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Toward a Theory of Reinsurance and Retrocession

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REINSURANCE AND RETROCESSION

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Abstract

In recent years, the global reinsurance market has undergone rapid evolution. A series of mergers and acquisitions has led to dramatic consolidation, while the development of insurance-based securities has begun offering new ways to enhance and/or compete with traditional reinsurance products. In this article, we employ the insurance market game framework of Powers, Shubik, and Yao (1994, 1998) and Powers and Shubik (1999) to study the design of an optimal reinsurance/retrocession market. Using price in the primary insurance market as our primary objective function, we analyze market configuration in terms of both the need for additional levels of reinsurance/retrocession, and the optimal number of firms at a given level.

1. Overview

In recent years, the global reinsurance market has undergone rapid evolution. A series of mergers and acquisitions has led to dramatic consolidation, while the development of insurance-based securities has begun offering new ways to enhance and/or compete with traditional reinsurance products. These changes naturally give rise to questions such as:

- Is a reinsurance market really necessary?
- What is the optimal number of reinsurers?
- What is the role of retrocession (i.e., the insuring of reinsurers)?
- Is there a theoretical upper bound on the optimal number of retrocession levels?

As a practical matter, it is difficult to assess the effects of the number and size of primary insurers on the price and availability of insurance in a given market. This is because empirical studies of scale economies can evaluate only the relative efficiencies of firms of different sizes under a fixed market configuration, and theoretical studies of market equilibrium are often restricted by traditional assumptions of competitive equilibrium with infinite numbers of buyers and sellers. The problem for reinsurance and retrocession markets is magnified by even greater limitations on industry data, and the general lack of a formal modeling structure.

1.1. Previous Work

In Powers, Shubik, and Yao (1994, 1998) and Powers and Shubik (1999), we proposed a game-theoretic model to study various effects of scale in a primary insurance market. Unlike conventional equilibrium-analysis models, in which buyers and sellers are assumed to be price-takers, our game-theoretic model permitted the analysis of market equilibrium with arbitrary numbers of buyers and sellers, so that marginal changes in competitive forces and insurer solvency could be studied as the numbers of players changed.

In our previous work, we were able to show that, under certain assumptions, there is a natural tradeoff between the positive and negative aspects of increasing the number of firms in a market with a fixed amount of capital. As the number of firms increases, the weakening of the oligopolistic structure of the market improves efficiency, causing price to decrease and customers to purchase more insurance. However, the increasing number of firms also diminishes the "quality" of insurance (by lowering the average capital per firm, thereby increasing the probability of insurer default) eventually causing the customers to purchase less insurance.[1] These two opposing influences determine an optimal number of firms– in terms of maximizing the amount of insurance purchased– when the marginal changes are equalized.

1.2. New Results

A natural way in which the dangers of default can be ameliorated, yet competition can still be preserved, is to introduce a level of reinsurers, even if the total capital invested in insurance/reinsurance underwriting remains the same. In this article, we develop a formal game-theoretic model to examine the potential value of

[1] A further effect of increasing the number of insurers is that the sellers restrict the amount of insurance they are willing to sell as the price continues to decrease.

adding a reinsurance structure to the insurance industry.

Specifically, we consider a market game, $G(r)$, with one level of primary insurance and r levels of reinsurance. At each level, we assume a simple, symmetric model in order to preserve analytical tractability. However, we are confident that our qualitative results are independent of this simplification, and we note that it is a straightforward matter to simulate results for non-symmetric cases.

Because of the complex notation and modeling throughout much of the rest of the text, we summarize our principal results in this section, as follows:

- Theorem 1 shows that, for an insurance/reinsurance market with an endogenously imposed configuration, both price and quantity decrease over the reinsurance levels $l = 1, 2, \ldots, r$.
- Corollary 1 shows that, for the insurance/reinsurance market of Theorem 1, with risk neutral reinsurers at and above reinsurance level L, equilibria exist up to, but not above, this level.
- One sufficient condition for the desirability of introducing a reinsurance level imposes a bounded interval on the primary insurer's risk aversion coefficient, $s_0 = s$; this interval tends to shrink if the exogenous probability of the primary insurer's insolvency is positive, or the losses are catastrophic (i.e., perfectly correlated) in nature.
- Given that the reinsurer's risk aversion coefficient, s_l, decreases over the reinsurance level, l, approaching zero (i.e., risk neutrality) as $l \to \infty$, it follows that

the sufficient “reinsurance desirability” condition eventually imposes an upper bound on the optimal number of reinsurance levels.

- A simple numerical criterion for the optimal saturation level of a reinsurance market is given by a monotonically increasing, concave-downward function of the number of primary insurers, each point of which is found as the solution to a cubic equation in the number of reinsurers.
- Corollary 2 reveals that, for the insurance / reinsurance market of Corollary 1, in which insurers / reinsurers are able to coordinate their bids and offers, equilibria do exist at levels above L, and these markets are characterized by the purchase of “over-insurance”.

2. Statement of the Problem

There are two major difficulties inherent in the study of reinsurance markets– difficulties that hinder both the collection of empirical data and the development of appropriate theoretical models. First, reinsurance is, and traditionally has been, the most international of insurance markets, with primary insurers in one country often purchasing reinsurance from both domestic and alien reinsurers. As a result, it is difficult to isolate data relating to transactions within only one country, and it is also difficult to obtain complete and consistent information regarding the global reinsurance market as a whole. A second problem is the hazy line that exists between the primary insurance and reinsurance markets in many countries, where large primary insurers frequently assume reinsurance business from other primary insurers.

2.1. U.S. Insurance/Reinsurance Markets

To simplify matters for our current study, we will restrict attention to the relatively tractable relationship between domestic U.S. primary property-liability (P-L) insurers and domestic U.S. "professional" reinsurers (i.e., reinsurers that write no primary insurance business of their own). In 1996, these combined insurance/reinsurance markets consisted of approximately 3,300 primary insurers[2] writing $250 billion in premiums, and 72 reinsurers (belonging to 65 reinsurance groups), writing $19 billion in premiums.[3] We note that this reinsurance market represents only about 20 percent of the global reinsurance market, and, most notably, excludes the London and Bermuda markets, which provide substantial capacity for U.S. primary insurers.[4]

In the past fifteen years, the reinsurance market has evolved rapidly. Mergers and acquisitions have led to dramatic consolidation, reflected in a 26 percent decline in the number of domestic U.S. reinsurers (from 97 to 72) in the period from 1985 to 1996.[5] Over the same time period, the (premium-volume) market share of the 10 largest reinsurers grew by 12.5 percent (from 60 percent to 65 percent of the total market).[6]

Since the early 1990s, the development of insurance-based securities, including various property catastrophe indexes (see Powers and Powers, 1997) and catastrophe

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[2] See Insurance Information Institute (1996), p. 5.
[3] See Conning and Company (1997), p. 30.
[4] In a sense, we are effectively making the coarse assumption that– for purposes of our analysis– the relationship between the global primary P-L market and the global professional reinsurance market is similar to that between the corresponding U.S. markets, apart from constant factors to recognize both the greater volume of the global markets, and the disproportionately smaller U.S. reinsurance market.
[5] See Conning and Company (1997), p. 22.

bonds (e.g., Nationwide Mutual's $400 million offering of 1995 and USAA/Residential Re's $477 million offering of 1997) have provided novel alternatives to traditional reinsurance products. The increasing viability and popularity of these alternative products is undoubtedly one competitive force underlying the consolidation of the traditional reinsurance market.

2.2. Studies of Insurance/Reinsurance Market Equilibrium

The theoretical study of insurance and reinsurance market equilibrium has been carried out under a variety of models. The earliest formal results were given by Arrow (1953) and Debreu (1953), who used contingent space to study economic equilibrium in a simple risk exchange model with two risk averse parties. By using contingent space, these authors were able to extend certain fundamental results of economic equilibrium from an exchange of goods to an exchange of risks. Specifically, they showed that competitive equilibrium exists, and proved that both the first and the second social welfare theorems hold in an economy with uncertainty. In other words, competitive equilibrium is Pareto optimal, and every Pareto optimal solution can be supported by a competitive equilibrium through the redistribution of endowments.

Though contingent space provides an elegant framework for economists to analyze uncertainty, it is far removed from the reality of most insurance markets. In presenting a risk exchange model of the reinsurance market, Borch (1962) argued that this market should contain only one price, rather than the multiplicity of prices associated with all possible states in contingent space. Borch proceeded to provide a

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[6] See Conning and Company (1997), p. 30.

price/quantity analysis, setting the price of reinsurance equal to expected losses plus a risk loading proportional to the variance of losses, but found that his competitive equilibrium results were not consistent with Pareto optimality.

As became clear from the work of Kihlstrom and Pauly (1971), Borch's model was over-specified because the form of his risk loading was not consistent with his assumption that the parties in the risk exchange had quadratic utility functions.[7] Kihlstrom and Pauly demonstrated that the single price of insurance is correlated with the prices of contingent claims, and that the competitive equilibrium of a risk exchange in price/quantity space is consistent with the competitive equilibrium of a risk exchange in contingent space.

Subsequently, Baton and Lemaire (1981a, 1981b) applied the Nash bargaining framework of cooperative game theory to the analysis of a reinsurance market, and Kihlstrom and Roth (1982) provided a similar analysis of risk transfer between one insured and one insurer. Focusing on the case in which the insurer is risk neutral, Kihlstrom and Roth showed that the equilibrium price of insurance will be actuarially fair under this assumption, and pointed out that additional results could be proved if the insurer were assumed to be risk averse.

As noted by Arrow (1996), the risk transfer– as opposed to risk exchange– model recognizes the reality that the parties in most traditional insurance markets are either buyers or sellers, and not eligible both to cede and to assume risk as in a risk

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[7] See Geanakoplos and Shubik (1990) for a discussion of necessary and sufficient conditions for the Pareto optimality of competitive equilibrium in a one-good CAPM.

exchange. In Powers, Shubik, and Yao (1994, 1998), we first applied the full process structure of a strategic market game to study equilibrium effects in a risk transfer model of a primary insurance market. For a one-period game in which the buyers and sellers of insurance make strategic bids and offers to determine market price and quantity, we were able to prove the existence and uniqueness of market equilibrium under certain conditions.

More recently, in Powers and Shubik (1999), we focused on the relationship between the law of large numbers (LLN) and the oligopolistic effect of the number of firms in the market. For the case of risk neutral insurers, we found that, for certain reasonable parameter values, there is a natural tradeoff between the effects of the LLN and oligopoly. This tradeoff causes both equilibrium quantity and the equilibrium payoff to customers to possess unique interior maxima over the number of insurance firms.

3. Modeling Considerations

Formal economic models require a high level of abstraction. The precision and consistency of the model comes at a high price, and one frequently has to guard against "throwing the baby out with the bath water"– i.e., simplifying the model to the point that critical features are omitted. If there are one or more precise questions to be asked, then it may be possible to build a highly abstract "stripped down" model that provides answers to these particular questions. However, there may be many other relevant questions for which such a model is not adequate.

In this article, we address the set of questions posed at the beginning of the Introduction, using a "minimalist" formal model. We recognize that our portrayal of the primary and reinsurance markets is somewhat unrealistic, but we believe that the model as a whole captures the essential statistical and strategic elements of an insurance market, thereby enabling us to characterize accurately the conditions under which reinsurance (and retrocession) is desirable, and optimal.

3.1. The Primary Insurance Market

We now review the formal model of a primary insurance market presented in Powers, Shubik, and Yao (1994, 1998). This model employs a Cournot price-formation mechanism with arbitrary numbers of buyers and sellers, so that marginal changes in insurer solvency and competitive forces can be studied directly as the numbers of players change.[8]

Consider a primary insurance market game with players consisting of m homogeneous customers, $i = 1, 2, \ldots, m$, and n homogeneous insurance firms, $i = 1, 2, \ldots, n$. At time 0, let each customer (buyer) i have initial endowment $B_i(0) = V + A$ consisting of one unit of property with replacement value V and $A\ (\geq V)$ dollars in cash. Furthermore, let each insurer (seller) j have initial endowment $S_j(0) = R/n$ dollars of net worth, where R is the total amount of capital supplied by investors to the insurance market.

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[8] In this sense, our work is conceptually similar to that of Venezian (1994), who developed a theory of "pseudo-supply" and "pseudo-demand" curves to account for changes in the "quality" of the insurance product (i.e., the financial soundness of the insurer) as the number of customers per insurer varies.

We assume that, during the policy period $[0,t]$, each customer's property is subject to a random loss with probability p, and that all losses are total. The random variable d_i equals 1 if customer i suffers a property loss during $[0,t]$, and equals 0 otherwise, where the d_i ~ i.i.d. Bernoulli(p).

3.2. Strategies

To insure against a potential property loss in $[0,t]$, each customer i has the option of purchasing insurance from some insurer by making a strategic bid, $x_i \in [0,V]$, that represents the amount that he or she is willing to pay for insurance. Simultaneously, each insurer j has the option of offering to sell insurance by making a strategic offer, $y_j \in [0, cR/n]$, that represents the total dollar amount of risk that j is willing to assume, where $c > 1$ is a solvency constraint imposed by government regulators.

We assume that all bids and offers are submitted to a central clearinghouse that:

- calculates an average market price of insurance per exposure unit,

$$P(\mathbf{x},\mathbf{y}) = \sum_{i'=1}^{m} x_{i'} \Big/ \sum_{j'=1}^{n} y_{j'};^{9}$$

However, whereas Venezian employed a contingent claims framework to determine a "pseudo-supply" curve, we use expected utilities to compute the payoffs of both insurers and their customers.

[9] Given that $y_j = 0$ is a permissible offer, it is theoretically possible--although highly unlikely in a real insurance market--that $\sum_{j'=1}^{n} y_{j'} = 0$, causing $P(\mathbf{x},\mathbf{y})$ to be undefined. To avoid this problem, as well as similar problems associated with $\sum_{i'=1}^{m} x_{i'} = 0$, we take the approach of Dubey and Shubik (1978) and assume that the clearinghouse furnishes at least one insurer, and one customer per insurer, that must make non-zero bids/offers.

- collects all premium bids, x_i, and distributes them to the n insurers in proportion to the insurers' respective coverage offers, y_j (i.e., insurer j receives the premium amount $y_j P(\mathbf{x},\mathbf{y})$);
- randomly assigns each customer i to an insurer $j(i)$ so that each insurer ends up with the same number of customers, m (i.e., it is assumed that n divides m exactly and that $m = m/n$).

Letting M_j denote the set of customers associated with insurer j, we assume that if customer $i \in M_j$ suffers a loss in $[0,t]$, then he or she will receive a loss payment in the amount $y_j\left(x_i \Big/ \sum_{h \in M_j} x_h\right)$ – i.e., an amount proportional not only to i's premium bid, x_i, but also to j's coverage offer, y_j. This loss payment will be bounded above by V to reduce problems of moral hazard.

To recognize the possibility of insurer insolvency during $[0,t]$, let h_j be a Bernoulli random variable that equals 1 if insurer j becomes insolvent, and equals 0 otherwise. If there is an insolvency, it is assumed that government guaranty funds will pay a fixed proportion $g \in [0,1]$ of all insurance claims made against the insolvent insurer.[10]

[10] This assumption is made to facilitate the analysis. In the U.S., the large majority of state guaranty funds have been set up in accordance with the National Association of Insurance Commissioners' Model Act, which provides for the payment of losses up to a dollar limit (often $300,000); see Duncan (1984).

3.3. Payoffs

Given the above development, we see that at time t customer i's wealth consists of

$$B_i(t) = (1-d_i)(A+V-x_i) + d_i\left(1-h_{j(i)}\right)\left[A - x_i + y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right]$$

$$+ d_i h_{j(i)}\left[A - x_i + g y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right],$$

and insurer j's wealth equals

$$S_j(t) = R/n + y_j P(\mathbf{x},\mathbf{y}) - y_j\left(\sum_{h \in M_j} d_h x_h \Big/ \sum_{h \in M_j} x_h\right).$$

Note that $B_i(t) \geq 0$, but $S_j(t)$ can take on both positive and negative values.

Now let $u_B(.) : \Re \to \Re$ denote the utility function of customer i, for all i, and $u_S(.) : \Re \to \Re$ denote the utility function of insurer j, for all j. It then follows that the payoffs to customer i and insurer j are given by

$$E\left[u_B(B_i(t))\right] = (1-p)u_B(A+V-x_i) + p\left(1-r_{j(i)}\right)u_B\left(A - x_i + y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right)$$

$$+ p r_{j(i)} u_B\left(A - x_i + g y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right)\right)$$

and

$$E\left[u_S(S_j(t))\right] = \sum_{r=0}^{m} \sum_{H_r \subseteq M_j} p^r (1-p)^{m-r} u_S\left(R/n + y_j P(\mathbf{x},\mathbf{y}) - y_j\left(\sum_{h \in H_r} x_h \Big/ \sum_{h \in M_j} x_h\right)\right),$$

respectively, where $r_{j(i)} = Pr\left\{h_{j(i)} = 1 \mid d_i = 1, \mathbf{x}, \mathbf{y}\right\}$ and $H_r = \{h_1, h_2, \ldots, h_r\}$.

Given that an insolvency of insurer $j(i)$ at time t is equivalent to the event $S_{j(i)}(t) \leq 0$, we note that

$$\rho_{j(i)} = Pr\left\{S_{j(i)}(t) \leq 0 \,\middle|\, \delta_i = 1, \mathbf{x}, \mathbf{y}\right\}$$

$$= Pr\left\{R/n + y_{j(i)}P(\mathbf{x},\mathbf{y}) - y_{j(i)}\left(x_i \Big/ \sum_{h \in M_{j(i)}} x_h\right) - y_{j(i)}\left(\sum_{h \in M_{j(i)}, h \neq i} \delta_h x_h \Big/ \sum_{h \in M_{j(i)}} x_h\right) \leq 0\right\},$$

which is difficult to evaluate because the random variable $\sum_{h \in M_{j(i)}, h \neq i} \delta_h x_h$ has a complicated probability distribution. To simplify matters, we assume that $\rho_{j(i)}$ is given exactly by the normal approximation for all i. Furthermore, since the loss payment $y_j\left(x_i \Big/ \sum_{h \in M_j} x_h\right) \leq V$ will typically be substantially less than $S_{j(i)}(0) = R/n$, it follows that the effect of any individual x_i on the ruin probability will generally be insignificant.[11] Thus, we make the additional assumption that $\frac{\partial \rho_{j(i)}}{\partial x_i} = 0$ for all i.

4. A Model of Reinsurance and Retrocession

An important aspect of insurance is the pooling of risk.[12] However, if all risk is pooled into a single firm, the customers face a monopolist. If a society wants to minimize the need for regulation, it may wish to find an industrial structure which achieves, or comes close to achieving, the benefits of the pooling of risk while

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[11] See Appendix A of Powers and Shubik (1999) for a more rigorous characterization of this assertion.

[12] Although pooling may not be a necessary component of all insurance transactions, the benefits of the LLN that arise from pooling tend to improve the economic efficiency of such transactions (see, e.g., Porat and Powers, 1999).

preserving a reasonable degree of competition in the market. We suggest that the creation of a reinsurance market contributes to achieving this goal.

We now extend the model presented in Powers, Shubik, and Yao (1994, 1998) by introducing one or more levels of reinsurers. Although the second and higher levels of reinsurance are commonly referred to as "retrocession", we will adopt a convention of simply denoting each level of reinsurance by its distance from the primary insurance market; thus, level "1" will denote the reinsurance of primary insurers, level "2" the reinsurance of level "1" reinsurers, etc.

In essence, we envision an $(r+1)$-stage strategic game in which there is first an interaction between the customers and the primary insurers, then an interaction between the primary insurers and the level 1 reinsurers, etc., through r levels of reinsurance. The solution to be considered here is a perfect pure strategy non-cooperative equilibrium (PSNE)– "perfect" in the sense that the equilibrium in the overall game is also an equilibrium in every sub-game.[13]

Let $G(r)$ denote an insurance market game with one primary insurance market and $r \in \{1,2,3,\ldots\}$ levels of reinsurance. The following four assumptions will provide the basic framework for our analysis.

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[13] The detailed specification of information conditions in the extensive form of the game is critical to the identification of perfect equilibria. In general, the greater the information, the larger the set of non-cooperative equilibria becomes. We conjecture that a PSNE exists for all twice-differentiable utility functions characterizing the risk aversion of customers, insurers, and various levels of reinsurers.

Assumption 1: There are

(i) m homogeneous customers (buyers) in the primary market, each with utility function $u_B(w) = \dfrac{1 - e^{-bw}}{b}$,

(ii) n_0 homogeneous insurers (sellers) in the primary market, each with utility function $u_{S^{(0)}}(w) = \dfrac{1 - e^{-s_0 w}}{s_0}$, and

(iii) n_l homogeneous reinsurers at level $l \in [1, r]$, each with utility function $u_{S^{(l)}}(w) = \dfrac{1 - e^{-s_l w}}{s_l}$,

where $m > n_0 > n_1 > \ldots > n_r > 1$ and $b > s_0 \geq s_1 \geq \ldots \geq s_r \geq 0$.

Assumption 2: The primary insurers make offers $y_{j_0}^{(0)}$ and bids $x_{j_0}^{(1)}$, the reinsurers at level $l \in [1, r-1]$ make offers $y_{j_l}^{(l)}$ and bids $x_{j_l}^{(l+1)}$, and the reinsurers at level r make offers $y_{j_r}^{(r)}$, where: (1) all primary insurers and reinsurers make their offers independently of their bids, and (2) price determinations, premium distributions, and customer assignments are made at each level by a central clearinghouse.

Assumption 3: Letting $M_k^{(1)}$ denote the set of primary insurers associated with level 1 reinsurer k, it follows that if insurer $j \in M_k^{(1)}$ suffers a loss in $[0, t]$, then it will receive a loss payment in the amount $y_k^{(1)} \dfrac{x_j^{(1)}}{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)}} \dfrac{\sum_{h \in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i' \in M_j^{(0)}} x_{i'}^{(0)}}$ (i.e., an amount proportional not only to j's premium bid, $x_j^{(1)}$, but also to k's coverage offer, $y_k^{(1)}$), and that an analogous loss payment rule is applied at each higher level of reinsurance $l \in [2, r]$.

Assumption 4: The conditional probability of insolvency of the primary insurer j, given a loss associated with customer $i \in M_j^{(0)}$, is $r_{j(i)} = \Pr\left\{h_{j(i)} = 1 \middle| d_i = 1, \mathbf{x}^{(1)}, \mathbf{y}^{(1)}\right\}$, and the reinsurers at all levels $l \in [1, r]$ remain solvent with probability 1.

For notational convenience, let

(i) $$r^* \equiv \Phi\left(\frac{\left(\frac{m_0}{Q_0^* - Q_1^*}\right)\left(-R_0 + P_1^* Q_1^* - P_0^* Q_0^*\right) + 1 + (m_0 - 1)p}{\sqrt{p(1-p)(m_0 - 1)}} \right),$$

and let

(ii) $Q_0^* = f(P_0^*)$ denote the solution of the equation

$$(1-p)e^{-bV} + p(1-r^*)e^{-\frac{bQ_0^*}{m}}\left(1 - \frac{m_0 - 1}{m_0 P_0^*}\right) + pr^* e^{-\frac{bgQ_0^*}{m}}\left[1 - \frac{g(m_0 - 1)}{m_0 P_0^*}\right] = 0,$$

in all subsequent results.

The following theorem presents the fundamental result for our insurance market game, $G(r)$.

Theorem 1: If Assumptions 1 through 4 hold,[14] then there exists a unique type-symmetric pure strategy equilibrium for $G(r)$ in which:

(i) the equilibrium price at reinsurance level $l \in [1, r]$ is given by

$$P_l^* = P_0^* \prod_{v=1}^{l} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right),$$

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[14] In addition to Assumptions 1 to 4, certain regularity conditions– analogous to conditions identified in the proposition of Section 4 of Powers, Shubik, and Yao (1998)– are assumed to hold in this and subsequent results.

where

$$P_0{*} = \frac{\left(\dfrac{n_r}{n_r - 1}\right) pe^{\frac{s_r Q_r *}{m}}}{\left[\prod_{v=1}^{r}\left(\dfrac{m_v - 1}{m_v}\right)\left(\dfrac{n_{v-1} - 1}{n_{v-1}}\right)\right]\left(pe^{\frac{s_r Q_r *}{m}} + 1 - p\right)}$$

denotes the equilibrium price in the primary insurance market; and

(ii) the equilibrium quantity at reinsurance level $l \in [1, r]$ is given by

$$Q_l{*} = Q_0{*} - \sum_{v=0}^{l-1} \frac{m}{s_v} \ln\left(\frac{\dfrac{1}{p} - 1}{\dfrac{1}{\left(\dfrac{n_v - 1}{n_v}\right) P_0{*} \prod_{z=1}^{v}\left(\dfrac{m_z - 1}{m_z}\right)\left(\dfrac{n_{z-1} - 1}{n_{z-1}}\right)} - 1} \right),$$

where $\prod_{z=1}^{0}\left(\dfrac{m_z - 1}{m_z}\right)\left(\dfrac{n_{z-1} - 1}{n_{z-1}}\right) \equiv 1$, and $Q_0{*} = f(P_0{*})$ denotes the equilibrium quantity in the primary insurance market.

Proof: The proof of this theorem is provided in the Appendix. ■

5. The Case of Risk Neutral Reinsurers

If the reinsurers at all levels $l \in [L, r]$ are risk neutral (i.e., $s_l \to 0$ for $l \in [L, r]$), then we are able: (1) to provide explicit analytical forms for $P_0{*}$ and $Q_0{*}$, and (2) to show that equilibria do not exist for reinsurance levels $l \in [L+1, r]$.

Corollary 1: If Assumptions 1 through 4 hold, and if the reinsurers at all levels $l \in [L, r]$ are risk neutral (i.e., $s_l \to 0$ for $l \in [L, r]$), then there exists a unique type-symmetric pure strategy equilibrium for $G(r)$ in which:

(i) the equilibrium price at reinsurance level $l \in [1, L]$ is given by

$$P_l^* = \frac{\left(\frac{n_L}{n_L - 1}\right)p}{\prod_{v=l+1}^{L}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)},$$

where $\prod_{v=L+1}^{L}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right) \equiv 1$, and

$$P_0^* = \frac{\left(\frac{n_L}{n_L - 1}\right)p}{\prod_{v=1}^{L}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)}$$

denotes the equilibrium price in the primary insurance market;

(ii) the equilibrium quantity at reinsurance level $l \in [1, L]$ is given by

$$Q_l^* = Q_0^* - \sum_{v=0}^{l-1}\frac{m}{s_v}\ln\left(\frac{\frac{1}{p} - 1}{\frac{\prod_{z=v+1}^{L}\left(\frac{m_z - 1}{m_z}\right)\left(\frac{n_z - 1}{n_z}\right)}{p} - 1}\right),$$

where $Q_0^* = f(P_0^*)$ denotes the equilibrium quantity in the primary insurance market; and

(iii) equilibrium price and quantity do not exist for reinsurance levels $l \in [L+1, r]$. ■

Proof: The proof is provided in the Appendix.

5.1. Analysis—When Is Reinsurance Desirable?

Using the results of Corollary 1, we are able to explore and characterize conditions under which it is desirable, on the margin, to introduce a level of reinsurance– a problem that is conceptually similar for all reinsurance levels $l \geq 1$. To this end, we compare the price of insurance in the primary insurance market under two alternatives. The first alternative, denoted by A, is a primary insurance market in which the number of primary insurers is increased by 2. The second alternative, B, is the same primary insurance market, except that the number of primary insurers remains fixed, while we add a reinsurance level with 2 risk neutral reinsurers.[15]

For both alternatives, we assume that there are m identical primary insurance customers (each with constant risk aversion coefficient b), and (initially) n_0 identical primary insurers (each with constant risk aversion coefficient $s_0 = s < b$). In addition, we assume that the primary insurers are subject to *exogenous* i.i.d. insolvency perils ~ Bernoulli(r), and that the customers receive no loss payments following an insurer's insolvency (i.e., $g = 0$). Finally, under alternative B, we assume that the reinsurers remain solvent with probability 1.

From Equations (8) and (7) of Powers and Shubik (1999), we obtain the following expressions for price and quantity, respectively, in the primary market under alternative A:

$$P_0^{(A)} = \frac{\left(\frac{n_0+2}{n_0+1}\right) pe^{sQ_0^{(A)}/m}}{pe^{sQ_0^{(A)}/m} + 1 - p}, \qquad (1)$$

[15] Note that we consider marginal changes of *two* firms, rather than *one* firm, simply to avoid problems of division by zero.

$$Q_0^{(A)} = -\frac{m}{b}\ln\left(\frac{(1-p)e^{-bV}+pr}{p(1-r)\left[\frac{\left(\frac{m}{n_0+2}\right)-1}{\left(\frac{m}{n_0+2}\right)P_0^{(A)}}-1\right]}\right). \qquad (2)$$

Similarly, from Corollary 1 above, we obtain the following expressions for price and quantity, respectively, under alternative B:

$$P_0^{(B)} = 2\left(\frac{n_0}{n_0-1}\right)\left(\frac{n_0}{n_0-2}\right)p, \qquad (3)$$

$$Q_0^{(B)} = -\frac{m}{b}\ln\left(\frac{(1-p)e^{-bV}+pr}{p(1-r)\left[\frac{\left(\frac{m}{n_0}\right)-1}{\left(\frac{m}{n_0}\right)P_0^{(B)}}-1\right]}\right). \qquad (4)$$

We now consider under what conditions alternative B (the creation of reinsurance) provides a lower price in the primary market than does alternative A (primary market expansion); i.e., when $P_0^{(B)} < P_0^{(A)}$, or equivalently,

$$2\left(\frac{n_0}{n_0-1}\right)\left(\frac{n_0}{n_0-2}\right)p < \frac{\left(\frac{n_0+2}{n_0+1}\right)pe^{sQ_0^{(A)}/m}}{pe^{sQ_0^{(A)}/m}+1-p}.$$

This inequality may be rewritten as

$$p < 1 - \frac{\left(\dfrac{1+e_{n_0}}{2}\right)}{1 - e^{-sQ_0^{(A)}/m}}, \qquad (5)$$

where $e_{n_0} = 1 - \left(\dfrac{n_0 - 1}{n_0}\right)\left(\dfrac{n_0 - 2}{n_0}\right)\left(\dfrac{n_0 + 2}{n_0 - 1}\right) > 0$. Plotting $\dfrac{Q_0^{(A)}}{m}$ on the horizontal axis, and p on the vertical axis, Inequality (5) may be denoted by the region between Curve (I) and the horizontal axis in Figure 1 below.

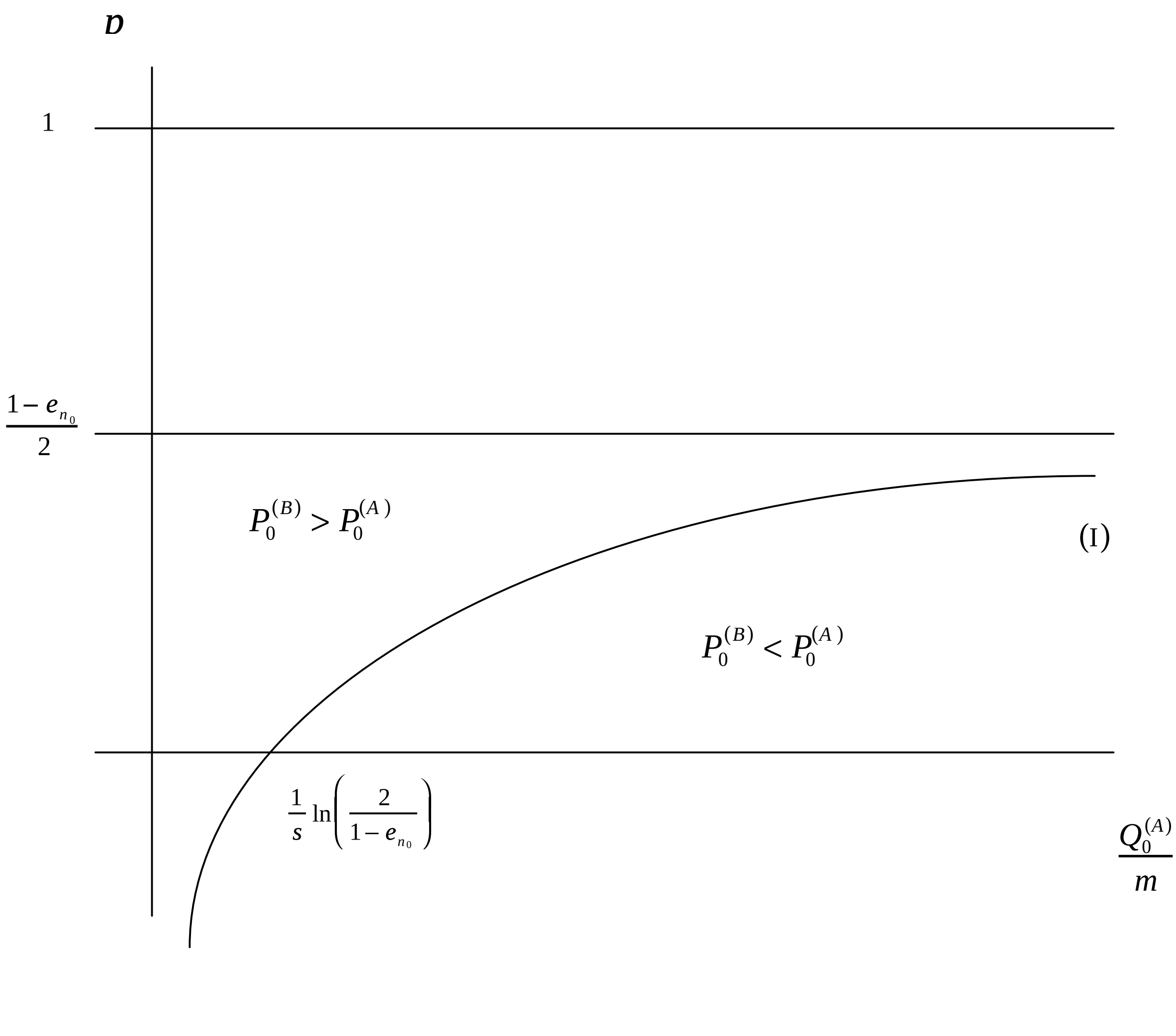

Figure 1. The Frontier of Reinsurance Desirability

We now consider the actual functional relationship between $\frac{Q_0^{(A)}}{m}$ and p in equilibrium. Rearranging Equation (2), we find that

$$p = \frac{1}{(1-r)\left(\frac{1-e_{m,n_0}}{P_0^{(A)}} - 1\right)e^{-b\left[\left(Q_0^{(A)}/m\right)-V\right]} + 1 - re^{bV}}, \tag{6}$$

where $e_{m,n_0} = \frac{n_0 + 2}{m} > 0$.

The Case of No Insolvencies

For the moment, let $r = 0$, so that

$$p = \frac{1}{\left(\frac{1-e_{m,n_0}}{P_0^{(A)}} - 1\right)e^{-b\left[\left(Q_0^{(A)}/m\right)-V\right]} + 1}.$$

This equation is then plotted as Curve (II) in Figure 2 below.

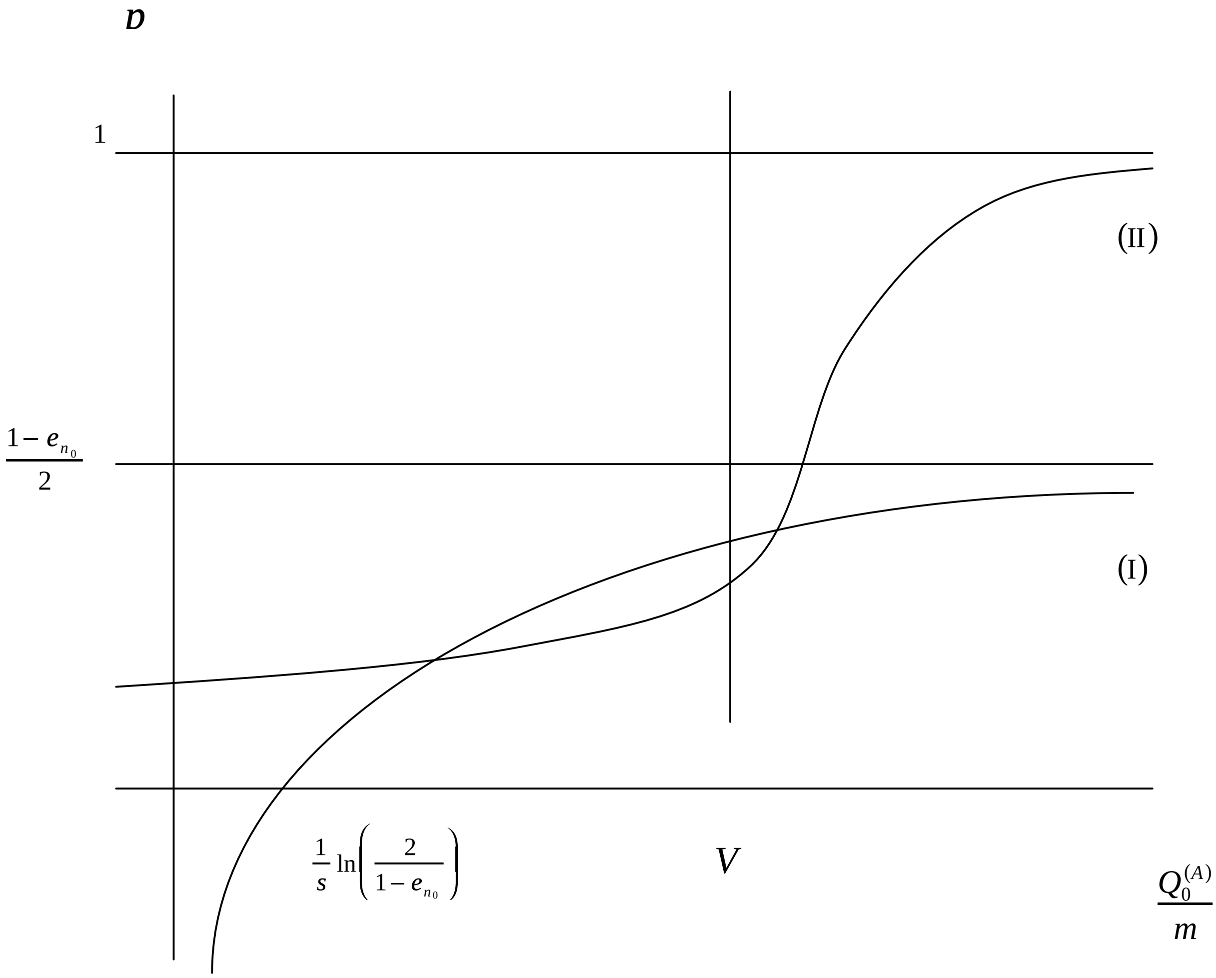

Figure 2. A Sufficient Condition for Reinsurance Desirability

From this second figure, it can be seen that a sufficient condition for there to exist a region such that $P_0^{(B)} < P_0^{(A)}$ is that Curve (II) be lower than Curve (I) at the upper bound of the feasible domain of $\frac{Q_0^{(A)}}{m}$; i.e., at $\frac{Q_0^{(A)}}{m} = V$. This sufficient condition is described by the inequality

$$\left.\frac{1}{\left(\frac{1-e_{m,n_0}}{P_0^{(A)}}-1\right)e^{-b\left[\left(Q_0^{(A)}/m\right)-V\right]}+1}<1-\frac{\left(\frac{1+e_{n_0}}{2}\right)}{1-e^{-sQ_0^{(A)}/m}}\right|_{\frac{Q_0^{(A)}}{m}=V}$$

$$\Leftrightarrow \frac{P_0^{(A)}}{1-e_{m,n_0}}<1-\frac{\left(\frac{1+e_{n_0}}{2}\right)}{1-e^{-sV}}, \tag{7}$$

where

$$P_0^{(A)}=\frac{\left(\frac{n_0+2}{n_0+1}\right)pe^{sV}}{pe^{sV}+1-p} \tag{8}$$

is given by Equation (1). Combining (7) and (8) yields

$$\frac{\left(\frac{n_0+2}{n_0+1}\right)}{\left(1-e_{m,n_0}\right)}\left[\frac{1}{1+\left(\frac{1-p}{p}\right)e^{-sV}}\right]<\frac{\left(1-e_{n_0}\right)\left(\frac{1}{2}\right)-e^{-sV}}{1-e^{-sV}}$$

$$\Leftrightarrow \frac{\left(\frac{n_0+2}{n_0+1}\right)}{\left(1-e_{m,n_0}\right)}\left(1-e^{-sV}\right)<\left[\left(1-e_{n_0}\right)\left(\frac{1}{2}\right)-e^{-sV}\right]\left[1+\left(\frac{1-p}{p}\right)e^{-sV}\right]$$

$$\Leftrightarrow k_1\left(1-e^{-sV}\right)<\left[k_2\left(\frac{1}{2}\right)-e^{-sV}\right]\left[1+\left(\frac{1-p}{p}\right)e^{-sV}\right]$$

$$\Leftrightarrow k_1-k_1e^{-sV}<\frac{k_2}{2}+\frac{k_2}{2}\left(\frac{1-p}{p}\right)e^{-sV}-e^{-sV}-\left(\frac{1-p}{p}\right)e^{-2sV}$$

$$\Leftrightarrow y^2+\left[\left(1-k_1\right)\left(\frac{p}{1-p}\right)-\frac{k_2}{2}\right]y+\left(k_1-\frac{k_2}{2}\right)\left(\frac{p}{1-p}\right)<0, \tag{9}$$

where $y=e^{-sV}$, $k_1=\frac{\left(\frac{n_0+2}{n_0+1}\right)}{\left(1-e_{m,n_0}\right)}$, and $k_2=1-e_{n_0}$.

To simplify the mathematics, we assume that $m >> n_0 >> 0$, so that $k_1 \approx 1$ and $k_2 \approx 1$. Inequality (9) then reduces to

$$y^2 - \frac{1}{2}y + \frac{1}{2}\left(\frac{p}{1-p}\right) < 0,$$

which has solution set

$$y \in \left(\frac{1}{4} - \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}, \frac{1}{4} + \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}\right)$$

$$\Leftrightarrow \quad -\frac{1}{V}\ln\left(\frac{1}{4} + \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}\right) < s < -\frac{1}{V}\ln\left(\frac{1}{4} - \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}\right), \qquad (10)$$

as long as $p < \frac{1}{9}$.[16]

The Case of Potential Insolvencies

If we now let $r > 0$, we easily can see from Equation (6) that this has the effect of shifting Curve (II) upward, thereby reducing the size of the potential interval in which $P_0^{(B)} < P_0^{(A)}$.

The Case of Catastrophe Losses

For the case of (perfectly correlated) catastrophe losses, the analysis is very similar to the non-catastrophe case. All of the expressions for price and quantity in the primary market remain the same, except for the expression for price under alternative A, which now becomes

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[16] We note that the restriction $p < \frac{1}{9}$ is easily satisfied for most primary property insurance markets.

$$P_0^{(A)} = \frac{\left(\frac{n_0+2}{n_0+1}\right) pe^{s m_0 Q_0^{(A)}/m}}{pe^{s m_0 Q_0^{(A)}/m} + 1 - p}, \tag{1'}$$

where $m_0 = \dfrac{m}{n_0+1}$.

Following the same analysis as before, we find that both Curves (I) and (II) are compressed to the left– although not in the same proportions– yielding the sufficient condition for $P_0^{(B)} < P_0^{(A)}$,

$$-\frac{1}{m_0 V}\ln\left(\frac{1}{4} + \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}\right) < s < -\frac{1}{m_0 V}\ln\left(\frac{1}{4} - \sqrt{\frac{1}{16} - \frac{p}{2(1-p)}}\right) \tag{10'}$$

(as long as $p < \dfrac{1}{9}$). Clearly, condition (10') is substantially more restrictive than condition (10), suggesting that the potential benefit of reinsurance on price in the primary market may be much more limited in the catastrophe context.

5.2. Analysis—When Are There Enough Reinsurers?

We now seek to identify conditions under which the reinsurance market is saturated– i.e., under which it is no longer desirable, on the margin, to introduce an additional risk neutral reinsurer rather than an additional primary insurer (with risk aversion coefficient s). As we will see, this problem is conceptually similar for all reinsurance levels $l \geq 1$.

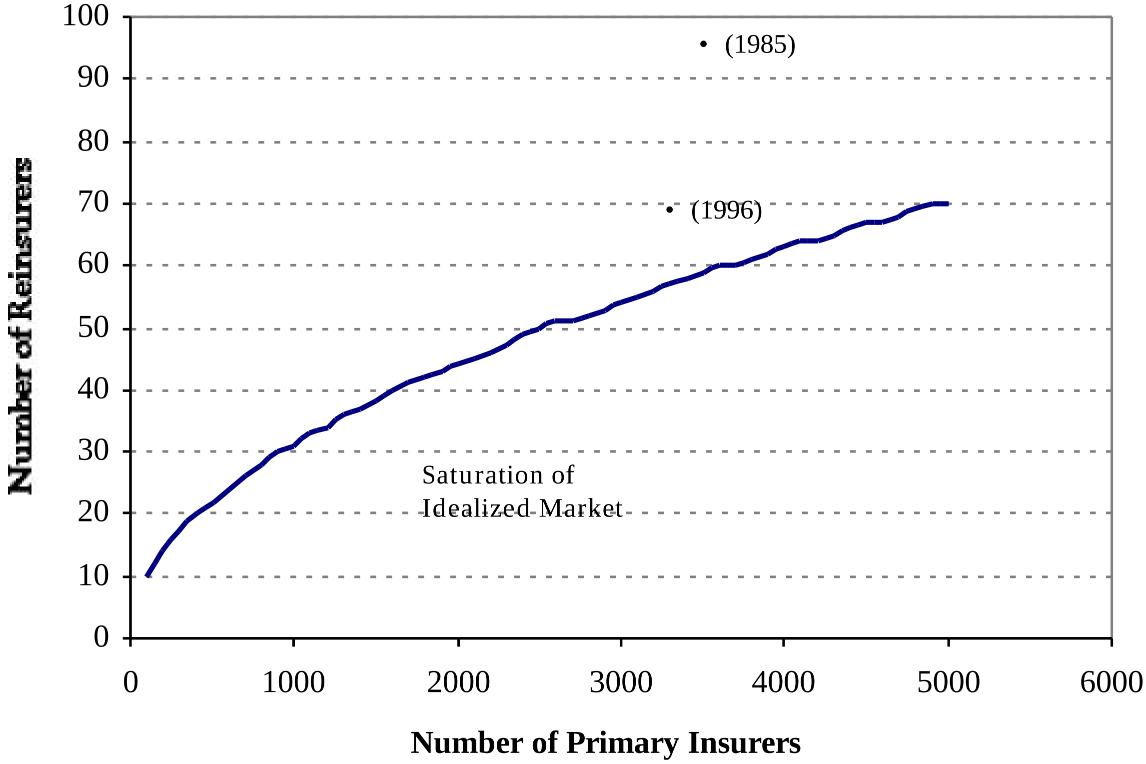
We begin by comparing the price of insurance in the primary insurance market under two alternatives. The first alternative, denoted by C, is a primary insurance market with one level of reinsurance, where the primary market has n_0 insurers, and the reinsurance market has n_1 reinsurers. The second alternative, D, is the same primary insurance market, except that the number of primary insurers is increased by one (to $n_0 + 1$), while the number of reinsurers is decreased by one (to $n_1 - 1$). To identify the point at which the number of reinsurers has reached its optimal saturation level, we solve for the maximum value of n_1 such that $P_0^{(C)} < P_0^{(D)}$; i.e.,

$$n_1{*} = Max\left\{n_1 : p \le P_0^{(C)} < P_0^{(D)} \le 1\right\}$$

$$= Max\left\{n_1 : \left(\frac{n_1}{n_1 - 1}\right)\left(\frac{n_0}{n_0 - 1}\right)\left(\frac{n_0}{n_0 - n_1}\right) - \left(\frac{n_1 - 1}{n_1 - 2}\right)\left(\frac{n_0 + 1}{n_0}\right)\left(\frac{n_0 + 1}{n_0 + 2 - n_1}\right) < 0\right\}. \quad (11)$$

Interestingly, this saturation level may be found as the solution to a cubic equation in n_1. In the figure below, we provide the saturation level for a wide range of primary market sizes, and, for comparison purposes, we also include the actual domestic U.S. insurance/reinsurance market figures for 1985 and 1996. Intriguingly, we observe that, as the U.S. market consolidates, its position on the graph appears to follow a trajectory toward the idealized market curve.

Figure 3. U.S. Insurance/Reinsurance Market



Finally, we note that for reinsurance levels $l \geq 2$, Equation (11) generalizes to

$$n_l{}^* = Max\left\{n_l : p \leq P_0^{(C)} < P_0^{(D)} \leq 1\right\}$$

$$= Max\left\{n_l : \left(\frac{n_l}{n_l - 1}\right)\left(\frac{n_{l-1}}{n_{l-1} - 1}\right)\left(\frac{n_{l-1}}{n_{l-1} - n_l}\right)\left(\frac{n_{l-2}}{n_{l-2} - n_{l-1}}\right) < \left(\frac{n_l - 1}{n_l - 2}\right)\left(\frac{n_{l-1} + 1}{n_{l-1}}\right)\left(\frac{n_{l-1} + 1}{n_{l-1} + 2 - n_l}\right)\left(\frac{n_{l-2}}{n_{l-2} - n_{l-1} - 1}\right)\right\},$$

which also may be found as the solution to a cubic equation in n_l.

5.3. A Market with Correlated Strategies

If we modify the model to permit the risk neutral reinsurer at level $l \in [L, r-1]$ to make its offer as an explicit function of its bid, then there do exist equilibria for levels

$l \in [L+1, r]$, characterized by the purchase of "over-insurance" by all risk neutral reinsurers.

Corollary 2: If the premises of Corollary 1 hold, but Assumption 2 is relaxed so that the risk neutral reinsurers at levels $l \in [L, r-1]$ are able to make their offers as explicit functions of their bids (i.e., $y_{j_l}^{(l)} = j_l\left(x_{j_l}^{(l+1)}\right)$), then there exists a (non-unique) type-symmetric pure strategy equilibrium for $G(r)$ in which:

(i) the equilibrium price at reinsurance level $l \in [1, r]$ is given by

$$P_l* = \frac{\left(\frac{n_r}{n_r - 1}\right)p}{\prod_{v=l+1}^{r}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)},$$

where $\prod_{v=r+1}^{r}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right) \equiv 1$, and

$$P_0* = \frac{\left(\frac{n_r}{n_r - 1}\right)p}{\prod_{v=1}^{r}\left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)}$$

denotes the equilibrium price in the primary insurance market; and

(ii) the equilibrium quantity at reinsurance level l is given by

$$Q_l* = Q_0* - \sum_{v=0}^{l-1}\frac{m}{s_v}\ln\left(\frac{\frac{1}{p} - 1}{\frac{\prod_{z=v+1}^{r}\left(\frac{m_z - 1}{m_z}\right)\left(\frac{n_z - 1}{n_z}\right)}{p} - 1}\right) \text{ for } l \in [1, L],$$

and by

$$Q_l^* = \left[\prod_{v=L+1}^{l}\left(\frac{m_v}{m_v-1}\right)\right]\left[Q_0^* - \sum_{v=0}^{L-1}\frac{m}{s_v}\ln\left(\frac{\frac{1}{p}-1}{\frac{\prod_{z=v+1}^{r}\left(\frac{m_z-1}{m_z}\right)\left(\frac{n_z-1}{n_z}\right)}{p}-1}\right)\right] \text{ for } l \in [L+1, r],$$

where $Q_0^* = f(P_0^*)$ denotes the equilibrium quantity in the primary insurance market.

Proof: The proof is provided in the Appendix. ■

Appendix

Proof of Theorem 1:

Payoff Expressions

First, note that the payoffs to the primary market customers, the primary market insurers, the level 1 reinsurers, etc., are given as follows:

$$E\left[u_B\left(B_i(t)\right)\right] = (1-p)u_B\left(A+V-x_i^{(0)}\right) + p\left(1-r_{j(i)}\right)u_B\left(A - x_i^{(0)} + y_{j(i)}^{(0)}\frac{x_i^{(0)}}{\sum_{h\in M_{j(i)}} x_h^{(0)}}\right)$$

$$+pr_{j(i)}u_B\left(A - x_i^{(0)} + gy_{j(i)}^{(0)}\frac{x_i^{(0)}}{\sum_{h\in M_{j(i)}} x_h^{(0)}}\right),$$

$$E\left[u_{S^{(0)}}\left(S_j^{(0)}(t)\right)\right] = \sum_{w=0}^{m_0}\sum_{H_j^{(0)}(w)\subseteq M_j^{(0)}} p^w(1-p)^{m_0-w}u_{S^{(0)}}\left(\frac{R_0}{n_0} + y_j^{(0)}P_0 - x_j^{(1)}\right.$$

$$\left.+\left(y_k^{(1)}\frac{x_j^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}} - y_j^{(0)}\right)\frac{\sum_{h\in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i'\in M_j^{(0)}} x_{i'}^{(0)}}\right),$$

$$E\left[u_{S^{(1)}}\left(S_k^{(1)}(t)\right)\right] = \sum_{w=0}^{m_0 m_1}\sum_{\substack{H_k^{(1)}(w)=\prod_{j'\in M_k^{(1)}} H_{j'}^{(0)}(w_{j'}),\\ s.t.\ w_{j'}\le m_0,\ \sum_{j'\in M_k^{(1)}} w_{j'}=w}} p^w(1-p)^{m_0 m_1-w}u_{S^{(1)}}\left(\frac{R_1}{n_1} + y_k^{(1)}P_1 - x_k^{(2)}\right.$$

$$\left.+\left(y_\ell^{(2)}\frac{x_k^{(2)}}{\sum_{k'\in M_\ell^{(2)}} x_{k'}^{(2)}} - y_k^{(1)}\right)\frac{\sum_{j'\in M_k^{(1)}}\frac{\sum_{h\in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i'\in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}}\right),$$

$$E\left[u_{S^{(2)}}\left(S_\ell^{(2)}(t)\right)\right] = \sum_{w=0}^{m_0 m_1 m_2}\sum_{\substack{H_\ell^{(2)}(w)=\prod_{k'\in M_\ell^{(2)}}\prod_{j'\in M_{k'}^{(1)}} H_{j'}^{(0)}(w_{j'}),\\ s.t.\ w_{j'}\le m_0,\ \sum_{k'\in M_\ell^{(2)}}\sum_{j'\in M_{k'}^{(1)}} w_{j'}=w}} p^w(1-p)^{m_0 m_1 m_2-w}u_{S^{(2)}}\left(\frac{R_2}{n_2} + y_\ell^{(2)}P_2 - x_\ell^{(3)}\right.$$

$$+\left(y_m^{(3)}\frac{x_\ell^{(3)}}{\sum_{\ell'\in M_m^{(3)}} x_{\ell'}^{(3)}} - y_\ell^{(2)}\right)\frac{\sum_{k'\in M_\ell^{(2)}} \frac{\sum_{j'\in M_{k'}^{(1)}} \frac{\sum_{h\in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i'\in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j'\in M_{k'}^{(1)}} x_{j'}^{(1)}} x_{k'}^{(2)}}{\sum_{k'\in M_\ell^{(2)}} x_{k'}^{(2)}}\Bigg),$$

etc.

First-Order Conditions—Primary Insurance Market

Now consider the first-order conditions for the primary insurance market. Given the assumption that $\frac{\partial r_{j(i)}}{\partial x_i^{(0)}} = 0$, it follows that:

$$\frac{\partial E\left[u_B\left(B_i(t)\right)\right]}{\partial x_i^{(0)}} = -(1-p)e^{-b\left(A+V-x_i^{(0)}\right)}$$

$$-p\left(1-r_{j(i)}\right)e^{-b\left(A-x_i^{(0)}+y_{j(i)}^{(0)}\frac{x_i^{(0)}}{\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}}\right)}\left[1-y_{j(i)}^{(0)}\frac{\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}-x_i^{(0)}}{\left(\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}\right)^2}\right]$$

$$-pr_{j(i)}e^{-b\left(A-x_i^{(0)}+gy_{j(i)}^{(0)}\frac{x_i^{(0)}}{\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}}\right)}\left[1-gy_{j(i)}^{(0)}\frac{\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}-x_i^{(0)}}{\left(\sum_{h\in M_{j(i)}^{(0)}} x_h^{(0)}\right)^2}\right].$$

Setting the above derivative equal to 0 yields

$$(1-p)e^{-bV}+p(1-r^*)e^{-\frac{bx^{*(0)}}{P_0^*}}\left(1-\frac{m_0-1}{m_0P_0^*}\right)+pr^*e^{-\frac{bgx^{*(0)}}{P_0^*}}\left[1-\frac{g(m_0-1)}{m_0P_0^*}\right]=0,$$

or equivalently,

$$(1-p)e^{-bV} + p(1-r^*)e^{-\frac{bQ_0^*}{m}}\left(1-\frac{m_0-1}{m_0 P_0^*}\right) + pr^* e^{-\frac{bgQ_0^*}{m}}\left[1-\frac{g(m_0-1)}{m_0 P_0^*}\right] = 0.$$

Furthermore,

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t)\right)\right]}{\partial y_j^{(0)}} =$$

$$\sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w)\subseteq M_j^{(0)}} p^w (1-p)^{m_0-w} e^{-s_0\left[\frac{R_0}{n_0} + y_j^{(0)} P_0 - x_j^{(1)} + \left(y_k^{(1)} \frac{x_j^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}} - y_j^{(0)}\right) \frac{\sum_{h\in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i'\in M_j^{(0)}} x_{i'}^{(0)}}\right]}$$

$$\times\left[\left(1-\frac{y_j^{(0)}}{\sum_{j'=1}^{n_0} y_{j'}^{(0)}}\right)P_0 - \frac{\sum_{h\in H_j^{(0)}(w)} x_h}{\sum_{i'\in M_j^{(0)}} x_{i'}}\right].$$

Setting this derivative equal to 0 implies

$$\sum_{w=0}^{m_0} \binom{m_0}{w} p^w (1-p)^{m_0-w} \left[\left(\frac{n_0-1}{n_0}\right) P_0^* e^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)w}{m_0}} - \frac{w}{m_0} e^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)w}{m_0}}\right] = 0,$$

or equivalently,

$$\left(\frac{n_0-1}{n_0}\right) P_0^* \left[pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}} + 1 - p\right] - pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}} = 0$$

$$\Rightarrow\; P_0^* = \frac{\left(\frac{n_0}{n_0-1}\right) pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}}}{pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}} + 1 - p} \;\Leftrightarrow\; P_0^* = \frac{\left(\frac{n_0}{n_0-1}\right) pe^{\frac{s_0(Q_0^*-Q_1^*)}{m}}}{pe^{\frac{s_0(Q_0^*-Q_1^*)}{m}} + 1 - p}, \qquad \text{(A1)}$$

where we have used the facts that, for $W \sim \text{Binomial}\,(m_0, p)$,

$$E\left[e^{zW}\right] = \left(pe^z + 1 - p\right)^{m_0}$$

and

$$E\left[We^{zW}\right] = \frac{\partial}{\partial z} E\left[e^{zW}\right] = m_0\left(pe^z + 1 - p\right)^{m_0-1} pe^z.$$

First-Order Conditions—Level 1 Reinsurance Market

We now turn to the first-order conditions for the level 1 reinsurance market. First, note that

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t)\right)\right]}{\partial x_j^{(1)}} =$$

$$\sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w)\subseteq M_j^{(0)}} p^w (1-p)^{m_0-w} e^{-s_0\left[\frac{R_0}{n_0} + y_j^{(0)}P_0 - x_j^{(1)} + \left(y_k^{(1)}\frac{x_j^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}} - y_j^{(0)}\right)\frac{\sum_{h\in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i'\in M_j^{(0)}} x_{i'}^{(0)}}\right]}$$

$$\times\left[-1 + y_k^{(1)} \frac{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)} - x_j^{(1)}}{\left(\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}\right)^2} \frac{\sum_{h\in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i'\in M_j^{(0)}} x_{i'}^{(0)}}\right].$$

Setting this derivative equal to 0 yields

$$\sum_{w=0}^{m_0}\binom{m_0}{w} p^w (1-p)^{m_0-w}\left[-e^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)w}{m_0}} + \frac{w}{m_0}\left(\frac{m_1-1}{m_1P_1*}\right)e^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)w}{m_0}}\right] = 0,$$

or equivalently,

$$-\left[pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}} + 1 - p\right] + \left(\frac{m_1-1}{m_1P_1*}\right)pe^{\frac{s_0\left(y^{*(0)}-\frac{y^{*(1)}}{m_1}\right)}{m_0}} = 0$$

$$\Rightarrow\; P_1* = \left(\frac{m_1-1}{m_1}\right)\left(\frac{n_0-1}{n_0}\right)P_0*. \qquad \text{(A2)}$$

Furthermore,

$$\frac{\partial E\left[u_{S^{(1)}}\left(S_k^{(1)}(t)\right)\right]}{\partial y_k^{(1)}} =$$

$$
\sum_{w=0}^{m_0 m_1} \sum_{\substack{H_k^{(1)}(w) = \prod_{j' \in M_k^{(1)}} H_{j'}^{(0)}(w_{j'}), \\ s.t.\ w_{j'} \le m_0,\ \sum_{j' \in M_k^{(1)}} w_{j'} = w}} p^w (1-p)^{m_0 m_1 - w} e^{-s_1 \left[\frac{R_1}{n_1} + y_k^{(1)} P_1 - x_k^{(2)} + \left(y_\ell^{(2)} \frac{x_k^{(2)}}{\sum_{k' \in M_\ell^{(2)}} x_{k'}^{(2)}} - y_k^{(1)} \right) \frac{\sum_{j' \in M_k^{(1)}} \frac{\sum_{h \in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i' \in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)}} \right]}
$$

$$
\times \left[\left(1 - \frac{y_k^{(1)}}{\sum_{k'=1}^{n_1} y_{k'}^{(1)}} \right) P_1 - \frac{\sum_{j' \in M_k^{(1)}} \frac{\sum_{h \in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i' \in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)}} \right].
$$

Setting this derivative equal to 0 implies

$$
\sum_{w=0}^{m_0 m_1} \binom{m_0 m_1}{w} p^w (1-p)^{m_0 m_1 - w} \left[\left(\frac{n_1 - 1}{n_1} \right) P_1 * e^{\frac{s_1 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right) w}{m_0 m_1}} - \frac{w}{m_0 m_1} e^{\frac{s_0 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right) w}{m_0 m_1}} \right] = 0
$$
,

or equivalently,

$$
\left(\frac{n_1 - 1}{n_1} \right) P_1 * \left[p e^{\frac{s_1 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right)}{m_0 m_1}} + 1 - p \right] - p e^{\frac{s_1 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right)}{m_0 m_1}} = 0
$$

$$
\Rightarrow\ P_1 * = \frac{\left(\frac{n_1}{n_1 - 1} \right) p e^{\frac{s_1 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right)}{m_0 m_1}}}{p e^{\frac{s_1 \left(y*^{(1)} - \frac{y*^{(2)}}{m_2} \right)}{m_0 m_1}} + 1 - p}. \qquad \text{(A3)}
$$

Seller's First-Order Condition—Level 2 Reinsurance Market

For the level 2 reinsurance market, we find that

$$
\frac{\partial E\left[u_{S^{(1)}} \left(S_k^{(1)}(t) \right) \right]}{\partial x_k^{(2)}} =
$$

$$
\sum_{w=0}^{m_0 m_1} \sum_{\substack{H_k^{(1)}(w)=\prod_{j'\in M_k^{(1)}} H_{j'}^{(0)}(w_{j'}),\\ s.t.\ w_{j'}\le m_0,\ \sum_{j'\in M_k^{(1)}} w_{j'}=w}} p^w (1-p)^{m_0 m_1 - w} e^{-s_1\left[\frac{R_1}{n_1}+y_k^{(1)}P_1-x_k^{(2)}+\left(y_\ell^{(2)}\frac{x_k^{(2)}}{\sum_{k'\in M_\ell^{(2)}} x_{k'}^{(2)}}-y_k^{(1)}\right)\frac{\sum_{j'\in M_k^{(1)}}\frac{\sum_{h\in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i'\in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}}\right]}
$$

$$
\times\left[-1+y_\ell^{(2)}\frac{\sum_{k'\in M_\ell^{(2)}} x_{k'}^{(2)}-x_k^{(2)}}{\left(\sum_{k'\in M_\ell^{(2)}} x_{k'}^{(2)}\right)^2}\frac{\sum_{j'\in M_k^{(1)}}\frac{\sum_{h\in H_{j'}^{(0)}(w_{j'})} x_h^{(0)}}{\sum_{i'\in M_{j'}^{(0)}} x_{i'}^{(0)}} x_{j'}^{(1)}}{\sum_{j'\in M_k^{(1)}} x_{j'}^{(1)}}\right].
$$

Setting this derivative equal to 0 yields

$$
\sum_{w=0}^{m_0 m_1}\binom{m_0 m_1}{w} p^w (1-p)^{m_0 m_1 - w}\left[-e^{\frac{s_1\left(y^{*(1)}-\frac{y^{*(2)}}{m_2}\right)w}{m_0 m_1}}+\frac{w}{m_0 m_1}\left(\frac{m_2-1}{m_2 P_2{}^*}\right)e^{\frac{s_1\left(y^{*(1)}-\frac{y^{*(2)}}{m_2}\right)w}{m_0 m_1}}\right]=0,
$$

or equivalently,

$$
-\left[pe^{\frac{s_1\left(y^{*(1)}-\frac{y^{*(2)}}{m_2}\right)}{m_0 m_1}}+1-p\right]+\left(\frac{m_2-1}{m_2 P_2{}^*}\right)pe^{\frac{s_1\left(y^{*(1)}-\frac{y^{*(2)}}{m_2}\right)}{m_0 m_1}}=0
$$

$$
\Rightarrow\ P_2{}^*=\left(\frac{m_2-1}{m_2}\right)\left(\frac{n_1-1}{n_1}\right)P_1{}^*. \tag{A4}
$$

Reinsurance Market, Level l

By induction on (A2) and (A4), we see that

$$
P_l{}^* = P_0{}^* \prod_{v=1}^{l}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_{v-1}-1}{n_{v-1}}\right) \tag{A5}
$$

for $l\in[1,r]$. Moreover, it follows from (A1) and (A3) that

$$y^{*(1)} = m_1 y^{*(0)} - m_1 \frac{m_0}{s_0} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_0 - 1}{n_0}\right) P_0^*} - 1} \right)$$

and

$$y^{*(2)} = m_2 y^{*(1)} - m_2 \frac{m_0 m_1}{s_1} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_1 - 1}{n_1}\right) P_1^*} - 1} \right),$$

and so by induction,

$$y^{*(l)} = m_l y^{*(l-1)} - \frac{\prod_{v=0}^{l} m_v}{s_{l-1}} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_{l-1} - 1}{n_{l-1}}\right) P_{l-1}^*} - 1} \right)$$

$$\Leftrightarrow\ y^{*(l)} = \left(\prod_{v=0}^{l} m_v\right)\left[\frac{y^{*(0)}}{m_0} - \sum_{v=0}^{l-1} \frac{1}{s_v} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_v - 1}{n_v}\right) P_0^* \prod_{z=1}^{v} \left(\frac{m_z - 1}{m_z}\right)\left(\frac{n_{z-1} - 1}{n_{z-1}}\right)} - 1} \right)\right]$$

$$\Leftrightarrow\ Q_l^* = Q_0^* - \sum_{v=0}^{l-1} \frac{m}{s_v} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_v - 1}{n_v}\right) P_0^* \prod_{z=1}^{v} \left(\frac{m_z - 1}{m_z}\right)\left(\frac{n_{z-1} - 1}{n_{z-1}}\right)} - 1} \right) \qquad \text{(A6)}$$

for $l \in [1, r]$.

For $l = r + 1$, we know that $y^{*(r+1)} = 0$, and so

$$0 = y^{*(r)} - \frac{\prod_{v=0}^{r} m_v}{s_r} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_r - 1}{n_r}\right) P_r^*} - 1} \right)$$

$$\Leftrightarrow \; Q_r^* = \frac{m}{s_r} \ln\left(\frac{\frac{1}{p} - 1}{\frac{1}{\left(\frac{n_r - 1}{n_r}\right) P_r^*} - 1} \right)$$

$$\Leftrightarrow \; P_r^* = \frac{\left(\frac{n_r}{n_r - 1}\right) p e^{\frac{s_r Q_r^*}{m}}}{p e^{\frac{s_r Q_r^*}{m}} + 1 - p}.$$

It then follows from (A5) that

$$P_0^* = \frac{\left(\frac{n_r}{n_r - 1}\right) p e^{\frac{s_r Q_r^*}{m}}}{\left[\prod_{v=1}^{r} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)\right]\left(p e^{\frac{s_r Q_r^*}{m}} + 1 - p\right)}.$$

Proof of Corollary 1:

Given that reinsurers at level L and above are risk neutral, we find that the equation analogous to (A1) or (A3) for reinsurance level L simplifies to

$$P_L^* = \left(\frac{n_L}{n_L - 1}\right) p.$$

It then follows from (A5) that

$$P_0^* = \frac{\left(\frac{n_L}{n_L - 1}\right) p}{\prod_{v=1}^{L} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)} \;\Rightarrow\; P_l^* = \frac{\left(\frac{n_L}{n_L - 1}\right) p}{\prod_{v=l+1}^{L} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)}$$

for $l \in [1, L]$, and then from (A6) that

$$Q_l{*} = Q_0{*} - \sum_{v=0}^{l-1} \frac{m}{s_v} \ln \left(\frac{\frac{1}{p} - 1}{\frac{\prod_{z=v+1}^{L} \left(\frac{m_z - 1}{m_z} \right) \left(\frac{n_z - 1}{n_z} \right)}{p} - 1} \right)$$

for $l \in [1, L]$.

To see what happens at reinsurance level $L+1$ (and above), consider the simple case in which $L = 0$ (i.e., all primary insurers are risk neutral). Then

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t) \right) \right]}{\partial y_j^{(0)}} = \sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w) \subseteq M_j^{(0)}} p^w (1-p)^{m_0 - w} \left[\left(1 - \frac{y_j^{(0)}}{\sum_{j'=1}^{n_0} y_{j'}^{(0)}} \right) P_0 - \frac{\sum_{h \in H_j^{(0)}(w)} x_h}{\sum_{i' \in M_j^{(0)}} x_{i'}} \right].$$

Setting this derivative equal to 0 yields

$$\sum_{w=0}^{m_0} \binom{m_0}{w} p^w (1-p)^{m_0 - w} \left[\left(\frac{n_0 - 1}{n_0} \right) P_0{*} - \frac{w}{m_0} \right] = 0,$$

or equivalently,

$$\left(\frac{n_0 - 1}{n_0} \right) P_0{*} - p = 0$$

$$\Rightarrow \; P_0{*} = \left(\frac{n_0}{n_0 - 1} \right) p,$$

as expected.

However, now consider

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t) \right) \right]}{\partial x_j^{(1)}} = \sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w) \subseteq M_j^{(0)}} p^w (1-p)^{m_0 - w} \left[-1 + y_k^{(1)} \frac{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} - x_j^{(1)}}{\left(\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} \right)^2} \frac{\sum_{h \in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i' \in M_j^{(0)}} x_{i'}^{(0)}} \right],$$

which implies

$$\sum_{w=0}^{m_0} \binom{m_0}{w} p^w (1-p)^{m_0 - w} \left[-1 + \frac{w}{m_0} \left(\frac{m_1 - 1}{m_1 P_1{*}} \right) \right] = -1 + \left(\frac{m_1 - 1}{m_1} \right) \left(\frac{n_1 - 1}{n_1} \right) < 0.$$

Thus, the optimal amount of reinsurance for the risk neutral primary insurer is 0. Analogous results hold for higher levels of reinsurance in which the reinsurers are risk neutral.

Proof of Corollary 2:

As in the proof of Corollary 1, consider the simple case in which $L = 0$, and let $y_j^{(0)} = j_0\left(x_j^{(1)}\right)$. It then follows that

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t)\right)\right]}{\partial y_j^{(0)}} =$$

$$\sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w) \subseteq M_j^{(0)}} p^w (1-p)^{m_0 - w} \left[\left(1 - \frac{y_j^{(0)}}{\sum_{j'=1}^{n_0} y_{j'}^{(0)}} \right) P_0 - \frac{dx_j^{(1)}}{dy_j^{(0)}} \right.$$

$$\left. + y_k^{(1)} \frac{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} - x_j^{(1)}}{\left(\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} \right)^2} \frac{\sum_{h \in H_j^{(0)}(w)} x_h}{\sum_{i' \in M_j^{(0)}} x_{i'}} \frac{dx_j^{(1)}}{dy_j^{(0)}} - \frac{\sum_{h \in H_j^{(0)}(w)} x_h}{\sum_{i' \in M_j^{(0)}} x_{i'}} \right].$$

Setting this derivative equal to 0 yields

$$\sum_{w=0}^{m_0} \binom{m_0}{w} p^w (1-p)^{m_0 - w} \left[\left(\frac{n_0 - 1}{n_0} \right) P_0 * - \frac{dx^{*(1)}}{dy^{*(0)}} + \left(\frac{m_1 - 1}{m_1 P_1 *} \right) \frac{w}{m_0} \frac{dx^{*(1)}}{dy^{*(0)}} - \frac{w}{m_0} \right] = 0,$$

or equivalently,

$$\left(\frac{n_0 - 1}{n_0} \right) P_0 * - \frac{dx^{*(1)}}{dy^{*(0)}} + \left(\frac{m_1 - 1}{m_1 P_1 *} \right) p \frac{dx^{*(1)}}{dy^{*(0)}} - p = 0$$

$$\Rightarrow \quad \frac{dx^{*(1)}}{dy^{*(0)}} = \frac{\left(\frac{n_0 - 1}{n_0} \right) P_0 * - p}{1 - \left(\frac{m_1 - 1}{m_1} \right) \left(\frac{p}{P_1 *} \right)}. \qquad \text{(A7)}$$

Furthermore,

$$\frac{\partial E\left[u_{S^{(0)}}\left(S_j^{(0)}(t)\right)\right]}{\partial x_j^{(1)}} =$$

$$\sum_{w=0}^{m_0} \sum_{H_j^{(0)}(w) \subseteq M_j^{(0)}} p^w (1-p)^{m_0 - w} \left[\left(1 - \frac{y_j^{(0)}}{\sum_{j'=1}^{n_0} y_{j'}^{(0)}} \right) P_0 \frac{dy_j^{(0)}}{dx_j^{(1)}} - 1 \right.$$

$$\left. + y_k^{(1)} \frac{\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} - x_j^{(1)}}{\left(\sum_{j' \in M_k^{(1)}} x_{j'}^{(1)} \right)^2} \frac{\sum_{h \in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i' \in M_j^{(0)}} x_{i'}^{(0)}} - \frac{\sum_{h \in H_j^{(0)}(w)} x_h^{(0)}}{\sum_{i' \in M_j^{(0)}} x_{i'}^{(0)}} \frac{dy_j^{(0)}}{dx_j^{(1)}} \right].$$

Setting this derivative equal to 0 implies

$$\sum_{w=0}^{m_0} \binom{m_0}{w} p^w (1-p)^{m_0 - w} \left[\left(\frac{n_0 - 1}{n_0} \right) P_0 * \frac{dy *^{(0)}}{dx *^{(1)}} - 1 + \left(\frac{m_1 - 1}{m_1 P_1 *} \right) \frac{w}{m_0} - \frac{w}{m_0} \frac{dy *^{(0)}}{dx *^{(1)}} \right] = 0,$$

or equivalently,

$$\left(\frac{n_0 - 1}{n_0} \right) P_0 * \frac{dy *^{(0)}}{dx *^{(1)}} - 1 + \left(\frac{m_1 - 1}{m_1 P_1 *} \right) p - p \frac{dy *^{(0)}}{dx *^{(1)}} = 0$$

$$\Rightarrow \quad \frac{dy *^{(0)}}{dx *^{(1)}} = j_0' \left(x *^{(1)} \right) = \frac{1 - \left(\frac{m_1 - 1}{m_1} \right) \left(\frac{p}{P_1 *} \right)}{\left(\frac{n_0 - 1}{n_0} \right) P_0 * - p} \quad . \tag{A8}$$

Solving (A8) as a differential equation (subject to the boundary condition $j_0(0) = 0$) yields

$$y *^{(0)} = j_0' \left(x *^{(1)} \right) x *^{(1)},$$

which generalizes to

$$y *^{(l)} = j_l' \left(x *^{(l+1)} \right) x *^{(l+1)}$$

$$\Leftrightarrow \; Q_l * = n_l j_l' \left(x *^{(l+1)} \right) x *^{(l+1)} \tag{A9}$$

for $l \in [L, r-1]$, where

$$j_l' \left(x^{(l+1)} \right) = \frac{1 - \left(\frac{m_{l+1} - 1}{m_{l+1}} \right) \left(\frac{p}{P_{l+1} *} \right)}{\left(\frac{n_l - 1}{n_l} \right) P_l * - p}.$$

The fact that (A7) and (A8) are equivalent implies that the entire system of first-order conditions is underspecified, with $r - L$ degrees of freedom. Thus, any equilibrium solution will not be unique. However, to maintain continuity between the solution for the case at hand (i.e., $s_l \to 0$ for $l \in [L,r]$) and the solution for the case in which only the reinsurer at level r is risk neutral (i.e., $s_l > 0$ for $l \in [L,r-1]$, but $s_r \to 0$), we may impose the $r-L$ conditions

$$P_l * = P_0 * \prod_{v=1}^{l} \left(\frac{m_v - 1}{m_v} \right) \left(\frac{n_{v-1} - 1}{n_{v-1}} \right),$$

for $l \in [L+1, r]$. These conditions, in conjunction with

$$P_r * = \left(\frac{n_r}{n_r - 1} \right) p$$

(which follows from an equation analogous to (A1) or (A3) for reinsurance level r) imply

$$P_l * = \frac{\left(\frac{n_r}{n_r - 1} \right) p}{\prod_{v=l+1}^{r} \left(\frac{m_v - 1}{m_v} \right) \left(\frac{n_{v-1} - 1}{n_{v-1}} \right)}$$

for $l \in [L+1, r]$. It then follows that (A9) may be rewritten as

$$Q_l * = n_l \left[\frac{\prod_{v=l+1}^{r} \left(\frac{m_v - 1}{m_v} \right) \left(\frac{n_v - 1}{n_v} \right)}{p} \right] \left(\frac{P_{l+1} * Q_{l+1} *}{n_l} \right) = \left(\frac{m_{l+1} - 1}{m_{l+1}} \right) Q_{l+1} *$$

for $l \in [L, r-1]$, or equivalently,

$$Q_l * = \left[\prod_{v=L+1}^{l} \left(\frac{m_v}{m_v - 1} \right) \right] Q_L *$$

for $l \in [L+1, r]$.

For this solution to be feasible, it must be true that all of the risk neutral reinsurers are better off by entering equilibrium than by remaining out of the market. In the simple case of $L = 0$, this means that

$$\frac{R_0}{n_0} + y *^{(0)} P_0 * - x *^{(1)} + \left(\frac{y *^{(1)}}{m_1} - y *^{(0)} \right) p > \frac{R_0}{n_0} + \tilde{y} *^{(0)} \tilde{P}_0 * - \tilde{y} *^{(0)} p$$

(where the tilde denotes the relevant quantity in the absence of reinsurance at the next highest level). More generally,

$$\frac{R_L}{n_L} + y^{*(L)} P_L^* - x^{*(L+1)} + \left(\frac{y^{*(L+1)}}{m_{L+1}} - y^{*(L)}\right) p > \frac{R_L}{n_L} + \tilde{y}^{*(L)} \tilde{P}_L^* - \tilde{y}^{*(L)} p \qquad \text{(A10)}$$

for $l = L$,

$$\frac{R_l}{n_l} + y^{*(l)} P_l^* - x^{*(l+1)} + \left(\frac{y^{*(l+1)}}{m_{l+1}} - y^{*(l)}\right) p > \frac{R_l}{n_l} \qquad \text{(A11)}$$

for $l \in [L+1, r-1]$, and

$$\frac{R_r}{n_r} + y^{*(r)} P_r^* - y^{*(r)} p > \frac{R_r}{n_r} \qquad \text{(A12)}$$

for $l = r$.

Inequality (A12) follows immediately from the fact that

$$P_r^* = \left(\frac{n_r}{n_r - 1}\right) p > p.$$

Rewriting (A10) yields

$$y^{*(L)} (P_L^* - p) - x^{*(L+1)} + \frac{y^{*(L+1)}}{m_{L+1}} p > \tilde{y}^{*(L)} \left(\tilde{P}_L^* - p\right)$$

$$\Leftrightarrow \; y^{*(L)} \left[\frac{\left(\frac{n_r}{n_r - 1}\right) p}{\prod_{v=L+1}^{r} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)} - p \right] - x^{*(L+1)} + \frac{x^{*(L+1)}}{P_{L+1}^*} p > \tilde{y}^{*(L)} \left[\left(\frac{n_L}{n_L - 1}\right) p - p\right]$$

$$\Leftrightarrow \; j_L'\left(x^{*(L+1)}\right) x^{*(L+1)} \left[\frac{\left(\frac{n_r}{n_r - 1}\right) p}{\prod_{v=L+1}^{r} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)} - p \right] - x^{*(L+1)}$$

$$+ \frac{x^{*(L+1)} p}{\left[\frac{\left(\frac{n_r}{n_r - 1}\right) p}{\prod_{v=L+2}^{r} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_{v-1} - 1}{n_{v-1}}\right)} \right]} > j_L'\left(x^{*(L+1)}\right) x^{*(L+1)} \left[\left(\frac{n_L}{n_L - 1}\right) p - p\right], \qquad \text{(A13)}$$

where we make use of the fact that

$$y^{*(L)} = \tilde{y}^{*(L)} = j_L'\left(x^{*(L+1)}\right) x^{*(L+1)} = \left[\frac{\prod_{v=L+1}^{r} \left(\frac{m_v - 1}{m_v}\right)\left(\frac{n_v - 1}{n_v}\right)}{p} \right] x^{*(L+1)}.$$

Inequality (A13) can then be rewritten as

$$\left[\frac{\prod_{v=L+1}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_v-1}{n_v}\right)}{p}\right]x^{*(L+1)}\left[\frac{\left(\frac{n_r}{n_r-1}\right)p}{\prod_{v=L+1}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_{v-1}-1}{n_{v-1}}\right)}-\left(\frac{n_L}{n_L-1}\right)p\right]-x^{*(L+1)}$$

$$+\frac{x^{*(L+1)}\,p}{\left[\frac{\left(\frac{n_r}{n_r-1}\right)p}{\prod_{v=L+2}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_{v-1}-1}{n_{v-1}}\right)}\right]}>0$$

$$\Leftrightarrow\left[\left(\frac{n_L}{n_L-1}\right)-\left(\frac{n_L}{n_L-1}\right)\prod_{v=L+1}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_v-1}{n_v}\right)-1\right.$$

$$\left.+\left(\frac{m_{L+1}}{m_{L+1}-1}\right)\prod_{v=L+1}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_v-1}{n_v}\right)\right]x^{*(L+1)}>0$$

$$\Leftrightarrow\left[\left(\frac{1}{n_L-1}\right)+\left(\frac{m_{L+1}}{m_{L+1}-1}-\frac{n_L}{n_L-1}\right)\prod_{v=L+1}^{r}\left(\frac{m_v-1}{m_v}\right)\left(\frac{n_v-1}{n_v}\right)\right]x^{*(L+1)}>0,$$

which is true because $\frac{m_{L+1}}{m_{L+1}-1}>\frac{n_L}{n_L-1}$ (from Assumption 1).

Finally, (A11) follows by substituting l for L in the proof of (A10), and noting that the right-hand side of (A11) is smaller than that of (A10).

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