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By

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March 2014

COWLES FOUNDATION DISCUSSION PAPER NO. 1938

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
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Computational Complexity of the Walrasian Equilibrium Inequalities

Donald J. Brown
Department of Economics,
Yale University
donald.brown@yale.edu

March 2014

Abstract

Recently Cherchye et al. (2011) reformulated the Walrasian equilibrium inequalities, introduced by Brown and Matzkin (1996), as an integer programming problem and proved that solving the Walrasian equilibrium inequalities is NP-hard. Following Brown and Shannon (2000), we reformulate the Walrasian equilibrium inequalities as the Hicksian equilibrium inequalities.

Brown and Shannon proved that the Walrasian equilibrium inequalities are solvable iff the Hicksian equilibrium inequalities are solvable. We show that solving the Hicksian equilibrium inequalities is equivalent to solving an NP-hard minimization problem. Approximation theorems are polynomial time algorithms for computing approximate solutions of NP-hard minimization problems.

The contribution of this paper is an approximation theorem for the NP-hard minimization, over indirect utility functions of consumers, of the maximum distance, over observations, between social endowments and aggregate Marshallian demands. In this theorem, we propose a polynomial time algorithm for computing an approximate solution to the Walrasian equilibrium inequalities, where explicit bounds on the degree of approximation are determined by observable market data.

Keywords: Rationalizable Walrasian markets, NP-hard minimization problems, Approximation theorems

JEL Classification: B41, C68, D46

1 Introduction

The Brown-Matzkin (1996) theory of rationalizing market data with Walrasian markets, where consumers are price-taking, utility maximizers subject to a budget constraint, consists of market data sets and the Walrasian equilibrium inequalities. A market data set is a finite number of observations on market prices, income distributions and social endowments. The Walrasian equilibrium inequalities are the Afriat inequalities for each consumer; the budget constraints for each consumer and the market clearing conditions in each observation. The unknowns in the Walrasian equilibrium inequalities are the utility levels, the marginal utilities of income and the Marshallian demands of individual consumers in each observation. The parameters are the observable market data: market prices, income distributions and social endowments in each observation. The Walrasian equilibrium inequalities are said to rationalize the observable market data if the Walrasian equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of individual consumers, where aggregate Marshallian demands are equal to the social endowments in every observation. This theory is intended to provide an empirical, nonparametric foundation for equilibrium in market economies, consistent with the Walrasian paradigm, as articulated by Arrow and Debreu (1954). As such, the Brown–Matzkin theory of rationalizing market data with Walrasian markets requires an efficient algorithm for solving the Walrasian equilibrium inequalities.

The Tarski–Seidenberg theorem, Tarski (1951), proposes an algorithm, "quantifier elimination," for deriving a finite family of multivariate polynomial inequalities from the Walrasian equilibrium inequalities, where the unknowns are the observable market data: market prices, income distributions and the social endowments in each observation. We define these inequalities as the "revealed Walrasian equilibrium inequalities." It follows from the Tarski–Seidenberg theorem that the revealed Walrasian equilibrium inequalities are solvable for the given market data iff the equilibrium inequalities are solvable for some family of utility levels, marginal utilities of income and Marshallian demands of consumers. The revealed Walrasian equilibrium inequalities exhaust the empirical content of equilibrium in the Walrasian model of market economies. An example is the special case of Afriat's (1967) seminal theorem on revealed consumer demand, where consumer's demands are observable. The Afriat inequalities are linear inequalities, hence they can be solved in polynomial time, using interior point methods. In Afriat's theory, $GARP$, introduced by Varian (1982), is the revealed Afriat equilibrium inequalities. Unfortunately, in general, the computational complexity of the Tarski–Seidenberg algorithm, is known to be doubly exponential in the worse case. See Basu (2011) for a discussion of the Tarski–Seidenberg theorem and the computational complexity of quantifier elimination.

A decision problem is a problem where the answer is "yes" or "no." In this paper, the decision problem is: Can the observed market data set be rationalized with Walrasian markets? That is, are the Walrasian equilibrium inequalities solvable if the values of the parameters are derived from the observed market data? A decision problem is said to have polynomial complexity, i.e., the problem is in class P, if there exists an algorithm that solves each instance of the problem in time that is polynomial in some measure of the size of the problem instance. For solving linear programming, interior-point methods are polynomial time algorithms, but the simplex method solves the worse case linear program in time that is exponential in some measure of the size of the worse case linear program. In the literature on computational complexity, polynomial time algorithms are referred to as “efficient” algorithms. A decision problem is said to be in NP, if there exists an algorithm that verifies, in polynomial time, if a proposal is a solution of the problem instance Clearly,

$$P \subset NP$$

but it is widely conjectured by computer scientists that

$$P \neq NP$$

What is the computational complexity of solving the Walrasian equilibrium inequalities? This important question was first addressed by Cherchye et al. (2011). They proved that solving the Walrasian equilibrium inequalities, reformulated as an integer programming problem is NP-*hard*. The decision problem A is said to be NP-*hard*, if every problem in NP can be reduced in polynomial time to A. That is, if we can decide the NP-*hard* problem A in polynomial time then we can decide every NP problem in polynomial time. In this case, contrary to the current beliefs of computer scientists,

$$P = NP.$$

Brown and Shannon (2000) proposed a “dual” theory of rationalizing market data with Walrasian markets, where demands of individual consumers are not observed, and the Afriat inequalities are replaced by the dual Afriat inequalities for minimizing the consumer’s monotone, convex, indirect utility function over prices subject to her budget constraint. This “dual” family of Walrasian equilibrium inequalities is called the Hicksian equilibrium inequalities in this paper. The Hicksian equilibrium inequalities are said to rationalize the observed market data if the Hicksian equilibrium inequalities are solvable for some family of indirect utility levels, marginal indirect utilities and Marshallian demands of individual consumers, derived from Roy’s identity, where the aggregate Marshallian demands are equal to the social endowments in every observation. Brown and Shannon proved that the Walrasian equilibrium inequalities are solvable iff the Hicksian equilibrium inequalities are solvable. The Hicksian equilibrium inequalities are unsolvable iff for very solution of the consumer’s dual Afriat inequalities there is excess aggregate Marshallian demand in some observation. Of course, if for some family of consumer’s indirect utility functions “markets almost clear” in every observation, then the market data is “almost rationalized” by Walrasian markets. Theorems of this kind for NP-*hard* minimization problems, where the degree of approximation is explicit are called "approximation theorems" in the literature on computational complexity. The contribution of this paper is an approximation theorem for the NP-*hard* minimization, over indirect utility functions of consumers, of the maximum distance, over observations, between social endowments and aggregate Marshallian demands. In this theorem, we propose a polynomial time algorithm for computing an approximate solution of the Walrasian equilibrium inequalities, where explicit bounds on the degree of approximation are determined by observable market data.

2 Solving the Walrasian Equilibrium Inequalities

For completeness, we recall the (strict) Afriat inequalities (1967) and the (strict) dual Afriat inequalities, introduced by Brown and Shannon (2000). Given solutions of the Afriat inequalities, we also recall Afriat's construction of a piece-wise linear, monotone, concave utility function for rationalizing Marshallian demands. This is the same construction used by Brown and Shannon to derive a piece-wise linear, monotone, convex indirect utility function, from solutions of the dual Afriat inequalities, to rationalize Marshallian demand.

Proposition 1 *If the market data set is*

$$D \equiv \{p_j, x_j, I_j\}_{j=1}^{N}$$

where

$$u(x_j) = \max_{p_j \cdot y \leq I_j} u(y)$$

and

$$V\left(\frac{p_j}{I_j}\right) = \min_{\frac{p}{I_j} \cdot x_j \leq 1} V\left(\frac{p}{I_j}\right)$$

then the (strict) Afriat inequalities for N observations are:

$$u_i < u_j + \lambda_j p_j \cdot (x_i - x_j) \text{ for } i, j \leq N$$

and the dual (strict) Afriat inequalities for N observations are:

$$V_i - V_j > \nabla_{\frac{p}{L}} V\left(\frac{p_j}{I_j}\right) \cdot \left(\frac{pi}{I_i} - \frac{p_j}{I_j}\right)$$

where (i) u_i and u_j are positive utility levels, (ii) λ_j are positive marginal utilities of income, (iii) p_j are the market prices in observation, (iv) I_j is the consumer's income in observation j, (v) x_i and x_j are the Marshallian demands in the respective budget sets, (vi) V_i and V_j are positive indirect utility level, (vii) $q_j \equiv \nabla_{\frac{p}{L}} V(p_j/I_j)$, and (viii) where $q_j \ll 0$

Proposition 2 *If the (strict) Afriat inequalities are solvable for utility levels,u_j, marginal utilities of income λ_j and Marshallian demands x_j for the given market data*

$$D \equiv \{p_j, x_j, I_j\}_{j=1}^{N}$$

then

$$\widehat{u}(x) \equiv \max_{j=1}^{j=N} [u_j + \lambda_j p_j \cdot (x - x_j)]$$

is a piece-wise linear, monotone, concave utility function that rationalizes $\{x_j\}_{j=1}^{j=N}$. *That is,*

$$\widehat{u}(x_j) = \{\max \widehat{u}(x) : p_j \cdot x \leq I_j\}$$

Proposition 3 *If the (strict) dual Afriat inequalities are solvable for indirect utility levels,*V_j *and marginal indirect utilities*

$$q_j \equiv \nabla_{\frac{p}{L}} V(\frac{p_j}{I_j})$$

for the given market data

$$D \equiv \{p_j, x_j, I_j\}_{j=1}^{N}$$

then

$$\widehat{V}\left(\frac{p}{I}\right) \equiv \min_{j=1}^{j=N} \left[\nabla_{\frac{p}{L}} V\left(\frac{p_j}{I_j}\right) \cdot \left(\frac{p}{I} - \frac{p_j}{I_j}\right)\right]$$

is a piece-wise linear, monotone, convex indirect utility function that rationalizes $\{p_j/I_j\}_{j=1}^{j=N}$. *That is,*

$$\widehat{V}\left(\frac{p_j}{I_j}\right) = \left\{\min \widehat{V}\left(\frac{p}{I}\right) : \frac{p}{I} \cdot x_j \leq 1\right\}.$$

In addition to the Afriat and the dual Afriat inequalities, our analysis is predicated on Roy's identity.

Proposition 4 *Roy's Identity [Theorem* 22.3 *in Simon and Blume* (1994)*]Let* $U(x)$ *be a* C^2*utility function that satisfies monotonicity and strict concavity. Let* $\xi(p, I)$ *be the Marshallian demand function for* U *and* $V(p, I)$ *the corresponding indirect utility function. If* (p, I) *and* $\xi(p, I)$ *are all strictly positive, then*

$$\xi_i(p, I) = -\frac{\frac{\partial V(p,I)}{\partial p_i}}{\frac{\partial V(p,I)}{\partial I}}$$

Definition 5 *If the market data set* $D = \{p_j, \omega_j, I_{ij}\}_{ij}^{MN}$ *then, the Hicksian equilibrium inequalities are defined as:(1) The strict, dual Afriat inequalities for each consumer.(2) Market clearing in each observation, where it follows from Roy's identity that the Marshallian demands are:*

$$x_{ij} \equiv -\frac{\nabla_p V_i(\frac{p_j}{I_{ij}})}{\nabla_I V_i(\frac{p_j}{I_{ij}})} = \frac{(-\frac{q_{ij}}{I_{ij}})}{-\frac{p_j \cdot q_{ij}}{I_{ij}^2}} = \frac{I_{ij} q_{ij}}{p_j \cdot q_{ij}} = \frac{(-q_{ij})}{I_{ij}\lambda_{ij}}.$$

That is,

$$x_{ij} = \frac{I_{ij} q_{ij}}{p_j \cdot q_{ij}}.$$

Hence markets clear in each observation iff for $j = 1, 2, ..., N$

$$\sum_{i=1}^{t=M} x_{ij} = \sum_{i=1}^{t=M} \frac{I_{ij} q_{ij}}{p_j \cdot q_{ij}} \leq \omega_j.$$

Rationalizability of the Hicksian equilibrium inequalities was established by Brown and Shannon (2000)*.*

Theorem 6 *(Brown and Shannon) If*

$$D \equiv \{p_j, \omega_j, I_{ij}\}_{ij}^{MN}$$

is the given market data set, where for $1 \leq j \leq N$

$$\sum_{i=1}^{i=M} I_{ij} = p_j \cdot \omega_j,$$

then the following statements are equivalent:

(a)There exists a strictly convex, monotone, smooth indirect utility function $V_i(p/I)$ *that rationalizes* D

(b)There exists numbers

$$V_{ij} \equiv V_i \left(\frac{p_j}{I_{ij}} \right)$$

and vectors

$$q_{ij} \equiv \nabla_{\frac{p}{I}} V_i \left(\frac{p_j}{I_{ij}} \right) \in R^L$$

for $j, k = 1, 2, ..., N$ *such that* $k \neq j$ *(i)*

$$V_{ik} - V_{ij} > q_{ij} \cdot \left(\frac{p_k}{I_{ik}} - \frac{p_j}{I_{ij}} \right)$$

for $1 \leq i \leq M$*;* $1 \leq j \leq N$*. where (ii)*

$$-\frac{p_j \cdot q_{ij}}{I_{ij}^2} \equiv \lambda_{ij} > 0, \;\; q_{ij} \ll 0.$$

Proof. See Theorem 1 and Lemma 1 in Brown and Shannon. ■

Corollary 7 *(Brown and Shannon) An equivalent family of dual Afriat inequalities, where prices are not normalized by incomes, is the following: for* $j, k = 1, 2, ..., N$ *such that* $k \neq j$

$$V_{ik} - V_{ij} > \frac{q_{ij}}{I_{ij}} \cdot (p_k - p_j) - \frac{p_j \cdot q_{ij}}{I_{ij}^2} (I_{ik} - I_{ij})$$

for $1 \leq i \leq M$*;* $1 \leq j \leq N$*. where*

$$\nabla_p V_i \left(\frac{p_j}{I_{ij}} \right) = \frac{q_{ij}}{I_{ij}}$$

for $1 \leq i \leq M$*;* $1 \leq j \leq N$*., where*

$$\nabla_I V_i \left(\frac{p_j}{I_{ij}} \right) = -\frac{p_j \cdot q_{ij}}{I_{ij}^2} \equiv \lambda_{ij}.$$

In Lemma 2, Brown and Shannon show that Hicksian equilibrium inequalities are equivalent to the Walrasian equilibrium inequalities. That is, the Walrasian equilibrium inequalities are solvable iff the Hicksian equilibrium inequalities are solvable.

3 An Approximation Theorem for Walrasian Markets

The best known uniform bound on the marginal utilities of income is the assumption that consumers are endowed with quasilinear utilities. In this case, we restrict attention to rationalizing market data with Hicksian economies where $\lambda_{ij} = 1$. That is, we assume the market data can be rationalized by a Hicksian quasilinear economy, where each consumer is endowed with a smooth, monotone, convex, indirect quasilinear utility function. The Hicksian quasilinear equilibrium inequalities consist of the Hicksian equilibrium inequalities and the linear equalities:

$$p_j \cdot (-q_{ij}) = I_{ij}^2,$$

The Hicksian quasilinear equilibrium inequalities is a family of linear inequalities in q_{ij}, where

$$\sum_{i=1}^{i=M} x_{ij} \equiv -\frac{\nabla_p V_i\left(\frac{p_j}{I_{ij}}\right)}{\nabla_I V_i\left(\frac{p_j}{I_{ij}}\right)} = \sum_{i=1}^{i=M} \frac{\left(-\frac{q_{ij}}{I_{ij}}\right)}{-\frac{p_j \cdot q_{ij}}{I_{ij}^2}} = \sum_{i=1}^{i=M} \frac{(-q_{ij})}{I_{ij}}.$$

Hence Γ, the optimal value of the linear program R, can be computed in polynomial time, where

$$\Gamma \equiv \left\{ \min_{1 \leq r; q_{ij} \leq 0} r : \sum_{\lambda_{ij} \neq 1} \frac{(-q_{ij})}{I_{ij}} \leq r\omega_j \right\} : R$$

$\Gamma = 1$ iff the market data is rationalized by a Hicksian quasilinear economy.

The importance of Hicksian quasilinear economies is that they allow us to approximate market economies, where we assume the consumer's marginal utilities of income $\lambda_{ij} \geq 1$. The empirical justification for this assumption follows from restricting incomes of consumers to to be bounded above and the empirical finding of Laydard (2008) and Elsas and Assmann (2012) that the marginal utility of income diminishes with income, as conjectured by the classical economists. Diminishing marginal utility of income is one of the theoretical justifications for progressive taxation. Hence w.o.l.o.g. we assume that the lower bound on the marginal utilities of income for each consumer is one. In this case, the Hicksian equilibrium inequalities are augmented by the linear inequalities:

$$p_j \cdot (-q_{ij}) \geq I_{ij}^2.$$

If we recast the dual Afriat inequalities as the first order conditions for minimizing a smooth, monotone, convex indirect utility function subject to a budget constraint defined by the Marshallian demand, then we can invoke Gauvin's (1977) theorem that the set of Lagrange multipliers for the budget constraint is bounded iff the consumers optimization problem satisfies the Mangasarian–Fromovitz constraint qualification, derived in Mangasarian–Fromovitz (1967). That is, for fixed (x_{ij}, I_{ij}) the ith consumer solves the following optimization problem (Q_{ij}):

$$\min_{\{p \in R_{++}^K | \frac{p}{I_{ij}} \cdot x_{ij} \leq 1\}} V_i\left(\frac{p}{I_{ij}}\right) = V_i\left(\frac{p_j}{I_{ij}}\right).$$

The Lagrangian for Q_{ij} is

$$L(p:\mu_{ij}) \equiv V_i\left(\frac{p}{I_{ij}}\right) + \mu_{ij}\left(\frac{p}{I_{ij}}\cdot x_{ij} - 1\right)$$

where

$$x_{ij} \equiv -\frac{\nabla_p V_i\left(\frac{p_j}{I_{ij}}\right)}{\nabla_I V_i\left(\frac{p_j}{I_{ij}}\right)} = -\frac{\frac{q_{ij}}{I_{ij}}}{-\frac{p_j\cdot q_{ij}}{I_{ij}^2}} = \frac{I_{ij}q_{ij}}{p_j\cdot q_{ij}} = \frac{\left(-\frac{q_{ij}}{I_{ij}}\right)}{\lambda_{ij}}.$$

Hence the first order conditions are:

$$\frac{q_{ij}}{I_{ij}} = \nabla_p V_i\left(\frac{p_j}{I_{ij}}\right) = -\mu_{ij}x_{ij} = \mu_{ij}\frac{\frac{q}{I_{ij}}}{\lambda_{ij}} = \frac{q_{ij}}{I_{ij}} \text{ iff } \mu_{ij} = \lambda_{ij}.$$

The Mangasarian–Fromovitz constraint qualification follows from Slater's constraint qualification for convex inequality constraints, e.g., the budget constraint. See the Fritz John Theorem, Theorem 19.1, in Simon and Blume (1994) for the Mangasarian–Fromovitz constraint qualification and Theorem 19.12 in Simon and Blume for Slater's constraint qualification.

Gauvin computes a bound Υ_{ij} on the Lagrange multipliers μ_{ij} by solving the following linear program:

$$\mu_{ij} \leq \Upsilon_{ij} \equiv \min_{\{y\in R^K | x_{ij}\cdot y \leq -1} \nabla_p V_i\left(\frac{p_j}{I_{ij}}\right)\cdot y = \nabla_p V_i\left(\frac{p_j}{I_{ij}}\right)\cdot y_{ij}.$$

We compute a universal upper bound Θ on the μ_{ij}, independent of $\nabla_p V(p_j/I_{ij})$ and x_{ij}, as the optimal value of the maxmin optimization problem, where Π_i is the family of dual Afriat inequalities for consumer i,

$$\Theta \equiv \max_{\frac{q_{ij}}{I_{ij}}\in\Pi_i} \min_{\{y\in R^K|\omega_j\cdot y\leq -1\}} \frac{(-q_{ij})}{I_{ij}}\cdot y.$$

If ω_j is the social endowment in observation j and $x_{ij} \leq \omega_j$ then $\Upsilon_{ij} \leq \Theta$. The results in Brown and Calsamiglia (2007) imply that the quasilinear solutions of the dual Afriat inequalities are characterized by the family of linear inequalities : $\lambda_{ij} = 1$, hence

$$1 \leq \Theta$$

$(-q_{ij})/I_{ij}$ and y lie in separable polyhedral constraint sets and the objective function is bilinear. Gosh and Boyd (2003) — see section 2 on bilinear problems — show that maximin problems with these properties can be reduced to a linear program and solved in polynomial time. Since $\mu_{ij} = \lambda_{ij}$, it follows that $\lambda_{ij} \leq \Theta$. For $1 \leq j \leq M$:

Definition 8 *Approximation Theorems If $OPT(\beta)$ is the optimal value of a NP-hard minimization problem for the input β and $\widehat{OPT(\beta)}$ is the optimal value of the* *approximating minimization problem for the input β, then the ratio $\frac{\widehat{OPT(\beta)}}{OPT(\beta)}$ is bounded above by the "approximation ratio" $\alpha(\beta) \geq 1$. Hence*

$$OPT(\beta) \leq \widehat{OPT(\beta)} \leq \alpha(\beta)OPT(\beta),$$

where $\widehat{OPT(\beta)}$ and $\alpha(\beta)$ can be computed in time polynomial in β.

Lemma 9 *[Approximation Lemma]: If $\Theta \geq \lambda_{ij} \geq 1$, then*

$$\Theta \sum_{i=1}^{i=M} x_{ij} \geq \sum_{\lambda_{ij}>1} \frac{(-q_{ij})}{I_{ij}} \geq \sum_{i=1}^{i=M} x_{ij} \text{ for } 1 \leq j \leq N.$$

Proof.

$$\sum_{i=1}^{i=M} x_{ij} \equiv -\frac{\nabla_p V_i(\frac{p_j}{I_{ij}})}{\nabla_I V_i(\frac{p_j}{I_{ij}})} = \sum_{i=1}^{i=M} \frac{\left(-\frac{q_{ij}}{I_{ij}}\right)}{-\frac{p_j \cdot q_{ij}}{I_{ij}^2}} = \sum_{i=1}^{i=M} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}}.$$

If $\Theta \geq \lambda_{ij} > 1$, then

$$\Theta \sum_{\lambda_{ij}\ >1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}} \geq \sum_{\lambda_{ij}>1} \frac{(-q_{ij})}{I_{ij}} > \sum_{\lambda_{ij}\ >1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}}.$$

That is,

$$\Theta \sum_{i=1}^{i=M} x_{ij} \geq \sum_{\lambda_{ij}>1} \frac{(-q_{ij})}{I_{ij}} \geq \sum_{i=1}^{i=M} x_{ij} \text{ for } 1 \leq j \leq N.$$

■

Theorem 10 *[Approximation Theorem]If (1) Δ is the optimal value of the nonconvex program S, where*

$$\Delta \equiv \left\{ \min_{1\leq s; q_{ij}\leq 0} s : \sum_{\lambda_{ij}\ >1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}} \leq s\omega_j \right\} : S.$$

(2) Γ is the optimal value of the linear program R,where

$$\Gamma \equiv \min_{1\leq r; q_{ij}\leq 0} r : \sum_{\lambda_{ij}>1} \frac{(-q_{ij})}{I_{ij}} \leq r\omega_j\} : R.$$

(3) Ψ is the optimal value of the nonconvex program T, where

$$\Psi \equiv \left\{ \min_{1\leq t; q_{ij}\leq 0} t : \Theta \sum_{\lambda_{ij}\ >1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}} \leq t\omega_j \right\} : T.$$

Then

$$(a)\ \Psi \geq \Gamma \geq \Delta$$

and

$$(b)\ \Psi = \Theta\Delta.$$

That is,

$$(c)\ \Theta\Delta \geq \Gamma \geq \Delta.$$

Proof. (a) follows from the approximation lemma. To prove (b) note the $1-1$ correspondence between s and t, where

$$t \to \frac{t}{\Theta} \equiv s \text{ and } s \to s\Theta \equiv t$$

That is,

$$\frac{\Psi}{\Theta} = \left\{ \min_{1 \leq t; q_{ij} \leq 0} t : \sum_{\lambda_{ij} > 1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}} \leq \frac{t}{\Theta}\omega_j \right\} = \left\{ \min_{1 \leq s; q_{ij} \leq 0} s : \sum_{\lambda_{ij} > 1} \frac{(-q_{ij})}{I_{ij}\lambda_{ij}} \leq s\omega_j \right\} = \Delta.$$

Hence

$$\Psi = \Theta\Delta.$$

■

$\Delta = 1$ iff the market data is rationalized by a Hicksian economy, where all consumers are endowed with indirect utility functions with marginal utilities of income, $\lambda_{ij} \geq 1$. $\Gamma = 1$ iff the market data is rationalized by a Hicksian quasilinear economy, where all consumers are endowed with quasilinear utility functions, i.e., the marginal utilities of income, $\lambda_{ij} = 1$. Θ and Γ are optimal values of linear programs, hence they can be computed in polynomial time using interior point methods. Δ is an $NP - hard$ minimization problem..

Utility functions with marginal utilities of income bounded below by one are interesting in their own right, e.g., in fields like public finance and development economics. We close with an extension of the Brown-Calsamiglia characterization of quasilinear utility functions, where the marginal utilities of income equal one, to utility functions where the marginal utilities of income are bounded below by one.

Proposition 11 *The market data* $\{p_j, I_j\}_{j=1}^{j=N}$ *is rationalized by an indirect utility function* $V_i(p, I)$*, where the marginal utilities of income*

$$\lambda_{ij} \equiv \frac{p_j \cdot (-q_{ij})}{I_{ij}^2}$$

are bounded below by one, iff the following family of linear inequalities are solvable,

$$(1)\ V_{ik} - V_{ij} > \frac{q_{ij}}{I_{ij}} \cdot (p_k - p_j) - \frac{p_j \cdot q_{ij}}{I_{ij}^2}(I_{ik} - I_{ij}),$$

$$(2)\ \lambda_{ij} \geq 1 \text{ or equivalently } p_j \cdot (-q_{ij}) \geq I_{ij}^2.$$

Proof. Necessity is immediate. We use Afriat's construction for sufficiency:

$$V_i(p, I) \equiv \vee_{j=1}^{j=N} \left[V_{ij} + \frac{q_{ij}}{I_{ij}} \cdot (p - p_j) - \frac{p_j \cdot q_{ij}}{I_{ij}^2} (I - I_{ij}) \right].$$

It follows from convex analysis that the subgradient of the max of a finite family of convex functions is a convex combination of the subgradients of the component functions. Since the marginal utilities of income of the component functions are bounded below by one, the marginal utility of income of $V_i(p, I)$ is bounded below by one. ■

Acknowledgement 12 *Dan Spielman suggested several excellent references on the computational complexity of algorithms. The comments of Felix Kubler on an earlier draft of this paper were very helpful. I thank them both.*

4 References

Afriat, S.: "The Construction of a Utility Function from Demand Data," *International Economic Review* (1967), 8: 67–77.

Arrow, K., and G. Debreu: "The Existence of an Equilibrium for a Competitive Economy," *Econometrica* (1954), 22: 265–290.

Basu, S.: "Algorithms in Real Algebraic Geometry: Survey. Department of Mathematics," Purdue University, West
Lafayette, 2011.

Brown, D.J., and C. Calsamiglia: "The Nonparametric Approach to Applied Welfare Analysis," *Econ Theory* (2007), 31: 181–188.

Brown, D.J., and R.L. Matzkin: "Testable Restrictions on the Equilibrium Manifold," *Econometrica* (1996), 64: 1249–1262.

Brown, D.J., and C. Shannon: "Uniqueness, Stability, and Comparative Statics in Rationalizable Walrasian Markets," *Econometrica* (2000), 68: 1529–1539.

Cheryche, L. et al.: "Testable Implications of General Equilibrium Models: An Integer Programming Approach," *J. Math. Econ.* (2011), 47: 564–575.

Debreu, G.: "The Coefficient of Resource Utilization," *Econometrica* (1951), 19: 273–292.

Elsas, S., and C. Assman: "The Diminishing of Marginal Utility of Income under Latent Individual Specific Heterogeneity," http://www.iariw.org (2012).

Gauvin, J.: "A Necessary and Sufficient Regularity Condition to Have Bounded Multipliers in Nonconvex Programming," *Mathematical Programming* (1977), 12: 136–138.

Gosh, A., and S. Boyd: "Minimax and Convex-Concave Games," EE3920, Stanford University, Autumn, 2003.

Laydard, R., et al.: "The Marginal Utility of Income," *Journal of Public Economics* (2008), 92(8–9): 1846–1857.

Mangasarian, O.L., and S. Fromowitz: "The Fritz John Necessary Optimality Conditions in the Presence of Equality and Inequality Constraints," *Journal of Mathematical Analysis and Applications* (1967), 17: 37–47.

Simon, C.P., and L.E. Blume: *Mathematics for Economists.* Norton and Company, 1994.

Tarski, A.: *A Decision Method for Elementary and Geometry*, 2nd rev. ed.. Rand Corporation, Berkeley and Los Angles, 1951.

Varian, H.: "The Nonparametric Approach to Demand Analysism" *Econometrica* (1982), 50: 945-973.