983) to get a sense of the magnitude of our welfare bounds relative to what was previously known.

of winning bids. We subsequently construct an equilibrium that attains the bound, thus verifying that it is indeed the minimum winning bid distribution.

We describe this proof strategy in more detail for this two bidder uniform common value example in Section 3. The result and argument can then be adapted for all distributions of (not necessarily common) values and any number of bidders. Without common values, it becomes important to characterize the efficiency of allocations at the minimizing equilibrium. It turns out that giving bidders more surplus in equilibrium relaxes the critical uniform upward deviation constraints. This implies that the allocation corresponding to the minimum equilibrium winning bid distribution must be efficient. This implies in turn that bidder surplus is maximized in the information structure and equilibrium where revenue is minimized. Thus, maximum bidder surplus is equal to the total surplus from the efficient allocation minus minimum revenue. We use our result to characterize minimum revenue and maximum bidder surplus for a variety of examples. We also take comparative statics as the number of bidders becomes large, in which case minimum revenue is bounded away from the total surplus and maximum bidder surplus is bounded away from zero.

We report a number of further results about bids, revenue, and bidder surplus. A straightforward upper bound on revenue is the efficient surplus, and we show that this bound is in fact tight. To see this, first suppose that there was a tie breaking rule allowing the bidder with the highest value to win if he was among the high bidders. Now consider the information structure in which all of the bidders observe the highest value, but not the identities of the bidders with that value. In that case, there is an equilibrium where everyone bids the highest value but the high value bidder wins. Now it is enough to adapt this information and strategy profile to one without the efficient tie breaking rule which attains revenue arbitrarily close to full surplus extraction. In the information structure and equilibrium that we construct, bidders end up frequently bidding above their own values, but they only win at bids below their values. Moreover, bidders are always uncertain about whether they will win or lose. It is therefore difficult to refine this bound using, say, weak dominance arguments.

So far, we allowed for the possibility that each bidder does not know his own value. We refer to this as the *unknown values* case. In common values models, if bidders knew their own values, they would also know others' values and the problem would be reduced to the degenerate complete information case. When values have an idiosyncratic component, however, even if one's own value is known, the specification of beliefs about others' values can have a big impact on the equilibrium outcome. We refer to this as the *known values* case, in which we maintain the assumption that each bidder knows at least his own value for the good, but allow for an arbitrary specification of bidders' information about others' values and beliefs.

We characterize maximum revenue for the known values case, which turns out to be strictly less than the maximum for unknown values. To establish this result, we first note an easy lower bound on bidder surplus.[4] It is always feasible for a bidder to choose his bid as a function of his value alone and not condition on any additional information about the other bidders. Even if the bidder is as pessimistic as possible about others' strategies—believing that other bidders will all bid their values[5]—such a bidder can guarantee himself a certain surplus as a function of his value. Now, an ex-ante lower bound on that bidder's surplus is the expectation of that value dependent minimum surplus. Moreover, an upper bound on revenue is the efficient surplus minus the sum of all of the bidders' surplus lower bounds. In fact, we construct an information structure and equilibrium where this bound is exactly attained. In this information structure, each bidder observes if he has the highest value or not, and in equilibrium those who do not have the highest value always lose and bid their values. If the highest value bidder knew nothing more, the optimal shading of his bid would mean that the allocation would be inefficient (and it would also mean that low value bidders would no longer have an incentive to bid their values). On the other hand, if the highest value bidder knew the second-highest value precisely, he would bid that value and get more surplus than his lower bound. Thus, in order to attain the bound, it is necessary to show that there is an intermediate amount of information for the highest value bidder such that he (i) always bids more than the second-highest value, and therefore wins the auction, but (ii) is always indifferent between his equilibrium bid and the bid associated with his surplus lower bound. The existence of such an information structure turns out to be a consequence of a third degree price discrimination result that we established in earlier work (Bergemann, Brooks, and Morris, 2015c).

A natural next question would be: what is minimum revenue in the known values case? We do not have a general analytic characterization of minimum revenue for known values, although a lower bound is given by the unknown values minimum revenue. In an earlier version of this paper (Bergemann, Brooks, and Morris, 2015b), we provided a complete characterization of minimum revenue for the case in which bidders have only two possible values, high or low. This result employs a methodology that is similar to what we will use to characterize unknown values minimum revenue. We also gave a detailed discussion of the limitations of this methodology for going beyond the case of two values. In the present paper, we will simply report computational results on other welfare outcomes in the known values model, and relate them to the analytic results described earlier. We also explore

[4]We are grateful to Satoru Takahashi for suggesting this bound to us.

[5]We will assume a version of weak dominance throughout the paper, which rules out bidding above one's value in the known values case.

the whole welfare space of possible revenue and bidder surplus outcomes, including those associated with inefficient equilibria. In the case of two bidders and independent values, we construct a maximally inefficient information structure and equilibrium strategy profile where the seller's revenue is minimized subject to feasibility constraints, so that his revenue is the expectation of the lower value, and the bidders receive zero surplus.

Our primary focus in this paper is on developing insights about how general information structures can affect outcomes in the first price auction and on the qualitative properties of the information structure that lead to different outcomes. However, our results can be used in a variety of applications, for example, to provide robust partial identification in settings where the information structure is unknown and to make informationally robust comparisons of mechanisms. We postpone discussion of these applications until the concluding section.

1.1 Related Literature

The game theoretic analysis of the first price auction was initiated by the seminal paper of Vickrey (1961). Kaplan and Zamir (2015) provide a survey of the ever growing literature. Our main results hold for the unknown values case, where bidders do not necessarily know their own values. We can relate this to the literature on common values (when bidders have the same value) or there is a common component of their values. Milgrom and Weber (1982) introduced a setting where bidders' signals are affiliated and symmetrically distributed, and a bidder's signal may be informative about his value and also the others' values. They characterized equilibria for the first price auction (as well as other mechanisms) in this setting. Engelbrecht-Wiggans, Milgrom, and Weber (1983) and, more generally, Syrgkanis, Kempe, and Tardos (2015) analyze the common value auctions where there are ex-ante asymmetries in information.

Much of the early literature on the first price auction focuses on the known values case, in which bidders know their own values for certain. Vickrey (1961) provided the first statement and analysis of the problem and Riley and Samuelson (1981) gave a general analysis for the symmetric and independent values case. Most work on known values assumes that bidders do not receive additional signals (beyond their own value) about others' values. Only a small number of papers study the case where bidders do receive additional information. Landsberger et al. (2001) consider the case where bidders know whether they have the highest value or not, but do not know anything about others' cardinal values. Kim and Che (2004) consider the case where bidders are divided into groups and each bidder knows the values of those in his group, but knows nothing about the values of those outside the group. Fang and Morris (2006a) and Azacis and Vida (2015) analyze a model with two bidders and two possible values, where each bidder observes a conditionally independent signal of the other bidder's value.[6]

Our main result assumes a symmetric distribution of values and allows for asymmetries in information. When the value distribution is asymmetric, we obtain bounds that are generally not tight. The unknown values model with an asymmetric prior is studied by Lizzeri and Persico (2000). A large literature examines asymmetric distributions of values in settings with known values, both giving conditions for existence (e.g., Reny and Zamir, 2004) and revenue rankings (e.g., Maskin and Riley, 2000).

We will subsequently describe the behavior of our welfare bounds as the number of bidders becomes large. There is a substantial literature on this subject. Wilson (1977) and Milgrom (1979) gave conditions in common value models under which revenue converges to the realized common value. Bali and Jackson (2002) show that revenue always converges to the expectation of the highest value in a more general setting where bidders' values have an idioyncratic component. We will discuss the relationship with these papers in greater detail in Section 4.

Bergemann and Morris (2013, 2015a) discuss a general methodology for simultaneously characterizing all equilibria under all information structures in an incomplete information game. They show that it is without loss of generality to focus on information structures where the signal space is equal to the action space, i.e., the bid space in the context of this paper. This observation means that the set of outcomes that can arise corresponds to a version of incomplete information correlated equilibrium, which they call *Bayes correlated equilibrium* (BCE). The BCE perspective will not be explicitly used in the body of the paper, as we will work directly with information structures and their equilibria, although we are implicitly applying this methodology. Bergemann and Morris (2013) and Bergemann, Heumann, and Morris (2015d) characterize symmetric BCE in games with linear best responses and normally distributed uncertainty. One contribution of this paper is to characterize BCE in a more challenging game, with non-linear and discontinuous best responses.

Other authors have also studied the first price auction under weaker solution concepts than Bayes Nash equilibrium. Lopomo, Marx, and Sun (2011) studied the set of communication equilibria when values are known and independent.[7] In a communication equilibrium, bidders can communicate with one another via a mediator but decide rationally what to

[6] In earlier versions of this paper (Bergemann, Brooks, and Morris, 2013, 2015b), we provide a complete analytic characterization of possible welfare outcomes in the setting of Fang and Morris (2006b) and Azacis and Vida (2015).

[7] See Forges (1986) for a definition of communication equilibrium. Forges (1993) and Bergemann and Morris (2015a) discuss the relationship between various extensions of correlated equilibrium to incomplete information games.

communicate. We obtain more permissive results because communication equilibria are a subset of BCE. Battigalli and Siniscalchi (2003) characterize rationalizable behavior in the affiliated known values first price auction, maintaining the assumption that bidders know no more than their own value. They characterize an upper bound on bids which is strictly higher than the unique equilibrium bid and strictly below values. Our analysis is both more permissive, in that we allow bidders to obtain extra information, and more restrictive, in that we assume common knowledge of the strategies that are being used.[8] Dekel and Wolinsky (2003) consider rationalizable behavior in the first price auction when the number of bidders becomes large. The give conditions on beliefs such that revenue converges to the expectation of the highest value.

One reason why rich information structures have not been studied in first price auctions is because the combination of discontinuous payoffs and continuous action spaces makes existence results relatively hard to establish (see Jackson and Swinkels (2005) for one set of sufficient conditions). In our analysis, we augment the bidding game with "custom" information structures. In a sense, the ability to design the information structure allows us to side step issues of existence and reverse engineer endogenous tie breaking through asymmetric information.

1.2 Outline of Paper

The rest of this paper proceeds as follows. In Section 2, we describe a general model of a first price auction. In Section 3, we preview our argument with a two bidder example in which there is a pure common value that is uniformly distributed on the interval $[0,1]$. In Section 4, we report our main result, a characterization of minimum winning bids, minimum revenue, and maximum bidder surplus over all possible specifications of bidders' beliefs and equilibrium strategies consistent with a given distribution of values. In Section 5, we describe further results on welfare outlined above. Section 6 concludes, with a discussion of applications of our results to identification and the robust comparison of alternative selling mechanisms. Omitted proofs are contained in the Appendix.

2 Model

We consider the sale of a single unit of a good by a first price auction. There are N individuals who bid for the good, each of whom has a value which lies in the compact

[8]Bergemann and Morris (2015b) provides a general approach analyzing the relation between the solution concepts used in these papers.

interval $V = [\underline{v}, \overline{v}] \subset \mathbb{R}_+ = [0, \infty)$. Values are jointly distributed according to a symmetric common prior $P \in \Delta\left(V^N\right)$.[9] By symmetry, we mean that the distribution of values is exchangeable: Let Ξ denote the set of permutations of the bidders identities, i.e., bijective mappings from $\{1, \ldots, N\}$ into itself. We associate each $\xi \in \Xi$ with a mapping from V^N into itself, where $v' = \xi(v)$ if $v'_i = v_{\xi(i)}$. The distribution P is exchangeable if $P(X) = P(\xi(X))$ for all Borel sets $X \subseteq V^N$ and for all permutations $\xi \in \Xi$.

We allow the symmetric common prior P to be a general measure (and not necessarily a density) in order to encompass the common value case in the same analytic framework as non-common values. All our results go through as stated if one lets P be a density (i.e., absolutely continuous with respect to Lebesgue measure) at the cost of excluding the common values case. We will however assume that certain marginal and conditional distributions derived from P are absolutely continuous. In particular, we assume that for each $x \in \mathbb{R}$, the event in which the $N-1$ lowest values sum to x has zero probability. Also, we will assume that for each v_i, the conditional distribution of $\max v_{-i}$ is non-atomic.

In the auction, each individual $i \in \{1, \ldots, N\}$ submits a bid $b_i \in B$, where $V \subseteq B$ and B is bounded below. The winner of the auction will be the bidder with the highest bid. For a profile of bids $b \in B^N$, let $W(b)$ be the set of high bidders,

$$W(b) = \{i \,|b_i \geq b_j \, \forall j = 1, \ldots, N\}.$$

There is a uniform tie breaking rule, so that the probability that bidder i receives the good if bids are $b \in B^N$ is

$$q_i(b) = \begin{cases} \frac{1}{W(b)} & \text{if } i \in W(b); \\ 0 & \text{otherwise.} \end{cases}$$

Bidders may receive additional information about the profile of values, beyond knowing the prior distribution. This information comes in the form of signals that are correlated with the profile of values. An *information structure* is a collection $\mathcal{S} = \left(\{S_i\}_{i=1}^N, \pi\right)$, where the S_i are complete and separable metric spaces and $\pi : V^N \to \Delta(S)$ is a measurable mapping from profiles of values to probability measures over $S = \times_{i=1}^N S_i$. The interpretation is that S_i is the set of bidder i's signals and that π describes the conditional joint distribution of signals given values.

This definition of an information structure allows for the possibility that bidders do not know their own values. We will sometimes wish to consider a model where bidders are known

[9] For a topological space X, we let $\Delta(X)$ denote the set of all Borel probability measures on X. When it is convenient and without risk of ambiguity, we may denote he cumulative distribution of a measure $F \in \Delta(X)$ with $X \subseteq \mathbb{R}^K$ by $F(x) = F([-\infty, x_1] \times \cdots \times [-\infty, x_K])$ for some $x \in X$.

to have precise knowledge of their own values but potentially noisy information about others' values, as in the classical model of independent private values. We say that $\mathcal{S}$ is a *known values* information structure if we can write $S_i = V \times S_i'$, with $\pi(\{v\} \times S'|v) = 1$ almost surely. Thus each bidder's signal can be decomposed as an observation of a value and an auxiliary signal, and with probability one, the value signal is equal to that bidder's true value.

For a fixed information structure $\mathcal{S}$, the first price auction is a game of incomplete information, in which bidders' strategies are measurable mappings $\sigma_i : S_i \to \Delta(B)$ from signals to distributions over bids. Let Σ_i denote the set of strategies for agent i. Fixing a profile of strategies $\sigma \in \Sigma = \times_{i=1}^{N} \Sigma_i$, bidder i's (ex-ante) surplus from the auction is

$$U_i(\mathcal{S}, \sigma) = \int_{v \in V^N} \int_{s \in S} \int_{b \in B^N} (v_i - b_i)\, q_i(b)\, \sigma(db|s)\, \pi(ds|v)\, P(dv),$$

where σ is the product measure $\sigma_1 \times \cdots \times \sigma_N$. When bidders play a pure strategy, we will identify $\sigma_i : S_i \to B$ with the strategy that puts probability one on the bid $\sigma_i(s_i)$.

Throughout our analysis, we restrict attention to strategies in which each bidder does not bid amounts that are greater than his value with probability one. In particular, we can write $P(v_i|s_i)$ for a version of the conditional distribution of bidder i's value given bidder i's signal. Then we restrict attention to strategies σ_i such that with probability one,

$$\operatorname{supp} \sigma_i(\cdot|s_i) \subseteq [0, \max \operatorname{supp} P(\cdot|s_i)].$$

This rules out weakly dominated behavior in which bidders knowingly bid strictly more than their maximum possible value. Our requirement is, however, strictly weaker than weak dominance, since we do not rule out players bidding $\max \operatorname{supp} P(\cdot|s_i)$ with positive probability, e.g., in the known values case, bidders can bid their own known value with positive probability. To rule out such high bids would eliminate the unique equilibrium in many complete information specifications.

The profile $\sigma \in \Sigma$ is a *Bayes Nash equilibrium*, or *equilibrium* for short, if and only if

$$U_i(\mathcal{S}, \sigma) \geq U_i(\mathcal{S}, \sigma_i', \sigma_{-i})$$

for all i and all $\sigma_i' \in \Sigma_i$.[10]

Our goal is to study how welfare outcomes and bidding behavior vary across information structures and equilibrium strategy profiles for a *fixed* distribution of values. The outcomes

[10]We do not address sufficient conditions for the existence of equilibrium. We provide constructive proofs of properties of equilibria throughout the paper.

that we will investigate include:

$$(i) \text{ Bidder surplus: } U(\mathcal{S},\sigma) = \sum_{i=1}^{N} U_i(\mathcal{S},\sigma);$$

$$(ii) \text{ Revenue: } R(\mathcal{S},\sigma) = \sum_{i=1}^{N} \int_{v\in V^N} \int_{s\in S} \int_{b\in B^N} b_i q_i(b)\, \sigma(db|s)\, \pi(ds|v)\, P(dv);$$

$$(iii) \text{ Total surplus: } T(\mathcal{S},\sigma) = U(\mathcal{S},\sigma) + R(\mathcal{S},\sigma).$$

Note that by bidder surplus we mean the sum of all of the bidders' individual surpluses. We denote the efficient surplus by

$$\overline{T} = \int_{v\in V^N} \max v\, P(dv).$$

A information structure $\mathcal{S}$ and equilibrium strategy profile σ are *efficient* if $T(\mathcal{S},\sigma) = \overline{T}$. We will study how these welfare objectives vary across information structures and equilibria.

3 A Pure Common Value Example

Before giving our general results, we will illustrate where we are headed with a simple example. Two bidders participate in a first price auction. They have a common value for the good, which is uniformly distributed between 0 and 1. The bidders receive signals about the common value. After observing their signals, they submit bids, and the high bidder wins and has to pay the amount he bid. Since we assume there is no reservation price in the auction, the good is always allocated, and as both bidders have the same value, all equilibria are socially efficient and result in a total surplus of $1/2$. However, there may be variation across information structures and equilibria in how this surplus is split between the seller and the bidders.

We allow bidders to observe arbitrary and possibly correlated signals about the common value. At one extreme, both bidders may be uninformed, so that the information structure entails a single "null" signal for each bidder. In this case, bidders always believe that the good has an expected value of $1/2$, and therefore the unique equilibrium is for both bidders to tie at a bid of $1/2$. Bidders get zero rents and the seller extracts all of the surplus as revenue. The opposite extreme is complete information, where both bidders observe the true value of the good. In this case, conditional on the true value being v, the unique equilibrium involves both bidders tying at a bid of v. Again, bidder surplus is zero and the seller extracts all of the value of the good. These examples illustrate our later general result that, unless we make additional assumptions about what the bidders know, a tight upper bound on revenue is the efficient surplus.

For intermediate amounts of information, however, the distribution of ex-ante surplus can be rather different. An important intermediate information structure has been studied by Engelbrecht-Wiggans, Milgrom, and Weber (1983) to which we referred in the introduction: one bidder (say, bidder 1) observes the true value while the other bidder (bidder 2) is uninformed and observes nothing. In the case of the uniform distribution, the resulting equilibrium involves bidder 1 bidding $v/2$ and the uninformed bidder randomizing uniformly over the interval $[0, 1/2]$. Let us briefly verify that this is an equilibrium. If bidder 2 bids b, she will win whenever bidder 1 bids less than b, which is when the true value is less than $2b$. The conditional distribution of v in the event that bidder 2 wins is therefore uniform on $[0, 2b]$, so that the expected value of v conditional on bidder 2 winning is exactly b. Thus, any bid of bidder 2 results in zero surplus in expectation. On the other hand, conditional on the true value being v, bidder 1 wins with a bid of $b \in [0, 1/2]$ with probability $2b$, which results in surplus $(v - b)\, 2b$, which is maximized at precisely $b = v/2$. This equilibrium results in a surplus of 1/6 for bidder 1, a surplus of 0 for bidder 2, and revenue of only 1/3.

But what can we say about intermediate information structures more generally? A simple lower bound on revenue is zero, but this bound cannot be tight: revenue could be zero only if both bidders bid zero with probability one, in which case they must be each getting a surplus of 1/4. However, either bidder could deviate up to a bid of $\epsilon > 0$ (independent of his signal) and obtain a surplus of $1/2 - \epsilon$. This suggests a general intuition that there cannot be too many bids too close to zero, lest the probability of winning increase too quickly as bidders deviate to higher bids. More generally, we might expect that the limit of how low bidding can go will be characterized by binding upward incentive constraints, i.e., bidders being indifferent to deviating to higher bids. Note that in the analysis of Engelbrecht-Wiggans et al., the informed bidder strictly prefers their equilibrium bid over any other bid. This suggests that it might be possible to construct other information structures in which revenue is even lower.

Let us provide one such construction. The two bidders will receive signals $s \in [0, 1]$ that are independent draws from the cumulative distribution $F(s) = \sqrt{s}$, so that the distribution of the maximum signal is standard uniform, the same as the common value. Moreover, we will correlate signals with values so that the maximum signal is always exactly the true common value, i.e., $v = \max\{s_1, s_2\}$. It turns out that this information structure admits a monotonic pure strategy equilibrium in which the bidders use the same strategies *as if* the signal were their true value. Under this as-if interpretation, the model is one of independent private values, and it is well-known that the equilibrium strategy is

$$\sigma\left(s\right)=\frac{1}{\sqrt{s}}\int_{x=0}^{s}x\frac{dx}{2\sqrt{x}}=\frac{s}{3}.$$

With this bidding function, the winning bid will always be $\max_i s_i/3 = v/3$, so that revenue is 1/6, and as the equilibrium is symmetric, each of the bidders obtains a surplus of 1/6 as well. Thus, this information structure doubles the total rents that the bidders receive relative to the proprietary information structure of Engelbrecht-Wiggans et al..

Let us now verify that this is an equilibrium. A bidder who deviates by bidding $s'/3$ for some $s' < s$ will only win when their own signal was the highest signal, in which case surplus is proportional to

$$\left(s-\frac{s'}{3}\right)\sqrt{s'},$$

which one can verify is increasing in s' for $s' < s$, since the derivative is

$$\left(s-\frac{s'}{3}\right)\frac{1}{2\sqrt{s'}}-\frac{1}{3}\sqrt{s'}=\frac{1}{2\sqrt{s'}}\left(s-s'\right).$$

On the other hand, if a bidder deviates up to $s'/3$ with $s' > s$, the bidder continues to win on the event that they had the high signal, and now wins on some events when it was the other bidder who had the high signal, which was the true value. Surplus will be

$$\left(s-\frac{s'}{3}\right)\sqrt{s}+\int_{x=s}^{s'}\left(x-\frac{s'}{3}\right)\frac{1}{2\sqrt{x}}dx.$$

Again, we can differentiate this expression with respect to s' to obtain

$$\frac{2}{3}s'\frac{1}{2\sqrt{s'}}-\frac{1}{3}\left(\sqrt{s'}-\sqrt{s}\right)-\frac{\sqrt{s}}{3}=0.$$

In other words, bidders are exactly indifferent to *all* upward deviations!

In fact, it turns out that no matter how one structures the information or the equilibrium strategies, it is impossible for revenue to fall below the level attained in this example, i.e., 1/6 is a tight lower bound on revenue when there are two bidders and a standard uniform common value. Moreover, not only is it impossible for revenue to fall below the level of the example, but the distribution of winning bids in any information structure and any equilibrium must always first-order stochastically dominate the winning bid distribution in the equilibrium we just constructed.

The full proof of this result will be established in Theorem 1 below. To develop intuition, we will give a partial proof that revenue cannot fall below 1/6. First, we notice that the equilibria in the no-information, complete information, and independent signal constructions all have the feature that the winning bid is a deterministic and increasing function $\beta(v)$ of the true value v. In the no-information case, the winning bid is always $\beta(v) = 1/2$; under complete information, the winning bid is $\beta(v) = v$; and in the independent signal construction, $\beta(v) = v/3$. Let us explore more generally what can happen in symmetric equilibria of this form, in which revenue will be

$$\int_{v=0}^{1} \beta(v)\, dv.$$

An equilibrium in which the winning bid is $\beta(v)$ must deter a large number of deviations, most of which we cannot assess without explicitly modeling the rest of the equilibrium. There is, however, one class of deviations which we can evaluate using only information about winning bids: for some $v \in [0,1]$, bid $\beta(v)$ whenever the equilibrium bid would have been some $x \leq \beta(v)$. We refer to this as a *uniform deviation up to* $\beta(v)$. Such a deviation results in a (potential) gain of surplus from winning when the bidder would have lost by following the equilibrium strategy:

$$\frac{1}{2}\int_{x=0}^{v} (x - \beta(v))\, dx.$$

This expression can be interpreted as follows: the upward deviator would have lost in equilibrium 1/2 of the time, and by deviating up to $\beta(v)$, this bidder will now win when they would have lost and the value is some $x \leq \beta(v)$ and pay a price of $\beta(v)$. There is also, however, a certain loss associated with paying extra when the bidder would have won anyway:

$$\frac{1}{2}\int_{x=0}^{v} (\beta(v) - \beta(x))\, dx.$$

In particular, with ex-ante probability 1/2, the upward deviator would have been the bidder to win in equilibrium, and thus for values less than x, this bidder still wins but pays an extra amount of $\beta(v) - \beta(x)$. Thus, the equilibrium deters uniform upward deviations if and only if the loss from the uniform upward deviation exceeds the gain, i.e.,

$$\frac{1}{2}\int_{x=0}^{v} (x - \beta(v))\, dx \leq \frac{1}{2}\int_{x=0}^{v} (\beta(v) - \beta(x))\, dx,$$

for all v. This condition rearranges to

$$\beta(v) \geq \frac{1}{v}\int_{x=0}^{v} (x + \beta(x))\, dx. \tag{1}$$

A relaxation of the original problem of minimizing revenue over all information structures and equilibria (of this particular form) is to minimize revenue over all bidding functions that satisfy (1) and the condition that $\beta(v) \geq 0$. The solution of this relaxed program has a simple form. Suppose that at the optimum, the incentive constraint (1) holds as a strict inequality for some v. Then it is possible to decrease $\beta(v)$ while still deterring a uniform deviation up to $\beta(v)$. Moreover, inspection of the right-hand side of (1) reveals that decreasing $\beta(v)$ actually relaxes the constraint even further and makes all uniform upward deviations weakly less attractive. Decreasing $\beta(v)$ is therefore feasible, and since revenue is increasing in β, the modification must lower revenue.

At the optimum, (1) must therefore hold as an equality for all v. The unique solution to this integral equation with the initial condition $\beta(0) = 0$ is the minimum winning bid function $\underline{\beta}(v) = v/3$. This bidding function minimizes the distribution of winning bids in the sense of first-order stochastic dominance, within the class of equilibria we considered and subject only to the incentive constraints associated with uniform upward deviations. In fact, we will subsequently show in Section 4 that this bound continues to hold even if one allows asymmetric equilibria and equilibria in which the winning bid is stochastic conditional on v. This implies that revenue cannot fall below 1/6 in any equilibrium under any information structure, and moreover provides a global lower bound on the distribution of winning bids. Thus, the independent signal information structure and its equilibrium attain a global lower bound on the distribution of winning bids.

4 Minimum Bidding, Minimum Revenue, and Maximum Bidder Surplus

We shall see that the characterization of minimum winning bids can be generalized to *any* symmetric prior distribution over values and *any* number of bidders. By minimum, we mean that this distribution will be first-order stochastically dominated by any distribution of equilibrium winning bids that arises from any equilibrium in any information structure. We will also construct an information structure and equilibrium under which the generalized bound is attained. We do not rely on any assumptions of independence or affiliation, and the distribution of values, while symmetric, is allowed to be correlated in an arbitrary manner. The qualitative features of the solution, and the methods used to characterize it, closely resemble the arguments in the uniform example of the previous section: the relevant incentive constraints that pin down the minimum are those corresponding to upward deviations, and in particular, uniform upward deviations. The winner's bid will be a deterministic function of the profile of values, and it will have to be monotonic in a sense described below.

4.1 Preview and Statement of Main Result

In Section 3, we derived a formula for minimum equilibrium winning bids in the common value case with two bidders, where the common value was drawn according to a standard uniform distribution. As a preview to stating our main result, we give a heuristic account of how this formula generalizes as we encompass progressively broader classes of models.

First, consider the case where the common value was drawn from an arbitrary continuous distribution $P(v)$ on the interval $[\underline{v}, \overline{v}]$. All of our analysis would continue to go through using the appropriate probabilities of winning derived from P, and we would conclude that the minimum winning bid function would have to satisfy the integral inequality

$$\frac{1}{2}\int_{x=\underline{v}}^{v}\left(\beta(v)-\beta(x)\right)P(dx) \geq \frac{1}{2}\int_{x=\underline{v}}^{v}\left(x-\beta(v)\right)P(dx), \tag{2}$$

which represents the incentive constraint that each bidder does not want to uniformly deviate up to the higher bid $\beta(v)$. The solution of this integral equation with the initial condition $\beta(\underline{v}) = \underline{v}$ turns out to be

$$\underline{\beta}(v) = \frac{1}{\sqrt{P(v)}}\int_{x=\underline{v}}^{v} x\frac{P(dx)}{2\sqrt{P(x)}}. \tag{3}$$

When $P(v) = v$ and $[\underline{v}, \overline{v}] = [0, 1]$, this formula reduces to $\underline{\beta}(v) = v/3$.

Second, consider the case where there are still two bidders, but where those bidders were allowed to have distinct values, drawn from a joint distribution $P(dv_1, dv_2)$. The relevant constraints will continue to be those associated with uniform upward deviations. In fact, these incentive constraints can be represented by precisely the same formula (2), but with a different interpretation. Recall that the right-hand side of this constraint represents a deviator's potential gains from winning when he would have lost in equilibrium. Thus, the values that we are integrating over are those belonging to the buyer who loses in equilibrium, and the distribution with respect to which we should integrate is the distribution of the loser's value. If we reinterpret the P in the formula as the distribution of the loser's value, then this inequality describes the incentive constraint corresponding to a uniform upward deviation, as long as the winner's bid is a deterministic and strictly increasing function of the losing buyer's value. Of course, the allocation of the good and the distribution of losing values are endogenously determined in equilibrium. It will turn out, however, that the allocation that minimizes the distribution of winning bids is efficient, i.e., the loser always has the lower of the two values. The reason is that the efficient allocation achieves the lowest expected value of the losing buyer, which results in weaker incentives to deviate upwards. Summing up, the minimum winning bid function continues to be described by (3), but with the reinterpretation of $P(v)$ as the cumulative distribution of the lower of the two values. If we write $Q(v) = 1 - P\left([v, \overline{v}]^2\right)$ for this cumulative distribution of the lowest value, then the minimum winning bid function is

$$\underline{\beta}(v) = \frac{1}{\sqrt{Q(v)}} \int_{x=\underline{v}}^{v} x \frac{Q(dx)}{2\sqrt{Q(x)}}. \tag{4}$$

Third, consider the general case, with N bidders whose values are jointly distributed according to the probability measure $P(dv_1, \ldots, dv_N)$. The equilibrium bid distribution continues to be characterized by a winning bid that is a deterministic function of the losing bidders' values, for the same reason as with two bidders: it is only the losing bidders' values that matter directly for uniform upward constraints. Moreover, for the same reasons as with two bidders, an efficient allocation minimizes the distribution of losing bidders' values and relaxes the uniform upward incentive constraints, which in turn lowers the distribution of winning bids. The remaining question is how the winning bid depends on the $N-1$ values that lose the auction in equilibrium.

To get some intuition for this, let us reason by analogy with the benchmark of complete information, in which all bidders see the entire profile of values. The equilibria in this information structure involve the bidder with the highest value winning the good, and paying a price which is equal to the maximum of the $N-1$ lowest values. There is a simple garbling of this information structure which will result in the same allocation but substantially lower revenue. Suppose that instead of knowing the entire profile of values, the bidders only learn (i) the identity of the bidder with the highest value and (ii) the distribution of values among the remaining bidders. Importantly, losing bidders do not learn who has which value, beyond knowing the identity of the winner. The equilibria in this information structure will still involve the high value bidder (whose identity is known to all of the bidders) winning the auction. However, the losing bidders no longer know who has which value, but only the distribution of realized values. Moreover, since the prior distribution P is symmetric, all of the value profiles that induce this distribution of realized values are equally likely. The losing buyers therefore believe their own value is drawn from the distribution of realized losing values, and expect their value to be

$$\mu(v) = \frac{1}{N-1}\left(\sum_{i=1}^{N} v_i - \max v\right). \tag{5}$$

The equilibria for this information structure will therefore involve the winner being the bidder with the highest value and paying a price of $\mu(v)$. This is an equilibrium because the losing buyers can only win by deviating to a bid higher than $\mu(v)$, which on average would be greater than the deviator's value. Revenue will be substantially lower, since the winner will pay the average, rather than the maximum, of the losing buyers' values.

Based on this intuition, we can guess that the generalized minimum winning bid will depend only on the average of the losing buyers' values. Let us now use Q to denote the distribution of the mean losing value $m = \mu(v)$ which is induced by P. For any Borel set $X \subseteq \mathbb{R}_+$, this distribution assigns a probability

$$Q(X) = P\left(\mu^{-1}(X)\right) \tag{6}$$

to the average losing value being in X. We can write $M = [\underline{m}, \overline{m}]$ for the convex hull of the support of Q. Now let $\underline{\beta}$ be defined by

$$\underline{\beta}(m) = \frac{1}{Q^{\frac{N-1}{N}}(m)} \int_{x=\underline{m}}^{m} x \frac{N-1}{N} \frac{Q(dx)}{Q^{\frac{1}{N}}(x)}, \tag{7}$$

and let $\underline{H}$ be defined by

$$\underline{H}(b) = Q\left(\underline{\beta}^{-1}(b)\right). \tag{8}$$

Our main result for this section will be:

Theorem 1 (Minimum Winning Bids)**.**

(i) Any distribution of winning bids H arising in some information structure and equilibrium must first-order stochastically dominate $\underline{H}$, in the sense that $H(b) \leq \underline{H}(b)$ for all b;

(ii) There exists an information structure and an efficient equilibrium in which the distribution of winning bids is exactly $\underline{H}$, and the winning bid is the deterministic and increasing function $\underline{\beta}(m)$ of the average losing value given by (7).

The minimum winning bid function $\underline{\beta}(m)$, as well as minimum revenue, have the following simple and remarkable interpretation. Consider a symmetric and independent private values model in which the bidders' values are independently drawn from the cumulative distribution $F(v) = Q^{1/N}(v)$. The standard Bayes Nash equilibrium involves each bidder bidding according to a symmetric and monotonic strategy, which turns out to be precisely $\underline{\beta}(v)$. Moreover, the maximum bid will be submitted by the bidder with the highest value, which has distribution $(F(v))^N = Q(v)$. Thus, the minimum distribution of winning bids and minimum revenue are equal to the winning bid distribution and revenue that would arise in a symmetric independent private values model in which the distribution of the highest "as if" value is equal to the distribution of the average of the $N-1$ lowest true values. We shall see that this interpretation is intimately connected to the information structure and equilibrium that we construct to attain the bounds.

Immediate implications of Theorem 1 are characterizations of minimum revenue and maximum bidder surplus. Let $\underline{R}$ be defined by

$$\begin{aligned}\underline{R} &= \int_{m=\underline{m}}^{\overline{m}} \underline{\beta}(m)\; Q(dm) \\ &= \overline{m} + \int_{m=\underline{m}}^{\overline{m}} \Big((N-1)\, Q(m) - N Q^{\frac{N-1}{N}}(m) \Big)\, dm. \end{aligned} \tag{9}$$

Corollary 1 (Minimum Revenue)**.**
Any equilibrium in any information structure must result in revenue that is at least $\underline{R}$*. Moreover, there exists an equilibrium for some information structure in which revenue is exactly* $\underline{R}$*.*

The formula for minimum revenue is simply obtained by plugging in $\underline{\beta}(m)$ and then integrating by parts. Recall that $\overline{T}$ is the total surplus that is generated by the efficient allocation, and let

$$\overline{U} = \overline{T} - \underline{R}.$$

Corollary 2 (Maximum Bidder Surplus)**.**
Any equilibrium in any information structure must result in bidder surplus that is less than $\overline{U}$*. Moreover, there exists an equilibrium for some information structure in which bidder surplus is exactly* $\overline{U}$*.*

The rest of this section will be devoted to the proof of Theorem 1. The proof will consist of two main pieces. We will first set up and solve a relaxed program which generalizes the one we introduced in the example of Section 3. As in the example, this relaxed program will entirely ignore the distribution of losing bids, and just track the statistical relationship between winning bids and the bidders' values. In addition, we will only require that these winning bid distributions deter uniform upward deviations. Unlike the example, however, we will allow for arbitrary and potentially asymmetric distributions of winning bids conditional on the realized profile of values. The key steps in our characterization will be to show that it is without loss of generality to restrict attention to solutions of the relaxed problem that (i) are symmetric, (ii) give rise to an efficient allocation, and (iii) involve winning bids that are a strictly increasing and deterministic function of the average of the realized losing values. Properties (i) - (iii) collapse the relaxed program into one that has all of the structure we assumed uniform example, and in which we are solving for a single strictly increasing function. The optimal winning bid function will then be characterized by bidders being indifferent to all uniform upward deviations.

The second piece of the proof is the construction of an equilibrium that exactly attains the solution to the relaxed program. This step again mirrors the construction for the uniform example of Section 3. This information structure and equilibrium turn out to be remarkably simple, and involve the bidders receiving independent signals and using a symmetric monotonic pure strategy which is equal to the minimum winning bid function. While the signals are independent of one another, they are correlated with the true values in a particular way so that the distribution of the highest of the independent bids will coincide with the solution of the relaxed program.

We will conclude the section with examples that illustrate the mechanics of minimum revenue in common value and interdependent value settings. We will also explore comparative statics as the number of bidders becomes large.

4.2 The Relaxed Program

We first describe a relaxed program, minimizing the winning bid distribution and thus revenue imposing only a subset of equilibrium conditions. For any given strategy profile σ, let

$$H_i\left(b|v\right) = \int_{s\in S} \int_{\{x\in B^N | x_j \leq x_i \leq b \ \forall j\neq i\}} \frac{1}{|\arg\max x|} \sigma\left(dx|s\right) \pi\left(ds|v\right) \tag{10}$$

denote the probability that bidder i wins with a bid $b \in B$ when the profile of values is v. In addition, let

$$H\left(b|v\right) = \sum_{i=1}^{N} H_i\left(b|v\right)$$

denote the total probability that the winning bid is less than b when the distribution of values is v. Finally, let

$$H\left(b\right) = \int_{v\in V^N} H\left(b|v\right) P\left(dv\right)$$

denote the aggregate distribution of winning bids. Note that the probability $H_i\left(b|v\right)$ is conditional on v, but not on the identity of the winner, so that $H_i\left(\infty|v\right)$ need not be 1. The H_i are, however, monotonically increasing, measurable with respect to V^N, and must satisfy $H_i\left(b|v\right) = 0$ for all $b < 0$ and $H\left(b|v\right) \leq 1$ for all b.

The $\{H_i\left(\cdot|\cdot\right)\}$ are a family of marginal distributions that are induced by the equilibrium, and they contain insufficient information to evaluate all potential deviations from equilibrium. Even so, we can use them to evaluate the class of uniform upward deviations that we introduced previously. In particular, bidders must not want to uniformly deviate up to any bid b by bidding b whenever their equilibrium bid would have been some $x \leq b$. As before, we can decompose the change in surplus from such a deviation into two pieces. First, it may have been that the bidder would have lost in equilibrium and obtained zero rents, whereas by deviating upwards, she will now win whenever a winning bid less than b would have been made by one of the other bidders. The change in surplus on this event is therefore

$$\int_{v \in V^N} (v_i - b) \left(H(b|v) - H_i(b|v) \right) P(dv).$$

On the other hand, it may have been that the deviator would have won anyway if she had followed the equilibrium strategy. By deviating, she will still win but will have to pay more, resulting in a loss of

$$\int_{v \in V^N} \int_{x=0}^{b} (b - x) H_i(dx|v) P(dv) = \int_{v \in V^N} \int_{x=0}^{b} H_i(x|v) \, dx P(dv).$$

In order for such a deviation to not be attractive, it must therefore be that

$$\int_{v \in V^N} (v_i - b) \left(H(b|v) - H_i(b|v) \right) P(dv) \leq \int_{v \in V^N} \int_{x=0}^{b} H_i(x|v) \, dx P(dv). \tag{11}$$

Note that this formulation of the uniform upward incentive constraints implicitly assumes that the upward deviator wins all ties at the cutoff bid b. This issue is addressed in the proof of Lemma 1 below.

Our relaxed problem is to minimize

$$\int_{b \in B} f(b) \ H(db), \tag{12}$$

over all families $\{H_i(\cdot|\cdot)\}$ of winning bid distributions that satisfy (11) where f is some weakly increasing function. Thus, we restrict attention to winning bid distributions that satisfy (11), i.e., that deter uniform upward deviations. If f was the identity function, the objective would correspond to revenue. However, we are also interested in the entire distribution of losing bids, which is why we allow for a general bid-weighting function. In fact, the solution will turn out to be independent of the choice of f. This implies that the solution is a lower bound on the distribution of winning bids satisfying (11) in the sense of first-order stochastic dominance, which is of course what we wish to show for Theorem 1.

The rest of this subsection will be devoted to the solution of the relaxed program. We first verify existence of a solution.

Lemma 1 (Relaxed Program)**.**
A solution to the relaxed program exists. Moreover, any equilibrium under any information structure must induce a winning bid distribution that is feasible for the relaxed program.

Our next three results show that it is without loss of generality to look at a relatively small family of candidate optima that are (i) symmetric, (ii) are associated with efficient allocations, and (iii) satisfy a particular form of monotonicity with respect to the average losing value. These results will justify the assumptions that we made above in the context of the uniform example that the winning bid was a deterministic and strictly increasing function of the common value. Once these properties have been established, we will use essentially the same argument as in Section 3 to characterize the optimal winning bid function, with the average of the $N-1$ lowest values playing the role of the common value.

First, say that a solution $\{H_i(\cdot|v)\}$ is *symmetric* if for all $\xi \in \Xi$, $H_{\xi(i)}(b|\xi(v)) = H_i(b|v)$. In other words, the probability of a bidder winning with bid less than b only depends on (i) the bidder's own value and (ii) the distribution of the bids of the other bidders, but it does not depend on how those values are matched to bidders' individual identities.

Lemma 2 (Symmetry)**.**
For any feasible solution to the relaxed program, there exists a symmetric feasible solution with the same aggregate distribution of winning bids.

The idea behind the proof is that if we had a feasible solution that was asymmetric, it is possible to "symmetrize" the solution by creating new winning bid distributions that are the average of the winning bid distributions over all permutations of the bidders identities. For example, suppose that there are only two bidders and that we have a potentially asymmetric solution $\{H_i(\cdot|\cdot)\}$. We can use this to create a new solution

$$\widetilde{H}_1(b|v_1,v_2) = \widetilde{H}_2(b|v_2,v_1) = \frac{1}{2}\left(H_1(b|v_1,v_2) + H_2(b|v_2,v_1)\right).$$

This new solution $\left\{\widetilde{H}_i(\cdot|\cdot)\right\}$ is symmetric, and since the objective and constraints for the relaxed program are all linear in the H_i, the constraints that were previously satisfied will still be satisfied at the symmetrized solution. In light of Lemma 2, we will henceforth assume that the solution we are working with is symmetric.

Second, say that a solution $\{H_i(\cdot|\cdot)\}$ to the relaxed program is *efficient* if $H_i(\cdot|v) = 0$ whenever $v_i < \max v$. The second property that we can assume without loss of generality is that the solution is efficient.

Lemma 3 (Efficiency)**.**
For any feasible solution to the relaxed program, there exists an efficient feasible solution with the same aggregate distribution of winning bids.

The result is somewhat surprising: generally speaking, bidders might not bid as aggressively if they thought there were less surplus to be obtained in equilibrium, which would be the case if the allocation were inefficient. In fact, the opposite is the case: increasing the efficiency of the allocation always weakens bidders' incentives to deviate upwards. The reason is the following. The change in a bidder's surplus that results from a uniform upward deviation does not depend at all on the bidder's value when they win in equilibrium, since they will still win after the upward deviation. The incentive to deviate depends very much, however, on the deviator's value when she would lose the auction in equilibrium, since by deviating upwards she may win on this event and obtain additional rents from consuming the good. The incentive to deviate up is therefore weaker when the expected value conditional on losing is lower, and all things equal, the losing value will be lower when the allocation is efficient.

Now that we can restrict attention to symmetric and efficient solutions, it is possible to collapse the relaxed program into a somewhat more compact form. Note that the value of the bidder who wins the auction only appears in the objective and constraints as a variable over which we integrate. It is only the losing bidders' values that enter directly in the program, through the incentive constraint (11). Thus, it is in principle possible to integrate out the winner's value and only condition the winning bid distributions on losing values. Thus, in the case of two bidders, there will be a one-dimensional family of conditional distributions of winning bids, indexed by the lower of the two values $v^{(2)}$:

$$H\left(b|v^{(2)}\right)=\int_{v^{(1)}\geq v^{(2)}} H\left(b|v^{(1)},v^{(2)}\right)P_{v^{(1)}|v^{(2)}}\left(dv^{(1)}|v^{(2)}\right),$$

where $P_{v^{(1)}|v^{(2)}}\left(\cdot|\cdot\right)$ is a version of the conditional distribution of the highest value given the lowest value.

In fact, we can go even further. When there are more than two bidders, there will be $N-1$ losing values, and in principle we might have to index winning bid distributions over all of these losing values. The fact that the H_i are symmetric implies that the distribution of winning bids will be the same for all permutations of the losing buyers' values. Thus, a bidder who uniformly deviates up to b will win with probability $H\left(b|v\right)$ when the profile of values is v, but he will also win with the same probability when the profile of values is $\xi\left(v\right)$ for any permutation $\xi\in\Xi$. In short, this bidder believes that he will win whenever the profile of values is in the equivalence class

$$[v] = \{\xi(v) | \xi \in \Xi\}.$$

Symmetry of the distribution P therefore implies that the deviator is equally likely to have any of the values in the profile v. In equilibrium, they would have lost when their value was not the highest, or in the case of ties, when their value was the highest but they lost the tie break. The expected value conditional on losing in equilibrium and conditional on $[v]$ is therefore $\mu(v)$ as defined by (5), which is the average of the $N-1$ lowest values in the profile v. This must also be the expected surplus that the bidder gains by winning when he would have lost in equilibrium on the event $[v]$.

We can use this observation to collapse the relaxed program down by integrating out the different profiles that induce the same average losing value. Let us write Q for the measure over the average losing value that is induced by P, which is defined by (6). The continuity assumption on P that we made in Section 2 implies that the events $\mu^{-1}(m)$ have zero probability, so that $Q(m)$ is absolutely continuous. We write $M \subseteq V$ for the convex hull of the support of Q, with $\underline{m} = \min M$ and $\overline{m} = \max M$.

Note that the conditional distribution $H(b|v)$ induces a joint distribution $\phi \in \Delta\left(V^N \times B\right)$, which also induces a probability measure $\psi \in \Delta(M \times B)$, where $\psi(X) = \phi\left((\mu \times \mathbf{I})^{-1}(X)\right)$, and $\mathbf{I} : B \to B$ is the identity mapping. The marginal of ψ on M will necessarily be Q, and the joint distribution ψ can be disintegrated to produce a probability transition kernel $H : M \to \Delta(B)$, where $H(\cdot|m)$ is the distribution of the winning bid conditional on the average losing value.

With this notation in hand, we can now rewrite the relaxed program as follows

$$\min \int_{m \in M} \int_{b \in B} f(b)\ H(db|m)\, Q(dm), \tag{13}$$

subject to

$$H(b|m) \in [0,1]\ \ \forall m \in V, b \in B, \tag{14}$$

and

$$\frac{N-1}{N} \int_{m \in M} (m-b)\, H(b|m)\, Q(dm) \leq \frac{1}{N} \int_{m \in M} \int_{x=0}^{b} H(x|m)\, dx Q(dm). \tag{15}$$

We say that the solution H is *monotonic* if $H(b|m) < 1$ implies that $H(b|m') = 0$ for all $m' > m$. In other words, the supports of the winning bid distributions are monotonically increasing in the average losing value. Our next result will show that it is without loss of generality to restrict attention to solutions that are monotonic. The reason is the following. Note that the incentive constraint (15) can be rewritten as

$$\frac{N-1}{N}\int_{m\in M} m\,H\left(b|m\right)Q\left(dm\right)\leq\frac{N-1}{N}b\,H\left(b\right)+\frac{1}{N}\int_{x=0}^{b}H\left(x\right)dx.$$

Let us consider how this constraint is affected by varying the solution but while maintaining a fixed aggregate distribution of winning bids $H(b)$. The only part of the expression which depends on how winning bids are correlated with values is the left-hand side, which is proportional to the expectation of the average losing value conditional on the winning bid being less than b. On the whole, decreasing this expectation is going to relax the constraint, which is intuitive, as lower losing values translate into smaller gains from a uniform upward deviation. Monotonicity essentially says that the lowest losing values should be associated with the lowest winning bids. In fact, this is the structure that will minimize pointwise, for every b, the expectation of m conditional on the winning bid being less than b, and thereby relax the constraint as much as possible while maintaining $H(b)$.

Lemma 4 (Monotonicity)**.**
For any feasible solution to the relaxed program, there exists a monotonic feasible solution with the same aggregate distribution of winning bids.

Incidentally, this result also shows that it is without loss of generality to consider solutions to the relaxed program that correspond to a deterministic winning bid $\beta(m)$ as a function of the average losing value, which is defined by

$$\beta\left(m\right)=\min\left\{b|H\left(b\right)\geq Q\left(m\right)\right\}.$$

We can therefore write down a final form for the relaxed program:

$$\min\int_{m\in M} f\left(\beta\left(m\right)\right)Q\left(dm\right), \tag{16}$$

subject to $\beta(m)$ being weakly increasing in m and

$$\frac{N-1}{N}\int_{x=\underline{m}}^{m}\left(x-\beta\left(m\right)\right)Q\left(dx\right)\leq\frac{1}{N}\int_{x=\underline{m}}^{m}\left(\beta\left(m\right)-\beta\left(x\right)\right)Q\left(dx\right). \tag{17}$$

It is now apparent that our situation is quite close to where we began in the example of Section 3. Our object of choice in the relaxed program is a deterministic winning bid as a function of a one-dimensional statistic. In the case of pure common values, this statistic was the true value of the good, and in the general model it is the average of the $N-1$ lowest values.

Lemma 5 (Binding Uniform Upward Constraints)**.**
An optimal solution to the relaxed program must satisfy $\beta(\underline{m}) = \underline{m}$ *and solves (17) with equality for all* $m \in M$*.*

One can verify that the solution to this formula is given by precisely (8). The solution of the relaxed program is summarized as follows:

Proposition 1 (Solution to the Relaxed Program)**.**
For any choice of weights f*, the solution to the relaxed program is the deterministic minimum winning bid function* $\underline{\beta}$ *given by (7). The resulting aggregate winning bid distribution* $\underline{H}$ *is given by (8). Since this solution is independent of the weights* f*,* $\underline{H}$ *must be first-order stochastically dominated by any feasible solution to the relaxed program, and therefore any distribution of winning bids that can arise in equilibrium under some information structure.*

4.3 A Minimum Bidding Information Structure and Equilibrium

At this point, we have already proven the first part of Theorem 1, which is that our $\underline{H}$ must be first-order stochastically dominated by any equilibrium winning bid distribution. To show that the bounds are tight, we will simply construct an efficient equilibrium in which the distribution of winning bids is precisely $\underline{H}$. This construction will generalize the independent signal information structure of Section 3.

In particular, the bidders will receive signals which are *independent* of one another and drawn from the same distribution

$$F(s) = Q^{\frac{1}{N}}(s)$$

on the support $S = M$. This distribution is chosen so that the highest signal is distributed according to Q. Signals will be correlated with values so that (i) the highest signal is always equal to the realized average losing value and (ii) it is the bidder with the highest value who receives the high signal. So that the reader is convinced that it is possible to correlate signals and values in this manner, we could have alternatively specified the information structure by first, drawing a profile of values v from $P(dv)$, then giving the highest value bidder a signal $m = \mu(v)$, and then giving the losing bidders signals which are independent draws from $L(s|m) = F(s)/F(m)$.

In equilibrium, all bidders use the symmetric strategy of bidding $\sigma(s) = \underline{\beta}(s)$. It is a well-known result in auction theory that these strategies are an equilibrium for a different model in which values are independent and symmetric draws from F. Specifically, in the independent private value (IPV) model, bidders must not want to deviate from their strategy by bidding $\sigma(m)$ when they receive some signal $s \neq m$. If we write $G(s) = F^{N-1}(s)$ for the distribution of the maximum of others' signals, such a bid would result in surplus

$$(s - \sigma(m))\, G(m),$$

so that the equilibrium σ satisfies the first-order condition

$$(s - \sigma(s))\, G(ds) - \sigma'(s)\, G(s) = 0.$$

When $G(s) = Q^{(N-1)/N}(s)$, the solution to this differential equation is precisely $\underline{\beta}$. In our model, the signals have a quite different interpretation and a complicated relationship with values. Nonetheless, we shall see that these strategies still constitute an equilibrium.

Indeed, the connection to the IPV model provides straightforward arguments that our construction is an equilibrium. Consider a bidder who receives a signal s and bids $\underline{\beta}(m)$ with $m < s$. Such a bidder would only win when he had the highest signal, in which case he expects his value to be the maximum of everyone's values conditional on $\mu(v) = s$, which we can denote by $\tilde{v}(s)$. Note that $\tilde{v}(s)$ must be weakly larger than s and is uncorrelated with others' signals, conditional on others' signals being less than s. Thus, a downward deviator receives surplus

$$\left(\tilde{v}(s) - \underline{\beta}(m)\right) G(m),$$

since the deviator will only win when all of the $N-1$ other bidders receive signals less than m. The derivative of this expression with respect to m is

$$\left(\tilde{v}(s) - \underline{\beta}(m)\right) G(dm) - \underline{\beta}'(m)\, G(m) \geq \left(s - \underline{\beta}(m)\right) G(dm) - \underline{\beta}'(m)\, G(m).$$

It is a well-known feature of the IPV equilibrium that the surplus of a bidder with value s is single-peaked in their bid. Thus, the right-hand side of this inequality is necessarily positive for $m < s$, and as such, a bidder who believes their value to be even larger than s will not want to deviate down as well.

Now let us consider upward deviations. Note that a bidder with signal s believes that by following the equilibrium strategy, they win with probability $G(s)$ and have value $\tilde{v}(s)$. On the other hand, by deviating upwards to some $\underline{\beta}(m)$ with $m > s$, the bidder will win when the highest signal of others is in $[s,m]$, and their expected value conditional on winning when $\max s_{-i}=x$ is precisely x, since the highest signal of others must equal the average losing value.[11] The surplus from the upward deviation is therefore

$$\left(\tilde{v}(s)-\underline{\beta}(m)\right)G(s)+\int_{x=s}^{m}\left(x-\underline{\beta}(m)\right)G(dx)\,. \tag{18}$$

The derivative with respect to m is

$$\left(m-\underline{\beta}(m)\right)G(dm)-\underline{\beta}'(m)\,G(m)\,.$$

But the first-order condition that defines $\underline{\beta}$ at m says that this quantity must be zero, so that bidders are indifferent to all upward deviations. Thus, we have proved the following:

Proposition 2 (Winning Bid Minimizing Equilibrium)**.**
There exists an information structure and efficient equilibrium in which the winning bid is a deterministic function of the average losing value and is given by $\underline{\beta}(m)$*.*

Proposition 2 completes the proof of Theorem 1. We have constructed an information structure and equilibrium in which the distribution of winning bids is precisely the solution to the relaxed program, so that minimum revenue must be attained. Moreover, it is always a bidder with a high value who receives the precise signal, so that the equilibrium allocation is efficient.

We have constructed a simple information structure and equilibrium in which the bounds from the relaxed program are attained. A natural question is: are there other information structures and equilibria that would also attain the bound, and more generally, what does this minimizing class look like? Except in special cases, the minimizing information structure and equilibrium is not unique. However, all constructions that attain the bounds must share a number of features. First, the induced allocation must be efficient, the winning bid must be a deterministic function $\underline{\beta}$ of the average losing value, and bidders must be indifferent to all uniform upward deviations. Second, we can, without loss of generality, restrict attention to information structures in which the signal space is the set of average losing values. Moreover, this can be done so that in equilibrium all bidders use the symmetric monotonic pure strategy $\underline{\beta}$. If we had an information structure with richer signals or different equilibrium strategies, then we could always construct another information structure equilibrium where bidders

[11] Note that for non-common value models, $\tilde{v}(s)$ is generally strictly larger than s. This means that a bidder's expectation of their own value is non-monotonic in others' signals: when $\max_{j\neq i}s_j<s_i$, bidder i expects his value to be $\tilde{v}(s_i)>s_i$, but when $\max_{j\neq i}s_j>s_i$, bidder i expects his value to be $\max_{j\neq i}s_j$. This non-monotonicity puts our information structure outside the affiliated values model of Milgrom and Weber (1982), though this violation is not so severe as to disrupt an equilibrium in monotonic pure strategies.

simply receive a "recommendation" to bid their equilibrium bid b in the form of the signal $s = \underline{\beta}^{-1}(b)$ (Bergemann and Morris, 2013, 2015a). Combined with the previous properties of information and equilibrium, we conclude that precisely one bidder must receive a signal equal to the average losing value, and this bidder must be the bidder with the highest valuation. The remaining bidders must all receive lower signals.

In fact, it turns out that the marginal distribution of each bidder's losing signals is uniquely pinned down in terms of Q. The reason is the following. From the relaxed program, we concluded that the lower bound is characterized by all uniform upward constraints binding. An implication of this result is that in an equilibrium that attains the bound, all *pointwise* upward constraints must bind as well. In other words, for any signal s, the bidder must be indifferent to deviating to $\underline{\beta}(m)$ for all $m > s$. The reason is that if there were a set of signals for which the bidder strictly preferred his equilibrium bid over deviating up to m, then since the bidder is indifferent to the uniform deviation up to m, there must be another set of signals on which the bidder would strictly prefer to deviate up.

Now, the fact that all pointwise upward constraints bind effectively pins down the marginal distribution of losing signals. Recall our notation $L(s|m)$ for the distribution of a losing bidder's signal conditional on the average losing value (and highest signal) being m. A bidder who receives signal s and deviates up to $\underline{\beta}(m)$ with $m > s$ expects to obtain surplus

$$\frac{1}{N}\left(\tilde{v}(s) - \underline{\beta}(m)\right) Q(ds) + \frac{N-1}{N}\int_{x=s}^{m}\left(x - \underline{\beta}(m)\right) L(ds|x)\, Q(dx).$$

The first term represents surplus on the event that s was the true average losing value and this bidder had the highest valuation. The second term represents the additional surplus on the event that the average losing value was in $[s, m]$ and this bidder did not have the highest value, so that they would have lost in equilibrium. Note that these events are weighted by the likelihood of receiving the signal s. For the bidder to be indifferent to deviating upwards, this surplus must be constant in m. The derivative with respect to m is

$$\left(m - \underline{\beta}(m)\right)\frac{N-1}{N} L(ds|m)\, Q(dm) - \underline{\beta}'(m)\left[\frac{N-1}{N}\int_{x=s}^{m} L(ds|x)\, Q(dx) + \frac{1}{N} Q(ds)\right]. \tag{19}$$

Note that we can use (17) to derive the following differential equation that equivalently defines $\underline{\beta}$:

$$\underline{\beta}'(m) = \frac{N-1}{N}\frac{Q(dm)}{Q(m)}\left(m - \underline{\beta}(m)\right).$$

Substituting this into (19) gives the integral equation

$$L\left(ds|m\right)Q\left(m\right)=\frac{N-1}{N}\int_{x=s}^{m}L\left(ds|x\right)Q\left(dx\right)+\frac{1}{N}Q\left(ds\right),$$

which has the solution

$$L\left(s|m\right)=\left(\frac{Q\left(s\right)}{Q\left(m\right)}\right)^{\frac{1}{N}}.$$

In other words, the losing bidder must receive a draw from the distribution $Q^{1/N}\left(s\right)$ truncated at m, which is precisely what happens in our construction.

Thus, the marginal distribution of each bidder's losing signals is uniquely pinned down. One remaining degree of freedom is the correlation structure among losing bidders' signals. In the construction for Proposition 2, we specified that losing signals were independent of one another. However, it may be possible to introduce correlation without violating any incentive constraints. Another degree of freedom is that losing bidders' signals may be correlated with the values in more complicated ways, for example by having losing signals be correlated with the winner's value. In the case of $N=2$ and pure common values, neither of these additional forms of correlation are possible, and in that setting the minimizing information and equilibrium are essentially unique. More generally, the construction that we have provided is, in a sense, the least informative information structure that attains the bounds.

Finally, we comment on the maintained assumption of a symmetric prior on values. Symmetry allowed us to collapse the relaxed program into the choice of a single winning bid distribution which does not depend on the identity of the winner. If the prior were asymmetric, then in general the optimal winning bid distribution would depend on who wins. There is, however, a simple way to apply our symmetric methodology to asymmetric models in order to obtain a lower bound on the distribution of winning bids: we can associate to any asymmetric prior P a "symmetrized" measure $\widetilde{P}$. This symmetrization is obtained via a two step randomization, where first we permute the identities of the bidders according to a randomly chosen $\xi\in\Xi$ and then draw values for the bidders under their permuted identities according to the original P. The distribution $\widetilde{P}$ generated in this manner will be symmetric, and our characterization yields a minimum winning bid distribution $\underline{H}$ for $\widetilde{P}$. The distribution $\underline{H}$ must also be a lower bound on any equilibrium winning bid distribution H with prior P: any information structure $\mathcal{S}$ and equilibrium σ for P can be used to construct an information structure $\widetilde{\mathcal{S}}$ and equilibrium $\widetilde{\sigma}$ for $\widetilde{P}$, by following the same symmetrizing procedure, i.e., permuting the players at random and then drawing values, signals, and bids from $(P,\mathcal{S},\sigma)$ according to the permuted identities. For each permutation, the distribution of winning bids will be the same H, so on average we will have H as well. The distribution H is therefore feasible for $\widetilde{P}$ and must exceed the minimum $\underline{H}$. The asymmetric model in effect imposes additional constraints on the information that the bidders have, since it is as if there is common knowledge amongst the bidders which permutation has obtained. For this reason, minima under the asymmetric prior will generally be greater than the corresponding symmetrized lower bound.

4.4 Examples

We report a few examples to develop intuition and to compare our results with what was previously established in the auction literature.

4.4.1 Pure Common Values

Let us first consider a more general version of the common values model that we studied in Section 3, in which the bidders have the same value which is distributed according to $P(v)$. Recall the information structure of Engelbrecht-Wiggans et al. (1983), in which one bidder knows the true value and the remaining bidders are uninformed. The corresponding equilibrium has the informed player bid

$$\sigma(v) = \frac{1}{P(v)} \int_{x=\underline{v}}^{v} x \, P(dx), \tag{20}$$

i.e., the expected value of the good conditional on it being below its true value. This bidding function ensures that the uninformed bidders must get zero rents in equilibrium, because no matter what they bid, they must pay the expected value conditional on winning. In equilibrium, the uninformed bidders bid independently of one another and independently of the true value so that the marginal distribution of the highest of the $N-1$ uninformed bids is equal to the marginal distribution of the informed bid.

Let us compare the welfare properties of the equilibrium under their information structure with our bounds for the family of power distributions with support equal to $[0,1]$ and the cumulative distribution

$$P(v) = v^{\alpha},$$

where $\alpha \geq 0$. For this family of distributions, the informed bidder's strategy reduces to a deterministic bid of

$$\sigma(v) = \frac{\alpha}{\alpha+1} v.$$

Given the interpretation of the informed bid, we can immediately conclude that the expected value of the object is

$$\overline{T} = \frac{\alpha}{\alpha + 1}.$$

We can think of the highest of the $N - 1$ uninformed bids as also being of the same form $\sigma(v)$, but for an independently draw of v from the same prior. Thus, the surplus obtained by the informed bidder is

$$U^{EMW} = \int_{v=0}^{1} \left(v - \frac{\alpha}{1 + \alpha} v \right) v^{\alpha} \alpha v^{\alpha - 1} dv = \frac{\alpha}{(\alpha + 1)(2\alpha + 1)}.$$

Given our calculation of total surplus, revenue must be

$$R^{EMW} = \frac{2\alpha^2}{(\alpha + 1)(2\alpha + 1)}.$$

On the other hand, when $N = 2$, the revenue minimizing winning bid function we obtained earlier is

$$\underline{\beta}(v) = \frac{\alpha}{\alpha + 2} v.$$

Minimum revenue is therefore

$$\underline{R} = \frac{\alpha^2}{(\alpha + 2)(\alpha + 1)}$$

and maximum bidder surplus is

$$\overline{U} = \frac{2\alpha}{(\alpha + 2)(\alpha + 1)}.$$

We can now compare the welfare outcome in the equilibrium with the informed bidder with our bounds for the parametrized family of distributions. Note that the ratio of the bidder surplus between these two information structures is

$$\frac{\overline{U}}{U^{EMW}} = 2 \left(\frac{2\alpha + 1}{\alpha + 2} \right).$$

This quantity is 2 when $\alpha = 1$, which corresponds to our earlier observation in the uniform example that the two bidders collectively earn twice as many rents in the bidder surplus maximizing equilibrium as does the informed bidder. As $\alpha \to 0$, the ratio converges to 1 so that the informed bidder asymptotically attains the lower bound on bidder surplus (which is zero). As $\alpha \to \infty$, the bidder surplus ratio converges to 4, meaning that as the distribution of the common value converges weakly to a point mass on $v = 1$, each of the two bidders receives twice as much surplus as the informed bidder in Engelbrecht-Wiggans et al.. We will revisit this comparison when we consider the many bidder limit.

4.4.2 Independent Private Values

For a second example, let us again take $N = 2$ and suppose the bidders' values are independent draws from the standard uniform, so that the efficient surplus is $\overline{T} = 2/3$. The distribution of the lowest of two independent uniform draws is $Q(v) = 1 - (1-v)^2$, so that the revenue minimizing bidding function is

$$\underline{\beta}(v) = \frac{1}{\sqrt{1-(1-v)^2}} \int_{x=0}^{v} \frac{x(1-x)}{\sqrt{1-(1-x)^2}} dx.$$

Minimum revenue in this case does not have a simple analytical expression, but it numerically integrates to $\underline{R} \approx 0.096$, so that maximum bidder surplus is $\overline{U} \approx 0.571$.

For comparison, in the independent private values model—when each bidder only knows his own value but maintains the common prior regarding the other bidder's value—the bidders' surplus is 1/3 and revenue is 1/3. This is the same outcome as would obtain in the complete information model where both bidders observe both values. Maximum bidder surplus is therefore approximately 1.7 times larger than that predicted by either the independent private value or the complete information structure. By contrast, in the zero information environment in which bidder knows nothing about the values except the prior distribution, each bidder believes that every bidder's expected value is 1/2. In equilibrium the bids attain that level, the allocation will be ex-post inefficient, revenue is 1/2, and bidder surplus is 0.

4.4.3 The Many Bidder Limit

Our model permits a clean analysis of minimum bidding when the number of bidders becomes large. Consider a sequence of economies indexed by N, each of which is associated with a joint distribution of the potential bidders' values. The preceding analysis tells us that the only features of the distribution of values which matter for minimum revenue and maximum bidder surplus are (i) the distribution of the average losing value, which determines the minimum winning bid distribution, and (ii) the total surplus that is generated by an efficient allocation. Thus, we can analyze behavior in the many bidder limit by analyzing the behavior of these two objects. Let us suppose that the distribution of the average losing value converges to a limit $Q(m)$ and the limit of total surplus converges to $\overline{T}$. The mininum winning bid is given by the limit as $N \to \infty$ of (7), i.e.,

$$\underline{\beta}(m) = \frac{1}{Q(m)} \int_{x=\underline{m}}^{m} xQ(dx), \tag{21}$$

which is the expected average lowest value of the good conditional on it being below its true value. Minimum revenue will in turn converge to

$$\underline{R} = \int_{m=\underline{m}}^{\overline{m}} \underline{\beta}(m) Q(dm),$$

and maximum bidder surplus converges to $\overline{U} = \overline{T} - \underline{R}$.

In the common value case, the distribution of the average losing value is just the distribution of the common value. If we hold this distribution fixed as the number of bidders becomes large, then limiting revenue and bidder surplus converge to the two expressions above where Q is the distribution of the common value. Thus, as N becomes large, revenue is *bounded away* from the total surplus and bidder surplus is bounded away from zero. This conclusion, while perhaps surprising, is not novel. For example, the same result obtained in the model of Engelbrecht-Wiggans et al. in which one bidder is informed while $N-1$ bidders are uninformed. In that case, bidder surplus is independent of the number of bidders; in equilibrium, the informed bidder always bids according to (20), and the uninformed bidders' strategies adjust so that the distribution of the maximum of the $N-1$ uninformed bids is equal to (but independent of) the distribution of the informed bid. As a result, the surplus of the informed bidder is also independent of N, and since the uninformed bidders obtain zero rents, bidder surplus is constant as well. From this example, we could have already concluded that minimum revenue is bounded away from the expectation of the highest value. In general, though, maximum bidder surplus is strictly higher than that obtained by the informed bidder. Note that the limiting winning bid function (21) is exactly equal to the strategy of the informed bidder (20). However, the informed bidder will lose to an uninformed bid with non-vanishing probability, whereas in the bidder surplus maximizing equilibrium, the player who bids (20) always wins. For example, in the case of a uniform distribution, bidder surplus in Engelbrecht-Wiggans et al. (1983) in the many bidder limit remains 1/6, whereas maximum bidder surplus converges to 1/4.

Another leading case is the one where bidders' values are independent draws from a prior $P \in \Delta([0,1])$. When the number of bidders is large, the distribution of the empirical distribution of the $N-1$ lowest values converges weakly to a Dirac measure on P, so that the distribution of the average losing value converges to the Dirac measure on the mean of

P:

$$\hat{v} = \int_{v=\underline{v}}^{\overline{v}} vP\,(dv)\,.$$

In the limit, the winning bid converges almost surely to $\hat{v}$, but the allocation is efficient so total surplus converges to $\overline{v}$ and bidder surplus converges to $\overline{v} - \hat{v}$.

In this independent values case, we could have derived these bounds without direct appeal to the finite population minimum revenue results. If the equilibria along the sequence of economies are efficient, then as N becomes large, total surplus must converge to $\overline{v}$. Now, suppose that revenue converged to a quantity x smaller than $\hat{v}$. As the number of bidders becomes large, it must be that there is some bidder's surplus that goes to zero, but that bidder could always switch to bidding $(x + \hat{v})/2$ all the time and obtain a rent of $(\hat{v} - x)/2$ with probability at least $1 - 2x/(x + \hat{v})$. Thus, revenue cannot fall below $\hat{v}$. On the other hand, there is a simple information structure and equilibrium in which revenue is $\hat{v}$, which is when the bidders learn nothing about their own value except that it came from the prior P. In this case, the bidders are essentially in Bertrand competition with one another, and they bid the price up to the ex-ante expected value. This proves that minimum revenue in the many bidder limit is exactly $\hat{v}$. Without additional information, however, this equilibrium would result in surplus of only $\hat{v}$. Our limit construction spreads out the bids by a small amount and correlates them with values so as to break ties in favor of the bidder with the highest value, with all of the additional surplus going to the bidders. This is reminiscent of the efficient tie breaking rules that are used elsewhere in the literature on first price auctions to ensure existence. We will have more to say on the tie breaking rule in the next section.

The conclusion that minimum revenue in the many bidder limit is bounded away from the efficient surplus stands in stark contrast with the literature on information aggregation in large markets (Wilson, 1977; Milgrom, 1979; Bali and Jackson, 2002). This literature considers two distinct but related issues: first, is information is aggregated in large markets, in the sense that the winning bid reveals the true efficient use value of the good? And second, does this revelation of information induce the bidders to compete away their rents, so that revenue converges to the efficient surplus? The positive results in this literature rely upon assumptions about information which our constructions violate. For example, Wilson (1977) assumes a particular parametric form for the information structure and Bali and Jackson (2002) assume that the information structure changes "smoothly" as the number of bidders becomes large. In contrast, our constructions exhibit discontinuities in the limit, in the sense that the marginal distribution of each losing bidder's signal converges to a point mass on the lowest possible average losing value. In the common value case, our model illustrates that the sale price can be fully revealing about the value without competition forcing the price up to the true value. With non-common values, however, we have both failure of information aggregation and full surplus extraction in the limit, although enough information is aggregated in the limit to ensure an efficient allocation.

The bottom line is that the minimum revenue characterization can be used to characterize bounds on behavior in large auctions. A robust conclusion is that information need not be aggregated in the selling price, and asymptotic revenue can be below total surplus. This phenomenon seems to rely on large informational asymmetries between the bidders, with a small number of bidders receiving precise information and most of the bidders receiving noisy information. Indeed, as the number of bidders becomes large, the quality of losing buyers' signals deteriorates and asymptotically they are completely uninformative. This suggests that in order for the selling price to be competed up to the highest value, there must be some uniform lower bound on the quality of the bidders' information as the number of bidders becomes large. Such a lower bound is not required for the sale price to be perfectly informative about the highest value or for the allocation itself to be efficient.

5 Further Results on Revenue and Bidder Surplus

In this section, we shall continue to explore the limits of bidder surplus and revenue. To describe our new results and their relation to Theorem 1, it is useful to report their implications in the independent private value example with a uniform distribution that we already discussed in Section 4.4.2. Thus, there are two bidders whose values are independently and uniformly distributed on the interval $[0, 1]$. Figure 1 illustrates results for the example. Possible combinations of bidder surplus (on the x-axis) and revenue (on the y-axis) are plotted. Maximum total surplus is $2/3$, so efficient allocations must correspond to the -45 degree line on the right of the picture. The worst case for efficiency is that the object is always sold to the bidder with the lowest value, which would generate a total surplus of $1/3$. Thus the set of revenue-bidder surplus pairs that are feasible is bounded by the green parallelogram. The minimum revenue bound identified in the previous section is represented by point A. We gave an analytic expression for minimum revenue in this example, which was approximately 0.096, and corresponded to an efficient allocation.

We will now consider how large revenue can be and how low bidder surplus can be across all possible information structures. We begin with the *unknown values* model, where there is an information structure and equilibrium where the seller extracts the total surplus leaving the bidders with zero surplus. This corresponds to point B in Figure 1. We then consider the *known values* model. Here, each bidder can always guarantee himself a strictly positive surplus due to the private information about his own values. We characterize the exact limits

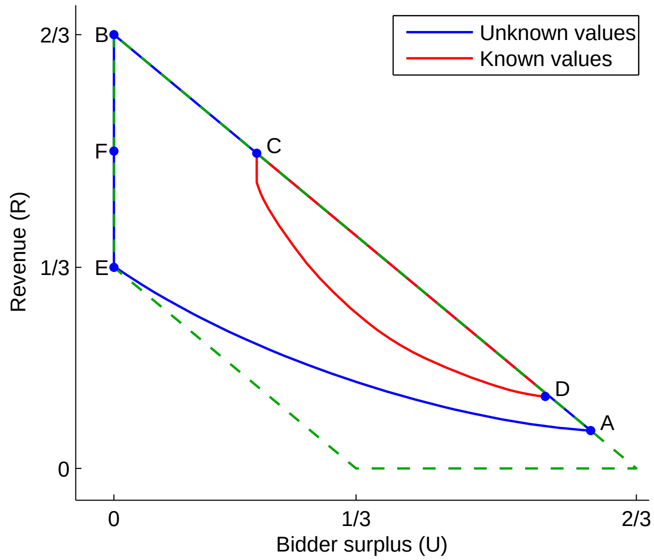

Figure 1: The set of revenue-bidder surplus pairs that can arise in equilibrium.

of how much surplus the bidders need to receive, which in turn pins down maximum revenue the seller can earn. This corresponds to point C. Next, we discuss minimum revenue in the known values case. This corresponds to point D. Here we do not have an analytic result, and point D is derived from a numerical analysis. We can, however, report partial results for this problem as well.

Away from the common value case, we can ask about bidder surplus-revenue pairs that may arise in inefficient equilibria. The area enclosed by the blue curves corresponds to all pairs that can arise in the case of unknown values while the area enclosed by the red curves corresponds to all pairs that can arise in the known values case.[12] A striking point in this picture is point E: here, there is zero bidder surplus but revenue is held down to the minimum feasible surplus. We show that an analogous but somewhat weaker inefficiency results holds for many bidders and independent values.

Because there is a linear programming characterization of bidder surplus and revenue pairs that can arise in equilibrium, it is computationally feasible to compute all bidder surplus and revenue pairs that can arise in discretized examples. While points A, B, C, and E are derived analytically, as described above, other points are computed numerically.

[12]The fact that the known values set (in red) is contained within the unknown values set (in blue) is a reflection of the general observation that adding more information for the bidders decreases the set of outcomes that can be rationalized as an equilibrium with even more information. In other words, the set of Bayes correlated equilibria is decreasing in the minimum information of the players. Bergemann and Morris (2015a) formalize the notion of "more information" and give a precise statement of this result, and Bergemann, Brooks, and Morris (2015a) describe the technical extension from finite type and action games to continuum type and action games.

Specifically, the above picture is computed for and independent uniform distribution with grids of 10 values and 50 bids between 0 and 1. The axes have been re-scaled to match moments with the continuum limit; for the discretized example, the efficient surplus and minimum surplus are respectively 41/60 and 19/60, as opposed to their limit values of 2/3 and 1/3.

5.1 Maximum Revenue with Unknown Values

To maximize revenue and minimize bidder surplus, we would like to generate a highly competitive environment where the highest bid is equal to the highest value. Consider the information structure where each bidder received a signal that was equal to the highest value, independent of who had the highest value:

$$s_i = \max v, \text{ for all } i.$$

Suppose that there was an efficient tie breaking rule where the highest value bidder always wins in case of ties. With efficient tie breaking, there would be an equilibrium where each bidder sets his bid equal to his signal. A bidder would then be indifferent between all bids less than or equal to his signal s_i: if he bid less than the signal, he would lose for sure, and if he bid his signal, he would only win when he has the highest value, in which case he is paying his value and getting a payoff of 0. Moreover, no one would want to deviate up, since this would result in winning for sure and paying a price which is greater than the highest value. Thus, bidding the signal is an equilibrium, and under these strategies the winning bid is the maximum value, so that revenue is the efficient surplus.

This argument relied on the endogenous tie breaking rule. However, it is possible to achieve approximately the same outcome with the uniform (and thus potentially inefficient) tie breaking rule by suitably perturbing the information structure:

Theorem 2 (Maximal Revenue and Minimum Bidder Surplus with Unknown Values)**.**
For all $\varepsilon > 0$, there exists an information structure and equilibrium such that revenue is at least $\overline{T} - \epsilon$ and bidder surplus is less than ϵ.

To establish the result under the uniform tie breaking rule, consider the information structure where the bidder with the highest value observes a signal that is a convex combination of the highest and second-highest values, $b = (1-x)v^{(1)} + xv^{(2)}$. The losing bidder with the second-highest value observes a signal which is a convex combination $b' = yv^{(1)} + (1-y)v^{(2)}$, where the weight y is drawn from the interval $[0, 1-x]$. All other bidders receive signals which are equal to their respective valuations. If each bidder bids his signal, then the resulting allocation is efficient. To ensure this is an equilibrium, assume that y is drawn from the distribution

$$y \sim F(y) = \frac{y}{1-y}\frac{1-x}{x}.$$

As we show in the Appendix, for any given x, these strategies form an equilibrium, and for x sufficiently small, the winning bid is arbitrarily close to the highest value of the object among the bidders. In the limit, revenue approaches the efficient surplus, and yet the bidders surplus is arbitrarily close to zero.

Thus, in an environment with unknown values, the private information of each bidder might be sufficiently confounding to induce very aggressive bidding behavior. It induces beliefs such that the bidders are willing to bid a large amount because they think that the bid is less than their value conditional on *winning*, although their value might be quite a bit lower than their bid conditional on *losing*. As a result, the strategy of bidding one's signal is weakly undominated.

We note that the construction of the bid distribution described above exploits symmetry among the bidders, but the argument could be extended to asymmetric distributions of values, assuming only that the asymmetric distribution of values is absolute continuous over a symmetric support.

5.2 Maximum Revenue with Known Values

In the environment with arbitrary interdependence in values, we have a stark characterization of maximum revenue and minimum bidder surplus: the seller can extract all the surplus, leaving the bidders with zero rents. One might ask how our results would change if we imposed additional restrictions on how much bidders can learn about their values from the outcome of the auction. An extreme assumption, but one which is commonly adopted in independent value models, is that each bidder knows his own value for sure. This is what we termed the *known values* case in Section 2.

The assumption that bidders know their own values substantially affects the set of possible outcomes, and it is no longer the case that bidder surplus can be driven all the way down to zero. In fact, we can derive a rather elementary lower bound on each bidder's surplus. As each bidder knows his value for the object, any weakly undominated strategy profile requires that the bidders never bid above their values. Thus, each bidder knows that their opponents cannot be using a more aggressive strategy than bidding their values. If this were in fact the strategy that others are using, bidder i would face a cumulative distribution of the highest of others bids

$$P_i^{(2)}(b|v_i) = \int_{\left\{v_{-i}\in V^{N-1}\middle|\max_{j\neq i} v_j \leq b\right\}} P\left(dv_{-i}|v_i\right).$$

Against this most aggressive bidding behavior by his competitors, bidder i would optimally bid

$$\sigma_i^*(v_i) = \max\left\{\underset{b\in\mathbb{R}_+}{\arg\max}\left\{(v_i - b)P_i^{(2)}(b|v_i)\right\}\right\}, \tag{22}$$

where we take without loss of generality the largest optimal bid in case there are multiple solutions. It follows that bidder i with value v_i must receive in any equilibrium at least the surplus

$$\underline{U}_i(v_i) = (v_i - \sigma_i^*(v_i))\, P_i^{(2)}(\sigma_i^*(v_i)|v_i).$$

For if the equilibrium surplus were lower, bidder i could deviate upwards to $b = \sigma_i^*(v_i) + \epsilon$ and guarantee himself surplus arbitrarily close to $\underline{U}_i(v_i)$. This implies that in ex-ante terms, bidder i must receive at least

$$\underline{U}_i = \int_{v\in V^N} \underline{U}_i(v_i)P\left(dv\right). \tag{23}$$

Theorem 3 below establishes that this lower bound is in fact tight: there is an information structure and equilibrium in which bidder i receives *exactly* $\underline{U}_i$ in surplus. The difficulty in establishing this result is that we must hold each highest value bidder down to the utility that he would get if he were completely uninformed about the other bidders (beyond his prior) and bid $\sigma_i^*(v_i)$ while other bidders bid their values. Of course, these strategies do not constitute an equilibrium since other bidders would prefer to shade as well. Nonetheless, it is possible for all bidders to receive $\underline{U}_i$ in the same equilibrium, so that bidder surplus is

$$\underline{U} = \sum_{i=1}^{N} \underline{U}_i.$$

Moreover, bidders can be held to this lower bound while maintaining an efficient allocation. Thus, this equilibrium simultaneously minimizes bidder surplus and maximizes the revenue of the seller at the level:

$$\overline{R} = \overline{T} - \underline{U}. \tag{24}$$

Theorem 3 (Maximum Revenue and Minimum Bidder Surplus with Known Values)**.** *Minimum bidder surplus and maximum revenue across all known values information structures and equilibria are respectively $\underline{U}$ and $\overline{R}$.*

We will give now a constructive proof under the efficient tie breaking rule in which ties are broken in favor of the bidders with higher values. In the formal argument in the Appendix, we drop this assumption and establish the same result under the symmetric tie breaking rule.

Our proof is constructive. Call the bidder with the highest value the "winning bidder" and all bidders who do not have the highest value "losing bidders", corresponding to what will eventually happen in equilibrium. Each bidder is assumed to know his own value. In the information structure that attains the bound, each bidder also knows if he is the winning bidder or a losing bidder. In addition, each losing bidder knows all the information of the winning bidder, i.e., the value but also the signal observed by the winning bidder. Since we will end up constructing a pure strategy equilibrium, this means that each losing bidder also knows the winner's bid, which will be weakly greater than the losers' values. Thus, each losing bidder will knows that his value less than the winner's bid, so it will be optimal for each losing bidder to bid his value and lose the auction.

The remaining step to complete the proof is to construct an information structure and equilibrium strategy for the winning bidder where he (i) always bids at least the second-highest value (and thus wins under our tie breaking rule) and (ii) is held down to surplus $\underline{U}_i$. Giving the winning bidder perfect information about the second-highest value will not achieve this pair of requirements: he will always win but his surplus will be the expectation of the difference between his value and the second-highest value, which is strictly more than $\underline{U}_i$. Giving the winner no information about the second-highest value will also not work: in this case he will be held down to surplus $\underline{U}_i$, but he will bid $\sigma_i^*(v_i)$ which will sometimes be lower than the second-highest value and so he will not win. So the required information structure must give the winner intermediate information about the second-highest value attaining the two requirements (i) and (ii) simultaneously.

The existence of such an intermediate information structure and equilibrium strategy is a consequence of our earlier work on third degree price discrimination (Bergemann, Brooks, and Morris, 2015c). Consider a seller of a single unit of a good, which has a given reservation value for for the seller. The seller can make a take-it-or-leave-it offer to a single buyer, whose value is not known precisely but it is drawn from a known distribution. If the seller only knows the distribution of the buyer's value, he will choose an optimal price which obtains for the seller some no-information surplus. The optimal price will generally exceed the seller's reservation value, so that trade will sometimes not occur even though the buyer's value exceeds that of the seller. We showed that one can construct an information structure for the seller such that his surplus will be held down to the no-information level but that the good will be sold whenever the buyer's value exceeds seller's value, leading to an efficient allocation.[13]

This earlier construction can be used to establish the existence of an information structure and equilibrium strategy satisfying requirements (i) and (ii) above. Let us translate the price discrimination result to the present auction setting. First, we can reverse the roles of the buyer and seller in monopoly setting of Bergemann et al., so that there is a buyer who has a fixed value and is making take-it-or-leave-it offers to a seller whose reservation value is unknown. Translated thus, the previous result shows that there is a way to give additional information to this buyer so that trade always takes place but the buyer obtains his no-information surplus. Now, we can relate this buyer-seller interaction to the auction setting by identifying the buyer with the bidder with the high value who is making a "take-it-or-leave-it" bid for the good. The seller is identified with the auctioneer himself, who has an implicit "reservation value" which is equal to the second-highest bid, and will "accept" any offer exceeding that outside option. If the high value bidder (the buyer) knew nothing about the the second-highest value (the seller's reservation), he would choose an optimal bid, $\sigma_i^*(v_i)$, that would sometimes prevent an efficient allocation but would deliver him a surplus $\underline{U}_i$. The result in Bergemann, Brooks, and Morris (2015c) now shows that we can construct an information structure for the winning bidder such that his surplus will be held down to $\underline{U}_i$ but he will always win, thus resulting in an efficient allocation. We have therefore constructed an information structure and equilibrium strategy delivering the two requirements, (i) and (ii), described above.

This is an essentially complete proof. There are three additional issues which are dealt with in the formal proof in the Appendix. First, we broke ties in favor of the winning bidder in the above argument. But by having losers bid close to but below their values, we can dispense with the high value tie breaking rule and establish the result under our maintained uniform and symmetric tie breaking rule. Second, the result of Bergemann, Brooks, and Morris (2015c) with a continuum of values (Theorem 1B) that we appeal to here was proved non-constructively by a limit argument. In contrast, our formal proof will give an explicit construction of the winning bidder's information, both for the sake of completeness and to make it clearer how the arguments are translated from the monopoly setting to the auction. We note that this is a new constructive proof of the result in Bergemann, Brooks, and

[13]In the present auction environment we consider a continuum of values, and thus the results correspond to Theorem 1B in Bergemann, Brooks, and Morris (2015c). The results there are stated in the language of price discrimination, so (i) the seller is a "monopolist"; (ii) the distribution over buyer values is a "demand curve" (i.e., distribution of consumer values in a continuum population); (iii) the seller's value is the "constant marginal cost of production" ; (iv) the optimal bid $\sigma_i^*(v_i)$ is the "uniform monopoly price"; and (v) the no-information bidder surplus is the "uniform monopoly profits". These changes are simply a relabeling of the relevant variables.

Morris (2015c) for the continuum case, which is of some independent interest. Third, in the monopoly setting, the seller only had a single known value (his cost of production). By contrast, here the winning bidder has many possible values, and thus we need to guarantee that we can construct an information structure and equilibrium for all realized values of the winning bidder simultaneously.

Finally, we note that under our maintained assumption of symmetry of values, the minimum surplus of each bidder, $\underline{U}_i$, is the same for each bidder. However, symmetry is not used in the argument, and Theorem 3 extends unchanged to asymmetric value distributions.

5.3 Minimum Revenue with Known Values

The unknown values minimum revenue is a lower bound for the known values minimum revenue. However, this bound is typically not tight, and we do not have a comprehensive analytical characterization of minimum revenue for known values. We briefly discuss the difficulties in establishing general results and describe the partial results we can obtain. The previous version of this paper, Bergemann, Brooks, and Morris (2015b), provides detailed analysis of both these issues.

We can pursue the same course that characterized revenue in the unknown values model, namely, (i) formulate and solve a relaxed program for revenue, and then (ii) extend this solution to a complete information structure and equilibrium. In the relaxed program with known values, we would again only keep track of the winning bid distributions and drop all incentive constraints except those corresponding to uniform upward deviations. For unknown values, there was a one-dimensional family of such constraints, indexed by the bid that the player deviates up to. With known values, however, there is a separate family of such constraints for each possible known value v_i. This introduces a second dimension to the problem, and for general known values models, the pattern of binding uniform upward incentive constraints becomes rather complicated. Moreover, while the solution to the relaxed program still generates a bound, we know by example that this bound is not tight.

There is one class of known values models for which we can provide a complete and tight characterization of the minimum revenue and maximum bidder surplus, namely when there are only two possible values, low and high. With binary values, the low type is strategically quite simple and can be solved out, leaving only the one-dimensional family of uniform upward constraints corresponding to the high type. Given this reduction, we can solve the relaxed program and extend the solution to an equilibrium, this providing sharp bounds on the minimum revenue and maximum bidder surplus. The case of the binary values can also inform us about the qualitative differences between the unknown and known values cases. In both models, uniform upward deviations are the key for characterizing the revenue minimizing equilibrium. However, the relevant information that is conveyed by the equilibrium bid is quite different in the two cases. With unknown values, the probability of being the winner when there is a given realized distribution of types is independent of that realized distribution, so that the only manner in which the realized type profile affects the incentives to deviate uniformly upwards is through inference about the average losing value, $\mu(v)$. With known values, bidders know their value when they lose the auction, thus rendering $\mu(v)$ irrelevant to evaluating uniform upward deviations. On the other hand, the probability of winning versus losing may depend on one's rank in the distribution of values: for example, in an efficient equilibrium, having a value below the maximum means that one will lose for sure, and the more ties there are for the highest value, the less likely is a high value bidder to win the auction. Thus, bidders know in equilibrium that certain bids may be associated with certain distributions of values and therefore certain probabilities of winning the auction in equilibrium, and this association complicates the inference about the costs and benefits of upward deviations. In the binary values case, distributions over values can be ordered one-dimensionally by the number of high types. We can use this to show that minimum revenue is characterized by the support of the winning bid being monotonically increasing in the number of bidders with high values.

5.4 Additional Welfare Outcomes

Thus far, our analyses have led us to equilibria in which the allocation of the good was efficient, so that the welfare outcome lay on the northeast frontier of Figure 1. As the figure plainly shows, however, there is a large number of possible outcomes in which the allocation of the good is inefficient. That some inefficiency might arise is obvious, for example when the bidders have no information about values except the prior, in which case the allocation is independent of the realizes values and welfare corresponds to point F. What is more striking is the extent of this potential inefficiency. In particular, point E attains a maximally inefficient outcome in which the good is always allocated to the buyer with the *lowest* value, all while giving the bidders zero rents.

For the model underlying Figure 1, with two bidders and independent uniform valuations, there is an extremely simple information structure and equilibrium which attains this outcome: each bidder observes the *other* bidder's valuation, and bids half of what they observe. To see that this is an equilibrium, consider a bidder i who has observed a signal s_i. Because of independence, this signal contains no information about the bidder's own value v_i, which has a posterior distribution that is uniform. Now, conditional on bidder i bidding some b_i, bidder i wins whenever bidder j's bid is less than b_i, which is when s_j is less than $2b_i$. But s_j is equal to v_i, so that the expectation of v_i conditional on winning with a bid of b_i is just the expectation of a uniform random variable conditional it being below $2b_i$, which is precisely b_i! No matter what bidder i bids, the expected value conditional on winning equals the bid. Thus, bidders must receive zero rents in equilibrium, and since bids are monotonic in the other bidder's value, it is the bidder with the lowest value who wins in equilibrium, thereby minimizing total surplus.

While we have not explored minimum efficiency in its full generality, we can report a class of information structures and equilibria which generalize this example to the case of many bidders and symmetric independent values. We note that there is no hope of attaining the maximally inefficient outcome for general models with many bidders: As the number of bidders becomes large in the independent and symmetric case, the expected lowest valuation converges to the minimum of the support. At the same time, minimum total surplus must be at least minimum revenue, and—as we observed in Section 4.4.3—the latter converges to the mean value under the prior. Nonetheless, we can describe a class of equilibria which holds the bidders to zero surplus while attaining a total surplus which is below the no-information outcome. In these equilibria, bidders are indifferent between *all* bids, both higher and lower than their equilibrium bids, and in the limit as N grows large, minimum total surplus is attained. When there are two bidders and values are independent and symmetric, this construction attains the maximally inefficient outcome, and more generally, we conjecture that this equilibrium minimizes total surplus subject to the constraint that the bidders obtain zero surplus.

Specifically, consider an information structure in which each bidder observes the maximum of the other bidders' values, i.e.,

$$s_i = \max v_{-i}. \tag{25}$$

In the case of two bidders and independent values, this of course means that each bidder learns the value of his competitor, but not his own value. Under this information structure and with independent private values, the bidder with the highest value receives the lowest signal, and all other bidders receive the same (higher) signal that reflects the highest value among all bidders. However, each individual bidder does not know whether his signal is lower or higher than the signals received by his competitors.

We now construct a symmetric bidding strategy that is monotonically increasing in the signal s_i. In the resulting equilibrium, the bidder with the highest value will never receive the object, and with equal probability, one of the $N-1$ low value bidders will win the object. Let $P \in \Delta([\underline{v}, \overline{v}])$ denote the independent prior over values. We claim that it is an equilibrium for all bidders to use the symmetric and monotonic pure strategy

$$\sigma(s) = \frac{1}{P(s)} \int_{x=\underline{v}}^{s} xP(dx). \tag{26}$$

In other words, each bidder bids the expectation of the value under the prior, conditional on it being below the signal that the bidder observed.

We shall now argue that these strategies constitute an equilibrium. First, consider a bidder i who observes a signal s_i. If bidder i follows the equilibrium strategy and bids $\sigma(s_i)$, then they will win with probability $1/(N-1)$ when they had a high signal, which is when some other bidder had a higher value. But since bidder i's value is independent of the highest of others' values, the posterior distribution of bidder i's value on this event is precisely the truncated prior $P(v_i)/P(s_i)$ with support equal to $[\underline{v}, s_i]$. Thus, the expected valuation conditional on winning is precisely $\sigma(s_i)$, and the bidder obtains zero rents in equilibrium.

Now, consider a bidder i who deviates down to some $\sigma(s')$ with $s' < s$. We can separately consider the case of $N = 2$ and $N > 2$. In the latter case, there is more than one bidder who sees a signal equal to the highest value, and therefore in equilibrum there is a tie for the highest bid at $\sigma(s_i)$. Thus, a downward deviator will always lose the auction and obtain zero rents. On the other hand, if $N = 2$, then the bidder wins whenever the other bidder's signal was less than s'. But since the other bidder's signal equal to v_i, the event where bidder i wins is precisely when v_i is in the range $[\underline{v}, s']$, so that the expectation conditional on winning is $\sigma(s')$, and the deviator still obtains zero rents.

Finally, let us consider a bidder i who deviates up to $\sigma(s')$ with $s' > s_i$. This bidder will now win outright whenever s_i was equal to the maximum valuation. Moreover, when s_i was a losing signal, the bidder will now win whenever others' signals were less than s'. But on this event, others signals are equal to $v_i = \max v$. Thus, the upward deviator will win whenever $v_i \leq s'$, and again the expected value conditional on winning is precisely $\sigma(s')$, so that the deviator's surplus is still zero.

We observe that the realized value among the winning bidders is exactly given by the average value among the $N-1$ bidders with the lowest values, or

$$\mu(v) = \frac{1}{N-1} \left(\sum_{i=1}^{N} v_i - \max v \right). \tag{27}$$

It follows that the revenue of the seller is given exactly by the expectation over the average value among the $N-1$ lowest valuations. We have therefore proven the following:

Theorem 4 (Inefficient Equilibrium)**.**
The equilibrium of the information structure (25) is given by (26). In this equilibrium, revenue and total surplus are both equal to

$$\int_{\underline{v}}^{\overline{v}} mQ(dm),$$

and bidder surplus is zero.

We note that this equilibrium construction can be extended well beyond the independent values case. In such a generalization, the equilibrium bid would be each bidder's expected value conditional on it being less than the observed maximum of others' values. As long as there is sufficient positive correlation between bidders' values, e.g., affiliation, this bidding function will be strictly increasing, and for this more general class, the upward incentive constraints will be strictly satisfied.

To sum up, we have characterized three "corners" of the unknown values set (for the two bidder case), and by convexity, we can generate both the western and northeastern flats of the blue region in Figure 1. The remaining feature of Figure 1, hitherto unexplained, is the apparently smooth southwestern frontier that runs from the maximally inefficient equilibrium to the efficient revenue minimizing equilibrium. In the working paper version, Bergemann, Brooks, and Morris (2015b) we give a complete description of the class of equilibria that generate this southwestern frontier. They are members of a class of "conditionally revenue minimizing" equilibria, which minimize revenue conditional on a fixed allocation of the good. As the allocation ranges from efficient to maximally inefficient, we move smoothly between points A and E.

The known values surplus set, depicted in red, is significantly smaller than the unknown values surplus set, and—except for the known values maximum revenue result (point C)—is derived from computations, since the same difficulties that arose in the analysis of minimum revenue with known values as described earlier also apply here. An interesting aspect of the inefficient boundary as displayed in Figure 1 is the fact that the inefficiencies that can arise in the known value model are relatively small, as visually expressed by the slimness of the red lens that describes the set of all possible equilibrium surplus realization. This observation is in line with the result of Syrgkanis and Tardos (2013) and Syrgkanis (2014) who show the efficiency loss in the independent private value auction expressed in terms of the ratio between realized surplus and efficient surplus in the first price auction is bounded below $1 - 1/e$.

6 Discussion

This paper has provided new and general characterizations of equilibrium bidding in the first price auction. Relative to the previous literature, our results rely on minimal assumptions on the structure of bidders' information. In particular, our results hold for any symmetric but arbitrarily correlated distribution of values and for any asymmetric and multidimensional signal structure. Many of our results also extend to asymmetric value distributions. For this very general class of models, we have characterized the range of possible revenues and bidder surpluses that might obtain under unknown values, and we have characterized maximum revenue and minimum bidder surplus under known values. More broadly, we have demonstrated methodologies, both analytical and computational, that could be fruitfully applied in other settings.

Many game theoretic models are used figuratively, to aid in the abstract discussion of a particular economic channel or mechanism. In that case, it might not be a critical failure if the model depends on somewhat ad hoc parametric assumptions. The comparative analysis of auction mechanisms does not fall into this category. Theoretical properties of mechanisms like the first price auction have been used to inform practical policy debates about how to design institutions (Milgrom, 2004). Moreover, there is a large and growing literature in the econometrics of auctions that attempts to identify values from observed bidding behavior (e.g., Laffont, Ossard, and Vuong, 1995, Athey and Haile, 2007, and Somaini, 2015). This work interprets the received model of the first price auction quite literally, and assumes that bids are generated by an equilibrium under a classical—i.e., affiliated—information structure. It is therefore important that we develop more general theories of bidding behavior that do not rely on such restrictive assumptions about aspects of the environment, e.g., beliefs, that are almost impossible to measure in a practical setting. The present paper is a contribution towards this goal.

Let us comment on some natural applications of our results along these lines. First, our main result on minimum bidding can be used for identification in auctions. Since our model generates set valued predictions for behavior, we cannot point identify the distribution of values and information structure that generated bidding behavior. We can, however, place bounds on certain moments of the prior distribution of values so that the empirical distribution of bids can be rationalized by an equilibrium for some information structure. In particular, let us suppose that we have observed a distribution of winning bids $H(b)$. If we fix a distribution of the average losing value, then we know that the corresponding minimum winning bid distribution must be stochastically dominated by H. This implies that we can identify a subset of distributions of average losing values that are consistent with the data.

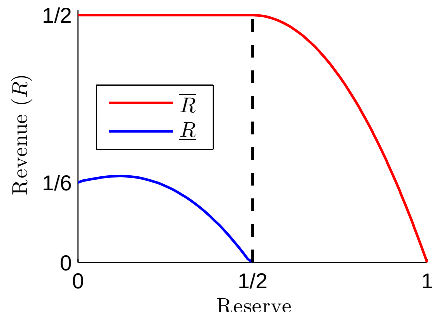

Figure 2: Minimum and maximum revenue with different reserve prices.

This identification result is especially powerful under the assumption of pure common values, as in this case, the distribution of average losing values is just the distribution of the common value.

Second, our approach can be used to show how maximum and minimum revenue varies across mechanisms, thus allowing informationally robust conclusions about performance. This paper is about the first price auction *without* a reserve price. We briefly discuss what happens when we allow a reserve price. For a given and fixed information structure, allowing a reserve price is known to increase revenue (Myerson, 1981).

Figure 6 reports maximum and minimum revenue of the first price auction with reserve prices for the uniform pure common value example we have discussed throughout the paper. With a zero reserve price, the mechanism is exactly the first price auction we characterized, and maximum and minimum revenue coincide with the theoretical predictions of $1/2$ and $1/6$. Since $1/2$ corresponds to full surplus extraction, maximum revenue can only go down. For reserve prices below $1/2$, the no-information setting still yields full surplus extraction. For higher reserve prices, the bidders all receive a signal for whether or not the value is in an interval $[2r-1, 1]$ when the reserve price is r. This signal tells the bidders that the expected value is exactly r, so that they are indifferent to tying at bids of r or 0, respectively. The revenue minimizing information and equilibria are more complicated, and we will not give a complete description. Reserve prices do increase minimum revenue, however, and the $\max\min$ reserve price is approximately $1/8$.

We have analytically characterized bounds on bidding behavior in the first price auction without a reserve price, and this example shows that the methodology can be applied computationally to characterize other mechanisms. For the auctions we have considered thus far, there are typically many welfare outcomes that are consistent with a given distribution of values. Thus, even if one adopts the view that bidders have a common prior and will play according to some equilibrium, there remains ambiguity as to which welfare outcome will obtain. In order to use our results to completely rank auction formats, it is necessary to put additional structure on preferences with respect to this ambiguity. For example, one could consider a seller who is concerned with revenue maximization but is averse to the ambiguity about information and equilibrium, in the sense of Gilboa and Schmeidler (1989). For such a seller, our results can be used to rank the first price auction relative to other commonly considered auction formats.

For example, a classic question is whether it is preferable to sell a good by a second price auction or a first price auction. In the independent symmetric private values model with risk neutral bidders, the revenue equivalence theorem tells us that these auctions generate equal revenue. If values are affiliated (Milgrom and Weber, 1982), then second price auctions generate higher revenue. With asymmetric value distributions, revenue comparisons may go either way (Maskin and Riley, 2000). However, broader arguments for favoring one mechanism over the other exist: the second price auction is said to be "safer" for a seller because bidders have a dominant strategy to bid their value if it is known, whereas the first price auction introduces strategic considerations. On the other hand, collusion may be easier in second price auctions than in first price auctions.[14]

Our results offer a perspective on this debate. With unknown values, there is no lower bound on revenue in the second price auction. It is well known that the second-price auction has equilibria in weakly dominated strategies in which one $N-1$ of the bidders submit the minimum bid, and one bidder bids a large amount. Moreover, we can add a small amount of incomplete information about values so that this behavior is not weakly dominated, as in our construction of the revenue maximizing equilibrium under unknown values. Intuitively, this relies on collusion among bidders.[15] In contrast, our main result shows that there is a strictly positive lower bound on revenue in the first price auction. On the other hand, with known values, second price auction revenue (under our maintained assumption of weakly undominated strategies) is equal to the expectation of the second-highest value, which exceeds the lower bound of Theorem 1.

[14]Rothkopf, Teisberg, and Kahn (1990) gives this and other reasons for the rarity of the second price auction in practise. Krishna (2009, chapter 11) summarizes the literature on collusion in the first and second price auction, supporting the intuition that collusion is easier in the second price auction.

[15]With exogenous information structures, there is well known multiplicity of equilibria of the second price auction, as documented by Milgrom (1979).

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A Proofs

Proof of Lemma 1. We can identify the set of candidate solutions with the N-fold product of the set $\Delta\left(B \times V^N\right)$ of Borel probability measures on $B \times V^N$ that have $P(dv)$ as the marginal over V^N, since any such distribution induces a probability transition kernel from V^N to B. Note that the set $\Delta\left(B \times V^N\right)$ is compact in the weak-* topology, and therefore is metrizable. The feasible set for the relaxed program is the subset of $\Delta\left(B \times V^N\right)$ that also satisfies (11). We wish to show that this set is compact. This is not trivial, however, since each of the individual constraints (11) is not closed in the weak-* topology. Let $\left\{H_i^k(db, dv)\right\}$ be a convergent sequence of measures in $\Delta\left(B \times V^N\right)$ that satisfy (11), and suppose that the limit $\{H_i(db, dv)\}$ violates (11) for some i and b, so that

$$\int_{v \in V^N} (v_i - b) \sum_{j \neq i} H_j([0, b], dv) - \int_{x=0}^{b} H_i\left([0, x] \times V^N\right) dx = \delta > 0.$$

Let $\epsilon > 0$. Since $H_i(db, V)$ has only countably many mass points, we can find a $\hat{b} \in (b, b+\epsilon)$ such that for all i, $\left[0, \hat{b}\right] \times V$ is a continuity set of H_i. It follows from weak-* convergence that

$$\begin{aligned}\lim_{k \to \infty} \int_{v \in V^N} \left(v_i - \hat{b}\right) H_j^k\left(\left[0, \hat{b}\right], dv\right) &= \int_{v \in V^N} \left(v_i - \hat{b}\right) H_j\left(\left[0, \hat{b}\right], dv\right) \\ &\geq \int_{v \in V^N} (v_i - b) H_j\left(\left[0, \hat{b}\right], dv\right) - \epsilon.\end{aligned}$$

In addition, countable additivity implies that

$$\lim_{\hat{b} \to b} \int_{v \in V^N} (v_i - b) H_j\left(\left[0, \hat{b}\right], dv\right) = \int_{v \in V^N} (v_i - b) H_j([0, b], dv)$$

Thus, we can find a K_1 sufficiently large and a ϵ_1 sufficiently small that $k > K_1$and $\hat{b} \in (b, b+\epsilon_1)$ implies that

$$\int_{v \in V^N} \left(v_i - \hat{b}\right) H_j^k\left(\left[0, \hat{b}\right], dv\right) > \int_{v \in V^N} (v_i - b) H_j([0, b], dv) - \frac{1}{N-1}\frac{\delta}{4}.$$

Moreover, if $\hat{b} \in (b, b+\epsilon)$, then

$$\int_{x=0}^{\hat{b}} H_i^k\left([0, x] \times V^N\right) \leq \int_{x=0}^{b} H_i^k\left([0, x] \times V^N\right) + \epsilon.$$

And since

$$\lim_{k\to\infty} H_i^k\left([0,x]\times V^N\right) \leq H_i\left([0,x]\times V^N\right),$$

we can take K_2 sufficiently large and ϵ_2 sufficiently small such that $k > K_2$ and $\hat{b} \in (b, b+\epsilon_2)$ implies that

$$\int_{x=0}^{\hat{b}} H_i^k\left([0,x]\times V^N\right) < \int_{x=0}^{b} H_i\left([0,x]\times V^N\right) + \frac{\delta}{4}.$$

Thus, by taking $k > \max\{K_1, K_2\}$ and $\epsilon < \min\{\epsilon_1, \epsilon_2\}$, it must be the case that we can find a $\hat{b} \in (b, b+\epsilon)$ such that

$$\int_{v\in V^N} \left(v_i - \hat{b}\right) H_j^k\left(\left[0,\hat{b}\right], dv\right) - \int_{x=0}^{\hat{b}} H_i^k\left([0,x]\times V^N\right) > \frac{\delta}{2},$$

which contradicts that $\left\{H_i^k\right\}$ satisfies (11). Finally, it is obvious that (12) is continuous in the weak-* topology, so that a minimum must exist.

It is straightforward to show that an equilibrium σ on an information structure $\mathcal{S}$ induces winning bid distributions according to (10), which in turn induce an objective according to (12). It is also obvious that the winning bid distributions are in fact probability measures. Indeed, the only subtle piece is that (11) must be satisfied. These constraints implicitly assume that the upward deviator wins all ties. We argue that in fact these stronger constraints must be satisfied by the winning bid distributions induced by the equilibrium, because the deviator could always deviate to $b+\epsilon$ for ϵ arbitrarily small and win any ties outright. In particular, a bidder who deviates uniformly up to $b+\epsilon$ obtains a surplus of at least

$$\int_{v\in V^N} \left[(v_i - b - \epsilon) H\left([0,b]\,|v\right) + \min\{v_i - b - \epsilon, 0\} H\left((b, b+\epsilon]\,|v\right)\right] P(dv)$$
$$+ \int_{v\in V^N}\int_{x=b+\epsilon}^{\infty} (v_i - x) H_i(dx|v) P(dv)$$

where we have omitted the event where the winning bid is exactly b and one of the other bidders makes the winning bid. By following the equilibrium strategy, the bidder obtains a surplus of exactly

$$\int_{v\in V^N}\int_{x=0}^{\infty} (v_i - x) H_i(dx|v) P(dv).$$

Thus, a necessary condition for upward deviations to be deterred is that

$$\int_{v\in V^N} \left[(v_i - b - \epsilon)\, H\left([0,b]\,|v\right) + \min\left\{v_i - b - \epsilon, 0\right\} H\left((b, b+\epsilon]\,|v\right)\right] P\left(dv\right)$$
$$\leq \int_{v\in V^N} \int_{x=0}^{b+\epsilon} (v_i - x)\, H_i\left(dx|v\right) P\left(dv\right).$$

Now, it is obvious that the right-hand side converges to the right-hand side of (11) as $\epsilon \to 0$, and

$$\lim_{\epsilon\to 0} \int_{v\in V^N} (v_i - b - \epsilon)\, H\left([0,b]\,|v\right) P\left(dv\right) = \int_{v\in V^N} (v_i - b)\, H\left([0,b]\,|v\right) P\left(dv\right)$$

Finally,

$$\left|\int_{v\in V^N} \min\left\{v_i - b - \epsilon, 0\right\} H\left((b, b+\epsilon]\,|v\right) P\left(dv\right)\right| \leq \overline{v}\, H\left((b, b+\epsilon] \times V^N\right)$$

converges to zero because of countable additivity. Combining these inequalities yields (11). □

Proof of Lemma 2. Given a feasible solution $\{H_i\left(\cdot|v\right)\}$, we can explicitly define a symmetrized solution by

$$\widetilde{H}_i\left(b|v\right) = \frac{1}{N!} \sum_{\xi\in\Xi} H_{\xi(i)}\left(b|\xi\left(v\right)\right).$$

It is clear that this solution is symmetric, since $\Xi = \{\xi \circ \xi' | \xi \in \Xi\}$ for each $\xi' \in \Xi$. Moreover, $\left\{\widetilde{H}_i\right\}$ will clearly still be be increasing and satisfy the probability bounds, since they are just obtained by averaging the H_i. In addition, since P is symmetric, it must be that

$$\int_{v\in V^N} \int_{b\in B} b\, \widetilde{H}\left(db|v\right) P\left(dv\right) = \frac{1}{N!} \sum_{\xi\in\Xi} \int_{v\in V^N} \int_{b\in B} b\, H\left(b|\xi\left(v\right)\right) P\left(dv\right)$$
$$= \frac{1}{N!} \sum_{\xi\in\Xi} \int_{v\in V^N} \int_{b\in B} b\, H\left(b|v\right) P\left(dv\right)$$

since both v and $\xi\left(v\right)$ have the same density under the prior, which is the same as revenue under the solution $\{H_i\}$. An analogous argument implies that (11) must be satisfied. □

Proof of Lemma 3. Suppose that we have an inefficient solution. We can define an alternative solution $\widetilde{H}_i$ by

$$\widetilde{H}_i\left(b|v\right) = \begin{cases} \frac{1}{|\arg\max v|} H\left(b|v\right) & \text{if } v_i = \max v; \\ 0 & \text{otherwise.} \end{cases}$$

In other words, this alternative reallocates all of the winning bids to the bidders with the highest value. It is clear that $\widetilde{H}\left(b|v\right) = H\left(b|v\right)$, so that revenue (12) is unchanged, and this solution will also be increasing, non-negative, and sum across bidders to probabilities of at most one. We therefore only have to check that $\widetilde{H}$ deters uniform upward deviations. Since H_i is symmetric, we conclude that

$$\begin{aligned} \int_{v\in V^N} \widetilde{H}_i\left(b|v\right) P\left(dv\right) &= \frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N} \widetilde{H}_j\left(b|v\right) P\left(dv\right) \\ &= \frac{1}{N}\int_{v\in V^N} H\left(b|v\right) P\left(dv\right), \end{aligned}$$

so that the right-hand side of (11) is unchanged. Again, using symmetry we conclude that

$$\begin{aligned} \int_{v\in V^N} v_i\widetilde{H}_i\left(b|v\right) P\left(dv\right) &= \frac{1}{N}\sum_{j=1}^{N}\int_{v\in V^N} v_j\widetilde{H}_j\left(b|v\right) P\left(dv\right) \\ &= \frac{1}{N}\int_{v\in V^N} \max v\, H\left(b|v\right) P\left(dv\right) \\ &= \int_{v\in V^N} \max v\, H_i\left(b|v\right) P\left(dv\right). \end{aligned}$$

The left-hand side can be rewritten as

$$\begin{aligned} &\int_{v\in V^N} \left(v_i - b\right)\left(H\left(b|v\right) - H_i\left(b|v\right)\right) P\left(dv\right) \\ &+ \int_{v\in V^N} \left(v_i - b\right)\left(H_i\left(b|v\right) - \widetilde{H}_i\left(b|v\right)\right) P\left(dv\right). \end{aligned}$$

But the second line reduces to

$$\int_{v\in V^N} \left(v_i - \max v\right) H_i\left(b|v\right) P\left(dv\right) \leq 0,$$

and we conclude that (11) must be satisfied for the solution $\widetilde{H}$. $\square$

Proof of Lemma 4. Let us write

$$H(b) = \int_{m \in V} H(b|m)\, Q(dm)$$

for the distribution of winning bids unconditional on the average losing value. Clearly, revenue only depends on $H(b)$ and not on how winning bids are correlated with average losing values. The incentive constraint (15) can be rewritten as

$$\frac{N-1}{N}\left(\int_{m \in V} mH(b|m)\, Q(dm) - bH(b)\right) \leq \frac{1}{N}\int_{x=0}^{b} H(x)\, dx.$$

Thus, the only piece of the incentive constraint that depends on how b is correlated with m is through the first term on the left-hand side, and all things equal, making $\int_{m \in V} mH(b|m)\, Q(dm)$ smaller relaxes the incentive constraint and makes uniform upward deviations less attractive. It is now obvious that fixing $H(b)$, there is a unique division into $H(b|m)$ that minimizes the left-hand side of (15). Let us define the function β by

$$\beta(m) = \min\{b \in B | H(b) \geq Q(m)\}.$$

Because $Q(m)$ is non-atomic, the minimum always exists. Then $\int_{m \in V} mH(b|m)\, Q(dm)$ is minimized pointwise and for all $b \in B$ by setting

$$H(b|m) = \begin{cases} 1 & \text{if } b \geq \beta(m); \\ 0 & \text{otherwise.} \end{cases}$$

□

Proof of Lemma 5. First, let us argue that any feasible solution must satisfy $\beta(m) \geq \underline{m}$ for all $m \in M$. This is essentially a consequence of (17). Without loss of generality, we can assume that $\beta(\underline{m}) = \lim_{m \to \underline{m}} \beta(m)$. Since β is increasing, we know that

$$\frac{1}{N}\int_{x=\underline{m}}^{\underline{m}+\epsilon} (\beta(\underline{m}+\epsilon) - \beta(x))\, Q(dx) \leq \frac{1}{N}(\beta(\underline{m}+\epsilon) - \beta(\underline{m}))\, Q(m+\epsilon)$$

and also that

$$\frac{N-1}{N}\int_{x=\underline{m}}^{\underline{m}+\epsilon} (x - \beta(\underline{m}+\epsilon))\, Q(dx) \geq \frac{N-1}{N}(\underline{m} - \beta(\underline{m}+\epsilon))\, Q(\underline{m}+\epsilon).$$

These two constraints imply that

$$(N-1)\left(\underline{m}-\beta\left(\underline{m}+\epsilon\right)\right)\leq\left(\beta\left(\underline{m}+\epsilon\right)-\beta\left(\underline{m}\right)\right).$$

Continuity of $\beta(m)$ at $\underline{m}$ in turn implies that $\underline{m}-\beta(\underline{m})\leq 0$.

Now suppose that (17) does not hold at a positive measure of m. We can define a new bidding function $\hat{\beta}$ by

$$\hat{\beta}(m)=\frac{\int_{x=\underline{m}}^{m}\left(\frac{N-1}{N}x+\frac{1}{N}\beta(x)\right)Q(dx)}{Q(m)}.$$

This function is obviously weakly increasing, and one can verify from L'Hôpital's rule that

$$\hat{\beta}(\underline{m})=\frac{N-1}{N}\underline{m}+\frac{1}{N}\beta(\underline{m}),$$

so as long as $\beta(m)\geq\underline{m}$, $\hat{\beta}(m)\geq\underline{m}$ as well. Moreover, (17) implies that $\hat{\beta}(m)$ must be weakly less than $\beta(m)$ everywhere and strictly less on a positive Q-measure of m, so that revenue (16) is strictly lower under the feasible solution $\hat{\beta}$ than under β. □

Proof of Theorem 2. We construct a sequence of information structures and associated equilibria, indexed by $x\in(0,1)$, such that for every $\epsilon>0$, there exists a sufficiently large x such that for all $x'\geq x$, revenue is within ϵ of the efficient social surplus.

The information structure is constructed as follows. Fix any $x\in(0,1)$. Given any realization of values, $v_1,...,v_N$, every bidder i with the highest valuation v_i, and hence $v^{(1)}=v_i$ is told to bid

$$\overline{b}=xv^{(1)}+(1-x)v^{(2)}. \tag{28}$$

For one of the bidders with value $v_i=v^{(2)}$, chosen at random, the recommended bid is:

$$\underline{b}=yv^{(1)}+(1-y)v^{(2)}, \tag{29}$$

where y is a random variable with $y\in[0,x]$ and a distribution function parametrized by x:

$$y\sim F(y\,|x)=\left(\frac{y}{1-y}\frac{1-x}{x}\right)^{1/(N-1)}. \tag{30}$$

All other bidders (whose values are weakly less than $v^{(2)}$) are told to bid their values. The bid distribution for the losing bidder are determined independently, and thus the highest y

among the losing bidders, the first order statistic out of $N-1$ is given by:

$$y^{(1)} \sim F^{(1)}(y\,|x\,) = \left(\frac{y}{1-y}\frac{1-x}{x}\right). \tag{31}$$

We claim that this information structure and associated bidding strategy given by (28) and (29) forms an equilibrium for every $x{\in}(0,1)$. Clearly, conditional on the highest value $v^{(1)}$, the distributions of both winning bids and losing bids are absolutely continuous and have support equal to $\left[v^{(2)}, v^{(1)}\right]$. Thus, bidders can never infer from their bid recommendation that they are bidding more than their own value, and the proposed equilibrium strategy is not weakly dominated.

We now verify that there is no profitable deviation for any bidder. We establish the absence of a profitable deviation pointwise, that is for every realized profile of values, $v_1, ..., v_N$. First, note that if there are several bidders with the highest valuation, then $v^{(1)} = v^{(2)}$, and by (28) and (29), it follows that $\underline{b} = \overline{b}$, and there are several winning bidders, and each one receives zero bidder surplus; yet clearly there is no profitable deviation for anybody. For the rest of the argument, it is then sufficient to consider the case of $v^{(1)} > v^{(2)}$.

Now, if the bid b is a recommendation for a losing bidder with value v_i, then it is never profitable to deviate to a higher bid since by construction $b > v_i$. Similarly, lowering the bid below b is not profitable either as it will not change the outcome of the auction. Next, if the bid b is a recommendation for a winning bidder i, then $b < v_i = v^{(1)}$ and a bid increase is not profitable as it does not change the outcome but rather leads to higher sale price. It remains to verify that the winning bidder has no incentive to lower his bid. Given the equilibrium bid, the payoff for winning bidder is:

$$v^{(1)} - \overline{b} = v^{(1)} - xv^{(1)} + (1-x)v^{(2)}.$$

By deviating to a lower bid b', the deviator will win whenever the realized y is below a critical level defined by $b' = yv^{(1)} + (1-y)v^{(2)}$. Given the distribution of y as defined by (30), the payoff from such a deviation is:

$$(v^{(1)} - (yv^{(1)} + (1-y)v^{(2)}))\left(\frac{y}{1-y}\frac{1-x}{x}\right) = (v^{(1)} - v^{(2)})\left(y\frac{1-x}{x}\right),$$

which is increasing in y and at $y = x$ equals:

$$(v^{(1)} - v^{(2)})\,(1-x)\,,$$

which is the winning bidder's surplus given x. Thus there is no profitable deviation either for the winning bidder.

Finally, for each x, the expected winning bid is simply a convex combination of the expected highest and the expected second-highest values, with weights x and $1-x$ respectively. As x approaches 1, the expected winning bid converges to the expected highest value, and bidder surplus must therefore converge to zero. $\square$

Proof of Theorem 3. We first describe the information structure. At every realized profile of values, $v_1, \ldots, v_N$, the losing bidders are informed about the entire profile of values. The winning bidder $v_i = v^{(1)}$ receives only partial information in terms of a recommended bid $b \leq v_i$. Every bid b will be associated with a conditional distribution of profiles of values by losing bidders such that we recover the entire distribution of values below the winning value $v^{(1)}$.

For the construction of the equilibrium it will be sufficient to recover the distribution of the second-highest value, $v^{(2)}$, as the second-highest provides the equilibrium constraints on the highest value bidder. We earlier defined $P^{(2)}\left(v^{(2)} \left| v^{(1)}\right.\right)$ as the conditional distribution of the second-highest value $v^{(2)}$ given the highest value $v^{(1)}$. The information structure and equilibrium will be constructed for every realization of the value of the winning bidder, $v^{(1)} = w$. For notational simplicity, within the current proof we therefore denote by $P(v)$ the conditional distribution of the second-highest value $v^{(2)} = v$ given the highest value $v^{(1)} = w$ of the "winner", or

$$P(v) = P^{(2)}\left(v^{(2)} \leq v \left| v^{(1)} = w\right.\right).$$

Our assumptions on the joint distribution of values implies that $P(v)$ is absolutely continuous, and we write $p(v)$ for a version of its associated density. We thus consider a fixed and given value of the winner, $v^{(1)} = w$. The (highest) optimal bid against the (conditional) distribution $P(v)$ is:

$$b^* = \max\left\{\arg\max_{b \in \mathbb{R}_+}(w-b)P(b)\right\},$$

assuming that others are bidding their values.

Now for every winning valuation w, we construct a distribution of winning bids. Every bid in the support of the winning bid distribution is associated with a conditional distribution over losing values that consists of a conditional point mass of a value equal to the bid b. In addition, there will be positive probability of values strictly below the bid b. These will be distributed proportional to the conditional probability distribution $P(v)$ restricted to the interval $[0, b)$. The distribution of values conditional on bid b, $F_b(v)$, thus (i) has zero mass above b, (ii) has a mass point of weight α at b, and (iii) is proportional to the prior distribution of the second order statistic below b, and (iv) the mass point at b is just large enough to support the indifference condition between b and b^* (implicitly assuming for now that the bidder with the high value wins all ties):

$$w - b = (1-\alpha)(w - b^*)P(b^*) \Leftrightarrow \alpha = 1 - \frac{P(v)}{P(b^*)}\frac{w-b}{w-b^*} \tag{32}$$

The unique solution to the above four conditions is given by:

$$F_b(v) = \begin{cases} \frac{P(v)}{P(b^*)}\frac{w-b}{w-b^*}, & \text{if} \quad 0 \le v < b; \\ 1, & \text{if} \quad b \le v. \end{cases} \tag{33}$$

In other words, we have guaranteed that

$$w - b = (w - b^*) F_b(b^*).$$

To complete the description of the information structure, we need to specify the distribution of bid recommendations for the winning bidder w and thus the distribution of losing values v across the bids. We write $H(b)$ for the distribution over the bids $b \in [b^*, w]$ for a winning value w. It will turn out that the $H(b)$ we construct is absolutely continuous and has an associated continuous density $h(b)$. Given the lower triangular structure of the losing values, and the fact that in each bid segment the probability distribution of values is proportional to the conditional distribution $P(b)$, it is sufficient to insist that for all $x \in [0, w]$ and all $w \in V$, we have the adding-up constraint of the second-highest value. Namely for $b^* \le x \le w$:

$$p(x) = \int_x^w \frac{(w-b)p(x)}{(w-b^*)P(b^*)} h(b)db + \left(1 - \frac{(w-x)P(x)}{(w-b^*)P(b^*)}\right) h(x)dx, \tag{34}$$

and for $0 \le x < b^*$:

$$p(x) = \int_x^w \frac{(w-b)p(x)}{(w-b^*)P(b^*)} h(b)db. \tag{35}$$

The above two equalities, (34) and (35), require that the distribution of bids, $H(b)$ and the distribution of losing values conditional on the bid b, $F_b(v)$ preserve the distribution of the second order statistic $P(v)$. The first condition, (34) reflects that all losing values x between b^* and w get contributions from a mass point and from a continuous component, whereas all losing values below b^* only get contributions for the continuous component (as they are not even competitive against the bid b^*).

We observe that if the adding up constraint (34) is satisfied at $x = b^*$, then the adding up constraint (35) is satisfied everywhere for $0 \leq x < b^*$. At $x = b^*$,condition (34) simply reads

$$p(b^*) = p(b^*) \int_{b^*}^{w} \frac{(w-b)}{(w-b^*) P(b^*)} h(b) db,$$

and hence guarantees that

$$\int_{b^*}^{w} \frac{(w-b)}{(w-b^*) P(b^*)} H(db) = 1.$$

We can thus focus entirely on condition (35). Now let us write

$$G(x) = \frac{1}{(w-b^*) P(b^*)} \int_{x}^{w} (w-b) h(b) db,$$

and

$$\lambda(x) = \frac{(w-x)p(x)}{(w-b^*) P(b^*) - (w-x)P(x)}.$$

We can then rewrite the adding up condition (34) as

$$\lambda(x)G(x) - g(x) = \lambda(x), \tag{36}$$

where

$$g(x) = -\frac{(w-x)h(x)}{(w-b^*) P(b^*)}, \tag{37}$$

is the derivative of $G(x)$. The adding up constraint can therefore be written as a separable ordinary differential equation, which has an explicit solution given by:

$$G(x) = 1 - \exp\left(-\int_{x}^{w} \lambda(b) db\right), \tag{38}$$

and corresponding derivative:

$$g(x) = -\lambda(x) \exp\left(-\int_{x}^{w} \lambda(b) db\right).$$

We can easily verify that the solution (38) satisfies the adding up constraint (34). From here we find from (37) that the distribution over values is:

$$h(x) = \frac{g(x)}{w-x} (w-b^*) P(b^*) = \frac{\lambda(x)}{w-x} (w-b^*) P(b^*) \exp\left(-\int_{x}^{w} \lambda(b) db\right),$$

or, expanding $\lambda(x)$, this is

$$h(x)=\frac{\left(w-b^*\right)P\left(b^*\right)p(x)}{\left(w-b^*\right)P\left(b^*\right)-(w-x)P(x)}\exp\left(-\int_x^w\frac{(w-b)p(b)}{\left(w-b^*\right)P\left(b^*\right)-(w-x)P(x)}db\right).$$

Now, we can write

$$h(x)=p(x)(1-G(x))-g(x)P(x).$$

To see this, we insert the expression for $g(x)$ and $G(x)$ in the right-hand side of the above equality and obtain:

$$[p(x)+P(x)\lambda(x)]\exp\left(-\int_{b=x}^w\lambda(b)db\right),$$

and

$$\begin{aligned}p(x)+P(x)\lambda(x)&=p(x)\left[\frac{(w-x)P(x)}{\left(w-b^*\right)P\left(b^*\right)-(w-x)P(x)}+1\right]\\&=p(x)\frac{\left(w-b^*\right)P\left(b^*\right)}{\left(w-b^*\right)P\left(b^*\right)-(w-x)P(x)},\end{aligned}$$

and thus we obtain $h(x)$. Hence,

$$H(x)=P(x)(1-G(x)),$$

and it is sufficient to verify that $G\left(b^*\right)=1$. For this, we simply need to show that the integral $\int_{b^*}^w\lambda(b)db$ diverges. But this is immediate as long as $P(x)$ is differentiable and if $p(x)$ is bounded away from zero. The reason is that b^* maximizes $(w-b)P(b)$, so that $(w-b)p(b)-P(b)$ must go to zero as $b\to b^*$. Thus, for ϵ small,

$$(w-b)p(b)\geq P(b)-(w-b)p(b)$$

for $b\in[b^*,b^*+\epsilon]$. This implies that

$$
\begin{aligned}
\int_{b^*}^{b^*+\epsilon} \lambda(b) db &= \int_{b}^{b^*+\epsilon} \frac{(w-b)p(b)}{(w-b^*)\,P\,(b^*)-(w-b)P(b)} db \\
&\geq \int_{b^*}^{b^*+\epsilon} \frac{P(b)-(w-b)p(b)}{(w-b^*)\,P\,(b^*)-(w-b)P(b)} db \\
&= \log\left((w-b^*)\,P\,(b^*)-(w-b)P(b)\right)\big|_{b^*}^{b^*+\epsilon} \\
&= \infty,
\end{aligned}
$$

so that $G\,(b^*) = 1$.

Finally, we note the distribution of the second-order statistic associated with $F_b(v)$ yields the desired equilibrium behavior if all losing values (and associated bidders) bid their value v under the efficient tie breaking rule. We need the efficient tie breaking rule as the distribution $F_b(v)$ has a mass point at b. We end this proof by constructing a bidding strategy for the second-highest value that maintains the equilibrium even under the symmetric tie breaking rule that does not favor the highest value $v^{(1)} = w$ at equal bids. We achieve this by asking the second-highest bidder with value b to bid below his value b. In particular, we redistribute the mass point at $v = b$ to bids below b governed by a continuous distribution. The resulting new distribution is defined through the indifference condition:

$$
(w-b) = (w-x)\widehat{F}_b(x) \Leftrightarrow \widehat{F}_b(x) = \frac{(w-b)}{(w-x)}
$$

with support $[b-\epsilon, b]$ for some small $\epsilon > 0$. The resulting distribution of bids

$$
\alpha\widehat{F}_b(x) + (1-\alpha)P(x),
$$

has the highest value bidder win with probability one under the symmetric tie breaking rule. The losing bidder with value b on the other hand is willing to randomize his bid and hence receive net value zero as winning the auction with a bid b would equally result in a zero net utility. $\square$