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Prices, Asset Markets and Indeterminacy

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Prices, asset markets and indeterminacy [1]

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1 Introduction

Individuals optimize under a multiplicity of constraints, of two types: constraints on expenditures and constraints on net trades.

Constraints on expenditures or budget constraints reflect the structure of assets for the reallocation of revenue across markets. [1] At each market, net expenditure is constrained not to exceed the payoff of assets carried over from preceeding markets.

Constraints on net trades reflect, most interestingly, the information available when net trades in commodities are determined. When capital, physical or human, requires "time to build," net trades are decided at a date - event prior to the date - event at which capital is employed in production. It is then natural that commodities may have non - zero prices at multiple date - events; relative to the consumption sets of individuals and the production sets of firms which capture the restrictions on net trades, [2] the price system may be extended.

This is not only a theoretical possibility. Payment for goods, services, and factors of production often extends to date - events past the date - event at which the good or service is consumed or the factor is employed. Pension plans, bonuses, even alimony payments are examples of deferred payment, contingent on uncertainty realized after trade has occured.

When the asset market is complete, an extension of the price system is of no consequence: via the implicit prices for elementary securities, the prices of a commodity at multiple date - events reduce to a price at the initial date - event, and only then. [3]

When the asset market is incomplete, the transfers of revenue across date - events are restricted. An extension of the price system need not reduce to a price system with a single price associated with each commodity. It is then evident that indeterminacy typically arises, as multiple, non - redundant prices effectively control each commodity market.

When the asset market is complete, competitive equilibrium allocations are typically locally unique or determinate. [4] Real assets, whose payoffs are denominated in commodities, preserve determinacy, even when the asset market is incomplete, [5] though not when the cardinality of the state space is infinite. [6]

We show, here, that, with real assets, competitive equilibrium allocations are typically indeterminate when the asset market is incomplete, while the net trades in commodities are constrained.

Nominal assets, whose payoffs are denominated in abstract units of account

[1] Radner (1972).
[2] Radner (1968).
[3] Arrow (1953), Debreu (1960).
[4] Debreu (1970).
[5] Geanakoplos and Polemarchakis (1986).
[6] Mas - Colell (1991).

typically lead to indeterminacy when the asset market is incomplete. [7] They can be understood as a particular extension of the price system.

In an aggregate model, the indeterminacy of competitive equilibrium allocations when the asset market is incomplete while the net trades in commodities are constrainted accounts for macroeconomic regularities.[8]

2 The Economy

Uncertainty is indexed by states of the world $s \in \mathcal{S} = \{1, \ldots, S\}$, a non - empty, finite set.

Commodities, consumption goods, are indexed by $l \in \mathcal{L} = \{1, \ldots, L\}$, a non - empty, finite set, and are traded in spot markets following the resolution of uncertainty. A commodity bundle in state s is $x_s = (\ldots, x_{(s,l)}, \ldots)'$, a column vector of dimension L, and a commodity bundle is $x = (\ldots, x_s', \ldots)'$, an element of the commodity space, $\mathcal{E}^{SL}$.

Individuals, consumer - investors, are indexed by $h \in \mathcal{H} = \{1, \ldots, H\}$, a non - empty finite set. An individual is described by the triple of characteristics $(\mathcal{X}^h, u^h, w^h)$, where $\mathcal{X}^h \subset \mathcal{E}^{SL}$ is the consumption set, a subset of the commodity space, $u^h : \mathcal{X}^h \to \mathcal{E}$ is the utility function over consumption bundles, elements of the consumption set, and $w^h \in \mathcal{X}^h$ is the endowment, a commodity bundle.

Assumption 1 *For every individual, (i) the consumption set coincides with the interior of the nonnegative orthant of the commodity space,* Int $\mathcal{E}_+^{SL}$; *(ii) the utility function is continuous, strictly monotonically increasing and strictly quasi - concave; on the interior of its domain of definition, it is twice continuously differentiable, differentiably strictly monotonically increasing:* $Du^h \gg 0$, *and differentiably strictly quasi - concave:* D^2u^h *is negative definite on the orthogonal complement of the gradient,* $[Du^h]^\perp; \lim_{n\to\infty}(\|Du^h(x_n)\|)^{-1}\ x_n' Du^h(x_n) = 0$, *for any sequence* $(x_n \in \mathcal{X}^h : n = 1, \ldots)$ *with* $\lim_{n\to\infty} x_n = \overline{x} \in \text{Bd } \mathcal{X}^h, \text{Bd } \mathcal{X}^h \setminus \{0\}$. *(iii) the endowment is a consumption bundle in the consumption set:* $w^h \in \mathcal{X}^h$.

This is strong, but standard. Strict monotonicity eliminates free goods. The boundary condition eliminates corner solutions. Twice differentiability and the curvature condition guarantee the differentiability of demand.

The net trade of an individual in order to consume $x^h \in \mathcal{X}^h$ is

$$z^h = (x^h - w^h) \in \mathcal{X}^h - \{w^h\}.$$

An allocation is an array, $x^{\mathcal{H}} = (\ldots, x^h, \ldots)$, such that, for every individual, $x^h \in \mathcal{X}^h$, while $\sum_{h\in\mathcal{H}} x^h = \sum_{h\in\mathcal{H}} w^h$.

[7] Following Cass (1985), Balasko and Cass (1989), Geanakoplos and Mas - Colell (1989), Polemarchakis (1988).

[8] Dutta and Polemarchakis (1995).

Assets are indexed by $a \in \mathcal{A} = \{1, \ldots, A\}$, a finite set, and are traded in the first period. They pay off in the second period after the realization of uncertainty, before commodity spot markets open. A portfolio is $y = (\ldots, y_a, \ldots)'$, a column vector, an element of the portfolio space, $\mathcal{E}^A$.

Assets are real. The payoffs of assets are denominated in a "numeraire" commodity, $\hat{\ell}$. Across states of the world, $r_a = (\ldots, r_{(s,a)}, \ldots)'$, a column vector of dimension S. The asset structures is described by the matrix of payoffs of assets, $R = (\ldots, r_a, \ldots)$, of dimension $(S \times A)$.

It is not difficult to allow the numeraire commodity to vary across states of the world, even to be a bundle of commodities; what matters is that it coincide across assets.

Assumption 2 *The matrix of payoffs of assets has full column rank, and $A < S$.*

This eliminates redundant assets, with no loss of generality. That $A < S$, guarantees that the asset market is incomplete.

Economies are parametrized by the allocation of endowments, $w^{\mathcal{H}} = (\ldots, w^h, \ldots)$. The space of economies, $\mathcal{O}$, can thus be identified with the interior of the positive orthant, $\times_{h \in \mathcal{H}} \mathcal{X}^h$. A property holds generically if, and only if, it holds for an open set of economies of full lebesgue measure.

Individuals face constraints in the net trade in commodities $l \in \overline{\mathcal{L}} = \{\bar{l}_1, \ldots, \bar{l}_{\overline{L}}\} \subset \mathcal{L}$, while net trades in commodities $l \in \hat{\mathcal{L}} = \{\hat{l}_1, \ldots, \hat{l}_{\hat{L}}\} \subset \mathcal{L}$ are unconstrained.

We write commodity bundles, in particular bundles of net trades, as $z = (\hat{z}, \overline{z})$, where $\hat{z} = (\ldots, \hat{z}_s, \ldots), \hat{z}_s = (\hat{z}_{(s,\hat{l})} : \hat{l} \in \hat{\mathcal{L}})$, is an element of the commodity space of unconstrained commodities, $\mathcal{E}^{S\hat{L}}$, while $\overline{z} = (\ldots, \overline{z}_s, \ldots), \overline{z}_s = (\overline{z}_{(s,\bar{l})} : \bar{l} \in \overline{\mathcal{L}})$, is an element of the commodity space of constrained commodities, $\mathcal{E}^{S\overline{L}}$.

The set of attainable net trades in the constrained commodities is described by the column span of a matrix M of dimension $(S\overline{L} \times K), 0 \leq K \leq S\hat{L}$. The set of attainable net trades of an individual is thus

$$\mathcal{Z}^h = \{z^h = (\hat{z}^h, \overline{z}^h) \in \mathcal{X}^h - \{w^h\} : \overline{z}^h \in [M]\}.$$

We write $M = (\ldots, M_s, \ldots)'$, where M_s is a matrix of dimension $(\overline{L} \times K)$.

Assumption 3 *The matrix M is in general position, in particular it has full column rank, $K, 1 \leq K < S\overline{L}$; furthermore, it is orthonormal: $M'M = I_K$.*

That the constraint matrix M be in general position excludes the case in which net trades are constrained only within each spot market and which is essentially of no consequence. For suppose the net trades of commodities are constrained only within each spot market. Then $M_s = (\ldots, 0, M_{(s,s)}, 0, \ldots)$, where $M_{(s,s)}$ is a matrix of dimension $\overline{L} \times K_s$, and $\Sigma_{s \in S} K_s = K$. Since $K < S\overline{L}, K_s < \overline{L}$, for some s. But then, the matrix M_s has $(K_s + 1) \leq K$ linearly dependent rows, which contradicts the general position of the matrix M. The dimension of the set of attainable net trades for each individual is of dimension at least K; and thus, in particular, if $K \geq \overline{L}$, all net trades are attainable within each spot market, but not independently across spot markets. That $K < S\overline{L}$ guarantees that there are effective constraints on net trades. That the matrix M be orthonormal is with no loss of generality.

Commodity prices in state s are $p_s = (\ldots, p_{(s,l)}, \ldots)$, a row vector of dimension L, and commodity prices are $p = (\ldots, p_s, \ldots) \in \mathcal{P}$, an element of the domain of commodity prices, a subset of $\mathcal{E}^{SL}$. We write prices as $p = (\hat{p}, \overline{p})$, where $\hat{p} = (\ldots, \hat{p}_s, \ldots) \in \hat{\mathcal{P}}$, with $\hat{p}_s = (\hat{p}_{(s,\hat{l})} : \hat{l} \in \hat{\mathcal{L}})$, is an element of the space of unconstrained commodity prices, the interior of the positive orthant, Int $\mathcal{E}_+^{S\hat{L}}$, while $\overline{p} = (\ldots, \overline{p}_s, \ldots) \in \overline{\mathcal{P}}$, with $\overline{p}_s = (\overline{p}_{(s,\overline{l})} : \overline{l} \in \overline{\mathcal{L}})$, is an element of the space of constrained commodity prices, $\mathcal{E}^{S\overline{L}}$.

Any constrained commodity prices $\overline{p} \in \overline{\mathcal{P}}$ can be decomposed uniquely as

$$\overline{p} = m(\overline{p})M' + n(\overline{p}),$$

where $m(\overline{p})$ is a row vector, an element of $\mathcal{E}^K$, while $n(\overline{p}) \in [M]^\perp$, and, moreover, since M is orthonormal, $m(\overline{p}) = \overline{p}M$.

At commodity prices p, the matrix of revenue payoffs of assets is $R(p) = (\ldots, r_a(p), \ldots)$, where $r_a(p) = (\ldots, p_s r_{(s,a)}, \ldots)$ are the revenue payoffs of asset a, a column vector of dimension S.

Assumption 4 *The numeraire commodity, $\hat{\ell}$, in which the payoffs of assets are denominated is unconstrained: $\hat{\ell} \in \hat{\mathcal{L}}$; in particular, $R(p) = R(\hat{p})$. Moreover, $[R(\hat{p})] \cap \mathcal{E}_+^S \backslash \{0\} \neq \emptyset$.*

In economies in which consumption in state $s = 1$ is interpreted as consumption in the first period and asset $a = 1$ as revenue in the first period, the condition $[R(\hat{p})] \cap \mathcal{E}_+^S \backslash \{0\} \neq \emptyset$ is evidently satisfied.

Assumption 5 $H \geq K + 1$.

This allows for sufficient heterogeneity.

At commodity prices p, asset prices are

$$q(p) = 1_S' R(\hat{p}).$$

This is a normalization, with no loss of generality.

The individual optimization problem at commodity prices p and endowment w^h is

$$\begin{array}{ll} \max & u^h(w^h + z), \\ \text{s.t.} & pz \leq 0, \\ & \overline{z} \in [M], \\ & p \otimes z \in [R(\hat{p})]. \end{array}$$

A competitive equilibrium allocation for the economy $w^{\mathcal{H}}$ is an allocation, $x^{\mathcal{H}*} = (\ldots, x^{h*}, \ldots)$, such that, for some commodity prices, p^*, for every individual, $z^{h*} = (x^{h*} - w^h)$ solves the individual optimization problem.

Competitive equilibrium allocations are indeterminate of degree d if and only if the set of competitive equilibrium allocations contains the image, under a one - to - one and continuously differentiable function, of a neighborhood of dimension $d \geq 1$.

A simple economy illustrates the indeterminacy of equilibrum allocations which arises from the interaction of constraints on expenditures and constraints on net trades.

This example is analytically equivalent to the "leading example" introduced to display the indeterminacy of equilibrium allocations when the asset market is incomplete and assets are nominal; [9] which illustrates that the nominal denomination of the payoffs of assets is a special case of an extension of the price system.

Example 1 States of the world are $s \in \{1, 2, 3\}$, and commodities are $l \in \{1, 2\}$.

There are no assets for the transfer of revenue across states.

Net trades in commodity 1 are unconstrained, $\hat{\mathcal{L}} = \{1\}$, while net trades in commodity 2 are constrained, $\overline{\mathcal{L}} = \{2\}$. In particular, the net trade of commodity 2 is restricted to be equal across states of the world. Equivalently, restrictions in net trades are described by the matrix

$$M = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It is pedantic to normalize so that $M'M = 1$. Commodity prices in state s are $p_s = (\hat{p}_{(s,1)}, \overline{p}_{(s,2)})$ and, across states, prices of commodities are $p = (\ldots, p_s, \ldots)$.

At commodity prices p, the constraints under which an individual optimizes are

$$\begin{aligned} &p_s z_s = 0, \quad s \in \{1, 2, 3\}, \\ &z_{(1,2)} = z_{(2,2)}, \\ &z_{(1,2)} = z_{(3,2)}. \end{aligned}$$

Without loss of generality, $\overline{p}_{(s,2)} = 1$, which leaves $\hat{p}_1$ to attain market clearing.

The indeterminacy of equilibrium allocations follows by observing that, since $z_{(s,2)} = z_2$, while $p_s z_s = 0$, if one commodity market clears all do. Thus, $\hat{p}_{(s,1)}$, for $s \in \{2, 3\}$, can be set arbitrarily, and $\hat{p}_{(1,1)}$ employed to attain equilibrium in all markets. Different choices for the commodity prices which can be set arbitrarily typically yield distinct equilibrium allocations: typically, $\hat{p}_{(s,1)} \neq \hat{\hat{p}}_{(s,1)}$ prevents $z = (\ldots, z_s, \ldots)$ from satisfying both $\hat{p}_s z_s = 0$ and $\hat{\hat{p}}_s z_s = 0$.

[9] Balasko and Cass (1989).

Claim 1 *An allocation, $x^{\mathcal{H}*} = (\ldots, x^{h*}, \ldots)$, is a competitive equilibrium allocation if and only if $z^{h*} = (x^{h*} - w^h)$ solves the individual optimization problem*

$$\begin{array}{ll} \max & u^h(\hat{w}^h + \hat{z}, \overline{w}^h + Mv), \\ s.t. & \hat{p}^* \hat{z} + m(\overline{p}^*)v = 0, \\ & (\hat{p}^* \otimes \hat{z} + (m(\overline{p}^*)M' + n(\overline{p}^*)) \otimes Mv) \in [R(\hat{p}^*)], \end{array}$$

for $h \in \mathcal{H} \setminus \{1\}$, and the individual optimization problem

$$\begin{array}{ll} \max & u^h(\hat{w}^h + \hat{z}^h, \overline{w}^h + Mv), \\ s.t. & \hat{p}^* \hat{z} + m(\overline{p}^*)v = 0, \end{array}$$

for $h = 1$, at some commodity prices

$$p^* = (\hat{p}^*, \overline{p}^*) = (\hat{p}^*, m(\overline{p}^*)M' + n(\overline{p}^*)).$$

Proof This is evident. □

Here, v serves to obtain the excess demand for constrained commodities, via $\overline{z} = Mv$.

The subspace of nonarbitrage prices is

$$\mathcal{P}_{NA} = \{p = (\hat{p}, \overline{p}) \in \mathcal{P} : (\hat{z}, Mv) > 0 \Rightarrow \hat{p}\hat{z} + m(\overline{p})v > 0\};$$

equivalently,

$$\mathcal{P}_{NA} = \{p = (\hat{p}, \overline{p}) \in \mathcal{P} : \hat{p} \gg 0 \text{ and } Mv > 0 \Rightarrow m(\overline{p})v > 0\}.$$

Claim 2 *For $(p, w^h) \in \mathcal{P}_{NA} \times \mathcal{X}^h$, a solution to the individual optimization problems, $z^h(p, w^h)$, exists, is unique and satisfies $z^h(p, w^h) + w^h \in \mathcal{X}^h$. As prices and the endowment vary, the excess demand function, $z^h : \mathcal{P}_{NA} \times \mathcal{X}^h \to \mathcal{E}^{SL}$, is continuously differentiable. For $h = 1$, along any sequence $((p_n, w_n^1) \in \mathcal{P}_{NA} \times \mathcal{X}^1 : n = 1, \ldots)$ with $\lim_{n \to \infty}(p_n, w_n^1) = (p, w^1) \in (\text{Bd } \mathcal{P}_{NA} \setminus \{(\hat{p}, m(\overline{p})) = 0\}) \times \mathcal{X}^1$, $\lim_{n \to \infty} \|z^1(p_n, w_n^1)\| = \infty$.*

Proof This is evident. □

The aggregate excess demand function is

$$z^a = \sum_{h \in \mathcal{H}} z^h : \mathcal{P}_{NA} \times \mathcal{O} \to \mathcal{E}^{SL},$$

and, from the excess demand for constrained commodities,

$$v^a = \sum_{h \in \mathcal{H}} v^h : \mathcal{P}_{NA} \times \mathcal{O} \to \mathcal{E}^{K}.$$

Also,

$$\tilde{z}^a = (\hat{z}^a_{\setminus 1}, v^a) : \mathcal{P}_{NA} \times \mathcal{O} \to \mathcal{E}^{S\hat{L}-1+K},$$

where $\hat{z}^a_{\backslash 1}$ is the aggregate excess demand function for unconstrained commodities other than $(1, \hat{l}_1)$.

In order to show the existence of competitive equilibria, we focus on the normalized domain of non - arbitrage prices, which, with no ambiguity, is

$$\mathcal{D}_N = \{(\hat{p}, m) : (\hat{p}, mM') \in \mathcal{P}_{NA}, \text{ and } \|\hat{p}, m\| = 1\}.$$

Observe that the normalization, in addition to fixing the absolute price level, $\|(\hat{p}, m)\| = 1$, eliminates the component of constrained commodity prices normal to the transactions constraints: $n(\overline{p}) = 0$.

The normalized equilibrium manifold is

$$\mathcal{W}_N(M) = \{(\hat{p}, m, w^{\mathcal{H}}) : (\hat{p}, m) \in \mathcal{D}_N \text{ and } z^a_N(p, m, w^{\mathcal{H}}) = 0\},$$

where z^a_N is the restriction of the aggregate excess demand function to the domain $\mathcal{D}_N \times \mathcal{O}$, and similarly for $\tilde{z}^a_N$; on the domain $\mathcal{D}_N \times [M]^\perp \times \mathcal{O}$, where $\mathcal{D}_N \times [M]^\perp = \{(\hat{p}, m, n) : (\hat{p}, m) \in \mathcal{D}_N, n \in [M]^\perp\}$, we write z^a and $\tilde{z}^a$.

Claim 3 *The normalized equilibrium set has the structure of a smooth manifold with* $\dim \mathcal{W}_N(M) = \dim \mathcal{O} = HSL$.

Proof It suffices to observe that $\mathcal{W}_N(M) = (\tilde{z}^a_N)^{-1}(0)$, while, by a standard argument, $\tilde{z}^a_N$ is transverse to 0. □

The natural projection from the normalized equilibrium manifold to the space of initial endowments is

$$\pi : \mathcal{W}_N \to \mathcal{O}.$$

Claim 4 *The map π is proper and surjective.*

Proof Properness follows from the boundary behavior of the excess demand of individual 1, as in claim 2.

To establish that π is surjective, let $w^{\mathcal{H}*}$ be a pareto optimal allocation of endowments. Let p^* be walrasian supporting prices, which are unique by assumption 1. From the modified optimization problem for individual 1, it follows that the unique normalized equilibrium prices are $(\hat{p}^*, m^*) = (\hat{p}^*, (M'M)^{-1}M'\overline{p}^*)$.

Moreover, $w^{\mathcal{H}*}$ is a regular value of π, since the jacobean matrix $D_{(\hat{p},m)}$ $\tilde{z}^a_N(\hat{p}^*, m^*, w^{\mathcal{H}*})$ is invertible.

The invariance of degree mod 2 concludes the argument. □

This claim implies the existence of competitive equilibria for every economy, $w^{\mathcal{H}} \in \mathcal{O}$. Alternatively, exploiting the boundary behavior of individual 1, a standard fixed point argument establishes the existence of equilibria. With the rank of the matrix of asset payoffs in terms of revenue invariant with respect to commodity prices, the existence of competitive equilibria is not surprising. The argument, here, simply shows that the restrictions on net trades or, equivalently, the structure of the effective consumption sets of individuals does not pose a problem.

The demand of individuals $h \in \hat{\mathcal{H}} = \{2, \ldots, K+1\} \subset \mathcal{H}$ defines the matrix

$$V(\hat{p}, m, w^{\mathcal{H}}) = (\ldots, v^h(\hat{p}, m, w^{\mathcal{H}}), \ldots),$$

of dimension $(K \times K)$.

Claim 5 *There exists an open, dense set of full lebesgue measure of endowments, $\mathcal{O}^*$, such that, for $(\hat{p}, m, w) \in \pi^{-1}(\mathcal{O}^*)$, the matrices*

$$V(\hat{p}, m, w^{\mathcal{H}})$$

and

$$D_{(\hat{p},m)} \tilde{z}^a(\hat{p}, m, w^{\mathcal{H}})$$

are invertible.

Proof Consider the map $\tilde{z}^a_N : \mathcal{D}_N \times \mathcal{O} \rightarrow \mathcal{E}^{S\hat{L}+K-1}$. The matrix $D_{w^1} \tilde{z}^a$ has rank equal to $S\hat{L}+K-1$. Hence, $\tilde{z}^a_N$ is transverse to 0. Therefore, the properness of π and the transversality theorem [10] imply that the map $\tilde{z}^a_{N,w^{\mathcal{H}}} : \mathcal{D}_N \rightarrow \mathcal{E}^{S\hat{L}+K-1}$ is transverse to 0 for $w^{\mathcal{H}} \in \mathcal{O}'$, an open subset of $\mathcal{O}$ of full lebesgue measure.

Let $\mathcal{S}^{K-1} = \{\xi \in \mathcal{E}^K : ||\xi|| = 1\}$ be the sphere of dimension $(K-1)$ and consider the funtion $\varphi : \mathcal{D}_N \times \mathcal{O} \times \mathcal{S}^{K-1} \rightarrow \mathcal{E}^{S\hat{L}+K-1} \times \mathcal{E}^K$ defined by

$$\varphi(\hat{p}, m, w^{\mathcal{H}}, \xi) = (\tilde{z}^a(\hat{p}, m, w^{\mathcal{H}}), \xi V(\hat{p}, m, w^{\mathcal{H}})).$$

We want show that φ is transverse to 0. Choose $(\hat{p}, m, w^{\mathcal{H}})$, such that $\varphi(\hat{p}, m$ $w^{\mathcal{H}}) = 0$. Since $\xi \in \mathcal{S}^{K-1}$, without loss of generality $\xi_1 \neq 0$. Let M^1 be the first column of the matrix M. Consider the perturbation of the economy, $\Delta w^{\mathcal{H}} = (\ldots, \Delta w^h, \ldots) = (\ldots, \Delta \hat{w}^h, \Delta \overline{w}^h, \ldots)$ defined by

$$\hat{p} \otimes \Delta \hat{w}^h = mM'M^1, \quad h \in \hat{\mathcal{H}}$$

$$\Delta \overline{w}^h = -M^1, \quad h \in \hat{\mathcal{H}}$$

$$\Delta w^1 = -\textstyle\sum_{h \in \hat{\mathcal{H}}} \Delta w^h,$$

$$\Delta w^h = 0, \quad h \in \mathcal{H} \setminus (\hat{\mathcal{H}} \cup \{1\}).$$

The perturbation $\Delta w^{\mathcal{H}}$ leaves the wealth of every individual in every spot market unchanged. Since $\overline{x}^h(\hat{p}, m, w^h) = Mv^h(\hat{p}, m, w^h) + \overline{w}^h$,

$$D_{(\Delta w^{\mathcal{H}})} \xi V(\hat{p}, m, w^h) = \text{diag}_K(\xi_1).$$

[10] Mas - Colell, A. (1985), *The Theory of General Economic Equilibrium*, Cambridge University Press, Proposition 8.3.1, 320.

Since $D_{w^1} z_N^a$ has full row rank,

$$D_{(w^1,\Delta w^{\mathcal{H}})}\varphi = \begin{pmatrix} D_{w^1} z_N^a & D_{\Delta w^{\mathcal{H}}} z_N^a \\ 0 & \mathrm{diag}_K(\xi_1) \end{pmatrix}$$

has full row rank and, hence, φ is transverse to 0. But then, since π is proper and $\mathcal{S}^{K-1}$ is compact, by the transversality theorem, there exists and open and dense subset of economies, $\mathcal{O}''$, of full lebesgue measure, such that, for all $w \in \mathcal{O}''$, the function

$$\varphi_{w^{\mathcal{H}}} : \mathcal{D}_N \times \mathcal{S}^{K-1} \to \mathcal{E}^{S\hat{L}+K-1} \times \mathcal{E}^K,$$

defined by $\varphi_{w^{\mathcal{H}}}(\hat{p}, m, \xi) = \varphi(\hat{p}, m, w^{\mathcal{H}}, \xi)$ is transverse to 0. Since the dimension of the domain, $(S\hat{L} + K - 1 + K - 1)$, is one less than the dimension of the range, $(S\hat{L} + K - 1 + K), \varphi_{w^{\mathcal{H}}}^{-1}(0) = \emptyset$, for all $w^{\mathcal{H}} \in \mathcal{O}''$. Hence, for $(\hat{p}, m) \in \pi^{-1}(w^{\mathcal{H}}), w^{\mathcal{H}} \in \mathcal{O}''$, $\xi V(\hat{p}, m, w^{\mathcal{H}}) \neq 0$, for all $\xi \in \mathcal{S}^{K-1}$; equivalently, $V(\hat{p}, m, w^{\mathcal{H}})$ is invertible. The subset of economies $\mathcal{O}^* = \mathcal{O}' \cap \mathcal{O}'' \subset \mathcal{O}$ is open and of full lebesgue measure and, for $w^{\mathcal{H}} \in \mathcal{O}^*$, both $V(\hat{p}, m, w^{\mathcal{H}})$ and $D_{(\hat{p},m)}\tilde{z}^a(\hat{p}, m, w^{\mathcal{H}})$ are invertible. □

3 Indeterminacy

Variations of prices of constrained commodities in the orthogonal complement of the span of the matrix of attainable net trades do not affect the value of the net trade of an individual; but the do affect the distribution of expenditures across states of the world. With an incomplete asset market, reallocations of revenue across states of the world are restricted to the span of the matrix of payoffs of assets. Variations of prices of constrained commodities in the orthogonal complement of the span of the matrix of attainable net trades are associated with distinct allocations of commodities at equilibrium, as long as they require variations in the allocation of revenue across states of the world not in the span of the matrix of payoffs of assets. The argument that follows formalizes this intuition.

Recall that $\tilde{z}^a$ is the aggregate excess demand function for commodities other than $(1, \hat{l}_1)$ on the domain $\mathcal{D}_N \times [M]^\perp \times \mathcal{O}$.

Claim 6 *For $w^{\mathcal{H}} \in \mathcal{O}^*$, there exists $(p^*, w^{\mathcal{H}}) = (\hat{p}^*, m^*, w^{\mathcal{H}}) \in \pi^{-1}(w^{\mathcal{H}})$, neighborhoods $\mathcal{B}(\hat{p}^*, m^*) \subset \mathcal{D}_N$ and $\mathcal{B}(0) \subset [M]^\perp$, and a continuously differentiable function $g : \mathcal{B}(0) \to \mathcal{B}(\hat{p}^*, m^*)$, such that*

$$(\hat{p}^*, m^*) = g(0),$$

and

$$\tilde{z}^a(\hat{p}, mM' + n, w^{\mathcal{H}}) = 0 \Leftrightarrow (\hat{p}, m) = g(n).$$

Proof By claim 5, for $w^{\mathcal{H}} \in \mathcal{O}^*, \tilde{z}^a(\hat{p}, m, 0, w^{\mathcal{H}}) = \tilde{z}^a_N(\hat{p}, m, w^{\mathcal{H}}) = 0$ has a solution, $(\hat{p}^*, m^*)$; equivalently, there exists $(\hat{p}^*, m^*, w^{\mathcal{H}}) \in \pi^{-1}(w^{\mathcal{H}})$.

By claim 5, for $w^{\mathcal{H}} \in \mathcal{O}^*$, the jacobean matrix $D_{(\hat{p},m)}\hat{z}^a(\hat{p}^*, m^*, 0, w^{\mathcal{H}}) = D_{(\hat{p},m)}\hat{z}^a_N(\hat{p}^*, m^*, w^{\mathcal{H}})$ is invertible.

The implicit function theorem then concludes the argument. □

For $w \in \mathcal{O}^*$, the allocation function $x : \mathcal{B}(0) \to \times_{h\in\mathcal{H}}\mathcal{X}^h$ is defined by

$$x(n) = (\ldots, w^h + z^h(\hat{p}^*(n), m(n)M' + n, w^h), \ldots),$$

where $(\hat{p}^*, (n), m(n)) = g(n)$. From claim 6, the allocation function is continuously differentiable.

Proposition 1 *Generically, competitive equilibrium allocations are indeterminate of degree $d \geq 1$.*

Proof The argument is based on the properties of $[M]$ and of its orthogonal complement, $[M]^{\perp}$, which we state and prove in claim 7.

Let N be a matrix of dimension $(S\overline{L} - K) \times S\overline{L}$, such that

$$[N'] \;=\; [M]^{\perp}.$$

Since, M is in general position, so is N. Let n_j, for $j = 1, \ldots, (S\overline{L} - K)$, be the jth row of the matrix N, and let

$$N \otimes M \;=\; (\ldots, n_j \otimes M, \ldots),$$

a matrix of dimension $(S \times K(S\overline{L} - K))$.

Claim 7 *The rank of the matrix $N \otimes M$ is $(S - 1)$.*

Proof Let $\lambda = (\ldots, \lambda_s, \ldots)$ be a row vector of dimension S. Observe that

$$\lambda N \otimes M \;=\; \mathbf{1}'_S N \Lambda \otimes M \;=\; (\ldots, n_j \Lambda M, \ldots),$$

where

$$\Lambda \;=\; \begin{pmatrix} \ddots & & \\ & \mathrm{diag}_{\overline{L}}(\lambda_s) & \\ & & \ddots \end{pmatrix}.$$

By the definition of the matrix N,

$$\mathbf{1}'_S N \otimes M \;=\; (\ldots, n_j M, \ldots) \;=\; 0,$$

and hence rank $(N \otimes M) \leq (S-1)$. Suppose that rank $(N \otimes M) < (S-1)$. Then, there exists a vector $\lambda \neq 0$, of dimension S, such that $\lambda \neq \mathbf{1}_S$ and $\lambda(N \otimes M) = 0$.

Equivalently, $n_j \Lambda M = 0$, for $j = 1, \ldots, (\overline{L}S - K)$, or $N\Lambda M = 0$. Since M has full column rank,

$$[\Lambda N'] = [N'],$$

which contradicts the general position of N'. □

Since the set $\mathcal{B}(0)$ is connected, while the equilibrium allocation function is continuously differentiable, it suffices to show that

$$x(n) \neq x(0), \quad \text{for some } n \in \mathcal{B}(0).$$

To derive a contradiction, suppose $x(n) = x(0)$, for all $n \in \mathcal{B}(0)$.

From the optimization problem, (3), of individual 1, $g(n) = g(0) = (\hat{p}^*, m^*)$ or

$$p(n) \;=\; (\hat{p}(n), m(n)M' + n) \;=\; (\hat{p}^*, m^*M' + n), \quad n \in \mathcal{B}(0).$$

For individuals $h \in \mathcal{H}\backslash\{1\}$, let $(\hat{z}^{h*}, \hat{v}^{h*}) = (z^h(p^*, m^*M', w^h), v^h(p^*, m^*M', w^h))$, and let $(\hat{z}^h(n), v^h(n)) = (\hat{z}^h(\hat{p}^*, m^*M' + n, w^h),\ v^h(\hat{p}^*, m^*M' + n, w^h))$, for $n \in \mathcal{B}(0)$. From the optimization problem, (2), of these individuals,

$$(\hat{p}^* \otimes \hat{z}^{h*} + m^*M' \otimes Mv^{h*}) \in [R(\hat{p}^*)].$$

Therefore, $(\hat{z}^{h*}, v^{h*}) = (\hat{z}^h(n), v^h(n))$ and $n \in \mathcal{B}(0)$ only if

$$(\hat{p}^* \otimes \hat{z}^h(n) + (m^*M' + n) \otimes Mv^h(n)) \in [R(p^*)], \quad n \in \mathcal{B}(0).$$

By claim 4, the matrix $V(\hat{p}^*, m^*, w^{\mathcal{H}})$ is invertible. Therefore,

$$[n \otimes M] \subset [R(\hat{p}^*)], \quad \text{for all } n \in \mathcal{B}(0).$$

Since $\mathcal{B}(0) \subset [M]^\perp$ is an open ball relative to $[M]^\perp$, there exist $(S\overline{L} - K)$ linearly independent vectors, $n_j \in \mathcal{B}(0)$, for $j = 1, \ldots, (S\overline{L} - K)$, that span $[M]^\perp$. Since $\mathbf{1}_S \notin [R(\hat{p}^*)]^\perp$ and $A < S$, it follows from claim 7 that there exists at least one j, such that

$$[n_j \otimes M] \not\subset [R(\hat{p}^*)], \quad n_j \in \mathcal{B}(0),$$

a contradiction. □

If $\overline{L} \geq K$, we can refine the result.

Corollary If $\overline{L} > K$, generically, equilibrium allocations are indeterminate of degree $d \geq \max\{1, (S - A - 1)K\}$.

Proof The argument evolves in several steps. Let $(w^{\mathcal{H}}, \hat{p}^*, m^*) \in \pi^{-1}(w^{\mathcal{H}})$. Select $\varepsilon > 0$ small enough to guarantee that $V(\hat{p}, mM' + n, w^{\mathcal{H}})$ is invertible, for $n \in B^{\varepsilon}(0) \subset [M]^{\perp}$, where $(\hat{p}, m) = g(n)$. The existence of such an ε follows from claims 5 and 6.

Step 1 Let n and $n' \in B^{\varepsilon}(0), n \neq n'$. Let $g(n) = (\hat{p}, m)$. If $(n-n') \otimes M \not\subset [R(\hat{p})]$ then $x(n) \neq x(n')$.

Proof Suppose by contradiction, that $x(n) = x(n')$. By the maximization problem of individual 1,

$$g(n) \;=\; g(n') \;=\; (\hat{p}, m),$$

while, by the maximization problems of $h \in \mathcal{H} \setminus \{1\}$,

$$\hat{p} \otimes \hat{z}^{h}(n) + (mM' + n) \otimes Mv^{h}(n) \;\in\; [R(\hat{p})],$$

and

$$\hat{p} \otimes \hat{z}^{h}(n) + (mM' + n') \otimes Mv^{h}(n) \;\in\; [R(\hat{p})].$$

Hence, claim (5) implies that

$$(n - n') \otimes M \;\subset\; [R(\hat{p})],$$

a contradiction. $\square$

Step 1 establishes as a sufficient condition for indeterminacy that $[(n - n') \otimes M] \not\subset [R(\hat{p})]$ or, equivalently, that $\text{proj}_{[R(\hat{p})]^{\perp}}[(n - n') \otimes M] \neq 0, n, n' \in B^{\varepsilon}(0), n \neq n'$.

Let

$$\mathcal{G} \;=\; \{G : G \text{ is a matrix of dimension } S \times K, \text{ and } \mathbf{1}_S' G = 0\}.$$

Since each column of a matrix $G \in \mathcal{G}$ is an element of $[\mathbf{1}_S]^{\perp}, \mathcal{G}$ is a linear subspace of dimension $(S-1)K$.

Since $M = (\ldots, M_s, \ldots)'$ is in general position and $\overline{L} \geq K, M_s$, a matrix of dimension $\overline{L} \times K$, has rank K. Hence, define

$$\overline{N} \;=\; \{n = (\ldots, n_s, \ldots) \in [M]^{\perp} : n_s \in [M_s]\},$$

a linear subspace of dimension $(S-1)K$. Evidently, for each $G \in \mathcal{G}$ there exists a unique $n \in \overline{N}$ such that $n \otimes M = G$ and for each $n \in \overline{N}$ there exists a unique $G \in \mathcal{G}$ such that $G = n \otimes M$. Let $w^{\mathcal{H}} \in \mathcal{O}^{*}$ and $(w, \hat{p}, m) \in \pi^{-1}(w)$. Suppose that $\mathbf{1}_S \notin [R(\hat{p})]$. Then each matrix $\overline{R}^{\perp}(\hat{p})$ of dimension $S \times (S - A - 1)$ that solves

$$\overline{R}^{\perp}(\hat{p})'(\mathbf{1}_S, R(\hat{p})) \;=\; 0$$

has rank equal to $(S-A-1)$. Moreover, for any matrix C of dimension $(S-A-1)\times K, \overline{R}^{\perp}(\hat{p})C$ is an element of G and can be, therefore, generated as $n\otimes M$, for some $n\in\overline{N}$.

As a first step we have, therefore, to show that typically $\mathbf{1}_S\notin[R(\hat{p})]$.

Step 2 For an open and dense and full lebesgue measure subset, $\overline{\mathcal{O}}$, and $(w^{\mathcal{H}},\hat{p},m)\in\pi^{-1}(w), w^{\mathcal{H}}\in\overline{\mathcal{O}}, \mathbf{1}_s\notin R(\hat{p})$.

Proof Since asset payoffs are denominated in an unconstrained numeraire commodity, $\hat{\ell}_1\in\hat{\mathcal{L}}$,

$$R(\hat{p}) \;=\mathrm{diag}_s(\hat{p}_{\hat{\ell}_1})R.$$

Let $\psi: D_N\times\mathcal{O}\times E^A\to E^{S\hat{L}+K-1}\times E^A$ be a smooth map defined by

$$\psi(\hat{p},m,w^{\mathcal{H}},y) \;=\; (\tilde{z}^a(\hat{p},m,w^{\mathcal{H}}),R(\hat{p})y-\mathbf{1}_S).$$

Let $(\hat{p}^*,m^*,w^{\mathcal{H}*},y^*)$ be a zero of ψ. Since $R(\hat{p}^*)y^*=\mathbf{1}_S, \hat{p}^*_{(s,\ell_1)}\sum_{a\in\mathcal{A}}r^*_{(s,a)}y_a=1$, for $s\in S$. Hence

$$D_{\hat{p}_{\hat{\ell}_1}}(R(\hat{p})y-\mathbf{1}_S)\bigg|_{(\hat{p}^*,y^*)} \;=\; (diag_s(\hat{p}_{\hat{\ell}_1}))^{-1}.$$

Then

$$D_{(w^{\mathcal{H}},\hat{p}_{\ell_1})}\psi\bigg|_{(w^*,\hat{p}^*,m^*,y^*)} \;=\; \begin{pmatrix} D_{w^{\mathcal{H}}}\tilde{z}^a, & D_{\hat{p}_{\hat{\ell}_1}}\tilde{z}^a \\ 0, & (\mathrm{diag}_s(\hat{p}_{\hat{\ell}_1}))^{-1}\end{pmatrix}.$$

Hence ψ is transverse 0. The properness of π and the transversality theorem imply that $\psi_{w^{\mathcal{H}}}$ is transverse to 0, for $w^{\mathcal{H}}\in\overline{\mathcal{O}}$, an open, dense and full lesbegue measure subset of $\mathcal{O}$. But $\psi_{w^{\mathcal{H}}}: D_N\times E^A\to E^{S\hat{L}+K-1+S}$ and, since $A<S$, the dimension of the domain of $\psi_{w^{\mathcal{H}}}$ is less than
$(S\hat{L}+K-1+S)$. Hence, $\psi^{-1}_{w^{\mathcal{H}}}(0)=\emptyset$, for $w\in\overline{\mathcal{O}}$. □

Let $\mathcal{O}^{**}=\overline{\mathcal{O}}\cap\mathcal{O}^*$ an open, dense and full lesbegue measure subset of $\mathcal{O}$. Let $(w^{\mathcal{H}},\hat{p}^*,m^*)\in\pi^{-1}(w^{\mathcal{H}}), w^{\mathcal{H}*}\in\mathcal{O}^*$. Select $\varepsilon>0$ and define

$$\begin{aligned}\overline{B}^{\epsilon}(0) \quad = \quad & \{n\in B^{\epsilon}(0)\cap\overline{N}: n\otimes M=\overline{R}^{\perp}(\hat{p}^*)'C, \text{ for} \\ & \text{some matrix } C \text{ of dimension } (S-A-1)\times K\}.\end{aligned}$$

By the definition of $\overline{B}^{\epsilon}(0)$, there exists $\eta>0$, arbitrarily small, such that to each matrix C of dimension $(S-A-1)\times K$, with $||C||<\eta$, there corresponds a unique $n\in\overline{B}^{\epsilon}(0)$ with $n\otimes M=\overline{R}^{\perp}(\hat{p}^*)C$, and vice versa.

Hence, $\overline{B}^{\epsilon}(0)$ is essentially diffeomorphic to an open ball of $E^{(S-A-1)K}$. Furthermore, ε can be chosen small enough to guarantee that the matrix $V(\hat{p}, mM' + n, w^{\mathcal{H}})$, defined by analogy with $V(\hat{p}, m, w^{\mathcal{H}})$, is invertible, for $n \in \overline{B}^{\epsilon}(0)$ and $(\hat{p}, m) = g(n)$.

Step 3 Let $S - A > 1$ and $w^{\mathcal{H}} \in \mathcal{O}^{**}$. Then, for n and $n' \in \overline{B}^{\epsilon}(0), n \neq n' \Rightarrow x(n) \neq x(n')$.

Proof Suppose by contradiction that $x(n) = x(n')$. Then, as we have already argued, it must be $g(n) = g(n') = (\hat{p}, m)$. Since assets are denominated in a numeraire commodity there exists a diagonal matrix of dimension $(S \times S) = \text{diag}_s(\hat{p}_{\mathcal{L}_1})(\text{diag}_s(\hat{p}^*_{\mathcal{L}_1}))^{-1}$, with strictly positive elements and such that

$$\Lambda R(\hat{p}^*) = R(\hat{p}), \quad (\hat{p}^*, m^*) = g(0).$$

Since $g(n) = g(n'), x(n) = x(n')$ implies that $(n - n') \otimes M \subset [R(\hat{p})]$ or, equivalently,

$$(n - n') \otimes M = R(\hat{p})B,$$

for some matrix B of dimension $(A \times K)$, $(R(\hat{p})B \neq 0$, since n and $n' \in \overline{N})$. But then, since n and $n' \in \overline{B}(0)$ and $n \neq n'$,

$$(n - n') \otimes M = \overline{R}^{\perp}(\hat{p}^*)'C, \quad C \neq 0.$$

Hence,

$$\Lambda R(\hat{p}^*)B = \overline{R}^{\perp}(\hat{p}^*)C \Rightarrow R(\hat{p}^*)'\Lambda R(\hat{p}^*)B = R(\hat{p}^*)'\overline{R}^{\perp}(\hat{p}^*)C.$$

But by the definition of $\overline{R}^{\perp}(\hat{p}^*), R(\hat{p}^*)'\overline{R}^{\perp}(\hat{p}^*)' = 0$, while $R(\hat{p}^*)'\Lambda R(p^*)B \neq 0$, since $B'R(\hat{p}^*)'\Lambda R(p^*)B \neq 0$, a contradiction. □

Hence by the proposition $d \geq 1$, while from step 3, if $(S - A) > 1$ and $\overline{L} > K$, $d \geq (S - A - 1)K$. Hence, $d \geq \max(1, (S - A - 1)K)$.

Remark 1 In the special case in which there are no assets, $A = 0$, the exact degree of indeterminacy generically, coincides with the dimension of $[M]^{\perp} : d = S\overline{L} - K$.

Remark 2 Consider the canonical model of an economy in which the asset market is incomplete, assets are nominal: their payoffs are denominated in abstract units of account, and there are no constraints on net trades. The matrix of asset payoffs is, without loss of generality $R = (R_1, I_A)'$, where R_1 is a matrix of dimension $(S - A) \times A$. A net trade is $z = (z_0, z_1, z_2)'$, where z_0 is the net trade in commodities in the first, asset trading period, z_1 is the net trade in the second period in states $s = 1, \ldots, (S - A)$, and z_2 is the net in the second period in states $s = (S - A + 1), \ldots, S$. Similarly, commodity prices are $p = (p_0, p_1, p_2)$. The price of units of account are $\delta = (\delta_1, \delta_2)$ and, by an abuse of notation, $\Delta = (\Delta_1, \Delta_2)$ is the associated diagonal matrix. The constraints in the individual optimization problem are

$$pz \leq 0,$$

$$p \otimes z \in [\Delta R].$$

Equivalently,

$$p_0 z_0 + (\mathbf{1}'_{(S-A)} \Delta_1 R_1 \Delta_2^{-1} + \mathbf{1}'_A) p_2 \otimes z_2 = 0,$$
$$p_1 \otimes z_1 - (\Delta_1 R_1 \Delta_2^{-1}) p_2 \otimes z_2 = 0.$$

Let $\hat{z} = (z_0, z_1)', \hat{p} = (p_0, p_1), M' = (S - A + 1)^{-1/2}(\ldots, I_{AL}, \ldots)', m = (S - A + 1)^{-1/2}(p_{2,(S-A+1)} \cdots, p_{2,S}), v = (S - A + 1)^{1/2} z_2$. The constraints take the form

$$\hat{p} \otimes \hat{z} + (mM' + n) \otimes Mv = 0,$$

where

$$n_s = -\delta_s\left(\frac{r_{(s,1)}}{\delta_{(S-A+1)}} p_{(2,S-A+1)}, \ldots, \frac{r_{(s,A)}}{\delta_S} p_{(2,S)}\right) - (S - A + 1)^{-1/2} m,$$

$$s = 1, \ldots, (S - A,)$$

and

$$n_0 = \left(\left(\sum_{s=1}^{(S-A)} \frac{r_{(s,1)}\delta_s}{\delta_{(S-A+1)}} + 1\right) p_{(2,(S-A+1))}, \ldots, \left(\sum_{s=1}^{(S-A)} \frac{r_{(s,A)}\delta_s}{\delta_S} + 1\right) p_{(2,S)}\right) +$$

$$(S - A)(S - A + 1)^{-1/2} m.$$

This is a special case of the model with constraints on net trades and no assets, in which case the constraint $pz \leq 0$ is redundant, since the constraint $p \otimes z \in [R(p)]$ reduces to $p \otimes z = 0$. Note that, in order to obtain the model with nominal assets, we restrict the variation of prices to a subset of the domain of restricted commodity prices.

Remark 3 It may seem that there is a "natural" restriction on prices, which eliminates the indeterminacy: the restriction of prices to the dual of the subspace on which net trades are restricted, $[M]$. Evidently, this is ad hoc. Perhaps more convincingly, such a restriction is not possible when the constraints on net trades differ across individuals, a generalization of the model.

Notation

- "$'$" denotes the transpose.
- "$\mathcal{E}^K$" denotes the euclidean space of dimension K; we write $\mathcal{E}$, for $\mathcal{E}^1$.
- "$\mathcal{E}_+^K$" denotes the non - negative orthant of $\mathcal{E}^K$.
- "$\mathcal{A} \setminus \mathcal{B}$ denotes the set $\{a \in \mathcal{A} : a \notin \mathcal{B}\}$.
- " $\mathcal{A} - \mathcal{B}$" denotes the set $\{c : c = a - b, a \in \mathcal{A}, b \in \mathcal{B}\}$.
- "[ ]" denotes the column space of a matrix and "$[\ ]^\perp$" its orthogonal complement.
- "Int" denotes the interior and "Bd" the boundary of subsets of euclidean space.
- "I_K" denotes the identity matrix of order K.
- For vectors $p = (\ldots, p_s, \ldots)$ and $z = (\ldots, z_s, \ldots)$, $p \otimes z = (\ldots, p_s z_s, \ldots)$, and similarly for $\hat{p} \otimes \hat{z}$ and $\overline{p} \otimes \overline{z}$; for matrices $N = (\ldots, n_k', \ldots)'$, where $n_k = (\ldots, n_{(k,s)}, \ldots)'$, and $M = (\ldots, M_s, \ldots)'$, $n_k \otimes M = (\ldots, n_s M_s, \ldots)'$, and $N \otimes M = (\ldots, (n_k \otimes M)', \ldots)$.
- "$\mathbf{1}_N$" denotes the column vector of 1's of dimension N.
- "$\text{diag}_K(a)$" denotes the diagonal matrix of dimension $(K \times K)$ with a across the diagonal.

References

[1] Arrow,K.J.(1953),"Le rôle des valeurs boursières pour la repartition la meilleure des risques," *Econometrie,* 11, 41 - 47.

[2] Balasko, Y. and D. Cass (1989), "The structure of financial equilibrium I: exogenous yields and unrestricted participations," *Econometrica,* 57, 135 - 162.

[3] Cass, D. (1985), "On the 'number' of equilibrium allocations with incomplete financial markets," mimeo, CARESS, University of Pennsylvania.

[4] Debreu, G. (1960), "Une Economie de l'incertain," *Economie Appliquée,* 13, 111 - 116.

[5] Debreu, G. (1970), "Economies with a finite set of equilibria," *Econometrica,* 38, 387 - 392.

[6] Dutta, J. and H.M. Polemarchakis (1995), "Time to build and equilibrium contracts," Discussion Paper no 9550, CORE, Universit Catholique de Louvain.

[7] Geanakoplos, J.D. and A. Mas - Colell (1989), "Real indeterminacy with financial assets," *Journal of Economic Theory,* 47, 22 - 38.

[8] Geanakoplos, J.D. and H.M. Polemarchakis (1986), "Existence, regularity and constrained suboptimality of competitive allocations when the asset market is incomplete," in W.P. Heller, R.M. Starr and D. Starrett (eds.), *Uncertainty, Information and Communication: Essays in Honor of K.J. Arrow, Vol. III,* Cambridge University Press, 65 - 96.

[9] Mas - Colell, A. (1991), "Indeterminacy in incomplete market economies," *Economic Theory,* 1, 45 - 61.

[10] Polemarchakis, H.M. (1988), "Portfolio choice, exchange rates and indeterminacy," *Journal of Economic Theory,* 46, 414 - 421.

[11] Radner, R. (1968), "Competitive equilibrium under uncertainty," *Econometrica,* 36, 31 - 58.

[12] Radner, R. (1972), "Equilibrium of plans, prices and price expectations," *Econometrica,* 40, 289 - 303.