3-1-1988

Gold, Liquidity and Secured Loans in a Multistage Economy. Part I: Gold as Money

Martin Shubik

Shuntian Yao

COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale University
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 871

GOLD, LIQUIDITY AND SECURED LOANS

IN A MULTISTAGE ECONOMY

PART I: GOLD AS MONEY

by

M. Shubik and S. Yao

March 31, 1988

GOLD, LIQUIDITY AND SECURED LOANS

IN A MULTISTAGE ECONOMY

PART I: GOLD AS MONEY

GOLD, LIQUIDITY AND SECURED LOANS IN A MULTISTAGE ECONOMY PART I: GOLD AS MONEY*

by

M. Shubik and S. Yao

1. INTRODUCTION

A multiperiod exchange economy is considered where n types of traders trade m goods each period using an m+1st good, a consumer durable, which we will call "gold" as the money. Traders of type α have an endowment of $a^{\alpha} = (a_1^{\alpha}, \ldots, a_m^{\alpha})$ of the consumer goods (which in the initial model are assumed to be perishable) at the start of each period $t = 1, \ldots, T$. At the start of $t = 1$ the initial stock of gold held by traders of type α is A^{α}.

All transactions are paid for in gold. Trade takes place at the start of each period, but individuals do not obtain payment for their goods sold until the end of the period. Gold can be either used for trade in any period or it can be used as jewelry (we assume, for simplicity, that the costs and time to transform jewelry to money and vice versa, are as a reasonable first order approximation, negligible).

---

*This work relates to Department of the Navy Contract N00014-86-K0220 issued by the Office of Naval Research under Contract Authority NR 047-006. However, the content does not necessarily reflect the position or the policy of the Department of the Navy or the Government, and no official endorsement should be inferred.

The preferences of a trader of type α in any period can be described generally by a utility function ϕ^α . The overall utility function is:

$$(1) \qquad \phi^\alpha = \sum_{t=1}^{\infty} \delta^{t-1} \varphi^\alpha(x^\alpha(t), X^\alpha(t))$$

where $x_j^\alpha(t)$ = the consumption by type α during t of consumer good j ; $X^\alpha(t)$ = the use by type α of gold as jewelry (a consumer durable), and δ may be interpreted as a time discount.

We assume that trade takes place by means of a bid-offer mechanism where quantities of goods are offered in exchange for quantities of gold.

A move by a trader of type α is described by: $(b^\alpha, B^\alpha, q^\alpha)$ where $0 < b_i^\alpha$ = the fraction of α's stock of gold bid for good i

$$(2) \qquad \sum_{j=1}^{m} b_j^\alpha = 1 - B^\alpha$$

where B^α = the fraction of α's stock used for jewelry; $0 < q_i^\alpha < a_i^\alpha$ is the amount of good i offered by α to the market.

Prices[1] will be formed as:

$$(3) \qquad p_i = (\sum_{\alpha=1}^{n} b_i^\alpha A^\alpha)/(\sum_{\alpha=1}^{n} q_i^\alpha) \ .$$

Reallocations of resources by the markets are:

$$(4) \qquad x_i^\alpha = a_i^\alpha - q_i^\alpha + (b_i^\alpha A^\alpha)/p_i$$

---

[1] In this model the price of gold is implicitly fixed at one. Jewelry is not traded but has an implicit price equal to the marginal worth of gold used as jewelry for one period.

and $X^{\alpha} = B^{\alpha}A^{\alpha}$.

In order to describe strategies and the full game we use substantially the same notation with time added explicitly. Thus a strategy is:

$$(b^{\alpha}(t),\ B^{\alpha}(t);\ q^{\alpha}(t)) \ , \quad t = 1, \ldots, T$$

and similarly for prices and allocations of goods.

For gold which is a durable, the updating of holdings is given by:

$$A^{\alpha}(t+1) = X^{\alpha}(t) + p(t)q^{\alpha}(t) \ . \tag{5}$$

2. PRELIMINARY DISCUSSION

Prior to presenting the theorems and proofs some heuristic observations are made concerning the motivations for the modeling and the relevance to the development of a satisfactory theory of money.

It is our belief that as the transactions technology of a society clearly plays an important role in linking the real and paper aspects of an economy it is desirable to be explicit (even if simplistic) in describing a complete process mechanism.

Commodity money is a halfway house between goods and the various forms of financial paper which constitute the control mechanism of virtually any modern economy. For this reason we believe that it is worth while to provide a precise physical formulation and analysis of a strategic market game with commodity money at a level of explicitness and simplicity such that the mechanism could serve to define a playable game.

Money and financial institutions are simultaneously peculiarly abstract and institutional; thus any attempt to formulate a process model runs the danger of being regarded as too institutional and special. Yet any failure to be specific in details such as how price is formed or how debts are incurred and paid, or when money or goods are received would leave us with an insufficiently defined mechanism.

We opt for selecting the simplest mechanisms we can devise. These have the properties of logical consistency and completeness but are institutionally simplistic. They are extensions of models of Cournot, Bertrand and Edgeworth. After the properties of the system can be established for a Cournot mechanism, such as the one used here we may then attempt to generalize for other classes of mechanisms.

2.1. A commodity money

Our approach is to isolate phenomena as much as possible, to study them independently and if possible to perform a conceptual sensitivity analysis, i.e., vary the assumptions concerning relevant features of our models.

In particular in this model we are concerned with the use of a commodity as a money. We wish to separate out problems involving its production and its uses for transactions and investment. Thus to begin with our model assumes exchange only where the commodity money is the only durable. Even at this level of simplicity distinctions must be made about the physical properties of the money. Jevons (1875) has provided a listing of the properties of a money such as durability, transportability and cognizability. Here we assume that these are perfectly satisfied. We concentrate on the transactions problem and the distinction between using a durable commodity money such as gold and one that is a consumption storable such as a brick of tea or bar of salt. This physical distinction matters when we consider the efficiency of using commodity money.

2.2. An aside on velocity

The model presented here has one trading session per period, thus the transactions velocity of the commodity money is less than or equal to one. It appears to be an economic fact of life that the velocity of money varies in the actual economy and that velocity may be an important factor in considering control. Our model could be modified to include the possibility for strategic choice of velocity, but even though it may be empirically important it can be separated out and treated independently (see Shubik, 1987).

2.3. Credit: secured and unsecured loans

Our first model does not involve the granting of credit. But by introducing a new financial instrument, the IOU note, we can bring in credit in two contrasting ways. The first is the totally secured loan. This model requires no failure-to-repay or bankruptcy rule as the commodity is always available at the repayment date. The second has unsecured loans. In order to well define the model with unsecured loans a failure-to-pay or bankruptcy rule must be fully specified. This includes considerations of whether the rollover or refinancing of debt is permitted, when the failure-to-pay penalty is enforced and what happens in an infinite horizon model.

In the design of a penalty one has to consider whether it is to be purely economic, or could it involve extra economic factors such as going to jail. Furthermore the administrative cost and physical feasibility of procedures such as garnishing wages or seizing goods must be taken into account. Partially secured loans will be considered in a further paper.

3. THE MULTISTAGE GAME WITH NO CREDIT

3.1. Existence of NEs in the finite horizon

The question of the existence of pure strategy NEs in the multistage game can be separated into two. The first is to establish the existence of pure strategy low information NEs where individuals bid ratios or percentages of unknown quantities. The second is to establish the existence of perfect equilibria. Our approach is to first consider a two period model and to establish the existence of a low information equilibrium (LNE), then to observe that this proof can be generalized to any finite number of periods. We may then show that there is a trigger threat strategy to move to another equilibrium which will support the same outcome as that of the LNE as a perfect equilibrium (PE).

We consider the multistage buy and sell strategic market game.

In the two-period model, the low information strategy for a trader α looks like

(6) $$s^{\alpha} = (q^{\alpha}(1), b^{\alpha}(1); q^{\alpha}(2), r^{\alpha}(2))$$

where $q_i^{\alpha}(t)$ $(t = 1, 2)$ is the amount of good i sent to market by α at stage t ; $b_i^{\alpha}(1)$ the amount of money gold α spends for good i at stage 1; and $r_i^{\alpha}(2)$ the ratio of the updated amount of gold α spends for good i at stage 2.

Let $p(t) = (p_1(t), \ldots, p_m(t))$ be the price vector at stage t $(t = 1, 2)$. We have

(7) $$p_i(1) = (\sum_{\alpha} b_i^{\alpha}(1))/(\sum_{\alpha} q_i^{\alpha}(1))$$

provided that the right-hand side of the above equality has positive denominator and positive numerator. (If either one of them is zero, $b_i^\alpha(1)$ and $q_i^\alpha(1)$ are all returned.)

As for $p(2)$, we have

$$(8) \qquad p_i(2) = (\sum_\alpha r_i^\alpha(2)A^\alpha(2))/(\sum_\alpha q_i^\alpha(2))$$

where $A^\alpha(2) = A^\alpha - \sum_j b_j^\alpha(1) + \sum_j q_j^\alpha(1)p_j(1)$ is α's updated holding of gold at the beginning of stage 2. If the right-hand side of (8) is not positive for some i, then the ith good and the money sent for it are returned.

The reallocation of resources is given by

$$x_i^\alpha(1) = \begin{cases} b_i^\alpha(1)/p_i(1) & p_i(1) \text{ is positive} \\ a_i^\alpha(1) & \text{otherwise} \end{cases}$$

$$X^\alpha(1) = A^\alpha - \sum_{j\in S} b_j^\alpha(1) , \quad \text{where } S = \{i : p_i(1) \text{ is positive}\}$$

For $x_i^\alpha(2)$ and $X^\alpha(2)$, we have similar formulae with $b_i^\alpha(2) = r_i^\alpha(2)A^\alpha(2)$.

Under the strategy selection $s = (s^1, \ldots, s^n)$, the payoff to α is

$$(9) \qquad \pi^\alpha(s) = \varphi^\alpha(x^\alpha(1), X^\alpha(1)) + \delta\varphi^\alpha(x^\alpha(2), X^\alpha(2)) + \frac{\delta^2}{1-\delta}\varphi^\alpha(0, A^\alpha(3)) .$$ [2]

The strategy set of α is

$$(10) \qquad \Sigma^\alpha = \Sigma^\alpha(1) \times \Sigma^\alpha(2)$$

---

[2] We use the term $(\delta^2/(1-\delta))\varphi^\alpha(0, A^\alpha(3))$ to denote the future utility of the amount $A^\alpha(3)$ of gold.

where $\Sigma^\alpha(1) = \{(q^\alpha(1), b^\alpha(1)) : q^\alpha(1) \le a^\alpha(1), \sum_i b_i^\alpha(1) \le A^\alpha\}$

$$\Sigma^\alpha(2) = \{(q^\alpha(2), r^\alpha(2)) : q^\alpha(2) \le a^\alpha(2), \sum_i r_i^\alpha(2) \le 1\} .$$

It is obvious that Σ^α is convex and compact.

Definition 1. A low information Nash equilibrium (LNE) of the market game described above is a strategy selection $\overset{*}{s} = (\overset{*}{q}(1), \overset{*}{b}(1); \overset{*}{q}(2), \overset{*}{r}(2))$ in $\Sigma = \prod_\alpha \Sigma^\alpha$ such that for any α

(11) $$\pi^\alpha(\overset{*}{s}) \ge \pi^\alpha(\overset{*}{s}|s^\alpha) , \quad \forall s^\alpha \in \Sigma^\alpha$$

where $\overset{*}{s}|s^\alpha$ is the strategy selection obtained from $\overset{*}{s}$ by replacing $\overset{*}{s}^\alpha$ with s^α .

To get rid of the singularity at the points with some prices undefined, we first prove the LNE existence for a modified game Γ_ε $(\varepsilon > 0)$ of the above game Γ . In Γ_ε , everything is the same as in Γ except the price vectors $p(1)$ and $p(2)$:

$$p_i(1) = (\sum_\alpha b_i^\alpha(1) + \varepsilon)/(\sum_\alpha q_i^\alpha(1) + \varepsilon)$$

(12) $$p_i(2) = (\sum_\alpha b_i^\alpha(2)A^\alpha(2))/(\sum_\alpha q_i^\alpha(2) + \varepsilon) .$$

(Note that consequently $A^\alpha(t+1)$, $x^\alpha(t)$, $X^\alpha(t)$ and $\pi^\alpha(s)$ are also dependent on ε .)

From now on, we always assume that the utility functions are differentiable, concave and strictly increasing in R^{m+1}_{++} .

Now assume that in Γ_ε ($\varepsilon > 0$ fixed), all strategies except that of α are fixed. Let P be the set of all price vector pairs $(p(1), p(2))$ which are achievable when α chooses some strategies. The following result is of primary importance.

Lemma 1. P is geometrically convex,[3] i.e. if $(p(1), p(2))$ and $(p'(1), p'(2))$ are in P, so is $(\tilde{p}(1), \tilde{p}(2))$, where $\tilde{p}_i(t) = (p_i(t)p'_i(t))^{1/2}$ $(t = 1, \ldots, m)$.

Remark. The conclusion of this lemma is similar to that in the one stage model, but the proof is much more complicated.

Proof of Lemma 1

1) Assume that $(q^\alpha(1), b^\alpha(1); q^\alpha(2), r^\alpha(2))$ leads to $(p(1), p(2))$ and $(q'^\alpha(1), b'^\alpha(1); q'^\alpha(2), r'^\alpha(2))$ to $(p'(1), p'(2))$. We first want to find some $(\tilde{q}^\alpha(1), \tilde{b}^\alpha(1))$ leading to $\tilde{p}'(1)$ at stage 1. Introduce the notations $q_i^{-\alpha}(t)$, $b_i^{-\alpha}(1)$, $r_i^{-\alpha}(2)$, ... etc., where $q_i^{-\alpha}(t) = \sum_{\nu\neq\alpha} q_i^\nu(t) + \varepsilon$, ... and so on. Choose

$$\begin{aligned} &\tilde{q}_i^\alpha(1) = \max\{0, -q_i^{-\alpha}(1) + b_i^{-\alpha}(1)/\tilde{p}_i(1)\} \\ &\tilde{b}_i^\alpha(1) = \max\{0, q_i^{-\alpha}(1)\tilde{p}_i(1) - b_i^{-\alpha}(1)\} . \end{aligned} \qquad (i = 1, \ldots, m) \tag{13}$$

It is not difficult to check (13) leads to $\tilde{p}(1)$ at stage 1. To see that $(\tilde{q}^\alpha(1), \tilde{b}^\alpha(1)) \in \Sigma^\alpha(1)$, we observe that for $\tilde{q}_i^\alpha(1) > 0$,

[3] If there is more than one resources which is durable, this conclusion is no longer true.

$$\tilde{q}_i^{\alpha}(1) \le -q_i^{-\alpha}(1) + \frac{1}{2}b_i^{-\alpha}(1)\left[\frac{1}{p_i(1)} + \frac{1}{p_i'(1)}\right]$$

$$= \frac{1}{2}(-q_i^{-\alpha}(1) + b_i^{-\alpha}(1)/p_i(1)) + \frac{1}{2}(-q_i^{\alpha}(1) + b_i^{-\alpha}(1)/p_i'(1))$$

$$\le \frac{1}{2}\, a_i^{\alpha}(1) + \frac{1}{2}\, a_i^{\alpha}(1)$$

$$= a_i^{\alpha}(1) \ .$$

Similarly, one can show that for $\tilde{b}_i^{\alpha}(1) > 0$,

(14) $$\tilde{b}_i^{\alpha}(1) \le \frac{1}{2}(q_i^{-\alpha}(1)p_i(1) - b_i^{-\alpha}(1)) + \frac{1}{2}(q_i^{-\alpha}(1)p_i'(1) - b_i^{-\alpha}(1))$$

$$\le \frac{1}{2}\, b_i^{\alpha}(1) + \frac{1}{2}\, b_i'^{\alpha}(1) \ .$$

Hence

$$\sum_{i=1}^{m} \tilde{b}_i^{\alpha}(1) \le \frac{1}{2}\sum_{i=1}^{m} b_i^{\alpha}(1) + \frac{1}{2}\sum_{i=1}^{m} b_i'^{\alpha}(1) \le \frac{1}{2}\, A^{\alpha} + \frac{1}{2}\, A^{\alpha} = A^{\alpha} \ .$$

2) Introduce the notations $A^{\alpha}(2)$, $A'^{\alpha}(2)$ and $\tilde{A}^{\alpha}(2)$ for α's updated holding of gold at the beginning of stage 2 when the price vectors at stage 1 are $p(1)$, $p'(1)$ and $\tilde{p}(1)$, respectively. Here $A^{-\alpha}(2) = A^{-\alpha} - \sum_i b_i^{-\alpha}(1) + \sum_i q_i^{-\alpha}(1)p_i(1) + m\varepsilon$, ... etc. Note that $A^{\alpha}(2) + A^{-\alpha}(2) = A'^{\alpha}(2) + A'^{-\alpha}(2) = \tilde{A}^{\alpha}(2) + \tilde{A}^{-\alpha}(2) = (A^{\alpha} + A^{-\alpha})$. Moreover, it is easy to see that $\tilde{A}^{-\alpha}(2) \le \frac{1}{2}(A^{-\alpha}(2) + A'^{-\alpha}(2))$. Hence $\tilde{A}^{\alpha}(2) \ge \frac{1}{2}(A^{\alpha}(2) + A'^{\alpha}(2))$.

We now want to find some $(\tilde{q}^{\alpha}(2), \tilde{b}^{\alpha}(2))$ such that $(\tilde{q}^{\alpha}(1), \tilde{b}^{\alpha}(1); \tilde{q}^{\alpha}(2), \tilde{b}^{\alpha}(2))$ leads to $(\tilde{p}(1), \tilde{p}(2))$. Define $\tilde{q}^{\alpha}(2)$, $\tilde{b}^{\alpha}(2)$ as follows:

$$\tilde{q}_i^{\alpha}(2) = \max\{0,\ -q_i^{-\alpha}(2) + [\sum_{\nu\neq\alpha} r_i^{\nu}(2)\tilde{A}^{\nu}(2) + \varepsilon]/\tilde{p}_i(2)\}$$

$$\tilde{b}_i^{\alpha}(2) = \max\{0,\ q_i^{-\alpha}(2)\tilde{p}_i(2) - [\sum_{\nu\neq\alpha} r_i^{\nu}(2)\tilde{A}^{\nu}(2) + \varepsilon]\}\ .$$

(Note that here $\tilde{b}_i^{\alpha}(2)$ is an <u>amount</u> of gold, not a ratio.) It is not difficult to show that $(\tilde{q}^{\alpha}(2), \tilde{b}^{\alpha}(2))$ leads to $\tilde{p}(2)$ at stage 2. What we still need to check is that $(\tilde{q}^{\alpha}(2), \tilde{b}^{\alpha}(2))$ is feasible for α .

To show $\tilde{q}^{\alpha}(2) \leq a^{\alpha}(2)$ is easy (in fact, it is the same as the proof of $\tilde{q}^{\alpha}(1) \leq a^{\alpha}(1)$). The difficult part is showing that $\sum_i \tilde{b}_i^{\alpha}(2) \leq \tilde{A}^{\alpha}(2)$. Observe that

$$\begin{aligned}
\tilde{b}_i^{\alpha}(2) &\leq \frac{1}{2}\, q_i^{-\alpha}(2)(p_i(2) + p_i'(2)) - [\sum_{\nu\neq\alpha} r_i^{\nu}(2)\tilde{A}^{\nu}(2) + \varepsilon] \\
&= \frac{1}{2}\, q_i^{-\alpha}(2)(p_i(2) + p_i'(2)) - \frac{1}{2}[\sum_{\nu\neq\alpha} r_i^{\nu}(2)(A^{\nu}(2) + A'^{\nu}(2)) + 2\varepsilon] \\
&\quad + \sum_{\nu\neq\alpha} r_i^{\nu}(2)\left[\frac{1}{2}(A^{\nu}(2) + A'^{\nu}(2)) - \tilde{A}^{\nu}(2)\right] \\
&\leq \frac{1}{2}(b_i^{\alpha}(2) + b_i'^{\alpha}(2)) + \sum_{\nu\neq\alpha} r_i^{\nu}(2)\left[\frac{1}{2}(A^{\nu}(2) + A'^{\nu}(2)) - \tilde{A}^{\nu}(2)\right]
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_i \tilde{b}_i^{\alpha}(2) &\leq \frac{1}{2}[\sum_i b_i^{\alpha}(2) + \sum_i b_i'^{\alpha}(2)] + \sum_{\nu\neq\alpha}\left[\frac{1}{2}(A^{\nu}(2) + A'^{\nu}(2)) - \tilde{A}^{\nu}(2)\right] \\
&\leq \frac{1}{2}[A^{\alpha}(2) + A'^{\alpha}(2)] + \frac{1}{2}[A^{-\alpha}(2) + A'^{-\alpha}] - \tilde{A}^{-\alpha}(2) \\
&= \tilde{A}^{\alpha}(2)
\end{aligned}$$

i.e. $(\tilde{q}^{\alpha}(2), \tilde{b}^{\alpha}(2))$ is feasible. To find $\tilde{r}^{\alpha}(2)$, just let $\tilde{r}_i^{\alpha}(2) = \tilde{b}_i^{\alpha}(2)/\tilde{A}^{\alpha}(2)$. Q.E.D.

Lemma 2. Assume that all the strategies except that of α are fixed in Γ_ε . The $(p(1), p(2)) \in P$ corresponding to α's best response is unique.

Proof. Since Σ^α is compact, π^α must obtain its maximum M somewhere. Assume there are two pairs $(p(1), p(2))$ and $(p'(1), p'(2))$ corresponding to M . We will show that $(\tilde{p}(1), \tilde{p}(2))$ leads to an improvement.

First it is obvious that $\tilde{x}^{-\alpha}(1) \le \frac{1}{2}(x^{-\alpha}(1) + x'^{-\alpha}(1))$. So $\tilde{x}^\alpha(1) \ge \frac{1}{2}(x^\alpha(1) + x'^\alpha(1))$. Then from (14), we can directly see that $\tilde{X}^\alpha(1) \ge \frac{1}{2}(X^\alpha(1) + X'^\alpha(1))$. Moreover, it is also easy to see that $\tilde{A}^\alpha(2) \ge \frac{1}{2}(A^\alpha(2) + A'^\alpha(2))$. For $\tilde{x}^\alpha(2)$, $\tilde{X}^\alpha(2)$ and $\tilde{A}^\alpha(3)$, similar inequalities can be obtained by appealing to the property of Arithmetic Mean and Geometric Mean.

In view of the concavity of φ^α , $(\tilde{p}(1), \tilde{p}(2))$ really leads to an improvement, which contradicts the assumption that M is the maximum.

Q.E.D.

Proposition 1. For any given $\varepsilon > 0$, Γ_ε has at least one LNE $\bar{s} = (\bar{s}^1, \ldots, \bar{s}^n)$.

Proposition 2. Under the general assumptions on the utility functions, assume, in addition, that the utility level surface of every trader α passing through $(a^\alpha(1), A^\alpha(1))$ does not touch the hyperplane $x_{m+1} = 0$. Then at any LNE of any modified game Γ_ε , the corresponding normalized price vector pair $(p(1), p(2))$ is uniformly bounded away from zero.

Theorem 1. Under all the assumptions on the utility functions mentioned above, the market game Γ has at least one LNE $\bar{s} = (\bar{s}^1, \ldots, \bar{s}^n)$.[4]

The proofs of Propositions 1 and 2, and the proof of Theorem 1 are similar to the corresponding proofs in Dubey and Shubik [1], Amir, Sahi, Shubik and Yao [2].

Remark. With the help of mathematical induction, a LNE existence theorem can be proved for any finite stage market games of this kind.

Remark on perfect equilibria

For the game with few players and with many time periods and complete information after each set of simultaneous moves it is evident that there is an enormous proliferation in the complexity of the strategy sets for each trader. It is by no means evident that any perfect equilibria exist. However observation of an almost pathological property of these games guarantees that associated with any LNE we can also find a PE. All we need to do is to construct a strategy for all traders which consists of their LNE strategy with the additional contingent condition that if anyone deviates then all select zero. But as no trade is always an equilibrium point in any stage this is enough to show that every LNE is also a PE.

Although this observation is mathematically correct we do not feel that this answers in a fully satisfactory manner from the point of view of the economics the question of the existence or nonexistence of other PEs. In

---

[4] In general we cannot guarantee this LNE is not the trivial one--the zero LNE. But we would like to point out that (1) all the ideas employed in our proofs can also be applied to the multi-period sell-all model, where a LNE can never be zero, and (2) in the bid-and-offer model, if the payoff to α is $\pi^\alpha = \varphi^\alpha(x^\alpha(1), A^\alpha(2)) + \varphi^\alpha(x^\alpha(2), A^\alpha(3))$, then the existence of a nontrivial LNE can be proved.

particular this argument does not hold for the sell-all model (see Shapley and Shubik, 1977).

In the two-stage model, the full information strategy of a trader α looks like

$$(15) \qquad \sigma^{\alpha} = (q^{\alpha}(1),\ b^{\alpha}(1);\ \varphi^{\alpha},\ \psi^{\alpha})$$

where $q^{\alpha}(1)$, $b^{\alpha}(1)$ are the same as in the low information strategy, whereas φ^{α} , ψ^{α} are functions from $\Sigma(1) = \prod_{\nu} \Sigma^{\nu}(1)$ into R^m_+ , subject to the following constraints:

$$(16) \qquad \varphi^{\alpha}(q(1),\ b(1)) \le a^{\alpha}(2) \ ;$$

$$\sum_i \psi_i^{\alpha}(q(1),\ b(1)) \le A^{\alpha}(2) = A^{\alpha}(1) - \sum_i b_i^{\alpha}(1) + \sum_i q_i^{\alpha}(1) p_i(1) \ .$$

Let $\Phi^{\alpha} = \{(\varphi^{\alpha},\ \psi^{\alpha}) : \varphi^{\alpha},\ \psi^{\alpha}$ are functions from $\Sigma(1)$ into R^m_+ subject to (16)$\}$.

Then the strategy set of α is

$$(17) \qquad \Lambda^{\alpha} = \Sigma^{\alpha}(1) \times \Phi^{\alpha} \ .$$

Note that Λ^{α} is convex but not compact in usual topology. The strategy selection set $\Lambda = \prod_{\alpha} \Lambda^{\alpha}$ has similar properties.

Definition 2. A subgame perfect Nash equilibrium (PNE) of the market is a strategy selection $\overset{*}{\sigma} = (\overset{*}{q}(1),\ \overset{*}{b}(1);\ \overset{*}{\varphi},\ \overset{*}{\psi})$ such that

(i) for any $(q(1),\ b(1)) \in \Sigma(1)$, $(\overset{*}{\varphi}(q(1),\ b(1)),\ \overset{*}{\psi}(q(1),\ b(1)))$ is a Nash equilibrium of the market game at stage 2 when the traders played $(q(1),\ b(1))$ at stage 1; and

(ii) for any trader α ,

$$\pi^{\alpha}(\overset{*}{\sigma}) \geq \pi^{\alpha}(\overset{*}{\sigma}|\sigma^{\alpha}) \ , \quad \forall\sigma^{\alpha} \in \Lambda^{\alpha} \ .$$

From Theorem 1, one can derive the existence of a special kind of PNE.

Theorem 2. Let $\overset{*}{s} = (\overset{*}{q}(1), \overset{*}{b}(1); \overset{*}{q}(2), \overset{*}{r}(2))$ be a LNE of the market game Γ . Define $\overset{*}{\sigma} = (\overset{*}{q}(1), \overset{*}{b}(1); \overset{*}{\varphi}, \overset{*}{\psi})$ as follows:

$$\overset{*}{\varphi}(q(1), b(1)) = \begin{cases} \overset{*}{q}(2) & \text{if } q(1) = \overset{*}{q}(1) \text{ and } b(1) = \overset{*}{b}(1) \\ 0 & \text{otherwise} \end{cases}$$

$$\overset{*}{\psi}(q(1), b(1)) = \begin{cases} \overset{*}{A}(2)\cdot\overset{*}{r}(2) & \text{if } q(1) = \overset{*}{q}(1) \text{ and } b(1) = \overset{*}{b}(1) \\ 0 & \text{otherwise} \end{cases}$$

where $\overset{*}{A}(2)\cdot\overset{*}{r}(2) = (\overset{*}{A}{}^{1}(2)\overset{*}{r}{}^{1}(2), \ldots, \overset{*}{A}{}^{n}(2)\overset{*}{r}{}^{n}(2))$

$$\overset{*}{A}{}^{\alpha}(2) = A^{\alpha} - \sum_{i} \overset{*}{b}{}^{\alpha}_{i}(1) + \sum \overset{*}{q}{}^{\alpha}_{i}(1)\overset{*}{p}_{i}(1) \ .$$

Sketch of Proof

1) If $q(1) = \overset{*}{q}(1)$ and $b(1) = \overset{*}{b}(1)$, at the second stage, the strategy selection $(\overset{*}{\varphi}(\overset{*}{q}(1), \overset{*}{b}(1)), \overset{*}{\psi}(\overset{*}{q}(1), \overset{*}{b}(1))) = (\overset{*}{q}(2), \overset{*}{A}(2)\cdot\overset{*}{r}(2))$ is indeed a Nash equilibrium (since $\overset{*}{s}$ is a LNE of the 2 stage game). If $q(1) \neq \overset{*}{q}(1)$ or $b(1) \neq \overset{*}{b}(1)$, $(\overset{*}{\varphi}(q(1), b(1)), \overset{*}{\psi}(q(1), b(1))) = (0,0)$, which is also a (trivial) Nash equilibrium of the coming second game. So in any case, $(\overset{*}{\varphi}(q(1), b(1)), \overset{*}{\psi}(q(1), b(1)))$ is an NE of the second stage when $(q(1), b(1))$ is played at first stage.

2) Assume all the strategies except that of α are given, say

$$\overset{*}{\sigma}^{-\alpha} = (\overset{*}{q}^{-\alpha}(1),\ \overset{*}{b}^{-\alpha}(1);\ \overset{*}{\varphi}^{-\alpha},\ \overset{*}{\psi}^{-\alpha}) .$$

We must show that

$$\overset{*}{\sigma}^{\alpha} = (\overset{*}{q}^{\alpha}(1),\ \overset{*}{b}^{\alpha}(1);\ \overset{*}{\varphi}^{\alpha},\ \overset{*}{\psi}^{\alpha})$$

is a best response of α .

Let $\sigma^{\alpha} = (q^{\alpha}(1),\ b^{\alpha}(1);\ \varphi^{\alpha},\ \psi^{\alpha})$ be any strategy of α . Denote the corresponding payoff to α by $f(\sigma^{\alpha}) = f(q^{\alpha}(1),\ b^{\alpha}(1);\ \varphi^{\alpha},\ \psi^{\alpha})$. Consider two different cases:

<u>Case 1</u>. $(q^{\alpha}(1),\ b^{\alpha}(1)) = (\overset{*}{q}^{\alpha}(1),\ \overset{*}{b}^{\alpha}))$.

Now $(q(1),\ b(1)) = (\overset{*}{q}(1),\ \overset{*}{b}(1))$. So at the second stage, the strategies for all other traders are

(18) $$(\overset{*}{\varphi}^{-\alpha}(\overset{*}{q}(1),\ \overset{*}{b}(1));\ \overset{*}{\psi}^{-\alpha}(\overset{*}{q}(1),\ \overset{*}{b}(1))) .$$

And α in the second stage can do no better than playing

(19) $$(\overset{*}{\varphi}^{\alpha}(\overset{*}{q}(1),\ \overset{*}{b}(1)),\ \overset{*}{\psi}^{\alpha}(\overset{*}{q}(1),\ \overset{*}{b}(1)))$$

for (19) is a best response to (18) at second stage. Thus in Case 1, we see that

(20) $$f(\overset{*}{q}^{\alpha}(1),\ \overset{*}{b}^{\alpha}(1);\ \varphi^{\alpha},\ \psi^{\alpha}) \le f(\overset{*}{q}(1),\ \overset{*}{b}^{\alpha}(1);\ \overset{*}{\varphi}^{\alpha},\ \overset{*}{\psi}^{\alpha}) .$$

<u>Case 2</u>. $(q^{\alpha}(1),\ b^{\alpha}(1)) \neq (\overset{*}{q}^{\alpha}(1),\ \overset{*}{b}^{\alpha}))$.

Now $(q(1),\ b(1)) \neq (\overset{*}{q}(1),\ \overset{*}{b}(1))$. So at second stage, all other traders play (0,0). And at second stage α can do no better than playing (0,0) also. The payoff to α is equal to his payoff in the low information case when all other traders play $(\overset{*}{q}^{-\alpha}(1), \overset{*}{b}^{-\alpha}(1); \overset{*}{q}^{-\alpha}(2), \overset{*}{r}^{-\alpha}(2))$ whereas he himself plays $(q^{\alpha}(1), b^{\alpha}(1); 0, 0)$. This payoff must be less than or equal to that when he plays $(\overset{*}{q}^{\alpha}(1), \overset{*}{b}^{\alpha}(1); \overset{*}{q}^{\alpha}(2), \overset{*}{r}^{\alpha}(1))$, which is a best response of α to $(\overset{*}{q}^{-\alpha}(1), \overset{*}{b}^{-\alpha}(1); \overset{*}{q}^{-\alpha}(2), \overset{*}{r}^{-\alpha}(2))$ in the two stage game. Thus in Case 2 we have

$$(21) \qquad f(q^{\alpha}(1), b^{\alpha}(1); \varphi^{\alpha}, \psi^{\alpha}) \leq f(\overset{*}{q}^{\alpha}(1), \overset{*}{b}^{\alpha}(1); \overset{*}{\varphi}^{\alpha}, \overset{*}{\psi}^{\alpha})$$

(20) and (21) imply that

$$(22) \qquad \pi^{\alpha}(\overset{*}{\sigma}) \geq \pi^{\alpha}(\overset{*}{\sigma}|\sigma^{\alpha}) \ , \quad \forall \sigma^{\alpha} \in \Lambda^{\alpha} \ .$$

Therefore, $\overset{*}{\sigma}$ is a PNE. Q.E.D.

3.2. <u>Existence of NEs in the infinite horizon</u>

It is possible to extend our results to the model with an infinite horizon and a discount factor.

Assume that now each trader α begins with endowments $(a^{\alpha}(1), A^{\alpha})$, then he obtains a new endowment $a^{\alpha}(t)$ at stage t . Let $a(t) = \sum_{\alpha} a^{\alpha}(t)$, $A = \sum_{\alpha} A^{\alpha}$. Assume that $\|a(t)\| \leq a$, $\forall t$ and for any α , $\sum_{t=1}^{\infty} \delta^{t-1} \varphi^{\alpha}(a(t), A) < \infty$.

<u>Theorem 3</u>. Under the general assumptions on the φ^{α} , if, in addition we assume that the utility level surface of every trader passing through $(a^{\alpha}(1), A^{\alpha})$ never touches the hyperplane $x_{m+1} = 0$, then the market game at the infinite horizon has at least one nontrivial low information Nash equilibrium.

<u>Sketch of Proof</u>. First consider the game $\Gamma(T)$ ended up at the end of stage T with the payoffs to α equal to

$$\sum_{t=1}^{T} \delta^{t-1}\varphi^{\alpha}(x^{\alpha}(t),\ X^{\alpha}(t)) + \frac{\delta^{t}}{1-\delta}\varphi^{\alpha}(0,\ A^{\alpha}(t+1)) \ .$$

From the Remark after Theorem 1, we know that $\Gamma(T)$ has a nontrivial LNE, say

$$(q_T(1),\ b_T(1);\ \ldots;\ q_T(T),\ b_T(T)) \ .$$

Let T run over 1, 2, ... First consider the sequence

$$\{(q_T(1),\ b_T(1)\}_{T\geq 1} \ .$$

This is a bounded sequence in a compact set. There must exist a convergent subsequence $\{(q_{T_1}(1),\ b_{T_1}(1))\}$ converging to, say $(q(1),\ b(1))$.

Now consider the sequence $\{(q_{T_1}(2),\ b_{T_1}(2))\}_{T_1\geq 2}$ in turn, there is a convergence subsequence $\{(q_{T_2}(2),\ b_{T_2}(2))\}$ such that $(q_{T_2}(2),\ b_{T_2}(2)) \xrightarrow{T_2} (q(2),\ b(2))$.

Continue the above process infinitely, one can find $s = (q(1),\ b(1);\ q(2),\ b(2);\ \ldots;\ q(t),\ b(t);\ \ldots)$.

<u>Claim</u>. s is a LNE of the market game at the infinite horizon.

For any α , assume that

$$s^{-\alpha} = (q^{-\alpha}(1),\ b^{-\alpha}(1);\ q^{-\alpha}(2),\ b^{-\alpha}(2);\ \ldots)$$

is given. Let $\tilde{s}^{\alpha} = (\tilde{q}^{\alpha}(1),\ \tilde{b}^{\alpha}(1);\ \tilde{q}^{\alpha}(2),\ \tilde{b}^{\alpha}(2);\ \ldots)$ be a response of α . Let $(\tilde{x}^{\alpha}(t),\ \tilde{X}^{\alpha}(t))$ be the holding of α at stage t when he plays

$\tilde{s}^{\alpha}$. Then the total payoff to him is

$$\pi^{\alpha}(\tilde{s}^{\alpha}, s^{-\alpha}) = \sum_{t=1}^{\infty} \delta^{t-1}\varphi^{\alpha}(\tilde{x}^{\alpha}(t), \tilde{X}^{\alpha}(t))$$

$\forall\varepsilon > 0$. Since the series at the right-hand side is convergent, there exist T_0 such that when $T \geq T_0$

$$\pi^{\alpha}(\tilde{s}^{\alpha}, s^{-\alpha}) - \sum_{t=1}^{T_0} \delta^{t-1}\varphi^{\alpha}(\tilde{x}^{\alpha}(t), \tilde{X}^{\alpha}(t)) < \frac{\varepsilon}{2} . \tag{23}$$

On the other hand, one can choose $T \geq T_0$ such that

$$\frac{\delta^T}{1-\delta} \varphi^{\alpha}(0, A^{\alpha}(T+1)) < \frac{\varepsilon}{2} . \tag{24}$$

Now in $\Gamma(T)$, since $s^{\alpha} = (q^{\alpha}(1), b^{\alpha}(1); \ldots; q^{\alpha}(T), b^{\alpha}(T))$ is a best response to $s^{-\alpha} = (q^{-\alpha}(1), b^{-\alpha}(1); \ldots; q^{-\alpha}(T), b^{-\alpha}(T))$, we must have

$$\begin{aligned} &\sum_{t=1}^{T} \delta^{t-1}\varphi^{\alpha}(x^{\alpha}(t), X^{\alpha}(t)) + \frac{\delta^T}{1-\delta} \varphi^{\alpha}(0, A^{\alpha}(T+1)) \\ &\geq \sum_{t=1}^{T} \delta^{t-1}\varphi^{\alpha}(\tilde{x}^{\alpha}(t), \tilde{X}^{\alpha}(t)) + \frac{\delta^T}{1-\delta} \varphi^{\alpha}(0, \tilde{A}^{\alpha}(T+1)) \\ &\geq \sum_{t=1}^{T} \delta^{t-1}\varphi^{\alpha}(\tilde{x}^{\alpha}(t), \tilde{X}^{\alpha}(t)) . \end{aligned} \tag{25}$$

From (23), (24) and (25) one obtains

$$\pi^{\alpha}(s^{\alpha}, s^{-\alpha}) > \pi^{\alpha}(\tilde{s}^{\alpha}, s^{-\alpha}) - \varepsilon . \tag{26}$$

Since $\varepsilon > 0$ is arbitrary, we have

(27) $\pi^{\alpha}(s^{\alpha}, s^{-\alpha}) \geq \pi^{\alpha}(\tilde{s}^{\alpha}, s^{-\alpha})$.

Therefore, s^{α} is a best response of α to $s^{-\alpha}$. That α is arbitrary implies s is a LNE of the game at infinite horizon. Q.E.D.

Remark

1. From Theorem 3, one can prove the existence of a PNE which is constructed from a LNE.

2. All the results about the existence of a LNE can be generalized to the market with a continuum of traders with countably many different types. There the LNE becomes a TSLNE.

3.3. Existence of a stationary state for the finite and infinite horizons

When considering the possibility of the existence of a stationary state it is natural to confine our model to a situation in which the same vector of resources is replenished each period and the utility functions are as in (1). It is easy to see by considering a backward induction that for the finite horizon with gold there cannot be a stationary state as in the last period although many may wish to offer gold in exchange for other items no one will accept gold as it will have no value.

We turn to the infinite horizon:

From now on we only consider the market game with a continuum of traders at infinite horizon. We always assume that the endowment of consumer good is stationary: $a^{\alpha}(t) = a^{\alpha}$ $(t = 1, 2, \ldots; \alpha = 1, \ldots, n)$, and at the beginning $(t = 1)$, α has an amount A^{α} of gold in hand. We will denote such a game by $\Gamma_{\infty}(a,A)$.

<u>Definition 3</u>. A utility function $u : R^k_+ \to R_+$ is said to have property CIS∞ if u is $C^1(R^k_{++})$, increasing, strictly concave and $\lim_{x_j \to 0} \partial_j u(x) = \infty$ for any $x \in R^k_{++}$ with x_ℓ fixed for all $\ell \neq j$.

In the following two sections, we will assume that the utility functions φ^α are gold separable:

$$\varphi^\alpha(x^\alpha, X^\alpha) = f^\alpha(x^\alpha) + g^\alpha(X^\alpha) . \tag{28}$$

<u>Lemma 3</u>. Assume that the exchange economy $E(a,f)$ $(a = (a^1, \ldots, a^n),$ $f = (f^1, \ldots, f^n))$ has an interior CE $\dot{a} = (\dot{a}^1, \ldots, \dot{a}^n) > 0$ with price vector $\dot{p} = (\dot{p}_1, \ldots, \dot{p}_m) > 0$. Assume that the g^α all have property CIS∞. Then for any amount $G > 0$ of gold given, there is a distribution of G, say $\dot{A} = (\dot{A}^1, \ldots, \dot{A}^n)$ such that the exchange economy $E((a,\dot{A}), u)$ with $u^\alpha(x^\alpha, X^\alpha) = f^\alpha(x^\alpha) + \frac{1}{1-\delta} g^\alpha(X^\alpha)$ has a CE $((\dot{a}^1, \dot{A}^1), \ldots, (\dot{a}^n, \dot{A}^n))$ with price vector $\dot{p}^+ = (\dot{p}_1, \ldots, \dot{p}_m, \dot{p}_{m+1})$.

<u>Proof</u>. First let $p_{m+1} > 0$ be a parameter. Consider the equations

$$\frac{\partial_1 f^\alpha(\dot{x}^\alpha)}{\dot{p}_1} = \frac{g^{\alpha\prime}(X^\alpha)}{p_{m+1}} \quad (\alpha = 1, \ldots, n) . \tag{29}$$

Let $\bar{p}_{m+1} = \max\left\{\lim_{X^\alpha \to \infty} \frac{\dot{p}_1 g^\alpha(X^\alpha)}{(1-\delta)\partial_1 f^\alpha(\dot{a}^\alpha)}\right\}$. Then for any $p_{m+1} > \bar{p}_{m+1}$, there are $X^\alpha(p_{m+1})$ $(\alpha = 1, \ldots, n)$ such that the $X^\alpha = X^\alpha(p_{m+1})$ is the unique solution of (29).

Let $X(p_{m+1}) = \sum_\alpha X^\alpha(p_{m+1})$. then it is easy to see that

$$\lim_{p_{m+1} \to \bar{p}_{m+1}} X(p_{m+1}) \to +\infty ; \quad \lim_{p_{m+1} \to +\infty} X(p_{m+1}) \to 0 . \tag{30}$$

It is not difficult to see that the $X^{\alpha}(p_{m+1})$ and hence $X(p_{m+1})$ depend on p_{m+1} continuously.[5] (32) implies that $\exists$ a unique $\dot{p}_{m+1}$ such that $X(\dot{p}_{m+1}) = G$. Let $\dot{A}^{\alpha} = X^{\alpha}(\dot{p}_{m+1})$. The distribution $(\dot{A}^1, \ldots, \dot{A}^n)$ is then as required. Q.E.D.

Lemma 4. Assume that the utility functions g^{α} have property CIS∞. Let $\dot{a} = (\dot{a}^1, \ldots, \dot{a}^n)$ be an interior CE of the exchange economy $E(a,f)$ with prices $(\dot{p}_1, \ldots, \dot{p}_m)$. Then for any amount $G > 0$ of gold, there is a distribution of G , say $\bar{A} = (\bar{A}^1, \ldots, \bar{A}^n)$ and a $\dot{p}_{m+1} > 0$ such that the exchange economy $E((a,\dot{A}),u)$ has a CE allocation $(\dot{a},\dot{A})$ with prices $(\dot{p}_1, \ldots, \dot{p}_m, \dot{p}_{m+1})$. Here $\dot{A} = (\dot{A}^1, \ldots, \dot{A}^n)$ and $\dot{A}^{\alpha} = \bar{A}^{\alpha} - \sum_{j=1}^{m} (\dot{p}_m/\dot{p}_{m+1})\max\{0, \dot{a}_j^{\alpha} - a_j^{\alpha}\}$.

Proof. For any $G' \geq G$, let $A'(G') = (A'^1(G'), \ldots, A'^n(G'))$ be the distribution of G' such that $(\dot{a}, A')$ is a CE allocation of $E((a, A'), u)$ with prices $(\dot{p}_1, \ldots, \dot{p}_m, p_{m+1}(G'))$.

Let

$$(31) \qquad \tilde{G}(G') = G' + \sum_{\alpha=1}^{n} \sum_{j=1}^{m} \frac{\dot{p}_j}{p_{m+1}(G')}\max\{0, \dot{a}_j^{\alpha} - a_j^{\alpha}\}$$

Note that $p_{m+1}(G')$ depends on G' continuously, and $G' \to 0 \Rightarrow p_{m+1}(G') \to 0 \Rightarrow \tilde{G}(G') \to 0$; $G' \to +\infty \Rightarrow \tilde{G}(G') \to \infty$. Therefore, there must exist a unique $\dot{G}$ such that $\tilde{G}(\dot{G}) = G$. Let $\dot{p}_{m+1} = p'_{m+1}(\dot{G})$. Let $\dot{A}^{\alpha} = A'^{\alpha}(\dot{G})$ and $\bar{A}^{\alpha} = \dot{A}^{\alpha} + \sum_{j=1}^{m} (\dot{p}_j/\dot{p}_{m+1})\max\{0, \dot{a}_j^{\alpha} - a_j^{\alpha}\}$. Done. Q.E.D.

---

[5] In fact, $X^{\alpha}(p_{m+1})$ is strictly decreasing with respect to p_{m+1} .

Theorem 4. Under the same assumptions as described in Lemma 4, for any given amount $G > 0$ of gold, there is a distribution of G, say $\bar{A} = (\bar{A}^1, \ldots, \bar{A}^n)$ such that in the market game $\Gamma_\infty(a,\bar{A})$ there is a stationary strategy selection $s = (s^1, \ldots, s^n)$ leading to a stationary allocation $(x(t), X(t)) = (\dot{a},\dot{A})$. Here $(\dot{a},\dot{A})$ is the CE allocation of the exchange economy $E((a,\bar{A}), u)$ as mentioned in Lemma 4.

Proof. Define

$$(32) \qquad \begin{cases} q_j^\alpha(t) = \max\{0,\ a_j^\alpha - \dot{a}_j^\alpha\} = q_j^\alpha \\ b_j^\alpha(t) = \max\{0,\ (\dot{p}_j/\dot{p}_{m+1})(\dot{a}_j^\alpha - a_j^\alpha)\} = b_j^\alpha . \end{cases}$$

The conclusion follows from Lemma 4.

Definition 4. The stationary allocation $(\dot{a},\dot{A})$ corresponding to the stationary strategy selection (32) in $\Gamma_\infty(a,\bar{A})$ is said to be a gold-supported CE allocation. (GCE)

Remark. Usually (32) is not an NE of $\Gamma_\infty(a,\bar{A})$!

Example 1. Two kinds of consumer goods: 1, 2; and gold. Set of traders = [0,1] [0, 1/2) of type α, [1/2, 1] type β utility functions (at each period)

$$\varphi^\alpha(x,y;z) = \ln(1+x) + 2\ln(1+y) + \ln(1+z)$$

$$\varphi^\beta(x,y;z) = 2\ln(1+x) + \ln(1+y) + 2\ln(1+z) .$$

Endowments $a^\alpha(t) = (2,1)$, $a^\beta(t) = (1,2)$.

Discount factor δ.

(1) Consider a pure exchange economy: $E(a;f)$ such that $a^{\alpha} = (2,1)$, $a^{\beta} = (1,2)$ and

$$f^{\alpha} = \ln(1+x^{\alpha}) + 2\ln(1+y^{\alpha})$$

$$f^{\beta} = 2\ln(1+x^{\beta}) + \ln(1+y^{\beta}) \ .$$

It is very easy to see that $((2/3,\ 7/3),\ (7/3,\ 2/3))$ with $(p_1, p_2) = (1,1)$ is a CE.

(2) Consider a pure exchange economy $E(a,\mathring{A},u)$ with same a as in (1), $\mathring{A}$ to be determined and

$$u^{\alpha}(x^{\alpha},y^{\alpha};\ z^{\alpha}) = \ln(1+x^{\alpha}) + 2\ln(1+y^{\alpha}) + \frac{1}{1-\delta}\ln(1+z^{\alpha})$$

$$u^{\beta}(x^{\beta},y^{\beta};z^{\beta}) = 2\ln(1+x^{\beta}) + \ln(1+y^{\beta}) + \frac{2}{1-\delta}\ln(1+z^{\beta}) \ .$$

In order that there exists a CE $(2/3,\ 7/3;\ \mathring{A}^{\alpha})$, $(7/3,\ 2/3;\ \mathring{A}^{\beta})$ with prices $(1,1;p)$. What we need are

$$\frac{\partial_z u^{\alpha}\left(\frac{2}{3},\ \frac{7}{3};\ \mathring{A}^{\alpha}\right)}{p} = \frac{\partial_x u^{\alpha}\left(\frac{2}{3},\ \frac{7}{3};\ \mathring{A}^{\alpha}\right)}{p} = \frac{3}{5}$$

$$\frac{\partial_z u^{\beta}\left(\frac{2}{3},\ \frac{7}{3};\ \mathring{A}^{\beta}\right)}{p} = \frac{\partial_x u^{\beta}\left(\frac{2}{3},\ \frac{7}{3};\ \mathring{A}^{\beta}\right)}{p} = \frac{3}{5} \ .$$

We then have

$$\frac{1}{1-\delta}\cdot\frac{1}{1+\mathring{A}^{\alpha}} = \frac{3}{5}\,p\ ,\quad \frac{2}{1-\delta}\cdot\frac{1}{1+\mathring{A}^{\beta}} = \frac{3}{5}\,p$$

from which one solves

$$\dot{A}^{\alpha} = \frac{1}{3(1-\delta)p} - 1 , \quad \dot{A}^{\beta} = \frac{10}{3(1-\delta)p} - 1 \qquad (*)$$

(3) Calculation of $\Delta^{\alpha} = \sum_{j=1}^{m} (p_j/p_{m+1})\max(0, \dot{a}_j^{\alpha} - a_j^{\alpha})$, etc.

$$\Delta^{\alpha} = (\dot{a}_2^{\alpha} - a_2^{\alpha})p_2/p = 4/3p$$

$$\Delta^{\beta} = (\dot{a}_1^{\beta} - a_1^{\beta})p_1/p = 4/3p$$

(4) The Market Game with SS

$$(a^{\alpha}; A^{\alpha}) = \left(2, 1; \frac{5}{3(1-\delta)p} - 1 + \frac{4}{3p}\right)$$

$$(a^{\beta}; A^{\beta}) = \left(2, 1; \frac{10}{3(1-\delta)p} - 1 + \frac{4}{3p}\right)$$

(5) Redistribution of gold guaranteeing the GCE

$$\frac{5}{3(1-\delta)p} - 1 + \frac{4}{3p} + \frac{10}{3(1-\delta)p} - 1 + \frac{4}{3p} = A$$

i.e. $$\frac{5}{(1-\delta)p} + \frac{8}{3p} = A + 2$$

$$p = \left[\frac{5}{(1-\delta)} + \frac{8}{3}\right](A+2)^{-1} \qquad (**)$$

From (*) and (**) one can see that: given any G , for $G \geq 1 + \frac{8}{5}(1-\delta)$, there is really a redistribution of G s.t. a GCE exists. In particular, if $G > 1 + \frac{8}{5} = 2\frac{3}{5}$, then for any δ , such a redistribution always exists; if $A > 1$, then for δ close to 1 sufficiently, such a redistribution exists.

Take $\delta = 1/2$, $G = 10\frac{2}{3}$ for example. Easy to see that $p = 1$.

$\bar{A}^{\alpha} = 11/3$, $\bar{A}^{\beta} = 21/3$; $\dot{A}^{\alpha} = 7/3$, $\dot{A}^{\beta} = 17/3$.

i.e. Endowments: $(2, 1; 11/3)$ for $\beta \in [0, 1/2)$;

$(1, 2; 21/3)$ for $\beta \in [1/2, 1]$.

Strategies: $(q^{\alpha}(t); b^{\alpha}(t)) = (4/3, 0; 0, 4/3)$

$(q^{\beta}(t); b^{\beta}(t)) = (0, 4/3; 4/3; 0)$

Reallocations: $(x^{\alpha}(t), X^{\alpha}(t)) = (2/3, 7/3; 7/3)$

$(x^{\beta}(t), X^{\beta}(t)) = (7/3, 2/3; 17/3)$

Payoffs: $\pi^{\alpha} = (1/(1 - 1/2)\varphi^{\alpha}(2/3, 7/3; 7/3) = 8.245488$

$\pi^{\beta} = (1/(1 - 1/2)\varphi^{\beta}(7/3, 2/3; 17/3) = 13.426022$

If α and β did not trade, the payoffs would be

$$\pi^{\alpha} = \frac{1}{1 - 1/2} \varphi^{\alpha}(2, 1; 11/3) = 8.0507033$$

$$\pi^{\beta} = \frac{1}{1 - 1/2} \varphi^{\beta}(2, 1; 21/3) = 13.287579$$

It is easy to see that even though they do not lose gold without trading, α and β prefer trading incurring the loss of use of gold for jewelry.

To see that $(q^{\alpha}(t), b^{\alpha}(t)) = (4/3, 0; 0, 4/3)$ and $(q^{\beta}(t), b^{\beta}(t)) = (0, 4/3; 4/3; 0)$ is not an NE, let an individual i of type α change his strategy to

$$s^{i} = ((1, 0; 0, 4/3), (4/3, 0; 0, 4/3)\cdot(4/3, 0; 0, 4/3), \ldots)$$

Then his holdings at each period (after trade but before he is paid) are

$$((1, 7/3, 7/3), (2/3, 7/3, 2), (2/3, 7/3, 2), \ldots) .$$

The payoff to i will be

$$\begin{aligned}\pi^i &= \varphi^i(1,\ 7/3,\ 7/3) + \frac{\delta}{1-\delta}\,\varphi^i(2/3,\ 7/3,\ 2) \\ &= \left[\ln 2 + \ln \frac{10}{3} + \ln \frac{10}{3}\right] + \left[\ln \frac{5}{3} + 2\ln \frac{10}{3} + \ln 3\right] \\ &= 8.322449\end{aligned}$$

i.e. i really makes an improvement by playing s^i .

Now we turn to the existence of stationary Nash equilibria. First look at $E(a,f)$.

<u>Definition 5</u>. A δ-CE of $E(a,f)$ is an allocation $\hat{a} = (\hat{a}^1, \ldots, \hat{a}^n)$ with price vector $\hat{p} = (\hat{p}_1, \ldots, \hat{p}^n) > 0$ such that for every α ,

$$(33)\qquad \begin{cases} \dfrac{\partial_j f^\alpha(\hat{a}^\alpha)}{\hat{p}_j} = \theta(j,k)\,\dfrac{\partial_k f^\alpha(\hat{a}^\alpha)}{\hat{p}_k} \\[2ex] \theta(j,k) = \begin{cases} 1\ , & \text{if } \hat{a}^\alpha_j - a^\alpha_j > 0 \text{ and } \hat{a}^\alpha_k - a^\alpha_k > 0 \\ \delta\ , & \text{if } \hat{a}^\alpha_j - a^\alpha_j < 0 \text{ and } \hat{a}^\alpha_k - a^\alpha_k > 0 \\ \in [\delta,\ 1/\delta]\ , & \text{otherwise} \end{cases} \\[2ex] \sum_{j=1}^{m} \hat{a}^\alpha_j \hat{p}_j = \sum_{j=1}^{m} a^\alpha_j \hat{p}_j\ . \end{cases}$$

<u>Proposition 2</u>. Assume that for every α ,

(i) f^α has the property CIS∞; and

(ii) for any given $p = (p_1, \ldots, p_m) > 0$, if x^α , x'^α are two different vectors with $x^\alpha \cdot p = x'^\alpha \cdot p = a^\alpha \cdot p$, then $\exists\ x^\alpha_j > x'^\alpha_j$ and $x^\alpha_k < x'^\alpha_k$ such that

$$(34) \qquad \frac{\partial_j f^\alpha(x^\alpha)}{\partial_k f^\alpha(x^\alpha)} < \frac{\partial_j f^\alpha(x'^\alpha)}{\partial_k f^\alpha(x'^\alpha)} \, .$$ [6]

Then $E(a,f)$ has a δ-CE.

Proof of Proposition 2. Given any $p = (p_1, \ldots, p_m) \geq 0$ with $\sum_{j=1}^{m} p_j = 1$, find $x^\alpha = (x_1^\alpha, \ldots, x_m^\alpha)$ such that

$$(35) \qquad \begin{cases} x_j^\alpha = M & \text{if } p_j = 0 \, , \ M > 0 \text{ sufficiently large;} \\ \dfrac{\partial_j f^\alpha(x^\alpha)}{p_j} = \theta(j,k)\dfrac{\partial_k f^\alpha(x^\alpha)}{p_k} & \text{if } p_j > 0 \text{ and } p_k > 0 \, ; \\ x^\alpha \cdot p = a^\alpha \cdot p & \end{cases}$$

where $\theta(j,k)$ is defined similar to the one in (33). For a given p, this x^α is unique due to assumptions (i) and (ii). Calculating x^α for $\alpha = 1, \ldots, n$ and consider the excess demand. We are in the situation as in the proof of the existence of CE, follow the same steps as in G. Owen [2], the proof can be completed without difficulty.

Theorem 5. Assume that the game $\Gamma_\infty(a,\bar{A})$ has an SNE with stationary allocation $(\hat{a}^\alpha, \hat{A}^\alpha) > 0$ and stationary price vector $\hat{p}^+ = (\hat{p}_1, \ldots, \hat{p}_m, \hat{p}_{m+1}) > 0$. Then $\hat{a}^\alpha$ with $\hat{p} = (\hat{p}_1, \ldots, \hat{p}_m)$ is a δ-CE of $E(a,f)$.

---

[6] Note that (ii) always hold for any m if the utility functions are separable.

Proof. Assume that $\hat{a}_j^\alpha < a_j^\alpha$, $\hat{a}_k^\alpha > a_k^\alpha$ and $\hat{a}_\ell^\alpha = a_\ell^\alpha$. We try to show that

$$(36) \qquad \begin{cases} \dfrac{\partial_j f^\alpha(\hat{a}^\alpha)}{\hat{p}_j} = \dfrac{\delta}{1-\delta}\,\dfrac{g^{\alpha\prime}(\hat{A}^\alpha)}{\hat{p}_{m+1}} \\[2ex] \dfrac{\partial_k f^\alpha(\hat{a}^\alpha)}{\hat{p}_k} = \dfrac{1}{1-\delta}\,\dfrac{g^{\alpha\prime}(\hat{A}^\alpha)}{\hat{p}_{m+1}} \\[2ex] \dfrac{\partial_\ell f^\alpha(\hat{a}^\alpha)}{\hat{p}_\ell} = \dfrac{\theta}{1-\delta}\,\dfrac{g^{\alpha\prime}(\hat{A}^\alpha)}{\hat{p}_{m+1}}, \quad \theta \in [\delta,1] . \end{cases}$$

If not, for example, say $\partial_j f^\alpha(\hat{a}^\alpha)/\hat{p}_j > (1/1-\delta)(g^{\alpha\prime}(\hat{A}^\alpha)/\hat{p}_{m+1})$, let i be an individual of type α . Imagine that i changes his strategy a little bid: at period $t = 1$, $\tilde{q}^\alpha(1) = q^\alpha(1) - (0, \ldots, \Delta_j, \ldots, 0)$, i.e. he keeps a small amount more of good j than he did. It is not hard to see that the change in his payoff is

$$f^\alpha(\hat{a}^\alpha + (0, \ldots, \Delta_j, \ldots, 0)) - f^\alpha(\hat{a}^\alpha) - \frac{\delta}{1-\delta}\left[g^\alpha(\hat{A}^\alpha) - g^\alpha\left(\hat{A}^\alpha - \frac{\Delta_j\hat{p}_j}{\hat{p}_{m+1}}\right)\right]$$

$$= \partial_j f^\alpha(\hat{a}^\alpha)\cdot\Delta_j - \frac{\delta}{1-\delta}\, g^{\alpha\prime}(\hat{A}^\alpha)\cdot\frac{\Delta_j\hat{p}_j}{\hat{p}_{m+1}} + o(\Delta_j)$$

$$= \Delta_j\hat{p}_j\left[\frac{\partial_j f^\alpha(\hat{a}^\alpha)}{p_j} - \frac{\delta}{1-\delta}\,\frac{g^{\alpha\prime}(\hat{A}^\alpha)}{\hat{p}_{m+1}}\right] + o(\Delta_j)$$

when $\Delta_j \to +0$, the change is positive. A contradiction. Similarly, one can show that all the equalities in (36) must hold, which imply that $(\hat{a},\hat{p})$ is a δ-CE of $E(a,f)$.

Lemma 5. Assume that all the g^α have the property CIS∞. Let $(\hat{a},\hat{p}) > 0$ be an δ-CE allocation of $E(a,f)$. Then for any given amount $G > 0$ of gold, there exist a distribution of G , say $\hat{A} = (\hat{A}^1, \ldots, \hat{A}^n)$ and a $\hat{p}_{m+1} > 0$ such that (36) holds.

Proof. The proof is similar to the proof of Lemma 3. We omit it.

Lemma 6. Assume that the g^α have the property CIS∞. Let $(\hat{a},\hat{p}) > 0$ be an δ-CE allocation of $E(a,f)$. Then for any given $G > 0$, there is a distribution of G , say $\tilde{A} = (\tilde{A}^1, \ldots, \tilde{A}^n)$, and a $\hat{p}_{m+1} > 0$ such that (36) holds for the $\hat{A}^\alpha$ satisfying

$$(37) \qquad \hat{A}^\alpha = \tilde{A}^\alpha - \sum_{j=1}^{m} \frac{\hat{p}_t}{\hat{p}_{m+1}} \max(0,\ \hat{a}_j^\alpha - a_j^\alpha) \quad (\alpha = 1, \ldots, n) .$$

Proof. The proof is similar to that of Lemma 4. We omit it.

Theorem 6 (Existence of SNE). Assume that $(\hat{a},\hat{p}) > 0$ is a δ-CE of $E(a,f)$. Assume that the g^α have property CIS∞. For $G > 0$ given, let $\tilde{A} = (\tilde{A}^1, \ldots, \tilde{A}^n)$ and $\hat{p}_{m+1}$ be as in Lemma 6. If

$$(38) \qquad \lim_{X^\alpha \to \infty} \frac{1}{1-\delta} \frac{g^{\alpha\prime}(X^\alpha)}{\hat{p}_{m+1}} < \frac{\partial_j f^\alpha(a^\alpha)}{\hat{p}_j} \quad (\text{all } j \text{ , all } \alpha \text{ }) .$$

Then the $(\hat{a}^\alpha, \hat{A}^\alpha)$ is a stationary Nash equilibrium allocation.

Proof. Define

$$(39)\qquad \begin{cases} q_j^{\alpha}(t) = \max(0,\ a_j^{\alpha} - \hat{a}_j^{\alpha}) = q_j^{\alpha} \\ b_j^{\alpha}(t) = \max(0,\ (\hat{p}_j/\hat{p}_{m+1})(\hat{a}_j^{\alpha} - a_j^{\alpha}) \end{cases} \quad (\alpha = 1,\ \ldots,\ n)\ .$$

Then it is easy to see that (39) leads to the stationary allocation $(\hat{a}^{\alpha}, \hat{A}^{\alpha})$. What we need to show is the (39) is an SNE.

Imagine that i is an individual of type α . Let the initial holding of gold A^i be a parameter. For A^i fixed, i has a unique best response [to the market prices $(\hat{p}_1, \ldots, \hat{p}_m, \hat{p}_{m+1})$], say

$$(40)\qquad s^i(A^i) = (q^i_{A^i}(1),\ b^i_{A^i}(1);\ q^i_{A^i}(2),\ b^i_{A^i}(2);\ \ldots)\ .$$

The uniqueness implies that $q^i_{A^i}(1)$, $b^i_{A^i}(1)$ and hence $A^i(2)$ (the updated holding of gold at $t = 2$) depends on A^i continuously.

Claim. When $A^i = 0$, $A^i(2) > A^i$; $\exists A^i$ sufficiently large such that $A^i(2) < A^i$.

In fact, when $A^i = 0$, obviously we have $A^i(2) > A^i$.

Now assume that A^i is sufficiently large. If $A^i(t+1) \geq A^i(t)$ all t , we will have

$$(41)\qquad X^i(t) \geq A^i - \hat{p}_{m+1}^{-1}(\hat{p}\cdot a^i) \quad (t = 1,\ 2,\ \ldots\ )$$

Since g has the property (38) for A^i large enough, we have

$$(42)\qquad \frac{\partial_j f^{\alpha}(a^i)}{\hat{p}_j} > \frac{g^{\alpha'}(X^{\alpha}(t))}{(1-\delta)\hat{p}_{m+1}} \quad (t = 1,\ 2,\ \ldots;\ j = 1,\ \ldots,\ m)\ .$$

But it is easy to see that i can make an improvement by keeping or buying a little bit more of some good j $(x_j^i(1) \le a_j^i)$ at the first period. If, on the other hand, there is some K such that $A^i(K+1) < A^i(K)$. Then when $A^i = A^i(K)$, we should have $A^i(2) = A(K+1) < A^i$.

Our claim is proved.

Now it is obvious that $\exists$ some $\mathring{A}^i$ such that $\mathring{A}^i(2) = \mathring{A}^i$, i.e. i has a stationary best response when he has gold endowment $\mathring{A}^i$. But on the other hand we can easily see that the stationary holding of i must be $\hat{a}^i$, $\hat{A}^i$, hence $\mathring{A}^i = \bar{A}^i$. Therefore we finally have shown that when i starts with $\bar{A}^i$, the strategy above is really the best response. Q.E.D.

Example 2. All the data are the same as in Example 1. We try to find an SNE.

By observation, E(a,f) has a δ-CE $(\delta = 1/2)$ allocation

(43) $((3/2,\ 3/2)(3/2,\ 3/2))$.

The associated prices are (1,1). If the price of gold is p_3 , from the equation

$$\frac{\partial f_2^{\alpha}(3/2,\ 3/2)}{1} = \frac{1}{1-\frac{1}{2}}\,\frac{g^{\alpha'}(A^{\alpha})}{p_3}$$

we obtain

$$\frac{2}{p_3(1+\hat{A}^{\alpha})} = \frac{2}{1+\frac{3}{2}}$$

or

$$\hat{A}^{\alpha} = \frac{5}{2p_3} - 1 .$$

Similarly $\hat{A}^{\beta} = 5/p_3 - 1$. If $G = 32/3$ as in Example 1, then we should have

$$\left(\frac{5}{2p_3} - 1 + \frac{1}{2p_3}\right) + \left(\frac{5}{p_3} - 1 + \frac{1}{2p_3}\right) = \frac{32}{3}$$

from which we solve $p_3 = 51/76$. By calculation one obtains $\hat{A}^{\alpha} = 2\frac{37}{51}$, $\hat{A}^{\beta} = 6\frac{23}{51}$; $\tilde{A}^{\alpha} = 3\frac{48}{102}$, $\tilde{A}^{\beta} = 7\frac{20}{102}$. Thus we have:
$\alpha \in [0, 1/2)$, $p \in [1/2, 1)$

initial endowment: $\left(2, 1, 3\frac{24}{51}\right)$, $\left(1, 2, 7\frac{10}{51}\right)$

strategies: $\left(\frac{1}{2}, 0; 0, \frac{76}{102}\right)$, $\left(0, \frac{1}{2}; \frac{76}{102}, 0\right)$

reallocations: $\left(\frac{3}{2}, \frac{3}{2}, 2\frac{37}{51}\right)$, $\left(\frac{3}{2}, \frac{3}{2}, 6\frac{23}{51}\right)$

payoffs: $\pi^{\alpha} = 2\left[\ln\frac{5}{2} + \ln\frac{5}{2} + \left(3\frac{37}{51}\right)\right] = 8.128141$

payoffs: $\pi^{\beta} = 2\left[\ln\frac{5}{2} + \ln\frac{5}{2} + \left(7\frac{23}{51}\right)\right] = 13.53113$.

It is interesting to note that in the SNE allocation, the sum of the payoffs is less than that in the GCE allocation. Another interesting thing can be observed from the following.

Example 3. In Example 2, assume that there is another type γ of traders (say $\gamma \in [1,2]$) who has the same utility function as type α , but a different endowment $(2/3, y^{\gamma}, 101/51)$. Then as long as $y^{\gamma} \in [7/3, 17/3]$

any individual of type γ prefer the zero strategy, i.e. $[(1/2, 0; 0, 76/102), (0, 1/2; 76/102, 0)(0, 0; 0, 0)]$ is always an SNE of the market game. The reason for this is when $y^\gamma \in [7/3, 17/3]$, we always have

$$\begin{cases} \dfrac{\partial_1 f^\nu(2/3, y^\nu)}{1} = \dfrac{1}{1-\frac{1}{2}} \dfrac{g^{\nu'}(101/51)}{51/76} \\ \dfrac{\partial_2 f^\nu(2/3, y^\nu)}{1} = \dfrac{\theta}{1-\frac{1}{2}} \dfrac{g^{\nu'}(101/51)}{51/76} \qquad \theta \in [1/2, 1] . \end{cases}$$

Remark. It is not difficult to see that

1. When $\delta \to 1-0$, the GCE allocation and SNE allocation tend to a same limit-- $(\dot{a}, \dot{A})$. But on the other hand, if $\delta = 1$, even if we use another definition of payoff: $\Phi^\alpha = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \varphi^\alpha(x^\alpha(t), X^\alpha(t))$, the only SNE is the zero SNE!
2. When δ becomes smaller and smaller, from Example 3 one can imagine that the traders become less eager to sell consumer goods. In the extreme case, as $\delta = 0$, no one wants to sell, and consequently the only NE is the zero NE.

3.4. The Meaning of Enough Money

Assume we have already seen, when all the utility functions are gold-separable and $\lim_{X^\alpha\to 0} g^{\alpha'}(X^\alpha) = \infty$, an SNE exists for any amount $G > 0$, provided G is properly distributed. If G is not properly distributed, from next section, an SNE usually can be achieved by introducing the measure of secured lending. So, " $\lim_{X^\alpha\to 0} g^{\alpha'}(X^\alpha) = \infty$ " in some sense eliminates the problem of "enough money."

On the other hand, if $\lim_{X^\alpha \to 0} g^{\alpha'}(X^\alpha) < \infty$, the problem of "enough money" does exist. For this we have

<u>Theorem 7</u>. Assume that the economy $E(a,f)$ has a δ-CE $(\hat{a},\hat{p}) > 0$. Assume that the g^α have the property CIS∞ and $\lim_{X^\alpha \to \infty} g^{\alpha'}(X^\alpha) = 0$.[7] Let

$$(44) \qquad \overset{\circ}{p}_{m+1} = \min_{\alpha,0} \left\{ \frac{\hat{p}_j \lim_{X^\alpha \to 0} g^{\alpha'}(X^\alpha)}{(1-\delta)\partial_j f^\alpha(\hat{a}^\alpha)} \right\} \quad (\hat{a}^\alpha_j - a^\alpha_j > 0) \ .$$

Assume that $\hat{X}^\alpha$ satisfies

$$(45) \qquad \frac{g^{\alpha'}(\hat{X}^\alpha)}{(1-\delta)\overset{\circ}{p}_{m+1}} = \frac{\partial_j f^\alpha(\hat{a}^\alpha)}{\hat{p}_j} \ .$$

Then the game has an SNE provided

$$(46) \qquad G \geq \sum_\alpha \hat{X}^\alpha + \overset{\circ}{p}{}^{-1}_{m+1} \sum_\alpha \hat{p}_j \max(0,\ \hat{a}^\alpha_j - a^\alpha_j)$$

and G is properly distributed.

<u>Sketch of Proof</u>. For any p_{m+1} with $\overset{\circ}{p}_{m+1} \geq p_{m+1} > 0$, let $X^1(p_{m+1}), \ldots, X^n(p_{m+1})$ be the solutions of

$$(47) \qquad \frac{g^{\alpha'}(X^\alpha(p_{m+1}))}{(1-\delta)p_{m+1}} = \frac{\partial_j f^\alpha(\hat{a}^\alpha)}{\hat{p}_j} \quad (\hat{a}^\alpha_j - a^\alpha_j > 0) \ .$$

Let $A^\alpha(p_{m+1}) = X^\alpha(p_{m+1}) + p^{-1}_{m+1} \sum_j \hat{p}_j \max\{0,\ \hat{a}^\alpha_j - a^\alpha_j\}$. Let

---

[7] We assume this just for simplicity of the proof.

$G(p_{m+1}) = \sum_\alpha A(p_{m+1})$. Then as $p_{m+1} \to 0$, $G(p_{m+1}) \to \infty$; as $p_{m+1} \to \mathring{p}_{m+1}$, $G(p_{m+1}) \to \sum_\alpha \hat{X}^\alpha + \mathring{p}_{m+1}^{-1} \sum_\alpha \sum_j \hat{p}_j \max(0, \hat{a}_j^\alpha - a_j^\alpha)$. From (46) one can see that $\exists \hat{p}_{m+1}$ such that $G(\hat{p}_{m+1}) = G$. The distribution of G is given by $A^\alpha = A^\alpha(\hat{p}_{m+1})$.

Remark. From (44), (45) and (46) one can see that as $\delta \to 1-0$, $G \to 0+0$ provided every type of trader has the same marginal utility of gold. Thus in this case when δ is very close to 1, a very small amount of gold is enough.

4. THE MULTISTAGE GAME WITH FULLY SECURED LOANS

Suppose that there is an economy where the n types of traders each obtain each period an endowment of all consumer goods of $a^i = (a_1^i, \ldots, a_m^i)$ for a trader of type i , $i = 1, \ldots, n$.

Initially each trader of type i has A^i units of gold. We have seen from Section 3 above that if there is enough gold properly distributed there could exist a stationary state NE for the infinite horizon game where each trader of type i tries to maximize

$$\sum_{t=1}^{\infty} \beta^{t-1} \varphi_i(x_1^i, \ldots, x_m^i) .$$

We consider the possibility of an economy where gold may not be distributed so that a stationary state exists without borrowing and lending. We establish here that there exists a robust class of games which involve borrowing and lending where the loans are fully secured in the sense made explicit below.

In period t individuals first trade gold against I.O.U. notes for gold to be paid next period. Suppose that at the start of t i holds A_t^i . He may offer v_t^i units of gold for loan where $0 \quad v_t^i \le A_t^i$. Or he may offer u_t^i units of I.O.U. notes to be paid in gold at the start of period $t+1$, where $0 \le u^i \le A_t^i$.

An individual i who borrows will have

$$(48) \qquad A_t^{*i} = A_t^i + u_t^i \frac{\int v_t^j}{\int u_t^j} .$$

An individual i who lends will have

$$(49) \qquad A_t^{*i} = A_t^i - v_t^i .$$

In the first case

$$(50) \qquad \sum_{j=1}^{m} b_{jt}^i \le A_t^{*i} - u_t^i .$$

In the second case

$$(51) \qquad \sum_{j=1}^{m} b_{jt}^i \le A_t^{*i} .$$

In period $t+1$ at the start an individual who has borrowed will have:

$$(52) \qquad A_{t+1}^i = A_t^{*i} - b_t^i - u_t^i + \sum_{j=1}^{m} p_{jt} q_t^i .$$

An individual who has loaned has:

$$(53)\qquad A^{i}_{t+1} = A^{*i}_{t} - b^{i}_{t} + v^{i}_{t}(1+\rho) + \sum_{j=1}^{m} p_{jt} q^{i}_{t}$$

where

$$(54)\qquad 1 + \rho = \frac{\int u^{j}_{t}}{\int v^{j}} .$$

The constraint (3) guarantees that an individual i will always keep enough gold to be able to pay back the loan at the start of period $t+1$ hence there is no need to specify a default penalty. Although implicit in the model is the assumption that if a debtor has the means to pay the rules can be enforced which make him pay.

Because gold is a durable and has a consumption value as jewelry, but does not yield this value during the period it is used for transitions an individual with little gold and a high marginal utility could borrow it to use as a consumer durable, but as it can be converted into money it serves to provide security for a loan up to its discounted value.

First we look at the following Example.

Example 4. Two types α , β of traders each of which has the same utility function as in Example 1.

$$\text{Initial endowment } \left(2,\ 1;\ \frac{556}{180}\right),\ \left(1,\ 2;\ \frac{651}{180}\right) .$$

Not hard to check the original distribution of gold does not lead to an SNE.

Now first let each individual i_α of type α lend an amount $v = 13/30 = 78/180$; and each individual i_β of type β borrows v for trade and return $u = 2v = 26/30$ (note $\delta = 1/2$, $1+\rho = 1/\delta = 2 \Rightarrow \rho = 1$)

at next period. Then

$$A^{\alpha} = \frac{556}{180} \Rightarrow \overset{*}{A}{}^{\alpha} = \frac{556}{180} - \frac{78}{180} = \frac{478}{180}$$

$$A^{\beta} = \frac{651}{180} \Rightarrow \overset{*}{A}{}^{\beta} = \frac{651}{180} + \frac{78}{180} = \frac{729}{180} .$$

The stationary strategy selection

$$\Big(\underset{q_1^{\alpha}}{\frac{7}{20}},\ \underset{q_2^{\alpha}}{0};\ \underset{b_1^{\alpha}}{0},\ \underset{b_2^{\alpha}}{\frac{141}{180}}\Big) \quad \Big(0,\ \frac{3}{4};\ \frac{7}{20},\ 0\Big)$$

is really an SNE.

By calculation we get

$$\text{Prices} \quad \Big(1,\ \frac{47}{45},\ 1\Big)$$

reallocation $\Big(\frac{33}{20}, \frac{7}{4}; \frac{337}{180}\Big)$	$\Big(\frac{27}{20}, \frac{5}{4}; \frac{666}{180}\Big)$
$A^{\alpha}(2) = X^{\alpha} + (1+\rho)v + q_1^{\alpha}p_1$	$A^{\beta}(2) = \frac{666}{180} - u + p_2q_2^{\beta}$
$= \frac{337}{180} + \frac{26}{30} + \frac{7}{20}$	$= \frac{666}{180} - \frac{26}{30} + \frac{47}{45} \times \frac{3}{4}$
$= \frac{556}{180} = A^{\alpha}$	$= \frac{651}{180} = A^{\beta}$

<u>Theorem 8</u>. Assume that there is a partition of the n types of traders: $I_1 = \{1, \ldots, k\}$, $I_2 = \{k+1, \ldots, n\}$ with two sets of nonnegative numbers: $u^1, \ldots, u^k$; $v^{k+1}, \ldots, v^n$ such that $\delta \sum_{i=1}^{k} u^i = \sum_{i=k+1}^{n} v^i$. Assume that there are x^i , X^i $(i = 1, \ldots, n)$ and $p_1, \ldots, p_m, p_{m+1}$ all greater than zero such that all the following requirements hold

$$\text{(i)}\quad \frac{\partial_j f^i(x^i)}{p_j} = \theta(j)\frac{g^{i\prime}(X^i)}{(1-\delta)p_{m+1}}\ , \quad \theta(j) = \begin{cases} 1 & \text{if } x_j^i > a_j^i \\ \delta & \text{if } x_j^i < a_j^i \\ [\delta, 1] & \text{if } x_j^i = a_j^i \end{cases}$$

$$\text{(ii)}\quad \sum_{j=1}^{m} p_j x_j^i + p_{m+1}X^i + \sum_{j=1}^{m} p_j \max(0,\ a_j^i - x_j^i)$$

$$= \begin{cases} \sum_{j=1}^{m} p_j a_j^i + p_{m+1}(A^i + \delta u^i) & (i \in I_1) \\ \sum_{j=1}^{m} p_j a_j^i + p_{m+1}(A^i - v^i) & (i \in I_2) \end{cases}$$

$$\text{(iii)}\quad X^i - A^i + p_{m+1}^{-1}\left(\sum_{j=1}^{m} p_j \max(0,\ a_j^i - x_j^i)\right)$$

$$= \begin{cases} u^i & (i \in I_1) \\ -\delta^{-1}v^i & (i \in I_2) \end{cases}$$

Then $[(x^i,\ X^i)\ (i = 1,\ \ldots,\ n)]$ is an SNE allocation with secured lending (SLSNE). The corresponding strategy selection is given by:

$$\text{(55)}\qquad s^i = \begin{cases} (u^i;\ q^i(t),\ b^i(t)) & (i \in I_1) \\ (v^i;\ q^i(t),\ b^i(t)) & (i \in I_2) \end{cases}$$

where $q_j^i(t) = \max(0,\ a_j^i - x_j^i)$, $b_j^i(t) = \max(0,\ (p_j/p_{m+1})(x_j^i - a_j^i))$.

<u>Proof</u>. It is easy to see that (55) leads to the stationary allocation $(x^i,\ X^i)$ $(i = 1,\ \ldots,\ n)$. To see that (55) is really an NE, one can use the similar argument as in the proof of Theorem 5.

Remark. We believe that generically (under the assumptions on the utility functions and the initial endowments mentioned above) the I_1, I_2, u^i, v^i, x^i, X^i, p do exist. For $n = 2$, we can really prove the existence.

Theorem 9. Assume that there are two types of traders: $\alpha \in [0, 1/2)$, $\beta \in [1/2, 1]$. Assume that the utility functions are C^1, strictly concave, increasing and gold separable. Assume that for each type of traders preference surface passing through the initial point does not touch the coordinate hyperplanes. Then the game Γ_∞ with endowment (a^α, A^α) and (a^β, A^β) has an SLSNE.

Proof. If (A^α, A^β) is a proper distribution, done.

If (A^α, A^β) is not proper, assume that $(\bar{A}^\alpha, \bar{A}^\beta)$ is a proper distribution of $G = A^\alpha + A^\beta$. WLOG, assume that $A^\alpha < \bar{A}^\alpha$, $A^\beta > \bar{A}^\beta$. Now for $v \in [0, A^\beta - \bar{A}^\beta]$ and $u = \delta^{-1}v$, consider the problem: find $(x^\alpha(v), X^\alpha(v))(x^\beta(v), X^\beta(v))$ and $p = (p_1, \ldots, p_m, p_{m+1})$ such that

$$(56) \qquad \frac{\partial_j f^i(x^i(v))}{p_j} = \theta(j)\frac{g^i(X^i(v))}{(1-\delta)p_{m+1}}$$

$$\text{S.T.} \quad \sum_{j=1}^m x^i_j(v)p_j + \sum_{j=1}^m p_j \max(0, a^i_j - x^i_j(v)) + X^i(v)p_{m+1}$$

$$= \begin{cases} \sum_{j=1}^m a^i_j p_j + (A^i + u)p_{m+1}, & i = \alpha \\ \sum_{j=1}^m a^i_j p_j + (A^i - v)p_{m+1}, & i = \beta \end{cases}$$

For $v \in [0, A^\beta - \bar{A}^\beta]$ given, (56) has a unique solution under the assumption of "gross substitute."

Define v' by

$$v' = \delta[v + \sum_{j=1}^{m} p_j(x_j^\alpha - a_j^\alpha)] . \tag{57}$$

Then v' depends on v continuously. Note that $v = 0 \Rightarrow v' > v$ and $v = A^\beta - \bar{A}^\beta \Rightarrow v' = \delta v$. There must exist $\overset{*}{v}$ such that

$$\overset{*}{v} = \delta[\overset{*}{v} + \sum_{j=1}^{m} p_j(x_j^\alpha - a_j^\alpha)] . \tag{58}$$

Now $(x^\alpha(\overset{*}{v}), X^\alpha(\overset{*}{v}))(x^\beta(\overset{*}{v}), X^\beta(\overset{*}{v}))$ and $p(\overset{*}{v})$ satisfy all the requirements of Theorem 7. Q.E.D.

5. CONCLUDING REMARKS

5.1. On Dynamics

We have not yet discussed the dynamic problem concerning the asymptotic behaviors of the NE paths when $a = (a^1, \ldots, a^n)$ and $G > 0$ are given. The mathematical difficulty with this problem comes from the nonuniqueness of the Nash equilibria of $\Gamma_\infty(a,A)$. We have already seen that this non-uniqueness makes the proof of the SNE existence rather complicated, because we cannot assert that along the NE paths $(A^1(2), \ldots, A^n(2))$ continuously depends on $(A^1(1), \ldots, A^n(1))$. Without the uniqueness of NE path of $\Gamma_\infty(a,A)$, when a and G are given, we do not have a well-defined dynamical system (for the mapping $(A^1(t), \ldots, A^n(t)) \mapsto (A^1(t+1), \ldots, A^n(t+1))$ is not well-defined). So the asymptotic behaviors of NE paths are hard to deal with. We leave this as an open problem.

It is likely that the mathematical difficulty due to nonuniqueness signals a problem with the inadequacy of the economic representation of the model. The presence of many long term assets and lags in production processes as well as other institutional givens (such as the bankruptcy laws) might, in a richer model give more structure to a dynamic process, making it possibly dependent upon initial conditions.

5.2. Special Properties of a Commodity Money

Gold and bricks of tea have both served as a money. All commodity moneys have been storable, but some like gold or silver are not consumable whereas bricks of tea or rice or cocoa beans are. Durability, portability, cognizability and several other physical features have long been recognized as desirable properties of a money. Are there any economic properties which should be considered? Tied in with the physical aspects of ease to turn into coin are the economic aspects of relative value. Gold coins for every day transactions would be too small and (as the Swedes found out) copper coins for large transactions are too bulky. Two questions come to mind. Would it matter if the commodity selected for the money were in some regions an inferior good? Would it be desirable if the good were a gross substitute in all regions?

5.3. Circulating Capital and Liquidity

By the simple device of having some time required for trade there is a loss in value by utilizing in trade gold which could be put to a different utilitarian use. In equilibrium there is an amount of gold in circulation which never yields its value in consumption. Yet its value serves as a noncooperative mechanism to facilitate trade.

Even a limited amount of lending is feasible without an elaborate bankruptcy law if borrowers can be required to hold gold up to the value of their borrowing, thus they are always in a position to repay. This condition is consistent with the general ideas about secured lending where the borrower is able to have the use of the asset while it is serving as a hostage for the lender. When assets other than gold are used to provide backing or security for a loan new difficulties appear in guaranteeing that their prices will remain sufficiently stable that they will always cover the loan. We propose to examine this problem elsewhere.

REFERENCES

Amir, R., S. Sahi, M. Shubik and S. Yao (1987). "A Strategic Market Game with Complete Markets," CFDP No. 814R, Yale University, New Haven, CT.

Dubey, P. and M. Shubik (1978). "The Noncooperative Equilibria of a Closed Trading Economy with Market Supply and Bidding Strategies," Journal of Economic Theory, 17, 1, 1-20.

Jevons, W. S. (1984). Money and the Mechanism of Exchange. New York: Garland, original 1875.

Shapley, L. S. and M. Shubik (1977). "Trade Using One Commodity as a Meawns of Payment," Journal of Political Economy, 85, 5, 937-968.

Shubik, M. (1987). "Silver and Gold and Liquidity," Cowles Foundation Discussion Paper No. 841, Yale University, New Haven, CT.