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5-1-1996

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COWLES FOUNDATION DISCUSSION PAPER NO. 1124

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ABSTRACT

This paper considers tests for seasonal and non-seasonal serial correlation in time series and in the errors of regression models. The problem of testing for white noise against multiplicative seasonal ARMA(1,1)–ARMA(1,1) alternatives is investigated. This testing problem is non-standard due to nuisance parameters that appear under the alternative but not under the null hypothesis. The likelihood ratio (LR), sup Lagrange multiplier (LM), and exponential average LM and LR tests are considered and are shown to be asymptotically admissible for multiplicative seasonal ARMA(1,1)–ARMA(1,1) alternatives. In addition, they are shown to be consistent against all (weakly stationary strong mixing) non-white noise alternatives. Simulation results compare the tests to several tests in the literature. The exponential average test, Exp–LR_∞, is found to be the best test overall. It performs substantially better than the Box–Pierce, Durbin–Watson, and Wallis tests.

KEYWORDS

Autoregressive moving average model, consistent test, Lagrange multiplier test, likelihood ratio test, multiplicative seasonal ARMA model, nonstandard testing problem, seasonality, test of white noise.

1. INTRODUCTION

There are numerous tests available in the literature for testing for nonseasonal serial correlation and some available for testing for purely seasonal serial correlation. There are no tests available, however, for testing for both seasonal and non-seasonal serial correlation — a scenario that is likely to be common in practice. The present paper addresses this deficiency in the literature. It develops tests that are designed for a flexible class of time series models that includes pure seasonal, pure non-seasonal, and, most importantly, partially seasonal-partially non-seasonal models.

Specifically, this paper considers tests of serial correlation that are designed for multiplicative seasonal doubly autoregressive moving average (ARMA) models of order (1,1) under the alternative hypothesis. It is natural to consider tests of this sort, because multiplicative seasonal ARMA(1,1)–ARMA(1,1) models provide parsimonious representations of a broad class of seasonal and non-seasonal stationary time series. The multiplicative seasonal model includes as special cases the purely non-seasonal AR(1) and MA(1) models for which the Durbin and Watson (1950) test is designed, the purely non-seasonal ARMA(1,1) model for which the tests of Andrews and Ploberger (1996) are designed, and the purely seasonal AR(1) and MA(1) models for which Wallis' (1972) seasonality test is designed. It does not encompass the less parsimonious purely non-seasonal AR(p) model for which the Box and Pierce (1970) test is designed.

Testing for serial correlation in a multiplicative seasonal ARMA(1,1)–ARMA(1,1) model is a non-standard testing problem, because the model reduces to a white noise model whenever the AR and MA seasonal coefficients are equal and the AR and MA non-seasonal coefficients are equal. In consequence, the testing problem is one in which a nuisance parameter is present only under the alternative hypothesis. Davies (1977, 1987), Andrews and Ploberger (1994, 1995), and Hansen (1996) have considered problems of this sort. The standard likelihood ratio (LR) statistic does not possess its usual chi-square asymptotic distribution or its usual asymptotic optimality properties (of the sort established by Wald (1943)) in such cases. Nevertheless, Andrews and Ploberger (1996) show that the standard LR test and an asymptotically equivalent "sup" Lagrange multiplier (LM) test do possess an asymptotic admissibility property. In addition, Andrews and

Ploberger (1994) derive a class of tests, denoted average exponential tests, that possess certain asymptotic optimality properties for testing problems of the sort discussed above.

This paper proceeds in an analogous manner to that of Andrews and Ploberger (1996), which considers tests of serial correlation in a purely non-seasonal ARMA(1,1) model. This allows us to utilize proofs in the latter paper, which streamlines the exposition of the present paper. In particular, this paper proceeds as follows. The results of Andrews and Ploberger (1994, 1995) are general results that impose "high-level" assumptions. In this paper, we show in Section 2 that these results apply to the problem of testing for serial correlation in multiplicative seasonal ARMA(1,1)–ARMA(1,1) models. We provide explicit expressions for the average exponential, sup LM, and LR test statistics for the problem at hand. We then show that the corresponding tests have the attractive feature of being consistent against all forms of serial correlation. In consequence, the average exponential tests possess asymptotic optimality properties for a parametric class of alternatives and the robustness property of consistency against all (weakly stationary strong mixing) alternatives.

In Section 3, we derive the LR and LM tests, denoted LR_1 and LM_1, for the multiplicative seasonal AR(1)–AR(1) model. These tests are of standard type and have an asymptotic chi-square null distribution. They are not, however, consistent against all forms of serial correlation. In Section 4, we show that the tests introduced in Sections 2 and 3 apply to testing for serial correlation in the errors of (non-dynamic) regression models.

Lastly, in Section 5, we compare by simulation the tests introduced in Sections 2 and 3 with several tests in the literature. The exponential average test, Exp–LR_∞, is found to be the best test overall. It beats the LR test (for the ARMA(1,1)–ARMA(1,1) model) by a narrow margin, the sup LM, LR_1, and LM_1 tests by a wider margin, and the Box–Pierce, Durbin–Watson, and Wallis tests by a very substantial margin.

All limits below are taken as $T \to \infty$ unless specified otherwise.

2. TESTS OF SERIAL CORRELATION FOR MULTIPLICATIVE SEASONAL ARMA(1,1)–ARMA(1,1) PROCESSES

2.1. Definition of the Model and Test Statistics

The model we consider here is the multiplicative ARMA(1,1)–ARMA(1,1) model:

$$(1-(\pi_s+\beta_s)L^s)(1-(\pi_n+\beta_n)L)Y_t = (1-\pi_s L^s)(1-\pi_n L)\varepsilon_t \quad \text{for} \quad t = s+1,\, s+2,\, \ldots\,, \tag{2.1}$$

where $\{Y_t : t = 1, \ldots, T\}$ are observed random variables (rv's), $\{\varepsilon_t : t = 1, 2, \ldots\}$ are unobserved innovations, s (> 1) is the integer-valued seasonal period — usually 4 or 12, L is the lag operator (i.e., $LY_t = Y_{t-1}$), π_n and π_s are the non-seasonal and seasonal moving average parameters, respectively, and $\pi_n + \beta_n$ and $\pi_s + \beta_s$ are the non-seasonal and seasonal autoregressive parameters respectively. We let

$$\pi = (\pi_n, \pi_s)' \quad \text{and} \quad \beta = (\beta_n, \beta_s)'\ . \tag{2.2}$$

The parameter space for π is Π and for β is B. Throughout the paper, we assume Π and B are such that the absolute values of the autoregressive coefficients $\pi_n + \beta_n$ and $\pi_s + \beta_s$ are bounded below one, Π is closed, and B contains a neighborhood of the zero vector. The former condition rules out unit root and explosive behavior of $\{Y_t : t = 1, 2, \ldots\}$.

We are interested in testing the null hypothesis of white noise against the alternative of serial correlation of $\{Y_t : t = 1, 2, \ldots\}$. These hypotheses are given by

$$\mathrm{H}_0 : \beta = 0 \quad \text{and} \quad \mathrm{H}_1 : \beta \neq 0\ . \tag{2.3}$$

Note that when $\beta = 0$, the model (2.1) reduces to $Y_t = \varepsilon_t$ and the parameter π is no longer present. In consequence, the above testing problem is non-standard. Also note that when $\beta_n = 0$, the model (2.1) reduces to a purely seasonal ARMA(1,1) model: $(1-(\pi_s+\beta_s)L^s)Y_t = (1-\pi_s L^s)\varepsilon_t$ and the parameter π_n is no longer present. Similarly, when $\beta_s = 0$, the model (2.1) reduces to a purely non-seasonal ARMA(1,1) model: $(1-(\pi_n+\beta_n)L)Y_t = (1-\pi_n L)\varepsilon_t$ and the parameter π_s is no longer present.

Under the following assumption, we derive the standard LR, sup LM, and the average exponential tests.

Assumption 1: *$\{\varepsilon_t : t = 1, 2, ...\}$ is a sequence of* iid *$N(0, \sigma^2)$ rv's for some $\sigma^2 > 0$ and $Y_1, ..., Y_s$ are non-random.*

Model (2.1) and Assumption 1 are used to generate the test statistics of interest, but we consider the asymptotic properties of these tests below under a much more general specification of the distribution of the time series $\{Y_t : t \geq 1\}$. The assumption on $Y_1, ..., Y_s$ is made for simplicity. With some added complexity, we could assume $\{Y_t : t = 1, 2, ...\}$ is part of a doubly infinite sequence of stationary rv's that satisfy (2.1) for all $t = ..., 0, 1, ...$.

The standard LR statistic equals minus two times the logarithm of the likelihood ratio. Because the parameter π only appears in the denominator of the ratio, the unrestricted maximum of the likelihood function with respect to this parameter can be calculated after the ratio has been computed for a given π. That is, let $LR_T(\pi)$ denote the standard LR statistic for testing H_0 versus H_1 when π is known under the alternative to equal π. Then, the LR test statistic for unknown π is

$$LR = \sup_{\pi \in \Pi} LR_T(\pi) , \quad \text{where} \tag{2.4}$$

$$LR_T(\pi) = T^* \log(\widetilde{\sigma}^2/\widehat{\sigma}^2(\pi)) ,$$

$$T^* = T - s ,$$

$$\widetilde{\sigma}^2 = \tfrac{1}{T^*}\Sigma_{t=s+1}^{T} Y_t^2 ,$$

$$\widehat{\sigma}^2(\pi) = \tfrac{1}{T^*}\Sigma_{t=s+1}^{T} \left(Y_t - \widehat{\beta}_n(\pi)D_{1t}(\pi_n) - \widehat{\beta}_s(\pi)D_{2t}(\pi_s) + \widehat{\beta}_n(\pi)\widehat{\beta}_s(\pi)D_{3t}(\pi)\right)^2 ,$$

$$\widehat{\beta}(\pi) = \left(\widehat{\beta}_n(\pi), \widehat{\beta}_s(\pi)\right)'$$

$$= \arg\min_{\beta \in B} \tfrac{1}{T^*}\Sigma_{t=s+1}^{T} \left(Y_t - \beta_n D_{1t}(\pi_n) - \beta_s D_{2t}(\pi_s) + \beta_n\beta_s D_{3t}(\pi)\right)^2 ,$$

$$D_{1t}(\pi_n) = \Sigma_{i=0}^{t-2}\pi_n^i Y_{t-i-1} ,$$

$$D_{2t}(\pi_s) = \Sigma_{i=0}^{[(t-s-1)/s]}\pi_s^i Y_{t-is-s} ,$$

$$D_{3t}(\pi) = \Sigma_{i=0}^{[(t-s-2)/s]}\pi_s^i \Sigma_{j=0}^{t-is-s-2}\pi_n^j Y_{t-is-s-j-1} ,$$

and $[\cdot]$ denotes the "integer part of." (By definition, $D_{3t}(\pi) = 0$ for $t = s+1$.)

The calculation of the LR statistic requires the calculation of $\widehat{\beta}(\pi)$ for each $\pi \in \Pi$. The latter are nonlinear optimization problems. More specifically, they are linear regression problems with three regressors and with the nonlinear restriction that the third regression coefficient equals the product of the first two. One can obtain starting values, say $\widehat{\beta}_1(\pi)$, by computing the unrestricted linear regression (i.e., regress Y_t on $D_{1t}(\pi_n)$, $D_{2t}(\pi_s)$, and $D_{3t}(\pi)$) and taking $\widehat{\beta}_1(\pi)$ to be the two-vector of least squares coefficients on $(D_{1t}(\pi_n), D_{2t}(\pi_s))'$. Given these starting values, one can compute $\widehat{\beta}(\pi)$ using a standard local optimization routine, such as a Newton–Raphson algorithm.

Note that the calculation of LR is simple if one has already computed the maximum likelihood (ML) estimator for the multiplicative seasonal ARMA(1,1)-ARMA(1,1) model. In this case, LR equals -2 times the logarithm of the ratio of the likelihood function evaluated at (β', π') $= (0, ..., 0)$ divided by its value at the ML estimates of (β, π). Equivalently, $LR = LR_T(\widehat{\pi})$, where $\widehat{\pi}$ is the ML estimate of π.

Next, we introduce a test statistic that is asymptotically equivalent to the LR statistic (see Andrews and Ploberger (1994, Proof of Theorem A–1)), but that does not require the solution of any nonlinear optimization problems. Obviously, this test statistic has computational advantages over the LR statistic. This statistic is the *sup LM* statistic:

$$\sup_{\pi \in \Pi} \ LM_T(\pi) \ , \tag{2.5}$$

where for the present testing problem

$$LM_T(\pi) = \begin{pmatrix} \frac{1}{\sqrt{T^*}} \Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{t-2} \pi_n^i Y_{t-i-1} \\ \frac{1}{\sqrt{T^*}} \Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{[(t-s-1)/s]} \pi_s^i Y_{t-is-s} \end{pmatrix}' \begin{bmatrix} \frac{1}{1-\pi_n^2} & \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} \\ \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & \frac{1}{1-\pi_s^2} \end{bmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sqrt{T^*}} \Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{t-2} \pi_n^i Y_{t-i-1} \\ \frac{1}{\sqrt{T^*}} \Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{[(t-s-1)/s]} \pi_s^i Y_{t-is-s} \end{pmatrix}' / \widetilde{\sigma}^4 \ . \tag{2.6}$$

The term $D_{1t}(\pi_n) = \Sigma_{i=0}^{t-2} \pi_n^i Y_{t-i-1}$, which appears in the definition of $LM_T(\pi)$ and $LR_T(\pi)$, can be computed recursively via the recursion $D_{12}(\pi_n) = Y_1$ and $D_{1t}(\pi_n) = Y_{t-1} + \pi_n D_{1t-1}(\pi_n)$ for $t = 3 ..., T$. The same is true for $D_{2t}(\pi_s)$. In consequence, $LM_T(\pi)$ can be computed using a single do loop. The sup LM and LR statistics can be computed by grid search over $\pi \in \Pi$ to avoid errors in maximizing the functions $LM_T(\pi)$ and $LR_T(\pi)$, which can be multimodal (especially at or near the null hypothesis, since π is unidentified under the null).

By verifying the conditions of Theorem 1 of Andrews and Ploberger (1995), we find that the LR and sup LM tests satisfy the following asymptotic admissibility property. Let Power (φ_T, β, π) denote the power of the test φ_T when the true parameters are β and π.

Let

$$\mathcal{I}(\pi) = \begin{bmatrix} \frac{1}{1-\pi_n^2} & \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} \\ \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & \frac{1}{1-\pi_s^2} \end{bmatrix} . \qquad (2.7)$$

Let $U(\cdot)$ denote the uniform distribution on the unit circle (in R^2).

PROPOSITION 1: *Let $\{\xi_T : T \geq 1\}$ be a sequence of asymptotically level α LR or sup LM tests. Under Assumption 1, given any sequence of asymptotically level α tests $\{\varphi_T : T \geq 1\}$ and any probability measure $J(\cdot)$ on Π whose support is Π, there exists a constant $r_{\varphi,J} < \infty$ such that for all $r \geq r_{\varphi,J}$ we have*

$$\begin{aligned} &\overline{\lim_{T\to\infty}} \int\int \text{Power}(\varphi_T,\ r\mathcal{I}^{-1/2}(\pi)h/\sqrt{T},\ \pi)dU(h)dJ(\pi) \\ \leq\ &\underline{\lim_{T\to\infty}} \int\int \text{Power}(\xi_T,\ r\mathcal{I}^{-1/2}(\pi)h/\sqrt{T},\ \pi)dU(h)dJ(\pi)\ . \end{aligned}$$

COMMENTS: 1. The result of Proposition 1 concerns the asymptotic local power of the LR and sup LM tests since it considers parameter values β that are proportional to $1/\sqrt{T}$. Proposition 1 shows that the LR and sup LM tests beat any given test in terms of weighted average power against alternatives that are local to, but sufficiently distant from, the null. The weighting is over ellipses of β values and is with respect to an arbitrary probability measure $J(\cdot)$ on Π.

2. Note that the stationarity conditions, $|\pi_n + \beta_n| < 1$ and $|\pi_s + \beta_s| < 1$, are satisfied for T sufficiently large in Proposition 1, provided Π only includes π_n and π_s values less than one in absolute value, since $\beta = r\mathcal{I}^{-1/2}(\pi)h/\sqrt{T}$.

Next, we discuss the average exponential tests that are introduced in Andrews and Ploberger (1994). These tests are asymptotically optimal in the sense that they minimize weighted average power for specific weight functions. The weight functions for the parameter β are mean zero bivariate normal densities with covariance matrices proportional to a scalar $c > 0$. For small c, most weight is placed on alternatives that are close to the null. For large c, weight is distributed more uniformly across β values. The weight function J for the parameter π is chosen by the investigator. For the simulation results of this paper, we take it to be uniform on Π.

For each $c \in (0, \infty)$, the *average exponential LM* test statistic is given by

$$\text{Exp–}LM_{cT} = (1+c)^{-1/2} \int \exp\left(\tfrac{1}{2}\tfrac{c}{1+c} LM_T(\pi)\right) dJ(\pi) \,, \tag{2.8}$$

where $LM_T(\pi)$ is as defined above and $J(\cdot)$ is a probability measure on Π, such as the uniform measure. The average exponential LR test statistic, Exp–LR_{cT}, is defined analogously with $LM_T(\pi)$ replaced by $LR_T(\pi)$.

The limiting average exponential LM test statistics (after suitable normalization, see Andrews and Ploberger (1994)) as $c \to 0$ and $c \to \infty$ are given by

$$\begin{aligned} \text{Exp–}LM_{0T} &= \int LM_T(\pi) dJ(\pi) \ \text{ and} \\ \text{Exp–}LM_{\infty T} &= \ln \int \exp\left(\tfrac{1}{2} LM_T(\pi)\right) dJ(\pi) \,. \end{aligned} \tag{2.9}$$

The statistics Exp–LR_{0T} and Exp–$LR_{\infty T}$ are defined analogously.

Under Assumption 1, Theorem 2 of Andrews and Ploberger (1994) can be applied to yield the following asymptotic local power optimality property for the Exp–LM_{cT} and Exp–LR_{cT} tests. Let ξ_{cT} denote a test based on the test statistic Exp–LM_{cT} or Exp–LR_{cT}. Let $\phi(\beta, \omega)$ denote the density at the point β ($\in R^2$) of a mean zero covariance matrix ω bivariate normal rv.

PROPOSITION 2: *Under Assumption 1, for any $0 < c < \infty$ and any sequence of asymptotically level α tests $\{\varphi_T : T \geq 1\}$, the sequence of asymptotically level α average exponential LM or LR tests $\{\xi_{cT} : T \geq 1\}$ satisfies*

$$\begin{aligned} &\overline{\lim_{T\to\infty}} \iint 1(|\pi_n + \beta_n/\sqrt{T}| < 1,\, |\pi_s + \beta_s/\sqrt{T}| < 1)\text{Power}(\varphi_T, \beta/\sqrt{T}, \pi)\phi(\beta,\, c\mathcal{I}^{-1}(\pi))d\beta dJ(\pi) \\ \leq\ &\underline{\lim_{T\to\infty}} \iint 1(|\pi_n + \beta_n/\sqrt{T}| < 1,\, |\pi_s + \beta_s/\sqrt{T}| < 1)\text{Power}(\xi_{cT},\, \beta/\sqrt{T}, \pi)\phi(\beta,\, c\mathcal{I}^{-1}(\pi))d\beta dJ(\pi) \,. \end{aligned}$$

COMMENT: The indicator function, $1(|\pi_n + \beta_n/\sqrt{T}| < 1,\, |\pi_s + \beta_s/\sqrt{T}| < 1)$, restricts the integrals in Proposition 2 to stationary parameter values, which are the ones for which the theory of

Andrews and Ploberger (1994) is applicable. In fact, these restrictions are superfluous because the Lebesgue measure of the set of parameter points that correspond to non-stationary processes converges to zero as $T \to \infty$. In consequence, the result of the Proposition is true whether or not the indicator functions are included.

2.2. Asymptotic Null Distribution of the Test Statistics

We establish the asymptotic null distribution of the test statistics introduced above using the following martingale difference assumption.

ASSUMPTION 2: *The* rv's $\{Y_t : t = 1, 2, \ldots\}$ *satisfy* $E(Y_t|\mathcal{F}_{t-1}) = 0$ a.s. $\forall t \geq s+1$, $E(Y_t^2|\mathcal{F}_{t-1}) = \sigma^2$ a.s. $\forall t \geq s+1$, *and* $\sup_{t\geq 1} E|Y_t|^{4+\delta} < \infty$ *for some* $\delta > 0$, *where* $\mathcal{F}_t$ *denotes the* σ*-field generated by* $Y_1, \ldots, Y_t$.

The asymptotic null distributions of the test statistics are established by showing that the sequences of stochastic processes $\{LM_T(\cdot) : T \geq 1\}$ and $\{LR_T(\cdot) : T \geq 1\}$ indexed by $\pi \in \Pi$ converge weakly to a stochastic process $G(\cdot)$ and then applying the continuous mapping theorem. Let $\Rightarrow$ denote weak convergence of a sequence of stochastic processes. (We define weak convergence using the uniform metric on the appropriate space of functions on Π, as in Pollard (1990).) Let $\xrightarrow{d}$ denote convergence in distribution of a sequence of rv's. Let $\{Z_i : i \geq 1\}$ be a sequence of iid $N(0,1)$ rv's. Define

$$G(\pi) = \begin{pmatrix} \Sigma_{i=0}^{\infty}\pi_n^i Z_{i+1} \\ \Sigma_{i=0}^{\infty}\pi_s^i Z_{is+s} \end{pmatrix}' \begin{bmatrix} \frac{1}{1-\pi_n^2} & \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} \\ \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & \frac{1}{1-\pi_s^2} \end{bmatrix}^{-1} \begin{pmatrix} \Sigma_{i=0}^{\infty}\pi_n^i Z_{i+1} \\ \Sigma_{i=0}^{\infty}\pi_s^i Z_{is+s} \end{pmatrix}. \tag{2.10}$$

THEOREM 1: *Under Assumption 2,*

(a) $LM_T(\cdot) \Rightarrow G(\cdot)$,

(b) $\sup_{\pi\in\Pi} LM_T(\pi) \xrightarrow{d} \sup_{\pi\in\Pi} G(\pi)$,

(c) Exp–$LM_{cT} \xrightarrow{d} (1+c)^{-1/2}\int \exp\left(\frac{1}{2}\frac{c}{1+c}G(\pi)\right) dJ(\pi)$ *for all* $0 < c < \infty$,

(d) Exp–$LM_{0T} \xrightarrow{d} \int G(\pi) dJ(\pi)$,

(e) Exp–$LM_{\infty T} \xrightarrow{d} \ln \int \exp\left(\frac{1}{2}G(\pi)\right) dJ(\pi)$, *and*

(f) *parts* (a)–(e) *hold with LM replaced by LR.*

COMMENT: The martingale difference condition in Assumption 2 is not essential for the results of Theorem 1 to hold. What is essential is that (i) $EY_t = 0\ \forall t \geq s+1$, (ii) $EY_t^2 = \sigma^2 > 0\ \forall t \geq s+1$, (iii) $EY_tY_u = 0\ \forall u > t \geq s+1$, (iv) $EY_tY_uY_vY_w = 0\ \forall w \geq v \geq u \geq t \geq s+1$ unless $t = u$ and $v = w$, and (v) $(\frac{1}{\sqrt{T}}\Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{t-2}\pi_n^i Y_{t-i-1},\ \frac{1}{\sqrt{T}}\Sigma_{t=s+1}^T Y_t \Sigma_{i=0}^{[(t-s-1)/s]}\pi_s^i Y_{t-is-s})$ satisfies a CLT for each $\pi \in \Pi$. Assumption 2 implies conditions (i)–(v). An alternative to Assumption 2, which avoids the martingale difference assumption, is to assume conditions (i)–(iv) hold and $\{Y_t : t \geq 1\}$ is strong mixing (defined below) with strong mixing numbers that satisfy $\Sigma_{j=0}^{\infty}\alpha(j)^{(\kappa-2)/\kappa} < \infty$ and $\sup_{t\geq 1} E|Y_t|^\kappa < \infty$ for some $\kappa \geq 4$. The CLT of condition (v) holds under these conditions by Corollary 1 of Herrndorf (1984).

Asymptotic critical values for the test statistics in Theorem 1 can be simulated quite easily by truncating the series $\Sigma_{i=0}^{\infty}\pi_n^i Z_{i+1}$ and $\Sigma_{i=0}^{\infty}\pi_s^i Z_{is+s}$ at large values $i = TR$ and $i = TR \times s$ respectively. The sample size ∞ rows of Table 1 provide such values for the Exp–LR_∞, LR, and Exp–LM_0 statistics for the seasonal periods $s = 4$ and $s = 12$ for the case where $\Pi = \Pi_{.05} \times \Pi_{.05}$, $\Pi_{.05} = \{-.80,\ -.75,\ \ldots,\ .75,\ .80\}$, and $J(\cdot)$ is the uniform distribution on Π. (The simulation results of Section 5 lead us to concentrate on these particular statistics.) The critical values are based on $TR = 50$ and 40,000 repetitions. We note that, although the asymptotic critical values are very similar for $s = 4$ and $s = 12$, the asymptotic distributions of the test statistics do depend on s. See equation (A.8) of the Appendix.

The accuracy of the asymptotic critical values is assessed in Table 2 by simulating the true sizes of the asymptotic Exp–LR_∞, LR, and Exp–LM_0 tests for seasonal periods $s = 4$ and $s = 12$ and for nominal significance level of .05. The null model considered is the iid normal model with intercept and sample sizes 25, 50, 100, 250, and 500. In this case, the test statistics are defined using deviations from the mean, $\widehat{Y}_t = Y_t - \overline{Y}_T$, rather than the time series itself. See Section 4 below for an analysis of such statistics. Forty thousand simulation repetitions are used. The simulation standard errors are approximately .001. The true sizes are pretty good for sample sizes greater than or equal to 100 and not too bad for the smaller sample sizes.

Although the asymptotic critical values of Table 1 are convenient to use, we note that finite sample critical values for the normal innovations case can be generated by simulation easily. The sample size 25, 50, 100, 250, and 500 rows of Table 1 provide such critical values for the Exp–LR_∞, LR, and Exp–LM_0 tests for $s = 4$ and $s = 12$, for the case where the model contains an intercept and the test statistics are defined using deviations from the mean. Forty thousand repetitions are used to generate the finite sample critical values of Table 1.

2.3. Consistency Properties

In this section we show that the LR, sup LM, and average exponential LR and LM tests are consistent against all deviations from the null hypothesis of white noise within a broad class of weakly stationary strong mixing sequences of rv's. This consistency property illustrates the robust power properties of the tests. It is not shared by other common tests such as the Durbin–Watson and Box–Pierce tests for non-seasonal serial correlation and the Wallis (1972) test for seasonal serial correlation. It is shared, however, by the LR, sup LM, and average exponential LR and LM tests for non-seasonal serial correlation, see Pötscher (1990) and Andrews and Ploberger (1996).

We first state several definitions. The sequence of rv's $\{Y_t : t \geq s+1\}$ is said to be *weakly stationary* if EY_tY_{t-i} does not depend on t for all $t-i \geq s+1$ and $i \geq 0$. The sequence $\{Y_t : t \geq s+1\}$ is said to be *strong mixing* if

$$\alpha(m) = \sup_{t\geq s+1} \sup_{A\in\mathcal{F}_{-\infty}^t, B\in\mathcal{F}_{t+m}^\infty} |P(A\cap B) - P(A)P(B)| \to 0 \text{ as } m \to \infty ,$$

where $\mathcal{F}_{-\infty}^t$ and $\mathcal{F}_{t+m}^\infty$ are the σ-fields generated by ..., Y_{t-1}, Y_t and Y_{t+m}, Y_{t+m+1}, ... respectively. A sequence of rv's $\{W_T : T \geq 1\}$ is said to *converge in probability to infinity* (denoted $W_T \xrightarrow{p} \infty$) if $P(W_T > M) \to 1\ \forall M < \infty$.

For the consistency results, we assume:

ASSUMPTION 3: *$\{Y_t : t \geq s+1\}$ is a mean zero weakly stationary strong mixing sequence of random variables with $EY_t^2 = \sigma_Y^2 > 0\ \forall t \geq s+1$ whose strong mixing numbers $\{\alpha(j) : j \geq 1\}$*

satisfy $\Sigma_{j=1}^{\infty}\alpha(j)^{\delta/(4+\delta)} < \infty$ *and for which* $\sup_{t\geq s+1} E|Y_t|^{4+\delta} < \infty$ *for some* $\delta > 0$.

Let $\gamma_i = EY_tY_{t-i}$ for $i \geq 1$. Define

$$H_1(\pi) = \begin{pmatrix} \Sigma_{i=0}^{\infty}\pi_n^i\gamma_{i+1} \\ \Sigma_{i=0}^{\infty}\pi_s^i\gamma_{is+s} \end{pmatrix}' \begin{bmatrix} \frac{1}{1-\pi_n^2} & \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} \\ \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & \frac{1}{1-\pi_s^2} \end{bmatrix}^{-1} \begin{pmatrix} \Sigma_{i=0}^{\infty}\pi_n^i\gamma_{i+1} \\ \Sigma_{i=0}^{\infty}\pi_s^i\gamma_{is+s} \end{pmatrix} /\sigma_Y^4 \text{ and} \tag{2.11}$$

$H_2(\pi) = \log(\sigma_Y^2/\sigma^2(\pi))$, where

$\sigma^2(\pi) = \inf_{\beta\in B} E(Y_t - \beta_n D_{1t}^*(\pi_n) - \beta_s D_{2t}^*(\pi_s) + \beta_n\beta_s D_{3t}^*(\pi))^2$,

$D_{1t}^*(\pi_n) = \Sigma_{i=0}^{\infty}\pi_n^iY_{t-i-1}$, $D_{2t}^*(\pi_s) = \Sigma_{i=0}^{\infty}\pi_s^iY_{t-is-i}$, and $D_{3t}^*(\pi) = \Sigma_{i=0}^{\infty}\pi_s^i\Sigma_{j=0}^{\infty}\pi_n^jY_{t-is-s-j-1}$.

THEOREM 2: *Suppose* $\{Y_t : t \geq s+1\}$ *satisfies Assumption* 3. *Also, in parts* (b) *and* (c) *below, suppose* $\gamma_i \neq 0$ *for some* $i \geq 1$. *Then,*

(a) $\sup_{\pi\in\Pi} |LM_T(\pi)/T - H_1(\pi)| \xrightarrow{p} 0$ *and* $\sup_{\pi\in\Pi} |LR_T(\pi)/T - H_2(\pi)| \xrightarrow{p} 0$,

(b) $\sup_{\pi\in\Pi} LM_T(\pi) \xrightarrow{p} \infty$ *and* $LR \xrightarrow{p} \infty$ *provided* Π_n *is an infinite set, where* $\Pi_n = \{\pi_n : (\pi_n, \pi_s)' \in \Pi$ *for some* $\pi_s\}$,

(c) Exp–$LM_{cT} \xrightarrow{p} \infty$ *and* Exp–$LR_{cT} \xrightarrow{p} \infty$ $\forall 0 \leq c \leq \infty$ *provided the support of* $J_n(\cdot)$ *is an infinite set, where* $J_n(\cdot)$ *denotes the marginal distribution for* $\pi_n \in \Pi_n$ *that corresponds to the joint distribution* $J(\cdot)$ *for* $(\pi_n, \pi_s)' \in \Pi$.

COMMENT: Theorem 2(b) and (c) show that the sup LM, LR, and average exponential LM and LR tests are consistent against processes that have some autocovariance not equal to zero.

3. TESTS OF SERIAL CORRELATION FOR MULTIPLICATIVE SEASONAL AR(1)–AR(1) PROCESSES

Here we consider the likelihood ratio (LR_1) and Lagrange multiplier (LM_1) tests of serial correlation in a multiplicative seasonal AR(1)–AR(1) model. This model is a special case of model (2.1) with $\pi = (\pi_n, \pi_s)' = 0$. The null hypothesis is $H_0 : \beta = 0$ and the alternative hypothesis is $H_1 : \beta \neq 0$. This is a standard testing problem, because there is no nuisance parameter that appears only under the alternative hypothesis. In consequence, the LR_1 and

LM_1 tests have asymptotic chi-square distributions with two degrees of freedom (χ_2^2) under the null hypothesis and possess the standard asymptotic optimality properties of the sort established by Wald (1943). In fact, the LM test for the multiplicative seasonal MA(1)–MA(1) model is the same as the LM_1 test. In consequence, the LR_1 and LM_1 tests also possess Wald (1943)-type asymptotic optimality properties for testing against local alternatives of the multiplicative seasonal MA(1)–MA(1) variety.

By definition,

$$LR_{1T} = LR_T(0) \text{ and } LM_{1T} = LM_T(0) \;, \tag{3.1}$$

where $LR_T(\pi)$ and $LM_T(\pi)$ are as defined in (2.4) and (2.6) respectively. Note that in the definition of $LR_T(\pi)$, when $\pi = 0$, we have the simplification that $D_{1t}(0) = Y_{t-1}$, $D_{2t}(0) = Y_{t-s}$, and $D_{3t}(0) = Y_{t-s-1}$. One still has to compute $\widehat{\beta}(\pi)$ iteratively even when $\pi = 0$. Similarly, in the definition of $LM_T(\pi)$, when $\pi = 0$, $\Sigma_{i=0}^{t-2}\pi_n^i Y_{t-i-1}$ and $\Sigma_{i=0}^{[(t-s-1)/s]}\pi_s^i Y_{t-is-s}$ simplify to Y_{t-1} and Y_{t-s}, respectively, and the weight matrix simplifies to I_2.

By Theorem 1, LR_{1T} and LM_{1T} converge in distribution to a χ_2^2 rv not only under Assumption 1, but under the broader Assumption 2. (This follows because $G(0) = Z_1^2 + Z_2^2 \sim \chi_2^2$ by (2.10).)

Table 1 provides finite sample critical values for LR_1 and LM_1 for several sample sizes, for $s = 4$ and $s = 12$, for the case of an iid model with intercept and normal errors. In this case, LR_1 and LM_1 are defined as in (3.1) with $\widehat{Y}_t = Y_t - \overline{Y}_T$ in place of Y_t. Forty thousand simulation repetitions are used for these results.

Table 2 assesses the accuracy of the χ_2^2 asymptotic distribution of LR_{1T} and LM_{1T} by reporting the true sizes of these tests for the same cases as described in Section 2 above. For sample sizes of 100 or greater, the true sizes are quite good. For the smaller sample sizes, they are not too bad.

The consistency of results of Section 2.3 do not carry over to the LR_1 and LM_1 tests. Thus, one would not expect the LR_1 and LM_1 tests to have as good all-around power properties as the tests discussed in Section 2.

4. TESTS OF SERIAL CORRELATION FOR REGRESSION ERRORS

In this section, we show that the tests introduced above can be used to test whether regression errors are serially correlated. This includes the important case of a single time series with an intercept. The tests are constructed using residuals rather than the errors themselves. Provided that the regressors are exogenous (defined below), the resultant LR, sup LM, and average exponential LM and LR test statistics have the same asymptotic distribution as when the actual errors are used to construct the statistics. In consequence, the asymptotic critical values given in Table 1 are applicable.

The model we consider is given by

$$W_t = g(X_t,\ \lambda_0) + Y_t \quad \text{for} \quad t = 1, ..., T, \tag{4.1}$$

where $\{Y_t : t \leq T\}$ are unobserved errors, $\{X_t : t \leq T\}$ are observed regressor p–vectors, $\{W_t : t \leq T\}$ are observed dependent variables, λ_0 is an unknown parameter, and $g(\cdot\,,\,\cdot)$ is a known function. We consider two cases concerning the properties of the regression function. In the first case, the regression function may be non-linear, but must be non-trending. In the second case, the regression function is linear, but may be deterministically trending. In either case, we assume we have a consistent estimator $\widehat{\lambda}$ of λ_0 that is used to define the residuals

$$\widehat{Y}_t = W_t - g(X_t,\ \widehat{\lambda}) \quad \text{for} \quad t = 1, ..., T. \tag{4.2}$$

Under the null hypothesis of no serial correlation we impose one or other of the following two assumptions depending upon the nature of the regression function.

ASSUMPTION 4: (i) *Assumption* 2 *holds with* $\mathcal{F}_t$ *equal to the* σ*-field generated by* $(X_1, X_2, ...)$ *and* $(Y_1, ..., Y_t)$,

(ii) $g(X_t,\ \lambda)$ *is twice differentiable in* λ a.s., $\sup_{t \geq 1} E\|\frac{\partial}{\partial \lambda} g(X_t,\ \lambda_0)\|^2 < \infty$, *and* $\sup_{t \geq 1} \sup_{\lambda : \|\lambda - \lambda_0\| < \varepsilon} E\|\frac{\partial^2}{\partial \lambda \partial \lambda'} g(X_t,\ \lambda)\|^{4/3} < \infty$ *for some* $\varepsilon > 0$, *and*

(iii) $T^{1/4}(\widehat{\lambda} - \lambda_0) \xrightarrow{p} 0$.

ASSUMPTION 5: (i) *Assumption* 2 *holds with* $\mathcal{F}_t$ *equal to the* σ*-field generated by* $(X_1, X_2, \ldots)$ *and* $(Y_1, \ldots, Y_t)$,

(ii) $g(X_t, \lambda) = X_t'\lambda \ \forall t \geq 1$, *and*

(iii) *For some sequence* $\{\Delta_T : T \geq 1\}$ *of non-stochastic* $p \times p$ *diagonal matrices,* $\Delta_T(\widehat{\lambda} - \lambda_0) = O_p(1)$, $\sup_{t \leq T} E\|\Delta_T^{-1} X_t\|^2 \to 0$, *and* $[\Delta_T]_{jj} \to \infty \ \forall j \leq p$.

Part (i) of Assumptions 4 and 5 requires exogeneity of $\{X_t : t \geq 1\}$ in the strong sense that the conditional mean of Y_t is zero given past values of Y_t and past and *future* values of X_t. This assumption rules out dynamic regression models that include lagged values of the dependent variable.

The least squares estimator of λ_0 typically satisfies the consistency and rate of convergence results required in part (iii) of Assumptions 4 and 5.

If $X_t = (1, t, t^2)'$, then $\Delta_T = \text{Diag}(T^{1/2}, T^{3/2}, T^{5/2})$ in Assumption 5(iii) and $\sup_{t \leq T} E\|\Delta_T^{-1} X_t\|^2 \to 0$ as required.

The following result justifies the use of the LR, sup LM, and average exponential LM and LR tests when constructed using residuals.

THEOREM 3: *Under Assumption* 4 *or* 5, *the results of Theorem* 1 *still hold when* $LM_T(\pi)$ *and* $LR_T(\pi)$ *are constructed using the residuals* $\{\widehat{Y}_t : t \leq T\}$ *defined in* (4.2) *rather than the* rv's $\{Y_t : t \leq T\}$.

COMMENT: Table 1 provides asymptotic critical values for several of the tests discussed above for any of the models covered by Assumptions 4 and 5.

5. MONTE CARLO POWER COMPARISONS

5.1. *Introduction*

In this section, we compare the finite sample power of the tests introduced above with several tests in the literature.

The model we consider is the location model with serially correlated errors:

$$W_t = \lambda + Y_t \quad \text{for} \quad t = 1, \ldots, T \ , \tag{5.1}$$

where the sample size T is equal to 100. The model for the errors $\{Y_t : t \leq T\}$ is the multiplicative seasonal ARMA(1,1)–ARMA(1,1) model of (2.1) with $s = 4$. For convenience, we adopt a more conventional notation for (2.1):

$$(1 - \rho_s L^4)(1 - \rho_n L)Y_t = (1 + \phi_s L^4)(1 + \phi_n L)\varepsilon_t \quad \text{for} \quad t = s{+}1,\, s{+}2, \ldots, \tag{5.2}$$

where $\rho_s = \pi_s + \beta_s$, $\rho_n = \pi_n + \beta_n$, $\phi_s = -\pi_s$, $\phi_n = -\pi_n$, and $\varepsilon_t \sim$ iid $N(0, \sigma^2)$. All of the tests considered are invariant with respect to λ and σ^2, so (λ, σ^2) is set arbitrarily to equal (0,1). Approximately stationary samples from (5.2) of size 100 are simulated by taking the last 100 observations from simulated samples of size 400 with the five initial values set to zero.

Our interest lies in the all-around power properties of the tests, so we consider a variety of parameter combinations that encompasses seasonal/non-seasonal (S/NS) models, see Table 3, pure seasonal (PS) models, see Table 4, and pure non-seasonal (PN) models, see Table 5. The S/NS models we consider have $\rho_n = \rho_s$ and $\phi_n = \phi_s$. Thus, the seasonal and non-seasonal serial correlation is equally balanced in these models. The PS models, by definition, have $\rho_n = \phi_n = 0$. The PN models, by definition, have $\rho_s = \phi_s = 0$.

The (ρ_j, ϕ_j) parameter values that we consider are points above and below the diagonal line with slope -1 in the (ρ_j, ϕ_j) parameter space for $j = n$ or $j = s$. The diagonal lines with slope -1 in (ρ_n, ϕ_n) and (ρ_s, ϕ_s) spaces constitute the null hypothesis. The parameter values are chosen so that the Exp–LR_∞ test has power in the range (.7, .85). Tables 3, 4, and 5 list the particular parameter combinations that are considered.

The parameter combinations in each of the three tables are broken up into four blocks. The first and second blocks consist of parameter combinations that lie above the diagonal with slope -1. The first block consists of positive ρ values of increasing magnitude. The second block consists of non-positive ρ values of increasing absolute magnitude. The third and fourth blocks consist of parameter combinations that lie below the diagonal with slope -1. The third block has non-negative ρ values, while the fourth block has negative ρ values.

The tests introduced above that we consider are LR, Exp–LR_c for $c = 0, 1, \infty$, Sup–LM, Exp–LM_c for $c = 0, 1, \infty$, LR_1, and LM_1. We take

$$\Pi = \Pi_{.05} \times \Pi_{.05} \ , \quad \text{where} \quad \Pi_{.05} = \{-.80, -.75, ..., .75, .80\} \ , \tag{5.3}$$

and we take the weight function $J(\cdot)$ to be uniform on Π. This yields

$$\begin{aligned} LR &= \sup_{\pi_n \in \Pi_{.05}, \pi_s \in \Pi_{.05}} LR_T(\pi) \ , \\ \text{Exp}-LR_0 &= \frac{1}{1089} \Sigma_{\pi_n \in \Pi_{.05}} \Sigma_{\pi_s \in \Pi_{.05}} LR_T(\pi) \ , \\ \text{Exp}-LR_1 &= 2^{-1/2} \frac{1}{1089} \Sigma_{\pi_n \in \Pi_{.05}} \Sigma_{\pi_s \in \Pi_{.05}} \exp(LR_T(\pi)/4) \ , \\ \text{Exp}-LR_\infty &= \ln\left(\frac{1}{1089} \Sigma_{\pi_n \in \Pi_{.05}} \Sigma_{\pi_s \in \Pi_{.05}} \exp(LR_T(\pi)/2)\right) \ , \end{aligned} \tag{5.4}$$

and likewise for Sup–LM and Exp–LM_c with $LM_T(\pi)$ in place of $LR_T(\pi)$. In (5.4), $LR_T(\pi)$ and $LM_T(\pi)$ are as defined in (2.4) and (2.6) with Y_t replaced by $\widehat{Y}_t$, where

$$\widehat{Y}_t = W_t - \overline{W}_T \quad \text{and} \quad \overline{W}_T = \tfrac{1}{T} \Sigma_{t=1}^T W_t \ . \tag{5.5}$$

The statistics LR_1 and LM_1 are defined as in (3.1) with Y_t replaced by $\widehat{Y}_t$.

It is very much quicker to calculate the LM statistics than the LR statistics, because the former are defined in closed form whereas the latter require iterative computation of $\widehat{\beta}(\pi)$ for each $\pi \in \Pi$. We compute all the tests that depend on π by grid search, because $LR_T(\pi)$ and $LM_T(\pi)$ are often multi-modal functions of π when the true model is in the null hypothesis (in which case π is unidentified) or near the null hypothesis.

The tests from the literature that we consider are the Box–Pierce (1970) test with four lags (BP4), two-sided Wallis (1972) test for seasonal serial correlation with $s = 4$ (WAL4), the two-sided Durbin–Watson (1950) test for non-seasonal serial correlation (DW), and the LR test for non-seasonal serial correlation in an ARMA(1,1) model (NS–LR), which is analyzed in Hannan (1982), Pötscher (1990), and Andrews and Ploberger (1996). The BP4 test has some asymptotic optimality properties for AR(4) models. The WAL4, DW, and NS–LR tests do likewise for purely seasonal AR(1) and MA(1) models with one lag at $s = 4$, purely non-seasonal AR(1) and MA(1) models, and purely non-seasonal ARMA(1,1) models respectively. All of the latter models are contained in the multiplicative ARMA(1,1)–ARMA(1,1) location model.

By definition,

$$\begin{aligned} \text{BP4} &= T\Sigma_{j=1}^{4} r_j^2 \ , \quad \text{where } \ r_j = \Sigma_{t=j+1}^{T} \widehat{Y}_t \widehat{Y}_{t-j} / \Sigma_{t=1}^{T} \widehat{Y}_t^2 \ , \\ \text{WAL4} &= \Sigma_{t=5}^{T} (\widehat{Y}_t - \widehat{Y}_{t-4})^2 / \Sigma_{t=1}^{T} \widehat{Y}_t^2 \ , \quad \text{and} \\ \text{DW} &= \Sigma_{t=2}^{T} (\widehat{Y}_t - \widehat{Y}_{t-1})^2 / \Sigma_{t=1}^{T} \widehat{Y}_t^2 \ . \end{aligned} \tag{5.6}$$

The BP4 test rejects when BP4 is sufficiently large. The two-sided WAL4 and DW tests reject when $|\text{WAL4} - 2|$ and $|\text{DW} - 2|$ are sufficiently large respectively.

The NS–LR test statistic is defined by

$$\text{NS}-\text{LR} = \sup_{\pi_n \in \Pi_{NS}} LR_T^{NS}(\pi_n) \ , \quad \text{where } \ LR_T^{NS}(\pi_n) = T \log(\widetilde{\sigma}_W^2 / \widehat{\sigma}_W^2(\pi_n)) \ , \tag{5.7}$$

$$\widetilde{\sigma}_W^2 = \tfrac{1}{T} \Sigma_{t=1}^{T} \widehat{Y}_t^2 \ , \ \widehat{\sigma}_W^2(\pi_n) = \widetilde{\sigma}_W^2 - \tfrac{1}{T} \left(\Sigma_{t=2}^{T} \widehat{Y}_t \Sigma_{i=0}^{t-2} \pi_n^i \widehat{Y}_{t-i-1} \right)^2 / \Sigma_{t=2}^{T} \left(\Sigma_{i=0}^{t-2} \pi_n^i \widehat{Y}_{t-i-1} \right)^2 \ , \ \text{and}$$

$$\Pi_{NS} = \{-.80, \ -.79, \ \ldots, \ .79, \ .80\} \ .$$

The NS–LR test rejects when NS–LR is sufficiently large.

For comparative purposes, Table 2 provides the finite sample sizes of the 5% asymptotic BP4, WAL4, DW, and NS–LR tests for the iid location model with normal errors using 40,000 simulation repetitions. The BP4 test tends to under-reject. The WAL4 test, especially, and also the NS–LR test tend to over-reject.

All of the power results given below are for size-corrected 5% tests. That is, finite sample critical values, obtained via simulation with 40,000 repetitions, are employed. Five thousand repetitions are used for each of the power results.

5.2. Monte Carlo Results

In conducting the Monte Carlo experiment, we are interested in (i) the sensitivity of the power of the Exp–LR_c tests to c, (ii) the best choice of c for the Exp–LR_c tests, (iii) analogous results to (i) and (ii) for the Exp–LM_c tests, (iv) the comparison of Sup–LM and LR to the best Exp–LM_c and Exp–LR_c tests, (v) comparison of the best LR test with the best LM test, and (vi) comparison of the best of the tests considered above with LR_1, LM_1, BP4, WAL4, DW, and NS–LR. We consider each of these in turn.

We start the power results by discussing the sensitivity of the power of Exp–LR_c to c. For brevity, we only discuss the results and do not give them explicitly in Tables 3, 4, and 5. The parameters combinations considered are those given in the latter three tables. For the S/NS model, the power differences between $c = 0, 1, \infty$ are less than or equal to .02 in 10/17 cases. The differences are .13, .10, .06, .06, .04, .04, .03 in the other seven cases. In all of the seven cases with larger differences, $c = \infty$ is best. In the PS model, the power differences between $c = 0, 1, \infty$ are less than or equal to .01 in 12/19 cases. The differences are .21, .10, .10, .04, .04, .03, .02 in the other 7 cases. In the five cases where the differences are largest, $c = \infty$ is best. In the PN model, the power differences between $c = 0, 1, \infty$ are less than or equal to .02 in 14/19 cases. The differences are .21, .12, .11, .05, .04 in the other five cases. In all of these five cases, $c = \infty$ is best. In sum, the pattern across all three models is very similar. In most cases, there is little difference between $c = 0, 1, \infty$, and in those cases where there are larger differences $c = \infty$ is always best. Thus, $c = \infty$ is best overall. The power of Exp–LR_∞ for all of the cases considered is given in Tables 3, 4, and 5.

Next, we discuss the sensitivity of Exp–LM_c to c. Again, for brevity, the actual power results are not reported in Tables 3, 4, and 5. For the S/NS model, the power differences between $c = 0, 1, \infty$ are .03 or less in 11/17 cases. They are .06, .05, .05, .05, .04, .04 in the other six cases. $c = 0$ is best in 11/17 cases. On average, $c = 0$ is .01 better than $c = \infty$ and $c = 0$ is equal to $c = 1$. For the PS model, the differences between $c = 0, 1, \infty$ are .02 or less in 14/19 cases and .03 or less in all cases. $c = 0$ is best in 14/19 cases. On average, $c = 0$ is .01 better than $c = \infty$ and $c = 0$ is equal to $c = 1$. For the PN model, the differences between $c = 0, 1, \infty$ are .02 or less in 15/19 cases and .03 or less in all 19 cases. $c = 0$ is best in 16/19 cases. On average, $c = 0$ is .01 better than $c = \infty$ and $c = 0$ is equal to $c = 1$. In sum, the results are again quite similar for all three models. The power differences between Exp–LM_c for $c = 0, 1, \infty$ are quite small. The best choice is $c = 0$ by a small margin. Results for Exp–LM_0 are given in Tables 3, 4, and 5.

We now compare Exp–LM_0 with Sup–LM. In the S/NS model, Exp–LM_0 is better in 13/17 cases. It is better by an average of .04. In the PS model, Exp–LM_0 is better than Sup–LM in 16/19 cases. It is better by an average of .06. In the PN model, Exp–LM_0 is better than Sup–LM in 17/19 cases. It is better by an average of .07. In sum, Exp–LM_0 is noticeably better than Sup–LM across all three models.

We now discuss the power results that are reported in Tables 3, 4, and 5. The tables each provide the power of the Exp–LR_∞ test, which is the test with the best overall power amongst all the tests. For ease of comparison, the tables provide the difference in power between Exp–LR_∞ and seven other tests. Positive entries for the seven other tests denote higher power for Exp–LR_∞; negative entries denote lower power for Exp–LR_∞. The last column in each table provides the average power for Exp–LR_∞ and the average power differences for the seven other tests, where the average is over all parameter combinations in the given table.

First, we compare Exp–LR_∞ with the best LM test Exp–LM_0. In the S/NS model, Exp–LR_∞ is better than or equal to Exp–LM_0 in 13/17 cases by an average of .06. In the PS model, Exp–LR_∞ is best in only 10/19 cases, but it is better on average by .05. Similarly, in the PN model, Exp–LR_∞ is best in only 9/19 cases, but it is better on average by .05. Overall, Exp–LR_∞ is clearly better than Exp–LM_0. Exp–LR_∞ beats Exp–LM_0 by a substantial margin in a number of cases, but is never worse than Exp–LM_0 by very much.

Next, we compare Exp–LR_∞ with LR. In the S/NS model, Exp–LR_∞ is better than or equal to LR in 13/17 cases and it is better on average by .01. In the PS model, Exp–LR_∞ is better than LR in 16/19 cases and it is better on average by .03. In the PN model, Exp–LR_∞ is better in 16/19 cases and it is better on average by .03. Overall, Exp–LR_∞ is better than LR by a small, but not insignificant, amount.

We now compare Exp–LR_∞ and LR_1. In the S/NS model, Exp–LR_∞ is better than LR_1 in 11/17 cases and it is better on average by .07. In the PS model, Exp–LR_∞ is better in only 10/19 cases, but it is better on average by .05. In the PN model, Exp–LR_∞ is better than LR_1 in 11/19 cases and it is better on average by .06. For each model, Exp–LR_∞ is vastly superior to

LR_1 for some parameter combinations and somewhat worse for AR(1)–AR(1) and MA(1)–MA(1) parameter combinations (for which LR_1 has asymptotic optimality properties). This leaves Exp–LR_∞ clearly superior to LR_1 in an all-around sense.

Tables 3, 4, and 5 do not provide results for LM_1, because they are quite close to those of LR_1 with LR_1 being slightly superior. In the S/NS model, LR_1 is better than or equal to LM_1 in 15/17 cases and it is better by an average of .02. In the PS model, LR_1 is best in all cases and it is better by an average of .01. In the PN model, LR_1 is better than or equal to LM_1 in 14/19 cases and it is equal to LM_1 on average.

The comparison of Exp–LR_∞ and BP4 is clear-cut. In the S/NS and PS models, Exp–LR_∞ is better in all cases by averages of .12 and .15 respectively. In the PN model, Exp–LR_∞ is better than or equal to BP4 in 13/19 cases and it is better on average by .07. Overall, then, Exp–LR_∞ is substantially better than BP4.

Next, we compare Exp–LR_∞ and WAL4. In the S/NS and PN models, Exp–LR_∞ is better in all cases but one and is better on average by the substantial margins of .31 and .64 respectively. In the PS model, Exp–LR_∞ is better in 11/19 cases and it is better on average by .01. Overall, then, Exp–LR_∞ is very much better than WAL4.

Not surprisingly, the comparison of Exp–LR_∞ and DW is quite similar to that of Exp–LR_∞ and WAL4 with the results of the PS and PN models reversed. In the S/NS and PS models, Exp–LR_∞ is better than DW in all cases and it is better on average by the substantial margins of .26 and .70 respectively. In the PN model, DW is better than Exp–LR_∞ in 13/19 cases and it is better on average by .03. Overall, Exp–LR_∞ is substantially better than DW.

Lastly, we compare Exp–LR_∞ and NS–LR. This comparison is similar to that of Exp–LR_∞ and DW except NS–LR fares better in the S/NS and PN models. In the S/NS and PS models, Exp–LR_∞ is better than NS–LR in all cases and it is better on average by the large margins of .18 and .63 respectively. In the PN model, on the other hand, NS–LR is better than Exp–LR_∞ in every case by an average of .07. The NS–LR test has asymptotic optimality properties for this model, so these results are not surprising. Overall, Exp–LR_∞ is clearly superior to NS–LR.

In conclusion, the Monte Carlo results for a broad range of multiplicative seasonal ARMA(1,1)–ARMA(1,1) processes show that Exp–LR_∞ is the best overall test. It is marginally better than LR, noticeably better than Exp–LR_c for $c = 0$, 1, Exp–LM_c for $c = 0$, 1, ∞, Sup–LM, LR_1, and LM_1, and substantially better than BP4, WAL4, DW, and NS–LR. The most common tests used in practice are the DW, BP4, and WAL4 tests. Thus, substantial improvements in power are available using the tests introduced in this paper over the most common tests used in practice.

ACKNOWLEDGEMENT

The first author gratefully acknowledges research support from the National Science Foundation.

APPENDIX OF PROOFS

PROOF OF PROPOSITION 1: It suffices to verify Assumptions 1–7 of Andrews and Ploberger (1995), denoted Assumptions AP1–AP7. Theorem 1 of Andrews and Ploberger (1995) then gives the results of Proposition 1.

Under Assumption 1, the likelihood function is given by

$$f_T(\theta,\,\pi) = (2\pi\sigma^2)^{-T^*/2}\exp\left(-\tfrac{1}{2\sigma^2}\Sigma_{t=s+1}^{T}(Y_t - \beta_n D_{1t}(\pi_n) - \beta_s D_{2t}(\pi_s) + \beta_n\beta_s D_{3t}(\pi))^2\right)\ . \quad \text{(A.1)}$$

Assumptions AP1(a)–(e) and AP3–AP7 hold by analogous arguments to those given in the proof of Proposition 1 of Andrews and Ploberger (1996) with $-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}$ of Assumption AP1(d) given by

$$\begin{aligned}
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{11} &= \tfrac{1}{\sigma^2 T}\Sigma_{t=s+1}^{T}(D_{1t}-\beta_s D_{3t})^2\ , &\text{(A.2)}\\
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{12} &= \tfrac{1}{\sigma^2 T}\Sigma_{t=s+1}^{T}[(D_{2t}-\beta_n D_{3t})(D_{1t}-\beta_s D_{3t})\\
&\quad + (Y_t-\beta_n D_{1t}-\beta_s D_{2t}+\beta_n\beta_s D_{3t})D_{3t}]\ ,\\
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{13} &= \tfrac{1}{\sigma^4 T}\Sigma_{t=s+1}^{T}(Y_t-\beta_n D_{1t}-\beta_s D_{2t}+\beta_n\beta_s D_{3t})(D_{1t}-\beta_s D_{3t})\ ,\\
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{22} &= \tfrac{1}{\sigma^2 T}\Sigma_{t=s+1}^{T}(D_{2t}-\beta_n D_{3t})^2\ ,\\
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{23} &= \tfrac{1}{\sigma^4 T}\Sigma_{t=s+1}^{T}(Y_t-\beta_n D_{1t}-\beta_s D_{2t}+\beta_n\beta_s D_{3t})(D_{2t}-\beta_n D_{3t})\ ,\\
\left[-B_T^{-1}D^2\ell_T(\theta,\pi)B_T^{-1}\right]_{33} &= \tfrac{1}{\sigma^6 T}\Sigma_{t=s+1}^{T}(Y_t-\beta_n D_{1t}-\beta_s D_{2t}+\beta_n\beta_s D_{3t})^2 - \tfrac{T^*}{2T\sigma^4}\ ,
\end{aligned}$$

where D_{1t}, D_{2t}, and D_{3t} abbreviate $D_{1t}(\pi_n)$, $D_{2t}(\pi_s)$, and $D_{3t}(\pi)$ respectively.

Assumption AP1(f) holds because

$$\inf_{\pi\in\Pi}\det(\mathcal{I}(\theta_0,\,\pi)) = \inf_{\pi\in\Pi}\det\left(\begin{bmatrix} \frac{1}{1-\pi_n^2} & \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & 0\\ \frac{\pi_n^{s-1}}{1-\pi_n^{2s}\pi_s^2} & \frac{1}{1-\pi_s^2} & 0\\ 0 & 0 & \frac{1}{2\sigma^4}\end{bmatrix}\right) \quad \text{(A.3)}$$

$$= \inf_{\pi\in\Pi}\frac{1}{2\sigma^4}\left(\frac{1}{(1-\pi_n^2)(1-\pi_s^2)} - \frac{\pi_n^{2(s-1)}}{(1-\pi_n^{2n}\pi_s^2)^2}\right) \geq \inf_{\pi\in\Pi}\frac{1-\pi_n^{2(s-1)}}{2\sigma^4(1-\pi_n^2)(1-\pi_s^2)} > 0\ ,$$

since π_n is bounded below one.

Assumption AP2 requires that

$$\sup_{\pi\in\Pi} \|\widehat{\beta}(\pi)\| \xrightarrow{p} 0 \quad \text{and} \quad \sup_{\pi\in\Pi} |\widehat{\sigma}^2(\pi) - \sigma^2| \xrightarrow{p} 0 \tag{A.4}$$

under the null hypothesis. To establish the former, we use Lemma A1 of Andrews (1993). It states that if (a) $\widehat{\beta}(\pi)$ minimizes a random real function $Q_T(\beta,\pi)$ over $\beta \in B$ for each $\pi \in \Pi$, (b) $\sup_{\pi\in\Pi}\sup_{\beta\in B}|Q_T(\beta,\pi)-Q(\beta,\pi)| \xrightarrow{p} 0$ for some real function Q on $B\times\Pi$, and (c) for every neighborhood B_0 of β_0, $\inf_{\pi\in\Pi}(\inf_{\beta\in B/B_0} Q(\beta,\pi)-Q(\beta_0,\pi)) > 0$, then $\sup_{\pi\in\Pi}\|\widehat{\beta}(\pi)-\beta_0\| \xrightarrow{p} 0$. We apply this result with $\beta_0 = 0$ and $Q_T(\beta,\pi)$ given by the function in (2.4) that defines $\widehat{\beta}(\pi)$. Condition (a) holds by definition of $\widehat{\beta}(\pi)$. Condition (b) holds with $Q(\beta,\pi) = \lim_{T\to\infty} EQ_T(\beta,\pi)$ by a uniform law of large numbers using an argument that is analogous to arguments given in the proof of Proposition 1 of Andrews and Ploberger (1996).

It remains to establish condition (c). Under the null hypothesis, $Q(\beta,\pi)$ can be written as

$$\begin{aligned}
Q(\beta,\pi) &= E(\varepsilon_t - \beta_n D_1(\pi_n) - \beta_s D_2(\pi_s) + \beta_n\beta_s D_3(\pi))^2 \qquad\qquad (A.5)\\
&= \sigma^2 + E(\beta_n D_1(\pi_n) + \beta_s D_2(\pi_s) - \beta_n\beta_s D_3(\pi))^2 \ , \quad \text{where}\\
D_1(\pi_n) &= \varepsilon_{t-1} + \Sigma_{i=1}^{\infty}\pi_n^i\varepsilon_{t-i-1} \ ,\\
D_2(\pi_s) &= \varepsilon_{t-s} + \Sigma_{i=1}^{\infty}\pi_s^i\varepsilon_{t-is-s} \ ,\\
D_3(\pi) &= \Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\pi_s^i\pi_n^j\varepsilon_{t-is-s-j-1} \ ,
\end{aligned}$$

and $\{\varepsilon_t : t = ..., 0, 1, ...\}$ are iid $N(0, \sigma^2)$. Now, $Q(\beta,\pi)$ equals σ^2 plus the variance of a linear combination of $D_1(\pi_n)$, $D_2(\pi_s)$, and $D_3(\pi)$. The summand ε_{t-1} of $D_1(\pi_n)$ is independent of all the summands of $D_2(\pi_s)$ and $D_3(\pi)$. Similarly, the summand ε_{t-s} of $D_2(\pi_s)$ is independent of the summands of $D_3(\pi)$. In consequence, for all $\pi \in \Pi$, $Q(\beta,\pi) > Q(0,\pi) = \sigma^2$ for all $\beta \neq 0$. Furthermore, if we let $Q(\beta,\gamma,\pi)$ denote the function $Q(\beta,\pi)$ with the coefficient $-\beta_n\beta_s$ on $D_3(\pi)$ replaced by an arbitrary coefficient $\gamma \in R$, then $Q(\beta,\gamma,\pi) > Q(0,\pi)$ for all $(\beta', \gamma) \neq 0$.

We use the latter result to show that $Q(\beta,\pi)$ is bounded above $Q(0,\pi)$ uniformly over $\{\beta : \|\beta\| \geq \varepsilon\} \times \Pi$. We have, for $\varepsilon > 0$,

$$\inf_{\pi\in\Pi} \inf_{\beta:||\beta||\geq\varepsilon} Q(\beta,\pi) \geq \inf_{\pi\in\Pi} \inf_{(\beta',\gamma):||(\beta',\gamma)||\geq\varepsilon} Q(\beta,\gamma,\pi) \tag{A.6}$$
$$= \inf_{\pi\in\Pi} \inf_{||(\beta',\gamma):||=\varepsilon} Q(\beta,\gamma,\pi) \ ,$$

where the inequality holds because $\{(\beta', -\beta_n\beta_s) : ||\beta|| \geq \varepsilon\} \subset \{(\beta', \gamma) : ||(\beta', \gamma)|| \geq \varepsilon\}$ and the equality holds because $Q(\beta,\gamma,\pi)$ is homogeneous of degree two in (β', γ) (i.e., $Q(c\beta, c\gamma, \pi) = c^2Q(\beta,\gamma,\pi)$ for all constants $c > 0$). The right-hand side (rhs) of (A.6) is the infimum of a continuous function over a compact set. In consequence, the infimum is attained at some point $(\beta^*, \gamma^*, \pi^*)$ with $(\beta^{*'}, \gamma^*) \neq 0$. Using the result of the previous paragraph, this yields the rhs of (A.6) greater than $Q(0,\pi) = \sigma^2$, as desired. □

PROOF OF PROPOSITION 2: Proposition 2 follows from Theorem 2 of Andrews and Ploberger (1994) provided Assumptions 1–3 and 5 of the latter paper can be verified. These assumptions are the same as Assumptions 1–3 and 5 in Andrews and Ploberger (1995) (except for minor and insignificant differences), which have just been verified. □

PROOF OF THEOREM 1: The proof of part (a) is analogous to that of Theorem 1 of Andrews and Ploberger (1996) with $\nu_T(\pi)$ and $\nu(\pi)$ defined by

$$\nu_T(\pi) = \begin{pmatrix} \frac{1}{\sqrt{T}}\Sigma_{t=s+1}^T Y_t D_{1t}(\pi_n) \\ \frac{1}{\sqrt{T}}\Sigma_{t=s+1}^T Y_t D_{2t}(\pi_s) \end{pmatrix} \text{ and } \nu(\pi) = \begin{pmatrix} \sigma^2\Sigma_{i=0}^\infty \pi_n^i Z_{i+1} \\ \sigma^2\Sigma_{i=0}^\infty \pi_s^i Z_{is+s} \end{pmatrix} . \tag{A.7}$$

In the present case, we have

$$\text{Cov}(\nu(\pi_1), \nu(\pi_2)) = \begin{pmatrix} \sigma^4\Sigma_{i=0}^\infty \pi_{1n}^i\pi_{2n}^i & \sigma^4\Sigma_{i=0}^\infty \pi_{1n}^{is+s-1}\pi_{2s}^i \\ \sigma^4\Sigma_{i=0}^\infty \pi_{1n}^{is+s-1}\pi_{2s}^i & \sigma^4\Sigma_{i=0}^\infty \pi_{1s}^i\pi_{2s}^i \end{pmatrix} \text{ and} \tag{A.8}$$
$$\lim_{T\to\infty} \text{Cov}(\nu_T(\pi_1), \nu_T(\pi_2)) = \lim_{T\to\infty} \tfrac{1}{T}\Sigma_{t=s+1}^T \Sigma_{u=s+1}^T EY_tY_u \begin{pmatrix} D_{1t}(\pi_{1n}) \\ D_{2t}(\pi_{1s}) \end{pmatrix} \begin{pmatrix} D_{1u}(\pi_{2n}) \\ D_{2u}(\pi_{2s}) \end{pmatrix}'$$
$$= \text{Cov}(\nu(\pi_1), \nu(\pi_2)) \ , \text{ where}$$
$$\pi_1 = (\pi_{1n}, \pi_{1s})' \text{ and } \pi_2 = (\pi_{2n}, \pi_{2s})' \ .$$

Parts (b)–(e) follow from part (a) and the continuous mapping theorem.

Part (f) follows from $\sup_{\pi \in \Pi} |LR_T(\pi) - LM_T(\pi)| \xrightarrow{p} 0$ and parts (a)–(e). The former can be established by (i) showing that $\sup_{\pi \in \Pi} \|\widehat{\beta}(\pi)\| \xrightarrow{p} 0$ under Assumption 2 using the argument of the proof of Proposition 1, (ii) taking a two term Taylor expansion of $\log \widehat{\sigma}^2(0, \pi)$ about $\widehat{\beta}(\pi)$, where

$$\widehat{\sigma}^2(\beta, \pi) = \frac{1}{T^*} \Sigma_{t=s+1}^{T} (Y_t - \beta_n D_{1t}(\pi_n) - \beta_s D_{2t}(\pi_s) + \beta_n \beta_s D_{3t}(\pi))^2 , \qquad \text{(A.9)}$$

(iii) noting that the first term of the Taylor expansion is zero because $\frac{\partial}{\partial \beta} \widehat{\sigma}^2(\widehat{\beta}(\pi), \pi) = 0$ by the first order condition for $\widehat{\beta}(\pi)$, (iv) taking element by element mean value expansions of $\frac{\partial}{\partial \beta} \widehat{\sigma}^2(0, \pi)$ about $\widehat{\beta}(\pi)$ and substituting the resulting expression for $\widehat{\beta}(\pi)$ into the Taylor expansion of (ii), and (v) showing that all the remainder terms are $o_p(1)$ uniformly over $\pi \in \Pi$ using (i). For brevity, the details are omitted. □

PROOF OF THEOREM 2: The proof of part (a) is analogous to the proof of part (a) of Theorem 2 of Andrews and Ploberger (1996). Parts (b) and (c) hold for the LM statistics because $h(\pi_n) = \Sigma_{i=0}^{\infty} \pi_n^i \gamma_{i+1}$ has only a finite number of zeros if $\gamma_i \neq 0$ for some $i \geq 1$ and the weight matrix of $H_1(\pi)$ is positive definite for all $\pi \in \Pi$. The former follows because the function $h(\pi_n)$ for π_n complex and $|\pi_n| < 1$ is analytic and analytical functions are either identically zero or have finite numbers of zeros, e.g., see Ahlfors (1966, p. 127).

Parts (b) and (c) hold for the LR statistics, because

$$\begin{aligned} H_2(\pi) &\geq \log(\sigma_Y^2 / \inf_{(\beta_n, 0) \in B} E(Y_t - \beta_n D_{1t}^*(\pi_n))^2) \qquad \text{(A.10)} \\ &= \log(\sigma_Y^2 / (\sigma_Y^2 - h^2(\pi_n)/E D_{1t}^*(\pi_n)^2)) \\ &\geq 0 , \end{aligned}$$

where the last inequality is strict for all but a finite number of values π by the argument above. □

PROOF OF THEOREM 3: The proof is analogous to that of the proof of Theorem 3 of Andrews and Ploberger (1996). □

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Table 1. Finite sample and asymptotic critical values

Seasonal	Sample	Exp–LR_∞			LR			Exp–LM_0			LR_1			LM_1		
		10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
	25	2.65	3.40	5.10	8.52	10.25	14.18	3.79	4.90	7.69	4.74	6.19	9.47	4.40	5.74	8.56
	50	2.57	3.29	4.99	8.28	10.04	13.97	3.96	5.09	7.94	4.71	6.14	9.46	4.54	5.87	8.88
4	100	2.53	3.25	4.97	8.16	9.87	13.80	4.09	5.25	8.10	4.69	6.11	9.63	4.62	6.06	9.44
	250	2.50	3.20	4.84	8.00	9.66	13.40	4.18	5.34	7.98	4.69	6.15	9.38	4.68	6.12	9.33
	500	2.50	3.17	4.84	8.02	9.73	13.48	4.15	5.24	7.92	4.65	6.03	9.31	4.64	6.00	9.30
	∞	2.46	3.14	4.66	7.88	9.51	13.14	4.17	5.33	7.79	4.68	6.12	9.27	4.68	6.12	9.27
	25	2.82	3.61	5.39	7.99	9.79	13.87	3.92	5.26	8.82	5.25	6.80	10.43	4.72	6.35	10.65
	50	2.53	3.28	4.95	7.98	9.73	13.63	3.87	5.00	7.67	4.69	6.14	9.42	4.50	5.87	8.90
12	100	2.49	3.19	4.94	7.95	9.63	13.48	4.00	5.14	7.80	4.62	6.03	9.50	4.56	5.94	9.22
	250	2.48	3.14	4.67	7.89	9.50	12.99	4.10	5.16	7.71	4.63	6.03	9.11	4.61	5.98	8.97
	500	2.44	3.11	4.81	7.81	9.45	13.33	4.09	5.16	7.86	4.62	5.99	9.18	4.62	5.97	9.19
	∞	2.44	3.11	4.61	7.84	9.48	12.90	4.13	5.23	7.70	4.63	5.98	9.10	4.63	5.98	9.10

Table 2. True sizes of 5% asymptotic tests

Seasonal period	Sample size	Exp–LR_∞	LR	Exp–LM_0	LR_1	LM_1	BP4	WAL4	DW	NS–LR
4	25	6.32	6.83	3.91	5.18	4.06	2.72	8.05	3.95	6.41
	50	5.77	6.22	4.37	5.04	4.40	3.49	6.56	4.59	5.92
	100	5.52	5.87	4.97	4.98	4.85	4.38	5.99	5.00	5.59
	250	5.37	5.34	5.04	5.10	4.99	4.85	5.48	5.23	5.49
	500	5.17	5.45	4.72	4.78	4.73	4.84	4.84	5.20	5.42
12	25	7.64	5.65	5.06	7.22	5.80				
	50	5.94	5.55	4.31	5.41	4.70				
	100	5.36	5.34	4.70	5.12	4.88				
	250	5.20	5.04	4.81	5.14	5.00				
	500	5.00	4.96	4.81	5.01	4.93				

Table 3. Difference in power between 5% Exp–$LR\infty$ test and several other tests for seasonal/non-seasonal ARMA(1,1)–ARMA(1,1) models

	ρ_n, ρ_s	.25	.4	.6	.8	.0	–.2	–.4	–.6	–.8	.0	.2	.4	.6	–.22	–.4	–.6	–.8	Avg
	ϕ_n, ϕ_s	0	–.15	–.4	–.63	.25	.45	.65	.85	1.6	–.22	–.42	–.61	–.8	0	.18	.4	.62	
Exp–LR_∞		.77	.80	.73	.79	.76	.79	.80	.77	.74	.76	.77	.78	.78	.77	.79	.74	.77	.77
Exp–LM_0		–.01	–.02	–.04	–.04	.02	.07	.12	.22	.01	.06	.11	.19	.27	.02	.00	.00	.01	.06
LR		.05	.05	.05	.04	.04	.03	–.02	–.12	–.01	.04	.02	.00	–.07	.05	.05	.03	.00	.01
LR_1		–.05	–.05	–.02	.02	–.02	.05	.17	.35	.12	.01	.06	.16	.26	–.03	–.02	.02	.13	.07
BP4		.04	.02	.02	.02	.06	.10	.12	.17	.11	.14	.19	.28	.35	.10	.09	.10	.12	.12
WAL4		.05	.04	.01	–.03	.07	.15	.26	.44	.47	.42	.46	.52	.55	.39	.41	.47	.60	.31
DW		.11	.11	.16	.25	.14	.21	.25	.31	.29	.25	.31	.43	.54	.23	.23	.25	.30	.26
NS-LR		.20	.17	.16	.16	.22	.22	.15	.02	.24	.18	.14	.14	.14	.22	.24	.25	.25	.18

Table 4. Difference in power between 5% Exp–LR_∞ test and several other tests for pure seasonal ARMA(1,1) models ($\rho_n = \phi_n = 0$)

	ρ_s	.2	.32	.4	.6	.8	.0	–.2	–.4	–.6	–.8	.0	.2	.4	.6	–.2	–.3	–.4	–.6	–.8	Avg
	ϕ_s	.15	.0	–.07	–.3	–.55	.35	.55	.75	1.0	1.75	–.32	–.52	–.72	–3.2	–.1	0	.1	.32	.55	
Exp–LR_∞		.81	.74	.78	.77	.76	.79	.81	.82	.73	.76	.78	.79	.80	.73	.73	.74	.77	.78	.83	.77
Exp–LM_0		.01	–.02	–.02	–.04	–.03	.04	.12	.22	.34	–.04	.07	.14	.24	–.04	.02	.00	–.01	–.03	–.03	.05
LR		.07	.07	.06	.05	.03	.05	.03	–.01	–.11	.03	.05	.03	–.01	.05	.06	.06	.06	.05	.03	.03
LR_1		–.02	–.04	–.04	–.02	.04	.01	.10	.24	.38	.04	.02	.10	.22	–.02	–.03	–.04	–.03	–.02	.03	.05
BP4		.07	.05	.04	.05	.09	.11	.21	.34	.43	.14	.16	.26	.39	.05	.11	.10	.10	.10	.13	.15
WAL4		–.13	–.17	–.15	–.14	–.09	–.13	–.08	.02	.15	.11	.12	.21	.35	–.15	.07	.06	.06	.07	.10	.01
DW		.74	.67	.70	.69	.67	.72	.74	.69	.67	.67	.71	.72	.74	.65	.66	.67	.70	.70	.72	.70
NS–LR		.58	.57	.59	.55	.51	.65	.69	.65	.65	.65	.68	.69	.71	.52	.63	.64	.67	.68	.71	.63

Table 5. Difference in power between 5% Exp-LR_∞ test and several other tests for pure nonseasonal ARMA(1,1) models ($\rho_s = \phi_s = 0$)

	ρ_n	.2	.33	.4	.6	.8	.0	−.2	−.4	−.6	−.8	.0	.2	.4	.6	−.2	−.3	−.4	−.6	−.8	Avg
	ϕ_n	.15	.0	−.07	−.3	−.53	.35	.55	.72	.9	1.75	−.32	−.51	−.7	−3.4	−.1	.0	.1	.32	.57	
Exp-LR_∞		.79	.75	.76	.75	.79	.79	.81	.78	.75	.78	.77	.76	.77	.77	.73	.74	.76	.78	.78	.77
Exp-LM_0		.01	−.02	−.03	−.03	−.05	.05	.13	.24	.38	−.03	.04	.11	.23	−.04	.01	−.01	−.02	−.03	−.04	.05
LR		.05	.06	.05	.04	.03	.04	.02	−.01	−.08	.03	.05	.03	−.01	.05	.07	.07	.05	.04	.04	.03
LR_1		−.02	−.04	−.04	−.02	.01	.03	.12	.26	.39	.04	.02	.10	.24	−.01	−.02	−.02	.03	−.01	.04	.06
BP4		.06	.03	.01	−.01	−.05	.10	.15	.23	.34	−.03	.11	.17	.23	−.01	.08	.06	−.05	.00	−.04	.07
WAL4		.72	.68	.68	.62	.39	.72	.75	.73	.70	.38	.70	.69	.72	.63	.66	.66	.67	.62	.38	.64
DW		−.12	−.15	−.14	−.12	−.07	−.10	−.03	.10	.28	.01	−.05	.03	.17	−.12	−.09	−.08	−.08	−.06	.00	−.03
NS-LR		−.06	−.06	−.06	−.05	−.06	−.11	−.09	−.12	−.16	−.05	−.07	−.09	−.13	−.05	−.06	−.05	−.05	−.05	−.05	−.07