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SEGMENTATION, AND LINEAR PROGRAMMING

By
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May 2003

COWLES FOUNDATION DISCUSSION PAPER NO. 1416

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS
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ON HOUSESWAPPING, THE STRICT CORE, SEGMENTATION, AND LINEAR PROGRAMMING

by

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Abstract

We consider the n-player houseswapping game of Shapley-Scarf (1974), *with indifferences in preferences allowed*. It is well-known that the strict core of such a game may be empty, single-valued, or multi-valued.

We define a condition on such games called "segmentability", which means that the set of players can be partitioned into a "top trading segmentation". It generalizes Gale's well-known idea of the partition of players into "top trading cycles" (which is used to find the unique strict core allocation in the model with no indifference). We prove that a game has a nonempty strict core if and only if it is segmentable.

We then use this result to devise an $O(n^3)$ algorithm which takes as input any houseswapping game, and returns either a strict core allocation or a report that the strict core is empty. Finally, we are also able to construct a linear inequality system whose feasible region's extreme points precisely correspond to the allocations of the strict core. This last result parallels the results of Vande Vate (1989) and Rothblum (1991) for the marriage game of Gale and Shapley (1962).

1. Introduction

One of the most important models in the theory of the trade of indivisible goods was put forward by Shapley and Scarf (1974). In this game, there are n players (or agents), each initially endowed with his own single indivisible good, e.g. a house. The houses are *differentiated*, meaning that different players may value different houses differently. Players' valuations for houses are expressed as complete, transitive preference orderings. In addition, it is assumed that each player only has use for one house. Hence the only moves possible for the players in the game are to swap houses amongst themselves, in some mutually beneficial way.

The simple market of trade outlined above gives rise to an ordinal preference game, and Shapley-Scarf's remarkable result is that this game always has a nonempty core. In fact, if no indifference is allowed in any of the players' preferences, it turns out that the *strict* core is also nonempty, consisting of a single allocation of the houses (Roth and Postlewaite, 1977). This allocation can be found using the **top trading cycle algorithm** (TTCA). In this procedure, we initially form a directed graph in which there are n vertices, one for each player. An edge exists from vertex i to vertex j if player i's most preferred house is the one originally owned by player j. It is easy to see that this digraph must have a cycle, and that this cycle defines a "houseswap" among some of the players of the game. Assign the houses according to this "top trading cycle", and remove the corresponding players (and their houses) from consideration. Now do the same thing with the remaining players: define a digraph in which edge ij exists if and only if i's most preferred house (out of the ones remaining in the game) is the one originally owned by j. Again there must be a top trading cycle, etc. At the end of this process there will be an ordered list of top trading cycles, and the corresponding assignment is the unique strict core allocation.

However, the above analysis on the strict core breaks down if we allow players to be indifferent between houses. Indeed, it is easy to construct examples (with indifferences allowed) in which there are no strict core allocations[1] or to construct examples with multiple strict core outcomes. So the question becomes: how can we tell if a houseswapping game (with indifference allowed) has an empty strict core, singleton strict core, or multi-valued strict core? This is the main question which concerns us in this paper.[2]

To answer this question, we define the notion of a **top trading segmentation** (TTS). A TTS is an ordered partition $\{T_1, ..., T_m\}$ of the players with the properties that for each

[1] Shapley-Scarf give such an example in their 1974 paper. We re-use their example as Example 4.3 in this paper.

[2] Wako (1984) provides a few other results for the model where indifference is allowed. In particular, he gives an example in which the strict core is a nonempty proper subset of the set of competitive allocations, and proves that the strict core is in general a subset of the set of competitive allocations.

segment T_j, (a) for each player in T_j, all of his most-preferred houses (among the ones originally owned by members of $T_j \cup T_{j+1} \cup ... \cup T_m$) are originally owned by members of segment T_j; (b) no proper subset of T_j satisfies (a); and (c) there is a cyclical swap of houses within segment T_j such that every player in T_j can actually get one of his "most-preferred" houses. We note that in the case where there are no indifferences in the players' preferences, the list of top-trading-cycles generated by the TTCA satisfies the definition of a TTS.

Our first main result is that in a houseswapping game (with indifference allowed), the strict core is nonempty if and only if a TTS exists. In addition, the cyclical swaps associated with the segments in a TTS give a strict core allocation. Hence, we generalize the previous results for the case where no indifference is allowed.

Second, we use the theory of TTS's to formulate an $O(n^3)$ algorithm which takes as input a houseswapping game, and outputs either a strict core allocation or else a report that the strict core is empty.

Third, we use the notion of TTS to re-prove the result of Wako (1991) and Ma (1994) that strict core outcomes, if they exist, are unique in terms of utility payoffs for the players.

Finally, we formulate a system of linear inequalities, called "CLIS", whose feasible region's extreme points exactly correspond to the strict core allocations of the game. Hence, in the case where no indifference is allowed (and thus the strict core contains precisely one allocation), the feasible region is exactly one point; in the cases where the strict core is empty, CLIS is an infeasible program. Hence we have a natural analogue to linear systems presented by Vande Vate (1989) and Rothblum (1991) for the marriage game of Gale-Shapley (1962).

2. Preliminaries in Graph Theory

We begin with some background in the theory of directed graphs. A **directed graph**, or **digraph** for short, is a pair (V, E) where V is a set of **vertices** and E a set of directed edges. A (directed) **edge** in turn is an ordered pair (i,j) with $i,j \in V$, and thus E is a subset of $\{(i,j) \mid i \in V \text{ and } j \in V\}$. Given a digraph (V,E) and a vertex $i \in V$, we denote by Γ_i the set of vertices pointed to by outgoing edges from i, i.e., $\Gamma_i := \{j \in N \mid (i,j) \in E\}$. In fact, we may alternatively define a digraph by the pair $(V, \{\Gamma_i\}_{i \in V})$.

Let $G = (V,E)$ be a digraph. A **subgraph** of G is any digraph (V', E') with $\emptyset \neq V' \subseteq V$ and $E' \subseteq E$.[3] If V' is a particular subset of V, then the **subgraph of G induced by** V' is the subgraph with vertex set V' and edge set $E' = \{(i,j) \in E : i \in V' \text{ and } j \in V'\}$.

A sequence of vertices[4] $\{i_1, ..., i_m\}$ is called a **path from i_1 to i_m** (or **$i_1 i_m$-path**) if

(1) $m \geq 1$,

(2) $i_1, \ldots, i_m$ are distinct (except for possibly $i_1 = i_m$) elements of V, and

(3) $(i_k, i_{k+1}) \in E$ for $k = 1, \ldots, m-1$.

A **cycle** is a path in which $m \geq 2$ and $i_1 = i_m$.[5] Both a path and a cycle can be regarded as subgraphs (but not necessarily induced subgraphs) of the whole digraph G.

If $G = (V,E)$ is a digraph, we say G is **strongly connected** if for any $i,j \in V$ there is an ij-path. If $V' \subseteq V$, we say V' is **strongly connected** if the subgraph of G induced by V' is strongly connected. Equivalently, V' is strongly connected if for any $i,j \in V'$ there is an ij-path $\{i_1 = i, ..., i_m = j\}$ in which $i_k \in V'$ $\forall k$.[6]

Proposition 2.1. *Let $G = (V,E)$ be a digraph, and suppose $V' \subseteq V$ is strongly connected in G. Let $i \in V'$. Then there exists a cycle (within V') which contains i.*

Next, we define a concept which is crucial in Section 4. A **minimal self-mapped set** of digraph $G = (V, \{\Gamma_i\}_{i \in V})$ is a nonempty subset C of V for which

[3] Since a subgraph is a digraph, this implies that all edges of E' consist of vertex pairs in V'.

[4] In some texts, a directed path is defined as an alternating sequence of vertices and edges (see e.g., Roberts (1984)). However, the list of edges is superfluous, and so one may simply define a path as just a sequence of vertices, as we do here.

[5] The $m \geq 2$ condition implies that a singleton vertex $\{i\}$ is *not* a cycle. But $\{i,i\}$ is a cycle if $(i,i) \in E$.

[6] Note that this definition does not rule out $i = j$; hence, in order for a singleton vertex $\{i\}$ to be considered as strongly connected, the edge (i,i) must be present.

(1) $C = \bigcup_{i \in C} \Gamma_i$, and [self-mappedness]

(2) $\nexists S$ with $\emptyset \neq S \subset C$ and $S = \bigcup_{i \in S} \Gamma_i$. [minimality]

Proposition 2.2. *Let $G = (V, \{\Gamma_i\}_{i \in V})$ be a digraph, and suppose $V' \subseteq V$ satisfies a) V' is strongly connected and b) $\Gamma_i \subseteq V'$ for each $i \in V'$. Then V' is a minimal self-mapped set of G.*

Proof. The strong connectedness of V' implies that $V' \subseteq \bigcup_{i \in V'} \Gamma_i$, while condition b) implies $\bigcup_{i \in V'} \Gamma_i \subseteq V'$; taken together these imply self-mappedness (1). Now suppose there was a nonempty proper subset of V'' of V' which was also self-mapped. Then there would be no path from vertex $i \in V''$ to vertex $j \in V' \setminus V''$, thereby violating V''s strong connectedness. Hence V' also satisfies (2). ∎

Lemma 2.3. *Let $G = (V, \{\Gamma_i\}_{i \in V})$ be a digraph with $\Gamma_i \neq \emptyset$ for each $i \in V$. Then G has at least one minimal self-mapped set.*

Proof. We prove the Lemma by presenting an algorithm which takes as input the directed graph G, and outputs a minimal self-mapped set:

<<*Algorithm* $\mathcal{MSMS}$ >>

Step 0 (Initialization) Given the digraph $G = (V, \{\Gamma_j\}_{j \in V})$, with $\Gamma_j \neq \emptyset$ for all j. Set $L(j) = \{j\}$ for each $j \in V$.

Step 1 (Finding a directed cycle)

(1.0) All vertices in V are colored “blue”. Pick any vertex in V and let i_1 denote it. Vertex i_1 is now “red”. Set $t = 1$.

(1.1) If $\Gamma_{i_t} = \{i_t\}$, then go to Step 3. Otherwise, let i_{t+1} be a vertex in Γ_{i_t} which is *different* from i_t, and set $f(i_t) = i_{t+1}$.

(1.2) If i_{t+1} is colored “red”, i.e. $i_{t+1} = i_k$ for some $k \in 1, ..., t-1$, trace out the cycle $C = \{i_k, ..., i_t\}$ using function f, and go to Step 2. If i_{t+1} is colored “blue”, recolor it "red", then set $t = t + 1$ and return to Step 1.1.

Step 2 (Contracting the digraph using input cycle $C = \{i_k, ..., i_t\}$*)*

(2.1) Set $L(i_k) = \bigcup_{r=k}^{t} L(i_r)$. For vertices $j \in V \setminus C$, $L(j)$ remains unchanged.

(2.2) Set $\Gamma_{i_k} = \{i_k\} \cup \Big((\bigcup_{j \in C} \Gamma_j) \setminus C\Big)$.

(2.3) For each $j \in V \setminus C$, if $\Gamma_j \cap C \neq \emptyset$, then let $\Gamma_j = \{i_k\} \cup (\Gamma_j \setminus C)$; otherwise, keep Γ_j unchanged.

(2.4) Set $V = \{i_k\} \cup (V \setminus C)$ and let $G = (V, \{\Gamma_j\}_{j \in V})$ denote the newly contracted digraph.

(2.5) Return to Step 1.

Step 3 (End) The elements of $L(i_t)$ are the vertices that form a minimal self-mapped set in the *original* digraph G. HALT.//

Let us analyze this algorithm. First, note that every time we enter Step 1, it is with a digraph satisfying $\Gamma_j \neq \emptyset$ for all $j \in V$ (we assume this to be true in the original digraph; also, the digraph-contraction steps (2.2)-(2.4) do not alter this property). Hence, if the algorithm does not immediately go to Step 3 to terminate, the "path following" sub-algorithm (1.1)-(1.2) must yield a cycle. By virtue of the $i_{t+1} \neq i_t$ assumption in (1.1), this cycle contains at least two vertices – and this in turn implies that at least one vertex is contracted out of the digraph during each pass through Step 2. Hence, after at most $n - 1$ iterations of Steps 1 and 2, we will be down to one vertex and so the algorithm will terminate.

Now let us consider the outcome of the algorithm produced at Step 3. First we note that the elements of $L(i_t)$ comprise either a single cycle or else a concatenation of cycles; hence $L(i_t)$ is strongly connected. Next note that in order to move to Step 3 we must have had $\Gamma_{i_t} = \{i_t\}$ in Step (1.1). Hence $L(i_t)$ must also satisfy condition b) of Proposition 2.2, and so is a minimal self-mapped set. ■

Lemma 2.4. *Let $G = (V, \{\Gamma_i\}_{i \in V})$ be a digraph with $\Gamma_i \neq \emptyset$ for each $i \in V$, and suppose S is a minimal self-mapped set of G. Then S is strongly connected.*[7]

Proof. Since S is a minimal self-mapped set, it has no subsets which are self-mapped. Hence, if we consider the subgraph G_S of G induced by the vertices in S, then the only minimal self-mapped set of G_S is S itself. This in turn implies that if we run algorithm $\mathcal{MSMS}$ with input digraph G_S, it must return S itself.

However, we argued in the proof of Lemma 2.3 that the output of algorithm $\mathcal{MSMS}$ is strongly connected; hence S is strongly connected. ∎

Corollary 2.5. *Let $G = (V, \{\Gamma_i\}_{i \in V})$ be a digraph with $\Gamma_i \neq \emptyset$ for each $i \in V$, and suppose S is a minimal self-mapped set of G. Then for each $i \in S$, there exists a cycle $C \subseteq S$ with $i \in C$.*[8]

Proof. Follows directly from Lemma 2.4 and Proposition 2.1. ∎

Corollary 2.6. *Let $G = (V, \{\Gamma_i\}_{i \in V})$ be a digraph with $\Gamma_i \neq \emptyset$ for each $i \in V$, and suppose S and T are distinct minimal self-mapped sets in G. Then $S \cap T = \emptyset$.*

Proof. Since $S \neq T$, we may assume without loss of generality that there exists a vertex j which is an element of S but not T. Now suppose the Corollary was false, i.e. $\exists i \in S \cap T$. Since i and j are both in S, Lemma 2.4 implies there is a path from i to j. But $i \in T$ and $j \notin T$; hence there is an edge along this path connecting a vertex of T to a vertex not in T. This contradicts T's self-mappedness. ∎

Lemma 2.7. *The computational complexity of algorithm $\mathcal{MSMS}$ is $O(|V|^2)$.*

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[7] From this lemma and Proposition 2.2 it is easy to show that the converse of Proposition 2.2 holds.

[8] Here C is a cycle and S is a subset of V. Although technically C is an *ordered* set of vertices and S is *unordered*, we shall use the notation $C \subseteq S$ to mean that the set of vertices in C is a subset of S. This notation comes up again in the "only if" portion of the proof of Theorem 5.5.

Proof. Let $v = |V|$. It is clear that the cycle-building Step 1 is $O(v)$ for each pass through Step 1; since the algorithm passes through Step 1 no more than n times, the number of total steps spent in this phase is $O(v^2)$. Now let us consider a particular single pass (say, the qth pass) through Step 2, with "input cycle" C_q. The re-labeling step (2.1) is $O(|C_q|v)$. Steps (2.2) and (2.3) are also $O(|C_q|v)$, step (2.4) is $O(|C_q|)$, and (2.5) is one operation. Hence the complexity for the qth pass is $O(|C_q|v)$. This in turn implies that the total number of operations in Step 2 (summed over all iterations) is $O(\sum_q |C_q|v)$, which is $O(v^2)$. Thus the entire algorithm is $O(v^2)$. ∎

3. The Houseswapping Market

We consider a market in which there are n players, each endowed with one indivisible good, e.g., a house. Let $N = \{1, ..., n\}$ be the set of players. The indivisible goods are differentiated, and the endowment of player i is called "house i". Thus N also denotes the set of houses in the market.

We assume that each player i wishes to consume exactly one of the houses. His prefences are expressed as a complete, reflexive, and transitive preference ordering $\succeq_i$ over the houses in N, with indifference between houses allowed. Let $x \succ_i y$ indicate that i strictly prefers house x to house y, $x \sim_i y$ indicate that i is indifferent between house x and house y, and $x \succeq_i y$ indicate either of the two cases. Let $\succeq$ denote the bundle $\{\succeq_1, ..., \succeq_n\}$ of the preference orderings of the players.

The players each try to engineer "house-swaps" with other players in an effort to obtain the best possible house in the market. The end result is an **allocation**, which is a one-to-one assignment of houses to players. Formally, an allocation is a bijection $\pi : N \longrightarrow N$, i.e., a permutation of N. If $\pi(i) = j$, this means that in allocation π, player i receives house j. We refer to the market above as **houseswapping market** $\mathcal{M}(N, \succeq)$, or briefly, as market $\mathcal{M}$.

A nonempty subset of players $S \subseteq N$ in market $\mathcal{M}$ is called a **coalition**. For each coalition S, we define an $S-$**allocation** to be any bijection $\pi_S : S \longrightarrow S$. An S-allocation is a way that the players in coalition S can redistribute their original houses amongst themselves. For the grand coalition N, we often say "an allocation" instead of "an N-allocation". Let Π_S denote the set of all S-allocations for a given S, and so Π_N denotes the set of allocations of market $\mathcal{M}$.

Suppose μ is an allocation. A coalition S is said to **block** μ if there exists an S-allocation π_S with $\pi_S(i) \succ_i \mu(i)$ for all $i \in S$. Coalition S **weakly blocks** μ if there is an S-allocation π_S for which $\pi_S(i) \succeq_i \mu(i)$ for all $i \in S$, with $\pi_S(i) \succ_i \mu(i)$ for at least one $i \in S$. A **core allocation** is an allocation which is not blocked by any S, while a **strict core allocation** is an allocation which is not weakly blocked by any S. The (regular) **core** of market $\mathcal{M}$ is the set of all core allocations, and the **strict core** of market $\mathcal{M}$ is the set of all strict core allocations. These are the usual core and strict core solution concepts from cooperative game theory.

Finally, we note the connection between the concept of S-allocation and those of digraph and cycle discussed in the last section. For each S-allocation π_S, we define the **corresponding digraph** to π_S to be digraph $G = (S, \{\Gamma_i\}_{i \in S})$ with $\Gamma_i = \{\pi_S(i)\}$ for each $i \in S$. This digraph consists solely of vertex disjoint cycles, covering the vertices in S. If the corresponding digraph to π_S consists of exactly one cycle, we call π_S a **simple** S-allocation. Let Π_S^0 be the set of simple S-allocations for a given S. Conversely, if digraph $G = (S, \{\Gamma_i\}_{i \in S})$ consists only of vertex disjoint cycles, one may define its **corresponding** S**-allocation** to be the S-allocation π_S given by $\{\pi_S(i)\} = \Gamma_i$ for all $i \in S$.

4. A Fundamental Concept: The PMSS

Let $\mathcal{M}(N, \succeq)$ be a houseswapping market. For each player $i \in N$ and each coalition $S \subseteq N$, we define $B_i(S)$ to be the set of player i's most preferred items among those items in S, formally, $B_i(S) := \{h \in S \mid h \succeq_i j \text{ for each } j \in S\}$. A **partition** of N is a finite ordered set $T = \{T_1, T_2, \ldots, T_m\}$ of nonempty disjoint subsets of N, with $\bigcup_{k=1}^m T_k = N$.

Let $\mathcal{P}$ denote the set of partitions of N.

Definition 4.1. *We call a partition $T = \{T_1, T_2, \ldots, T_m\} \in \mathcal{P}$ a* **partition by minimal self-mapped sets** *(***PMSS***) if each $T_k \in T$ satisfies the following conditions:*

(1) $T_k = \bigcup_{i \in T_k} B_i(N \setminus \cup_{l=1}^{k-1} T_l)$,

(2) $\nexists S \subset T_k$ *s.t.* $S = \bigcup_{i \in S} B_i(N \setminus \cup_{l=1}^{k-1} T_l)$.

In words, a PMSS is a division of the players in the game into ordered groups $T_1, T_2, \ldots, T_m$. Condition (1) implies that for any player i in any group, the items he prefers most among those *not in lower-index-numbered groups* are *all* owned by players in his group. Hence, in this sense, we may regard groups with lower indices as "better" (or "more desirable"). In addition, the fact that (1) holds with equality implies that i's own item is on some member of his group's "most preferred" list.

Example 4.2. *Let $N = \{1, \ldots, 6\}$ be the set of players who have the following preferences:*

(1) $2 \succ_1 3 \succ_1 5 \succ_1 4 \succ_1 1 \succ_1 6$	(2) $1 \sim_2 3 \succ_2 4 \succ_2 6 \succ_2 5 \succ_2 2$
(3) $1 \succ_3 2 \succ_3 3 \succ_3 4 \succ_3 5 \succ_3 6$	(4) $2 \succ_4 5 \succ_4 6 \succ_4 3 \succ_4 4 \succ_4 1$
(5) $1 \sim_5 4 \succ_5 5 \succ_5 3 \succ_5 6 \succ_5 2$	(6) $3 \succ_6 6 \succ_6 1 \succ_6 2 \succ_6 4 \succ_6 5$.

This example has two PMSS's:

$$T = \{T_1 = \{1,2,3\},\ T_2 = \{4,5\},\ T_3 = \{6\}\}$$

$$T' = \{T'_1 = \{1,2,3\},\ T'_2 = \{6\},\ T'_3 = \{4,5\}\}$$

Partition T satisfies condition (1) for a PMSS, because (a) $\cup_{i \in T_1} B_i(N) = T_1$ follows from $B_1(N) = \{2\}$, $B_2(N) = \{1,3\}$, and $B_3(N) = \{1\}$; (b) $\cup_{i \in T_2} B_i(N \setminus T_1) = T_2$ follows from $B_4(\{4,5,6\}) = \{5\}$ and $B_5(\{4,5,6\}) = \{4\}$; and (c) $\cup_{i \in T_3} B_i(N \setminus (T_1 \cup T_2)) = T_3$ follows from $B_6(\{6\}) = 6$. It is also easy to verify from the listing of these B_i's that T satisfies condition (2) for a PMSS. Thus T is a PMSS. A similar argument shows that T' is also a PMSS. These are the only two PMSS's for this example.

Example 4.3 (Shapley and Scarf, 1974). *Let* $N = \{1, 2, 3\}$ *be the set of players who have the following preferences:*

(1) $2 \succ_1 3 \succ_1 1$

(2) $1 \sim_2 3 \succ_2 2$

(3) $2 \succ_3 1 \succ_3 3$

Here we see that there is only one nonempty set S for which $\cup_{i \in S} B_i(N) = S$, namely $S = N$. Hence the only possible "T_1" in a PMSS would be $T_1 = N$. Indeed, $T = \{N\}$ is the only PMSS in this example.

Proposition 4.4. *For every market* $\mathcal{M}$*, there exists at least one PMSS.*

Proof. We prove the Proposition by presenting an algorithm which takes as input any market $\mathcal{M}$, and returns a PMSS for that market:

<<*Algorithm* $\mathcal{PMSS}$ >>

Step 0 *(Initialization)* The market $\mathcal{M} = (N, \succeq)$ is given. Set $V = N$ and $k = 1$.

Step 1 *(Defining a digraph)* For each $j \in V$, find the set $B_j(V)$ of player i's most preferred items in V. Set $\Gamma_j = B_j(V)$ for each $j \in V$ and let G be the digraph $(V, \{\Gamma_j\}_{j \in V})$. [Note that $\Gamma_j \neq \emptyset$ for each $j \in V$.]

Step 2 *(Finding a minimal self-mapped set)* Find a minimal self-mapped set in digraph G by using the algorithm $\mathcal{MSMS}$. Let T_k denote the minimal self-mapped set.

Step 3 *(Updating)*

(3.1) Set $V = V \setminus T_k$.

(3.2) If $V = \emptyset$, then go to Step 4. Otherwise, set $k = k + 1$ and return to Step 1.

Step 4 *(End)* Let T be the ordered set $\{T_1, \ldots, T_{k^*}\}$ of the minimal self-mapped sets obtained hitherto. Here k^* is the index number of the last minimal self-mapped set. Then T is a PMSS for houseswapping market $(N, \succeq)$. HALT.//

The algorithm must terminate because on each iteration the subraction of the minimal self mapped set T_k shrinks the vertex set V. Also, note that by definition of V and $\Gamma_i = B_i(V)$, on iteration k the minimal self-mapped set T_k consists of all of its members' most preferred items in $N \setminus \bigcup_{l=1}^{k-1} T_l$. Hence $T = \{T_1, ..., T_{k^*}\}$ is in fact a PMSS. ∎

Lemma 4.5. *Algorithm $\mathcal{PMSS}$ has computational complexity $O(n^3)$.*

Proof. Consider any particular iteration of the algorithm. Step 1 takes $O(n^2)$ operations, while, by Lemma 2.7, running algorithm $\mathcal{MSMS}$ in Step 2 is $O(n^2)$. Step 3 is $O(n)$. Since $\mathcal{PMSS}$ terminates after no more than n iterations, it must have complexity $O(n^3)$. ∎

5. Segmentability of the Market

In this section we state necessary and sufficient conditions for a market to have a nonempty strict core.

Definition 5.1. *i) A* **top trading segmentation** (**TTS**) *of market $\mathcal{M}$ is a partition $T = \{T_1, T_2, \ldots, T_m\} \in \mathcal{P}$ that satisfies the following conditions:*

(1) for each $T_k \in T$, $B_i(N \setminus \cup_{l=1}^{k-1} T_l) \subseteq T_k$ for all $i \in T_k$,

(2) for each $T_k \in T$, there is no proper subset of T_k which satisfies (1) above.

(3) for each $T_k \in T$, there exists a T_k-allocation $\pi_k \in \Pi_{T_k}$ with $\pi_k(i) \in B_i(N \setminus \cup_{l=1}^{k-1} T_l)$ for all $i \in T_k$.

ii) A T_k-allocation in (3) above is called a **supporting T_k-allocation** *for* **segment** *T_k.*

The reader will note that conditions (1) and (2) above are almost by themselves the definition of PMSS (see Definition 4.1). In fact, we may more clearly see the relationship between the two concepts by noting the following:

Lemma 5.2. *If $T = \{T_1, T_2, \ldots, T_m\}$ is a TTS, then we have $\bigcup_{i \in T_k} B_i(N \setminus \cup_{l=1}^{k-1} T_l) = T_k$ for each $T_k \in T$.*

Proof. From comparing the Lemma's conclusion with condition (1) for a TTS, it is sufficient to show that for each k, $\bigcup_{i\in T_k} B_i(N\setminus \cup_{l=1}^{k-1} T_l)$ cannot be a *proper* subset of T_k. But this follows, due to the presence of the supporting T_k-allocation for segment T_k. ■

Corollary 5.3. *Every TTS is a PMSS. If T is a PMSS, then T is also a TTS if it satisfies condition (3) in the definition of TTS.*

Corollary 5.3 suggests that in order to check whether a given PMSS T is a TTS, all we need do is see if each of its segments T_k has a supporting T_k-allocation. To do this, define for each T_k the bipartite graph in which both parts have $|T_k|$ vertices, and an edge from i in the first part to j in the second part exists if and only if $j \in B_i(T_k)$. Now treat this as a "cardinality matching problem" (Lawler, p. 183,195). If the "maximal matching" contains $|T_k|$ edges, the edges give the supporting T_k-allocation for segment T_k. If it contains less than $|T_k|$ edges, there is no supporting T_k-allocation.

It is well-known that if both parts of a bipartite graph have m vertices, there are algorithms to solve the cardinality matching problem in $O(m^3)$ time (Lawler, p. 195). Hence, we may check if T_k has a supporting T_k-allocation in $O(|T_k|^3)$ time. This in turn implies that we may check if T is a TTS in $O(\sum_k |T_k|^3)$ time. But $\sum_k |T_k|^3 \leq (\sum_k |T_k|)^3 = n^3$, so we have

Lemma 5.4. *The computational complexity to check if a given PMSS is a TTS is $O(n^3)$.*

In section 4, we saw that we can think of a PMSS as a partition of players into groups, with lower-indexed groups being "better". The same interpretation holds for a TTS. Again, condition (1) means that for any player in any group, the items he prefers most among those *not in better groups* are *all* owned by players in his group. But in addition, condition (3) means that every player can get one of his most preferred items (among the items not in better groups) through a feasible exchange within his own group.

If a TTS exists in a market, then a player i within a certain group in the TTS would not have an incentive to trade with players in "worse" groups because such players' houses are strictly less preferred to what i can obtain by feasible exchange within his own group. Nor would i trade with players in better groups because by the same logic, such players would not want to trade with him. He will thus accomplish a (second) best trade within his own group. In this sense, the market is *segmented* into distinct groups.

We should note that in the case where no player exhibits indifference, the partition of the market into "top trading cycles" fits the definition of a TTS.

At this point it is instructive to revisit Examples 4.2 and 4.3. In Example 4.2, we see that PMSS $T = \{T_1, T_2, T_3\}$ is also a TTS, because the supporting T_k-allocations of segments T_k, $k = 1, 2, 3$, are those corresponding to the cycles $\{1, 2, 3, 1\}$, $\{4, 5, 4\}$, and $\{6, 6\}$ respectively. Note that in this example, there is exactly one strict core allocation, namely the N-allocation corresponding to the cycles $\{1, 2, 3, 1\}$, $\{4, 5, 4\}$, and $\{6, 6\}$. In the proof of Theorem 5.5 we will formalize this connection between strict core allocations and supporting allocations for TTS's.

In Example 4.3, we saw that there is only one PMSS, namely $T = \{N\} = \{\{1, 2, 3\}\}$. But it is clear that since $B_1(N) = B_3(N) = \{2\}$ and $B_2(N) = \{1, 3\}$, there is no N-allocation π_N with $\pi_N(i) \in B_i(N)$ for $i = 1, 2, 3$. Hence there are is no TTS in this market. We also remark that Shapley and Scarf pointed out that the strict core is empty in this example.

In the following, we show that the existence of a TTS, i.e. the **segmentability** of a market, is a necessary and sufficient condition for the strict core to be nonempty.

Theorem 5.5. *The strict core of market $\mathcal{M}(N, \succeq)$ is nonempty if and only if a TTS exists.*

Proof: (If part) We first prove that if a TTS exists, then the strict core is nonempty.

Let $T = \{T_1, T_2, \ldots, T_m\}$ be a TTS and $\pi_k \in \Pi_{T_k}$, $k = 1, \ldots, m$, supporting T_k-

allocations for the respective segments T_k. Then for each $k \in \{1, \ldots, m\}$,

$$T_k = \bigcup_{i \in T_k} B_i(N \setminus \cup_{l=1}^{k-1} T_l)$$

$$\pi_k(i) \in B_i(N \setminus \cup_{l=1}^{k-1} T_l) \qquad \text{for each} \;\; i \in T_k.$$

Let x be the allocation such that for each $i \in N$, $x(i) = \pi_k(i)$ if $i \in T_k$. We claim that x is a strict core allocation. To prove this claim, we suppose that some coalition S weakly blocks x via the S-allocation π_S. There are two cases: first, if π_S maps each member of S to a house within that member's "group" of the TTS, then it is clearly impossible for π_S to assign any member of S a better house than he gets from x; second, if π_S maps some member of S to a house from a *different* group of the TTS, then it must assign to some member $i \in S$ a house from a *worse* group. But then necessarily $x(i) \succ_i \pi_S(i)$, so it is impossible for S to weakly block x via π_S after all. ▪

(Only-if part) We suppose that the strict core is nonempty, and show this implies the existence of a TTS.

Let x be any strict core allocation and $T = \{T_1, T_2, \ldots, T_m\}$ be a PMSS of the market (a PMSS exists by virtue of Proposition 4.4). We now make the following claim:

Claim 5.6. *For each $T_k \in T$, $x(i) \in B_i(N \setminus \cup_{l=1}^{k-1} T_l)$ for each $i \in T_k$.*

Proof. To show the Claim, assume on the contrary that there exist $k^* \in \{1, \ldots, m\}$ and player $i^* \in T_{k^*}$ such that

$$x(i) \in B_i(N \setminus \cup_{l=1}^{k-1} T_l) \text{ for each } i \in T_k \text{ with } k \in \{1, \ldots, k^* - 1\}, \text{ and} \tag{3}$$

$$x(i^*) \notin B_{i^*}(N \setminus \cup_{l=1}^{k^*-1} T_l). \tag{4}$$

It follows from condition (1) of Definition 5.1 and (3) that for each $k \in \{1, \ldots, k^* - 1\}$,

$$x(T_k) = \bigcup_{i \in T_k} \{x(i)\} \subseteq \bigcup_{i \in T_k} B_i(N \setminus \cup_{l=1}^{k-1} T_l) \subseteq T_k.$$

Since x is a permutation of N, it must hold that

$$x(T_k) = T_k \qquad \text{for each } k \in \{1, \ldots, k^* - 1\}.$$

So, for $i \in T_{k^*}$, we have $x(i) \in \cup_{l=k^*}^{m} T_l = N \setminus \cup_{l=1}^{k^*-1} T_l$. Therefore,

$$\text{for each } i \in T_{k^*}, \quad b \succeq_i x(i) \quad \text{for } b \in B_i(N \setminus \cup_{l=1}^{k^*-1} T_l). \tag{5}$$

Now define a digraph $G = (V, \{\Gamma_i\}_{i \in V})$ by $V := N \setminus \cup_{l=1}^{k^*-1} T_l$ and $\Gamma_i = B_i(V)$ for each $i \in V$. We see that $T_{k^*} \subseteq V$ and furthermore that T_{k^*} is a minimal self-mapped set of G. Since each Γ_i $(= B_i(V))$, $i \in V$, is nonempty, we can apply Corollary 2.5 to conclude that there exists a cycle $C \subseteq T_{k^*}$ (see Footnote 8 in Section 2) as a subgraph of G, with $i^* \in C$. Let S be set of vertices in C, and let π_S be the corresponding simple S-allocation to C (see the discussion in Section 3). We have

$$\pi_S(i) \in B_i(V) = B_i(N \setminus \cup_{l=1}^{k^*-1} T) \quad \text{for all } i \in S, \text{ including } i = i^*. \tag{6}$$

It follows now from (4), (5), and (6) that

$$\pi_S(i^*) \succ_{i^*} x(i^*), \text{ and}$$

$$\pi_S(i) \succeq_i x(i) \text{ for each } i \in S,$$

namely, strict core allocation x is weakly blocked by coalition S. This is impossible. Thus the Claim must hold. ∎

From Claim 5.6 and the fact that each $T_k \in T$ satisfies condition (1) of Definition 4.1, we see that for each $i \in T_k$,

$$x(i) \in B_i(N \setminus \cup_{l=1}^{k-1} T_k) \subseteq T_k.$$

Hence, since x is an allocation, we have $x(T_k) = T_k$. So, for each T_k, we can use x restricted to T_k as a supporting T_k-allocation for segment T_k. This, together with the fact that T is a PMSS, shows that partition T is a TTS. The proof of Theorem 5.5 is complete. ∎ ∎

Corollary 5.7. *Suppose market $\mathcal{M}(N, \succeq)$ has a TTS $T = \{T_1, ..., T_m\}$, with supporting T_k-allocations for each segment $T_k \in T$ that define an allocation x. Then x is a strict core allocation of $\mathcal{M}$. Conversely, if x is a strict core allocation of market $\mathcal{M}(N, \succeq)$, then each PMSS is actually a TTS, in which the supporting T_k-allocations are given by x.*

Proof. This is clear from the proof of Theorem 5.5. ∎

Corollary 5.8. *Let $\mathcal{M}(N, \succeq)$ be a houseswapping market. Then exactly one of the following two conditions must hold:*

(1) No PMSS is a TTS.

(2) Every PMSS is a TTS.

Proof. If the strict core of $\mathcal{M}$ is empty, Theorem 5.5 tells us that there are no TTS's, so no PMSS is a TTS. Alternatively, if the strict core of $\mathcal{M}$ is nonempty, then Corollary 5.7 tells us that every PMSS is a TTS. ■

6. An $O(n^3)$ Algorithm for Strict Core Analysis

It is now a simple matter to present an algorithm which takes as input a houseswapping market $\mathcal{M}$, and outputs a strict core allocation or else a report that the strict core is empty:

<<*Algorithm* $\mathcal{STRICTCORE}$ >>

(1) Run algorithm $\mathcal{PMSS}$ to generate a PMSS $T = \{T_1, ..., T_m\}$.

(2) Determine if T is a TTS by checking if every T_k has a supporting T_k-allocation π_k.

(A) If "yes", by Corrolary 5.7 the allocation x defined by the π_k's is in the strict core.

(B) If "no", then T is not a TTS, Corollary 5.8 implies that there are no TTS's, and so the strict core is empty.

Theorem 6.1. *The computational complexity of algorithm $\mathcal{STRICTCORE}$ is $O(n^3)$.*

Proof. It is clear that Step 1 and Step 2 both are $O(n^3)$ (Lemmas 4.5 and 5.4), so the Theorem follows.

7. Another Result regarding PMSS's

We now state another lemma concerning PMSS's, which will enable us to independently derive a result of Wako (1991) and Ma (1994):

Lemma 7.1. *Suppose that $T = \{T_1, \ldots, T_m\}$ and $T' = \{T'_1, \ldots, T'_{m'}\}$ are PMSS's of a given market $\mathcal{M} = (N, \succeq)$. Then*

(1) $m = m'$,

(2) for each $T_k \in T$*, there exists a unique* $T'_{l_k} \in T'$ *with* $T_k = T'_{l_k}$.

In short, Lemma 7.1 says that any two PMSS's must consist of the same sets, with only the *order* of those sets possibly being different. This again generalizes what we know to be true in the no-indifference case, where the listing of top trading cycles is unique, except possibly for changes in the order of the cycles.

In Example 4.2, it is easy to see that (1) and (2) hold, just from noticing that the two PMSS's $T = \{T_1 = \{1, 2, 3\},\ T_2 = \{4, 5\},\ T_3 = \{6\}\}$ and $T' = \{T'_1 = \{1, 2, 3\},\ T'_2 = \{6\},\ T'_3 = \{4, 5\}\}$ consist of the same three sets.

Proof of Lemma 7.1. Before proceeding with the proof, we need to define a new concept. Suppose $\mathcal{M} = (N, \succeq)$ is a houseswapping market, and suppose $S \subseteq N$. Then the **submarket** $\mathcal{M}_{-S}$ is the market $(N', \succeq')$, where $N' = N \setminus S$ and $\succeq'$ is formed by restricting $\succeq$ to the elements of N'. It is clear that

Proposition 7.2. *Suppose* $T = \{T_1, ..., T_m\}$ *is a PMSS in market* $\mathcal{M}$*, and suppose* $k \in 1, ..., m$*. Then* $T \setminus T_k$ *is a PMSS of submarket* $\mathcal{M}_{-T_k}$.

Now we prove Lemma 7.1 by induction on m. If $m = 1$, then it is clear that the only minimal self-mapped set of digraph $(V, \{\Gamma_i\}_{i \in V}) = (N, \{B_i(N)\}_{i \in N})$ is N itself, and so the conclusion follows immediately.

So now suppose that the Lemma is true for $m \leq k$, and we are given two PMSS's T and T', with $T = \{T_1, ..., T_{k+1}\}$. First suppose $T_1 = T'_1$. Then $T \setminus T_1$ and $T' \setminus T'_1$ are both PMSS's for market $\mathcal{M}_{-T_1}$, and so by the inductive hypothesis consist of the same sets (except possibly ordered differently). The conclusion follows.

Now suppose $T_1 \neq T'_1$. Then, since T_1 and T'_1 are both minimal self-mapped sets in digraph $(N, \{B_i(N)\}_{i \in N})$, Corollary 2.6 gives $T_1 \cap T'_1 = \emptyset$. Thus T'_1 is also a minimal self-mapped set in digraph $(N \setminus T_1, \{B_i(N \setminus T_1)\}_{i \in N \setminus T_1})$. This in turn implies that, using algorithm $\mathcal{PMSS}$[9] we can form a PMSS T'' for market $\mathcal{M}_{-T_1}$, with $T''_1 = T'_1$. On the other hand, the set $T''' = \{T_2, ..., T_{k+1}\}$ is also a PMSS for market $\mathcal{M}_{-T_1}$, and T''' consists of k elements. Hence, by the inductive hypothesis T'' and T''' consist of the same sets, except possibly ordered differently. In particular, T'_1 is an element of T'''.

Since $T''' \subseteq T$, $T'_1 \in T'''$ implies $T'_1 \in T$. Hence, by Proposition 7.2 $T \setminus T'_1$ is a PMSS for market $\mathcal{M}_{-T'_1}$, and this PMSS contains k elements. But $T' \setminus T'_1$ is also a PMSS for $\mathcal{M}_{-T'_1}$ – so by the inductive hypothesis again the PMSS's $T \setminus T'_1$ and $T' \setminus T'_1$ must consist of the same elements. Hence T and T' also consist of the same elements. ∎

Definition 7.3. *Let $T = \{T_1, \ldots, T_m\}$ be a PMSS of market $\mathcal{M}(N, \succeq)$. The* **top preference digraph** $G(\mathcal{M})$ *of $\mathcal{M}$ is the digraph given by $V := N$ and $\Gamma_i := B_i(N \setminus \cup_{l=1}^{k-1} T_l)$ for each $i \in N$ with $i \in T_k \in T$.*

We note that the concept of top preference digraph is well-defined due to Proposition 4.4 and Lemma 7.1. The figures below show the top preference digraphs for Examples 4.2 and 4.3:

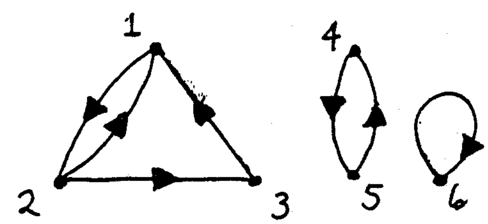

FIGURE 1

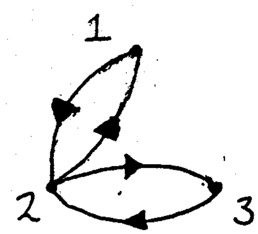

FIGURE 2.

We also observe that a market has a TTS (or, equivalently, a strict core allocation) if and only if its top preference digraph contains a subgraph which corresponds to an

[9] Actually, we'd be using a modification of algorithm $\mathcal{PMSS}$, where, instead of using algorithm $\mathcal{MSMS}$ to find an arbitrary minimal self-mapped set on the first pass through Step 2, we'd just take T'_1.

N-allocation. Such a subgraph exists in Figure 1, but does not in Figure 2.

Theorem 7.4 (Wako 1991, Ma 1994). *Let x and y be two distinct strict core allocations of market $\mathcal{M}$. Then $x(i) \sim_i y(i)$ for all $i \in N$.*

Proof. From Corollary 5.7, strict core allocations x and y have their respective TTS's, which in turn are PMSS's. Lemma 7.1 implies these PMSS's are the same list of sets, say $\{T_1, ..., T_m\}$, but not in the same order. Now choose any $i \in N$, and let T_k^i be the common element of the two PMSS's which contains i. From Claim 5.6, we have $x(i) \in T_k^i$ and $y(i) \in T_k^i$, and so $x(i) \in B_i(T_k^i)$ and $y(i) \in B_i(T_k^i)$. Hence $x(i) \sim_i y(i)$. ∎

8. Strict Core and Linear Inequalities

In the study of Gale-Shapley's (1962) marriage game, some of the most interesting theoretical results are those which use polyhedral combinatorics to characterize the core. In particular, given any marriage game $\mathcal{G}$ one may define a "marriage polytope" $P(\mathcal{G})$ whose extreme points exactly correspond (in the natural way) with the elements of the game's core (see Vande Vate 1989, Rothblum 1991, Roth-Rothblum-Vande Vate 1993, and Abeledo-Rothblum 1994).

In this section we extend this idea to the realm of houseswapping games. Specifically, given a houseswapping market $\mathcal{M}$, we define a "corresponding linear inequality system" $CLIS(\mathcal{M})$. We show $CLIS(\mathcal{M})$ is feasible if and only if the strict core of $\mathcal{M}$ is nonempty. Furthermore, if $CLIS(\mathcal{M})$ is feasible, the set of extreme points of the feasible set consists entirely of 0-1 integral solutions, and exactly corresponds to the set of strict core allocations.

Let $\mathcal{M}(N, \succeq)$ be a houseswapping market. Denote by 2^N the set of nonempty subsets of N, and recall that Π_S^0 denotes the set of simple S-allocations for a given $S \in 2^N$ (see Section 3). Now define the **corresponding linear inequality system** for $\mathcal{M}$, or

$CLIS(\mathcal{M})$, to be the linear inequality system in $x = (x_{ij})_{i \in N, j \in N}$ given by:

$$\sum_{i \in S} \left(\sum_{j : j \succ_i \pi(i)} x_{ij} + \frac{1}{|S|} \sum_{j : j \sim_i \pi(i)} x_{ij} \right) \geq 1 \quad \text{for each } S \in 2^N \text{ and } \pi \in \Pi_S^0 \tag{7a}$$

$$\sum_{i \in N} x_{ij} = 1 \quad \text{for each } j \in N \tag{7b}$$

$$\sum_{j \in N} x_{ij} = 1 \quad \text{for each } i \in N \tag{7c}$$

$$x_{ij} \geq 0 \quad \text{for each } i, j \in N. \tag{7d}$$

We see that any integer vector x that satisfies (7b), (7c) and (7d) has the following properties:

(1) $x_{ij} =$ *0 or 1 for each* $(i, j) \in N \times N$

(2) for each $i \in N$, *there is a unique* $j \in N$ *with* $x_{ij} = 1$

(3) for each $j \in N$, *there is a unique* $i \in N$ *with* $x_{ij} = 1$.

These properties show that x can be regarded as the N-allocation that maps each $i \in N$ to the unique element $j \in N$ with $x_{ij} = 1$. We call the conditions (7b),(7c) and (7d) the **allocation conditions** or **permutation conditions**, and an integer vector satisfying these conditions an **allocation vector** or a **permutation vector** depending on the context.[10] Given a permutation vector $x = (x_{ij})_{i \in N, j \in N}$, we can write it in *function form* by defining each $x(i)$, $i \in N$, to be the element $j \in N$ with $x_{ij} = 1$. Conversely, given $x \in \Pi_N$, we can represent it in *vector form* by defining $x_{ij} := 1$ for $(i, j) \in N \times N$ with $x(i) = j$ and $x_{ij} := 0$ for $(i, j) \in N \times N$ with $x(i) \neq j$. We use both notations together in an effort to clarify our exposition.

Lemma 8.1. *Suppose that* $y = (y_{ij})_{i \in N, j \in N}$ *is a strict core allocation vector of a market* $\mathcal{M} = (N, \succeq)$. *Then* y *is a feasible solution of* $CLIS(\mathcal{M})$.

[10] In the economics/game theory context, the term "allocation" is natural because x represents a way to redistribute the indivisible goods in a market. In a mathematics context, the term "permutation" is natural because x represents a bijection of the set $N = \{1, ..., n\}$ to itself.

Proof. Since y is an allocation vector, it satisfies the permutation conditions, i.e., (7b), (7c) and (7d). Thus the proof is complete if we can show that (7a) holds with y. We prove this by contradiction.

Suppose that there exist a coalition $S \in 2^N$ and a simple S-allocation $\pi \in \Pi_S^0$ with

$$\sum_{i \in S} \left(\sum_{j: j \succ_i \pi(i)} y_{ij} + \frac{1}{|S|} \sum_{j: j \sim_i \pi(i)} y_{ij} \right) < 1. \tag{8}$$

We see that it is impossible for (8) to hold if there exists $i \in S$, $j \succ_i \pi(i)$ with $y_{ij} = 1$. Hence, since y is a 0-1 vector, we have $y_{ij} = 0$ for all (i, j) for which $j \succ_i \pi(i)$. But this implies that (switching to the function form)

$$\pi(i) \succeq_i y(i) \text{ for all } i \in S. \tag{9}$$

In addition, we note that it is impossible for $\pi(i) \sim_i y(i)$ for *all* $i \in S$, because in that case (8) would hold with *equality*. Hence there must be some $i^* \in S$ for which $\pi(i^*) \succ_{i^*} y(i^*)$. But this in combination with (9) implies that coalition S weakly blocks y via S-allocation π, so we have a contradiction. ■

Lemma 8.2. *Suppose that x is a feasible solution of the CLIS of market $\mathcal{M}(N, \succeq)$. Then x is a convex combination of strict core allocation vectors.*

Proof. [1] First, let $T = \{T_1, \ldots, T_m\}$ be a PMSS for market $\mathcal{M}$; a PMSS exists by virtue of Proposition 4.4. Next, since x (even though not necessarily integral) satisfies the permutation conditions, it follows from the Birkhoff-Von Neumann Theorem (in Birkhoff, 1946) that there exist a finite set of permutation vectors $F = \{x^1, \ldots, x^K\}$ and a strictly positive probability vector $\lambda = (\lambda_1, \ldots, \lambda_K)$ with $x = \Sigma_{k=1}^K \lambda_k x^k$. The proof will be complete if we show that each $x^k \in F$ is a strict core allocation vector.

[2] To show this we suppose that there is some nonempty subset $\hat{K}$ of $\{1, ..., K\}$ for which x^k is *not* a strict core vector if $k \in \hat{K}$, but x^k *is* a strict core vector if $k \notin \hat{K}$. Then, for

each $k \in \hat{K}$, we claim there exists $l^*(k) \in \{1, ..., m\}$ such that

$$\begin{aligned} & x^k(t_k) \notin B_{t_k}(N \setminus \cup_{h=1}^{l^*(k)-1} T_h) \text{ for some } t_k \in T_{l^*(k)}, \text{ and} \\ & x^k(t) \in B_t(N \setminus \cup_{h=1}^{l-1} T_h) \text{ for all } l \in \{1, ..., l^*(k) - 1\} \text{ and } t \in T_l \end{aligned} \tag{10}$$

To see (10), suppose it were not true, i.e.,

$$x^k(t) \in B_t(N \setminus \cup_{h=1}^{l-1} T_h) \text{ for all } l \in \{1, ..., m\} \text{ and } t \in T_l. \tag{11}$$

Since T is a PMSS, we know $B_t(N \setminus \cup_{h=1}^{l-1} T_h) \subseteq T_l$ for each $l \in \{1, ..., m\}$ and $t \in T_l$. By (11), we would have $x^k(t) \in T_l$ for all $l \in \{1, ..., m\}$ and $t \in T_l$. Hence $x^k(T_l) = T_l$ for each $l \in \{1, ..., m\}$. But this in combination with (11) would imply that T was a TTS whose segments are supported by allocation x^k. Thus it follows from Corollary 5.7 that x^k was a strict core allocation. This is a contradiction, and so (10) must be true.

[3] Now let $k^* \in \hat{K}$ be an element with $l^*(k^*) = \min_{k \in \hat{K}} l^*(k)$. Without loss of generality, suppose $k^* = 1$ and denote $l^* := l^*(1)$ for simplicity. We then see the following:

$$x^k(i) \in B_i(N \setminus \cup_{h=1}^{l-1} T_h) \text{ for each } k \in \{1, ..., K\},\ l \in \{1, \ldots, l^* - 1\} \text{ and } i \in T_l \tag{12}$$

$$x^1(i^*) \notin B_{i^*}(N \setminus \cup_{l=1}^{l^*-1} T_l) \text{ for some } i^* \in T_{l^*} \tag{13}$$

$$x^k(T_l) = T_l \text{ for each } k \in \{1, ..., K\} \text{ and } l \in \{1, \ldots, l^* - 1\}. \tag{14}$$

[(14) holds by essentially applying the argument following (11) above to each k and l.]

[4] It follows from Corollary 2.5 that there exists a set $S^* \subseteq T_{l^*}$ and a simple S^*-allocation $\pi^* \in \Pi^0_{S^*}$ such that

$$i^* \in S^* \quad \text{and} \quad \pi^*(i) \in B_i(N \setminus \cup_{l=1}^{l^*-1} T_l) \text{ for each } i \in S^*. \tag{15}$$

Now (12), (14), and the second statement in (15) together imply that for each $x^k \in F$,

$$x^k_{ij} = 0 \quad \text{for all } i \in S^* \text{ and } j \in N \text{ with } j \succ_i \pi^*(i). \tag{16}$$

And, (13), (14), and (16) together imply that

$$x^1_{i^*j} = 0 \quad \text{for all } j \in N \text{ with } j \succeq_{i^*} \pi^*(i^*). \tag{17}$$

[5] From (16) and the fact that each x^k satisfies (7c) and (7d), we obtain the following for each $x^k \in F = \{x^1, ..., x^K\}$:

$$\begin{aligned}\sum_{i \in S^*} \left(\sum_{j: j \succ_i \pi^*(i)} x^k_{ij} + \frac{1}{|S^*|} \sum_{j: j \sim_i \pi^*(i)} x^k_{ij} \right) &= \sum_{i \in S^*} \frac{1}{|S^*|} \sum_{j: j \sim_i \pi^*(i)} x^k_{ij} \\ &\leq \sum_{i \in S^*} \frac{1}{|S^*|} \sum_{j \in N} x^k_{ij} \;=\; 1. \end{aligned} \tag{18}$$

[6] And, using (17) as well, we obtain a *strict* inequality for x^1, i.e.,

$$\begin{aligned}\sum_{i \in S^*} \left(\sum_{j: j \succ_i \pi^*(i)} x^1_{ij} + \frac{1}{|S^*|} \sum_{j: j \sim_i \pi^*(i)} x^1_{ij} \right) &= \sum_{i \in S^*} \frac{1}{|S^*|} \sum_{j: j \sim_i \pi^*(i)} x^1_{ij} \\ &= \frac{1}{|S^*|} \left(\sum_{j: j \sim_{i^*} \pi^*(i^*)} x^1_{i^*j} + \sum_{i \in S^*: i \neq i^*} \sum_{j: j \sim_i \pi^*(i)} x^1_{ij} \right) \\ &\leq \frac{1}{|S^*|} \left(0 + (|S^*| - 1) \right) \\ &< 1. \end{aligned} \tag{19}$$

[7] Since $x = \Sigma_{k=1}^K \lambda_k x^k$, it follows from inequalities (18) and (19) that

$$\sum_{i \in S^*} \left(\sum_{j: j \succ_i \pi^*(i)} x_{ij} + \frac{1}{|S^*|} \sum_{j: j \sim_i \pi^*(i)} x_{ij} \right) < 1.$$

Thus x does not satisfy inequality (7a) with $S = S^*$ and $\pi = \pi^*$. However, this contradicts that x is a feasible solution of $CLIS(\mathcal{M})$. The contradiction is due to the assumption that at least one $x^k \in F$ is not a strict core allocation vector. Therefore each $x^k \in F$ must be a strict core allocation vector. As indicated at the end of [1], this completes the proof. ■

We should note that the proof for Lemma 8.2 does not assume the existence of a strict core allocation, but it only assumes the existence of a feasible solution of $CLIS(\mathcal{M})$. Hence it is clear from Lemma 8.2 that the following corollary holds:

Corollary 8.3. *If the CLIS of a market $\mathcal{M}(N, \succeq)$ is feasible, then the strict core of market $\mathcal{M}$ is nonempty.*

Lemmas 8.1 and 8.2 and the corollary above imply the following theorem.

Theorem 8.4. *Let $\mathcal{M}$ be a houseswapping market. Then*

(1) the strict core of $\mathcal{M}$ is nonempty if and only if $CLIS(\mathcal{M})$ is feasible;

(2) the feasible set of $CLIS(\mathcal{M})$ is the convex hull of the set of strict core allocation vectors.

Proof. First, claim (1) follows from Lemma 8.1 and Corollary 8.3. Next we prove claim (2). Lemma 8.1 implies that the convex hull of the set of strict core allocation vectors is a subset of the feasible set of the $CLIS$. The inverse inclusion follows from Lemma 8.2, since any (fractional) feasible solution of the $CLIS$ can be represented by a convex combination of strict core allocation vectors. Hence we obtain claim (2). ■

We remark that feasible (but not necessarily integral) solutions to (7a)-(7d) can be thought of as "fractional strict core outcomes", much in the spirit of Roth-Rothblum-Vande Vate (1993) for marriage games. The idea is that if x is such a solution, we can think of x_{ij} as the percentage of time (in a time-sharing scenario) or the probability (in a lottery) that i receives house j.[11]

9. The "Regular" Core

At this point one might wonder whether a result similar to Theorem 8.4 might hold for the "regular" core (see Section 3 for the definition). For the core of market $\mathcal{M}$, a natural

[11] In the Roth-Rothblum-Vande Vate paper, the authors go on to show that under a suitably defined partial order, the set of fractional (strict) core matchings is a lattice. This partial order is interesting in the two-sided setup because the players in general rank different core outcomes differently. However, in our one-sided setup the players each rank all strict core outcomes the same (Theorem 7.4), so studying such a lattice structure is not an interesting problem.

analogue of the $CLIS$ (7a)-(7d) would be the following linear inequality system:

$$\sum_{i \in S} \left(\sum_{j : j \succeq_i \pi(i)} x_{ij} \right) \geq 1 \quad \text{for each } S \in 2^N \text{ and } \pi \in \Pi_S^0 \tag{20a}$$

$$\sum_{i \in N} x_{ij} = 1 \quad \text{for each } j \in N \tag{20b}$$

$$\sum_{j \in N} x_{ij} = 1 \quad \text{for each } i \in N \tag{20c}$$

$$x_{ij} \geq 0 \quad \text{for each } i, j \in N. \tag{20d}$$

Proposition 9.1. *A vector $y = (y_{ij})_{i \in N, j \in N}$ corresponds to a core allocation of market $\mathcal{M}(N, \succeq)$ if and only if y is an integral feasible solution of linear inequality system (20a)-(20d).*

Proof. The "if" part is proved by a similar argument to the proof of Lemma 8.1. To prove the "only-if" part, suppose on the contrary that there exists a integral feasible solution y to linear inequality system (20a)-(20d) which is not a core allocation vector. Since y is an integer vector satisfying (20b)-(20d), y is an allocation vector. Since y does not give a core allocation, however, there must exist a coalition S^* and a simple S^*-allocation $\pi^* \in \Pi_{S^*}$ such that (writing in function form) $\pi^*(i) \succ_i y(i)$ for all $i \in S^*$. Since y is a permutation vector, this implies that $y_{ij} = 0$ for all $(i, j) \in S^* \times N$ with $j \succeq \pi^*(i)$. However, this means that y does not satisfy (20a) for S^* and π^*, namely, $\sum_{i \in S^*} \left(\sum_{j : j \succeq_i \pi^*(i)} y_{ij} \right) = 0 < 1$. This is a contradiction. Hence y must be a core allocation vector. ∎

Proposition 9.1 means that each of the core allocation vectors corresponds to an extreme point of the feasible region of system (20a)-(20d). However, it is *not* true that all of the extreme points of the feasible region necessarily correspond to core allocations. We show this by the following example:

Example 9.2. *Let $N = \{1, 2, 3\}$ be the set of players who have the following preferences:*

(1) $3 \succ_1 2 \succ_1 1$

(2) $1 \succ_2 3 \succ_2 2$

(3) $2 \succ_3 1 \succ_3 3$.

The system (20a)-(20d) in this example turns out to be the allocation conditions plus the following inequalities:

$$x_{11} + x_{12} + x_{13} \geq 1$$
$$x_{21} + x_{23} + x_{22} \geq 1$$
$$x_{32} + x_{31} + x_{33} \geq 1$$
$$x_{12} + x_{13} + x_{21} \geq 1$$
$$x_{13} + x_{32} + x_{31} \geq 1$$
$$x_{21} + x_{23} + x_{32} \geq 1$$
$$x_{13} + x_{21} + x_{32} \geq 1$$
$$x_{13} + x_{21} + x_{32} + x_{12} + x_{23} + x_{31} \geq 1$$

In this example there is a *unique* core allocation, which is described by allocation vector y with $y_{13} = y_{21} = y_{32} = 1$ and $y_{11} = y_{12} = y_{22} = y_{23} = y_{31} = y_{33} = 0$. Needless to say, vector y satisfies the system above. Hence, if there were an exact correspondence between the extreme points and the core allocations, then y would be the *only* feasible solution to the system above. However, we can easily see that isn't the case, because vectors such as $y_{11} = y_{12} = ... = y_{33} = \frac{1}{3}$ are also feasible.

Acknowledgements

The authors thank TS Michael and Alvin Roth for helpful discussions, and also thank Jeff Mortensen for technical assistance. We especially thank an anonymous referee, who suggested the ideas behind the result of Section 6.

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