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KNIGHTIAN DECISION THEORY, PART II:

INTERTEMPORAL PROBLEMS

by

Truman F. Bewley

May 1987

ABSTRACT

The theory of choice proposed in "Knightian Decision Theory, Part I" is here applied to intertemporal problems. An analogue of dynamic programming called maxmin programming is developed. Also, it is shown that detailed contingent planning may not be needed in order to achieve maximality, a program being maximal if no other program is preferred to it. In certain circumstances, a maximal program can be achieved by making a finite calculation in each period. This calculation ignores distant future states and could also ignore unlikely contingencies. A decision maker making such calculations would behave much like a satisficer.

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INTRODUCTION

In this paper, I apply the theory of Knightian decision to intertemporal problems. Knightian decision theory is described in the companion paper "Knightian Decision Theory, Part I" (Bewley, 1986). This theory is obtained from the usual Bayesian theory by dropping the assumption that the preference ordering on lotteries is complete and by adding what I term an inertia assumption. The incomplete preferences are represented by a set of personal probability distributions rather than by a single distribution. One lottery is preferred to another if its expected utility is higher according to all the distributions. The inertia assumption asserts that in some circumstances one can define a status quo, which the decision maker abandons in favor of an alternative only if doing so leads to an improvement. The theory is meant to apply to contexts involving Knightian uncertainty, that is, to decision problems where the probabilities of various outcomes are not known objectively.

In this paper, I investigate how intertemporal decision theory is affected by the shift from the Bayesian to the Knightian point of view. Perhaps the main insight gained by this shift is that in Knightian programming one can in certain circumstances achieve an undominated program in an infinite horizon problem by making in each period only finitely many calculations. These calculations ignore distant future periods. They could also ignore future states of low probability. If a decision maker calculated a program in this way, his observed behavior would be similar to that of a Herbert Simon (1955, 1959) satisficer.

Another main result is that in certain circumstances one can define an analogue of dynamic programming, which I term maxmin programming.* This form of programming would allow one to calculate a program dominating a given initial program and would be appropriate for a decision maker who wished to satisfy the inertia assumption.

The plan of the paper is as follows. In Section 1, I discuss briefly the representation of Knightian preferences when there are infinitely many states of nature. The infinity of states arises naturally in intertemporal problems because many such problems are most easily formulated using infinite horizon models.

In Section 2, I describe two types of decision problems. The first type is similar to the usual Markov decision problem. In the second type, a distinction is made between exogenous and endogenous states. The evolution of exogenous states is not affected by the actions of the decision maker. Endogenous states evolve in response to his actions and to change in the exogenous states. The rest of the paper analyzes only problems of the two types.

Section 3 contains basic and unsurprising theorems for the two types of problem. One theorem asserts the existence of maximal programs. A program is maximal if no other program is preferred to it. The other theorem asserts that any maximal program is optimal with respect to one of the personal probability distributions.

Sections 4-6 are devoted to maxmin programming, which applies only to the second type of problem. One can apply the usual dynamic programming to

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*Henig (1985) develops dynamic programming with rewards which are only partially ordered. He does not define maxmin programming. I owe this reference to Donald Brown.

either type of problem simply by optimizing with respect to a fixed probability distribution. However, this technique of optimization would not necessarily yield a program dominating a particular initial program.

Sections 7-9 contain the main theorems that maximal programs may be realized by finite algorithms. These theorems are motivated by the criticism of decision theory made by behavioral theorists, who claim that actual decision makers do not make the detailed contingent plans that optimization requires. (For a critique of this nature, see Nelson and Winter (1982).) One answer to this criticism is that if computational costs are significant, one should be interested only in approximately optimal programs. Such an answer implies that detailed contingent planning should become more common as technological improvements reduce computational costs. Another way to meet the behavioralist criticism is to replace optimality with Knightian maximality. The Knightian approach implies that detailed contingent planning might not be necessary. The behavior suggested by the Knightian theory resembles that which behavioral theorists claim to observe.

1. Representation of Preferences Given Infinitely Many States

Yakar Kannai (1963) extended Aumann's representation of incomplete preferences over lotteries to the case of infinite dimensional alternatives. Here, I adapt his results to the context of the models in this paper.

The infinite set of states of the world is denoted Ω . Ω is assumed to be a compact metric space. M denotes the Borel σ-field generated by the open subsets of Ω . The set of lotteries is $C(\Omega)$, the set of continuous functions on Ω . If $x \in C(\Omega)$, $x(\omega)$ is the payoff in utility in state ω . The vector space $C(\Omega)$ is given the maximum norm, $\|\cdot\|_\infty$, defined by

$\|x\|_\infty = \max\{|x(\omega)| : \omega \in \Omega\}$. The function $e \in C(\Omega)$ denotes the constant function everywhere equal to 1. If x and y belong to $C(\Omega)$, " $x \geq y$ " means " $x(\omega) \geq y(\omega)$," for all ω in Ω . There is a preference ordering on $C(\Omega)$, denoted by $\succ$, which satisfies the following assumptions.

<u>Assumption 1.1</u>. $x \succ y \succ z$ implies $x \succ z$, and for no x is $x \succ x$.

<u>Assumption 1.2</u>. For all x , $\{y | y \succ x\}$ is open.

<u>Assumption 1.3</u>. $e > 0$ and if $y \leq x$, then $z \prec y$ implies $z \prec x$.

<u>Assumption 1.4</u>. For all x , y and z in $C(\Omega)$ and for all α such that $0 < \alpha < 1$, $y \succ z$ if and only if $\alpha x + (1-\alpha)y \succ \alpha x + (1-\alpha)z$.

<u>Proposition 1.1</u>. The set $K = \{x \in C(\Omega) : x \succ 0\}$ is open and convex. Also, $x \succ y$ if and only if $x-y \in K$. If $y \geq x \in K$, then $y \in K$ and $e \in K$.

This proposition is an easy consequence of the assumptions. Its proof is contained in the proof of Theorem 1.1 in Bewley (1986).

The set of continuous linear functions on $M(\Omega)$ is $rca(\Omega)$, the set of regular, countably additive set functions on M (see Dunford and Schwartz (1957), p. 265). If $\mu \in rca(M)$, the corresponding functional on $C(\Omega)$ is defined by $\mu \cdot x = \int x(\omega)\mu(dw)$, for $x \in M(\Omega)$. The weak topology on $rca(\Omega)$ is the weakest topology on $rca(\Omega)$ such that for each $x \in C(\Omega)$, the function carrying $\mu \in rca(\Omega)$ to $\mu \cdot x$ is continuous. A set function $\mu \in rca(\Omega)$ is called a probability measure if $\mu(A) \geq 0$, for all $A \in M$ and if $\mu(\Omega) = 1$. If μ is a probability measure, then $E_\mu x$ denotes $\int x(\omega)\mu(d\omega)$. $E_\mu x$ is the expected value with respect to μ .

Corollary 1.2. There is a set Π of probability measures in $rca(\Omega)$ such that $x \succ y$ if and only if $E_\pi x > E_\pi y$, for all $\pi \in \Pi$. Π is convex and weakly compact.

The set Π is simply $\{\pi \in rca(\Omega) : \pi(\Omega) = 1$ and $\int x d\pi > 0$, for all $x \in K\}$, where K is as in Theorem 1.1. The measures in Π may be termed personal probability measures. The corollary follows easily from the separation theorem for Banach spaces and the Banach-Alaoglu theorem (Dunford and Schwartz (1957), p. 417 and Schaeffer (1970), p. 84, respectively).

2. Two Types of Decision Problem

The decision problems to be discussed are now described. A decision problem is described by a set of observable states of the environment, sets of possible actions, reward functions and classes of possible transition probabilities. Throughout, it is assumed that a reward is received each period and future rewards are discounted at rate δ, where $0 < \delta < 1$.

Type I Problems. X denotes the set of states of the environment. X is assumed to be a tree with root x_0, x_0 being the initial state in period zero. The set of states possible in period t is $X_t = \{x \in X | x$ is connected to x_0 by t arcs$\}$. A member of X_t is denoted by x_t. For each $x \in X$, there is a set of possible actions, $A(x)$. The reward function is r : graph $A \to [0,\infty]$, where graph $A = \{(x,a) | x \in X,\ a \in A(x)\}$. If $x \in X$, let $I(x) = \{y \in X | y$ immediately succeeds $x\}$ and let $G(x) = \{g : A(x) \to I(x)\}$. The set of states on which personal probabilities are defined is $\Omega = \times_{x \in X} G(x)$. States in Ω are termed probability states in order to distinguish them from the environment states x_t. Give Ω the product of the discrete topologies on the $G(x)$ and let $\mathcal{M}$ be the

Borel σ-field generated by this topology. Since the $G(x)$ are finite sets, Ω is compact by the Tychonoff product theorem. Since Ω is metrizable, corollary 1.2 applies, and one may assume that the set of personal probabilities is a subset, Π , of $rca(\Omega)$.

A program is a member of $\underset{\sim}{A} = \times_{x \in X} A(x)$. A program $\underset{\sim}{a}$ and an $\omega \in \Omega$ together imply a unique state $x_t(\underset{\sim}{a},\omega)$ for each period t , where $x_t(\underset{\sim}{a},\omega)$ is defined by induction on t as follows. $x_0(\underset{\sim}{a},\omega) = x_0$ and, for all $t \geq 0$, $x_{t+1}(\underset{\sim}{a},\omega) = \omega(x_t(\underset{\sim}{a},\omega), \underset{\sim}{a}(x_t(\underset{\sim}{a},\omega)))$. In the last equation, $\omega(x,\cdot)$ denotes the x^{th} component of ω , which is itself a function from $A(x)$ to $I(x)$. Given $\underset{\sim}{a} \in \underset{\sim}{A}$ and $\omega \in \Omega$, $\hat{r}(\underset{\sim}{a},\omega)$ denotes $\sum_{t=0}^{\infty} \delta^t r(x_t(\underset{\sim}{a},\omega), \underset{\sim}{a}(x_t(\underset{\sim}{a},\omega)))$.

I next define random programs. Give $\underset{\sim}{A} = \times_{x \in X} A(x)$ the product of the discrete topologies on the $A(x)$ and let $\mathcal{A}$ be the Borel σ-field generated by this topology. A random program is a σ-additive probability measure on $\mathcal{A}$. The set of random programs is denoted Γ . $\underset{\sim}{A}$ may be considered to be a subset of Γ in the obvious way. If $\gamma \in \Gamma$, $\hat{r}(\gamma,\omega)$ denotes $\int_{\underset{\sim}{A}} \hat{r}(\underset{\sim}{a},\omega)\gamma(d\underset{\sim}{a})$.

If $\pi \in \Pi$ and $\gamma \in \Gamma$, $E_\pi \hat{r}(\gamma)$ denotes $\int \hat{r}(\gamma,\omega)\pi(d\omega)$. If γ and γ' are random programs, then γ is said to dominate γ' if $E_\pi \hat{r}(\gamma) > E_\pi \hat{r}(\gamma')$, for all $\pi \in \Pi$. A program is <u>maximal</u> if no program dominates it. A program $\bar{\gamma} \in \Gamma$ is optimal with respect to $\pi \in \Pi$ if it solves $E_\pi \hat{r}(\bar{\gamma}) = \max_{\gamma \in \Gamma} E_\pi \hat{r}(\gamma)$.

The following is assumed throughout the rest of the paper.

<u>Assumption 2.1</u>. For all $x \in X$, if a and a' belong to $A(x)$ and $a \neq a'$, then $\pi(\omega | \omega(x,a) = \omega(x,a')) = 0$, for all $\pi \in \Pi$.

If this assumption is not satisfied it may be obtained by replacing X by $\{(x_0, (a_0, x_1), \ldots, (a_{t-1}, x_t)) | \text{for all } n, x_n \in X_n, x_{n+1} \text{ follows } x_n \text{ and } a_n \in A(x_n)\}$.

Assumption 2.1 implies that knowledge of x_t implies knowledge of all actions taken previously. It follows that transition probabilities are Markov. That is, for all $\pi \in \Pi$, the probability of transition from x_t to x_{t+1} when action $a \in A(x_t)$ is taken depends only on x_t , a and π .

The fact that transition probabilities are Markov is a notational convenience, but is otherwise of no use since states never recur. Of course, in reality conditions do recur and as a result people learn about the probability laws governing their environment. All such learning is assumed to be incorporated in the definition of Π .

Type II Problems. The set X of environmental states is now written as $S \times W$. The components s and w of $(s,w) \in S \times W$ should be thought of as exogenous and endogenous, respectively. S is assumed to be a tree with root s_0 , and the set of exogenous states possible in period t is $S_t = \{S \in S | s \text{ is connected with } s_0 \text{ by } t \text{ arcs}\}$. A member of S_t is denoted by s_t . W is assumed to be a non-empty subset of some Euclidean space R^N . An initial endogenous state, w_0 , is specified. Actions are assumed to be members of an Euclidean space R^K . The set of possible actions is defined by a correspondence $A : S \times W \to R^K$. The reward function is $r : \text{graph } A \to [0,\infty)$.

The evolution of the endogenous state is determined by a function $h : Y \to W$, where $Y = \{(s,w,a) | s \in S, w \in W, \text{ and } a \in A(s',w), \text{ where } s' \text{ immediately precedes } s\}$. Thus, if action $a \in A(s_t, w_t)$ is taken in state (s_t, w_t) , and if the succeeding exogenous state is s_{t+1} , then the suc-

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The states on which personal probabilities are defined is $\Omega = \{(s_0, s_1, \ldots) | \forall t, s_t \in S \text{ and } s_{t+1} \text{ succeeds } s_t\}$. Given each S_t the discrete topology and $\times_{t=0}^{\infty} S_t$ the product of these topologies. Give Ω the relative topology as a subset of $\times_{t=0}^{\infty} S_t$. Ω is metrizable and is compact, being a closed subset of the compact set $\times_{t=0}^{\infty} S_t$. Let $\mathcal{M}$ be the Borel σ-field on Ω generated by this topology. The set of personal probabilities is a subset, Π , of $rca(\Omega)$. Because S is a tree, each $\pi \in \Pi$ is a Markov process on S .

The set $Z = \times_{s \in S} R^K$ may be considered to be the set of potential programs. Given $\underset{\sim}{a} \in Z$, the endogenous state in period t , $w_t(\underset{\sim}{a}, s_t)$, is defined as follows, if it may be defined at all. The definition proceeds by induction on t . For $t = 0$, $w_0(\underset{\sim}{a}, s_0) = w_0$. Suppose that $w_t(\underset{\sim}{a}, s_t)$ has been defined and that s_{t+1} succeeds s_t . If $\underset{\sim}{a}(s_t) \in A(s_t, w_t(\underset{\sim}{a}, s_t))$, let $w_{t+1}(\underset{\sim}{a}, s_{t+1}) = h(s_{t+1}, w_t(\underset{\sim}{a}, s_t), \underset{\sim}{a}(s_t))$. Otherwise, $w_{t+1}(\underset{\sim}{a}, s_{t+1})$ is not defined. A potential program, $\underset{\sim}{a}$, is called a program if $w_t(\underset{\sim}{a}, s_t)$ is defined for all t and all s_t . The set of programs is denoted $\underset{\sim}{A}$, or by $\underset{\sim}{A}(s_0, w_0)$ if it is necessary to indicate the dependence on the initial state.

If $\underset{\sim}{a} \in \underset{\sim}{A}$ and $\omega = (s_0, s_1, \ldots) \in \Omega$, then $\hat{r}(\underset{\sim}{a},\omega)$ denotes $\sum_{t=0}^{\infty} \delta^t r(s_t, w_t(\underset{\sim}{a}, s_t), \underset{\sim}{a}(s_t, w_t(\underset{\sim}{a}, s_t)))$. If $\underset{\sim}{a}$ and $\underset{\sim}{a}'$ are programs, $\underset{\sim}{a}$ <u>dominates</u> $\underset{\sim}{a}'$ if $E_\pi \hat{r}(\underset{\sim}{a}) > E_\pi \hat{r}(\underset{\sim}{a}')$, for all $\pi \in \Pi$, where $E_\pi \hat{r}(\underset{\sim}{a}) = \int \hat{r}(\underset{\sim}{a},\omega)\pi(d\omega)$. $\underset{\sim}{a}$ is <u>maximal</u> if no program dominates it. A program $\bar{\underset{\sim}{a}} \in \underset{\sim}{A}$ is optimal with respect to $\pi \in \Pi$ if it solves $E_\pi \hat{r}(\bar{\underset{\sim}{a}}) = \max_{\underset{\sim}{a} \in \underset{\sim}{A}} E_\pi \hat{r}(\underset{\sim}{a})$.

The states s_t are assumed to be observable, so that at any moment one can calculate what past and current returns would have been had one followed an alternative program. Such an assumption is appropriate for problems involving investment in securities with published prices and dividends. It might not be appropriate for problems involving investment in machinery and equipment, where one might know the return only for the type and scale of production process actually used. Such problems might better be modeled as problems of Type I.

3. Basic Theorems

I here state conditions under which a decision problem of either type has a maximal program and is such that any maximal program is optimal with respect to some $\pi \in \Pi$.

Assumptions Applying to Both Types of Problem

Assumption 3.1. The reward function r : graph $A \to [0,\infty)$ is bounded.

Assumption 3.2. Π is non-empty, convex and weakly compact.

Assumption Applying to Type I Problems

Assumption 3.3. For each $x \in X$, $I(x)$ and $A(x)$ are finite non-empty sets.

Assumptions Applying to Type II Problems

Assumption 3.4. Each $s \in S$ has a finite and non-empty set of immediate successors. W is a closed, convex and subset of R^N with non-empty interior. For all $(s,w) \in S \times W$, $A(s,w)$ is a non-empty compact subset of R^K.

If $s \in S$ and $s \neq s_0$, let $Y(s) = \{(s',w,a) \in Y | s' = s\}$, where Y is as defined in Section 2. For $n = 1, \ldots, N$, let h_n be the n^{th} component of h.

Assumption 3.5. For all $s \in S$, graph $A(s,\cdot)$ is convex and closed and $r(s,\cdot)$: graph $A(s,\cdot) \to [0,\infty)$ is concave and continuous. For $s \neq s_0$ and for $n = 1, \ldots, N$, $h_n(s,\cdot) : Y(s) \to W$ is concave and continuous.

Assumption 3.6. If w and w' belong to W and $w \geq w'$, then $A(s,w) \supseteq A(s,w')$ for all s. For all s and a and for $n = 1, \ldots, N$, the functions of w, $r(s,w,a)$ and $h_n(s,w,a)$, are non-decreasing.

Throughout the rest of the paper, it is assumed that the above assumptions apply to the respective types of problems. The following theorems apply to either type of problem.

Theorem 3.1. A maximal program exists.

Theorem 3.2. A program is maximal if and only if it is optimal with respect to some $\pi \in \Pi$.

Proof of Theorem 3.1 for Type I Problems

As in the previous section, for each $x \in X$, let $A(x)$ have the discrete topology and let $\underset{\sim}{A} = \times_{x \in X} A(x)$ have the product topology. Since each $A(x)$ is finite by assumption 3.3, $A(x)$ is compact and so by the Tychonoff product theorem $\underset{\sim}{A}$ is compact.

Since by assumption 3.1, r is bounded, the function $\hat{r} : \underset{\sim}{A} \times \Omega \to [0,\infty)$ is continuous with respect to the product of the topologies on $\underset{\sim}{A}$ and Ω. Since $\underset{\sim}{A} \times \Omega$ is compact, $\hat{r}$ is uniformly continuous. (The topologies on $\underset{\sim}{A}$ and Ω are metrizable, so that uniform continuity may be defined.) Therefore, for any $\pi \in \Pi$, the function $E_{\pi}\hat{r} : \underset{\sim}{A} \to [0,\infty)$ is continuous. Since $\underset{\sim}{A}$ is compact, $E_{\pi}\hat{r}$ achieves its maximum. Any program achieving the maximum is maximal in the sense of being undominated. Q.E.D.

Proof of Theorem 3.1 for Type II Problems

Give the set of potential programs, $Z = \underset{s \in S}{\times} R^K$, the product of the usual topologies on each copy of R^K. Give $\underset{\sim}{A}$ the relative topology as a subset of Z. It is routine to verify that $\underset{\sim}{A}$ is non-empty and compact.

It is not hard to verify that $\hat{r} : \underset{\sim}{A} \times \Omega \to [0,\infty)$ is continuous. As in the previous proof, it follows that there exists a maximal program. Q.E.D.

Proof of Theorem 3.2 for Type I Problems

Clearly, any program optimal with respect to some $\pi \in \Pi$ is maximal. In order to prove the converse, it must be shown that if $\bar{\gamma} \in \Gamma$ is maximal, there exists $\bar{\pi} \in \Pi$ such that $E_{\bar{\pi}}\hat{r}(\bar{\gamma}) \geq E_{\bar{\pi}}\hat{r}(\gamma)$, for all $\gamma \in \Gamma$.

Let $D = \{\hat{r}(\gamma,\cdot) : \Omega \to [0,\infty) | \gamma \in \Gamma\} \subset C(\Omega)$. D is convex, for if γ_1 and γ_2 belong to Γ and $0 < \alpha < 1$, then $\alpha\gamma_1 + (1-\alpha)\gamma_2 \in D$ and $\hat{r}(\alpha\gamma_1 + (1-\alpha)\gamma_2) = \alpha\hat{r}(\gamma_1) + (1-\alpha)\hat{r}(\gamma_2)$.

Let $K = \{x \in C(\Omega) | E_{\pi}\cdot x > E_{\pi}\hat{r}(\bar{\gamma})$, for all $\pi \in \Pi\}$. K is convex and has non-empty interior with respect to the maximum norm on $C(\Omega)$.

Since $\bar{\gamma}$ is maximal and Π is non-empty, $D \cap K = \phi$. By the separation theorem for Banach spaces, there exists $\bar{\pi} \in C(\Omega)$ such that $\bar{\pi} \neq 0$ and $\int (x(\omega) - y(\omega))\bar{\pi}(d\omega) > 0$, for all $x \in K$ and $y \in D$. It follows that $\bar{\pi}(\Omega) > 0$, so that one may assume $\bar{\pi}(\Omega) = 1$. Also, if $E_{\pi}x > 0$, for all $\pi \in \Pi$, then $E_{\bar{\pi}}x > 0$. Since Π is weakly compact, it follows easily that $\bar{\pi} \in \Pi$. Clearly, $E_{\bar{\pi}}\hat{r}(\bar{\gamma}) \geq E_{\bar{\pi}}\hat{r}(\gamma)$, for all $\gamma \in \Gamma$. Q.E.D.

Proof of Theorem 3.2 for Type II Problems

The following lemma is easy to verify.

Lemma 3.3. Let $\underset{\sim}{a}'$ and $\underset{\sim}{a}''$ belong to $\underset{\sim}{A}$ and $0 < \alpha < 1$. Let $\underset{\sim}{a}$ be defined by $\underset{\sim}{a}(s) = \alpha\underset{\sim}{a}'(s) + (1-\alpha)\underset{\sim}{a}''(s)$, for all $s \in S$. Then, $\underset{\sim}{a} \in \underset{\sim}{A}$ and $w_t(\underset{\sim}{a}, s_t) \geq \alpha w_t(\underset{\sim}{a}', s_t) + (1-\alpha)w_t(\underset{\sim}{a}'', s_t)$, for all t and all s_t and $\hat{r}(\underset{\sim}{a},\omega) \geq \alpha\hat{r}(\underset{\sim}{a}', \omega) + (1-\alpha)\hat{r}(\underset{\sim}{a}'', \omega)$, for all $\omega \in \Omega$.

The proof now proceeds as for Type I problems with D defined to be $\{x \in C(\Omega) | x \leq \hat{r}(\underset{\sim}{a}), \text{ for some } \underset{\sim}{a} \in \underset{\sim}{A}\}$. Q.E.D.

4. Recursivity in Problems of Type II

I now describe a special assumption on Π , which applies to problems of type II. First of all, the following minor assumption is needed.

Assumption 4.1. For all $s \in S$, $\pi(s) > 0$, for all $\pi \in \Pi$.

For each $s \in S$, let $I(s)$ be the set of immediate successors of s . If $s_t \in S$ and $\pi \in \Pi$, let $\pi_T(s_{t+1}|s_t)$ be the probability according to π of s_{t+1} conditional on the occurrence of s_t . Let $\Pi_T(s) = \{\pi_T(\cdot|s) | \pi \in \Pi\}$. ("T" stands for "transition.") By assumption 4.1, $\Pi_T(s)$ is a well-defined, closed convex set of probabilities on $I(s)$. Let $F : \Pi \to \times_{s \in S} \Pi_T(s)$ be the map defined by $F(\pi)_s = \pi_T(\cdot|s)$ for all $s \in S$.

Definition 4.2. Π is recursive if the map F is subjective.

This assumption plays the same role in maxmin programming as the Markov assumption in the usual dynamic programming. However, it is not a generalization of independence, as is the usual Markov assumption. If it were believed that the factors influencing transition from s_t to s_{t+1} were independent of s_t, for all t, then the sets $\Pi_T(s_t)$ should, for each t, be independent of s_t and, say, equal to Π_{Tt}. Also, if $\pi_t \in \Pi_{Tt}$, for all t, then there should be $\pi \in \Pi$ such that $\pi_T(\cdot|s_t) = \pi_t$, for all s_t and t. However, if $\pi_{s_t} \in \Pi_{Tt}$, for each s_t and t, it does not follow that there is $\pi \in \Pi$ such that $\pi_T(\cdot|s_t) = \pi_{s_t}$, for all t and s_t, so that Π is not recursive.

Nevertheless, even if the s_t are independent in the sense just described, Π may contain a recursive subset, so that recursivity may be used for some purposes, such as the result of Section 9 below. If the sets Π_{Tt} are sufficiently large, then Π contains a recursive subset. (In the extreme case in which each Π_{Tt} consists of all possible transition probabilities, Π consists of all regular, countably additive probability measures on M. The set of all such probability measures is certainly recursive.)

5. Maxmin Programming for Problems of Type II

In order to describe maxmin programming, some additional notation is needed. For any $(s_t, w) \in S \times W$ and any t, there is a decision subproblem $P(s_t, w)$ with initial state (s_t, w). The tree of exogenous states for $P(s_t, w)$ is $S(s_t) = \{s \in S | s = s_t \text{ or } s \text{ follows } s_t\}$. The set of personal probabilities for $P(s_t, w)$ is $\Pi(s_t) = \{\pi[\cdot|s_t] | \pi \in \Pi\}$. The probability measures in $\Pi(s_t)$ are defined on the measurable subsets of $\Omega(s_t)$, where $\Omega(s_t) = \{w = (s_0', s_1', \ldots) \in \Omega | s_t' = s_t\}$. Let $\underline{A}(s_t, w)$ be the set of programs for $P(s_t, w)$. If $\underline{a} \in \underline{A}(s_t, w)$, let $w_{t+n}(s_t, w, \underline{a}, s_{t+n})$ be defined just as $w_t(\underline{a}, s_t)$ was defined in Section 2. That is, $w_t(s_t, w, \underline{a}, s_t) = w$ and $w_{t+n+1}(s_t, w, \underline{a}, s_{t+n+1})$

$= h(s_{t+n+1}, w_{t+n}(s_t, w, \underset{\sim}{a}, s_{t+n}), \underset{\sim}{a}(s_{t+n}, w_{t+n}(s_t, w, \underset{\sim}{a}, s_{t+n})))$. Finally, if $\underset{\sim}{a} \in \underset{\sim}{A}(s_t, w)$ and $\omega = (s_0, s_1, \dots) \in \Omega(s_t)$, let $\hat{r}(s_t, w, \underset{\sim}{a}, \omega)$ $= \sum_{n=0}^{\infty} \delta^n r(s_{t+n}, w_{t+n}(s_t, w, \underset{\sim}{a}, s_{t+n}), \underset{\sim}{a}(s_{t+n}, w_{t+n}(s_t, w, \underset{\sim}{a}, s_{t+n})))$.

If $\underset{\sim}{a} \in \underset{\sim}{A}$, let $R_t(\underset{\sim}{a}, s_t) = r(s_t, w_t(\underset{\sim}{a}, s_t), \underset{\sim}{a}(s_t))$, where $w_t(\underset{\sim}{a}, s_t)$ is as defined in Section 2. Also, if $\omega \in \Omega(s_t)$, let $\hat{R}(\underset{\sim}{a}, s_t, \omega) = \sum_{n=0}^{\infty} \delta^n R_{t+n}(\underset{\sim}{a}, s_{t+n})$, where $\omega = (s_0, s_1, \dots, s_t, s_{t+1}, \dots)$.

The value function for maxmin programming is defined relative to a fixed program $\bar{\underset{\sim}{a}} \in \underset{\sim}{A}$, the value function being

$$V(\bar{\underset{\sim}{a}},s,w) = \max_{a \in A(s,w)} \min_{\pi \in \Pi(s)} E\,[\hat{r}(\underset{\sim}{a}) - \hat{R}(\bar{\underset{\sim}{a}},s)]\ .$$

Clearly, $V(\bar{\underset{\sim}{a}}, s_0, w_0) \geq 0$ and $\bar{\underset{\sim}{a}}$ is maximal if and only if $V(\bar{\underset{\sim}{a}}, s_0, w_0) = 0$. Also, if $\bar{\underset{\sim}{a}}$ is maximal, then $V(\bar{\underset{\sim}{a}}, s_t, w_t(\bar{\underset{\sim}{a}}, s_t)) = 0$, for all s_t .

<u>Theorem 5.1</u>. $V(\bar{\underset{\sim}{a}})$ is well-defined and satisfies the equation $V(\bar{\underset{\sim}{a}}, s_t, w)$

$$= \max_{a \in A(s_t,w)} [r(s_t,w,a) - R(\bar{\underset{\sim}{a}},s_t) + \delta \min_{\pi \in \Pi_1(s_t)} E_\pi V(\bar{\underset{\sim}{a}}, s_{t+1}, h(s_{t+1},w,a))]\ .$$

<u>Proof</u>. It is first shown that $V(\bar{\underset{\sim}{a}},s,w)$ is well-defined. Because $\Pi(s)$ is weakly compact and $\hat{r}(\underset{\sim}{a}) - \hat{R}(\bar{\underset{\sim}{a}},s) \in C(\Omega(s))$, the minimum over $\Pi(s)$ exists. As in the proof of theorem 3.1, the function $\hat{r} : \underset{\sim}{A}(s,w) \times \Omega(s) \to [0,\infty)$ is continuous and $\underset{\sim}{A}(s,w)$ and $\Omega(s)$ are compact and metrizable. It follows that $\hat{r}$ is uniformly continuous and so $\min_{\pi \in \Pi(s)} E_\pi \hat{r}(\underset{\sim}{a})$ is continuous with respect to $\underset{\sim}{a}$. Hence, the maximum over $\underset{\sim}{a}$ exists in the definition of V .

The recursion equation for V follows from the following equations.

$$V(\underline{\bar{a}}, s_t, w) = \max_{\underline{a}\in\underline{A}(s_t,w)} \min_{\pi\in\Pi(s_t)} E_\pi[\hat{r}(\underline{a}) - \hat{R}(\underline{\bar{a}}, s_t)]$$

$$= \max_{\underline{a}\in\underline{A}(s_t,w)} \{r(s_t, w, \underline{a}(s_t)) - R_t(\underline{\bar{a}}, s_t) + \min_{\pi\in\Pi_T(s_t)} \sum_{s_{t+1}} \pi(s_{t+1}) \min_{\pi'\in\Pi(s_{t+1})}$$

$$E_{\pi'}[\hat{r}(\underline{a}) - r(s_t, w, \underline{a}(s_t)) - \hat{R}(\underline{\bar{a}}, s_{t+1})]\}$$

$$= \max_{a\in A(s_t,w)} \{r(s_t, w_t, a) - R(\underline{\bar{a}}, s_t) + \min_{\pi\in\Pi_T(s_t)} \sum_{s_{t+1}} \pi(s_{t+1})$$

$$\max_{\underline{a}\in\underline{A}(s_{t+1},h(s_{t+1},w,a))} \min_{\pi'\in\Pi(s_{t+1})} E_{\pi'}[\hat{r}(a) - \hat{R}(\bar{a}, s_{t+1})]\}$$

$$= \max_{a\in A(s_t,w)} [r(s_t, w, a) - R_t(\underline{\bar{a}}, s_t) + \min_{\pi\in\Pi_T(s_t)} E_\pi V(\underline{\bar{a}}, s_{t+1}, h(s_{t+1}, w, a))] .$$

The first and last equations follow from the definition of V . The second equation follows from the definition of $\hat{r}(\underline{a})$ and $\hat{R}(\underline{\bar{a}}, s_t)$ and the recursivity of Π . The third equation should be obvious. Q.E.D.

It is not hard to obtain an analogue of the policy improvement method using V . In order to do so, it is necessary to characterize V as the unique fixed point of the obvious contraction mapping. Give $S \times W$ the product of the discrete topology on S and the usual topology on W as a subset of R^N . Let $C_b(S \times W)$ be the set of all continuous bounded functions on $S \times W$. If $v \in C_b(S \times W)$, let $H(\underline{\bar{a}})v$ be the function on $S \times W$ defined by $H(\underline{\bar{a}})v(s_t, w) = \max_{a\in A(s_t,w)} \{r(s_t, w, a) - R_t(\underline{\bar{a}}, s_t) + \delta \min_{\pi\in\Pi_T(s_t)} E_\pi v(s_{t+1}, h(s_{t+1}, w, a))]$. Then, $H(\underline{\bar{a}})v \in C_b(S \times W)$, and $H(\underline{\bar{a}}) : C_b(S \times W) \to C_b(S \times W)$ is a contraction with respect to this supremum norm on $C_b(S \times W)$. The next theorem is an immediate consequence of the contraction mapping theorem and theorem 5.1.

<u>Theorem 5.2</u>. $V(\bar{\underline{a}})$ is the unique fixed point of $H(\bar{\underline{a}})$ and so belongs to $C_b(S \times W)$. If $v \in C_b(S \times W)$, then $H^n(\bar{\underline{a}})v$ converges uniformly to $V(\bar{\underline{a}})$ as n goes to infinity. $V(\bar{\underline{a}},s,w)$ is a concave function of w , for each $\bar{\underline{a}}$ and s .

Once $V(\bar{\underline{a}})$ is known, it is possible to describe how to compute $\underline{a}$ such that $\min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}) - \hat{r}(\bar{\underline{a}})] = V(\bar{\underline{a}}, s_0, w_0)$. The program $\underline{a}$ must, for each s_t , solve the equation $V(\bar{\underline{a}}, s_t, w_t(\underline{a},s_t)) = r(s_t, w_t(\underline{a},s_t), \underline{a}(s_t))$ $- R_t(\bar{\underline{a}},s_t) + \delta \min_{\pi\in\Pi_T(s_t)} E_\pi V(\bar{\underline{a}}, s_{t+1}, h(s_{t+1}, w_t(\underline{a}, s_t), \underline{a}(s_t)))$. Since $A(s_t, w_t(\underline{a}, s_t))$ is compact and $V(\bar{\underline{a}})$ is continuous, this equation has a solution, provided $w_t(\underline{a}, s_t)$ is known. Since $w_0(\underline{a}, s_0) = w_0$ is known, one can build up $\underline{a}(s_t)$ by induction on t , starting from $t = 0$.

It is now possible to describe the analogue of the policy improvement method. The improvement step proceeds as follows. Suppose one is given a program $\underline{a}_n$. Compute $V(\underline{a}_n)$. If $V(\underline{a}_n, s_0, w_0) = 0$, $\underline{a}_n$ is maximal and no improvement is possible. If $V(\underline{a}_n, s_0, w_0) > 0$, let $\underline{a}_{n+1}$ be such $\min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}_{n+1}) - \hat{r}(\underline{a}_n)] = V(\underline{a}_n, s_0, w_0)$.

If one starts with an arbitrary $\underline{a}_0$, then successive application of the improvement step yields a sequence $\underline{a}_0, \underline{a}_1, \ldots$. If this sequence continues indefinitely, it has a limit point, since $\underline{A}$ is compact. Let $\bar{\underline{a}}$ be either the last member of the sequence or a limit point.

<u>Theorem 5.3</u>. $\bar{\underline{a}}$ is maximal and dominates $\underline{a}_0$ unless $\bar{\underline{a}} = \underline{a}_0$.

The improvement method just described is not, of course, an algorithm, for it may not be possible to carry out any of the steps exactly. However, the method suggests how to obtain an approximately maximal program dominating a given program, for each of the steps can be carried out approximately.

Proof of Theorem 5.3. If $\bar{\underline{a}} = \underline{a}_n$, for some n, then $V(\bar{\underline{a}}, s_0, w_0) = 0$, so that $\bar{\underline{a}}$ is maximal. Suppose that $\bar{\underline{a}}$ is a limit point of $\underline{a}_0, \underline{a}_1, \ldots$. For each n, let $\varepsilon_n = \min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}_{n+1}) - \hat{r}(\underline{a}_n)]$. For all n, $\varepsilon_n > 0$ and $\sum_{n=0}^{N} \varepsilon_n \leq \min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}_{N+1}) - \hat{r}(\underline{a}_0)]$. Since the right-hand-side of this inequality is bounded, $\lim_{n\to\infty} \varepsilon_n = 0$.

For each n, let $f_n : A(s_0, w_0) \to (-\infty, \infty)$ be defined by

$$f_n(a) = r(s_0, w_0, a) - R_0(\underline{a}_n, s_0) + \delta \min_{\pi\in\Pi_T(s_0)} E_\pi V(\underline{a}_n, s_1, h(s_1, w_0, a)) .$$

Similarly, let f be defined by the same formula with $\underline{a}_n$ replaced by $\bar{\underline{a}}$. Notice that $\varepsilon_n = f_n(\underline{a}_{n+1}(s_0)) = \max_{a\in A} f_n(a)$, for all n.

There is a subsequence $\underline{a}_{n(k)}$, $k = 1, 2, \ldots$ such that $\lim_{k\to\infty} \underline{a}_{n(k)}(s) = \bar{\underline{a}}(s)$, for all s. Then, $\lim_{k\to\infty} f_{n(k)}(a) = f(a)$, for all a.

Suppose that $\bar{\underline{a}}$ is not maximal. Then, $V(\bar{\underline{a}}, s_0, w_0) > 0$, so that $f(a) > 0$, for some a. Therefore, $\varepsilon_{n(k)} \geq f_{n(k)}(s) > \frac{1}{2}f(a)$, for k sufficiently large. This contradicts $\lim_{k\to\infty} \varepsilon_{n(k)} = 0$. This proves that $\bar{\underline{a}}$ is maximal.

It remains to be shown that $\bar{\underline{a}}$ dominates $\underline{a}_0$ when $\bar{\underline{a}} \neq \underline{a}_0$. If $\bar{\underline{a}} = \underline{a}_N$, for some N, then $\min_{\pi\in\Pi} E_\pi[\hat{r}(\bar{\underline{a}}) - \hat{r}(\underline{a}_0)] \geq \sum_{n=0}^{N-1} \varepsilon_n > 0$, so that $\bar{\underline{a}}$ dominates $\underline{a}_0$. If $\bar{\underline{a}} = \lim_{k\to\infty} \underline{a}_{n(k)}$, for some subsequence $n(k)$, then $\min_{\pi\in\Pi} E_\pi[\hat{r}(\bar{\underline{a}}) - \hat{r}(\underline{a}_0)] = \lim_{k\to\infty} \min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}_{n(k)}) - \hat{r}(\underline{a}_0)]$, and for all k such that $n(k) > 0$, $\min_{\pi\in\Pi} E_\pi[\hat{r}(\underline{a}_{n(k)}) - \hat{r}(\underline{a}_0)] \geq \varepsilon_0 > 0$. Q.E.D.

6. First Order Conditions for Type II Problems

Subgradients of the maxmin value function give rise to first order conditions just as they do in the usual dynamic programming.

Theorem 6.1. A program $\underline{a}$ is maximal if and only if for each s_t there exists $p(s_t) \in R_+^N$ and $\pi_T(\cdot|s_t) \in \Pi_T(s_t)$ such that, for all s_t,

1) $p(s_t)$ is a subgradient of $V(\underline{a}, s_t, w)$ at $w = w_t(\underline{a}, s_t)$ and

2) $\underline{a}(s_t)$ solves $\max\limits_{a \in A(s_t, w_t(\underline{a}, s_t))} [r(s_t, w_t(\underline{a}, s_t), a)$

$$+ \delta E_{\pi_T(\cdot|s_t)}(p(s_{t+1}) \cdot h(s_{t+1}, w_t(\underline{a}, s_t), a))] .$$

The following example should clarify the meaning of the theorem. The example represents the saving problem of an immortal consumer.

Example. Let $W = [0,\infty)$, $A(s,w) = [0,w]$, $r(s_t, w, a) = u(s_t, a)$, and $h(s_{t+1}, w, a) = (1 + R(s_{t+1}))(w-a) + y(s_{t+1})$, where $y(s_{t+1}) > 0$, for all s_{t+1}. Also, w is wealth, a is consumption and $w-a$ is saving. $R(s_{t+1})$ is the real interest rate on saving, and u is the utility function.

A program $\underline{a}$ is maximal if and only if for each s_t, there exists $\lambda(s_t)$ such that for all s_t,

1) $\frac{du}{da}(s_t, \underline{a}(s_t)) \leq \lambda(s_t)$, with equality if $\underline{a}(s_t) > 0$, and

2) $\underline{\lambda}(s_t) \leq \lambda(s_t)$, and $\lambda(s_t) \leq \overline{\lambda}(s_t)$ if $\underline{a}(s_t) < w_t(\underline{a}, s_t)$, where

$$\underline{\lambda}(s_t) = \delta \min_{\pi_T(\cdot|s_t) \in \Pi_T(s_t)} E_{\pi_T(\cdot|s_t)}((1 + R(s_{t+1}))\lambda(s_{t+1})) , \text{ and}$$

$$\overline{\lambda}(s_t) = \delta \max_{\pi_T(\cdot|s_t) \in \Pi_T(s_t)} E_{\pi_T(\cdot|s_t)}((1 + R(s_{t+1}))\lambda(s_{t+1})) .$$

The numbers $\overline{\lambda}(s_t)$ and $\underline{\lambda}(s_t)$ are, respectively, upper and lower marginal utilities of saving.

Proof of Theorem. Suppose that $\underset{\sim}{a}$ is maximal. By theorem 3.2, there is $\pi \in \Pi$ such that $\underset{\sim}{a}$ solves $\max_{\underset{\sim}{a}' \in \underset{\sim}{A}} E_\pi \hat{r}(\underset{\sim}{a}')$. As in Section 4, for each s_t, let $\pi_T(\cdot|s_t) \in \Pi_T(s_t)$ for the transition probability defined by π. For each $(s,w) \in S \times W$, let $V_\pi(s,w) = \max\{E_\pi \hat{r}(\underset{\sim}{a}') | \underset{\sim}{a}' \in \underset{\sim}{A}(s,w)\}$. By the usual first order conditions for a concave programming problem, there exists for each s_t, $p(s_t) \in R^N_+$ such that for all s_t,

1) $p(s_t)$ is a subgradient of $V_\pi(s_t, w)$ at $w = w_t(\underset{\sim}{a}, s_t)$ and

2) $\underset{\sim}{a}(s_t)$ solves $\max_{a \in A(s_t, w_t(\underset{\sim}{a}, s_t))} [r(s_t, w_t(\underset{\sim}{a}, s_t), a)$

$$+ \delta E_{\pi_T(\cdot|s_t)}(p(s_{t+1}) \cdot h(s_{t+1}, w_t(\underset{\sim}{a}, s_t), a))] .$$

That $p(s_t)$ is a subgradient of $V(\underset{\sim}{a}, s_t, w)$ at $w = w_t(\underset{\sim}{a}, s_t)$ follows from the following inequalities,

$$p(s_t) \cdot (w - w(\underset{\sim}{a}, s_t)) \geq V_\pi(s_t, w) - V_\pi(s_t, w_t(\underset{\sim}{a}, s_t))$$

$$= \max_{\underset{\sim}{a}' \in \underset{\sim}{A}(s_t, w)} E_\pi[\hat{r}(\underset{\sim}{a}') - \hat{R}(\underset{\sim}{a}, s_t)]$$

$$\geq \max_{\underset{\sim}{a}' \in \underset{\sim}{A}(s_t, w)} \min_{\pi' \in \Pi(s_t)} E_{\pi'}[\hat{r}(\underset{\sim}{a}') - \hat{R}(\underset{\sim}{a}, s_t)]$$

$$= V(\underset{\sim}{a}, s_t, w) = V(\underset{\sim}{a}, s_t, w) - V(\underset{\sim}{a}, s_t, w_t(\underset{\sim}{a}, s_t)) .$$

This completes the proof that the conditions of the theorem are necessary for maximality.

Now assume that conditions 1 and 2 of the theorem apply. Fix $s_t \in S$ and let $\underset{\sim}{a}' \in \underset{\sim}{A}(s_t, w_t(\underset{\sim}{a}, s_t))$. Then,

$$\min_{\pi\in\Pi(s_t)} E_\pi[\hat{r}(\underline{a}') - \hat{R}(\underline{a},\ s_t,\ w_t(\underline{a},\ s_t))]$$

$$\leq r(s_t,\ w_t(\underline{a},\ s_t),\ \underline{a}'(s_t)) - r(s_t,\ w_t(\underline{a},\ s_t),\ \underline{a}(s_t))$$

$$+ \delta E_{\pi_T(\cdot|s_t)} V(\underline{a},\ s_{t+1},\ h(s_{t+1},\ w_t(\underline{a},\ s_t),\ \underline{a}'(s_t)))$$

$$\leq r(s_t,\ w_t(\underline{a},\ s_t),\ \underline{a}'(s_t)) - r(s_t,\ w_t(\underline{a},\ s_t),\ \underline{a}(s_t))$$

$$+ \delta E_{\pi_T(\cdot|s_t)} [V(\underline{a},\ s_{t+1},\ h(s_{t+1},\ w_t(\underline{a},\ s_t),\ \underline{a}(s_t))$$

$$+ p(s_{t+1})\cdot(h(s_{t+1},\ w_t(\underline{a},\ s_t),\ \underline{a}'(s_t)) - h(s_{t+1},\ w_t(\underline{a},\ s_t),\ \underline{a}(s_t)))]$$

$$\leq \delta E_{\pi_T(\cdot|s_t)} V(\underline{a},\ s_{t+1},\ w_{t+1}(\underline{a},\ s_{t+1}))\ .$$

The second and third inequalities above follow from conditions 1 and 2, respectively. It now follows from the definition of $V(\underline{a})$ that $V(\underline{a},\ s_t,\ w_t(\underline{a},\ s_t)) \leq \delta E_{\pi_T(\cdot|s_t)} V(\underline{a},\ s_{t+1},\ w_{t+1}(\underline{a},\ s_{t+1}))$. Since the numbers $V(\underline{a},\ s_t,\ w_t(\underline{a},\ s_t))$, for $s_t \in S$, are non-negative and bounded, it must be that $V(\underline{a},\ s_0,\ w_0) = 0$. Hence, $\underline{a}$ is maximal. Q.E.D.

7. Recursivity in Problems of Type I

In order to define recursivity for problems of type I, I assume the following.

Assumption 7.1. For all t and $x_t \in X_t$, there is a program $\underline{a} \in \underline{A}$ such that $\pi(\omega|x_t(\underline{a},\omega) = x_t) > 0$, for all $\pi \in \Pi$.

By assumption 2.1, $\pi(\omega|x_t(\underline{a},\omega) = x_t)$ is independent of $\underline{a}$. Define $\pi(x_t)$ to be $\pi(\omega|x_t(\underline{a},\omega) = x_t)$ for some $\underline{a} \in \underline{A}$ such that this quantity is positive. Also if $a \in A(x_t)$, let $\pi_T(x_{t+1}|x_t,\ a) = \pi(\omega|x_t(\underline{a},\omega) = x_t$ and $\omega(x_t,\ a) = x_{t+1})(\pi(x_t))^{-1}$, where $\underline{a}$ is some program such that

$\pi(\omega | x_t(\underline{a},\omega) = x_t) > 0$. Again by assumption 2.1, the transition probability $\pi_T(x_{t+1} | x_t, a)$ is well-defined. Let $\pi_T(\cdot | x_t)$ be the function, f , from $A(x_t)$ to the set of probability measures on $I(x_t)$ defined by $f(a) = \pi_T(\cdot | x_t, a)$. For each $x \in X$, let $\Pi_T(x) = \{\pi_T(\cdot | x) : \pi \in \Pi\}$. Finally, let $F : \Pi \to \times_{x \in X} \Pi_T(x)$ be defined by $F(\pi)_x = \pi(\cdot | x)$.

<u>Definition 7.2</u>. Π is recursive if F is subjective.

Just as in the case of problems of type II, Π may contain a recursive subset even if it itself is not recursive.

There is an alternative definition of recursivity which is very similar to that of Section 4. Let $\Omega_t = \times_{x \in X_t} G(x)$, and, for $t > 0$, let $S_t = \{(\omega_0, \ldots, \omega_{t-1}) | \omega_n \in \Omega_n, \text{ for all } n\}$. Letting s_0 be an arbitrary point, one can define, in an obvious way, $S = \{s_0\} \cup \bigcup_{t=1}^{\infty} S_t$ to be a tree with root s_0 . One could define Π to be recursive if it were recursive relative to S in the sense of definition 4.2. This definition of recursivity is not useful because one cannot observe the s_t in S . If one could observe the s_t , one could use this definition of recursivity to define maxmin programming. Neither of the alternative definitions of recursivity implies the other.

8. Maximality by Approximation in Problems of Type I

The object of this section is to show that under certain conditions one may realize a maximal program for a type I problem by making a finite computation at each state at which one arrives. The intuition behind the result is that if one does not know how to evaluate precisely probabilities of future events, then it is pointless to carry contingent planning beyond a certain level of detail.

Assume that assumptions 2.1 and 7.1 apply. More notation is required in order to describe two additional assumptions. Let $I(x,a)$ be the set of those immediate successors of x which occur with positive probability if action $a \in A(x)$ is taken. By assumption 7.1, $I(x,a)$ is well-defined. Let $\Delta^{I(x,a)}$ be the set of all probability measures on $I(x,a)$ and let $\Delta(x) = \times_{a \in A(x)} \Delta^{I(x,a)}$. This set contains $\Pi_I(x)$. Give $\Delta(x)$ the usual topology as a subset of a Euclidean space.

Assumption 8.1. For all x, $\Pi_I(x)$ has non-empty interior in the topology of $\Delta(x)$.

This assumption asserts that the decision maker is uncertainty averse and is uncertain about all possible random events.

For each $x \in X$ there is a decision subproblem, $P(x)$, with initial state x. This subproblem is defined much as $P(s_t, w)$ is defined at the beginning of Section 5. The notation applying to $P(x)$ is like that of Section 5. The set of states is $X(x) = \{x' \mid x' \text{ equals or follows } x\}$. The set of probabilistic states is $\Omega(x) = \times_{x' \in X(x)} G(x')$, and $M(x)$ is the set of measurable subsets of $\Omega(x)$. $M(x)$ may be considered to be a subset of M by means of the natural projection from Ω to $\Omega(x)$. By assumptions

2.1 and 7.1, $\pi(x)$ is well-defined in $\pi \in \Pi$, so that $\pi(C|x) = \pi(C)(\pi(x))^{-1}$ is well-defined, for $C \in M(x)$. The set of personal probabilities for $P(x)$ is $\Pi(x) = \{\pi(\cdot|x) | \pi \in \Pi\}$. The set of deterministic programs is $\underline{A}(x)$. If $\underline{a} \in \underline{A}(x)$ and $\omega \in \Omega(x)$, the total reward is $\hat{r}(x,\underline{a},\omega)$. If $\pi \in \Pi(x)$, let $V_\pi(x) = \max\{E_\pi \hat{r}(x,\underline{a}) | \underline{a} \in \underline{A}(x)\}$. $V_\pi(x)$ is the value of $P(x)$ if the personal probability measure π is used to evaluate programs. If x' follows x and $\pi \in \Pi(x)$, then $\pi(\cdot|x') \in \Pi(x')$ is well-defined.

<u>Assumption 8.2</u>. For all $(x,a) \in$ graph A and $\pi \in \Pi(x)$, $V_{\pi(\cdot|x')}(x')$ is not constant as x' varies over $I(x,a)$.

This assumption asserts that random variations always matter.

I now define what it is for a program to be calculable. If $x_t \in X$ and n is a positive integer, let $X(x_t, n) = \{x_{t+k} \in X(x_t) | 0 \le k \le n\}$. There corresponds to $X(x_t, n)$ a decision subproblem, $P(x_t, n)$, with states $X(x_t, n)$. $P(x_t, n)$ is defined much as $P(x_t)$ is defined. Because $P(x_t, n)$ has only finitely many deterministic programs, a maximal program for $P(x_t, n)$ can be calculated in finitely many steps. A program $\bar{\underline{a}} \in \underline{A}$ is <u>calculable</u> if at each state x_t, there is a finite procedure for choosing a positive integer $n(x_t)$ and a program $\underline{a}_{x_t}$ for $P(x_t, n(x_t))$ such that $\bar{\underline{a}}(x_t) = \underline{a}_{x_t}(x_t)$.

The procedure used to calculate $\bar{\underline{a}}$ ignores distant future states. The procedure could be improved by ignoring states of very low probability as well.

<u>Theorem 8.1</u>. If assumptions 8.1 and 8.2 apply and Π is recursive in the sense of definition 7.2, then there exists a calculable maximal program $\bar{\underline{a}}$.

Proof. First of all, I need notation applying to the problem $P(x_t, n)$. The set of probabilistic states is $\Omega(x_t, n) = \times\{G(x) | x \in X(x_t, n)\}$. The sets of personal probabilities and of deterministic programs are denoted, respectively, by $\Pi(x_t, n)$ and $\underline{A}(x_t, n)$. If $\underline{a} \in \underline{A}(x_t, n)$ and $\omega \in \Omega(x_t, n)$ then $\hat{r}(x_t, n, \underline{a}, \omega)$ denotes the total reward. If $\pi \in \Pi(x_t, n)$, then $V_\pi(x_t, n) = \max\{E_\pi \hat{r}(x_t, n, \underline{a}) | \underline{a} \in \underline{A}(x_t, n)\}$. If $\pi \in \Pi(x_t, n)$, then $\pi_T(\cdot|x, a)$ denotes the corresponding vector of transition probabilities.

The pair $(P(x_t, n), \pi)$, where $\pi \in \Pi(x_t, n)$, is said to be satisfactory if it meets the following conditions.

Condition 8.3. For all $x_{t+k} \in X(x_t, n)$ with $k < n$, $\pi_T(\cdot|x_{t+k})$ belongs to the interior of $\Pi_T(x_{t+k})$.

Condition 8.4. If $\underline{a} \in \underline{A}(x_t, n)$ solves $V_\pi(x_t, n) = E_\pi \hat{r}(x_t, n, \underline{a})$, then there is $\pi'_T(\cdot|x_t) \in \Pi_T(x_t)$ such that $\pi'_T(\cdot|x_t, \underline{a}(x_t)) = \pi_T(\cdot|x_t, \underline{a}(x_t))$ and $V_\pi(x_t, n) - B(1-\delta)^{-2}\delta^n$

$$> r(x_t, a) + \delta \sum_{x_{t+1}} \pi'_T(x_{t+1}|x_t, a) V_{\pi(\cdot|x_{t+1})}(x_{t+1}, n-1), \text{ for all}$$

$a \neq \underline{a}(x_t)$, where $B > \sup\{r(x,a) | (x,a) \in \text{graph } A\}$.

Lemma 8.2. Fix $\bar{\pi}_T(\cdot|x_t)$ in the interior of $\Pi_T(x_t)$. If $(P(x_t, n), \pi)$ is such that $\pi_T(\cdot|x_t) = \bar{\pi}_T(\cdot|x_t)$, then $(P(x_t, n), \pi)$ satisfies condition 8.4 for all n sufficiently large.

Proof. It is enough to show that if $(P(x_t, n_k), \pi_k)$ is any sequence such that $\pi_{kT}(\cdot|s_t) = \bar{\pi}_T(\cdot|x_t)$, for all k, and $\lim_{k\to\infty} n_k = \infty$, then there is a subsequence, denoted $(P(x_t, n_k), \pi_k)$ again, such that $(P(x_t, n_k), \pi_k)$ satisfies condition 8.4 for k sufficiently large. By the compactness of

$\Pi(s_t)$, one may assume that $\lim_{k\to\infty} \pi_{kT}(\cdot|x) = \pi_T(\cdot|x)$, for all $x \in X(x_t)$, where $\pi \in \Pi(x_t)$. Therefore, $\lim_{k\to\infty} V_{\pi_t}(x_{t+1}, n_k - 1) = V_\pi(x_{t+1})$, for any x_{t+1} following x_t . Let $\underline{a}_k \in \underline{A}(x_t, n_k)$ satisfy $V_{\pi_t}(x_t, n_k) = E_{\pi_k}\hat{r}(x_t, n_k, \underline{a}_k)$ and let $\underline{a} \in \underline{A}(x_t)$ satisfy $V_\pi(x_t) = E_\pi \hat{r}(x_t, \underline{a})$. Since $A(x_t)$ is finite, one may assume that $\underline{a}_k(x_t) = \underline{a}(x_t)$, for all k .

By assumption 8.2, $V_{\pi(\cdot|x_{t+1})}(x_{t+1})$ is not constant on $I(x_t, a)$, for all $a \in A(x_t)$. Since $\pi_T(\cdot|x_t) = \bar{\pi}_T(\cdot|x_t)$ belongs to the interior of $\Pi_T(x_t)$, there is $\pi'_T(\cdot|x_t) \in \Pi_T(x_t)$ such that $\pi'_T(\cdot|x_t, \underline{a}(x_t)) = \bar{\pi}_T(\cdot|x_t, \underline{a}(x_t))$ and for some $\varepsilon > 0$,

$$V_\pi(x_t) - \varepsilon \geq \sum_{x_{t+1}} \pi'_T(x_{t+1}|x_t, a)V_{\pi(\cdot|x_{t+1})}(x_{t+1}) , \quad \text{for all } a \neq \underline{a}(x_t) .$$

Hence, for sufficiently large k , $V_{\pi_k}(x_t, n_k) - B(1-\delta)^{-2}\delta^{n_k}$

$$\geq \sum_{x_{t+1}} \pi'_T(x_{t+1}|x_t, a)V_{\pi_k(\cdot|x_{t+1})}(x_{t+1}, n_k - 1) , \quad \text{for all}$$

$a \neq \underline{a}(x_t) = \underline{a}_k(x_t)$.

This completes the proof of lemma 8.2.

The pair $(P(x_t, n'), \pi')$ is said to <u>extend</u> $(P(x_t, n), \pi)$ if $n' > n$ and π equals π' restricted to $\Omega(x_t, n)$. Assumption 8.1 and lemma 8.2 imply that one can find in finitely many steps a satisfactory extension of any pair $(P(x_t, n), \pi)$ which satisfies condition 8.3.

The procedure for calculating a maximal program is as follows. Choose $\pi_T(\cdot|x_0)$ in the interior of $\Pi_T(x_0)$. By assumption 8.1, such a choice is possible. Let $(P(x_0, n(x_0)), \pi_{x_0})$ be a satisfactory extension of $(P(x_0, 1), \pi_T(\cdot|x_0))$.

One now constructs by induction on t satisfactory pairs $(P(x_t, n(x_t)), \pi_{x_t})$ such that whenever x_{t+1} follows x_t ,

$(P(x_{t+1}, n(x_{t+1})), \pi_{x_{t+1}})$ extends $(P(x_{t+1}, n(x_t) - 1), \pi_{x_t}(\cdot|x_{t+1}))$. The induction step is as follows. Suppose the construction has been made for x_k with $k \leq t$. Let x_{t+1} follow x_t. Because π_{x_t} satisfies condition 8.3, $\pi_{x_t}(\cdot|x_{t+1})$ does so as well. Let $(P(x_{t+1}, n(x_{t+1})), \pi_{x_{t+1}})$ be a satisfactory extension of $(P(x_{t+1}, n(x_t) - 1), \pi_{x_t}(\cdot|x_{t+1}))$.

For each x_t, let $\underline{a}_{x_t} \in \underline{A}(x_t, n(x_t))$ be optimal with respect to π_{x_t}. Let $\bar{\underline{a}} \in \underline{A}$ be defined by $\bar{\underline{a}}(x_t) = \underline{a}_{x_t}(x_t)$.

I next show that the calculable program $\bar{\underline{a}}$ just defined is maximal. For each x_t, let $\pi'_T(\cdot|x_t)$ be a transition probability satisfying condition 8.4 for $(P(x_t, n(x_t)), \pi_{x_t})$. By the recursivity of Π, there is $\bar{\pi} \in \Pi$ such that $\bar{\pi}_T(\cdot|x_t) = \pi'_T(\cdot|x_t)$, for all x_t. It is sufficient to show that $\bar{\underline{a}}$ is optimal with respect to $\bar{\pi}$. By the standard theorem of dynamic programming, $\bar{\underline{a}}$ is optimal with respect to $\bar{\pi}$ if

8.5) for all x_t and t, $E_{\bar{\pi}(\cdot|x_t)}\hat{r}(x_t,\bar{\underline{a}})$

$$= \max_{a \in A(x_t)} [r(x_t,a) + \delta \sum_{x_{t+1}} \bar{\pi}_T(x_{t+1}|x_t,a) E_{\bar{\pi}(\cdot|x_{t+1})}\hat{r}(x_{t+1},\bar{\underline{a}})] .$$

In order to prove equation 8.5, I need the following lemma. Let $\pi \in \Pi$ be such that $\pi_T(\cdot|x_t) = \pi_{x_t T}(\cdot|x_t)$, for all x_t. Since Π is recursive, π exists.

<u>Lemma 8.3</u>. For any x_t and any $x_{t+1} \in X(x_t, n(x_t))$,

$E_{\pi(\cdot|x_{t+k})}\hat{r}(x_{t+k}, \bar{\underline{a}}) \geq V_{\pi_{x_t}(\cdot|x_{t+1})}(x_{t+k}, n(x_t) - k)$, and

$E_{\pi(\cdot|x_{t+k})}\hat{r}(x_{t+k}, \bar{\underline{a}}) - V_{\pi_{x_t}(\cdot|x_{t+k})}(x_{t+k}, n(x_t) - k) \leq B(1-\delta)^{-2}\delta^{n(x_t)-k}$.

Proof of Lemma. For notational convenience, the proof is given for the case $k = 0$. Let the increasing sequence $X_0', X_1', \ldots$ of subtrees of $X(x_t)$ be defined as follows by induction on t. $X_0' = X(x_t, n(x_t))$. Given X_n', $X_{n+1}' = X_n' \cup \cup(X(x_{t+n+1}, n(x_{t+n+1})) | x_{t+n+1} \in X_n')$. Let $\underset{\sim}{a}_n$ be the program on X_n' defined as follows. $\underset{\sim}{a}_n(x_{t+k}) = \bar{\underset{\sim}{a}}(x_{t+k})$, if $k \leq n$. For $k > n$, $\underset{\sim}{a}_n(x_{t+k}) = \underset{\sim}{a}_{x_{t+n}}(x_{t+k})$, where x_{t+n} is the element of X_{t+n} preceding x_{t+k}. Notice that $\underset{\sim}{a}_0 = \underset{\sim}{a}_{x_t}$.

It should be clear that $V_{\pi_{x_t}}(x_t, n(x_t)) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_0)$, that $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_n) \leq E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_{n+1})$, for all n, and that $\lim_{n\to\infty} E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_n) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$. Also, $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_{n+1}) - E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}}) \leq B(1-\delta)^{-1}\delta^{n(x_t)+n}$. Therefore, $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}) - V_{\pi_{x_t}}(x_t, n(x_t)) \leq B(1-\delta)^{-2}\delta^{n(x_t)}$.

This completes the proof of lemma 8.3.

I may now prove equation 8.5 and hence the theorem. By condition 8.4 and lemma 8.3,

8.6) for all x_t and t, $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$

$$> r(x_t, a) + \delta \sum \pi_T'(x_{t+1}|x_t, a)E_{\pi(\cdot|x_{t+1})}\hat{r}(x_{t+1}, \bar{\underset{\sim}{a}}),$$

for a in $\underset{\sim}{A}(x_t)$ not equal to $\bar{\underset{\sim}{a}}(x_t)$.

Recall that $\bar{\pi}_T(\cdot|x_t) = \pi_T'(\cdot|x_t)$ and $\pi_T(\cdot|x_t) = \pi_{x_tT}(\cdot|x_t)$, for all x_t. Since by condition 8.4, $\pi_{x_tT}(\cdot|x_t, \bar{\underset{\sim}{a}}(x_t)) = \pi_T'(\cdot|x_t, \bar{\underset{\sim}{a}}(x_t))$, it follows that $E_{\bar{\pi}(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}}) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$, for all x_t and t.

Proof of Lemma. For notational convenience, the proof is given for the case $k = 0$. Let the increasing sequence X_0', X_1', ... of subtrees of $X(x_t)$ be defined as follows by induction on t . $X_0' = X(x_t, n(x_t))$. Given X_n', $X_{n+1}' = X_n' \cup \cup(X(x_{t+n+1}, n(x_{t+n+1}))|x_{t+n+1} \in X_n')$. Let $\underset{\sim}{a}_n$ be the program on X_n' defined as follows. $\underset{\sim}{a}_n(x_{t+k}) = \bar{\underset{\sim}{a}}(x_{t+k})$, if $k \leq n$. For $k > n$, $\underset{\sim}{a}_n(x_{t+k}) = \underset{\sim}{a}_{x_{t+n}}(x_{t+k})$, where x_{t+n} is the element of X_{t+n} preceding x_{t+k} . Notice that $\underset{\sim}{a}_0 = \underset{\sim}{a}_{x_t}$.

It should be clear that $V_{\pi_{x_t}}(x_t, n(x_t)) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_0)$, that $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_n) \leq E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_{n+1})$, for all n , and that $\lim_{n\to\infty} E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_n) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$. Also, $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}_{n+1}) - E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}}) \leq B(1-\delta)^{-1}\delta^{n(x_t)+n}$. Therefore, $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \underset{\sim}{a}) - V_{\pi_{x_t}}(x_t, n(x_t)) \leq B(1-\delta)^{-2}\delta^{n(x_t)}$.

This completes the proof of lemma 8.3.

I may now prove equation 8.5 and hence the theorem. By condition 8.4 and lemma 8.3,

8.6) for all x_t and t , $E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$

$$> r(x_t, a) + \delta \; \Sigma \; \pi_T'(x_{t+!}|x_t, a)E_{\pi(\cdot|x_{t+1})}\hat{r}(x_{t+1}, \bar{\underset{\sim}{a}}) ,$$

for a in $\underset{\sim}{A}(x_t)$ not equal to $\bar{\underset{\sim}{a}}(x_t)$.

Recall that $\bar{\pi}_T(\cdot|x_t) = \pi_T'(\cdot|x_t)$ and $\pi_T(\cdot|x_t) = \pi_{x_tT}(\cdot|x_t)$, for all x_t . Since by condition 8.4, $\pi_{x_tT}(\cdot|x_t, \bar{\underset{\sim}{a}}(x_t)) = \pi_T'(\cdot|x_t, \bar{\underset{\sim}{a}}(x_t))$, it follows that $E_{\bar{\pi}(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}}) = E_{\pi(\cdot|x_t)}\hat{r}(x_t, \bar{\underset{\sim}{a}})$, for all x_t and t .

Substituting $\bar{\pi}$ for π in inequality 8.6, one obtains equation 8.5.

This completes the proof of the theorem. Q.E.D.

The procedure just described for calculating a maximal program reminds one of Simon's (1955, 1959) satisficing. A decision maker following the procedure outlined would not be interested in refining his calculations once he had achieved a satisfactory incomplete program. In this sense, he would behave as if he had achieved a predetermined aspiration level.

9. Maximality by Approximation in Problems of Type II

This section is devoted to an analogue of theorem 8.1 for type II problems. The main difference with the previous section is that the analogue of assumption 8.2 must be stated in terms of derivatives rather than levels, because type II problems have continuous state and action variables.

Assume that assumption 4.1 applies, so that $\Pi_T(s)$ is well-defined, for all s. $\Pi_T(s)$ is a subset of the set of all probability vectors on the immediate successors of s, call it $\Delta^{I(s)}$. The analogue of assumption 8.1 is the following.

Assumption 9.1. For all s, $\Pi_T(s)$ has non-empty interior in the usual topology on $\Delta^{I(s)}$.

If $(s,w) \in S \times W$, let $P(s,w)$ be the decision subproblem defined at the beginning of Section 5. If $\pi \in \Pi(s)$, let $V_\pi(s,w)$ be the value of $P(s,w)$ according to π, as defined in Section 6. If n is a positive integer, let $S(s_t, n) = \{s_{t+k} \in S(s_t) | 0 \leq k \leq n\}$. If $w \in W$, let $P(s_t, w, n)$ be the decision subproblem corresponding to $S(s_t, n)$ with initial state (s_t, w). The problem $P(s_t, w, n)$ is defined in the obvious way. Let $\underline{A}(s_t, w, n)$, $\Omega(s_t, n)$ and $\Pi(s_t, n)$ be, respectively, the sets of programs, probability states and personal probabilities for $P(s_t, w, n)$. If $\underline{a} \in \underline{A}(s_t, w, n)$ and $\omega \in \Omega(s_t, n)$, let $\hat{r}(s_t, w, n, \underline{a}, \omega)$ be the total return. Similarly, if $\pi \in \Pi(s_t, n)$, then $V_\pi(s_t, w, n) = \max\{E_\pi \hat{r}(s_t, w, n, \underline{a}) | \underline{a} \in \underline{A}(s_t, w, n)\}$. If $\pi \in \Pi(s)$, $V_\pi(s_t, w, n)$ is defined to be $V_{\pi'}(s_t, w, n)$, where π' is the restriction of π to $\Omega(s_t, n)$. In order to be able to deal with derivatives, the following assumptions are made.

<u>Assumption 9.2</u>. For all s , the functions r : graph $A(s,\cdot) \to [0,\infty)$ and $h : Y(s) \to W$ are continuously differentiable and there is $B_1 > 0$ such that $\|Dh(s,w,a)\| \leq B_1$, for all s , w and a .

The derivatives of r and h are denoted Dr and Dh , respectively.

<u>Assumption 9.3</u>. For every (s,w) and n and every $\pi \in \Pi(s)$, the functions $V_\pi(s,w)$ and $V_\pi(s,w,n)$ are continuously differentiable with respect to w . There are $B_2 > 0$ and $\eta > 0$ such that $\eta < 1$ and $\|DV_\pi(s,w,n)\| \leq B_2$ and $\|DV_\pi(s,w,n) - DV_\pi(s,w)\| \leq B_2\eta^n$, for all s , w , and n .

It is important that B_1 , B_2 , and η be known to the decision maker. Otherwise, he would not know how to calculate a maximal program.

Many economic programming problems satisfy assumption 9.3, though general conditions on r , h , A , W and Π guaranteeing these assumptions are awkward to state.

The analogue of assumption 8.2 is the following.

Assumption 9.4. For any $(s,w) \in S \times W$ and $\pi \in \Pi(s)$, if $\underset{\sim}{a}$ is any program for $P(s,w)$ which is optimal with respect to π, then the convex hull of $\{DV_{\pi(\cdot|s')}(s', h(s', w, \underset{\sim}{a}(s)))Dh(s', w, \underset{\sim}{a}(s)) | s' \in I(s)\}$ has non-empty interior in the affine subspace of R^{N+K} spanned by $\{p(s')Dh(s', w, \underset{\sim}{a}(s)) | s' \in I(s), p(s') \in R^N, \text{ for all } s'\}$.

The following assumption is also needed.

Assumption 9.5. For any program $\bar{a}$ and any $\pi \in \Pi$,
$\lim_{t\to\infty} \delta^t E_\pi w_t(\underset{\sim}{a}, s_t) = 0$.

A program $\bar{\underset{\sim}{a}} \in \underset{\sim}{A}$ is said to calculable if at each state (s_t, w_t) arrived at, there is a finite procedure for choosing a positive integer $n(s_t)$ and a program $\underset{\sim}{a}_{s_t}$ for $P(s_t, w_t, n(s_t))$ and if $\bar{\underset{\sim}{a}}(s_t) = \underset{\sim}{a}_{s_t}(s_t)$, for all s_t.

Theorem 9.1. If assumptions 9.1-9.5 apply and if Π is recursive in the sense of definition 4.2, then there exists a calculable maximal program $\bar{\underset{\sim}{a}}$.

Proof. The proof is much like that of theorem 8.1. In a pair $(P(s_t, w, n), \pi)$, it is understood that $\pi \in \Pi(s_t)$. Also, $(P(s_t, w, n'), \pi')$ extends $(P(s_t, w, n), \pi)$ if $n' > n$ and π equals π' restricted to $\Omega(s_t, n)$. The pair $(P(s_t, w, n), \pi)$ is satisfactory if it meets the following conditions.

Condition 9.6. For all $s_{t+k} \in S(s_t, n)$ with $k < n$, $\pi_T(\cdot|s_{t+k})$ belongs to the interior of $\Pi_T(s_{t+k})$.

Condition 9.7. Suppose that $\underline{a} \in \underline{A}(s_t, w)$ solves the equation $V_\pi(s_t, w, n) = E_\pi \hat{f}(s_t, w, n, \underline{a})$. If for each $s_{t+1} \in I(s_t)$, $p(s_{t+1}) \in R^N$ satisfies $\|p(s_{t+1}) - DV_{\pi(\cdot|s_{t+1})}(s_{t+1}, h(s_{t+1}, w, \underline{a}(s_t), n)\| < 2B_2\eta^{n-1}$, then there is $\pi'_T(\cdot|s_t) \in \Pi_T(s_t)$ such that

$$\sum_{s_{t+1}} \pi'_T(s_{t+1}|s_t)p(s_{t+1})Dh(s_{t+1}, w, \underline{a}(s_t))$$

$$= \sum_{s_{t+1}} \pi_T(s_{t+1}|s_t)DV_{\pi(\cdot|s_{t+1})}(s_{t+1}, h(s_{t+1}, w, \underline{a}(s_t)), n)Dh(s_{t+1}, w, \underline{a}(s_t)) .$$

Lemma 9.2. Fix $\bar{\pi}_T(\cdot|s_t)$ in the interior of $\Pi_T(s_t)$. If $(P(s_t, w, n), \pi)$ is such that $\pi_T(\cdot|s_t) = \bar{\pi}_T(\cdot|s_t)$, then $(P(s_t, w, n), \pi)$ satisfies condition 9.7, for all n sufficiently large.

Proof. It is enough to show that if $(P(s_t, w, n_k), \pi_k)$ is any sequence such that $\pi_{kT}(\cdot|s_t) = \bar{\pi}_T(\cdot|s_t)$, for all k, and $\lim_{k\to\infty} n_k = \infty$, then there is a subsequence, denoted $(P(s_t, w, n_k), \pi_k)$ again, such that $(P(s_t, w, n_k), \pi_k)$ satisfies condition 9.7 for k sufficiently large. Let $\underline{a}_k \in \underline{A}(s_t, w, n_k)$ solve $V_{\pi_k}(s_t, w, n_k) = E_{\pi_k} \hat{f}(s_t, w, n_k, \underline{a}_k)$. By compactness, one may assume that $\lim_{k\to\infty} \pi_{kT}(\cdot|s) = \pi_T(\cdot|s)$ and $\lim_{k\to\infty} \underline{a}_k(s) = \underline{a}(s)$, for all $s \in S(s_t)$, where $\pi \in \Pi(s_t)$ and $\underline{a} \in \underline{A}(s_t)$. Then, $V_\pi(s_t, w) = E_\pi \hat{f}(s_t, w, \underline{a})$, $\lim_{k\to\infty} DV_{\pi_k}(s_t, w, n_k) = DV_\pi(s_t, w)$ and $\lim_{k\to\infty} DV_{\pi_k}(s_{t+1}, h(s_{t+1}, w, \underline{a}_k(s_t))) = DV_\pi(s_{t+1}, h(s_{t+1}, w, \underline{a}(s_t)))$, for all $s_{t+1} \in I(s_t)$.

By assumption 9.4, there is $\varepsilon > 0$ such that if for each $s_{t+1} \in I(s_t)$, $p(s_{t+1}) \in R^N$ satisfies $\|p(s_{t+1}) - DV_\pi(s_{t+1}, h(s_{t+1}, w, \underline{a}(s_t)))\| < \varepsilon$, then there is

$\pi'_T(\cdot|s_t) \in \Pi_T(s_t)$ such that $\sum_{s_{t+1}} \pi'_T(\cdot|s_t)p(s_{t+1})Dh(s_{t+1}, w, \underline{a}(s_t))$

$= \sum_{s_{t+1}} \pi_T(\cdot|s_t)DV_{\pi(\cdot|s_t)}(s_{t+1}, h(s_{t+1}, w, \underline{a}(s_t)))Dh(s_{t+1}, w, \underline{a}(s_t))$. By assumption 9.3, it now follows that if n_k is so large that $4B_2\eta^{n_k-1} < \epsilon$, then $(P(s_t, w, n_k), \pi_k)$ satisfies condition 9.7. This completes the proof of the lemma.

The procedure for calculating a maximal program is as follows. Choose $\pi_T(\cdot|s_0)$ in the interior of $\Pi_T(s_0)$. By assumption 9.1, such a choice is possible. By assumption 9.1 and lemma 9.2, one may find in finitely many steps a satisfactory extension of $(P(s_0, w_0, 1), \Pi_T(\cdot|s_0))$, call it $(P(s_0, w_0, n(s_0)), \pi_{s_0})$. Let $\underline{a}_{s_0}$ be a program for $P(s_0, w_0, n(s_0))$ which is optimal with respect to π_{s_0} . Let $\bar{\underline{a}}(s_0) = \underline{a}_{s_0}(s_0)$. Then $w_1(\bar{\underline{a}}, s_1) = h(s_1, w_0, \bar{\underline{a}}(s_0))$ is well-defined.

One now continues by induction on t to construct a sequence of satisfactory pairs $(P(s_t, w_t(\bar{\underline{a}}, s_t), n(s_t)), \pi_{s_t})$ and the actions $\bar{\underline{a}}(s_t) = \underline{a}_{s_t}(s_t)$. These are such that if s_{t+1} follows s_t , then $(P(s_{t+1}, w_{t+1}(\bar{\underline{a}}, s_{t+1}), n(s_{t+1})), \pi_{s_{t+1}})$ extends $(P(s_{t+1}, w_{t+1}(\bar{\underline{a}}, s_{t+1}), n(s_t) - 1), \pi_{s_t}(\cdot|s_{t+1}))$. Thus, suppose by induction that $\bar{\underline{a}}(s_k)$ and $(P(s_k, w_k(\bar{\underline{a}}, s_k), n(s_k), \pi_{s_k})$ are defined for $k \leq t$. Then $w_{t+1}(\bar{\underline{a}}, s_{t+1})$ is well-defined. By assumption 9.1 and lemma 9.2, $(P(s_{t+1}, w_{t+1}(\bar{\underline{a}}, s_{t+1}), n(s_t) - 1), \pi_{s_t}(\cdot|s_{t+1}))$ has a satisfactory extension, call it $(P(s_{t+1}, w_{t+1}(\bar{\underline{a}}, s_{t+1}), n(s_{t+1})), \pi_{s_{t+1}})$. Let $\underline{a}_{s_{t+1}}$ be a program for $P(s_{t+1}, w_{t+1}(\bar{\underline{a}}, s_{t+1}), n(s_{t+1}))$ optimal with respect to $\pi_{s_{t+1}}$ and let $w_{t+1}(\bar{\underline{a}}, s_{t+1}) = h(s_{t+1}, w_t(\bar{\underline{a}}, s_t), \bar{\underline{a}}(s_t))$.

It remains to be shown that the calculable program $\bar{\underset{\sim}{a}}$ is maximal. By condition 9.7 and assumption 9.3, for each s_t, there is $\pi'_T(\cdot|s_t) \in \Pi_T(s_t)$ such that

9.8) $$\sum_{s_{t+1}} \pi'_T(s_{t+1}|s_t) DV_{\pi_{s_{t+1}}}(s_{t+1}, w_{t+1}(\bar{\underset{\sim}{a}}, s_{t+1}), n(s_{t+1})) \cdot Dh(s_{t+1}, w_t(\bar{\underset{\sim}{a}}, s_t), \bar{\underset{\sim}{a}}(s_t))$$

$$= \sum_{s_{t+1}} \pi_T(s_{t+1}|s_t) DV_{\pi_{s_t}(\cdot|s_{t+1})}(s_{t+1}, w_{t+1}(\bar{\underset{\sim}{a}}, s_{t+1}), n(s_t) - 1) \cdot Dh(s_{t+1}, w_t(\bar{\underset{\sim}{a}}, s_t), \bar{\underset{\sim}{a}}(s_t)) .$$

For each s_t and t, let $p(s_t) = DV_{\pi_{s_t}}(s_t, w_t(\bar{\underset{\sim}{a}}, s_t), n(s_t))$. Because $\underset{\sim}{a}_{s_t}$ is optimal with respect to π_{s_t} in $P(s_t, w_t(\bar{\underset{\sim}{a}}, s_t), n(s_t))$, it follows that the pair $(w_t(\bar{\underset{\sim}{a}}, s_t), \bar{\underset{\sim}{a}}(s_t))$ solves the problem

$$\max_{(w,a) \in \text{graph } A(s_t, \cdot)} [r(s_t, w, a) + \delta \sum_{s_{t+1}} \pi_{s_t T}(s_{t+1}|s_t) DV_{\pi_{s_t}(\cdot|s_{t+1})}(s_{t+1}, w_{t+1}(\bar{\underset{\sim}{a}}, s_{t+1}), n(s_t) - 1) h(s_t, w, a) - p(s_t) \cdot w] .$$

Therefore,

9.9) $(w_t(\bar{\underset{\sim}{a}}, s_t), \bar{\underset{\sim}{a}}(s_t))$ solves the problem

$$\max_{(w,a) \in \text{graph } A(s_t, \cdot)} [r(s_t, w, a) + \delta \sum_{s_{t+1}} \pi'_T(s_{t+1}|s_t) p(s_{t+1}) h(s_t, w, a) - p(s_t) \cdot w],$$

for by equation 9.8, the first order optimality conditions are the same in these two problems.

By the recursivity of Π, there exists $\bar{\pi} \in \Pi$ such that $\bar{\pi}_T(\cdot|s_t) = \pi'_T(\cdot|s_t)$, for all s_t. Conditions 9.9 are one form of the first order conditions for the problem $\max\{E_{\bar{\pi}} \hat{r}(\underset{\sim}{a}) | \underset{\sim}{a} \in \underset{\sim}{A}\}$. This problem is concave and by assumption 9.5, the transversality conditions are satisfied, so that $\bar{a}$ is optimal with respect to $\bar{\pi}$ and hence is maximal. Q.E.D.

The following example shows that the conditions of theorem 9.1 are not empty.

Example. Consider an investment problem with objective functions $E_{\pi} \sum_{t=0}^{\infty} \delta^{t} u(a_t)$, where u is differentiable, concave and increasing and $0 < \delta < 1$. Assume that $\frac{d}{da}u(0) < \infty$. The variable a_t represents consumption in period t and is the action variable. The endogenous state variable is wealth, w. If wealth is w, the set of possible actions is $[0,w]$. The set of exogenous states is the tree S, and Π is a set of probability distributions over paths in S. Assume that Π satisfies assumptions 4.1 and 9.1 and is recursive. If the current state is (s_t, w) and the action is $a \in [0,w]$, then the succeeding state is $(s_{t+1}, h(s_{t+1}, w, a)) = (s_{t+1}, b(s_{t+1})(w-a) + y(s_{t+1}))$, where $b(s_{t+1}) \geq 0$ and $y(s_{t+1}) > 0$. Assume that there is γ such that $0 < \gamma < 1$ and for every s_t, $0 < \delta \sum_{s_{t+1}} \pi_T(s_{t+1}|s_t) b(s_{t+1}) \leq \gamma$, for all $\pi_T(s_{t+1}|s_t) \in \Pi_T(s_t)$. Assume also that for every s_t, there is $s_{t+1} \in I(s_t)$ such that $b(s_{t+1}) = 0$.

This example satisfies the assumptions of theorem 9.1. In particular, if $V_{\pi}(s,w)$ is the value function, V_{π} is differentiable and $\frac{d}{dw}V_{\pi}(s,w)$ is positive. Notice that for every (s,w,a), $\frac{d}{dw}h(s', w, a) = 0$, for some $s' \in I(s)$ and $\frac{d}{dw}h(s', w, a) > 0$ for some other $s' \in I(s)$. Also, $dh(s', w, a)/dw = -[dh(s', w, a)/da]$, for every (s', w, a). It follows that the set of derivatives appearing in assumption 9.4 is a non-trivial interval of the line $\{(x_1, x_2) | x_2 = -x_1\}$ in R^2, so that this assumption is satisfied.

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