8-1-2007

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Eduardo Faingold

Yuliy Sannikov

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By

Eduardo Faingold and Yuliy Sannikov

August 2007

COWLES FOUNDATION DISCUSSION PAPER NO. 1624

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
YALE UNIVERSITY
Box 208281
New Haven, Connecticut 06520-8281

http://cowles.econ.yale.edu/

Reputation Effects and Equilibrium Degeneracy in Continuous-Time Games*

Eduardo Faingold
Department of Economics
Yale University
`eduardo.faingold@yale.edu`

Yuliy Sannikov
Department of Economics
University of California, Berkeley
`sannikov@econ.berkeley.edu`

August 27, 2007

Abstract

We study a class of continuous-time reputation games between a large player and a population of small players in which the actions of the large player are imperfectly observable. The large player is either a *normal type*, who behaves strategically, or a *behavioral type*, who is committed to playing a certain strategy. We provide a complete characterization of the set of sequential equilibrium payoffs of the large player using an ordinary differential equation. In addition, we identify a sufficient condition for the sequential equilibrium to be unique and Markovian in the small players' posterior belief. An implication of our characterization is that when the small players are certain that they are facing the normal type, intertemporal incentives are trivial: the set of equilibrium payoffs of the large player coincides with the convex hull of the set of static Nash equilibrium payoffs.

1 Introduction.

Reputation plays an important role in long-run relationships. A firm, for instance, can benefit from reputation to fight potential entrants, to provide high quality to consumers, or to generate good returns to investors. Governments can benefit from commitment to non-inflationary monetary policy, low capital taxation and efforts to fight corruption. Sometimes, bad reputation can also create perverse incentives that lead to market breakdown. In this paper, we study reputation dynamics in a repeated game between a large player and a population of small players in which the actions of the large player are imperfectly observable. For example, the quality of a firm's products may be a noisy outcome of a firm's hidden effort to maintain quality standards. The inflation rate can be a noisy signal of money supply.

*We are grateful to Daron Acemoglu, Kyna Fong, Drew Fudenberg, David K. Levine, George J. Mailath, Eric Maskin, Stephen Morris, Bernard Salanie, Paolo Siconolfi, Lones Smith and seminar participants at Bocconi, Columbia, Duke, Georgetown, the Institute for Advanced Studies at Princeton, UCLA, UPenn, UNC at Chapel Hill, Washington Univ. in St. Louis, Yale, the 2005 Summer Workshop at the Stanford Institute of Theoretical Economics, the 2006 Meetings of the Society for Economic Dynamics, the Cowles Foundation 75^{th} Anniversary Conference and the 2007 Game Theory Conference in SUNY, Stony Brook for many helpful comments and suggestions.

Our setting is a continuous-time analogue of the repeated game of Fudenberg and Levine (1992), hereafter FL. In the class of games that we study, the public signals about the large player's actions are distorted by a Brownian motion. The small players are *anonymous*, that is, the public information includes the aggregate distribution of the small players' actions but not the actions of any individual small player. Hence, as in FL, the small players behave myopically in every equilibrium, acting to maximize their instantaneous expected payoffs. We model reputation assuming that the large player can be either a *normal type*, who behaves strategically, or a *behavioral type*, who is committed to playing a certain strategy. The reputation of the large player is interpreted as the posterior probability the small players assign to the behavioral type.

Discrete-time methods offer two main limit results about reputation games. First, FL show that when the large player gets patient, in every equilibrium his payoff becomes at least as high as the *commitment payoff*, which is the payoff he would receive if he could credibly commit to the strategy of the behavioral type. Second, Cripps, Mailath, and Samuelson (2004) show that reputation effects are temporary: in any equilibrium, the type of the large player is gradually revealed in the long run. Apart from these two limits, not much is known about equilibrium behavior in reputation games, particularly when actions are imperfectly observable. The explicit construction of even one sequential equilibrium appears to be a hard problem.

In our continuous-time framework we are able to provide a complete characterization of the set of sequential equilibrium payoffs of the large player *for any fixed discount rate*. The characterization is in terms of an ordinary differential equation. We also identify an interesting class of reputation games that have a unique sequential equilibrium, which is Markovian in the large player's reputation. In a Markov perfect equilibrium (see Maskin and Tirole (2001)) behavior is determined by the small players' posterior belief, the payoff-relevant state variable. The main sufficient condition for uniqueness is expressed in terms of a family of auxiliary one-shot games in which the payoffs of the large player are adjusted by suitably defined "reputational weights." We show that whenever these auxiliary one-shot games have unique Bayesian Nash equilibria, the reputation game will have a unique sequential equilibrium, which is Markovian and continuous in the small players' posterior belief.

We then examine how the large player's reputation affects his equilibrium payoffs and incentives. Under our sufficient condition for uniqueness, we show that whenever the large player's static Bayesian Nash equilibrium payoff increases in reputation, his sequential equilibrium payoff in the repeated game also increases in reputation. In this case, reputation is good. The normal type of large player benefits from imitating the behavioral type and building his reputation, but in equilibrium this imitation is necessarily imperfect: if it were perfect, the public signals would be uninformative about the large player's type, so imitation would have no value. The normal type obtains his maximum payoff when the small players are certain that they are facing the behavioral type. In this extreme case the population's beliefs never change and the normal type "gets away" with any action.

We also extend our characterization to general environments with multiple sequential equilibria. Consider the correspondence of sequential equilibrium payoffs of the large player as a function of the small players' prior belief. We show that this correspondence is convex-valued and that its upper boundary is the maximal solution of a *differential inclusion* (see, e.g., Aubin and Cellina (1984)), with an analogous characterization for the lower boundary.

One implication of our characterization is that when reputation effects are absent (i.e., when the small players are *certain* that they are facing the normal type), the large player cannot attain better payoffs than in *static* Nash equilibrium (Theorem 2). This result has no counterpart in the discrete-time setting of FL, where non-trivial equilibria of the complete information game are known to exist (albeit with payoffs bounded away from efficiency). Our equilibrium degeneracy result reflects the fact that discrete-time intertemporal incentives break down when the large player is able to respond to public information quickly. In the best discrete-time equilibrium, the large player's incentives arise from the threat of a punishment phase, which is triggered when the signal about his actions is sufficiently bad. The intuition from Abreu, Milgrom, and Pearce (1991), who study discrete-time games in the limit as actions become more frequent, explains why such incentives unravel. In games with frequent actions, the information that players observe within each time period becomes excessively noisy, and so the statistical tests that trigger the punishment regimes produce false positives too often. This effect is especially strong when information arrives continuously via a Brownian motion, as shown in Sannikov and Skrzypacz (2006a) for discrete-time games with frequent actions.[1] We carry out our arguments directly in continuous time.

The Markovian property of reputational equilibria is connected to the collapse of intertemporal incentives in the repeated game without reputation effects. When the static game has a unique Nash equilibrium, the only equilibrium of the continuous-time game without reputation is the repetition of the static Nash equilibrium, which is trivially Markovian. In our setting, continuous time prevents non-Markovian incentives created by rewards and punishments from enhancing the incentives naturally created by reputation dynamics.

We conclude the introduction by discussing the related literature. The asymmetric information approach to reputations was introduced in the early papers of Kreps and Wilson (1982) and Milgrom and Roberts (1982), who analyze the chain store paradox, and Kreps, Milgrom, Roberts, and Wilson (1982), who study cooperation in the finitely repeated prisoners' dilemma. Arbitrarily small amounts of incomplete information (in the form of behavioral types) give rise to behaviors that cannot be supported by the equilibria of the underlying complete information games: entry deterrence in the chain store game and cooperation in the finitely repeated prisoners' dilemma.

FL study payoff bounds from reputation effects in repeated games in which the actions of the long-run player are imperfectly observable. FL show that when the set of behavioral types is sufficiently rich and the monitoring technology satisfies a statistical identification condition, the upper and lower bounds coincide with the long-run player's Stackelberg payoff, that is, the payoff he obtains from credibly committing to the strategy to which he would like to commit the most. A related paper, Faingold (2006), extends the Fudenberg-Levine payoff bounds to a class of continuous-time games that includes the games we study in this paper. In addition, Faingold (2006) shows that such payoff bounds also hold for discrete-time games with frequent

[1] See also the more recent studies of Fudenberg and Levine (2006) and Sannikov and Skrzypacz (2006b) into the qualitative differences between Poisson and Brownian information.

actions uniformly in the period length.

We use methods related to those of Sannikov (2006a) and Sannikov (2006b) to derive the connection between the large player's incentives and the law of motion of the large player's continuation value, which forms a part of the recursive structure of our games. The other part, which is new to our paper, comes from the evolution of the posterior beliefs. The consistency and sequential rationality conditions for sequential equilibria are formulated in terms of these two variables.

The paper is organized as follows. Section 2 presents our leading example. Section 3 introduces the continuous-time model. Section 4 provides a recursive characterization of public sequential equilibria. Section 5 examines the underlying complete information game. Section 6 provides the ODE characterization when equilibrium is unique. Section 7 extends the characterization to games with multiple equilibria.

2 Example: The Game of Quality Standards.

Consider a monopolist who provides a service to a continuum of identical consumers. At each time $t \in [0, \infty)$, the monopolist chooses a level of investment in quality, $a_t \in [0, 1]$, and each consumer $i \in I \equiv [0, 1]$ chooses a service level, $b_t^i \in [0, 3]$. The monopolist does not observe each consumer individually, but only the *average* level of service, $\bar{b}_t$, over the population of consumers. Likewise, the consumers do not observe the monopolist's investment. Instead, they publicly observe the quality of the service, dX_t, which is a noisy signal of the monopolist's investment:

$$dX_t = a_t(4 - \bar{b}_t)\, dt + (4 - \bar{b}_t)\, dZ_t\,,$$

where $(Z_t)_{t\geq 0}$ is a standard Brownian motion. The drift, $a_t(4 - \bar{b}_t)$, is the expected quality flow at time t, and $4 - \bar{b}_t$ is the magnitude of the noise. Hence, the technology features a *congestion effect:* the expected quality flow per customer deteriorates with greater usage. Note that the noise also decreases with usage: the more customers use the service the better they learn its quality.

The unit price for the service is exogenously fixed and normalized to unity. The overall surplus of consumer $i \in I$ is

$$r \int_0^\infty e^{-rt}(b_t^i\, dX_t - b_t^i\, dt),$$

where $r > 0$ is a discount rate. We emphasize that in equilibrium the consumers behave *myopically*, that is, they act to maximize their expected flow payoff, because the service provider can only observe their *aggregate* consumption.

The discounted profit of the monopolist is

$$r \int_0^\infty e^{-rt}(\bar{b}_t - a_t)\, dt\,. \tag{1}$$

In the unique static Nash equilibrium of this game, the service provider makes zero investment and the consumers choose zero service level. As we show in Section 5,s in the repeated game without reputation effects (i.e., when the consumers are *certain* that the monopolist's payoff is

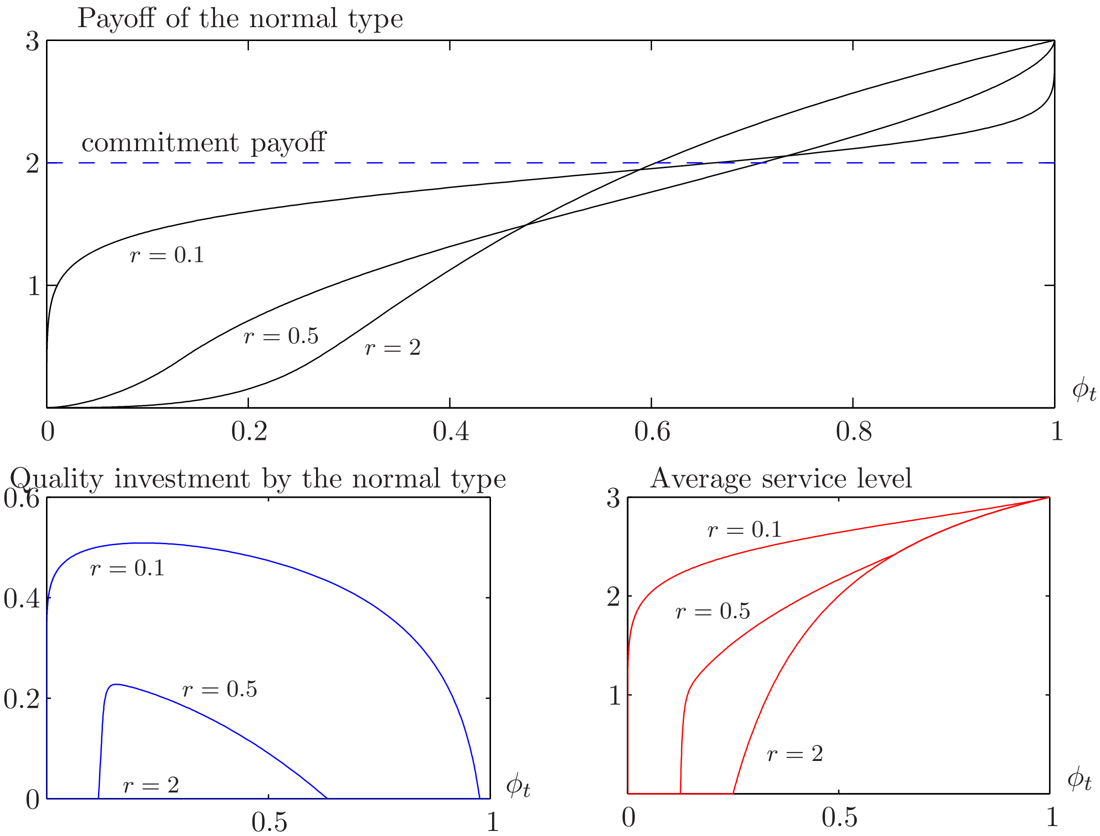

Figure 1: Equilibrium payoffs and actions in the game of quality standards.

given by (1)), the only equilibrium is the repetition of the static Nash equilibrium. Hence, unlike in discrete-time repeated games, here it is impossible for the consumers to create intertemporal incentives for the monopolist to invest, despite the fact that the monopolist's investment can be *statistically identified* (i.e., different investment levels induce different drifts for the quality signal X_t).

However, if the large player were able to credibly commit to investment level $a^* \in [0, 1]$, he would be able to influence the consumers' decisions and get a better payoff. Each consumer's choice, b^i, would maximize their expected flow payoff $b^i(a^*(4 - \bar{b}) - 1)$, and in equilibrium all consumers would choose the same level $b^{*i} = \max\{0, 4 - 1/a^*\}$. The service provider would earn a profit of $\max\{0, 4 - 1/a^*\} - a^*$ and at $a^* = 1$ this function achieves its maximal value of 2, the monopolist's *Stackelberg payoff*.

Following these observations, it is interesting to explore the repeated game with reputation effects. Assume that at time zero the consumers believe that with probability $p \in (0, 1)$ the service provider is a *behavioral type*, who always chooses investment $a^* = 1$, and with probability $1 - p$ he is a *normal type*, who chooses a_t to maximize his expected discounted profit. What happens in equilibrium?

The top panel of Figure 1 displays the unique sequential equilibrium payoff of the normal type as a function of the population's belief p, for different discount rates r. In equilibrium the consumers continually update their belief ϕ_t, the probability assigned to the behavioral type, using the observations of the public signal X_t. The equilibrium is Markovian in the posterior belief ϕ_t, which uniquely determines the equilibrium actions of the normal type (bottom left panel) and the consumers (bottom right panel).

Consistent with the asymptotic payoff bound from Faingold (2006, Theorem 3.1), the computation shows that as $r \to 0$, the large player's payoff converges to his commitment payoff of 2. We also see from Figure 1 that the customer usage level $\bar{b}$ increases towards the commitment level of 3 as the discount rate r decreases towards 0. While the normal type chooses action 0 for all levels of ϕ_t when $r = 2$, as r is closer to 0, his action increases towards $a^* = 1$. However, the imitation of the behavioral type by the normal type is never perfect, even for very low discount rates.

In this example for every discount rate $r > 0$ the equilibrium action of the normal type is exactly 0 near $\phi = 0$ and 1 and the population's action is 0 near $\phi = 0$ (not visible in Figure 1 for $r = 0.1$). The normal type of the large player imitates the behavioral type only for intermediate levels of reputation.

Using our characterization from Section 6, we can not only compute equilibria in examples, but also prove comparative statics results analytically. For example, consider a variation of our game in which the payoff flow of the small players is given by $\alpha b_t^i \, dX_t - b_t^i \, dt$, where $\alpha > 0$ is a parameter. In Appendix C.4 we show that for every prior $p \in (0, 1)$, the equilibrium payoff of the large player weakly increases in α.

3 The Repeated Game.

A *large player* participates in a repeated game with a continuum of *small players* uniformly distributed on $I = [0, 1]$. At each time $t \in [0, \infty)$, the large player chooses an action $a_t \in A$ and each small player $i \in I$ chooses an action $b_t^i \in B$ based on their current information. Action spaces A and B are compact subsets of an Euclidean space. The small players' moves are *anonymous*: at each time t, the large player observes the aggregate distribution $\bar{b}_t \in \Delta(B)$ of the small players' actions, but does not observe the action of any individual small player. There is *imperfect monitoring*: the large player's moves are not observable to the small players. Instead, the small players see a noisy public signal $(X_t)_{t \geq 0}$ that depends on the actions of the large player, the aggregate distribution of the small players' actions and noise. Specifically,

$$dX_t = \mu(a_t, \bar{b}_t) \, dt + \sigma(\bar{b}_t) \cdot dZ_t,$$

where (Z_t) is a d-dimensional Brownian motion, and the drift and the volatility of the signal are continuous functions $\mu : A \times B \to \mathbb{R}^d$ and $\sigma : B \to \mathbb{R}^{d \times d}$, which are linearly extended to $A \times \Delta(B)$ and $\Delta(B)$ respectively.[2] For technical reasons, assume that there exists $c > 0$ such that $|\sigma(b) \cdot y| \geq c|y|$, $\forall y \in \mathbb{R}^d$, $\forall b \in B$. Denote by $(\mathcal{F}_t)_{t \geq 0}$ the filtration generated by (X_t).

[2]Functions μ and σ are extended to distributions over B via $\mu(a, \bar{b}) = \int_B \mu(a, b) \, d\bar{b}(b)$ and $\sigma(\bar{b})\sigma(\bar{b})^\top = \int_B \sigma(b)\sigma(b)^\top \, d\bar{b}(b)$.

Our assumption that only the drift of X depends on the large player's action corresponds to the *constant support* assumption that is standard in discrete time repeated games. By Girsanov's Theorem the probability measures over the paths of two diffusion processes with the same volatility but different bounded drifts are *equivalent*, that is, they have the same zero-probabilty events. Since the volatility of a continuous-time diffusion process is effectively observable, we do not allow $\sigma(\bar{b})$ to depend on a.[3]

Small players have identical preferences.[4] The payoff of each small player depends only on his own action, the aggregate distribution of the small players' actions, and the sample path of the signal (X_t). A small player's payoff is

$$r \int_0^\infty e^{-rt} \left(u(b_t^i, \bar{b}_t)\, dt \,+\, v(b_t^i, \bar{b}_t) \cdot dX_t \right)$$

where $u : B \times B \to \mathbb{R}$ and $v : B \times B \to \mathbb{R}^d$ are continuously differentiable functions that are extended linearly to $B \times \Delta(B)$. Then the expected payoff flow of the small players $h : A \times B \times \Delta(B) \to \mathbb{R}$ is given by

$$h(a, b, \bar{b}) = u(b, \bar{b}) + v(b, \bar{b}) \cdot \mu(a, \bar{b}).$$

The small players' payoff functions are common knowledge.

The small players are uncertain about the type θ of the large player. At time 0 they believe that with probability $p \in [0, 1]$ the large player is a *behavioral type* ($\theta = \mathtt{b}$) and with probability $1 - p$ he is a *normal type* ($\theta = \mathtt{n}$). The behavioral type mechanically plays a fixed action $a^* \in A$ at all times. The normal type plays strategically to maximize his expected payoff. The payoff of the normal type of the large player is

$$r \int_0^\infty e^{-rt}\, g(a_t, \bar{b}_t)\, dt\,,$$

where the payoff flow is defined through a continuously differentiable function $g : A \times B \to \mathbb{R}$ that is extended linearly to $A \times \Delta(B)$.

In the dynamic game the small players update their beliefs about the type of the large player by Bayes rule from their observations of X. Denote by ϕ_t the probability that the small players assign to the large player being a behavioral type at time $t \geq 0$.

A pure *public strategy* of the normal type of large player is a progressively measurable (with respect to $(\mathcal{F}_t)$) process $(a_t)_{t \geq 0}$ with values in A. Similarly, a pure public strategy of small player $i \in I$ is a progressively measurable process $(b_t^i)_{t \geq 0}$ with values in B. We assume that jointly the strategies of the small players and the aggregate distribution satisfy appropriate measurability properties.

[3]If some dimensions of the large player's actions were observable, as the price in the game of quality standards, the normal type would have to imitate the behavioral type perfectly along those dimensions, or else reveal himself. Our results can be generalized to such a setting with minor modifications (e.g. the large player may sometimes mix between revealing himself or not).

[4]The characterization of our paper can be easily extended to a setting where the small players observe the same public signal, but have heterogeneous preferences.

Definition 1. A *public sequential equilibrium* consists of a public strategy $(a_t)_{t\geq 0}$ of the normal type of large player, public strategies $(b^i_t)_{t\geq 0}$ of small players $i \in I$, and a progressively measurable belief process $(\phi_t)_{t\geq 0}$, such that at all times t and after all public histories:

(a) the strategy of the normal type of large player maximizes his expected payoff

$$\mathsf{E}_t\left[r\int_0^\infty e^{-rt}\, g(a_t,\bar{b}_t)\,dt \;\middle|\; \theta = \mathtt{n}\right],$$

(b) the strategy of each small player maximizes his expected payoff

$$(1-\phi_t)\mathsf{E}_t\left[r\int_0^\infty e^{-rt}\, h(a_t,b^i_t,\bar{b}_t)\,dt \;\middle|\; \theta = \mathtt{n}\right] + \phi_t\,\mathsf{E}_t\left[r\int_0^\infty e^{-rt}\, h(a^*,b^i_t,\bar{b}_t)\,dt \;\middle|\; \theta = \mathtt{b}\right]$$

(c) beliefs $(\phi_t)_{t\geq 0}$ are determined by Bayes rule given the common prior $\phi_0 = p$.

A strategy profile that satisfies conditions (a) and (b) is called *sequentially rational.* A belief process (ϕ_t) that satisfies condition (c) is called *consistent*.

In Section 4 we explore these properties in detail and characterize them in our setting (Theorem 1). We use this characterization in Section 5 to explore the game with prior $p = 0$, and in Section 6 to present a set of sufficient conditions under which the sequential equilibrium for any prior is unique and Markovian in the population's belief. In this case, we characterize the sequential equilibrium payoffs of the normal type as well as the equilibrium strategies via an ordinary differential equation. In Section 7 we characterize the large player's sequential equilibrium payoffs when multiple equilibria exist.

Remark 1. Although the aggregate distribution of the small players' actions is publicly observable, our requirement that public strategies depend only on the sample paths of X is without loss of generality. In fact, for a given strategy profile, the public histories along which there are observations of $\bar{b}_t$ that differ from those on-the-path-of-play correspond to deviations by a positive measure of small players. Therefore our definition of public strategies does not alter the set of public sequential equilibrium outcomes.

Remark 2. All our results hold for public sequential equilibria in mixed strategies. A mixed public strategy of the large player is a random process $(\bar{a}_t)_{t\geq 0}$ progressively measurable with respect to $\mathcal{F}_t$ with values in $\Delta(A)$. The drift function μ should be extended linearly to $\Delta(A)\times\Delta(B)$ to allow for mixed strategies. Because there is a continuum of anonymous small players, the assumption that each of them plays a pure strategy is without loss of generality.

Remark 3. For both pure and mixed equilibria, the restriction to public strategies is without loss of generality in our games. For pure strategies, it is redundant to condition a player's current action on his private history, as it is completely determined by the public history. For mixed strategies, the restriction to public strategies is without loss of generality in repeated games in which the signals have a product structure, as in our games.[5] Informally, to form a belief about

[5]In a game with a product structure each public signal depends on the actions of only one large player. See the definition in Fudenberg and Levine (1994, Section 5)

his opponent's private histories, in a game with product structure a player can ignore his own past actions because they do not influence the signal about his opponent's actions. Formally, a mixed private strategy of the large player in our game is a random process (a_t) with values in A that is progressively measurable with respect to a filtration $(\mathcal{G}_t)$, which is generated by the public signals X and the large player's private randomization. For any private strategy of the large player, an equivalent mixed public strategies is defined by letting $\bar{a}_t$ be the conditional distribution of a_t given $\mathcal{F}_t$. Strategies a_t and $\bar{a}_t$ induce the same probability distributions over public signals and give the large player the same expected payoff (given $\mathcal{F}_t$).

4 The Structure of Sequential Equilibria.

This section provides a characterization of public sequential equilibria of our game, which is summarized in Theorem 1. In equilibrium, the small players always choose a static best response given their belief about the large player's actions. The behavioral type of the large player always chooses action a^*, while the normal type chooses his actions strategically taking into account his expected future payoff, which depends on the public signal X. The dynamic evolution of the small players' belief is also determined by X.

The equilibrium play has to satisfy two conditions: the beliefs must be consistent with the players' strategies, and the strategies must be sequentially rational given beliefs. For the consistency of beliefs, Proposition 1 presents equation (1) that describes how the small players' belief evolves with the public signal X. Sequential rationality of the normal type's strategy is verified by looking at the evolution of his continuation value W_t, the future expected payoff of the normal type given the history of public signals X up until time t. Proposition 2 presents a necessary and sufficient condition for the law of motion of a random process W, under which W is the continuation value of the normal type. Proposition 3 presents a condition for sequential rationality that is connected to the law of motion of W. Propositions 2 and 3 are analogous to Propositions 1 and 2 from Sannikov (2006b).

Subsequent sections of our paper use the equilibrium characterization of Theorem 1. Section 5 uses Theorem 1 to show that in the complete-information repeated game in which the small players are certain that they are facing the normal type, the set of public sequential equilibrium payoffs of the large player coincides with the convex hull of his static Nash equilibrium payoffs. Section 6 analyzes a convenient class of games in which the public sequential equilibrium turns out to be unique and Markovian in the population's posterior belief. Section 7 characterizes the set of public sequential equilibrium payoffs of the large player generally.

We begin with Proposition 1, which explains how the small players use Bayes rule to update their beliefs based on the observations of the public signals.[6]

Proposition 1 (Belief Consistency). *Fix a public strategy profile $(a_t, \bar{b}_t)_{t\geq 0}$ and a prior $p \in [0,1]$*

[6] A simpler version of equation (2) for history-independent drifts has been used in the literature on strategic experimentation in continuous time. See, e.g., Bolton and Harris (1999, Lemma 1) and Moscarini and Smith (2001, equation 2). For a more general filtering equation, which allows the unknown parameter θ to follow a Markov process, see Liptser and Shiryaev (1977, Theorem 9.1).

on the behavioral type. Belief process $(\phi_t)_{t\geq 0}$ is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$ if and only if it satisfies

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t) \cdot dZ_t^{\phi} \tag{2}$$

with initial condition $\phi_0 = p$, where

$$\begin{aligned}
\gamma(a, \bar{b}, \phi) &= \phi(1-\phi)\sigma(\bar{b})^{-1}\left(\mu(a^*, \bar{b}) - \mu(a, \bar{b})\right), && (3)\\
dZ_t^{\phi} &= \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), \text{ and} && (4)\\
\mu^{\phi}(a, \bar{b}) &= \phi\mu(a^*, \bar{b}) + (1-\phi)\mu(a, \bar{b}). && (5)
\end{aligned}$$

Proof. The strategies of the two types of large player induce two different probability measures over the paths of the signal (X_t). From Girsanov's Theorem we can find the ratio ξ_t between the likelihood that a path $(X_s : s \in [0, t])$ arises for type $\mathtt{b}$ and the likelihood that it arises for type $\mathtt{n}$. This ratio is characterized by

$$d\xi_t = \xi_t\, \rho_t \cdot dZ_s^{\mathtt{n}}, \quad \xi_0 = 1, \tag{6}$$

where $\rho_t = \sigma(\bar{b}_t)^{-1}\left(\mu(a^*, \bar{b}_t) - \mu(a_t, \bar{b}_t)\right)$ and $(Z_t^{\mathtt{n}})$ is a Brownian motion under the probability measure generated by type $\mathtt{n}$'s strategy.

Suppose that belief process (ϕ_t) is consistent with $(a_t, \bar{b}_t)_{t\geq 0}$. Then, by Bayes' rule, the posterior after observing a path $(X_s : s \in [0, t])$ is

$$\phi_t = \frac{p\xi_t}{p\xi_t + (1-p)}. \tag{7}$$

By Ito's formula,

$$\begin{aligned}
d\phi_t &= \frac{p(1-p)}{(p\xi_t + (1-p))^2} d\xi_t - \frac{2p^2(1-p)}{(p\xi_t + (1-p))^3} \frac{\xi_t^2 \rho_t \cdot \rho_t}{2}\, dt \\
&= \phi_t(1-\phi_t)\rho_t \cdot dZ_t^{\mathtt{n}} - \phi_t^2(1-\phi_t)(\rho_t \cdot \rho_t)\, dt && (8)\\
&= \phi_t(1-\phi_t)\rho_t \cdot dZ_t^{\phi},
\end{aligned}$$

which is equation (2).

Conversely, suppose that (ϕ_t) is a process that solves equation (2) with initial condition $\phi_0 = p$. Define ξ_t using expression (7), i.e.,

$$\xi_t = \frac{1-p}{p} \frac{\phi_t}{1-\phi_t}.$$

By another application of Ito's formula, we conclude that (ξ_t) satisfies equation (6). This implies that ξ_t is the ratio between the likelihood that a path $(X_s : s \in [0, t])$ arises for type $\mathtt{b}$ and the likelihood that it arises for type $\mathtt{n}$. Hence, ϕ_t is determined by Bayes rule and the belief process is consistent with $(a_t, \bar{b}_t)$. □

In the equations of Proposition 1, (a_t) is the strategy that the normal type is supposed to follow. If the normal type deviates, his deviation affects only the drift of X, but not the other terms in equation (2).

Coefficient γ in equation (2) is the volatility of beliefs: it reflects the speed with which the small players learn about the type of the large player. The definition of γ plays an important role in the characterization of public sequential equilibria presented in Sections 6 and 7 (Theorems 3 and 4). The intuition behind equation (2) is as follows. If the small players are convinced about the type of the large player, then $\phi_t(1-\phi_t) = 0$, so they never change their beliefs. When $\phi_t \in (0,1)$ then $\gamma(a_t, \bar{b}_t, \phi_t)$ is larger, and learning is faster, when the noise $\sigma(\bar{b}_t)$ is smaller or the drifts produced by the two types differ more. From the small players' perspective, (Z_t^ϕ) is a Brownian motion and their belief (ϕ_t) is a martingale. From equation (8) we see that, conditional on the large player being the normal type, the drift of ϕ_t is non-positive: in the long run, either the small players learn that they are facing the normal type, or the normal type plays like the behavioral type.

We turn to the analysis of the second important state descriptor of the interaction between the large and the small players, the continuation value of the normal type. A player's continuation value is his future expected payoff after a given public history for a given profile of continuation strategies. We derive how the large player's incentives arise from the law of motion of his continuation value. We will find that the large player's strategy is optimal if and only if a certain local incentive constraint holds at all times $t \geq 0$.

For a given strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$, the *continuation value* $W_t(S)$ of the normal type is his expected payoff at time t when he plans to follow strategy $(a_s)_{s\geq 0}$ from time t onwards, i.e.

$$W_t(S) = \mathsf{E}_t \left[r \int_t^\infty e^{-r(s-t)} g(a_s, \bar{b}_s)\, ds \;\middle|\; \theta = \mathrm{n} \right] \tag{9}$$

Proposition 2 below characterizes the law of motion of W_t.

Throughout the paper we will write $\mathcal{L}^*$ for the space of progressively measurable processes $(\beta_t)_{t\geq 0}$ with $\mathsf{E}\big[\int_0^T \beta_t^2\, dt\big] < \infty$ for all $0 < T < \infty$.

Proposition 2 (Continuation Values)**.** *A bounded process $(W_t)_{t\geq 0}$ is the continuation value of the normal type under the public-strategy profile $S = (a_t, \bar{b}_t)_{t\geq 0}$ if and only if for some d-dimensional process (β_t) in $\mathcal{L}^*$, we have*

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt \;+\; r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt). \tag{10}$$

Proof. First, note that $W_t(S)$ is a bounded process by (9), and let us show that $W_t = W_t(S)$ satisfies (10) for some d-dimensional process β_t in $\mathcal{L}^*$. Denote by $V_t(S)$ the average discounted payoff of the normal type conditional on the public information at time t, i.e.,

$$V_t(S) = \mathsf{E}_t \left[r \int_0^\infty e^{-rs} g(a_s, \bar{b}_s)\, ds \;\middle|\; \theta = \mathtt{n} \right] = r \int_0^t e^{-rs} g(a_s, \bar{b}_s)\, ds + W_t(S) \tag{11}$$

Then V_t is a martingale when the large player is of normal type. By the Martingale Representation Theorem, there exists a d-dimensional process β_t in $\mathcal{L}^*$ such that

$$dV_t(S) = re^{-rt} \beta_t \cdot \sigma(\bar{b}_t) dZ_t^{\mathtt{n}}, \tag{12}$$

where $dZ_t^{\mathtt{n}} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t)\, dt)$ is a Brownian motion from the point of view of the normal type of the large player.

Differentiating (11) with respect to time yields

$$dV_t(S) = re^{-rt}g(a_t, \bar{b}_t)\,dt \;-\; re^{-rt}W_t(S)\,dt \;+\; e^{-rt}dW_t(S) \tag{13}$$

Combining equations (12) and (13) yields (10).

Conversely, let us show if W_t is a bounded process that satisfies (10) then $W_t = W_t(S)$. When the large player is normal, the process

$$V_t = r\int_0^t e^{-rs}g(a_s, \bar{b}_s)\,ds + e^{-rt}W_t$$

is a martingale under the strategies $S = (a_t, \bar{b}_t)$ because $dV_t = re^{-rt}\beta_t \cdot \sigma(\bar{b}_t)dZ_t^{\mathfrak{n}}$ by (10). Moreover, martingales V_t and $V_t(S)$ converge because both $e^{-rt}W_t$ and $e^{-rt}W_t(S)$ converge to 0. Therefore,

$$V_t = \mathsf{E}_t[V_\infty] = \mathsf{E}_t[V_\infty(S)] = V_t(S) \quad \Rightarrow \quad W_t = W_t(S)$$

for all t, as required. □

Representation (10) describes how $W_t(S)$, defined above, evolves with the public history. It is valid independently of the large player's actions until time t, which caused a given history $(X_s, s \in [0,t])$ to realize. This fact is important in the proof of Proposition 3 below, which deals with incentives.

Next, we derive conditions for sequential rationality. The condition for the small players is straightforward: they maximize their static payoff because a deviation of an individual small player does not affect future equilibrium play. The situation of the normal type of large player is more complicated: he acts optimally if he maximizes the sum of his current payoff flow and the expected change in his continuation value.

Proposition 3 (Sequential Rationality)**.** *A public strategy profile $(a_t, \bar{b}_t)_{t\ge 0}$ is sequentially rational with respect to a belief process (ϕ_t) if and only if for all times $t \ge 0$ and after all public histories,*

$$a_t \;\in\; \arg\max_{a' \in A}\; g(a', \bar{b}_t) \;+\; \beta_t \cdot \mu(a', \bar{b}_t)\,, \tag{14}$$

$$b \;\in\; \arg\max_{b' \in B}\; u(b', \bar{b}_t) + v(b', \bar{b}_t)\cdot \mu^{\phi_t}(a_t, \bar{b}_t), \quad \forall b \in \text{support}\,\bar{b}_t\,. \tag{15}$$

Proof. Consider a strategy profile $(a_t, \bar{b}_t)$ and an alternative strategy $(\tilde{a}_t)$ of the normal type. Denote by W_t the continuation payoff of the normal type when he follows strategy (a_t) after time t, while the population follows $(\bar{b}_t)$. If the normal type of large player plays strategy $(\tilde{a}_t)$ up to time t and then switches back to (a_t), his expected payoff conditional on the public information at time t is given by

$$\tilde{V}_t = r\int_0^t e^{-rs}g(\tilde{a}_s, \bar{b}_s)\,ds \;+\; e^{-rt}W_t\,.$$

By Proposition 2 and the expression above,

$$\begin{aligned} d\tilde{V}_t &= re^{-rt}\left(g(\tilde{a}_t, \bar{b}_t) - W_t\right)dt \;+\; e^{-rt}dW_t \\ &= re^{-rt}\left(\left(g(\tilde{a}_t, \bar{b}_t) - g(a_t, \bar{b}_t)\right)dt + \beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\,dt)\right)\,, \end{aligned}$$

where the process $\beta \in \mathcal{L}^*$ is given by representation (10).

Hence the profile $(\tilde{a}_t, \bar{b}_t)$ yields the normal type expected payoff

$$\begin{aligned}\tilde{W}_0 &= \mathsf{E}[\tilde{V}_\infty] = \mathsf{E}\left[\tilde{V}_0 + \int_0^\infty d\tilde{V}_t\right] \\ &= W_0 + \mathsf{E}\left[r\int_0^\infty e^{-rt}\left(g(\tilde{a}_t,\bar{b}_t) - g(a_t,\bar{b}_t) + \beta_t \cdot (\mu(\tilde{a}_t,\bar{b}_t) - \mu(a_t,\bar{b}_t)\right) dt\right],\end{aligned}$$

where the expectations are taken under the probability measure induced by $(\tilde{a}_t, \bar{b}_t)$, and so (X_t) has drift $\mu(\tilde{a}_t, \bar{b}_t)$.

Suppose that strategy profile $(a_t, \bar{b}_t)$ and belief process (ϕ_t) satisfy the incentive constraints (14) and (15). Then, for every $(\tilde{a}_t)$, one has $W_0 \geq \tilde{W}_0$, and so the normal type is sequentially rational at time 0. By a similar argument, the normal type is sequentially rational at all times t, after all public histories. Note also that the small players are maximizing their instantaneous expected payoffs. Since the small players are anonymous, no unilateral deviation by a small player can affect the future course of play. Therefore each small player is also sequentially rational.

Conversely, suppose that incentive constraint (14) fails. Choose a strategy $(\tilde{a}_t)$ such that $\tilde{a}_t$ attains the maximum in (14) for all $t \geq 0$. Then $\tilde{W}_0 > W_0$ and the large player is not sequentially rational at $t = 0$. Likewise, if condition (15) fails, then a positive measure of small players is not maximizing their instantaneous expected payoffs. Since the small player's actions are anonymous, their strategies would not be sequentially rational. □

We can now summarize our characterization of sequential equilibria.

Theorem 1 (Sequential Equilibrium)**.** *A profile $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium with continuation values (W_t) for the normal type if and only if*

(a) (W_t) is a bounded process that satisfies

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt) \tag{16}$$

for some process $\beta \in \mathcal{L}^*$,

(b) belief process (ϕ_t) follows

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\, \sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\, dt), \quad \textit{and} \tag{17}$$

(c) strategies $(a_t, \bar{b}_t)$ satisfy the incentive constraints

$$\begin{gathered} a_t \in \arg\max_{a' \in A}\; g(a', \bar{b}_t) + \beta_t \mu(a', \bar{b}_t), \quad \textit{and} \\ b \in \arg\max_{b' \in B} u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t),\ \forall\, b \in \text{support}\, \bar{b}_t\,. \end{gathered} \tag{18}$$

Theorem 1 provides a characterization of public sequential equilibria which can be used to derive many of its properties. In Section 5 we apply Theorem 1 to the repeated game with prior $p = 0$, the complete information game. In Sections 6 and 7 we characterize the correspondence $\mathcal{E} : [0,1] \rightrightarrows \mathbb{R}$ that maps the prior probability $p \in [0,1]$ on the behavioral type into the set of public sequential equilibrium payoffs of the normal type in the repeated game with prior p. Theorem 1 implies that $\mathcal{E}$ is the largest bounded correspondence such that a controlled process (ϕ_t, W_t), defined by (16) and (17), can be kept in Graph($\mathcal{E}$) by controls $(a_t, \bar{b}_t)$ and (β_t) that satisfy (18).[7]

4.1 Gradual Revelation of the Large Player's Type.

To end this section, we apply Theorem 1 to show that Condition 1 below is necessary and sufficient for the reputation of the normal type to decay to 0 with probability 1 in any public sequential equilibrium (Proposition 4). Condition 1 states that in any Nash equilibrium of the static game with just the normal type, the large player cannot appear committed to action a^*.[8] Naturally, this condition plays an important role in Sections 6 and 7, where we characterize sequential equilibria with reputation.

Condition 1. For every Nash equilibrium $(a^N, \bar{b}^N)$ of the static game with prior $p = 0$, $\mu(a^N, \bar{b}^N) \neq \mu(a^*, \bar{b}^N)$.

In discrete time, Cripps, Mailath, and Samuelson (2004) show that the reputation of the normal type converges to zero in any sequential equilibrium under stronger conditions than Condition 1. Among other assumptions, they also require that the small players' best reply to the commitment action be strict. In discrete time, an analogue of Condition 1 alone would not be sufficient. (See Cripps, Mailath, and Samuelson (2004, p. 414).)

Proposition 4. *If Condition 1 fails, then for any $p \in [0, 1]$ the stage game has a Bayesian Nash equilibrium (BNE) in which the normal and the behavioral types look the same to the population. The repetition of this BNE is a public sequential equilibrium of the repeated game with prior p, in which the population's belief stays constant.*

If Condition 1 holds, then in any public sequential equilibrium $\phi_t \to 0$ as $t \to \infty$ almost surely under the normal type.

Proof. If Condition 1 fails, then there is a static Nash equilibrium $(a^N, \bar{b}^N)$ of the complete-information game with $\mu(a^N, \bar{b}^N) = \mu(a^*, \bar{b}^N)$. It is easy to see that $(a^N, \bar{b}^N)$ is also a BNE of the stage game with any prior p. The repetition of this BNE is a public sequential equilibrim of the repeated game, in which the beliefs $\phi_t \in p$ remain constant. With these beliefs (17) and (18) hold, and $W_t = g(a^N, \bar{b}^N)$ for all t.

Conversely, if Condition 1 holds there is no BNE $(a, \bar{b})$ of the static game with prior $p > 0$ in which $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$. Otherwise, $(a, \bar{b})$ would be a Nash equilibrium of the static game with prior $p = 0$, since the small players' payoffs depend on the actions of the large player only through the drift, a contradiction to Condition 1.

We present the rest of the proof in Appendix A, where we show that for some constants $C > 0$ and $M > 0$, in every sequential equilibrium at all times t either

[7] This means that there does not exist a bounded correspondence with such property whose graph contains the graph of $\mathcal{E}$ as a proper subset.

[8] Note that the action of the large player affects the small players' payoffs only through the drift of X.

(a) the absolute value of the volatility of ϕ_t is at least $C\phi_t(1-\phi_t)$ or

(b) the absolute value of the volatility of W_t is at least M.

To see this intuitively, note that if the volatility of ϕ_t at time t is 0, i.e. $\gamma(a_t, \bar{b}_t, \phi_t) = 0$, then $(a_t, \bar{b}_t)$ is *not* a BNE of the stage game by Condition 1. Then the incentive constraints (18) imply that $\beta_t \neq 0$. In Appendix A we rely on the fact that W_t is a bounded process to show that under conditions (a) and (b), ϕ_t eventually decays to 0 when the large player is normal. □

Proposition 4 also implies that players never reach an absorbing state in any public sequential equilibrium if and only if Condition 1 holds. Players reach an absorbing state at time t if their actions as well as the population's beliefs remain fixed after that time. We know that in continuous-time games between two large players, equilibrium play sometimes *necessarily* reaches an absorbing state, as shown in Sannikov (2006b). This possibility requires special treatment in the characterization of equilibria in games between two large players.

5 Equilibrium Degeneracy under Complete Information.

In this section we examine the structure of the set of equilibrium payoffs of the large player in the complete information game ($p = 0$), that is, in the game in which it is common knowledge that the large player is the normal type.

Theorem 2. *Suppose the small players are certain that they are facing the normal type, that is, $p = 0$. Then in every public sequential equilibrium of the repeated game the large player cannot achieve a payoff outside the convex hull of his stage-game Nash equilibrium payoffs, i.e.*

$$\mathcal{E}(0) \;=\; \text{co}\left\{ g(a,\bar{b}) \;:\; \begin{array}{l} a \in \arg\max_{a' \in A}\; g(a',\bar{b}) \\ b \in \arg\max_{b' \in B} u(b',\bar{b}) + v(b',\bar{b}) \cdot \mu(a,\bar{b}), \quad \forall b \in \text{support}\,\bar{b} \end{array} \right\}.$$

Proof. Let $\bar{v}$ be the highest Nash equilibrium payoff of the large player in the static game. We will show that it is impossible to achieve a payoff higher than $\bar{v}$ in any public equilibrium. (A proof for the lowest Nash equilibrium payoff is similar). Suppose there was a public equilibrium in which the large player's continuation value W_0 was greater than $\bar{v}$. By Proposition 3, for some random process (β_t) in $\mathcal{L}^*$, the large player's continuation value satisfies

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt),$$

where a_t maximizes $g(a', \bar{b}_t) + \beta_t \mu(a', \bar{b}_t)$ over all $a' \in A$. Denote $\bar{D} = W_0 - \bar{v}$.

We claim that there exists $\delta > 0$ such that, so long as $W_t \geq \bar{v} + \bar{D}/2$, either the drift of W_t is greater than $r\bar{D}/4$ or the norm of the volatility of W_t is greater than δ. To prove this claim we need the following lemma, whose proof is in Appendix A:

Lemma 1. *For any $\varepsilon > 0$ there exists $\delta > 0$ (independent of t or the sample path) such that $|\beta_t| \geq \delta$ whenever $g(a_t, \bar{b}_t) \geq \bar{v} + \varepsilon$.*

Letting $\varepsilon = \bar{D}/4$ in the lemma above we obtain $\delta > 0$ such that $|\beta_t| \geq \delta$ whenever $g(a_t, \bar{b}_t) \geq \bar{v} + \bar{D}/4$. Moreover, if $g(a_t, \bar{b}_t) < \bar{v} + \bar{D}/4$ then, so long as $W_t \geq \bar{v} + \bar{D}/2$, the drift of W_t is greater than $r\bar{D}/4$, concluding the proof of the claim.

It follows directly from the claim that with positive probability W_t becomes arbitrarily large, which is a contradiction since W_t is bounded. □

The intuition behind this result is as follows. In order to give incentives to the large player to take an action that results in a payoff better than in static Nash equilibrium, his continuation value must respond to the public signal X_t. When his continuation value reaches its upper bound, such incentives cannot be provided. In effect, if at the upper bound the large player's continuation value were sensitive to the public signal process (X_t), then with positive probability the continuation value would escape above this upper bound, which is not possible. Therefore, at the upper bound, continuation values cannot depend on the public signal and so, in the best equilibrium, the normal type must be playing a myopic best response.

While Theorem 2 does not hold in discrete time,[9] it is definitely *not* just a result of continuous-time technicalities. The large player's incentives to depart from a static best response become fragile when he is flexible to respond to public information quickly. The foundations of this result are similar to the deterioration of incentives due to the flexibility to respond to new information quickly in Abreu, Milgrom, and Pearce (1991) in a prisoners' dilemma with Poisson signals and, especially, in Sannikov and Skrzypacz (2006a) in a Cournot duopoly with Brownian signals.

Borrowing intuition from the latter paper, suppose that the large player must hold his action fixed for an interval of time of length $\Delta > 0$. Suppose that the large player's equilibrium incentives to take the Stackelberg action are created through a statistical test that triggers an equilibrium punishment if the signal is sufficiently bad. A profitable deviation has a gain on the order of Δ, the length of a time period. Therefore, such a deviation is prevented only if it increases the probability of triggering punishment by at least $O(\Delta)$. Sannikov and Skrzypacz (2006a) show that with Brownian signals, the log likelihood ratio for a test against any particular deviation is normally distributed. A deviation shifts the mean of this distribution by $O(\sqrt{\Delta})$. Then, a successful test against a deviation would generate a false positive with probability of $O(\sqrt{\Delta})$. This probability, which reflects the value destroyed in each period through punishments, is disproportionately large for small Δ compared to the value created during a period of length Δ. This intuition implies that in equilibrium the large player cannot sustain payoffs above static Nash as $\Delta \to 0$. Figure 2 illustrates the densities of the log likelihood ratio under the 'recommended' action of the large player and a deviation, and the areas responsible for the large player's incentives and for false positives.

Apart from this statistical intuition, the analysis of the game in Sannikov and Skrzypacz (2006a), as well as in Abreu, Milgrom, and Pearce (1991), differ from ours. Those papers look at the game between two large players, either focusing on symmetric equilibria or assuming

[9] Fudenberg and Levine (1994) show that equilibria with payoffs above static Nash often exist in discrete time, but they are always bounded from efficiency.

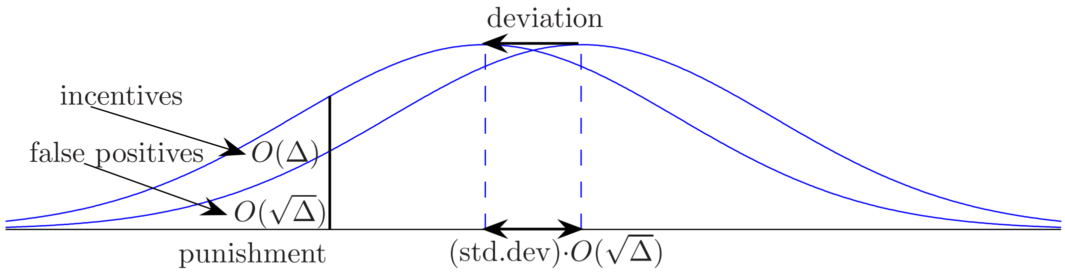

Figure 2: A statistical test to prevent a given deviation.

a failure of pairwise identifiability to derive their results.[10] In contrast, our result is proved directly in continuous time and for games from a different class, with small players but without any failure of identifiability.

Motivated by our result, a recent paper by Fudenberg and Levine (2006) studies the differences between Poisson and Brownian signals by taking the period between actions to zero in a moral hazard game between a large and a population of small players. They allow the large player's action to affect the variance of the Brownian signal and show that nontrivial equilibria exist whenever the variance is decreasing in the large player's effort level.[11]

6 Reputation Games with a Unique Sequential Equilibrium.

In many games, including the game of quality standards from Section 2, for every prior $p \in (0,1)$ the public sequential equilibrium is unique and Markovian in the population's belief. That is, the current belief ϕ_t uniquely determines the players' actions $a_t = a(\phi_t)$ and $\bar{b}_t = b(\phi_t)$, as well as the continuation value of the normal type $W_t = U(\phi_t)$. This section presents a sufficient condition for the equilibrium to be unique and Markovian, and characterizes Markov perfect equilibria using an ordinary differential equation.

First, we derive our characterization informally. Proposition 1 implies that in equilibrium the population's belief evolves according to

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\phi} = -\frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t} dt + \gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\mathtt{n}}\,, \tag{19}$$

where $dZ_t^{\mathtt{n}} = \sigma(\bar{b}_t)^{-1}(dX_t - \mu(a_t, \bar{b}_t)\, dt)$ is a Brownian motion under the strategy of the normal type. If the equilibrium is Markovian, then by Ito's lemma the continuation value $W_t = U(\phi_t)$

[10]The assumption of pairwise identifiability, introduced to repeated games by Fudenberg, Levine, and Maskin (1994), states that deviations by different players can be statistically distinguished given the observations of the public signals.

[11]They also find the surprising result that if the variance of the Brownian signal is increasing in the large player's effort level, then equilibrium must collapse to static Nash. The equilibrium collapse occurs despite the fact that in the continuous-time limit the actions of the large player would be effectively observable.

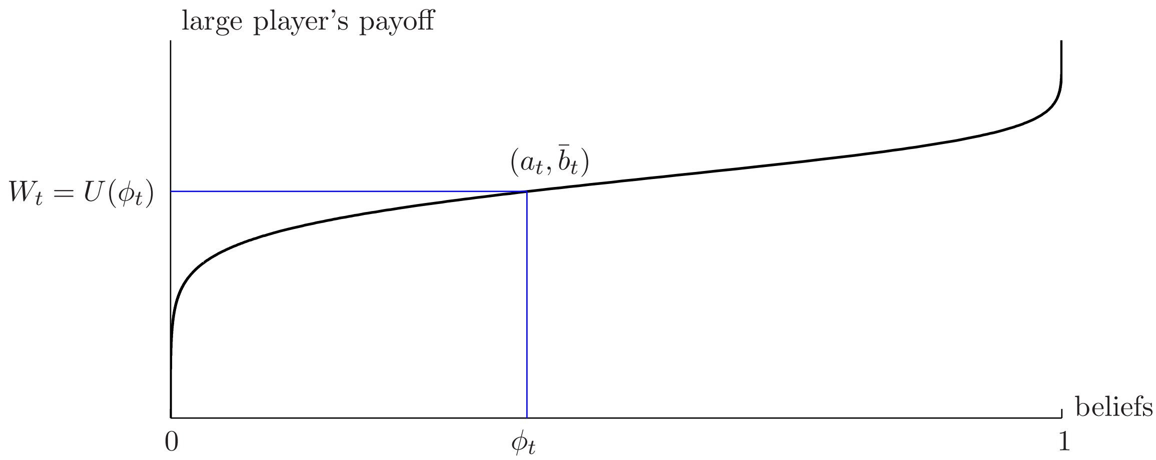

Figure 3: The large player's payoff in a Markov perfect equilibrium.

of the normal type follows

$$dU(\phi_t) = |\gamma(a_t, \bar{b}_t, \phi_t)|^2 \left(\frac{U''(\phi_t)}{2} - \frac{U'(\phi_t)}{1-\phi_t} \right) dt + U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\mathtt{n}}. \tag{20}$$

At the same time, Proposition 2 gives an alternative equation for the motion of $W_t = U(\phi_t)$,

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt \; + \; r\beta_t \sigma(\bar{b}_t)\, dZ_t^{\mathtt{n}}. \tag{21}$$

By matching the drifts and volatilities of (20) and (21), we can characterize Markov perfect equilibria. From the drifts we obtain the differential equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(a(\phi), \bar{b}(\phi)))}{|\gamma(a(\phi), \bar{b}(\phi), \phi)|^2}\,, \tag{22}$$

for the value function U. We can get the equilibrium actions $a(\phi)$ and $b(\phi)$ by matching the volatilities. Since

$$r\beta_t\sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t) \quad \Rightarrow \quad r\beta_t = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\sigma^{-1}(\bar{b}_t),$$

Proposition 3 implies that

$$(a(\phi), b(\phi)) \in \Psi(\phi, \phi(1-\phi)U'(\phi)),$$

where

$$\Psi(\phi, z) = \left\{ (a, \bar{b}) : \begin{array}{l} a \in \arg\max_{a' \in A} \; rg(a', \bar{b}) + z(\mu(a^*, \bar{b}) - \mu(a, \bar{b}))\sigma(\bar{b})^{-2}\mu(a', \bar{b}) \\ b \in \arg\max_{b' \in B} u(b', \bar{b}) + v(b', \bar{b}) \cdot \mu^{\phi}(a, \bar{b}), \quad \forall b \in \text{support } \bar{b} \end{array} \right\},$$

for each $(\phi, z) \in [0,1] \times \mathbb{R}$. We show that the public sequential equilibrium is unique and Markovian under Condition 2 below, which requires that the correspondence Ψ be single-valued. Then the equilibrium actions are uniquely determined by $(a(\phi), \bar{b}(\phi)) = \Psi(\phi, \phi(1-\phi)U'(\phi))$, where U satisfies equation (22). Figure 3 illustrates the function $U : [0,1] \to \mathbb{R}$ for the game of quality standards of Section 2.

These simple properties of equilibria follow from the continuous-time formulation. As the reader may guess, the logic behind this result is similar to that in Section 5. It is impossible to create incentives to sustain greater payoffs than in a Markov perfect equilibrium. Informally, in a public sequential equilibrium that achieves the largest difference $W_0 - U(\phi_0)$ across all priors, the joint volatility of (ϕ_0, W_0) has to be parallel to the slope of $U(\phi_0)$, since $W_t - U(\phi_t)$ cannot increase for any realization of X at time 0. It follows that $r\beta_0\sigma(\bar{b}_0) = U'(\phi_0)\gamma(a_0, \bar{b}_0, \phi_0)$. Thus, when Ψ is single-valued the players' actions at time zero must be Markovian, which leads to $W_t - U(\phi_t)$ having a positive drift at time zero, a contradiction.

In discrete-time reputation games equilibrium behavior is typically not determined uniquely by the population's posterior, and Markov perfect equilibria may not even exist. Our result, presented in Theorem 3 below, shows that continuous time provides an attractive way of modeling reputation.[12] Theorem 3 assumes Condition 1 from Section 4 and:

Condition 2. Ψ is a nonempty, single-valued, Lipschitz-continuous correspondence that returns an atomic distribution of small players' actions for all $\phi \in [0,1]$ and $z \in \mathbb{R}$.

Effectively, the correspondence Ψ returns the Bayesian Nash equilibria of an auxiliary static game in which the large player is a behavioral type with probability ϕ and the payoffs of the normal type are perturbed by a reputational weight of z. In particular, with $\phi = z = 0$ Condition 2 implies that the stage game with a normal large player has a unique Nash equilibrium. Moreover, by Theorem 2, the complete information *repeated game* also has a unique equilibrium, the repeated play of the static Nash.

While Condition 2 is fairly essential for the uniqueness result, Condition 1 is not. If Condition 2 holds but Condition 1 fails, then the repeated game with prior p would have a unique public sequential equilibrium $(a_t = a^N, \bar{b}_t = \bar{b}^N, \phi_t = p)$, which is trivially Markovian. Here $(a^N, \bar{b}^N)$ denotes the unique Nash equilibrium of the stage game, in which $\mu(a^N, \bar{b}^N) = \mu(a^*, \bar{b}^N)$ when Condition 1 fails.[13]

Theorem 3. *Under Conditions 1 and 2, $\mathcal{E}$ is a single-valued correspondence that coincides with the unique bounded solution of the* optimality equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi, \phi(1-\phi)U'(\phi))))}{|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)|^2}\,. \tag{23}$$

At $p = 0$ and 1, $\mathcal{E}(\phi)$ satisfies the boundary conditions

$$\lim_{\phi \to p} U(\phi) = \mathcal{E}(p) = g(\Psi(p, 0)), \quad \text{and} \quad \lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0. \tag{24}$$

[12] We expect our methods to apply broadly to other continuous-time games, such as the Cournot competition with mean-reverting prices of Sannikov and Skrzypacz (2006a). In that model the market price is the payoff-relevant state variable.

[13] When Condition 1 fails but Condition 2 holds, by an argument similar to the proof of Theorem 2 we can show that the large player cannot achieve any payoff other than $g(a^N, \bar{b}^N)$. Note that Theorem 1 implies that either $(a_t, \bar{b}_t) = (a^N, \bar{b}^N)$ or $|\beta_t| \neq 0$ at all times t.

For any prior $p \in (0,1)$ the unique public sequential equilibrium is a Markov perfect equilibrium in the population's belief. In this equilibrium, the players' actions at time t are given by

$$(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))\,, \tag{25}$$

the population's belief evolves according to

$$d\phi_t = \gamma(a_t, \bar{b}_t, \phi_t)\,\sigma(\bar{b}_t)^{-1}(dX_t - \mu^{\phi_t}(a_t, \bar{b}_t)\,dt)\,, \tag{26}$$

and the continuation values of the normal type are given by $W_t = U(\phi_t)$.

Proof. Proposition 8 from Appendix C.2 shows that under Conditions 1 and 2, there exists a unique continuous function $U : [0,1] \to \mathbb{R}$ that stays in the interval of feasible payoffs of the large player, satisfies equation (22) on $(0,1)$ and boundary conditions (38), which include (24).

We need to prove that for any prior $p \in (0,1)$ there are no public sequential equilibria with a payoff to the normal type different from $U(p)$, and that the unique equilibrium with value $U(p)$ satisfies the conditions of the theorem.

Let us show that for any prior $p \in (0,1)$, there are no equilibria with a payoff to the large player other than $U(p)$. Suppose, towards a contradiction, that for some $p \in [0,1]$, $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium that yields the normal type a payoff of $W_0 \neq U(p)$. Without loss of generality, consider the case when $W_0 > U(p)$.

Then by Theorem 1, the population's equilibrium belief follows (19), the continuation value of the normal type follows (21) for some process (β_t), and equilibrium actions and beliefs satisfy the incentive constraints (18). Then, using (21) and (20), the process $D_t = W_t - U(\phi_t)$ has drift

$$\underbrace{rD_t + rU(\phi_t)}_{rW_t} - rg(a_t, \bar{b}_t) + |\gamma(a_t, \bar{b}_t, \phi_t)|^2\left(\frac{U'(\phi_t)}{1-\phi_t} - \frac{U''(\phi_t)}{2}\right) \tag{27}$$

and volatility

$$r\beta_t\sigma(\bar{b}_t) - \gamma(a_t, \bar{b}_t, \phi_t)\,U'(\phi_t). \tag{28}$$

Lemma 13 from Appendix C.3 shows that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t \geq 0$, either

(a) the drift of D_t is greater than $rD_t - \varepsilon$ or

(b) the absolute value of the volatility of D_t is greater than δ

Here we provide a crude intuition behind Lemma 13. When the volatility of D_t is exactly 0, then $r\beta_t\sigma(\bar{b}_t) = \gamma(a_t, \bar{b}_t, \phi_t)\,U'(\phi_t)$, so

$$a_t \in \arg\max\nolimits_{a' \in A}\; rg(a', \bar{b}_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\sigma^{-1}(\bar{b}_t)}_{r\beta_t}\mu(a', \bar{b}_t)$$
$$b \in \arg\max\nolimits_{b' \in B}\, u(b', \bar{b}_t) + v(b', \bar{b}_t)\cdot\mu^{\phi_t}(a_t, \bar{b}_t) \quad \forall\, b \in \text{support}\,\bar{b}_t$$

and $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$. Then by (22), the drift of D_t is exactly rD_t.

In order for the drift of D_t to be lower than rD_t, the volatility of D_t has to be different from zero. Lemma 13 from Appendix C.3 presents a continuity argument to show that in order for the drift to be below $rD_t - \varepsilon$, the volatility of D_t has to be uniformly bounded away from 0.

By (a) and (b) above it follows that D_t would grow arbitrarily large with positive probability, a contradiction since W_t and $U(\phi_t)$ are bounded processes. The contradiction shows that for any prior $p \in [0,1]$, there cannot be an equilibrium that yields the normal type a payoff larger than $U(p)$. In a similar way, it can be shown that no equilibrium yields a payoff below $U(p)$.

Next, let us construct an equilibrium for a given prior p with value $U(p)$ to the normal type of the large player. Let (ϕ_t) be a solution to the stochastic differential equation (26) with the actions defined by (25). We will show that $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium in which the bounded process $W_t = U(\phi_t)$ is the large player's continuation value.

By Proposition 1 the beliefs (ϕ_t) are consistent with the strategy profile $(a_t, \bar{b}_t)$. Moreover, since $W_t = U(\phi_t)$ is a bounded process with drift $r(W_t - g(a_t, \bar{b}_t))dt$ by (20) and (22), Proposition 2 implies that (W_t) is the process of continuation values of the normal type under the strategy profile $(a_t, \bar{b}_t)$. The process (β_t) associated with the representation of W_t in Proposition 2 is given by $r\beta_t\sigma(b_t) = U'(\phi_t)\gamma(a_t, b_t, \phi_t)$. To see that the public-strategy profile $(a_t, \bar{b}_t)$ is sequentially rational with respect to beliefs (ϕ_t), recall that $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$ and so[14]

$$\begin{aligned} a_t &= \arg\max_{a' \in A} \; rg(a', b_t) + \underbrace{U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)\sigma^{-1}(\bar{b}_t)}_{r\beta_t} \mu(a', b_t)\,, \\ \bar{b}_t &= \arg\max_{b' \in B} u(b', \bar{b}_t) + v(b', \bar{b}_t) \cdot \mu^{\phi_t}(a_t, \bar{b}_t)\,. \end{aligned} \tag{29}$$

From Proposition 3 it follows that the strategy profile $(a_t, \bar{b}_t)$ is sequentially rational. We conclude that $(a_t, \bar{b}_t, \phi_t)$ is a public sequential equilibrium.

Finally, let us show that the actions of the players are uniquely determined by the population's belief in any public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)$ by (25). Let W_t be the continuation value of the normal type. We know that the pair (ϕ_t, W_t) must stay on the graph of U, because there are no public sequential equilibria with values other than $U(\phi_t)$ for any prior ϕ_t. Therefore, the volatility of $D_t = W_t - U(\phi_t)$ must be 0, i.e. $r\beta_t\sigma(\bar{b}_t) = U'(\phi_t)\gamma(a_t, \bar{b}_t, \phi_t)$. Then Proposition 3 implies that (29) holds and so $(a_t, \bar{b}_t) = \Psi(\phi_t, \phi_t(1-\phi_t)U'(\phi_t))$, as claimed. □

The game of quality standards of Section 2 satisfies Conditions 1 and 2, and so its equilibrium is unique and Markovian for every prior. For that game, the correspondence Ψ is given by

$$a = \begin{cases} 0 & \text{if } z \le r, \\ 1 - r/z & \text{otherwise,} \end{cases} \quad \text{and} \quad b = \begin{cases} 0 & \text{if } \phi a^* + (1-\phi)a \le 1/4, \\ 4 - 1/(\phi a^* + (1-\phi)a) & \text{otherwise.} \end{cases}$$

The example illustrates a number of properties that follow from Theorem 3:

(a) The players' actions, which are determined from the population's belief ϕ by $(a, \bar{b}) = \Psi(\phi, \phi(1-\phi)U'(\phi))$, vary continuously with ϕ. In particular, when the belief gets close to 0, the actions converge to the static Nash equilibrium. Thus, there is no discontinuity

[14] Recall that Ψ is a single-valued correspondence that returns an atomic distribution of the small players' actions.

for very small reputations, which is typical for infinitely repeated reputation games with perfect monitoring.

(b) The incentives of the normal type to imitate the behavioral type are increasing in $\phi(1-\phi)U'(\phi)$. However, imitation is never perfect, which is true for all games that satisfy conditions 1 and 2. Indeed, since the actions are defined by (25), $(a_t = a^*, \bar{b}_t)$ would be a Bayesian Nash equilibrium of the stage game with prior ϕ_t if the normal type imitated the behavioral type perfectly at time t. However, Condition 1 implies that the stage game does not have Bayesian Nash equilibria in which the normal type takes action a^*.

The actions of the players are often non-monotonic in beliefs. The large player's actions converge to static best responses at $\phi = 0$ and 1, creating $\cap$-shaped dependence on reputation in the quality standards game. Although not visible in Figure 1, the small players' actions are also non-monotonic for some discount rates.[15] Nevertheless, the large player's equilibrium payoff U is monotonic in the population's belief in this example. This fact, which does not directly follow from Theorem 3, holds generally under additional mild conditions.

6.1 The Effect of Reputation on the Large Player's Payoff.

Proposition 5. *Assume Conditions 1 and 2 and suppose that the static Bayesian Nash equilibrium payoff of the normal type is weakly increasing in the population's prior belief p. Then, the sequential equilibrium payoff $U(p)$ of the normal type is also weakly increasing in p.*

Proof. The static Bayesian Nash equilibrium payoff of the normal type is given by $g(\Psi(\phi, 0))$, where ϕ is the prior on the behavioral type. Recall that $U(0) = g(\Psi(0, 0))$ and $U(1) = g(\Psi(1, 0))$.

Suppose U is not weakly increasing on $[0, 1]$. Take a maximal subinterval $[\phi_0, \phi_1]$ on which U is strictly decreasing. Since $U(0) \leq U(1)$, it follows that $[\phi_0, \phi_1] \neq [0, 1]$. Without loss of generality, assume that $\phi_1 < 1$.

Since ϕ_1 is a local minimum, $U'(\phi_1) = 0$. Also, $U(\phi_1) \geq g(\Psi(\phi_1, 0))$ because otherwise

$$U''(\phi_1) = \frac{2r(U(\phi_1) - g(\Psi(\phi_1, 0)))}{|\gamma(\Psi(\phi_1, 0), \phi_1)|^2} < 0.$$

Since

$$U(\phi_0) > U(\phi_1) \geq g(\Psi(\phi_1, 0)) \geq g(\Psi(0, 0)) = U(0),$$

it follows that $\phi_0 > 0$. Therefore, $U'(\phi_0) = 0$ and

$$U''(\phi_0) = \frac{2r(U(\phi_0) - g(\Psi(\phi_0, 0)))}{|\gamma(\Psi(\phi_0, 0), \phi_0)|^2} > 0,$$

and so ϕ_0 is a strict local minimum, a contradiction. □

[15] For small discount rates r, not far from $\phi = 0$ the slope of U gets very high as it grows towards the commitment payoff. This can cause the normal type to get very close to imitating the behavioral type, producing a peak in the small players' actions.

The result of Proposition 5 is not obvious. Even in games in which the functions $\Psi(\phi, z)$ and $g(\Psi(\phi, z))$ are highly irregular and non-monotonic, the large player's equilibrium payoff $U(\phi)$ is increasing in reputation ϕ as long as the static Bayesian Nash equilibrium payoff of the large player is increasing in reputation.

Remark 4. If the static Bayesian Nash equilibrium payoff of the normal large player is increasing in the small players' belief p, then the conclusion of Theorem 3 holds even if the correspondence Ψ is single-valued and Lipschitz-continuous only for $z \geq 0$.[16] Indeed, if we construct a new correspondence $\hat{\Psi}$ from Ψ by replacing the values for $z < 0$ by an arbitrary Lipschitz-continuous function, then the optimality equation with $\hat{\Psi}$ replacing Ψ would have a unique solution U with boundary conditions $U(0) = g(\Psi(0, 0))$ and $U(1) = g(\Psi(1, 0))$ by Theorem 3. By Proposition 5 this solution must be monotonically non-increasing, and therefore it satisfies the original equation with correspondence Ψ. All other arguments of Theorem 3 apply to the function U constructed in this alternative way.

7 General Characterization.

In this section we extend the characterization of Section 6 to environments with multiple equilibria. When correspondence Ψ is not single-valued (so Condition 2 is violated), the correspondence of sequential equilibrium payoffs, $\mathcal{E}$, may also not be single-valued either. Theorem 4 below characterizes $\mathcal{E}$ for the general case.

Throughout this section, we maintain Condition 1 but relax Condition 2 to:

Condition 3. $\Psi(\phi, z)$ is non-empty for all $(\phi, z) \in [0, 1] \times \mathbb{R}$.

This is a weak assumption on the primitives of the game and it is automatically satisfied when the action spaces are finite and Ψ is replaced by its mixed-action extension (see Remark 2).

Consider the optimality equation from Section 6 (see Theorem 3). When Ψ is a multi-valued correspondence, there may exist multiple bounded functions that solve the differential equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(a(\phi), \bar{b}(\phi)))}{|\gamma(a(\phi), \bar{b}(\phi), \phi)|^2}\,, \tag{30}$$

corresponding to different measurable selections $\phi \mapsto (a(\phi), \bar{b}(\phi)) \in \Psi(\phi, \phi(1-\phi)U'(\phi))$. An argument similar to the proof of Theorem 3 can be used to show that for every such solution U and every prior p, there exists a sequential equilibrium that achieves payoff $U(p)$ for the normal type. Therefore, a natural conjecture is that the correspondence of sequential equilibrium payoffs, $\mathcal{E}$, contains all values between its upper boundary, the largest solution of (30), and its lower boundary, the smallest solution of (30). Accordingly, the pair $(a(\phi), \bar{b}(\phi)) \in \Psi(\phi, \phi(1 - \phi)U'(\phi))$ should minimize the right-hand side of (30) for the upper boundary, and maximize it for the lower boundary.

[16] Such conclusion has practical value because under typical concavity assumptions on payoffs, the large player's objective function in the definition of Ψ may become convex instead of concave for $z < 0$.

However, the differential equation

$$U''(\phi) \;=\; H(\phi, U(\phi), U'(\phi))\,, \tag{31}$$

with

$$H(\phi,u,u') \;\equiv\; \min\left\{\frac{2u'}{1-\phi} + \frac{2r(u-g(a,\bar{b}))}{|\gamma(a,\bar{b},\phi)|^2} \;:\; (a,\bar{b}) \in \Psi(\phi,\phi(1-\phi)u')\right\}, \tag{32}$$

may fail to have a solution in the classical sense. In general, Ψ is upper hemi-continuous, but not necessarily continuous, and so the right-hand side of (31) is lower semi-continuous but may fail to be continuous.

Due to this difficulty, we rely on a generalized notion of solution called *viscosity solution* (see Definition 2 below), which is suitable to deal with discontinuous equations. (For an introduction to viscosity solutions we refer the reader to Crandall, Ishii, and Lions (1992).) We show that the upper boundary $U(\phi) \equiv \sup \mathcal{E}(\phi)$ is the largest viscosity solution of the *upper optimality equation* (31), and that the lower boundary $L(\phi) \equiv \inf \mathcal{E}(\phi)$ is the smallest solution of the *lower optimality equation*, defined by replacing the minimum by the maximum in the expression of H.

While in general viscosity solutions may fail to be differentiable, we show that the upper boundary U is a continuously differentiable function with absolutely continuous derivative. When Ψ is single-valued in a neighborhood of $(\phi, \phi(1-\phi)U'(\phi))$ for some $\phi \in (0,1)$, and H is Lipschitz-continuous in a neighborhood of $(\phi, U(\phi), U'(\phi))$, any viscosity solution is a classical solution of (31) in a neighborhood of ϕ. Otherwise, we show that $U''(\phi)$, which exists almost everywhere since U' is absolutely continuous, can take any value between $H(\phi, U(\phi), U'(\phi))$ and its upper semi-continuous envelope $H^*(\phi, U(\phi), U'(\phi))$ (Note that H is lower semi-continuous, that is, $H = H_*$.)

Definition 2. A bounded function $U : (0,1) \to \mathbb{R}$ is a *viscosity super-solution* of the upper optimality equation if for every $\phi_0 \in (0,1)$ and every twice continuously differentiable *test function* $V : (0,1) \to \mathbb{R}$,

$$U_*(\phi_0) = V(\phi_0) \text{ and } U_* \geq V \quad\Longrightarrow\quad V''(\phi_0) \leq H^*(\phi, V(\phi_0), V'(\phi_0))\,.$$

A bounded function $U : (0,1) \to \mathbb{R}$ is a *viscosity sub-solution* if for every $\phi_0 \in (0,1)$ and every twice continuously differentiable test function $V : (0,1) \to \mathbb{R}$,

$$U^*(\phi_0) = V(\phi_0) \text{ and } U^* \leq V \quad\Longrightarrow\quad V''(\phi_0) \geq H_*(\phi, V(\phi_0), V'(\phi_0)).$$

A bounded function U is a *viscosity solution* if it is both a super-solution and a sub-solution.[17]

Appendix D presents the details of our analysis, which we summarize here. Propositions 9 and 10 show that U, the upper boundary of $\mathcal{E}$, is a bounded viscosity solution of the upper optimality equation. Lemma 16 shows that every bounded viscosity solution is a $\mathcal{C}^1$ function with absolutely continuous derivative (so its second derivative exists almost everywhere). Finally, Proposition 11 shows that U is the largest viscosity solution of (31), and that

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi))\,,\; H^*(\phi, U(\phi), U'(\phi))] \quad \text{a.e.} \tag{33}$$

[17]This is equivalent to Definition 2.2 in Crandall, Ishii, and Lions (1992).

In particular, when H is continuous at $(\phi, U(\phi), U'(\phi))$ then U satisfies (31) in the classical sense.

We summarize our characterization in the following theorem.

Theorem 4. *Assume Conditions 1 and 3 and that a public sequential equilibrium exists for every prior. Then $\mathcal{E}$ is a compact-, convex-valued correspondence with an arcwise connected graph. The upper boundary U of $\mathcal{E}$ is a C^1 function with absolutely continuous derivative (so $U''(\phi)$ exists almost everywhere). Moreover, U is characterized as the maximal bounded function that satisfies the differential inclusion*

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)),\; H^*(\phi, U(\phi), U'(\phi))] \quad a.e., \tag{34}$$

where the lower semi-continuous function H is defined by (32) and H^* denotes the upper semi-continuous envelope of H. The lower boundary of $\mathcal{E}$ is characterized analogously.

To see an example of such equilibrium correspondence $\mathcal{E}(p)$, consider the following game, related to our example of quality commitment. Suppose that the large player, a service provider, chooses investment in quality $a_t \in [0,1]$, where $a^* = 1$ is the action of the behavioral type, and each consumer chooses a service level $b_t^i \in [0,2]$. The public signal about the large player's investment is

$$dX_t = a_t\, dt + dZ_t.$$

The large player's payoff flow is $(\bar{b}_t - a_t)\, dt$ and consumer i receives payoff $b_t^i \bar{b}_t\, dX_t - b_t^i\, dt$. The consumers' payoff functions capture positive network externalities: greater usage $\bar{b}_t$ of the service by other consumers allows each individual consumer to enjoy the service more.

The unique Nash equilibrium of the stage game is $(0,0)$. The correspondence $\Psi(\phi, z)$ defines the action of the normal type uniquely by

$$a = \begin{cases} 0 & \text{if } z \le r \\ 1 - r/z & \text{otherwise.} \end{cases} \tag{35}$$

The consumers' actions are uniquely $\bar{b} = 0$ only when $(1-\phi)a + \phi a^* < 1/2$. If $(1-\phi)a+\phi a^* \ge 1/2$ then the game among the consumers, who face a coordination problem, has two pure equilibria with $\bar{b} = 0$ and $\bar{b} = 2$ (and one mixed equilibrium when $(1-\phi)a + \phi a^* > 1/2$). Thus, the correspondence $\Psi(\phi, z)$ is single-valued only on a subset of its domain.

How is this reflected in the equilibrium correspondence $\mathcal{E}(p)$? Figure 4 displays the upper boundary of $\mathcal{E}(p)$ for three discount rates $r = 0.1$, 0.2 and 0.5. The lower boundary for this example is identically zero, because the game among the consumers has an equilibrium with $\bar{b} = 0$.

For each discount rate, the upper boundary U is divided into three regions. In the region near 0, where the upper boundary is a solid line, the correspondence $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is single-valued and U satisfies the upper optimality equation in the classical sense. In the region near 1, where the upper boundary is a dashed line, the correspondence Ψ is continuous and has three values (two pure and one mixed). There, U also satisfies the upper optimality equation

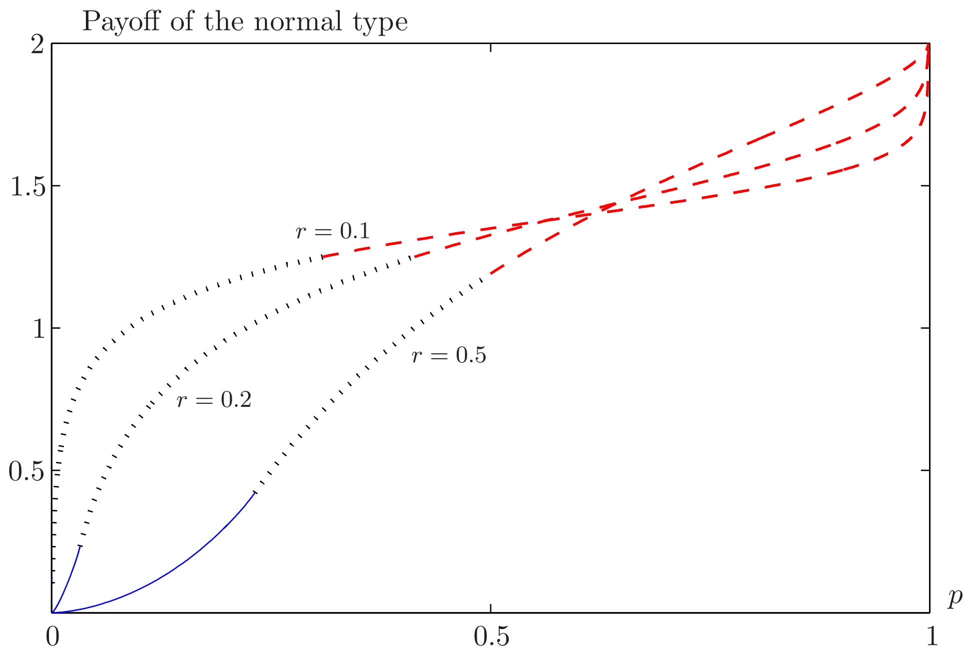

Figure 4: The upper boundary of $\mathcal{E}(p)$.

with the population's action $\bar{b} = 2$. In the middle region, where the upper boundary is a dotted line we have

$$U''(\phi) \in \left(\frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 2 + a)}{|\gamma(a, 2, \phi)|^2}, \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - 0 + a)}{|\gamma(a, 0, \phi)|^2} \right),$$

where a is given by (35) and 0 and 2 are two values of $\bar{b}$ that the correspondence Ψ returns. In that range, the correspondence $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is discontinuous in its arguments: if we lower $U(\phi)$ slightly the equilibrium among the consumers with $\bar{b} = 2$ disappears. These properties of the upper boundary follow from the fact that it is the *largest* solution of the upper optimality equation.

Remark 5. The assumption in Theorem 4 that a sequential equilibrium exists requires explanation. First note that standard fixed-point arguments do not apply to our games, because the set of histories at any time t is uncountable.[18] Second, observe that while the differential inclusion (34) is guaranteed to have a solution under Conditions 1 and 3, this does not imply the existence of a sequential equilibrium, since a measurable selection $(a(\cdot), \bar{b}(\cdot))$ satisfying (30) may fail to exist. However, if public randomization is allowed then existence is restored. In the supplemental appendix Faingold and Sannikov (2007) we develop the formalism of public randomization in continuous time and show that a sequential equilibrium in public randomized

[18]See Harris, Reny, and Robson (1995) for a related problem in the context of extensive-form games with continuum action sets.

strategies exists under Condition 1 and finite action sets (so that Condition 3 is automatically satisfied).

A Bounds on $\gamma(a, \bar{b}, \phi)$.

Throughout this appendix we will maintain Condition 1.

Lemma 2. *There exist $M > 0$ and $C > 0$ such that whenever $|\beta| \leq M$, and $(a, \bar{b}, \phi)$ satisfies the incentive constraints (18), we have*

$$|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)\,.$$

Proof. Consider the set Φ of 4-tuples $(a, \bar{b}, \phi, \beta)$ such that the incentive constraints (18) hold and $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$. Φ is a closed set that does not intersect the compact set $A \times \Delta(B) \times [0,1] \times \{0\}$, and therefore the distance $M' > 0$ between those two sets is positive. It follows that $|\beta| \geq M'$ for any $(a, \bar{b}, \phi, \beta) \in \Phi$.

Now, let $M = M'/2$. Let Φ' be the set of 4-tuples $(a, \bar{b}, \phi, \beta)$ such that the incentive constraints (18) hold and $|\beta| \leq M$. Φ' is a compact set, and so the continuous function $|\mu(a^*, \bar{b}) - \mu(a, \bar{b})|$ must reach a minimum C_1 on Φ'. We have $C_1 > 0$ because $|\beta| \geq 2M$ whenever $|\mu(a^*, \bar{b}) - \mu(a, \bar{b})| = 0$. Since for some $k > 0$, $|\sigma(\bar{b}) \cdot y| \leq k|y|$ for all y and $\bar{b}$, we have

$$|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)$$

whenever $|\beta| \leq M$ and $(a, \bar{b}, \phi)$ satisfies the incentive constraints (18), where $C = C_1/k$. This concludes the proof of the lemma. □

Lemma 3. *For all $\varepsilon > 0$ there exists $K > 0$ such that for all $\phi \in [0,1]$ and $u' \in \mathbb{R}$,*

$$|u'||\gamma(a, \bar{b}, \phi)| \geq K\,,$$

whenever $\phi(1-\phi)|u'| \geq \varepsilon$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)u')$.

Proof. As shown in the proof of Proposition 4, given Condition 1 there is no Bayesian Nash equilibium $(a, \bar{b})$ of the static game with prior $p > 0$ in which $\mu(a, \bar{b}) = \mu(a^*, \bar{b})$.

If the statement of the lemma were false, there would exist a sequence $(a_n, \bar{b}_n, u'_n, \phi_n)$, with $(a_n, \bar{b}_n) \in \Psi(\phi_n, \phi_n(1-\phi_n)u'_n)$ and $\phi_n(1-\phi_n)|u'_n| \geq \varepsilon$ for all n, for which $|u'_n||\gamma(a_n, \bar{b}_n, \phi_n)|$ converged to 0. Let $(a, \bar{b}, \phi) \in A \times \Delta B \times [0,1]$ denote the limit of a convergent subsequence. By upper hemi-continuity, $(a, \bar{b})$ is a BNE of the static game with prior ϕ. Hence, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$ and therefore $\liminf_n |u'_n||\gamma(a_n, \bar{b}_n, \phi_n)| \geq \varepsilon|\sigma(b)^{-1}(\mu(a, \bar{b}) - \mu(a^*, \bar{b}))| > 0$, a contradiction. □

Lemma 4. *For all $M > 0$ there exists $C > 0$ such that*

$$|\gamma(a, \bar{b}, \phi)| \geq C\,\phi(1-\phi)\,,$$

whenever $\phi(1-\phi)|u'| < M$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)u')$.

Proof. Fix $M > 0$. By Lemma 3, for all $\varepsilon \in (0, M)$ there exists $K > 0$ such that

$$|\gamma(a, \bar{b}, \phi)| \geq \frac{K}{|u'|} \geq \frac{K}{M}\,\phi(1-\phi)$$

whenever $\phi(1-\phi)|u'| \in (\varepsilon, M)$ and $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)u')$.

Therefore, Lemma 4 can be false only if

$$\frac{|\gamma(a_n, \bar{b}_n, \phi_n)|}{\phi_n(1-\phi_n)} = \sigma(\bar{b}_n)^{-1}(\mu(a^*, \bar{b}_n) - \mu(a_n, \bar{b}_n))$$

converges to 0 for some sequence $(a_n, \bar{b}_n, u'_n, \phi_n)$, with $(a_n, \bar{b}_n) \in \Psi(\phi_n, u'_n)$, $\phi_n \in (0,1)$, and $\phi_n(1-\phi_n)|u'_n| \to 0$. Let $(a, \bar{b}, \phi) \in A \times \Delta(B) \times [0,1]$ denote the limit of a convergent subsequence. By upper hemi-continuity, $(a, \bar{b})$ is a BNE of the static game with prior ϕ. Hence, $\mu(a, \bar{b}) \neq \mu(a^*, \bar{b})$ and so $|\gamma(a_n, \bar{b}_n, \phi_n)|/(\phi_n(1-\phi_n))$ cannot converge to 0, a contradiction. □

Proof of Lemma 1. Pick any constant $M > 0$. Consider the set Φ_0 of triples $(a, b, \beta) \in A \times \Delta B \times [0,1] \times \mathbb{R}^d$ that satisfy

$$a \in \arg\max_{a' \in A} g(a', b), \quad b \in \arg\max_{b' \in B} h(a, b', \bar{b}), \ \forall b \in \text{support}\, \bar{b}, \quad g(a, \bar{b}) \geq \bar{v} + \varepsilon \tag{36}$$

and $|\beta| \leq M$. Note that Φ_0 is a compact set, as it is a closed subset of the compact space $A \times \Delta(B)$. Therefore, the continuous function $|\beta|$ achieves its minimum δ on Φ_0, and $\delta > 0$ because of the condition $g(a, \bar{b}) \geq \bar{v} + \varepsilon$. It follows that $|\beta| \geq \min(M, \delta) > 0$ for any triple (a, b, β) satisfying conditions (36). □

B Proof of Proposition 4.

Fix a public sequential equilibrium $(a_t, \bar{b}_t, \phi_t)$ and $\varepsilon > 0$. Consider the function $f_1(W) = e^{K_1(W - \underline{g})}$. Then, by Ito's lemma, $f_1(W_t)$ has drift

$$K_1 e^{K_1(W-\underline{g})}(rW_t - g(a_t, \bar{b}_t)) + K_1^2/2 e^{K_1(W-\underline{g})} r^2 \beta_t^2,$$

which is always greater than or equal to

$$-K_1 e^{K_1(\bar{g}-\underline{g})} r(\bar{g} - \underline{g}),$$

and greater than or equal to

$$-K_1 e^{K_1(W-\underline{g})} r(\bar{g} - \underline{g}) + K_1^2/2 e^{K_1(W-\underline{g})} r^2 M^2 > 1$$

when $|\beta_t| \geq M$ (choosing K_1 sufficiently large).

Consider the function $f_2(\phi_t) = K_2(\phi_t^2 - 2\phi_t)$. We have

$$d\phi_t = -\frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t}\, dt + \gamma(a_t, \bar{b}_t, \phi_t)\, dZ_t^{\mathtt{n}}$$

and so by Ito's lemma $f_2(\phi_t)$ has drift

$$-K_2 \frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{1-\phi_t}(2\phi_t - 2) + K_2 \frac{|\gamma(a_t, \bar{b}_t, \phi_t)|^2}{2} 2 = 3K_2 |\gamma(a_t, \bar{b}_t, \phi_t)|^2 \geq 0$$

When K_2 is sufficiently large, then the drift of $f_2(\phi_t)$ is greater than or equal to $K_1 e^{K_1(\bar{g}-\underline{g})} r(\bar{g} - \underline{g}) + 1$, whenever $\phi_t \in [\varepsilon, 1-\varepsilon]$ and $|\beta_t| \leq M$, so that $|\gamma(a, \bar{b}, \phi)| \geq C\phi(1-\phi)$ by Lemma 2.

It follows that until the stopping time τ when ϕ_t hits an endpoint of $[\varepsilon, 1-\varepsilon]$, the drift of $f_1(W_t) + f_2(\phi_t)$ is greater than or equal to 1.

But then for some constant K_3, since f_1 is bounded on $[\underline{g}, \bar{g}]$ and f_2 is bounded on $[\varepsilon, 1-\varepsilon]$, it follows that for all t

$$K_3 \geq \mathsf{E}[f_1(W_{\min(\tau,t)}) + f_2(\phi_{\min(\tau,t)})] \geq f_1(W_0) + f_2(\phi_0) + \int_0^t \text{Prob}(\tau \geq s)\, ds$$

and so $\text{Prob}(\tau \geq s)$ must converge to 0 as $s \to \infty$.

But then ϕ_t must converge to 0 or 1 with probability 1, and it cannot be 1 with positive probability if the type is normal. This completes the proof of Proposition 4.

C Appendix for Section 6.

Throughout this appendix we will maintain Conditions 1 and 2.

C.1 Existence of a bounded solution of the optimality equation.

In this subsection we will prove the following Proposition.

Proposition 6. *The optimality equation has at least one solution that stays within the interval of all feasible payoffs of the large player on* $(0, 1)$.

The proof of Proposition 6 relies on several lemmas.

Lemma 5. *The solutions to the optimality equation exist locally for* $\phi \in (0, 1)$ *(that is, until a blowup point when* $|U(\phi)|$ *or* $|U'(\phi)|$ *become unboundedly large) and are unique and continuous in initial conditions.*

Proof. First, implies that the right-hand side of optimality equation is locally Lipschitz. It follows directly from the standard theorem on existence, uniqueness and continuity of solutions of ordinary differential equations in initial conditions, as the right hand side of the optimality equation is locally Lipschitz-continuous, by Lemma 4. □

Lemma 6. *Consider a solution* $U(\phi)$ *of the optimality equation. If there is a blowup at point* $\phi_1 \in (0, 1)$ *then both* $|U(\phi)|$ *and* $|U'(\phi)|$ *become unboundedly large near* ϕ_1.

Proof. By Lemma 3, there exists a constant $k > 0$ such that

$$|U'(\phi)||\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)| \geq k > 0$$

in a neighborhood of ϕ_1, when $|U'(\phi)|$ is bounded away from 0. Suppose, towards a contradiction, that $U(\phi)$ is bounded from above by K near ϕ_1. Without loss of generality assume that $U'(\phi)$ (as opposed to $-U'(\phi)$) becomes arbitrarily large near ϕ_1, and that ϕ_1 is the right endpoint of the domain of the solution U. Then let us pick points $\phi_3 < \phi_2 < \phi_1$ such that $U'(\phi)$ stays positive on the interval (ϕ_3, ϕ_2) and $U'(\phi_2) - U'(\phi_3)$ is sufficiently large.

Consider the case when $U'(\phi)$ is monotonic on (ϕ_2, ϕ_3), and let us parameterize the interval (ϕ_3, ϕ_2) by $u' = U'(\phi)$. Denote

$$\xi(u') = \frac{dU(\phi)}{dU'(\phi)} = \frac{U'(\phi)}{U''(\phi)} > 0$$

Note that

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi, \phi(1-\phi)U'(\phi))))}{|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)|^2} \leq k_1 U'(\phi) + k_2 U'(\phi)^2$$

for some constants k_1 and k_2 that depend on ϕ_1, K and the range of stage-game payoffs of the large player, so that $\xi(u') \geq 1/(k_1 + k_2 u')$.

Then

$$U(\phi_3) - U(\phi_2) = \int_{U'(\phi_2)}^{U'(\phi_3)} \xi(u') du' \geq \int_{U'(\phi_2)}^{U'(\phi_3)} 1/(k_1 + k_2 u') du' \tag{37}$$

This quantity grows arbitrarily large, leading to a contradiction, when $U'(\phi_3) - U'(\phi_2)$ gets large while $U'(\phi_2)$ stays fixed (this can be always guaranteed even if $U'(\phi)$ flips sign many times near ϕ_1.)

When $U'(\phi)$ is not monotonic on (ϕ_2, ϕ_3), a conclusion similar to (37) can be reached by splitting the integral into subintervals where $U'(\phi)$ is increasing (on which the bound (37) holds) and the rest of the subintervals (on which $U(\phi)$ is increasing). □

One consequence of Lemma 6 is that starting from any initial condition with $\phi_0 \in (0,1)$ the solution of the optimality equation exists until $\phi = 0$ and 1, or until $U(\phi)$ exits the range of feasible payoffs of the large player.

Lemma 7. *(Monotonicity) If two solutions U_1 and U_2 of the optimality equation satisfy $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \leq U_2'(\phi_0)$ with at least one strict inequality, then $U_1(\phi) \leq U_2(\phi)$ and $U_1'(\phi) \leq U_2'(\phi)$ for all $\phi > \phi_0$ until the blowup point. Similarly, if $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \geq U_2'(\phi_0)$ with at least one strict inequality, then $U_1(\phi) < U_2(\phi)$ and $U_1'(\phi) > U_2'(\phi)$ for all $\phi < \phi_0$ until the blowup point.*

Proof. Suppose that $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) < U_2'(\phi_0)$. If $U_1'(\phi) < U_2'(\phi)$ for all $\phi > \phi_0$ until the blowup point then we also have $U_1(\phi) < U_2(\phi)$ on that range. Otherwise, let

$$\phi_1 = \inf_{\phi \geq \phi_0} U_1'(\phi) \geq U_2'(\phi).$$

Then $U_1'(\phi_1) = U_2'(\phi_1)$ by continuity and $U_1(\phi_1) < U_2(\phi_1)$ since $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1(\phi) < U_2(\phi)$ on $[\phi_0, \phi_1)$. From the optimality equation, it follows that $U_1''(\phi_1) < U_2''(\phi_1) \Rightarrow U_1'(\phi_1 - \varepsilon) > U_2'(\phi_1 - \varepsilon)$ for sufficiently small ε, which contradicts the definition of ϕ_1.

For the case when $U_1(\phi_0) < U_2(\phi_0)$ and $U_1'(\phi_0) = U_2'(\phi_0)$ the optimality equation implies that $U_1''(\phi_0) < U_2''(\phi_0)$. Therefore, $U_1'(\phi) < U_2'(\phi)$ on $(\phi_0, \phi_0 + \varepsilon)$, and the argument proceeds as above.

The monotonicity argument for $\phi < \phi_0$ when $U_1(\phi_0) \leq U_2(\phi_0)$ and $U_1'(\phi_0) \geq U_2'(\phi_0)$ with at least one strict inequality is similar. □

Proof of Proposition 6. Denote by $[\underline{g}, \bar{g}]$ the interval of all feasible payoffs of the large player. Fix $\phi_0 \in (0,1)$.

(a) Note that if $|U'(\phi_0)|$ is sufficiently large then the solution U must exit the interval $[\underline{g}, \bar{g}]$ in a neighborhood of ϕ_0. This conclusion can be derived using an inequality similar to (37): $|U'(\phi)|$ cannot become small near ϕ_0 without a change in $U(\phi)$ of $\int_{|U'(\phi)|}^{|U'(\phi_0)|} 1/(k_1 + k_2|x|)dx$.

(b) Also, note that if a solution U reaches the boundary of the region of feasible payoffs, it must exit the region and never reenter. Indeed, it is easy to see from the optimality equation that when $U'(\phi) = 0$, $U''(\phi) \geq 0$ if $U(\phi) \geq \bar{g}$, and $U''(\phi) \leq 0$ if $U(\phi) \leq \underline{g}$. Therefore, $U'(\phi)$ never changes its sign when $U(\phi)$ is outside $(\underline{g}, \bar{g})$.

(c) For a given level $U(\phi_0) = u$, consider solutions of the optimality equation for $\phi \leq \phi_0$ for different values of $U'(\phi_0)$. When $U'(\phi_0)$ is sufficiently large, the resulting solution will reach $\underline{g}$ at some point $\phi_1 \in (0, \phi_0)$ by (a). As $U'(\phi_0)$ decreases, ϕ_1 also decreases by Lemma 7, until for some value $U'(\phi_0) = L(u)$ the solution never reaches the lower boundary of the set of feasible payoffs for any $\phi_1 \in (0, \phi_0)$. Note that this solution never reaches the upper boundary of the set of feasible payoffs for any $\phi_1 \in (0, \phi_0)$: if it did, then the solution with slope $U'(\phi_0) = L(u) + \varepsilon$ would also reach the upper boundary by Lemma 5, and by (b) it would never reach the lower boundary. We conclude that the solution of the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = L(u)$ stays within the range of feasible payoffs for all $\phi_1 \in (0, \phi_0)$.

(d) Similarly, define $R(u)$ as the smallest value of $U'(\phi_0)$ for which the resulting solution never reaches the largest feasible payoff of the large player at any $\phi \in (\phi_0, 1)$. Then the solution of the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = R(u)$ stays within the range of feasible payoffs for all $\phi_1 \in (\phi_0, 1)$, by the same logic as in (c).

(e) Now, Lemma 7 implies that $L(u)$ is increasing in u and $R(u)$ is decreasing in u. Moreover, $L(\underline{g}) \leq 0 \leq L(\bar{g})$ and $R(\underline{g}) \geq 0 \geq R(\bar{g})$. Therefore, there exists a value of u for which $L(u) = R(u)$. The solution to the optimality equation with boundary conditions $U(\phi_0) = u$ and $U'(\phi_0) = L(u) = R(u)$ must stay within the interval of feasible payoffs for all $\phi \in (0,1)$.

This completes the proof of Proposition 6. □

C.2 Regularity conditions at the boundary and uniqueness.

Proposition 7. *If U is a bounded solution of equation (22)on $(0,1)$, then U satisfies the following* boundary conditions *at $p = 0, 1$:*

$$\lim_{\phi \to p} U(\phi) = g(\Psi(p,0)) \,, \ \lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0 \,, \ \lim_{\phi \to p} \phi^2(1-\phi)^2 U''(\phi) = 0 \,. \tag{38}$$

Proof. Direct from Lemmas 10, 11 and 12 below. Lemmas 8 and 9 are intermediate steps. □

Lemma 8. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then U has bounded variation.*

Proof. Suppose there exists a bounded solution U of the optimality equation with unbounded variation near $p = 0$ (the case $p = 1$ is similar). Then let ϕ_n be a decreasing sequence of consecutive local maxima and minima of U, such that ϕ_n is a local maximum for n odd and a local minimum for n even.

Then for n odd we have $U'(\phi_n) = 0$ and $U''(\phi_n) \leq 0$. From the optimality equation it follows that $g(\Psi(\phi_n, 0)) \geq U(\phi_n)$. Likewise, for n even we have $g(\Psi(\phi_n, 0)) \leq U(\phi_n)$. Thus, the total variation of $g(\Psi(\phi, 0))$ on $(0, \phi_1]$ is no smaller than the total variation of U and therefore $g(\Psi(\phi, 0))$ has unbounded variation near zero. However, this is a contradiction, since $g(\Psi(\phi, 0))$ is Lipschitz continuous. □

Lemma 9. *Let $U : (0,1) \to \mathbb{R}$ be a bounded, continuously differentiable function. Then*

$$\liminf_{\phi \to 0} \phi U'(\phi) \;\leq\; 0 \;\leq\; \limsup_{\phi \to 0} \phi U'(\phi)\,, \quad \textit{and}$$

$$\liminf_{\phi \to 1} (1-\phi) U'(\phi) \;\leq\; 0 \;\leq\; \limsup_{\phi \to 1} (1-\phi) U'(\phi)\,.$$

Proof. Suppose, towards a contradiction, that $\liminf_{\phi \to 0} \phi U'(\phi) > 0$ (the case $\limsup_{\phi \to 0} \phi U'(\phi) < 0$ is analogous). Then for some $c > 0$ and $\bar{\phi} > 0$, for all $\phi \in (0, \bar{\phi}]$, $\phi U'(\phi) \geq c \;\Rightarrow U'(\phi) \geq c/\phi$. But then U cannot be bounded since the anti-derivative of $1/\phi$, $\log \phi$, tends to ∞ as $\phi \to 0$, a contradiction. The proof for the case $\phi \to 1$ is analogous. □

Lemma 10. *If U is a bounded solution of the optimality equation, then $\lim_{\phi \to p} \phi(1-\phi)U'(\phi) = 0$ for $p \in \{0, 1\}$.*

Proof. Suppose, towards a contradiction, that $\phi U'(\phi) \nrightarrow 0$ as $\phi \to 0$. Then, by Lemma 9,

$$\liminf_{\phi \to 0} \phi U'(\phi) \;\leq\; 0 \;\leq\; \limsup_{\phi \to 0} \phi U'(\phi)\,,$$

with at least one strict inequality. Without loss of generality, assume $\limsup_{\phi \to 0} \phi U'(\phi) > 0$. Hence there exist constants $0 < k < K$, such that $\phi U'(\phi)$ crosses levels k and K infinitely many times in a neighborhood of 0.

By Lemma 4 there exists $C > 0$ such that

$$|\gamma(a, \bar{b}, \phi)| \geq C\,\phi\,,$$

whenever $\phi U'(\phi) \in (k, K)$ and $\phi \in (0, \frac{1}{2})$. Hence, by the optimality equation, we have

$$|U''(\phi)| \leq \frac{L}{\phi^2}\,,$$

for some constant $L > 0$. This bound implies that for all $\phi \in (0, \frac{1}{2})$ with $\phi U'(\phi) \in (k, K)$, we have

$$|(\phi U'(\phi))'| \;\leq\; |\phi U''(\phi)| + |U'(\phi)| \;=\; (1 + \frac{|\phi U''(\phi)|}{|U'(\phi)|})\,|U'(\phi)| \;\leq\; (1 + \frac{L}{k})\,|U'(\phi)|\,,$$

which yields

$$|U'(\phi)| \geq \frac{|(\phi U'(\phi))'|}{1 + L/k}\,.$$

It follows that on every interval where $\phi U'(\phi)$ crosses k and stays in (k, K) until crossing K, the total variation of U is at least $(K-k)/(1+L/k)$. Since this happens infinitely many times in a neighborhood of $\phi = 0$, function U must have unbounded variation in that neighborhood, a contradiction (by virtue of Lemma 8.)

The proof that $\lim_{\phi \to 1}(1-\phi)U'(\phi) = 0$ is analogous. □

Lemma 11. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then for $p \in \{0,1\}$,*

$$\lim_{\phi \to p} U(\phi) = g(\Psi(p,0))\,.$$

Proof. First, by Lemma 8, U must have bounded variation and so the $\lim_{\phi\to p} U(\phi)$ exists. Consider $p = 0$ and assume, towards a contradiction, that $\lim_{\phi\to 0} U(\phi) = U_0 < g(a^N, b^N)$, where $(a^N, b^N) = \Psi(0,0)$ is the Nash equilibrium of the stage game (the proof for the reciprocal case is similar). By Lemma 10, $\lim_{\phi\to 0} \phi U'(\phi) = 0$, which implies that the function $\Psi(\phi, \phi(1-\phi)U'(\phi))$ is continuous at $\phi = 0$. Recall the optimality equation

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\Psi(\phi,\phi(1-\phi)U'(\phi)))}{|\gamma(\Psi(\phi,\phi(1-\phi)U'(\phi)),\phi)|^2} = \frac{2U'(\phi)}{1-\phi} + \frac{h(\phi)}{\phi^2},$$

where $h(\phi)$ is a continuous function that converges to

$$\frac{2r(U_0 - g(a^N, b^N))}{|\sigma(b^N)^{-1}(\mu(a^*, b^N) - \mu(a^N, b^N))|^2} < 0.$$

as $\phi \to 0$. Since $U'(\phi) = o(1/\phi)$ by Lemma 9, it follows that for some $\bar{\phi} > 0$, there exists a constant $K > 0$ such that

$$U''(\phi) < -\frac{K}{\phi^2}$$

for all $\phi \in (0, \bar{\phi})$. But then U cannot be bounded since the second-order anti-derivative of $1/\phi^2$ $(-\log\phi)$ tends to ∞ as $\phi \to 0$.

The proof for the case $p = 1$ is analogous. □

Lemma 12. *If $U : (0,1) \to \mathbb{R}$ is a bounded solution of the optimality equation, then*

$$\lim_{\phi\to p} \phi^2(1-\phi)^2 U''(\phi) = 0\,, \quad \textit{for } p \in \{0,1\}\,.$$

Proof. Consider $p = 1$. Fix an arbitrary $M > 0$ and choose $\underline{\phi} \in (0,1)$ so that $(1-\phi)|U'(\phi)| < M$ for all $\phi \in (\underline{\phi}, 1)$. By Lemma 4 there exists $C > 0$ such that $|\gamma(\Psi(\phi, \phi(1-\phi)U'(\phi)), \phi)| \geq C(1-\phi)$ for all $\phi \in (\underline{\phi}, 1)$. Hence, by the optimality equation, we have for all $\phi \in (\underline{\phi}, 1)$:

$$\begin{aligned}(1-\phi)^2|U''(\phi)| &\leq 2(1-\phi)|U'(\phi)| + (1-\phi)^2 \frac{2r|U(\phi) - g(\Psi(\phi,\phi(1-\phi)U'(\phi)))|}{|\gamma(\Psi(\phi,\phi(1-\phi)U'(\phi)),\phi)|^2} \\ &\leq 2(1-\phi)|U'(\phi)| + 2rC^{-2}|U(\phi) - g(\Psi(\phi,\phi(1-\phi)U'(\phi)))| \longrightarrow 0\,,\end{aligned}$$

as required. The case $p = 0$ is analogous. □

Proposition 8. *The optimality equation has a unique bounded solution over the interval* $(0,1)$.

Proof. Proposition 6 implies that at least one such solution $U : (0,1) \to \mathbb{R}$ exists. Suppose V were another bounded solution. Assuming that $V(\phi) > U(\phi)$ for some $\phi \in (0,1)$, let ϕ_0 be the point where the difference $V(\phi_0) - U(\phi_0)$ is maximized. It follows from Proposition 7 that $\phi_0 \in (0,1)$. But then by Lemma 7 the difference $V(\phi) - U(\phi)$ must be increasing for $\phi > \phi_0$, a contradiction. □

C.3 A uniform lower bound on volatility.

Lemma 13. *Let $U : (0,1) \to \mathbb{R}$ be the unique bounded solution of the optimality equation and let d and f be the continuous functions defined by:*

$$d(a,\bar{b},\phi) = \begin{cases} rU(\phi) - rg(a,\bar{b}) - \frac{|\gamma(a,b,\phi)|^2}{1-\phi} U'(\phi) - \frac{1}{2}|\gamma(a,b,\phi)|^2 U''(\phi) & : \quad \phi \in (0,1) \\ 0 & : \quad \phi = 0 \text{ or } 1 \end{cases} \tag{39}$$

and

$$f(a,\bar{b},\phi,\beta) = r\beta\sigma(\bar{b}) - \underbrace{\phi(1-\phi)\sigma(\bar{b})^{-1}(\mu(a^*,\bar{b}) - \mu(a,\bar{b}))}_{\gamma(a,b,\phi)} U'(\phi)\,. \tag{40}$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $(a,\bar{b},\phi,\beta)$ that satisfy

$$\begin{aligned} a &\in \arg\max_{a' \in A}\; rg(a',b) + r\beta\mu(a',b) \\ \bar{b} &\in \arg\max_{b' \in B}\; u(b',\bar{b}) + v(b',\bar{b}) \cdot \mu^{\phi}(a,\bar{b})\,, \quad \textit{for all } \; b \in \operatorname{support}\bar{b}, \end{aligned} \tag{41}$$

either $d(a,\bar{b},\phi) > -\varepsilon$ or $f(a,\bar{b},\phi,\beta) \geq \delta$.

Proof. Since $\phi(1-\phi)U'(\phi)$ is bounded (by Lemma 10) and there exists $c > 0$ such that $|\sigma(\bar{b}) \cdot y| \geq c|y|$ for all $y \in \mathbb{R}^d$ and $\bar{b} \in \Delta B$, there exist constants $M > 0$ and $m > 0$ such that $|f(a,\bar{b},\phi,\beta)| > m$ for all $\beta \in \mathbb{R}^d$ with $|\beta| > M$.

Consider the set Φ of 4-tuples $(a,b,\phi,\beta) \in A \times \Delta B \times [0,1] \times \mathbb{R}^d$ with $|\beta| \leq M$ that satisfy (41) and $d(a,\bar{b},\phi) \leq -\varepsilon$. Since U satisfies the boundary conditions (38), d is a continuous function and Φ is a closed subset of the compact set

$$\{(a,b,\phi,\beta) \in A \times \Delta B \times [0,1] \times \mathbb{R}^d \;:\; |\beta| \leq M\},$$

Φ is compact.[19]

Since U satisfies the boundary conditions (38), the function $|f(a,\bar{b},\phi,\beta)|$ is continuous. Hence, it achieves its minimum, η, on Φ. We have $\eta > 0$, because, as we argued in the proof of Theorem 3, $d(a,\bar{b},\phi) = 0$ whenever $f(a,\bar{b},\phi,\beta) = 0$. It follows that for all $(a,\bar{b},\phi,\beta)$ that satisfy (41), either $d(a,\bar{b},\phi) > -\varepsilon$ or $|f(a,\bar{b},\phi,\beta)| \geq \min(m,\eta) \equiv \delta$. □

C.4 Comparative statics for the quality game.

The correspondence $\Psi^{\alpha}(\phi,z)$ for our example is given by

$$a = \begin{cases} 0 & \text{if } z \leq r, \\ 1 - r/z & \text{otherwise,} \end{cases} \qquad \text{and} \qquad b = \begin{cases} 0 & \text{if } \phi a^* + (1-\phi)a \leq 1/(4\alpha), \\ 4 - 1/(\alpha(\phi a^* + (1-\phi)a)) & \text{otherwise.} \end{cases}$$

Note that $\gamma(\Psi^{\alpha}(\phi,z),\phi) = \phi(1-\phi)(a^* - a)$ does not depend on α for our example, and that $g(\Psi^{\alpha}(\phi,z)) = b - a$ weakly increases in α. Consider parameter values $\alpha > \alpha' > 0$, and let us show that the large player's payoffs for these parameters satisfy

$$U^{\alpha}(\phi) \geq U^{\alpha'}(\phi) \tag{42}$$

[19] Since B is compact, the set $\Delta(B)$ is compact in the topology of weak convergence of probability measures.

for all $\phi \in (0,1)$. Note that

$$U^{\alpha}(p) = g(\Psi^{\alpha}(p,0)) > g(\Psi^{\alpha'}(p,0)) = U^{\alpha'}(p)$$

for $p = 0,1$. Suppose that (42) fails for some $\phi \in (0,1)$. Letting ϕ_0 be the point where $U^{\alpha'}(\phi) - U^{\alpha}(\phi)$ is maximized, we have

$$U^{\alpha'}(\phi_0) > U^{\alpha}(\phi_0), \quad U^{\alpha'}(\phi_0)' = U^{\alpha}(\phi_0)' = z \quad \text{and} \quad U^{\alpha'}(\phi_0)'' \leq U^{\alpha}(\phi_0)''.$$

But then $g^{\alpha}(\Psi^1(\phi_0, z)) \geq g^{\alpha'}(\Psi^2(\phi_0, z))$ and so

$$U^{\alpha}(\phi_0)'' = \frac{2z}{1-\phi} + \frac{2r(U^{\alpha}(\phi_0) - g(\Psi^{\alpha}(\phi,z)))}{|\gamma(\Psi^{\alpha}(\phi,z),\phi)|^2} < \frac{2z}{1-\phi} + \frac{2r(U^{\alpha'}(\phi_0) - g(\Psi^{\alpha'}(\phi,z)))}{|\gamma(\Psi^{\alpha'}(\phi,z),\phi)|^2} = U^{\alpha}(\phi_0)'',$$

a contradiction (recall that $\gamma(\Psi^{\alpha'}(\phi,z),\phi) = \gamma(\Psi^{\alpha}(\phi,z),\phi)$).

D Appendix for Section 7.

Throughout this appendix, we will maintain Conditions 1 and 3. Write U and L for the upper and lower boundaries of the correspondence $\mathcal{E}$ respectively, that is,

$$U(p) = \sup \mathcal{E}(p)\,, \quad L(p) = \inf \mathcal{E}(p)$$

for all $p \in [0,1]$.

Proposition 9. *The upper boundary $U : (0,1) \to \mathbb{R}$ is a viscosity sub-solution of the Upper Optimality equation.*

Proof. If U is not a sub-solution, there exists $q \in (0,1)$ and a $\mathcal{C}^2$-function $V : (0,1) \to \mathbb{R}$ such that $0 = (V - U^*)(q) < (V - U^*)(\phi)$ for all $\phi \in (0,1) \setminus \{q\}$, and

$$H_*(q, V(q), V'(q)) = H_*(q, U^*(q), V'(q)) > V''(q)\,.$$

Since H_* is lower semi-continuous, U^* is upper semi-continuous and $V > U^*$ on $(0,1) \setminus \{q\}$, there exist ε and $\delta > 0$ small enough such that for all $\phi \in [q-\varepsilon, q+\varepsilon]$,

$$H(\phi, V(\phi) - \delta, V'(\phi)) > V''(\phi), \tag{43}$$

$$V(q-\varepsilon) - \delta > U^*(q-\varepsilon) \geq U(q-\varepsilon) \quad \text{and} \quad V(q+\varepsilon) - \delta > U^*(q+\varepsilon) \geq U(q+\varepsilon). \tag{44}$$

Figure 5 displays the configuration of functions U^* and $V - \delta$. Fix a pair $(\phi_0, W_0) \in \texttt{Graph}\,\mathcal{E}$ with $\phi_0 \in (q-\varepsilon, q+\varepsilon)$ and $W_0 > V(\phi_0) - \delta$. (Such pair (ϕ_0, W_0) exists because $V(q) = U^*(q)$ and U^* is u.s.c.) Let $(a_t, \bar{b}_t, \phi_t)$ be a sequential equilibrium that attains the pair (ϕ_0, W_0). Denoting by (W_t) the continuation value of the normal type, we have

$$dW_t = r(W_t - g(a_t, \bar{b}_t))\, dt + r\beta_t \cdot (dX_t - \mu(a_t, \bar{b}_t)\, dt)$$

for some $\beta \in \mathcal{L}^*$. Next, we will show that, with positive probability, eventually W_t becomes greater than $U(\phi_t)$, leading to a contradiction since U is the upper boundary of $\mathcal{E}$.

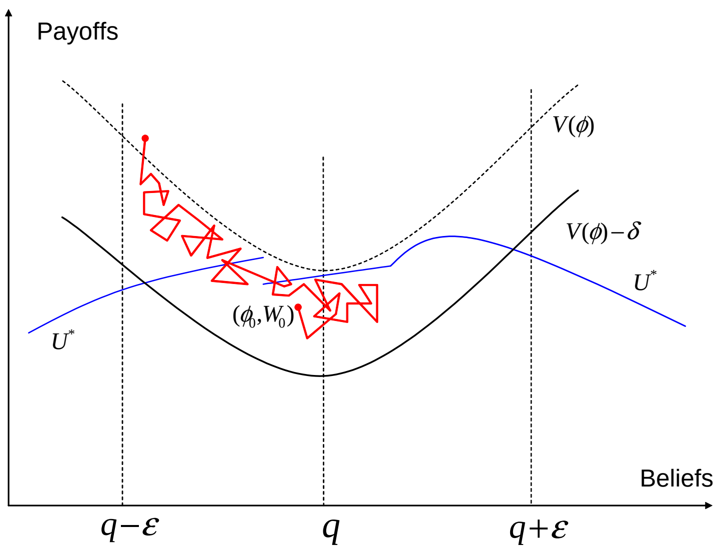

Figure 5: A viscosity sub-solution.

Let $D_t = W_t - (V(\phi_t) - \delta)$. By Itô's formula,

$$dV(\phi_t) = |\gamma_t|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\right) dt \; + \; \gamma_t V'(\phi_t)\, dZ_t^{\mathrm{n}} \,,$$

where $\gamma_t = \gamma(a_t, \bar{b}_t, \phi_t)$, and hence,

$$dD_t = \big(rD_t + r(V(\phi_t) - \delta) - rg(a_t, \bar{b}_t) - |\gamma_t|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\right)\big)\, dt \; + \; \big(r\beta_t \sigma(\bar{b}_t) - \gamma_t V'(\phi_t)\big)\, dZ_t^{\mathrm{n}} \,.$$

Therefore, so long as $D_t \geq D_0/2$,

(a) ϕ_t cannot exit the interval $[q-\varepsilon, q+\varepsilon]$ by (44), and

(b) there exists $\eta > 0$ such that either the drift of D_t is greater than $rD_0/2$ or the norm of the volatility of D_t is greater than η, because of inequality (43) using an argument similar to the proof of Lemma 13.[20]

This implies that with positive probability D_t stays above $D_0/2$ and eventually reaches any arbitrarily large level. Since payoffs are bounded, this leads to a contradiction. We conclude that U must be a sub-solution of the upper optimality equation. □

[20] While the required argument is similar to the one found in the proof of Lemma 13, there are important differences, so we outline them here. First, the functions d and f from Lemma 13 must be re-defined, with the test function V replacing U in the new definition. Second, the definition of the compact set Φ also requires change: Φ would now be the set of all $(a, \bar{b}, \phi, \beta)$ with $\phi \in [q-\varepsilon, q+\epsilon]$ and $|\beta| \leq M$ such that the incentive constraints (41) are satisfied and $d(a, \bar{b}, \phi) \leq 0$. Since ϕ is bounded away from 0 and 1, the boundary conditions will play no role here. Finally, note that U is assumed to satisfy the optimality equation in the lemma, while here V satisfies the strict inequality 43. Accordingly, we need to modify the last part of that proof as follows: $f(a, \bar{b}, \phi, \beta) = 0$ implies $d(a, \bar{b}, \phi) > 0$, and therefore we have $\eta > 0$.

The next lemma is an auxiliary result used in the proof of Proposition 10 below.

Lemma 14. *The correspondence $\mathcal{E}$ of public sequential equilibrium payoffs is convex-valued and has an arc-connected graph.*

Proof. We shall first prove that $\mathcal{E}$ is convex-valued. Fix $p \in (0,1)$, w^*, $w_* \in \mathcal{E}(p)$ (with $w^* > w_*$) and $v \in (w_*, w^*)$. We will prove that $v \in \mathcal{E}(p)$. Consider the set $\mathsf{V} = \{(\phi, w) \,|\, w = \alpha|\phi - p| + v\}$ where $\alpha > 0$ is chosen large enough so that $\alpha|\phi - p| + v > U(\phi)$ for all ϕ sufficiently close to 0 and 1. Let $(\phi_t, W_t)_{t\geq 0}$ be the belief / continuation value process of a public sequential equilibrium that yields the normal type a payoff of w^*. Let $\tau \equiv \inf\{t > 0 \,|(\phi_t, W_t) \in \mathsf{V}\}$. Proposition 4 implies that $\tau < \infty$ almost surely. If $\phi_\tau = p$ with probability 1, then $W_\tau = v$ and nothing remains to be shown. Otherwise, by the martingale property, we have $\phi_\tau < p$ with positive probability and $\phi_\tau > p$ also with positive probability. Hence, there exists a continuous curve $\mathcal{C} \subset \texttt{graph}\ \mathcal{E}$ with endpoints (p_1, w_1) and (p_2, w_2) such that $p_1 < p < p_2$, and for all $(\phi, w) \in \mathcal{C}$ we have $w > v$ and $\phi \in (p_1, p_2)$. Pick $0 < \varepsilon < p - p_1$. We will now construct a continuous curve $\mathcal{C}' \subset \texttt{graph}\,\mathcal{E}|_{(0,\,p_1+\varepsilon]}$ that has (p_1, w_1) as an endpoint and satisfies $\inf\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}'\} = 0$. Fix a public sequential equilibrium of the dynamic game with prior p_1 that yields the normal type a payoff of w_1. Let $\mathsf{P^n}$ denote the probability measure over the sample paths of X induced by the strategy of the normal type. By Proposition 4 we have $\phi_t \to 0$ $\mathsf{P^n}$-almost surely. Moreover, since (ϕ_t) is a supermartingale under $\mathsf{P^n}$, the maximal inequality for non-negative supermartingales yields:

$$\mathsf{P^n}\left[\sup_{t\geq 0} \phi_t \leq p_1 + \varepsilon\right] \;\geq\; 1 - \frac{p_1}{p_1 + \varepsilon} > 0\,.$$

Choose a sample path $(\bar{\phi}_t, \bar{W}_t)$ with the property that $\bar{\phi}_t \to 0$ and $\bar{\phi}_t \leq p_1 + \varepsilon$ for all $t \geq 0$. Define the curve $\mathcal{C}'$ as the image of the sample path $t \mapsto (\bar{\phi}_t, \bar{W}_t)$. By a similar argument we can construct a continuous curve $\mathcal{C}'' \subset \texttt{Graph}\,\mathcal{E}|_{[p_2-\varepsilon,1)}$ that has (p_2, w_2) as an endpoint and satisfies $\sup\{\phi \,|\, \exists w \text{ s.t. } (\phi, w) \in \mathcal{C}''\} = 1$.

Thus, we have constructed a continuous curve $\mathcal{C}^* \equiv \mathcal{C}' \cup \mathcal{C} \cup C'' \subset \texttt{Graph}\,\mathcal{E}|_{(0,1)}$ that projects onto $(0,1)$ and satisfies $\inf\{w \,|\, (p, w) \in \mathcal{C}^*\} > v$. By a similar argument, there exists a continuous curve $\mathcal{C}_* \subset \texttt{Graph}\,\mathcal{E}|_{(0,1)}$ that projects onto $(0,1)$ and satisfies $\sup\{w \,|\, (p, w) \in \mathcal{C}_*\} < v$.

Let $\phi \mapsto (a(\phi), \bar{b}(\phi))$ be a measurable selection from the correspondence of static Bayesian Nash equilibrium. Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2}{1 - \phi_t} + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)\, dZ_t^n$$

with initial condition $\phi_0 = p$.[21] Let (W_t) be the unique solution of

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t)))\, dt$$

with initial condition $W_0 = v$, up to the stopping time $T > 0$ when (ϕ_t, W_t) first hits either $\mathcal{C}^*$ or $\mathcal{C}_*$. Define a strategy profile $(a_t, \bar{b}_t)$ as follows: for $t < T$, $(a_t, \bar{b}_t) \equiv (a(\phi_t), \bar{b}(\phi_t))$; from $t = T$

[21] Condition 1 ensures that $\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)$ is bounded away from zero, hence standard results for existence/uniqueness of weak solutions apply.

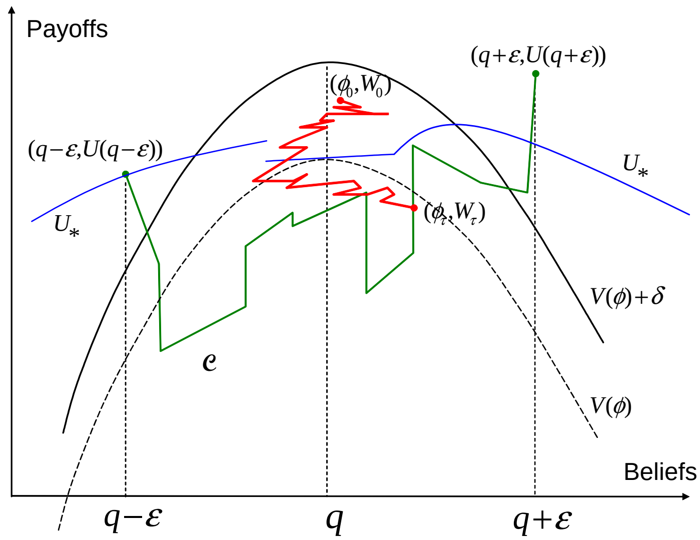

Figure 6: A viscosity super-solution.

onwards $(a_t, \bar{b}_t)$ follows an equilibrium of the game with prior ϕ_T. By Theorem 1 $(a_t, \bar{b}_t, \phi_t)$ is a sequential equilibrium of the game with prior p that yields the normal type a payoff of v. Hence $v \in \mathcal{E}(p)$, concluding the proof that $\mathcal{E}$ is convex-valued.

We will now prove that the graph of $\mathcal{E}$ is arc-connected. Fix $p < q$, $v \in \mathcal{E}(p)$ and $w \in \mathcal{E}(q)$. Consider a sequential equilibrium of the game with prior q that yields the normal type a payoff of w. Since $\phi_t \to 0$, there exists a continuous curve $\mathcal{C} \subset \texttt{graph}\,\mathcal{E}$ with endpoints (q, w) and (p, v') for some $v' \in \mathcal{E}(p)$. Since $\mathcal{E}$ is convex-valued, the straight line $\mathcal{C}'$ connecting (p, v) to (p, v') is contained in the graph of $\mathcal{E}$. Hence $\mathcal{C} \cup \mathcal{C}'$ is a continuous curve connecting (q, w) and (p, v) that is contained in the graph of $\mathcal{E}$, concluding the proof that $\mathcal{E}$ has an arc-connected graph. $\square$

Proposition 10. *The upper boundary $U : (0, 1) \to \mathbb{R}$ is a viscosity super-solution of the Upper Optimality equation.*

Proof. If U is not a super-solution, there exists $q \in (0, 1)$ and a $\mathcal{C}^2$-function $V : (0, 1) \to \mathbb{R}$ such that $0 = (U_* - V)(q) < (U_* - V)(\phi)$ for all $\phi \in (0, 1) \setminus \{q\}$, and

$$H^*(q, V(q), V'(q)) = H^*(q, U_*(q), V'(q)) < V''(q)\,.$$

Since H^* is upper semi-continuous, U_* is lower semi-continuous and $U_* > V$ on $(0, 1) \setminus \{q\}$, there exist ε, $\delta > 0$ small enough such that for all $\phi \in [q - \varepsilon, q + \varepsilon]$,

$$H(\phi, V(\phi) + \delta, V'(\phi)) < V''(\phi), \tag{45}$$

$$V(q - \varepsilon) + \delta < U(q - \varepsilon) \quad \text{and} \quad V(q + \varepsilon) + \delta < U(q + \varepsilon). \tag{46}$$

Figure 6 displays the configuration of functions U_* and $V + \delta$. Fix a pair (ϕ_0, W_0) with $\phi_0 \in (q - \varepsilon, q + \varepsilon)$ and $U(\phi_0) < W_0 < V(\phi_0) + \delta$. We will now construct a sequential equilibrium that attains (ϕ_0, W_0), and this will lead to a contradiction since $U(\phi_0) < W_0$ and U is the upper boundary of $\mathcal{E}$.

Let $\phi \mapsto (a(\phi), \bar{b}(\phi)) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ be a measurable selection of action profiles that minimize

$$rV(\phi) - rg(a, \bar{b}) - |\gamma(a, \bar{b}, \phi)|^2 (\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi}) \tag{47}$$

over all $(a, \bar{b}) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$, for each $\phi \in (0, 1)$.

Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2}{1-\phi_t} \, dt \; + \; \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t) \cdot dZ_t^{\mathtt{n}} \tag{48}$$

on the interval $[q-\varepsilon, q+\varepsilon]$, with initial condition ϕ_0.[22] Next, let (W_t) be the unique strong solution of

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t))) \, dt + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t) V'(\phi_t) \cdot dZ_t^{\mathtt{n}}, \tag{49}$$

with initial condition W_0, until the stopping time when ϕ_t first exits $[q-\varepsilon, q+\varepsilon]$.[23]

Then, by Ito's formula, the process $D_t = W_t - V(\phi_t) - \delta$ has zero volatility and drift given by

$$rD_t + rV(\phi_t) - rg(a(\phi_t), \bar{b}(\phi_t)) - |\gamma(a(\phi_t), \bar{b}(\phi_t)), \phi_t)|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t} \right).$$

By (45) and the definition of $(a(\phi), \bar{b}(\phi))$, the drift of D_t is strictly negative so long as $D_t \leq 0$ and $\phi_t \in [q-\varepsilon, q+\varepsilon]$. (Note that $D_0 < 0$.) Therefore, the process (ϕ_t, W_t) remains under the curve $(\phi_t, V(\phi_t) + \delta)$ from time zero onwards so long as $\phi_t \in [q-\varepsilon, q+\varepsilon]$.

By Lemma 14 there exists a continuous path $\mathcal{C} \subset \mathcal{E}|_{[q-\varepsilon, q+\varepsilon]}$ that connects the points $(q - \varepsilon, U(q-\varepsilon))$ and $(q+\varepsilon, U(q+\varepsilon))$. By (46), the path $\mathcal{C}$ and the function $V + \delta$ bound a region in $[q-\varepsilon, q+\varepsilon] \times \mathbb{R}$ that contains (ϕ_0, W_0), as shown in Figure 6. Since the drift of D_t is strictly negative while $\phi_t \in [q-\varepsilon, q+\varepsilon]$, the pair (ϕ_t, W_t) eventually hits the path $\mathcal{C}$ at a stopping time $\tau < \infty$ before ϕ_t exits the interval $[q-\varepsilon, q+\varepsilon]$.

We will now construct a sequential equilibrium of the repeated game with prior ϕ_0 that yields the normal type payoff W_0. Consider the strategy profile and belief process that coincides with $(a(\phi_t), \bar{b}(\phi_t), \phi_t)$ up to time τ, and follows a sequential equilibrium of the game with prior ϕ_τ at all times after τ. Since W_t is bounded, $(a(\phi_t), \bar{b}(\phi_t)) \in \Psi(\phi_t, \phi_t(1-\phi_t)V'(\phi_t))$, and the processes (ϕ_t, W_t) follow (48) and (49), Theorem 1 implies that the strategy profile $(a(\phi_t), \bar{b}(\phi_t))$ and belief process (ϕ_t) form a sequential equilibrium of the game with prior ϕ_0. It follows that $W_0 \in \mathcal{E}(\phi_0)$, leading to a contradiction since $W_0 > U(\phi_0)$. This contradiction shows that U must be a super-solution of the upper optimality equation. $\square$

Lemma 15. *Every bounded viscosity solution of the upper optimality equation is locally Lipschitz continuous.*

[22] Existence of a weak solution on a closed sub-interval of $(0, 1)$ follows from the fact that V' is bounded and therefore γ is bounded away from zero (Lemma 4). Uniqueness is also granted, because ϕ_t is a one-dimensional process (see Remark 4.32 on Karatzas and Shreve (1991, p. 327)).

[23] Existence and uniqueness of a strong solution follows from the Lipschitz and linear growth conditions in W, and the boundedness of $\gamma(a(\phi), \bar{b}(\phi), \phi)V'(\phi)$ on $[q-\varepsilon, q+\varepsilon]$.

Proof. En route to a contradiction, suppose U is a bounded viscosity solution that is not locally Lipschitz. That is, for some $p \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2})$ satisfying $[p-2\varepsilon, p+2\varepsilon] \subset (0,1)$ the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous. Let $M = \sup |U|$. By Lemmas 3 and 4 there exists $K > 0$ such that for all $(\phi, u, u') \in [p-2\varepsilon, p+2\varepsilon] \times [-M, M] \times \mathbb{R}$,

$$|H^*(\phi, u, u')| \leq K(1 + |u'|^2)\,, \tag{50}$$

Since the restriction of U to $[p-\varepsilon, p+\varepsilon]$ is not Lipschitz continuous, there exist ϕ_0, $\phi_1 \in [p-\varepsilon, p+\varepsilon]$ such that

$$\frac{|U_*(\phi_1) - U_*(\phi_0)|}{|\phi_1 - \phi_0|} \geq \max\{1, e^{2M(4K+\frac{1}{\varepsilon})}\}\,. \tag{51}$$

Hereafter we will assume $\phi_1 > \phi_0$ and $U_*(\phi_1) > U_*(\phi_0)$. The proof for the reciprocal case is analogous and will be omitted.

Let $V : I \to \mathbb{R}$ be the solution of the differential equation

$$V''(\phi) = 2K(1 + V'(\phi)^2)\,, \tag{52}$$

with initial conditions given by

$$V(\phi_1) = U_*(\phi_1) \text{ and } V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0}\,, \tag{53}$$

where the interval I is the maximal domain of V.

We claim V has the following two properties:

(a) There exists $\phi^* \in I \cap (p - 2\varepsilon, p + 2\varepsilon)$ such that $V(\phi^*) = -M$ and $\phi^* < \phi_0$. In particular, $\phi_0 \in I$.

(b) $V(\phi_0) > U_*(\phi_0)$.

We shall first prove property (a). For all $\phi \in I$ such that $V'(\phi) > 1$, we have $V''(\phi) < 4KV'(\phi)^2$ or, equivalently, $(\log V')'(\phi) < 4KV'(\phi)$, which yields

$$V(\hat{\phi}) - V(\tilde{\phi}) > \frac{1}{4K}(\log(V'(\hat{\phi})) - \log(V'(\tilde{\phi})))\,, \quad \forall \hat{\phi}, \tilde{\phi} \in I \text{ s.t. } V'(\hat{\phi}) > V'(\tilde{\phi}) > 1\,. \tag{54}$$

By (51) and (53), we have $\frac{1}{4K}\log(V'(\phi_1)) > 2M$, and therefore a unique $\tilde{\phi} \in I$ exists such that

$$\frac{1}{4K}(\log(V'(\phi_1)) - \log(V'(\tilde{\phi}))) = 2M\,. \tag{55}$$

Since $V'(\tilde{\phi}) > 1$, it follows from (54) that $V(\phi_1) - V(\tilde{\phi}) > 2M$ and so $V(\tilde{\phi}) < -M$. Since $V(\phi_1) > U(\phi_0) \geq -M$, there exists some $\phi^* \in (\tilde{\phi}, \phi_1)$ such that $V(\phi^*) = -M$. Moreover ϕ^* must belong to $(p - 2\varepsilon, p + 2\varepsilon)$, because the convexity of V implies

$$\phi_1 - \phi^* < \frac{V(\phi_1) - V(\phi^*)}{V'(\phi^*)} < \frac{2M}{V'(\hat{\phi})} < \frac{2M}{\log(V'(\hat{\phi}))} = \frac{2M}{\log(V'(\phi_1)) - 8KM} < \varepsilon\,,$$

where the equality follows from (55) and the rightmost inequality follows from (51). Finally, we have $\phi^* < \phi_0$, otherwise the inequality $V(\phi^*) \leq U_*(\phi_0)$ and the initial conditions (53) would imply

$$V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0} \leq \frac{V(\phi_1) - V(\phi^*)}{\phi_1 - \phi^*}\,,$$

which would violate the strict convexity of V. This concludes the proof of property (a).

Turning to property (b), the strict convexity of V and the initial conditions (53) imply

$$\frac{U_*(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} = \frac{V(\phi_1) - V(\phi_0)}{\phi_1 - \phi_0} < V'(\phi_1) = \frac{U_*(\phi_1) - U_*(\phi_0)}{\phi_1 - \phi_0},$$

and therefore $V(\phi_0) > U_*(\phi_0)$, as required.

Define

$$L = \max\left\{V(\phi) - U_*(\phi) \,|\, \phi \in [\phi^*, \phi_1]\right\}.$$

By property (b), we have $L > 0$. Let $\hat{\phi}$ be a point at which the maximum above is attained. Since $V(\phi_*) = -M$ and $V(\phi_1) = U_*(\phi_1)$, we have $\hat{\phi} \in (\phi^*, \phi_1)$ and therefore $V - L$ is a test function that satisfies

$$U_*(\hat{\phi}) = V(\hat{\phi}) - L,$$

and

$$U_*(\phi) \geq V(\phi) - L \quad \text{for every } \phi \in (\phi^*, \phi_1).$$

Since U is a viscosity supersolution,

$$V''(\hat{\phi}) \leq H^*(\hat{\phi}, V(\hat{\phi}) - L, V'(\hat{\phi})),$$

and hence, by (50),

$$V''(\hat{\phi}) \leq K(1 + V'(\hat{\phi})^2) < 2K(1 + V'(\hat{\phi})^2)),$$

which is a contradiction, since by construction V satisfies equation (52). □

Lemma 16. *Every bounded viscosity solution of the upper optimality equation is continuously differentiable with absolutely continuous derivatives.*

Proof. Let $U : (0,1) \to \mathbb{R}$ be a bounded solution of the upper optimality equation. By Lemma 15, U is locally Lipschitz and hence differentiable *almost* everywhere. We will now show that U is differentiable. Fix $\phi \in (0,1)$. Since U is locally Lipschitz, there exist $\delta > 0$ and $k > 0$ such that for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function $V : (\phi - \delta, \phi + \delta) \to \mathbb{R}$ satisfying $V(p) = U(p)$ and $V \geq U$ we have

$$|V'(p)| \leq k.$$

It follows from Lemma 4 that there exists some $M > 0$ such that

$$|H(p, U(p), V'(p))| \leq M$$

for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function V satisfying $V \geq U$ and $V(p) = U(p)$.

Let us now show that for all $\varepsilon \in (0, \delta)$ and $\varepsilon' \in (0, \varepsilon)$

$$-M\varepsilon'(\varepsilon - \varepsilon') < U(\phi + \varepsilon') - \left(\frac{\varepsilon'}{\varepsilon}U(\phi + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon}U(\phi)\right) < M\varepsilon'(\varepsilon - \varepsilon'). \tag{56}$$

If not, for example if the second inequality fails, then we can choose $K > 0$ such that the C^2 function (a parabola)

$$f(\phi + \varepsilon') = \left(\frac{\varepsilon'}{\varepsilon}U(\phi + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon}U(\phi)\right) + M\varepsilon'(\varepsilon - \varepsilon') + K$$

is completely above $U(\phi+\varepsilon')$ except for a tangency point at $\varepsilon'' \in (0, \varepsilon)$. But this contradicts the fact that U is a viscosity subsolution, since $f''(\phi+\varepsilon'') = -2M < H(\phi+\varepsilon'', U(p+\varepsilon''), U'(\phi+\varepsilon''))$.

We conclude that the bounds in (56) are valid, and so for all $0 < \varepsilon' < \varepsilon < \delta$,

$$\left| \frac{U(\phi + \varepsilon') - U(\phi)}{\varepsilon'} - \frac{U(\phi + \varepsilon) - U(\phi)}{\varepsilon} \right| \leq M\varepsilon.$$

It follows that as ε converges to 0 from above,

$$\frac{U(\phi + \varepsilon) - U(\phi)}{\varepsilon}$$

converges to a limit $U'(\phi+)$. Similarly, if ε converges to 0 from below, the quotient above converges to a limit $U'(\phi-)$.

We claim that $U'(\phi+) = U'(\phi-)$. Otherwise, if for example $U'(\phi+) > U'(\phi-)$, then the function

$$f_1(\phi + \varepsilon') = U(\phi) + \varepsilon' \frac{U'(\phi-) + U'(\phi+)}{2} + M\varepsilon'^2$$

is below U in a neighborhood of ϕ, except for a tangency point at ϕ. But this leads to a contradiction, because $f_1''(\phi) = 2M > H(\phi, U(\phi), U'(\phi))$ and U is a super-solution. Therefore $U'(\phi+) = U'(\phi-)$ and we conclude that U is differentiable at every $\phi \in (0, 1)$.

We will now show that U' is locally Lipschitz. Fix $\phi \in (0, 1)$ and, arguing just as we did above, choose $\delta > 0$ and $M > 0$ so that

$$|H(p, U(p), V'(p))| \leq M$$

for every $p \in (\phi - \delta, \phi + \delta)$ and every smooth test function V satisfying $V(p) = U(p)$ and either $V \geq U$ or $V \leq U$. We affirm that for any $p \in (\phi - \delta, \phi + \delta)$ and $\varepsilon \in (0, \delta)$

$$|U'(p) - U'(p + \varepsilon)| \leq 2M\varepsilon.$$

If not, e.g. if $U'(p + \varepsilon) > U'(p) + 2M\varepsilon$ for some $p \in (\phi - \delta, \phi + \delta)$ and $\varepsilon \in (0, \delta)$, then the test function

$$f_2(p + \varepsilon') = \frac{\varepsilon'}{\varepsilon} U(p + \varepsilon) + \frac{\varepsilon - \varepsilon'}{\varepsilon} U(p) - M\varepsilon'(\varepsilon - \varepsilon')$$

must be above U at some $\varepsilon' \in (0, \varepsilon)$ (since $f_2'(p + \varepsilon) - f_2'(p) = 2M\varepsilon$.) Therefore, there exists a constant $K > 0$ such that $f_2(p + \varepsilon') - K$ stays below U for $\varepsilon' \in [0, \varepsilon]$, except for a tangency at some $\varepsilon'' \in (0, \varepsilon)$. But then

$$f_2''(\phi + \varepsilon'') = 2M > H(\phi + \varepsilon'', U(\phi + \varepsilon''), U'(\phi + \varepsilon'')),$$

contradicting the fact that U is a viscosity super-solution. □

Proposition 11. *The upper boundary U is a continuously differentiable function, with absolutely continuous derivatives. In addition, U is characterized as the maximal bounded solution of the following differential inclusion:*

$$U''(\phi) \in [H(\phi, U(\phi), U'(\phi)) \,,\; H^*(\phi, U(\phi), U'(\phi))] \quad a.e.\,. \tag{57}$$

Proof. First, note that by Propositions 9, 10 and 16, the upper boundary U is a differentiable function with absolutely continuous derivative that solves the differential inclusion (57).

If U is not a maximal solution, then there exists another bounded solution V of the differential inclusion (57) that is strictly above U at some $p \in (0,1)$. Choose $\varepsilon > 0$ such that $V(p) - \varepsilon > U(p)$. We will show that $V(p) - \varepsilon$ is the payoff of a public sequential equilibrium, which is a contradiction since U is the upper boundary.

From the inequality

$$V''(\phi) \geq H(\phi, V(\phi), V'(\phi)) \quad \text{a.e.}$$

it follows that a measurable selection $(a(\phi), \bar{b}(\phi)) \in \Psi(\phi, \phi(1-\phi)V'(\phi))$ exists such that

$$rV(\phi) - rg(a(\phi), \bar{b}(\phi), \phi) - |\gamma(a(\phi), \bar{b}(\phi), \phi)|^2 \left(\frac{V''(\phi)}{2} - \frac{V'(\phi)}{1-\phi}\right) \leq 0\,, \tag{58}$$

for almost every $\phi \in (0,1)$.

Let (ϕ_t) be the unique weak solution of

$$d\phi_t = -\frac{|\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2}{1-\phi_t} + \gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)\, dZ_t^{\mathtt{n}}\,,$$

with initial condition $\phi_0 = p$.

Let (W_t) be the unique strong solution of

$$dW_t = r(W_t - g(a(\phi_t), \bar{b}(\phi_t)))\, dt + V'(\phi_t)\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)\, dZ_t^{\mathtt{n}}\,,$$

with initial condition $W_0 = V(p) - \varepsilon$.

Consider the process $D_t = W_t - V(\phi_t)$. It follows from Itô's formula for differentiable functions with absolutely continuous derivatives that:

$$\frac{dD_t}{dt} = rD_t + rV(\phi_t) - rg(a(\phi_t), \bar{b}(\phi_t), \phi_t) - |\gamma(a(\phi_t), \bar{b}(\phi_t), \phi_t)|^2 \left(\frac{V''(\phi_t)}{2} - \frac{V'(\phi_t)}{1-\phi_t}\right).$$

Therefore, by (58) we have

$$\frac{dD_t}{dt} \leq rD_t\,,$$

and since $D_0 = -\varepsilon < 0$ it follows that $W_t \searrow -\infty$.

Let τ be the first time that (ϕ_t, W_t) hits the graph of U. Consider a strategy profile / belief process that coincides with $(a_t, \bar{b}, \phi_t)$ up to time τ and, after that, follows a public sequential equilibrium of the game with prior ϕ_τ with value $U(\phi_\tau)$. It is immediate from Theorem 1 that the strategy profile / belief process constructed is a sequential equilibrium that yields the large player payoff $V(p) - \varepsilon > U(p)$, a contradiction. □

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Reputation Effects and Equilibrium Degeneracy in Continuous-Time Games: Supplemental Appendix

Eduardo Faingold
Department of Economics
Yale University
eduardo.faingold@yale.edu

Yuliy Sannikov
Department of Economics
University of California, Berkeley
sannikov@econ.berkeley.edu

August 17, 2007

In this supplemental appendix, we will show that in the reputation games from Section 7 of Faingold and Sannikov (2007) a sequential equilibrium is guaranteed to exist as long as public randomization is allowed. For simplicity, we will consider the case of finite action sets A and B (allowing for mixed strategies) and we will normalize σ to unity. We will maintain Condition 1 throughout.

A *publicly randomized strategy profile* is a random process $(\pi_t)_{t\geq 0}$ with values in $\Delta\big(A \times B\big)$ progressively measurable w.r.t. $(\mathcal{F}_t)$. Given a publicly randomized strategy profile (π_t), the public signal X follows:

$$dX_t = \mu(\pi_t)\, dt + dZ_t,$$

where μ is extended to $\Delta\big(A \times B\big)$ by linearity.

A *deviation* for the large player is a progressively measurable process (α_t) with values in the set of functions from $A \times B$ into A. A deviation for the small players is defined similarly. Given a publicly randomized strategy π and a deviation α, write:

$$\pi_t^\alpha(a, \bar{b}) = \pi_t(\alpha_t(a, \bar{b}), \bar{b}),$$

for all $(a, \bar{b}) \in A \times B$. A publicly randomized strategy is *sequentially rational* for the large player if for every deviation α, for every t and every public history,

$$\mathsf{E}_t \left[r \int_t^\infty e^{-r(s-t)} g(\pi_s)\, ds \;\middle|\; \theta = \mathtt{n} \right] \geq \mathsf{E}_t \left[r \int_t^\infty e^{-r(s-t)} g(\pi_s^\alpha)\, ds \;\middle|\; \theta = \mathtt{n} \right]$$

where the expectation on the LHS is taken w.r.t. to the measure generated by π and the one on the right is taken w.r.t. to the measure generated by π^α. Given a belief process (ϕ_t), sequential rationality for the small players is defined analogously. A *publicly randomized sequential equilibrium* is a publicly randomized strategy (π_t) together with a belief process (ϕ_t) such that (π_t) is sequentially rational w.r.t. (ϕ_t), and (ϕ_t) is consistent with (π_t).

Notice that we allow for a rich set of deviations in which the relabeling can depend on the recommendations to the opponent. Otherwise we would get correlated equilibrium instead of sequential equilibrium with public randomization.

For each $\phi \in [0,1]$ and $\beta \in \mathbb{R}^d$, let $\mathcal{N}(\phi,\beta) \subset \Delta A \times \Delta B$ denote the set of Nash equilibria of the static game in which the normal type has payoffs given by $g(a,\bar{b}) + \beta\mu(a,\bar{b})$ and the small players have payoffs given by $(1-\phi)h(a,b^i,\bar{b}) + \phi h(a^*,b^i,\bar{b})$.

We now state the appropriate modification of Theorem 1 for publicly randomized strategies.

Theorem. *A publicly randomized profile (π_t) together with a belief process (ϕ_t) is a public sequential equilibrium with continuation values (W_t) for the normal type if and only if*

(a) (W_t) is a bounded process that satisfies

$$dW_t = r(W_t - g(\pi_t))\,dt \;+\; r\beta_t \cdot (dX_t - \mu(\pi_t)\,dt) \tag{1}$$

for some process $\beta \in \mathcal{L}^*$,

(b) belief process (ϕ_t) follows

$$d\phi_t = \gamma(\pi_t,\phi_t)\,(dX_t - \mu^{\phi_t}(\pi_t)\,dt)\,, \quad \textit{and} \tag{2}$$

(c) strategy (π_t) satisfies the incentive constraints $\pi_t \in \mathrm{co}\,\mathcal{N}(\phi_t,\beta_t)$.

The proof is a straightforward modification of Theorem 1 from Faingold and Sannikov (2007) and will be omitted.

We shall now redefine the correspondence Ψ from Section 6 to allow for public randomization:

$$\Psi(\phi,z) = \Big\{\pi \in \Delta(A \times \Delta B) \,:\, \pi \in \mathrm{co}\,\mathcal{N}\big(\phi, z\,(\mu(a^*, \mathrm{marg}_{\Delta(B)}\,\pi) - \mu(\pi))/r\big)\Big\}\,.$$

Note that $\Psi(\phi,z)$ is connected for all (ϕ,z).

Next, consider the convex-valued, u.h.c. correspondence $\Gamma : [0,1] \times \mathbb{R}^2 \rightrightarrows \mathbb{R}$,

$$\Gamma(\phi,u,u') = \mathrm{co}\,\left\{\frac{2u'}{1-\phi} + \frac{2r(u-g(\pi))}{|\gamma(\pi,\phi)|^2} \;:\; \pi \in \Psi(\phi,\phi(1-\phi)u')\right\}$$

for all $(\phi,u,u') \in [0,1] \times \mathbb{R}^2$.

We will now show that a publicly randomized sequential equilibrium exists whenever Condition 1 is satisfied.[1] First, the lemma below shows that the differential inclusion

$$U''(\phi) \in \Gamma(\phi, U(\phi), U'(\phi)) \;\text{ a.e.} \tag{3}$$

has a bounded solution U. Therefore, since Ψ is connected-valued, by the Intermediate Value Theorem there exists a measurable selection $\phi \mapsto \pi(\phi) \in \Psi(\phi,\phi(1-\phi)U'(\phi))$ such that

$$U''(\phi) = \frac{2U'(\phi)}{1-\phi} + \frac{2r(U(\phi) - g(\pi(\phi)))}{|\gamma(\pi(\phi),\phi)|^2}\,,$$

and hence, by an argument similar to the penultimate paragraph of the proof of Theorem 3, we conclude that a sequential equilibrium in publicly randomized strategies exists.

[1] Condition 3 is redundant, for here we work in the finite action setup.

Before turning to the existence lemma we note that, by a straightforward extension of Lemmas 3 and 4 to the public randomization setup we have the following *quadratic growth condition* for Γ: given any closed interval $I \subset (0,1)$ there exists $C > 0$ such that

$$\sup |\Gamma(\phi, u, u')| \leq C(1 + |u'|^2)$$

for all $(\phi, u, u') \in I \times [\underline{g}, \bar{g}] \times \mathbb{R}$.

Lemma. *A bounded solution of the differential inclusion (3) exists.*

Proof. In five steps:

STEP 1 (Existence away from 0 and 1): Given any closed interval $[p,q]$ contained in $(0,1)$, the differential inclusion has a C^2 solution on $[p,q]$ that stays in the set of feasible payoffs. This follows from an existence theorem for boundary value problems for second-order differential inclusions that satisfy a certain growth condition, called the Nagumo condition, which is implied by our quadratic growth condition. The theorem can be found in Bebernes and Kelley (1973)

STEP 2 (A priori bounds on derivatives): Given any interval $[p,q] \subset (0,1)$ there exists $R > 0$ such that every solution $U : [p,q] \to \mathbb{R}$ that stays in the set of feasible payoffs on $[p,q]$ satisfies $|U'| < R$ on $[p,q]$. This can be shown using our quadratic growth condition and an argument similar to the proof of Lemma 6 from Faingold and Sannikov (2007, Appendix C.1).

STEP 3 (Extension to $(0,1)$): For each n, let $U_n : [1/n, 1-1/n] \to \mathbb{R}$ be a C^2 solution of the differential inclusion on $[1/n, 1-1/n]$ and let $R_n > 0$ denote the corresponding a priori bound on derivatives from STEP 2. Fix n and consider $m \geq n$. The restriction of U_m to $[1/n, 1-1/n]$ is a solution of the differential inclusion that stays in the set of feasible payoffs on $[1/n, 1-1/n]$. Hence STEP 2 applies and we have $|U_m'(\phi)| < R_n$ for all $\phi \in [1/n, 1-1/n]$, which implies a uniform bound on the second derivative U_m'' because of the growth condition. By the Arzel-Ascoli Theorem, for every n the sequence $(U_m)_{m>n}$ has a sub-sequence that converges in the C^1 topology on $[1/n, 1-1/n]$. Hence, by a standard diagonalization argument, there exists a subsequence, which we still call U_n, that converges pointwise to some function $U : (0,1) \to \mathbb{R}$. Moreover, on any closed sub-interval I of $(0,1)$, the convergence takes place in $C^1(I)$, i.e., U is C^1 and $(U_n, U_n') \to (U, U')$ uniformly on I.

STEP 4 (Absolute continuity of U'): Note that:

(i) $U_n' \to U'$ pointwise, and

(ii) for every $[p,q] \subset (0,1)$, U_n' is Lipschitz continuous on $[p,q]$ uniformly over all $n > \max\{1/p, 1/(1-q)\}$. This is because by STEP 2 there exists $R > 0$ such that for all n with $1/n < \min\{p, 1-q\}$ we have $|U_n'| \leq R$ on $[p,q]$, which implies a uniform bound on U_n'' by the growth condition.

Therefore, U' is locally Lipschitz, and hence absolutely continuous.

STEP 5 (U is a solution): Fix an arbitrary $\varepsilon > 0$ and a ϕ_0 at which U'' exists. Since Γ is u.h.c. and $(U_n, U_n') \to (U, U')$ uniformly on a nbd of ϕ_0 (by STEP 3), there exist δ and $N > 0$ such that for all $n \geq N$ and all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ we have

$$\Gamma(\phi, U_n(\phi), U_n'(\phi)) \subset \Gamma(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

In particular, for almost all $\phi \in (\phi_0 - \delta, \phi_0 + \delta)$ and all $n \geq N$ we have

$$U_n''(\phi) \in \Gamma(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon]. \tag{4}$$

On the other hand, for all $h > 0$ and $n \geq 1$ we have

$$(U_n'(\phi_0 + h) - U_n'(\phi_0))/h = \frac{1}{h} \int_{\phi_0}^{\phi_0 + h} U_n''(x)\, dx.$$

It then follows from the inclusion (4) and the fact that Γ has convex, closed values that

$$(U_n'(\phi_0 + h) - U_n'(\phi_0))/h \in \Gamma(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon]$$

for all $n \geq N$ and $|h| < \delta$. Thus, letting $n \to \infty$ first and then $h \to 0$ yields

$$U''(\phi_0) \in \Gamma(\phi_0, U(\phi_0), U'(\phi_0)) + [-\varepsilon, \varepsilon].$$

Finally, taking ε to zero yields the desired result.

□

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